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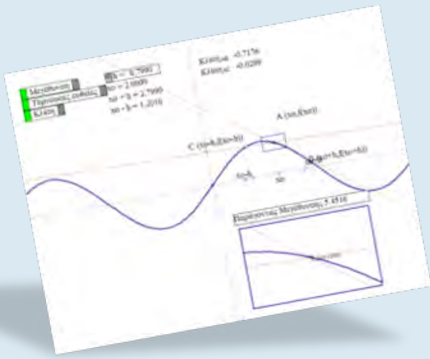
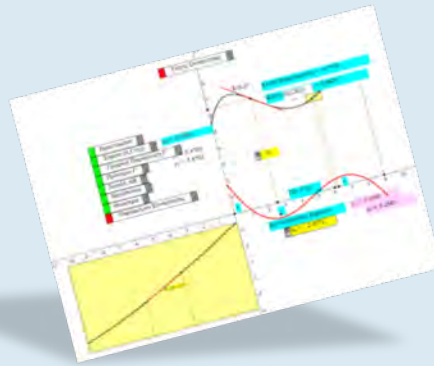
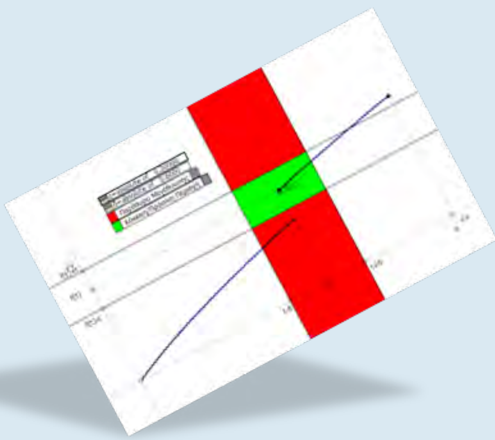
CAL GEO

Teaching Calculus using Dynamic Geometric Tools







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University of Sofia, Bulgaria
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SOUTHAMPTON 2007

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CONTENTS

PREFACE

The aim of this book and the accompanying CD is to help the teaching of Calculus/Analysis in secondary education. The material presented is a product of the project *Calgeo: Teaching Analysis with the use of tools of dynamic Geometry* in which five Universities from four countries participated. The participants in the program were:

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INTRODUCTION

Many years were required for most of the topics we teach today in mathematics to reach their present state. In Calculus/Analysis in particular, even though the roots of its founding concepts lie in Ancient Greek Mathematics, only in the 19th Century did it become possible to express the formal definitions with the requisite mathematical precision.

During the centuries, and in the course of evolution of these concepts, many mistakes were made and false attitudes were established that later were revised until they reached their present-day form. This long-term and difficult course proves that these concepts are, by their very nature, difficult and predictably not easily understood by students.

As a large number of international research papers show (see bibliography), a very high percentage of secondary education students and college students have serious problems comprehending the concepts of Calculus/Analysis.

As a result, the teaching of Calculus/Analysis constitutes a major problem for mathematics education. A key question thence is how the teaching of Calculus/Analysis might become more effective.

There is no easy answer to this question. There are no general ‘recipes’ to ensure effective teaching. Still, there are certain elements that can help students towards a conceptual understanding. One such element is the use of technology in class.

Infinite processes comprise the basis of Calculus/Analysis. An unknown quantity can be calculated arbitrarily close by other known quantities. A process like that involves motion. In other words, it is a dynamic process. This kind of processes can be more easily understood through a learning environment that can bring out the essential ingredients of the mathematical processes; that is, through a dynamic environment. This only can be done by using new technology. Yet this raises the question of which dynamic environment is the most appropriate for teaching Calculus/Analysis.

The roots of Calculus/Analysis, and the problems that led to its genesis, are problems that are connected to Geometry (calculating the area, the volume, the tangent at a curve) and with Physics (e.g. instantaneous velocity, optics). The combination of the dynamic nature of the concepts of Analysis and its historic roots leads to the conclusion that teaching Analysis aided by the use of tools of dynamic Geometry can augment its better understanding. The goals of the project *CalGeo: Teaching of Analysis with the use of dynamic Geometry tools* are the preparation of didactic activities and the creation of a programme for education of secondary education mathematics teachers.

In this project, which was materialized under the framework of European programme SOCRATES, action COMMENIUS 2.1, that concerns the education of secondary education teachers, five European universities collaborated; namely, the Universities of Athens (the co-ordinating institution), of Crete, of Cyprus, of Sofia, and of Southampton.

The didactic activities that were created under the *CalGeo* framework concern the introduction of concepts and the teaching of Analysis theorems with the use of dynamic Geometry software. Each activity consists of one or more worksheets addressed to the teacher.

The *Worksheet analysis* has the same content as the corresponding *worksheet analysis for the student* plus the subject, the goals, rationale, the way it can be integrated into curriculum, and an estimation for the time required for its realization.

The *Worksheet analysis* also contains added elements that aim to help the teacher to implement the activity in the class. Each activity is accompanied by one or more electronic files necessary for the implementation of the activity. These files are constructed with the aid of Dynamic Geometry software. In particular, for the activities that are presented in this book the *EucliDraw* software has been used. *EucliDraw* is Greek Dynamic Geometry software that provides tools for managing functions.

In this book we present the *CalGeo* project and the educational material produced thereby. In the first part we describe the *CalGeo* project, its goals and its theoretical background.

W*orksheet analysis* is contained in the second part, which consists of five sections. The first section is an introduction to infinite processes and to the concept of sequence convergence to a real number. The second section concerns the concept of the limit of a function when the variable is approaching a real number. The third section is about continuity of a function at a given point. The fourth section is an introduction to the concept of derivative through the tangent of the graph of a function and it contains also Fermat's theorem, mean value theorem and monotonicity theorems. The fifth section concerns the introduction of definite integral through the calculation of areas.

Finally the appendix contains the *student worksheets.* The book is accompanied by CD which contains all the contents of the book in electronic form, the *EucliDraw* files that are required for the realization of these activities and a demo of the software in order to make possible the use of the files.

The whole material can also be found at the URL address: www.math.uoa.gr/calgeo

It goes without saying that the subject of teaching Calculus/Analysis is not exhausted within the framework of this book. Furthermore, the activities that are included here can be subject to improvements. We are more than glad to accept the comments, remarks, suggestions and thoughts of teachers regarding not only the material of this book but in general the teaching of Calculus/Analysis.

DESCRIPTION OF *CALGEO* PROJECT

Analysis is a branch of mathematics with a broad field of applications that range from Science to Humanities. This is the reason why the introduction to Analysis has been a main part of mathematics curriculum taught at high school for many years. Nevertheless, many research papers indicate that the vast majority of students who has been taught Analysis continue to face serious problems of comprehending even the basic concepts of the material.

The root of this problem rests not only with the innate difficulty of the concepts involved but also with the way the subject is taught. Teaching is often focused primarily on learning algorithms and procedures, without necessarily paying attention to the building up of intuition and imaginary necessary for grasping the concepts. Teaching concepts in Analysis requires a different approach that aims at conceptual understanding. Using a similar approach complemented with new technologies, would help students develop the appropriate intuitions and mental images on which they could base their understanding.

The project *CalGeo: teaching of Analysis with the aid of Dynamic Geometry tools* intends to contribute towards this line of thought. Specifically, the goals of the project are:

a) Collaboration among European partners for the development of common teaching proposals inside E.U.

b) Formation of a didactic proposal concerning the teaching of Analysis in secondary education with the development of applications in Dynamic Geometry environment

c) Elaboration of a programme of in-service education of mathematics teachers in the framework of a didactic proposal that incorporates:

   i) recent research results regarding the teaching of mathematics that suggest new didactic approaches and a change of the teacher's role from a tool of transferring knowledge to a factor of facilitating the learning process.

ii) utilization of new technologies as a pedagogical tool and a means of supporting learning process and collaboration

iii) teaching schedules in secondary education of concepts and theorems of Analysis with the exploitation of dynamic geometry environments

The didactic approach on this subject has two strands: development and exploitation of geometric problems that establish the necessity of introducing the concepts of Calculus, and the usage of dynamic geometry environments that aim to enable students creating appropriate images. The project emphasizes intuition and inspection. Often, the usual didactic approaches used with basic Analysis concepts are disjointed and via typical formal language. The spontaneous ideas of students, usually the cause of misconceptions and misunderstandings, are not taken into account. The designed didactic approaches in the framework of the *CalGeo* project are taking into account previous student knowledge from everyday and school-life experience in order to activate them into the learning process. The point of departure for the proposed didactic activities is the informal – spontaneous beliefs of students that have been formed by their everyday and school-life experience and are connected to the concepts to be taught.

Through suitably designed environments, the objective of the *CalGeo* project IS to develop delicate images and proper intuitions for these concepts, facilitating the transition to formal mathematical knowledge and its comprehension.

The pedagogical and didactic approach of this plan is developed through the lens of learning theory of social constructionism. The project began on 1st October 2004 and concluded on 31st December 2007. During this period the following activities took place:

1. The current situation in the countries that are participating in the project regarding teaching of functions and Analysis in the secondary education was studied.

2. A proposal, based on the results of the previous study, was designed regarding teaching Analysis in the secondary education with the aid of Dynamic Geometry software

3. New applications in a Dynamic Geometry environment were designed in addition to their necessary accompanying material.

4. A program for educating teachers of secondary school mathematics was designed, accordingly to the aforementioned proposal, as well as

all the material in electronic and paper form for the backing of the program. In addition to this worksheets and evaluation sheets were also produced.

5. A experimental realization of the programme took place in the countries participating in the program. Furthermore, the educated teachers took part in practical implementation of the didactic proposals in a class. These were recorded and the results evaluated.

6. Workshops and a series of talks for teachers, schoolmasters and education officials were organised in order to facilitate the diffusion of knowledge. The results were also presented in some mathematics education conferences.

7. A website on the internet has been created which provides information regarding the project, its goals and the material produced, see: www.math.uoa.gr/calgeo

THE TEACHING OF ANALYSIS

Due to the great significance of Mathematical Analysis for a broad spectrum of sciences, its teaching constitutes one of the central topics of mathematical education. Many international studies indicate that most students finish secondary education displaying significant problems in the comprehension of the Analysis concepts taught. Many misconceptions regarding limits, continuity, tangent lines, etc, inflict serious problems to the continuation of their undergraduate studies.

According to some studies, the cause of these problems is the rational that rules the teaching of the introductory courses of Analysis which take place during the last years of secondary education. Through these courses, students perceive Analysis as a series of skills which they should learn. What is asked from them is to able to solve exercises, to draw graphs, to calculate quantities, using known methods. They are very rarely involved in tackling problems whose method of solving is not already known. Most exercises that appear in textbooks can be handled with superficial knowledge without a deeper conceptual understanding being necessary. Researchers conclude that the classic instructional schedule of definition - theorem - proof causes fear in the students, creates a misleading view of the nature of mathematics, does not show the procedure of thinking in Mathematics, and does not help students to use their intuition when considering these concepts. The result of all the above is that students learn less than what is possible for them to learn.

Researchers have studied the role of intuitions and of students' tacit knowledge in the answers which they give. It is found that erroneous intuitions or unconscious models might lead students to wrong interpretations, misconceptions, and contradictions. Another important element directly related to the understanding of a notion is the image of the concept that a student has. The name of a notion, when we see it or hear it, causes an irritation to our memory. Usually it is not the formal definition of the concept. What comes in our memory when we hear or see the name of a concept is what is called the *concept image*. This could be a visual representation of the concept, in cases where the concept has visual representations. It could also be a collection of impressions or experiences. The understanding of a concept pre-supposes the formation of a concept image. The memorization of the formal definition does not guarantee its comprehension. In order to comprehend it we should have a correct image.

The teaching of Analysis in secondary education focuses primarily on students learning procedures and algorithms, without always paying the necessary attention to intuition and to the creation of several representations of concepts which contribute to their substantial understanding.

In order to develop the correct intuitions from the students, in respect to a specific concept, the already formed perceptions of this concepts by the students, play an important role. These perceptions might have been formed either from the use of the mathematical term of the concept in everyday life or from the instruction of special cases of the concept in previous courses. Characteristic examples are the notion of limit, continuity and tangent. Far earlier than instruction about limits and the continuity of functions, students are likely to have heard and used in their everyday life the terms “limit” and “continuity”. The result of this use is the formation of perceptions concerning the meaning of these terms, perceptions which many times do not reconcile with the corresponding mathematical notion. For example, by using the term “limit” in everyday life, as “speed limit” etc the perception that “the limit is an insuperable upper boundary” is created. If, during the teaching of these concepts, the pre-existing perceptions are not taken into account, they remain and create obstacles in students understanding. A similar case is where students have been taught a special case of the concept and the characteristic properties of the concept in the special case do not apply in the general case; for instance, the tangent concept. Students are taught the circle tangent in Euclidean Geometry. If the previous knowledge is ignored during the teaching of the tangent to a graph in Analysis, obstacles will be created in the understanding of the concept by the students.

