A SENSOR FOR STIFFNESS CHANGE SENSING BASED ON THREE WEAKLY COUPLED RESONATORS WITH ENHANCED SENSITIVITY

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ABSTRACT
This paper reports on a novel MEMS resonant sensing device consisting of three weakly coupled resonators that can achieve an order of magnitude improvement in sensitivity to stiffness change, compared to current state-of-the-art resonator sensors with similar size and resonant frequency. In a 3 degree-of-freedom (DoF) system, if an external stimulus causes change in the spring stiffness of one resonator, mode localization occurs, leading to a drastic change of mode shape, which can be detected by measuring the modal amplitude ratio change. A 49 times improvement in sensitivity compared to a previously reported 2DoF resonator sensor, and 4 orders of magnitude enhancement compared to a 1DoF resonator sensor has been achieved.

INTRODUCTION
Over the last couple of decades, micro- and nano-fabricated resonant devices have been widely used to sense small changes in the properties of the resonator [1]. Among these, sensing devices that detect stiffness change have been used for many applications, such as accelerometers [2], imaging microscopy [3] and others.

For sensing a change in stiffness, an amplitude modulation sensing paradigm with two weakly coupled resonators [4] was previously proposed to enhance the sensitivity compared to conventional single resonator sensors with frequency shift as output [5]. By combining two identical resonators and a weak coupling element in between, the change in mode shapes is more pronounced than the shift in frequency for the same stiffness perturbation [6].

The device reported here employed a novel approach based on three weakly coupled resonators arranged in a chain. Unlike previous work using 2DoF resonators, for which identical resonators were used, we intentionally designed the suspension system of the middle resonator stiffer than that of the other two identical resonators; in this way, an enhancement in sensitivity could be achieved [7].

THEORY
System Model
The lumped parameter block diagram of a 3DoF resonator sensing device is shown in Fig. 1. Each resonator is modelled as a mass and spring; damping is neglected for the analysis. The springs between the resonators are the coupling springs.

FIGURE 1: Mass-damper-spring lumped parameter model of a 3DoF resonator sensing device

Suppose the mass of all resonators are identical, i.e. M1=M2=M3=M, the two coupling springs are also identical, Kc1=Kc2=Kc, whereas the spring stiffness of the resonators are asymmetrical with K1=K, K3=K+ΔK. In addition, the stiffness of the resonator in the middle is K2. Further, assuming all springs are linear, and no movement in the y and z-axis, the equations of motions in the x-axis after Laplace transform are given by:

\[ H_1(s)X_1(s) = F_1(s) + KcX_2(s) \]  
\[ H_2(s)X_2(s) = F_2(s) + Kc[X_1(s) + X_3(s)] \]  
\[ H_3(s)X_3(s) = F_3(s) + KcX_2(s) \]

where the transfer functions are defined as:

\[ cKMKsHsKc22)(1++= \]  
\[ cKMKsHsKc222)(2++= \]  
\[ \Delta KcKMKsHsKc222)(3+= \]

If the system is actuated by F1(s) only, the displacement X1(s) and X3(s) can be computed as a function of F1(s):

\[ X_1(s) = \frac{F_1(s)[H_2(s)H_3(s) - K_c^2]}{H_1(s)H_2(s)H_3(s) - K_c^2[H_1(s) + H_3(s)]} \]  
\[ X_3(s) = \frac{F_1(s)K_c^2}{H_1(s)H_2(s)H_3(s) - K_c^2[H_1(s) + H_3(s)]} \]

In the ideal case with negligible damping and ΔK=0, there are three distinctive modes: in the first mode, all three resonators vibrate in-phase; in the second mode, resonators 1 and 3 are out-of-phase, with the resonator in the middle being stationary; in the third mode, resonators 1 and 3 are in-phase, but are out-of-phase with respect to resonator 2 [8]. When a perturbation in stiffness is introduced, ΔK≠0, the three modes are disturbed resulting in amplitude changes and mode localization occurs [9]. The modes of interest in this work are the first two modes.
due to higher sensitivity than the third mode, which will be referred to as in-phase and out-of-phase modes, respectively.