The correct intuitions and the understanding of a concept are taken to evolve through the creation of multiple representations of the concept. It is important in teaching to use multiple representations. This usage contributes not only to the understanding of the didactic object but also in general to the development of students’ mathematical thinking. Teaching that uses multiple representations might help students to learn to use, by themselves, multiple representations when they try to understand a concept or a proof or even when they try to prove an argument. Multiple representations might help students learn to study the images, to understand what they "say", and to be able to convert symbolic relations into images and to interpret images into symbolic Mathematics. This capability helps and evolves mathematical thought. The transition from the symbols to the images (that is, from the abstract to the concrete) allows us to better understand the abstract and to reflect on something that we perceive with our senses. The capability of rendering the images by formal Mathematics is necessary because only by this way, namely by strictly formal proof, a Mathematical truth is established and accepted. Conclusions that are based exclusively on the image might be erroneous.

Below we refer to specific characteristic stages of the teaching in which the use of multiple representations might contribute essentially to the understanding of the didactical object.

Use of representations in the teaching of concepts

Mathematical concepts, except perhaps geometrical ones, are abstract concepts. As mentioned above, knowledge of a formal definition does not imply comprehension of the mathematical concept. This results in the incapability, or erroneous uses, in solving an exercise. The capability of representing the concepts in a way that they are understood through the senses can help the student to better understand them and to use them correctly. An example of a concept that creates problems to students as it is introduced in the first years of the senior secondary school (or *Gymnasium*) and whose essential understanding is a perquisite for the conceptual understanding in Analysis is the concept of absolute value. Students learn the formal definition instinctively and this can result in mistakes. Also it impedes them from understanding more difficult concepts which they will meet later and for whose definition the absolute value has a significant role, as for example the notion of limit. On the contrary, if students have understood the representation of the real numbers on an axis and can perceive the absolute value as its distance from zero, and the absolute value of the difference of two numbers as their distance on the axis, they will be able to regard topics related to the absolute value by a geometrical view. This point of view will be very useful in many cases.

Use of representations for creating conjectures

The role of conjectures in the development of the mathematical thought is very important. In order to prove a mathematical proposition it must have first arisen as a conjecture; namely, after suitable thoughts to be driven to the conclusion that the specific proposition might be valid. This raises the question of in which way could a brainstorming environment leading to a conjecture be created in the classroom. It is likely that visual representations play a significant role. Especially nowadays with the use of technology, visual representations can be done in better terms.

We quote, as an example, an important theorem of Analysis. It is the theorem that relates the monotonicity of a differentiable function to the sign of its first derivative. But how did we think that the monotonicity of a differentiable function is related to the sign of its derivative and we were led to stating the specific theorem? This step is a very important step for the development of the mathematical thinking of the student. The study of the motion of the tangent on the graph of a differentiable function (see Figure 1) might lead to the remark that in the intervals in which the function is increasing (or decreasing), the tangent forms an acute (or obtuse) angle with the xx′ axis. This study can be done far better with the use of the computer where the student can watch the motion of the tangent.

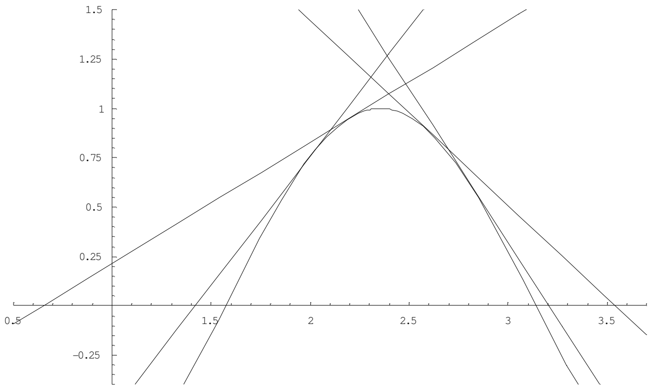


Figure 1

The above remark leads to relating the monotonicity to the derivative. Afterwards, a more careful observation of graphs and with appropriate questions that the teacher will pose, the following conjecture could be created: if the derivative of a differentiable function is positive (or negative) at an interval, then the function is strictly increasing (or strictly decreasing) at this interval.

Use of representations for describing Mathematical deductions

A mathematical proposition is stated in a purely symbolic form. To the student, this formulation can often seem obscure and strange. The description of the statement by a visual representation might help the student understand it better. A classic example is the following theorem, known as the Mean Value theorem:

"If f:[a, b] $\rightarrow$ **R** a function, continuous at [a, b] and differentiable at (a,b) then there exists $\xi \in$ (a,b) such that f(ξ) = $\frac{f(b)-f(a)}{b-a}$. „

The geometrical representation of the theorem (see Figure 2) helps the student understand what the theorem actually "says".

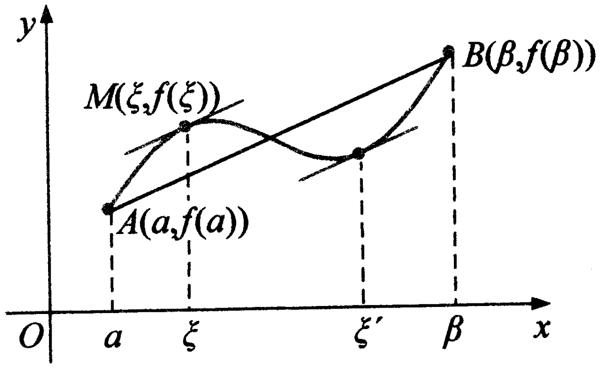


Figure 2

Use of representations for describing procedures and proofs

Many times, procedures or proofs seem difficult to students. They cannot understand what "they say" and so they cannot learn them. Even with procedures or proofs which students consider easy, many times students can only apply them or reproduce them only when asked. They have not understood their essence. They have not understood the essential mathematical idea which is hidden behind the formal presentation. The capability of describing such procedures or proofs with the use of multiple representations helps the students understand them better. An example of such a proof is the proof to the theorem of intermediate values:

"If f:[a, b] $\rightarrow$ **R** continuous function and f(a) < h < f(b) then there is $x_0 \in$ (a, b) such that f(x_0) = h."

The proof of this theorem results easily from the special case for h=0, which is known as the Bolzano theorem. Indeed if we set g:[a, b] $\rightarrow$ **R**, with g(x)=f(x)-h for every x$\in$ [a, b], it is easy to ascertain that function g satisfies the perquisites of Bolzano theorem. Therefore there exists $x_0 \in$ (a, b) such that $g(x_0)=0$. So $f(x_0)=\eta$.

The above proof is a simple proof that does not usually create problems for many students. Yet the question remains about how many students comprehend its essence. How many understand that we make a transition of function g in order to satisfy the perquisites of Bolzano theorem? The description of the above proof with Figure 3, or ever better with the use of a computer where the motion can be shown, embodies this idea.

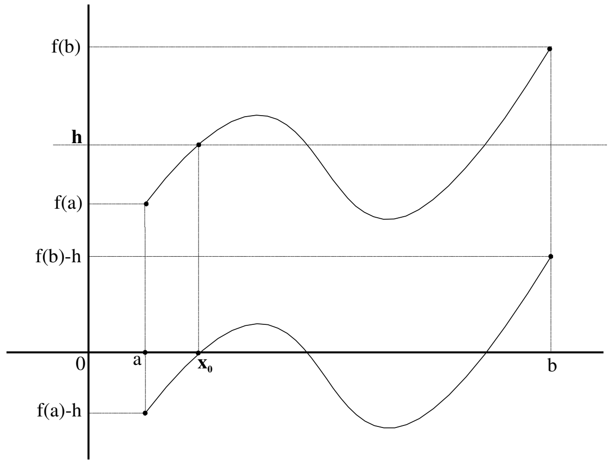


Figure 3

In the following we describe two general frameworks which concern the teaching of concepts and theorems and which constituted the base of the planning of *CalGeo* activities. These frameworks are based on the principle that we should try, in the teaching of Mathematics, to approach in a satisfactory point the following aims:

α) To show to students the evolution of the mathematical thinking that leads to the result.

β) To give to students the opportunity to participate actively to this evolution.

A framework for teaching concepts

All mathematical results have, as a starting point, the solution of problems. Hence when introducing a fundamental concept it is important to start with a problem that cannot be solved by the students with their existing state of knowledge.

The discussion regarding the solution will create the need for introducing the new concept. The symbolic formulation of the definition and its graphical and verbal descriptions help to construct the proper images and intuitions enhancing students' understanding of the concept.

Nevertheless, since Analysis concepts are quite complicated, it is required to clarify some of its aspects in order to avoid misconceptions by the students. This could be done by using appropriate examples.

The teacher, combining teaching experience and theoretical knowledge (historical, mathematical, didactical) in the particular concept, during the preparation of the course, should locate possible obstacles that students could face when encountering the new concept, and find the appropriate examples in order to overcome them.

Figure 4 shows this process in a graphical form.

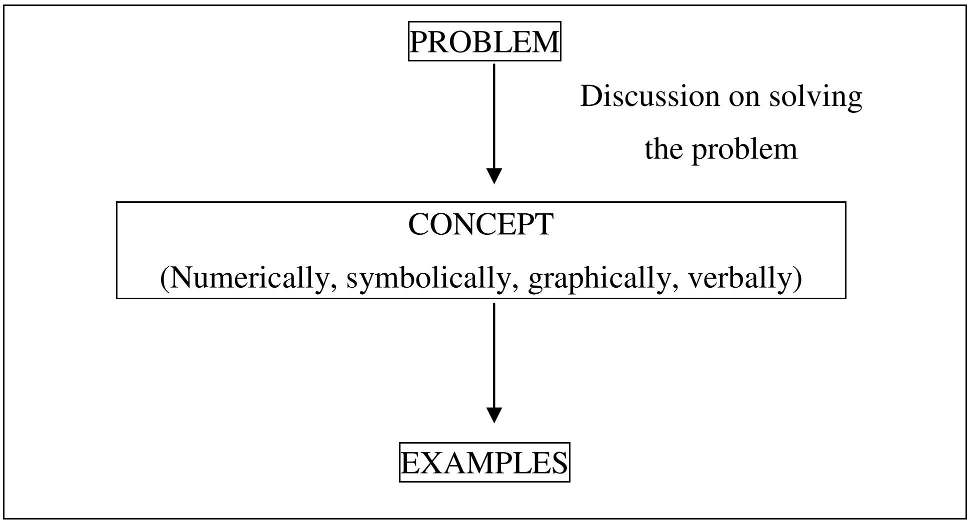


Figure 4

A framework for teaching theorems

When teaching a theorem, as when teaching concepts, it's important to start with a problem that cannot be solved by the students with their existing state of knowledge. The discussion regarding the solution will lead to the formation of a conjecture. Initially it may be phrased in a general form. Following that, an examination of the effectiveness of the conjecture is taking place. Thinking about the conjecture leads to a more careful formation of its statement that is essentially the theorem, and also to the belief that it holds true.

Hence we are driven to the theorem's formulation and its proof, provided it is contained in the curriculum.

Then we examine the necessity of the theorem's assumptions.

A theorem has certain assumptions and a conclusion. The assumptions appear to be necessary since they are used in its proof. However, the fact that these assumptions are required for a proof of the theorem doesn't mean that they are really necessary for the conclusion. Probably we could reach the same conclusion with another proof, using fewer assumptions than we already have used.

In order to be certain that each of the assumptions in a theorem's formulation is necessary (that is, the conclusion of the theorem does not hold if we drop anyone of these) we have to find a counter-example that satisfies the remaining assumptions and still the theorem does not hold.

This means that there cannot be other assumptions that lead to the same conclusion.

For example, Bolzano's theorem can be expressed as :
"Let a function $f:[a, b] \to \mathbf{R}$. If f is continuous and $f(a)\cdot f(b) < 0$, then there is a $x_0 \in (a, b)$ such that $f(x_0) = 0$."
The two assumptions of the theorem are the continuity of f and that it's values at its endpoints have opposite signs. In order to be sure that both assumptions are necessary, we have to find two counterexamples. One that satisfies only the continuity assumption and one that satisfies only the assumption that $f(a)\cdot f(b) < 0$, so that in both cases the function is discontinuous. This step could precede the formulation of the theorem during the thinking process of forming the conjecture.

After completing the checking on the necessity of the assumptions, we examine if the inverse holds. This is a basic part of mathematical thinking that usually comprises the starting point of a path towards other important theorems. Finally, we solve the initial problem, if it can be solved, with the theorem proved and then solve some more of its applications. Although the theorem could not necessarily solve the initial problem, it constitutes a step towards its solution. This could probably give us the opportunity to continue thinking about it and may lead us to other theorems. Furthermore, the applications of the theorem that will follow and the exercises that will be given to the student for solving have to be carefully chosen in order to highlight the importance of the theorem and to encompass definite mathematical and didactical goals.

Sometimes the profound study of the theorem we just proved, entails new questions that could lead in turn to new results. Figure 5 depicts the aforementioned process.

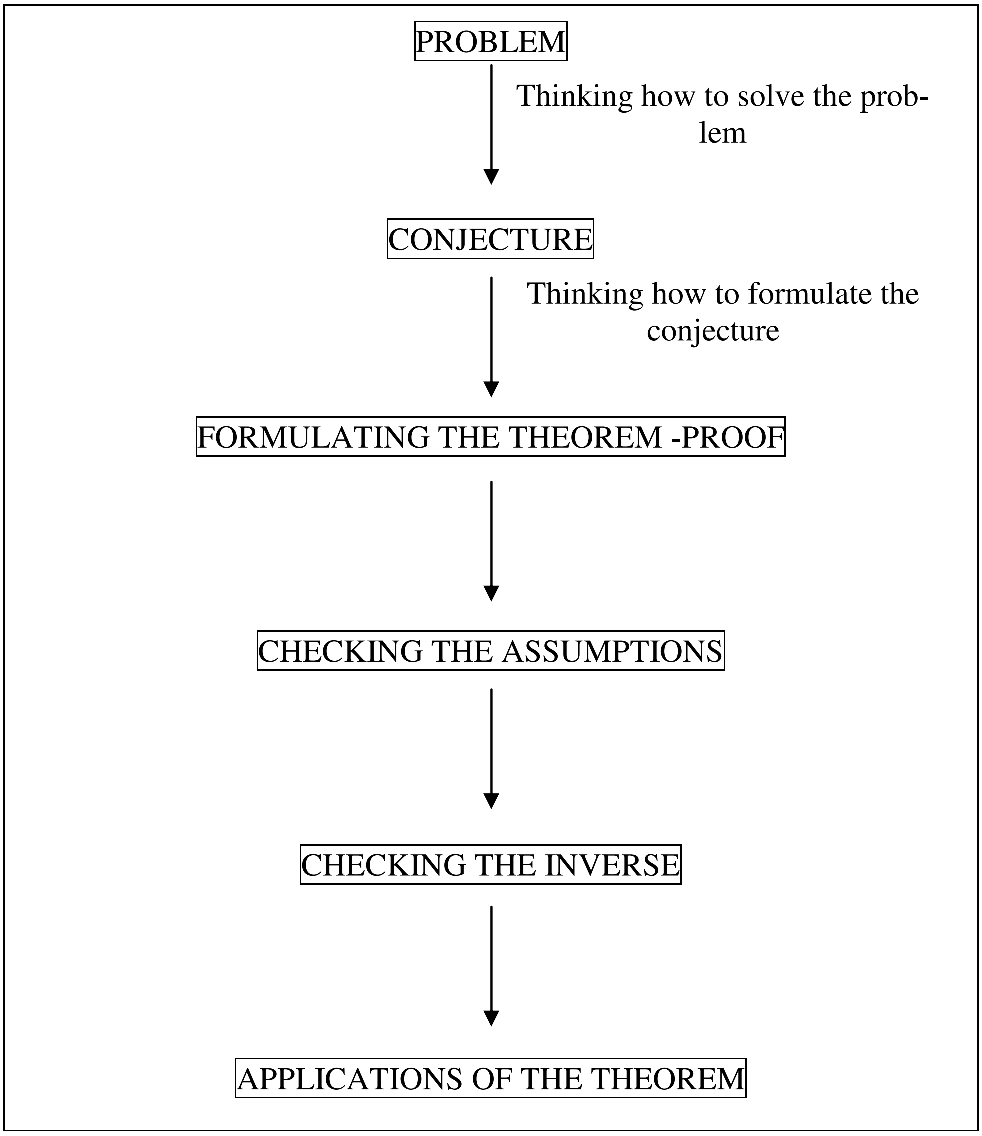


Figure 5

These discussions concerning the teaching of concepts and theorems are indicative and not for general implementation. Nevertheless, they can provide specific clues regarding teaching that can be used sporadically in several cases. Certainly time constrains constitute an objective factor that cannot be ignored. The necessity for finishing the material remains an obstacle in the development of teaching methods like the one mentioned above. Still, there is time during the process of teaching fundamental concepts and major theorems to make use of the ideas described above, instead of exhausting the class time into solving a number of identical exercises. In this way the student can see not only the final mathematical result but also the process leading to it.

ACTIVITIES

1. INTRODUCTION TO INFINITE PROCESSES – LIMIT OF SEQUENCES

1.1 Activity: Introduction to infinite processes

Subject of the activity

This activity is an introduction to infinite processes. This is achieved by applying the Eudoxos-Archimedes' method of exhaustion for the calculation of the area of the unit circle.

Goals of the activity:

This activity intends:

- to introduce students in a natural way to the use of infinite processes for solving problems that cannot be solved in another way.
- to help students encounter the method of exhaustion (and the idea of infinity behind it) in an environment free of algebraic calculations.
- to have students acquaint anted to the interchange of representations through the given environment where graphical and numerical representations are combined harmonically .

The rationale of the activity:

This activity introduces the students to infinite processes, using the problem of the unit circle area calculation. This problem cannot be solved with the use of students' prior knowledge as the circle cannot be divided into polygons. However, students can use their knowledge on polygons in order to approach the unknown area of the circle. The concept of an infinite process emerges naturally as a tool that allows the approach of unknown quantities no matter how close they are, by an infinite number of known ones. This idea prepares the field for the introduction of students to the limit concept at some later stage. The use of dynamic geometry software allows not only for the graphical representation of the problem, but it also eases the computation difficulties that are inherent in the calculation of the polygon areas, giving students the opportunity to focus on the infinite process

Activity and Curriculum:

This activity can be offered as an introduction to pre-calculus or calculus. Students at this level are not familiar with infinite processes. It is intended to last an hour.

1.1.1 Worksheet Analysis
Introduction to Infinite Processes

PROBLEM
How can we calculate the area of the circle having radius R=1?

This is the general problem which will introduce the student to infinite processes.
We expect that some of the students know the formula giving the area of the circle and may answer that the requested area is equal to π.
In this case there could be a discussion about the following questions:

- Why is this area equal to π?
- How did the formula $E=\pi R^2$ arise?
- How can we calculate π?

In order to solve this problem, we start with three questions related to the basic area measurement knowledge

Q1. What does it mean that the area of a triangle equals 4.5?

This means that four and a half squares with sides of length 1 fit this triangle.
There could be some discussion about area measurement and area measurement unit.

Q2: Find geometrical figures whose area can be measured with the previous method.

We can calculate polygon areas using the previous method.
There could be a discussion about how we can calculate the area of a polygon by diving it into smaller rectilinear shapes with easy to calculate areas.

Q3: Can we divide the circle into a certain number of figures with measurable area?

This question is equivalent to the following question: can we divide the circle into polygons? The polygon sides are straight line segments. The circle can not be subdivided into polygons. The negative answer leads us to the next question.

Q4: In which way is it possible to link the area under question with polygons areas?

Some discussion can arise from the students' answers.
The target of the discussion is to be led to the idea that it is possible to find polygons whose area is either smaller or bigger than the circle area.
(That is polygons inscribed in the circle as well as polygons circumscribed around it)

Draw two squares: One inscribed in the circle and the other one circumscribed around it.

Try to answer the question with the use of *1.1.1.activity.en.euc* file.

The environment provides us with the following:

We can see the circle.

The *sides* button controls the number n of the sides of the regular inscribed and circumscribed polygons.

When the *circumscribed* button is pressed, the circumscribed polygon appears. Another click results to the polygon disappearance.

The *inscribed* button acts in a similar way.

The areas of the polygons and their difference are also displayed.

The *magnification* button displays a window around a predetermined circle point. By enlarging *magnification scale* we achieve a more delicate focus.

Q5: What is the relation between the circle area E and the areas of these two squares?

Q6: What is the difference of these squares areas?

Using the aforementioned areas we can achieve the first estimation of the area E in question. This estimation obviously is not good enough. So we are driven to the next question.

Q7: Through which process is it possible to obtain a better approximation of E?

This question is the critical point where students can pass to successive approximations. The students can probably guess that an increase in the number of sides, results to better estimations.

The instructor can prompt the students to focus on the area difference between the inner and the outer polygons as n gets bigger. The difference indicates how close the areas of the polygons and the circle are.

By constructing both the internal and the external regular pentagons we can get a better approximation of the circle area. Students may experiment with a greater number of sides.

After a certain increase in the number of sides, the polygons seem to coincide with the circle although this really doesn't happen. In case some students have difficulties, we can use the zoom-in option. The interest moves now to the numerical results.

Q8: Complete the following table:

n	Inscribed n-gon area	Circumscribed n-gon area	Areas difference lees or equal to
4			
5			
6			
10			
12			
(18)			0.09
(23)	3.1…	3.1…	
(56)			0.009
(114)	3.14…	3.14….	
(177)			0.0009
(187)	3.141…	3.141…	
(243)			
(559)			0.00009

In this question the student fills in the empty cells of the table. The numerical results in the table intend to allow students to make some conjectures on the limits of these three sequences.
In the case of numbers such as 0.09 that are given in the table above, the students are expected to find a value of n such that the difference of the two areas is less than 0.09.
3.14… means that the number of sides is such that both the internal and external polygons have areas with the first two decimals equal. In the above questions students may give different answers.

Q9: Is there a step in this process in which the inscribed or the circumscribed polygon has the same area with that of the circle?

To answer the question you can use the *zoom-in* tool and focus on a circle point. By enlarging the *zooming scale* we can achieve better focus
Obviously no polygon can coincide with the circle.

Q10: Will this process come to an end?

Since the process doesn't terminate it can be continued indefinitely. That is we can always get polygons of more and more sides. At this point we pass to the infinite processes.

Q11: Which is the number approached by the area difference?

The difference approaches 0 as the number of sides gets bigger.

Q12: How close to this number can the area difference ever get?

We can make the difference as close to 0 as we want as long as a properly big number of sides is chosen.

Q13: How close to the circle area can we approach?

In conjunction with the previous question we can be as close as we want to the circle area.

1.2 Activity: Introduction to Sequence Limit

Subject of the activity

This activity introduces the notion of Sequence Limit. Two worksheets lead the student to the definition of convergence of a null sequence. The worksheets that appear in the further exploration section lead to the definition of non-zero limit.

Goals of the Activity

This activity intends:

- to introduce students to the definition of the limit of sequence.
- to get students acquainted with the graphical, algebraic and numerical representations of the sequence limit
- to broaden gradually the students' image of the definition of convergent sequence in order to avoid possible misconceptions (e.g every sequence converges to zero, every convergent sequence is monotonic).

The rationale of the activity

The problem used in the 1st worksheet is based on one of Zeno's paradoxes. The problem has been chosen such that through the description of an infinite process the students could be introduced to the concept of sequence convergence.

The problem of the 2nd worksheet also has a strong visual representation which becomes even stronger with the use of the dynamic geometry software. The processes which are presented in the worksheets are similar and both of them can be described by the convergence of a sequence. The questions posed are chosen in a way to orient the students to the definition of the convergent sequence.

Activity and Curriculum:

Two independent worksheets are suggested that can lead to the definition of null sequence. The instructor can choose whether he will use one or both of them. Each of the worksheets can be looked upon within an hour lesson, while the further exploration ideas will need another hour lesson. Worksheet 1 and further exploration tasks do not use dynamic geometry software.

1.2.1. Worksheet Analysis
Introduction to Sequence Limit I

We start with a line segment AB having a length equal to 1. A point moves from A towards B in the following way:

During the first day the point covers the interval AA_1 equal to the half of the interval AB.

During the second day the point covers the interval A_1A_2 equal to the half of the interval A_1B.

Keeping on this way, during the n^{th} day it covers the interval $A_{n-1}A_n$ equal to the half of the interval $A_{n-1}B$.

(That is every day the point covers half of the distance left to reach B).

Q1: Will this point arrive at B?