In this work, the amplitude ratio \( \frac{X_1(s)}{X_3(s)} \) is used to gauge the mode localization caused by stiffness perturbation.

**Amplitude Ratio and Sensitivity Analysis**

Assuming a weak coupling stiffness of \( K_c < K/10 \) and the stiffness of resonator 2 being more than twice than that of resonator 1, so that the following condition is satisfied:

\[
|K_c| < \frac{K_2 - K}{10}
\]  

(9)

Let \( s = j\omega \), the frequencies of the in-phase and out-of-phase modes can be approximated as:

\[
\omega_{ip} = \sqrt{\frac{K + K_c + \frac{1}{2}(\Delta K - \alpha - \sqrt{\Delta K^2 + \alpha^2})}{M}}
\]  

(10)

\[
\omega_{op} = \sqrt{\frac{K + K_c + \frac{1}{2}(\Delta K - \alpha + \sqrt{\Delta K^2 + \alpha^2})}{M}}
\]  

(11)

where \( \omega_{ip} \) and \( \omega_{op} \) denote the frequencies of the in-phase and out-of-phase modes, respectively, and

\[
\alpha = \frac{2K_c^2}{K_2 - K + K_c}
\]  

(12)

Substituting (10) and (11) into (7) and (8), we can estimate the amplitude ratios for the in-phase and out-of-phase modes as:

\[
\frac{X_1(j\omega_{ip})}{X_3(j\omega_{ip})} \approx \frac{\sqrt{2\gamma_3(\Delta K/K)^2 + 4 + \gamma_3(\Delta K/K)}}{2}
\]  

(13)

\[
\frac{X_1(j\omega_{op})}{X_3(j\omega_{op})} \approx \frac{\sqrt{2\gamma_3(\Delta K/K)^2 + 4 - \gamma_3(\Delta K/K)}}{2}
\]  

(14)

where,

\[
\gamma_3 = \frac{2K}{\alpha} = \frac{K_2 - K + K_c}{K_c^2}
\]  

(15)

To verify the results, an equivalent electrical RLC model was constructed as shown in Figure 2 [8].

![Figure 2: Equivalent RLC model of a 3DoF resonator system](image)

The electrical model was simulated using values listed in Table 1 representing our designed device. Small resistors were added so that the PSpice simulation converges. The resulting simulated quality factor was 10\(^2\), which is a good approximation of the undamped system. The simulated resonant frequencies are compared to theoretical values calculated using (10) and (11) in Figure 3, and the theoretical amplitude ratios computed by (13) and (14) are verified in Figure 4.

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
<th>Mechanical model equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>0.489MH</td>
<td>M</td>
</tr>
<tr>
<td>C</td>
<td>0.254mF</td>
<td>K</td>
</tr>
<tr>
<td>C_2</td>
<td>84.8aF</td>
<td>K_2/K=3</td>
</tr>
<tr>
<td>C_c</td>
<td>19.07F</td>
<td>K/K_c=75, ( \gamma_3=11324 )</td>
</tr>
<tr>
<td>R</td>
<td>0.44M(\Omega)</td>
<td>Q=10(^2)</td>
</tr>
<tr>
<td>( \omega_0 )</td>
<td>14.27kHz</td>
<td>Resonant frequency of single resonator</td>
</tr>
</tbody>
</table>

Figure 3: Simulation results showing the in-phase (black) and out-of-phase (red) mode frequencies as a function of a normalized stiffness perturbation. The theoretically calculated mode frequencies match well with the simulated values.

Figure 4: Simulated and calculated (using (13) and (14)) amplitude ratios of in-phase and out-of-phase modes as a function of normalized stiffness perturbation. The theoretically calculated amplitude ratios match well with simulated values.