Probably students will hold different views. Some might claim that the point will finally arrive while others that it won't. The instructor could ask for an explanation of each opinion expressed. The goal of the discussion is to help the class reach the conclusion that the moving point will never reach point B. This can be justified by using the method *ad absurdum*. If we suppose that it arrives at day n, this means that the $n-1^{th}$ day it was beyond B, let C be the point where it was on day n-1. Then on the n^{th} day it was on the midpoint A_n of the interval CB, but this is different from B.

Q2: Calculate the length of the intervals A_nB for *n*=1,2,…

$$A_1B = \frac{1}{2}$$

$$A_2B = \frac{1}{4}$$

...

$$A_nB = \frac{1}{2^n}$$

From the above it can be claimed numerically that the moving point will never reach B because there is no n such that $A_nB = 0$.

Q3: Let C_1 be a point of AB so that $C_1B = 10^{-6}$.
Will the moving point go beyond C_1?

We know from the previous question that $A_nB = \frac{1}{2^n}$. So, the question is equivalent to the following: Is there a natural number n such that $\frac{1}{2^n} < 10^{-6}$ or $2^n > 10^6$?

This is true because the set of natural numbers does not have an upper bound. So there exists a natural number n such that $n > 10^6$ and so $2^n > n > 10^6$.

Some students could probably try to solve it by using the monotony of the logarithmic function. It would be better to focus on the fact that we are interested in the existence of such an n, not in the determination of a specific one. Using this as a pretext, the instructor could mention the difference between proving the existence of a number n with a specific property and finding a certain number n with this property.

Q4: Consider the same question for points C_2, C_3 such that $C_2B = 10^{-100}$, $C_3B = 10^{-1000}$

An argument similar to that of Q3 can show that the point will get beyond C_2 and C_3.

Q5: Let C be a random point between A and B. Will the moving point get beyond C?

Let ε be the length of CB. The question is equivalent to the following:

Is there a natural number n such that $\frac{1}{2^n} < \varepsilon$ or $2^n > \frac{1}{\varepsilon}$?

This can be proved in a similar way to Q3 and Q4.

Q6: Can you describe the result you reached in question C5?

The students could postulate a proposition like the following :

For any ε>0 there is a natural n such that $a_n < \varepsilon$

Q7: Complete your answer to the question Q6 in such a manner to include the information that if some day the point surpasses C then the same will hold for all the next days .

For any ε >0 there is a natural number n such that for every $N \geq n$ we have $a_N < \varepsilon$.

Although in this problem the information added to Q7 is obvious due to the special feature of the problem (monotonic convergence) we know that in the general case convergent sequences are not monotonic. In the questions of further exploration appropriate examples are given.

1.2.2 Worksheet Analysis
Introduction to Sequence Limit II

Let ABCD be a square having side equal to 1 and K the intersection of the two diagonals.

Let A_1, B_1, C_1 and D_1 be the midpoints of KA, KB, KC and KD respectively.

We construct the square $A_1B_1C_1D_1$.

Let A_2, B_2, C_2 and D_2 be the midpoints of KA_1, KB_1, KC_1, and KD_1 respectively.

We construct the square $A_2B_2C_2D_2$.

In general if A_n, B_n, C_n, D_n are the midpoints of KA_{n-1}, KB_{n-1}, KC_{n-1}, and KD_{n-1} respectively, we construct the square

$A_nB_nC_nD_n$, for n = 2, 3, …

This means that each square has its vertices on the midpoints of the line segments that connect K with the vertices of the square in the previous step .

Q1: Are there any other common points except K in the interior of all constructed squares ?

Open *1.2.1.activity.en.euc* file and experiment.

The number labeled "area" represents the area E_n of the square $A_nB_nC_nD_n$.

If needed, you can use the magnification tool.

The infinite intersection of the internal of all the squares equals {K}. This can be proved by supposing that another point L different than K belongs to the intersection and letting n grow such that the diagonal leaves L out of the square $A_nB_nC_nD_n$. It implies that L doesn't belong to the square $A_nB_nC_nD_n$.

Q2: Calculate the square area E_n of $A_nB_nC_nD_n$, n = 1, 2, 3, …

The student can observe that at each step the side of the square equals to the half side of the previously constructed square (Thales theorem) and reach by induction the conclusion that $E_n = \left(\frac{1}{2^n}\right)^2 = \frac{1}{4^n}$

Q3: Is there any of these squares with area less than 10^{-60}?

This question can be answered both algebraically (similarly to activity 1.2.1) and by experiment on *EucliDraw*. The convergence of E_n is really fast so that the student can experiment in *EucliDraw* by changing the decimal points precision .

Q4: Consider the same question for the number $10^{-1.000.000}$.

The dynamic geometry environment cannot be used in answering this question, so here becomes the need of an algebraic argument.

Q5: Let ε > 0. Is there a square, whose area is less than ε?

Questions Q3 and Q4 prepare question Q5 where there is a generalization and ε is a variable leading the student to the ε definition of a convergent sequence.

Q6: Can you find a description for the result you reached on Q5 ?

The students might write something like the following:

For every $\varepsilon > 0$ there exists $n \in \mathbb{R}$ such that $a_n < \varepsilon$.

Q7: Supplement your answer on Q6 taking into account the information that if at some step the square area is less than ε, then the same holds for all the remaining steps.

For every $\varepsilon > 0$ there exists $n \in \mathbb{R}$ such that, for every $N \geq n$ we have $a_N < \varepsilon$.

Further Exploration

1. Let the sequence $\alpha_n = (-1)^n \frac{1}{n}$, n = 1, 2, …

(i) Complete the following table:

n	1	2	3	10^3	11^6	10^{100}
α_n						

(ii) Is there any real number λ being approached by the terms of α_n as n increases?

(iii) Is there a term of the sequence, so that the distance of λ from it and its successors is less than 10^{-6}?

Consider the same question for numbers 10^{-100} and 10^{-1000} respectively.

(iv) Let ε > 0. Is there a term of the sequence, so that the distance of λ from it and its successors is less than ε?

(v) Could you describe the conclusion you reached in question (iv) in a formal way?

2. Consider the same questions for the sequence $\beta_n = \frac{n+1}{n+2}$, n = 1, 2,…

2. FUNCTION LIMIT AT A POINT

2.1 Activity: Introduction to function limit at a point

Content of the activity:

This activity, following the calculation of instanteous velocity, introduces function limit at a point.

The goals of the activity:

With this activity students should

- be introduced intuitively to the ε-δ definition of function limit at a point.
- connect harmonically numerical and graphical representations of the problem in order to clarify the concept of function limit.

The rationale of the activity:

The activity uses as a pretext an instanteous velocity problem. From everyday life the students are familiar with the notion of velocity although the instantaneous velocity includes (hidden though) a limiting process. In this process the dynamic geometry software functions in two different ways. In the first part of the worksheet it provides numerical results. This enables us to avoid time consuming calculations. In the second part of the worksheet, the numerical data are represented graphically. The student can then visualize the convergence of function and move in a natural way to ε-δ definition. The use of the ε and δ zones in the dynamic geometry environment, gives the student the opportunity to handle in a dynamic way the basic parameters of the problem, in order comprehend the relation between ε and δ. The green and red colors are used not as a visual effect, but as a tool that allows for the verbal representation of complex expressions.

For example the expression "All (x,f(x)) such that $x \in (T-\delta, T+\delta)$ and $f(x) \in (U-\varepsilon, U+\varepsilon)$" is transformed to "the part of the graph which lies in the green region".

Activity and Curriculum:

This activity can be taught in an hour lesson, as an introduction to the definition of the function limit concept.

Depending on the student's level and the underlying didactical goals, the activity can either lead to an intuitive approach of the definition of function limit at a point or to the ε-δ definition.

2.1.1 Worksheet Analysis
Introduction to function limit at a point

PROBLEM
A camera has recorded a 100m race.
How could the camera's recording assist in calculating a runner's velocity at T=6sec?

- Open *activity2.1.euc EucliDraw* file. In this environment we can get the camera's recordings.
- When changing the values of t, the values of s(t), that represent the distance the runner has covered up to t, also change.
- t can approach T from less and greater values.
- Display the average velocity.

The yellow box displays the average velocity $\frac{s(T)-s(t)}{T-t}$ in the interval defined by t and T.

Q1: Fill the empty cells in the following table.

t	$\frac{s(T)-s(t)}{T-t}$	t	$\frac{s(T)-s(t)}{T-t}$
4		8	
5		7	
5.5		6.5	
5.8		6.3	
5.9		6.1	
5.93		6.07	
5.95		6.03	
5.99		6.01	
5.995		6.005	
5.999		6.001	
5.9999		6.0001	
5.99999		6.00001	

There can be conversation on the notion of average velocity.
It is possible for students to remark that in the *EucliDraw* environment the average velocity does not change value when we reach the value of 6 either from some smaller or

bigger numbers. This is due to the fact that the quantity T-t equals zero and the average velocity has no meaning since denominator is zero.

Q1: Which number does the average velocity approach as t approaches T=6sec?

As t approaches T from both greater and less values, we can observe that the average velocity approaches 10m/sec. It's useful to make clear to the students that t can be arbitrarily close, but never equal, to T.

Q2: What is the runner's velocity at T=6sec?

- Display the Average velocity Function U(t) in *EucliDraw* and confirm your findings graphically.
- Display the ε-zone in the *EucliDraw* file. The points in the ε-zone have ordinate that is bigger than L-ε and less than L+ε.
- Move t so that (t,U(t)) lies inside the epsilon zone, and observe the values of the average velocity.

Although we have determined the limit, we now show the function and try to see the convergence on the graph.

Q3: For which values of t is the point (t,U(t)) inside the ε=0.8 zone?

You may have some assistance on answering this question by displaying the delta zone. Points inside the δ-zone have abscissa bigger than T-δ and smaller than T+δ. Points simultaneously inside epsilon and delta zones are coloured in green. Points outside the epsilon zone are coloured in red.
The student can experiment by moving t and observing the change of U(t).

Q4: Try to find a δ such that no points of the graph lie in the red area.

If we have δ equal to 0.7 we get what we wanted.

Q5: Decrease ε to 0.5 and find a δ such that the points (t,U(t)) do not lie inside the red area.

e.g δ=0.4.

Q6: If ε=0.05 can you find such a δ?

You can display the magnification window. It can assist you in viewing inside a small area around (T,L).
e.g δ=0.05.

Q7: If ε gets less and less, will we be always able to find a suitable δ with the abovementioned property?

The students can experiment with less and less values of ε and conclude that they will always be able to find a δ.

Q8: Fill in the blank with a suitable colour in the following statement in order to express the conclusion of Q7.

"For every ε>0 we can find a δ>0 such that the function does not lie in the ……red………. area."

We need to pay attention to this proposition because it may lead students to a misunderstanding. The student may believe that even if the function was defined at point T then the value L=f(T) should not lie in the red area. This activity doesn't make clear that what interests us is not what happens at T but only what happens around T.
In some further activity it should be good to clarify that when examining the existence of a limit at a point, the value of the function (in case it exists) could lie inside the red area.

Q9: Fill in the blanks so that the following statement bears the same conclusion as Q7

The ………………..U(t) ………………………. can be arbitrarily close to …………..L……………. as long as the ………t…………….are close enough to ………T……………….. and different than …..T………..

Q10: Try to formulate the conclusion of Q7 using mathematical symbols

This is the most critical step and a conversation may take place taking into account the students' answers. The teacher can remind his/her students that the distance between two numbers equals the absolute value of their difference. The goal of this question is to give the ε-δ definition of the limit of a function at a point.
For every ε>0 there is a δ>0 such that if $0 < |t - T| < \delta$ then $|U(t) - L| < \varepsilon$.

3. CONTINUITY OF A FUNCTION AT A POINT

3.1 Activity: Introduction to continuity of a function at a point

Content of the activity:

This activity introduces the students to the notion of continuity (and discontinuity) of a function at a point. Through the study of a problem, students are leaded to an informal verbal representation of the continuity definition, in a way that it enables a smooth passing to the algebraic representation of the ε-δ definition.

Goals of the activity:

This activity intends:

- To introduce students intuitively to the ε-δ definition.
- To familiarize students to the "find a suitable δ whenever given an ε" kind of process, within a dynamic geometry environment, free of algebraic manipulation.
- To connect different representations of the concept of continuity.
- To realize the ε-δ definition's failure as a jump on the graph.

The rationale of the activity:

The ε-δ definition of continuity presents serious difficulties. Through this activity we seek to introduce students to continuity of function at a point, in an intuitive way free of algebraic manipulations.

The activity is divided into three steps. In the first step, through a problem, some concrete values of epsilon are given and the student is asked to find the corresponding δ. In the second step ε becomes a parameter. In the third step the student connects discontinuity to the existence of a jump in the graph. In each step the questions are first implemented graphically and then algebraically.

Activity and Curriculum:

Depending on the level of the target group, the activity can be taught in two different contexts. It can either be used as a tool introduction to the idea of continuity or –under suitable adjustment - can be also used as an intuitive introduction to the main idea of continuity

Stages 1 and 2 can be discussed in an hour lesson. Stage 3 can be discussed in a separate one-hour lesson, combined with examples which will dispel certain misunderstandings that could arise (for example δ we find is unique, "I do not lift up my pen while drawing a continuous function").

The concept of continuity appears independent from the limit concept and it does not presuppose the limit understanding.

3.1 Worksheet Analysis
Introduction to continuity of a function at a point

FIRST STEP

A chemical and health care corporation is about to produce a new antibiotic pill which will be able to cure a certain disease.

It is known that the pill should contain 3gr in order to provide the patient with the right dose of medicine.

The function $f(x)=\sqrt{x+1}-1$ gives the amount $f(x)$ of the antibiotic which is detected in the blood, when a patient gets a pill with x mgr of the substance.

According to current research results, if the antibiotic detected in blood is equal or less than 0.8gr, there will be no effect on the patient's health and if it is more than 1.2gr the patient is in danger due to overdose.

Q1: Which amount of medicine is presumably detected in the patient's blood?

$f(3)=1$

Q2: Which is the allowed error ε of divergence of the detected amount of medicine from the ideal value, such that the pills stay safe and effective?

The number ε=0.2 sets a boundary $(1-0.2, 1+0.2)=(0.8, 1.2)$ on the accepted quantities of antibiotic around f(3)=1. We will call ε the allowed error.

Some questions that familiarise students with the allowed error could be useful since epsilon plays a vital role throughout the activity. It is very useful to introduce the terms "allowed error" and "precision" in our language as soon as possible.

The machine available to the corporation that produces pills of t=3gr has an accuracy level adjusted to δ=1.1

This means that although the machine is programmed to produce 3gr pills, the pills are not always 3gr but their weights vary between 3 – 1.1gr and 3 + 1.1gr.

Q3: Is the machine suitably adjusted to produce safe and effective pills?

- Open *EucliDraw* file 3.1.1a.activity.gr.euc and try to answer question E3 using it.
- In this environment we get the graph of f(x). By changing ε we can alter the allowed error, and by changing δ we can alter machine's accuracy.
- The magnification window allows us to focus in a neighbourhood of point (3,1).

The student is expected to observe that there are some parts of the graph which lie inside the delta zone, but outside the epsilon zone. If we set the machine to produce pills, some of them will be ineffective and some dangerous.

The machine can be adjusted to another accuracy level.

Q4: Can the machine be adjusted in order to produce pills within the allowed error?

We expect the students to change delta in order to fix the problem of the previous question. The students should work from both sides in order to find a delta of about 0.8.
Some discussion may be held in order to emphasize that δ is not unique.
Every δ less than the one already found can work fine.
Another point that needs extra attention is the fact that we are not interested in finding a better δ.

The results of a new research indicated that the error level should be reduced to ε=0.1.

Q5: Is there any problem with this change in pill production? Does the accuracy level have to be adjusted anew?

Each student's answer depends on which δ has given in the preceding question.
For example if a student has given δ=0.8 before, the machine after the change of ε does not work right. If a student's δ was 0.2, then the machine still does not have a problem.
A δ of 0.3 or less does work.

SECOND STEP

Q1: If the results of another research suggest that ε should be lessen more, will the corporation be always able to adjust suitably the machine?

Open *activity3.1.1.euc* file and check whether we can always find an adequate δ>0, as ε gets smaller and smaller. Experiment graphically.

Display the Red/Green region and describe what it means for the function graph to lay in the green, red or white region.

This question offers the student the opportunity for a transition from a visual representation to a verbal one.
The green region represents the allowed (x,f(x)).
Algebraically, the green region is the set of those points (x,f(x)) of the plane for which, $|x-5|<\delta$ and $|f(x)-f(5)|<\varepsilon$
When some part of the function lies in the red region then there is a problem in the pill production. The machine is programmed in such a way, that it may produce pills which are ineffective (red part bellow the green region), or harmful (red part above the green region).
No point in the white region can be produced by the current state of the machine.

If necessary, use the magnification window.

In this question, the student plays an ε-δ game giving less and less epsilons. When ε lessens enough the zones are indistinguishable, so the interest moves to the magnification window.

Q2: In each of the following sentences, fill in the blanks with the correct colours:

a. Whenever we are given an ε, we can find a δ,
such that the function does not lie in the .....red........ region.

b. For every ε we can find a δ such that for every x in the accuracy limits of the machine, (x,f(x)) lies in the ....green........... region.

Q3: Write sentence b, replacing the colours with algebraic relations.

The teacher initially leads the students in a verbal description of the result like :
"For every arithmetic value of the error, there is a machine accuracy so that if x lies inside the bounds of the allowed precision, f(x) lies inside the bounds of the allowed error".
Hereupon he can ask them to demonstrate the description above with the use of mathematical symbols. That is
"For every ε we can find a δ such that for every $x \in (3-\delta, 3+\delta)$ we have: $f(x) \in (1-\varepsilon, 1+\varepsilon)$ ".

Or
For every ε we can find a δ such that for every x such that $|x-3|<\delta$,
we have $|f(x)-1|<\varepsilon$.
Under suitable circumstances the last proposition can drive to the introduction of ε-δ definition of continuity in the case of a random function.

THIRD STEP

Another research shows that the formula given by the function above which gives the quantity of drug traced in blood works well for values less than 3mgr. When values of x exceed or equal 3 it shows 0.06mgr less than the real amount detected in blood.

Q1. Find out the formula of the new function that gives the real quantity of drug detected in blood, taking into account the results of the last research.

$$g(x)=\begin{cases}\sqrt{x+1}-1 & , \ x<3 \\ \text{.................} & , \ x\geq 3\end{cases}$$

The second branch should be $\sqrt{x+1}-0.94$

Q2: Can the machine be adjusted properly to produce effective and safe pills?
Which δ should work for ε=0.1?

E.g. δ=0.1 works.

Open *activity3.1.2.euc* file
Give your answer by giving suitable values δ.
If needed use the magnification window.

ε is chosen in a way that a δ can be found.

Q3: What will happen if ε is reduced to 0.06? Can you find an adequate δ?

No δ can make the machine produce only effective pills since whenever a pill is less than 3mgr the quantity detected will be ineffective.

Q4: What causes this failure?

This question may lead the class to some very interesting discussion.
Some idea that can lead to the discontinuity concept is the fact that the gap which exists in the graph of f(x) is what causes the failure in finding δ, as long as the points g(x) around 3 appear distanced from each other.

Q5. In each of the following sentences, fill in the blanks with the correct colours:

a. For a given an ε, no δ could prevent the function from lying in the ....red............ region.

b. There is an ε, such that for every δ, there is an x in the accuracy field of the machine such that (x,g(x)) lies in the .....red......... region.

Q6. Write sentence Q5b using algebraic relationships instead of colours.

There is an ε such that for every δ, there is an $x \in (3-\delta, 3+\delta)$, such that $g(x) \notin (1-\varepsilon, 1+\varepsilon)$. Or There is an ε such that for every δ there is an x such that $|x-3| < \delta$, for which we have $|g(x)-1| \geq \varepsilon$.
Last proposition in correspondence to the last one of the second step, can lead to the generalization of the negation of the definition of continuity for a random function.

4. DERIVATIVE

4.1 Activity: The notion of derivative and the tangent

Content of the activity

This activity aims to introduce the students to the notion of the derivative of a function at a point x_0 through the geometric representation of the tangent line of the curve at the point $(x_0, f(x_0))$. In designing this activity students' previous knowledge concerning the tangent of the circle, was taken into account.

Goals of the Activity:

Through this activity we aim students:

- Generalise their previous knowledge concerning the tangent line grounded in the context of Euclidean Geometry, to other more general types of curves.
- Be introduced to the notion of the tangent line at point $(x_0, f(x_0))$, as the linear approximation of the curve at this point.
- Be introduced to the notion of derivative at a point x_0.
- Understand the geometric representation of the derivative at a point x_0 as the slope of the tangent line at the point $(x_0, f(x_0))$.
- Connect the symbolic representation of the derivative of a function at a point with the geometric one.
- Recognise from the curve of a function at which points the function is differentiable.

The rationale of the Activity:

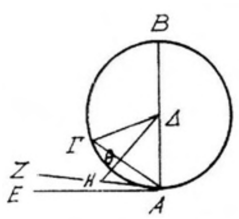
This activity starts with the notion of circle tangent that is already known to the students from the Euclidean Geometry courses. The already known properties of the circle tangent as the line "which has only one common point with the circle" or "is perpendicular to the diameter" are connected with other properties which are generalised to any curve, such as the property of "the best linear approximation of the curve at a point" or the "limiting position of the secant line". These properties which are new for the students are approached intuitively through the "local straightness" and the "magnification" of the curve. This new way offers the opportunity for a transition to the general definition of the tangent line and consequently to the introduction of the notion of the derivative of a curve at a point.

As a motivation to examine the new properties of the tangent line the students are asked to consider the truth of the following proposition:

«If we have a circle and its tangent line at one of its points A, there exists no half-line Ax which lies between the tangent line and the circle. »

This proposition is a simplified version of a property of the tangent line which appears in proposition III 16 of the *Elements* of Euclid and states the following:

«The straight line which is perpendicular to the extremity of a diameter of a circle will fall outside the circle and into the space between this line and the circle another straight line cannot be interposed. Furthermore the angle of the semicircle is greater than any other acute rectilinear angle while the remaining angle is less (than any acute angle) »[1]



This activity is divided into three steps. The first one is developed in the Euclidean Geometry context, in the second there is a transition to the Calculus context aiming to introduce the notion of derivative and finally the third one refers to points at which the function is not differentiable.

With the completion of the first step students should have understood that the tangent of a circle at a point *A* is the limiting position of the secants

[1]This is not a strict translation from the original. The explanation is added in order to make the text more comprehensible. The English translation made by Heath (1956, vol. 2, p. 37) was "The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle."

AB, as *B* approaches *A,* and that the circle within a small area around point *A looks as if it is identical* to its tangent.

The second step is an investigation of the properties observed during the first step. At first, the case of the semicircle as the curve of a function is investigated. It aims at the transition from the geometrical context to the calculus one and to helping the students in the calculation of the equation of the tangent. Afterwards, the students deal with the case of the tangent to a curve and are introduced to the notion of the derivative. After the completion of the second step, students should have understood the definition of the derivative of a function at a point and should have related it to the slope of the tangent at this point. Finally, during the third step students investigate cases of curves of functions which are not differentiable at some points. Furthermore, with the magnification of the curve we approach a visual representation of a curve which is not "smooth" at a point.

When the third step is completed the students should be able to recognise when a function is or is not differentiable at a point. Specifically if it is not differentiable they should know that if we magnify the graph at this point, it looks like a line.

In addition to the above introducing activity, more worksheets have been designed in order to help students investigate different perspectives of differentiation. This activity uses the environment of *EucliDraw* which provides tools of dynamic manipulation of geometrical shapes as well as functions. In the second and third step the electronic environment has already been constructed and the student benefits in order to follow the steps of the worksheet 4.1.1. In the first step, if the students are already familiar with such environments they could perform by themselves the constructions asked in the corresponding worksheets. Otherwise, the already constructed environment could be given. The same applies to the other worksheets.

If the introduction to the notion of the derivative has preceded this activity, then this activity could be used in introducing the tangent of a curve. In this case the worksheet should be properly adjusted. The instructor should connect the already known notion of the derivative with the slope of the tangent of the semicircle and the graph, in the second step.

Activity and Curriculum

This activity (all the worksheets) could be used in introducing the notion of the derivative in an elementary Calculus course. The perquisites for a student to be involved to this activity are the elementary knowledge of Euclidean Geometry and in specific, knowledge concerning the circle and its properties as well as general knowledge of functions, graphs of functions and limits. Particularly the first step of the worksheet 4.1.1 could be used in an Euclidean Geometry course since the knowledge of functions and limits is not necessary.

The time needed for the two first steps of worksheet 4.1.1, suggested to be handled together, is approximately two instruction hours and a third one for the third step. For the other worksheets the suggested time is one instruction hour.

4.1.1 Worksheet Analysis

Introduction to the notion of derivative

FIRST STEP "The circle tangent"

Euclid in «Elements» states that if we have a circle and its tangent at one of his points A, there exists no half-line Ax which lies between the tangent and the circle.
Let's investigate the validity of this proposition.