It can be seen from Figures 3 and 4 that the theoretical estimations of mode frequencies and amplitude ratios match well with the simulated results (within 1%).

Due to the symmetry as shown in Figure 4, without loss of generality, the amplitude ratio of the out-of-phase mode for \( \Delta K/K < 0 \) is chosen for the following sensitivity analysis.

It can be seen from Figure 4 that for negative stiffness perturbations the amplitude ratio is approximately a linear function of stiffness perturbation. Assuming \( |\gamma_3\Delta K/K|>10 \), the mathematical amplitude ratio
The system consists of three resonators. Electrostatic springs were used as coupling elements between the resonators [6], allowing variable coupling strength. Identical bias voltages of 30V were applied to resonators 1 and 3, whereas resonator 2 was grounded, to ensure $K_{c1}=K_{c2}$. To demonstrate the sensitivity of the 3DoF device to stiffness perturbations, another variable DC voltage was applied on the electrode on the right. Actuation of the resonators was realized by applying an AC voltage to the electrode on the left. Differential motional currents were obtained through the differential sensing comb fingers attached to resonators 1 and 3. The configuration of the device for characterization is shown in Figure 7.

![Experimental set up for 3DoF sensor characterization](image)

**RESULTS AND DISCUSSION**

**Frequency Response**

The frequency response of the device was measured with various perturbation voltages applied. A typical frequency response of resonators 1 and 3 of the sensing device is shown in Figure 9. The measured 3-dB bandwidth and the quality factor of the out-of-phase mode...
were 2.40Hz and 6221, respectively. The frequency difference between the in-phase and out-of-phase modes was 4.99Hz, which is greater than twice of the 3-dB bandwidth of the out-of-phase mode, indicating weak damping, which thus can be neglected.

**Figure 9:** Typical frequency response of resonators 1 and 3, with 30V coupling voltage and 4.15V perturbation voltage.

**Sensitivity**

Upon finding the mode frequency of the out-of-phase mode, mode amplitudes of resonators 1 and 3 were averaged and recorded. The amplitude ratios of the out-of-phase mode were then computed. Figure 10 shows the measured amplitude ratio (quotient of modal amplitudes of resonators 1 and 3) at the out-of-phase mode for different stiffness perturbations with 30V coupling voltage. The measurement results are presented together with a linear fit. The linear sensitivity to normalized stiffness change extracted from the measured data was found to be 13558, whereas the theoretical calculated value was 17073. The discrepancy was due to fabrication variances. Table 1 lists a comparison of sensitivity between state-of-the-art resonator sensors (1Dof and 2DoF) for stiffness change sensing and our work.

**Figure 10:** Measured amplitude ratio at the out-of-phase mode of the 3DoF resonator sensor, for different stiffness perturbations.

**Table 2:** Sensitivity comparison

<table>
<thead>
<tr>
<th>Reference</th>
<th>Sensor output</th>
<th>Measured sensitivity</th>
<th>Sensor type</th>
</tr>
</thead>
<tbody>
<tr>
<td>[5]</td>
<td>Frequency shift</td>
<td>0.5</td>
<td>Single resonator</td>
</tr>
<tr>
<td>[6]</td>
<td>Eigenstate shift</td>
<td>275</td>
<td>Two resonators</td>
</tr>
<tr>
<td>This work</td>
<td>Amplitude ratio change</td>
<td>13558</td>
<td>Three resonators</td>
</tr>
</tbody>
</table>

**CONCLUSIONS AND OUTLOOK**

In this paper, we have reported a novel 3DoF resonator device for stiffness change sensing applications. The measured sensitivity of a prototype sensor represents an improvement by over 49 times compared to the state-of-the-art stiffness change sensors consisting of two weakly coupled resonators. In the future, the effect of damping will be included in the analysis. In addition, other specifications of the sensor, such as dynamic range, linearity and resolution will also be investigated.

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**REFERENCES**


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