As a motivation of the students in this activity we used a simplified property of Proposition III 16 of the *Elements* of Euclid which states: "The straight line which is perpendicular to the extremity of a diameter of a circle will fall outside the circle and into the space between this line and the circle another straight line cannot be interposed. Furthermore the angle of the semicircle is greater than any other acute rectilinear angle while the remaining angle is less (than any acute angle)". If the instructor decides that his students could handle this proposition he could start this activity stating the whole proposition.

In a new file of *EucliDraw* sketch a circle with centre *O*, a point of it *A* and a line *l* through *A* and perpendicular to the radius *OA,* that is the tangent of the circle at *A*.

In this stage the students construct the figure by his/her own following the instructions. In this construction it could be better to add the point *A* on the circumference after the construction of the circle in order to keep the circle stable in potential changes of the position of the point *A.* If the students are not familiar with environments like *EucliDraw* we could offer them the already constructed archive *4.1.1.a.activity.en.euc* and adjust the worksheet properly.

Q1: Check if there is a line *xx΄* through point *A*, different than *l*, such as at least one the halflines *Ax* ή *Ax΄* to be between the line *l* and the circle.

(Hint: Draw a line *xx΄* through *A* and, if it is needed, magnify the region of your figure around *A* using the tool of magnification to check if your drawn line has this property. Try different positions of the line *xx΄* and check them in the magnification window.)

To construct this line it could be better to construct a free point *x* and after that a line *xx΄* such that passes through *x* and *A*. In order for the line *x΄x* to move in different positions it is enough to move point *x*. Students can colour the line *xx΄* with different colour than the ones of *l* and of the circle in order to distinguish the three lines when these are getting closer and closer.

It is possible that some students could claim that they have found a line with the above property. This could happen since the image is not clear as the line *xx΄* approaches *l*. Even in the case where all the students could claim that there is no line with the above property it is very difficult for them to support their statement with valid arguments. In any case, it could be good to encourage students to use the magnification tool and check different magnification factors big enough to realise that the drawn line is not adequate. As *xx´* approaches the tangent it gets more and more difficult to visualise the difference. In this case students can open another magnification window around a free point *K*. By moving the point *K* closer to the point *A* they could realise that the line *xx´* doesn't coincide with *l* and that there is another common point of *xx´* and the circle (except *A*). In any case when *xx´* is so close to *l,* the problem of visualisation still exists but students could make some conjectures about the position of this line and proceed to the following question.

Q2: How does the circle look in the magnification window?

The expected answer is that the circle looks like the tangent line. The main aim of this question is to lead the students intuitively to the property of the "local straightness" that characterises the tangent line.

Q3: If the line *xx´* does not coincide with *l* how many are the common points of *xx´* and circle?

The answer to this question is obvious. Through the previous tasks it is clear that any line passing through *A* and is not the tangent line, will have another common point with the circle (except *A*). This question facilitates the transition to the next task of the worksheet.

Supposing that *xx´* is a line different than *l*, passing through point *A*, name *B* the other common point of this line and the circle. Move point B so as to approach point A.

Q4: What could you say about the line AB if the point B gets closer and closer to point A?

Q5: Can you write another definition of the circle tangent line to a point of it *A*?

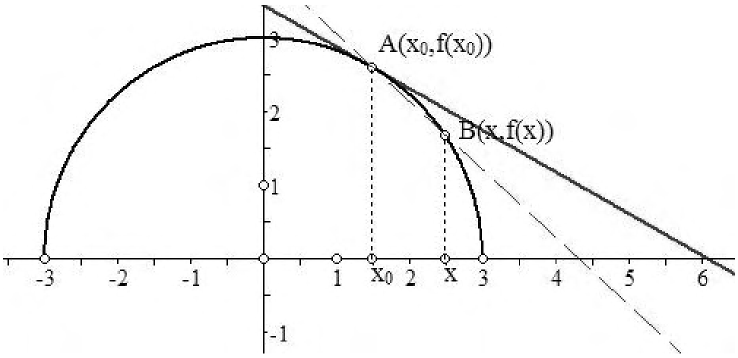
These questions aim to help students to express explicitly that "the tangent line is the limiting position of the secant lines *AB* as *B* approaches *A*». This is a new property of the tangent line that can be applied in the cases of the graph of a function. In the next step we will attempt a symbolic expression of this property. The aim of the next question is the transition to the next step.

WHAT COULD YOU SAY IF INSTEAD OF A CIRCLE, THE CURVE WAS A FUNCTION GRAPH?

SECOND STEP "Tangent line of a graph: Derivative"

This stage of the activity is connected to the previous one with the question "What could you say if, instead of a circle, the curve was a function graph?" First we consider a semicircle as the graph of a function designed in an axes system. This is the transitory stage in order to pass to the general case of the graph of a function. At first the students are asked to make some thoughts related to the way of calculating the equation of the tangent of the semicircle. This part of the activity is materialised in the worksheet and not necessarily in the environment of *EucliDraw*. Then the students make some conjectures about the shape of the tangent at a specific point of a given graph. The next step is to give an already constructed file of *EucliDraw* and the proper instructions through the worksheet for using it. If the students are familiar with *EucliDraw* environment they could proceed with the various constructions on their own.

In the following figure we have drawn the graph of the function $f(x)=\sqrt{9-x^2}, x\in[-3,3]$ which corresponds to a semicircle of radius 3 and centre the origin of the axes. The tangent of the semicircle at a point *A* and a random secant *AB* are also drawn.



Try to answer the following questions:

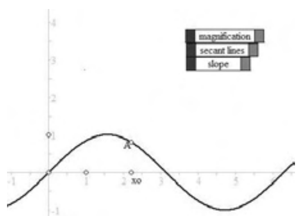
Q6: Which is the slope of the line *AB*?

The expected answer is: $\lambda = \dfrac{f(x)-f(x_0)}{x-x_0}$

Q6: Which is the slope of the tangent at *A*?

At this point we have a first introduction to the calculation of the slope via the limit: $\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}$. If the students have already been introduced to the notion of the derivative then at this point the students could move on to connecting the slope of the tangent with the derivative notion.

Open *EucliDraw* file *4.1.1.b.activity.en.euc* in which the graph of the function *f*(*x*) = sinx, has been drawn, as it can be seen in the next picture. Press the red square of the *magnification* button. On the graph you will see the



points $B(x_0+h, f(x_0+h))$ and $C(x_0-h, f(x_0-h))$. You can change the h to move these points. As h decreases the magnification factor increases. Decrease h to move points B and C closer to A and observe what is changing in the construction. Keep notes of your observations.

In this construction we have drawn the graph of the function $f(x)=\sin(x)$ as well as one of its points $A(x_0, f(x_0))$. Point A can move along the graph. In addition there are three different buttons that hide the constructions needed in the following questions. By pressing the red square the needed constructions will reveal. It would be wise for the instructor to encourage the students to make conjectures about the possible shape of the tangent line of the graph at a specific point, before the end of the activity.

Q7: What do you observe with respect to the behaviour of the curve in the interval $[x_0-h, x_0+h]$ as h becomes smaller and smaller?

This stage aims to familiarise students to the specific environment of *EucliDraw*. In this construction we can alter which affects the range of the interval $[x_0-h, x_0+h]$ as well as the magnification factor. The magnification factor k is the inverse number of h ($k=1/h$). Consequently, a decrease of the absolute value of h increases the absolute value of k, in order to have bigger magnification as the range of the region of x_o gets smaller. The values of h might be negative numbers. In this case the points B and C change their relative positions concerning the point A. Although h seems to take the value zero, this has no effect to the magnification factor ($=1/h$) since $h=0.0000$ is an approximation of a non zero value and $1/h$ is defined.

Press the red square of the button: *secant lines* to display the secants AB and AC of the points $B(x_0-h, f(x_0-h))$ and $C(x_0+h, f(x_0+h))$ of the curve. Decrease h (absolute value) and observe what is happening with these lines.

Q8: What do you notice about the behaviour of the lines *AB* and *AC* as the absolute value of *h* becomes smaller and smaller?

At this stage we sketch the secant lines in order to approach the tangent not only via magnification at a neighbourhood of the tangency point but also as the limiting position of the secants as h approaches zero.

Press the red square of the button *slope* in order to appear the slopes of the lines AB and AC. Decrease h and observe what happens with the slopes of the lines AB and AC. In the following table write the slopes of the lines AB and AC which correspond to the given values of h:

h	**Slope of *AB***	**Slope of *AC***
1		

0.1000		
0.0100		
0.0010		
0.0001		

Q9: What do you observe with respect to the slopes of *AB* and *AC* as *h* becomes smaller and smaller?

The slopes of the secants are calculated by the formula $\frac{y_2 - y_1}{x_2 - x_1}$ and the tool of *formula* offered by the software as shown in the next image. These calculations are in the *Hidden Objects* outside the working area of the students, so that the student will not be confused. We use the above table to keep some notes about the different values of *h* and the corresponding slopes. Different groups of students might have chosen different values of x_0 and will therefore have different values for the slopes. In any case the slopes will converge to the same number and the different values of x_0 will strengthen the conjecture of the existence of this convergence.

(y2-y1)/(x2-x1)
xo-h = 0.9010 cm
f(xo-h) = 0.7840 cm
xo+h = 3.0990 cm
f(xo+h) = 0.0426 cm
xo = 2.0000 cm
f(xo) = 0.9093 cm

Let *f* be a function and *A* $(x, f(x))$ a point of its graph

Q10: Can you define the tangent of the graph of the function at point *A*?

Q11: Can you write a formula for the calculation of the slope of this line?

Q12: Can you write the equation of this line?

Through these questions the notions of tangent line and derivative are introduced. It would be very interesting at this step to discuss the differences and similarities of the tangent of the circle as it was known from the Euclidean Geometry and the tangent of the graph of the function. Also it is important for the students to understand that the definition of the tangent is a "local" and not a "global" property. The next question leads to the next step.

CAN YOU ALWAYS FIND A LINE WITH THE ABOVE PROPERTY AT ANY POINT OF THE GRAPH OF ANY FUNCTION?

THIRD STEP "Non-differentiable function"

This step of the activity aims to the investigation of cases of non-differentiable functions and is connected to the previous step through the question: "Can you always find a line with the above property at any point of the graph of any function?"

At the previous file of *EucliDraw 4.1.1.b.activity.en.euc* change the type of the function to $f(x)$= abs(sin(x)).

(Hint: With *right click* on the graph select *Parameters*, the window for the handling of functions will appear. In this you will be able to define the new function after having changed the type to abs(sin(x)) from sin(x) and then select the button *Redefine Function*.)

Q13: Move point *A* to different places of the graph. Do you think that there is a tangent line at any place of the point *A*?

The students by moving point *A* could observe the "peculiar behaviour of the secants at the points (x_0, 0).

Let's examine what happens when point *A* is at the origin of the axes *O*(0,0). Move point *A* at the origin of the axes *O*. Decrease the absolute value of *h* and write down your observations concerning :

i. **the secants *AB* and *AC***

ii. **the behaviour of the graph in a small area of *A*.**

It is very difficult to place point *A* exactly at the origin of the axes. If point *A* is not exactly at the origin of the axes the students will realise it when *h* will take values very close to 0. In this case they will have to alter the position of *A* by moving it nearer to *O* and to write down their observations.

Q14: What do you notice about the limiting values of the slopes of the secants?

Q15: Is there a tangent of the graph of the function *f*(*x*)=abs(sin(x)) at point *O*? Justify your answer.

Through these questions we can introduce the conditions under which a function is not differentiable at a point, as for example, when the partial limits of the rate of change do not exist, when they exist but they are not equal, when they exist but they are not real numbers, when the function is not continuous (see more explorations in the following worksheets).

4.1.2 Worksheet Analysis
Non differentiability / differentiability and continuity

The approach in this activity is both algebraic (calculation of the derivative through the definition) and graphical (through the investigation of different values of the parameters in the environment of *EucliDraw*).

Suppose a function with formula:

$$f(x)=\begin{cases} x^2-5 & , \quad x\le a \\ cx^2+a^2-b-ca^2, & x>a \end{cases}$$

Where *α, b* and *c* are real numbers.

Q1: Find the proper values of the parameters *b* and *c* in order for the function *f* to be differentiable at *x=a,* for every value of the real number *a*.

This question is handled in a symbolic context. Students are asked to examine for which values of the parameters *b* and *c* the function is differentiable at *x=a* for every value of the real number *a*. Students have to examine first the continuity of the function at the point *x=a* and then the differentiability at this point. At this activity we expect the appearance of some sort of misunderstanding concerning the *concept* of the derivative for example not checking the continuity at *α* and the calculation of the derivative at *α* by replacement at the resulted derivatives for $x<\alpha$ and $x>\alpha$.

The correct answer to Q1 is *b*=5 and *c*=1. If the students answer wrongly, the teacher does not tell them the correct answer but he proceeds to the next step where the question is investigated visually in the environment of the software. This investigation aims to clarify the misunderstandings (whatever they are) and to offer alternative visual representations to the processes of the handling of symbols. Afterwards the students may return to E1 in order to prove symbolically their conjectures. If all the students answer correctly to E1 they can verify (visually) their answers in the electronic environment.

Open the *EucliDraw* file *4.1.2.activity.en.euc* in which the above function is sketched. Check the validity of your results by changing the values of the parameters. Afterwards write down your observations.

a. **The function is continuous at *x=a,* for every value of the real number *a*, when *b*=…**5**… and *c* =…**any real number**…**

b. **The function is differentiable at *x=α*, for every value of the real number α, when b=…**5**… and c =…**1**…**

Suppose the function with the formula:

$$f(x)=\begin{cases} x^2-5 & , \quad x\le a \\ cx^2+a^2-5-ca^2, & x>a \end{cases}$$

where *a* and *c* are real numbers with $c \neq 1$.

Q2: In the environment of the software examine if there exists a value of *a* for which the function f is differentiable regardless the value of *c*.

a=…0…

Q3: Can you prove the above result?

4.1.3 Worksheet Analysis
More about the tangent line I

In this activity students explore the properties of the tangent line as the linear approximation of the curve. This means that if $f:(m,n)\to R$ is a function , $x_0 \in (m,n)$ and l a line with equation $g(x)=ax+b$ which passes through the point $A(x_0, f(x_0))$, then the line l is the tangent line of the curve with equation $y=f(x)$ at the point A if and only if $\lim_{x\to x_0}\frac{f(x)-g(x)}{x-x_0}=0$.

In the environment of *EucliDraw* (file *4.1.3.activity.en.euc*) the graph, the tangent line K at $A(x_0, f(x_0))$ and a line L which passes through A with slope s, have been sketched. We can change the values of s and move the point A. Through the magnification tool we can magnify a neighbourhood around A by changing the magnification factor h. The h changes depending of the magnification factor since when the latter increases the region represented in the magnification window and h decrease.

For every value of h we can calculate the differences: $f(x_0+h)-L(x_0+h)$ and $f(x_0+h)-K(x_0+h)$ as well as the ratios: $\frac{f(x_0+h)-L(x_0+h)}{h}$ and $\frac{f(x_0+h)-K(x_0+h)}{h}$ for these two lines. Through these calculations students can compare the results as the values of h get smaller and smaller.

We suppose a function with the formula $f(x)=ax^2+bx+c$, where *α, b* and *c* are real numbers. Let $A(x_0, f(x_0))$ be a point of the graph of the above function and *L* a line passing through *A* with slope *s*.

Write the equation of the line *L*: *L*(*x*) = … $y=s(x-x_0)+f(x_0)$ …

Show that $\lim_{h\to 0}(f(x)-L(x))=0$

Is line *L* the tangent line?

If YES, why? …a potential answer would be that the difference between f and L tends to zero. As it is known this is not enough as there are infinite lines passing through A with slope different than $f'(x_0)$ which are not tangent lines. The following activity will clarify this difference . …

If NOT, why? …students might answer NO but the following activity will help the clarification of the difference between these two lines …

Can you calculate the correct formula of the tangent line :

***K*(*x*)=**… $y = (2ax_0 + b)(x - x_0) + f(x_0)$ …

Open the *EucliDraw* file *4.1.3.activity.en.euc*) in which the graph of the above function *f* is sketched. You may change the slope *s* of the line *L* and the software will calculate the differences and the ratios of the differences at every case. Try different values of the magnification factor and write down your observations.

4.1.4 Worksheet Analysis
More about the tangent line II

In this activity students realise the fact that a line that has only one common point with a graph is not necessary a tangent.

For this reason we investigate two different graphs of the functions: $f(x)=x^2$ and $h(x)=|x|$. The $x'x$ axe ($g(x)=0$) has with both graphs one common point the origin $O(0,0)$) but it the tangent only of the first one at this point. The difference of these two graphs concerning the line $y=0$ is the following: in the first one the limiting position of the secants halflines *OB* and *OC* are in the same line whereas in the second, the half-lines *OD* and *OE* are not. The first function satisfies the equality $\lim_{x\to x_0}\frac{f(x)-g(x)}{x-x_0}=0$ for $x_0=0$ while the second does not satisfy the $\lim_{x\to x_0}\frac{h(x)-g(x)}{x-x_0}=0$.

We suppose functions *f* and *h* with formulas: $f(x)=x^2$ and $h(x)=|x|$, for $x\in\mathbb{R}$. Open the *EucliDraw* file *4.1.4.activity.en.euc* in which the above functions *f* and *h* are sketched. Move the point *A* closer to the origin *O*.

Q1: What do you observe regarding the slopes of the half-lines *OB, OC* and *OD, OE*?

The slopes of *OB*, *OC* tend to zero while the slope of *OD*, *OE* remain constant (equal to 1 and -1, respectively)

Q2: What do you observe about the derivative of *f* and of h at *x*=0?

Press the red square of *Ratios* in order to see how the ratios $\left|\frac{f(x)}{x}\right|$ and $\left|\frac{h(x)}{x}\right|$ change. The red and green segments correspond to the values of *f*(*x*) and *h*(*x*), respectively. Move the point *A* closer to the origin *O*.

What do you observe with respect to:

a. the ratios?

b. the values of *f*(*x*) and *h*(*x*)?

Via the second equation students could formulate some conjectures about how quickly *f* tends to zero in reference to *x* (*f* tends to zero “as many times as we want” quicker than *x*).

4.1.5 Worksheet Analysis
Vertical tangent line

This is an activity in which students explore cases of vertical tangent line. This exploration aims to help students understand that the differentiability of a function, although is a sufficient condition is not necessary for the existence of the tangent line of a graph. In the contrary, the existence of limiting positions of the secants is a necessary and sufficient condition for the existence of tangent lines in all cases and in this way the tangent can be defined.

We consider the function with formula: $f(x)=\sqrt{|x|}$, where x is a real number.

Q1: Check whether f is differentiable at x=0.

Q2: If $O(0,0)$ and $B(h, f(h))$, $h>0$, what happens to the line OB as h approaches zero ?

Open *EucliDraw* file *4.1.5.activity.en.euc* in which the graph of f is sketched. Check the correctness of your results by choosing small absolute values of h and by changing the magnification factor. What do you observe?

h changes independently to the magnification factor. When h becomes small we may increase the magnification by choosing bigger values for the factor.

4.1.6 Worksheet Analysis
Geometric interpretation of the derivation of the inverse function

This activity is based on the geometric interpretation of the derivation of the inverse function. We can easily plot the graph of the inverse function by reflection with respect to the diagonal (*y*=*x*). Thus the derivative of the inverse function is calculated from the limit of the ratio of differences for which is valid:

$$\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\frac{x-x_0}{f(x)-f(x_0)}=\frac{1}{\dfrac{f(x)-f(x_0)}{x-x_0}},$$

provided that the ratios are defined. By plotting the tangents to the graph of the given function and the inverse function we notice that the tangents are symmetrical with respect to the bisector.

All the constructions in the software (*EucliDraw*) can be made by the students. If they are not familiar with the software the may use the file : *4.1.6.activity.en.euc.*

We consider the function with formula:

$$f(x)=\tan\left(\frac{x}{3}\right), x\in\left(-\frac{3\pi}{2},\frac{3\pi}{2}\right).$$

Q1: Prove that the inverse f^{-1} exists.

(Hint: Check if *f* is 1-1 in its domain).

The function $f(x)=\tan\left(\frac{x}{3}\right), x\in\left(-\frac{3\pi}{2},\frac{3\pi}{2}\right)$ is given and students are asked to check whether the inverse function exists by checking if it is 1-1.

In a new *EucliDraw* file, sketch the graphs of f and f^{-1}.

(Hint: For the construction of the graph of f^{-1} plot the line *y*=*x* and Reflection of the graph of *f* on the line *y*=*x*. If we you have any difficulty with the construction you may use the already constructed file: *4.1.6.activity.en.euc*.)

The graph of the inverse function is constructed geometrically by the following steps : we plot line *y*=*x*, take a point A of the graph of *f*, plot the *Reflection of B* of *A* with respect to *y*=*x*, and construct the locus of *B* as *A* is moving on the graph of *f*.

Plot the tangents of C_f and $C_{f^{-1}}$ at points *A*(*x, f*(*x*)) and *B*(*f*(*x*), *x*), respectively (or press the red square of *tangent line*).

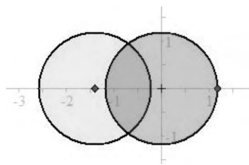
Q2: What do you observe about the slopes of the two curves? Justify your answers.

The product of the slopes equals 1. This can be justified with arguments from Calculus or from Geometry. Analytical arguments are based of the equality: $\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$, provided that the ratios are defined, as written above. As far as geometrical arguments are concerned students ascertain that the angles of these lines are complementary (angles add up to 90^o) and consequently the slopes are inverse numbers.

The teacher can lead students to observe that both tangent lines either intersect line $y=x$ at the same point $C(a,a)$ or are parallel to this line. In the first case the one tangent is the line CA whereas the other is the line CB. If CA or CB is not parallel to $x'x$ then the product of their slopes equals 1. In the second case the product of the slopes is also equal to 1 as the two slopes are equal to 1.

Further exploration

1. Let $f:\Re \to \Re$ function given by $f(x) = x^n \sin\left(\frac{1}{x}\right)$, $x \neq 0$, $f(0)=0$ and n natural number. Does the tangent line of f at point $A(0, f(0))$ for different values of n exist?



2. Let C_x be a circle with centre $(x, 0)$ and radius 1, for all $x \in R$. Calculate the area of the intersection of the two circles C_x and C_0. How does this area change for the different values of x?

4.2 Activity: Global and local extrema

Content of the activity:

This activity introduces the notions of global and local extreme.

Goals of the Activity:

Through this activity we intend that students:

- Can approach at first intuitively the notions of global and local extrema and then to be lead to their formal definitions.
- Can reflect on the previous notions, by creating with the help of the software some examples and counter-examples concerning the various cases of global and local extrema.
- Can clarify the relation between local and global extrema, also the fact that it is possible that extrema do not exist and finally to understand that if this happens, they are not unique.

The rationale of the activity

The logical structure of the activity is the following:

In the first step (4.2.1), starting with a problem concerning the population of a herd of deer, the necessity of determining the global and local extrema of a function, is presented. Their intuitive recognition at the graph of the function follows, as well as an approximate determination. The introduction of the definitions arises as a formalization of the situations appearing in the specific example. In the second step (4.2.2) these notions as well as their relation are extensively examined, with the help of various examples and counter-examples which can be produced by the software.

Activity and curriculum

This activity can be used by the students of grades 11 and 12 in the existing curriculum.

The detailed examination of all the cases of extrema can be restricted to the teaching level of Mathematical Analysis in secondary education, but can also be extended for the needs of a first year course on infinitesimal calculus at University. The actual time necessary for carrying out the activity is estimated to one instructive hour.

4.2.1 Worksheet Analysis
Use of the graph for the introduction of the notions of global and local extrema

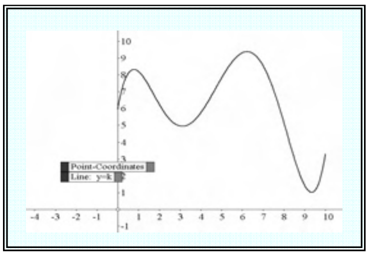
PROBLEM

The foreseen population y (in hundreds) of a herd of deer in a forest is described approximately by a function $y = P(x)$ with $0 \le x \le 10$, where x represents the years during the period from 1/1/2000 to 1/1/2010.
An environmental agency is interested to know at which moment of the period we are studying the herd had the maximum number of deer and at which moment it had the minimum.

This question refers to the local maxima.

Open the *4.2.1.activity.en.euc* file and on its screen press the button *Graph*, in order to see the graph of the function $y = P(x)$, where 0 of the $x'x$ axe corresponds to the year 2000.

The file is given constructed to the students in order to save time.
The graph presents evidence not only for the time evolution of the population of the deer but also for its change. Based on this fact a discussion could take place in relation to the potentials of the graphs and their help in better understanding a phenomenon.



Using the button *Point - Coordinates* a point M with its coordinates can appear on the graph of the function. You can change the abscissa of the point x_M, in order to shift its place on the graph and observe the corresponding ordinate y_M in various positions. Also, with the help of the parameter k *Line* $y = k$ you can shift in parallel the line $y = k$. The points of intersection of the above line with the graph, when they exist, are marked.

Q1: Is there a moment at which the herd has the maximum quantity of deer?

If yes, when does this occur and how many deer are there in the herd at that moment?

The intuitive approach from the students with the help of the software, for the notion of the global extreme is desirable. For the answer to the question the tool *Line* $y = k$ can be used. We expect students to notice that when the line passes through an extreme then this is the unique common point of the line with the curve at a neighborhood including the extreme point.

The following questions aim at the formal definition of the notion of global maximum.

Q2: We designate x_0 the moment which resulted from E1. Let $x \in [0,16]$. In what way are $P(x)$ and $P(x_0)$ related?

It is desirable that the students are lead to the notion of global maximum.

At the point x_0 we say that the function $P(x)$ presents *global maximum.*

Q3: Try to complete the following definition:
Definition: Let f be a function with domain A. f presents global maximum at point x_0 of A the value $f(x_0)$, if...........

The formal definition is expected from the students, with the help of the teacher:

Let A be a set .

«We say that the function $f : A \to \mathbb{R}$ presents global maximum at $x_0 \in A$, when $f(x) \le f(x_0)$ for every $x \in A$ ».

The following questions refer respectively to the global minimum.

Q4: Is there a moment at which the herd has the minimum population?
If yes, when does this happen and how many deer do you estimate that exist at that moment in the herd?

The approach of the notion of the global minimum by the students with the help of the software is desirable.

Q5: Can you, respectively to E3, give a definition for the global minimum?

The formal definition is expected from the students:

Let A be a set.

«We will say that the function $f : A \to \mathbb{R}$ presents global minimum at $x_0 \in A$, if $f(x) \geq f(x_0)$ for every $x \in A$ ».

After the notion of global extreme the teacher can stimulate students about the meaning of the upper and lower peaks at the graph of a function and orient the discussion towards local extrema. A basic point which will be later emphasized through questions is the appearance of local extreme as global, within a suitable open interval.

Q6: During the time period 2000-2002 is there a moment x_0 at which the deer population becomes maximum?
How many deer do you estimate that exist in the herd at that moment?

Here we try to give an intuitive image of the local extreme, which could be afterwards formalized by the students with the help of the teacher.

Q7: In what way are $P(x)$ and $P(x_0)$ related for $x \in (0,2)$;

The goal is that students would relate the extrema point with a neighborhood of x_0 or an open interval containing it, in which the local extreme is global.
More generally the teacher can, depending of the desirable goals and the available time, lead the students with the help of simple graphs or/and verbally, at the formulation of formal definitions for the local extrema.

At the point x_0 we say that the function $P(x)$ presents *local maximum*.

Q8: Try to complete the following definition:
Definition: Let f be a function with domain A. f presents local maximum at point x_0 of A the value $f(x_0)$, if...........

The formal definition is expected from the students, with the help of the teacher:

We say that the function $f : A \to \mathbb{R}$ presents local maximum at $x_0 \in A$, when there is an interval (α, β) with $x_0 \in (\alpha, \beta)$ so that $f(x) \leq f(x_0)$ for every $x \in A \cap (\alpha, \beta)$.
The teacher can add that the open interval could be of the form $(x_0 - \delta, x_0 + \delta)$. That is, an open interval with center x_0 and radius δ.

Q9: During the time period 2002-2005 is there a moment at which the population of the herd becomes minimum?
How many deer do you estimate that exist at that moment in the herd?

A first intuitive approach of the notion of local minimum through the graph is desirable.

Q10: Can you, respectively with E8, give a definition for the local minimum?

The formal definition is expected from the students:
We say that the function $f : A \to \mathbb{R}$ *presents local minimum at* $x_0 \in A$*, when there is an interval* (α, β) *with* $x_0 \in (\alpha, \beta)$*, so that* $f(x) \leq f(x_0)$ *for every* $x \in A \cap (\alpha, \beta)$.

It is necessary that there is a first contact of the students with potential local extrema, which could be at the endpoints of the intervals of the domain of a function. We could start a discussion with questions referring to the above function, such as: *Do you think that a local maximum should always be an interior point of the domain? In which cases can we have an extreme also at the endpoint of an interval?*

Q11: In the year 2009 is there a moment in which the population of the herd becomes maximum?
How many deer do you estimate that exist at that moment in the herd?

We expect students to answer that at the end of the year 2009 or on 1/1/2010 the population becomes maximum. The teacher can, in combination with the following question E12, lead to the formalization of the statement: *the function has local maximum at the endpoint* $x_0 = 10$ *of its domain* $[0,10]$.

Q12: Do you think that the previous definitions which you gave for the local extrema also include the case where x_0 is the endpoint of the interval in which the function is defined?

A discussion is desirable, starting from the local maximum at the endpoint $x_0 = 10$ of the question E11 and the local minimum at $x_0 = 0$.

Q13: Do you think that a local maximum or minimum, when it exists, is necessarily unique at a function?

The students state freely their points of view, taking under consideration the graph of the function.
Using the answers of the students in question E13 as a motivation, the discussion could be lead to consider the connection of local with global extrema.
In this direction the teacher could, by graphing on the blackboard some simple graphs, lead the students to the following:

- Local extreme means that there is a neighborhood of the point, independently from its range, in which this extreme is global (use of the global condition in order to understand the local).
- If in a close interval there are many different local extrema of the same kind, like for instance local maxima, then the global maximum (which exists from the maximum-minimum value theorem) is designated as the greater of these (from the local conditions to the global one). But if this interval is open at one of its end points at least, then it is not necessary that a local or global extreme exists. This last remark will be enriched with the examples and counter-examples of worksheet 4.2.2.

Q14: Can you deduce, if there are other local extrema, which you did not notice before, by observing the graph of the function $y = P(x)$?

We expect that all students will agree as to which are the local and which the global extrema of the function.

Q15: Can you write down all the local and global extrema of the function $y = P(x)$ that you found?

In the table a simple recording of the extrema can be done, in increasing order, as well as of their kind, so that the students will not omit any.

x						
P(x)						
Type of extreme (LM/LM, GM/GM)						

Q16: Do you think that the values which you found with the help of the software are absolutely exact? Why?

Motivated by the calculations of the program we could start a discussion concerning its potentials. If for example, one of the extrema is at $x_0 = \sqrt{2}$, how can we obtain accuracy? This discussion can lead to the necessity of finding other mathematical instruments which will allow us the accurate calculation of the values, for which the function has local or global extrema. As next step, activity 4.3 follows, introducing Fermat theorem.

4.2.2 Worksheet Analysis
Further exploration of local and global extrema

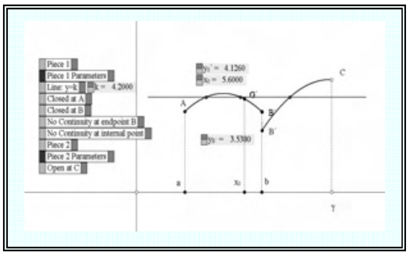
The file that follows presents a graph of two-piecewise function, which can be changed dynamically through their parameters (namely that they change, without the relations of the other objects depending from them being changed). The goal is the production of many and different cases of local and global extrema, something necessary for the enrichment of the notions formed in4.2.1.

Open the *4.2.2.activity.en.euc* file. After having opened the existing tools and the graphics that they contain, you may alter at will the parameters and make observations concerning the local or global extrema of the appearing graphs.

In order for the students to have the whole picture, they should handle simple graphs, apart from the one of the previous example. These graphs should be able to change through the parameters, so that many and different cases of local and global extrema can be produced.

From the graphs of the functions which you can construct try to locate in each case the local and global extrema, if they exist.

At this point the teacher should explain to the students the buttons and the parameters of the files, so that they can handle them. Afterwards he can lead them to construct various snapshots of graphs of continuous and discontinuous functions, with domains open or closed intervals, and they should determine whether there exist or not local or global extrema. A useful tool for this is the changing horizontal *line* with the points of intersection with the graph, which could give a first intuitive picture for the tangent at the extreme and the introduction to Fermat theorem.



Through students' answers we expect to confirm that local extrema could be the end points of the (closed) interval which is the domain of a function. Artificial cases, where Δ consists of multiple unions of intervals, are not considered here, since we consider them advanced in the context of secondary education. Nevertheless the parameterization already existing for the function of 4.2.1. allows also the division of the subintervals of its domain and consequently the extension of the problem to functions defined in union of intervals of , having as a final goal the possibility of extending the research in a different instructive/learning level.

With the help of the parameters you can alter the previous graphs. After making your observations you could answer the following:

Q1: Do you think that a local maximum is always greater than a local minimum (or that a local minimum is always less than a local maximum)?
You can make a construction with the help of the program or you can draw a graph that supports your claim.

Students can work in teams and construct various functions in order to support their claims in this and the following questions. Also they can refer to the original graph of 4.2.1.

Q2: Do you think that a function has always a global maximum or minimum?
When this exists, is it unique for a function?

Here, examples as well as counter-examples of specific not bounded functions could be mentioned. Aiming to clarify the previous notions the following examples could be given: $f(x) = \frac{1}{x}$ defined in different domains such as: $[1,4], [1,3), (0,3], \mathbb{R}\setminus\{0\}$ referring to the existence of extrema and $g(x) = \sin x$ which shows that the extrema are not unique. The Weierstrass theorem (or Maximum-Minimum value theorem) for a continuous function defined in a closed interval could also be mentioned or/and questions like the following:
Does the existence of a global extreme depend on the domain of a function? On its properties? Which ones?

Q3: Do you think that, if a function has a unique local maximum, this is always a global maximum as well?

A discussion could take place concerning the extrema in an open or closed interval and furthermore questions like the following could be posed: *When does this happen? Can you explain your claim with a suitable graph?*

See also the original graph (File 4.2.1) combined with the previous file 4.2.2, in order to answer the following questions.

Q4: If a function has many local maxima, then the greater of these is also a global maximum?
For the previous population function is something like this valid?
Do you think that this is always valid? Under which conditions?

Obviously function P is continuous in a closed interval and therefore it has global extrema. In contrast to the examples that can result from the file 4.2.2 it is expected to enrich students' meditation on the notions of local and global extrema.

Q5: If a function has many local minima then the least of these is also a global minimum?
For the previous population function is something like this valid?
Do you think that this is always valid? Under which conditions?

The comments of E4 are valid. In addition, the teacher can remind graphs which have already been studied or propose to the students to present their own examples which they can draw with the help of the software or on the worksheet.

4.3 Activity: Fermat Theorem

Content of the activity

This activity first introduces Fermat theorem and afterwards, its proof. Then we examine how it can be used in locating possible local extrema of a function.

Goals of the activity

Through this activity we aim that students:

- Be lead to a conjecture of Fermat theorem through the use of symbolical and graphical representations, to the formulation of the theorem and finally to its proof.
- Comprehend the necessity of the prerequisites of the theorem.
- Understand that the inverse theorem is not valid.
- Use the theorem for the determination of possible local extrema.

The rationale of the activity

With the help of the graphs, local extrema are related with the slope of the tangent, so that students could be lead to making a conjecture of the theorem. Then, through the definition, the local maximum is related with the slopes of the chords of the graph which have one of their ends at the local maximum. This procedure leads to the slope of the tangent at the local extreme and consequently to the proof of the theorem. With the help of examples and counter-examples we emphasize the necessity of the prerequisites and the fact that the inverse theorem is not valid. Finally we consider the potential of using the theorem in determining the local extrema.

Activity and curriculum

Depending on the partial didactical goals, this activity can be used as it is or by omitting some parts, for example those with the typical proof of the theorem. In order for the activity to be completed, one hour is needed.

4.3.1 Worksheet Analysis
Fermat theorem

The teacher could start with the need of finding the local extrema of a function referring to the problem of the activity 4.2. Except that this time, the population of the herd is given by a function with a much simpler formula, in order to be able at the end, to study the function algebraically as to the roots of the derivative.

PROBLEM

How could we find some general conditions related to the local extrema of a function, which could also help us in determining them?

Open the file *4.3.1.activity.en.euc*. In this appears the graph of the function:

$$f(x) = -0,0451\left(\frac{1}{4}\cdot x^4 - \frac{14}{3}\cdot x^3 + \frac{53}{2}\cdot x^2 - 40x\right) + 5,4 \text{ with } 0 \le x \le 10$$

which expresses the population of a deer herd related to time? Observe its shape and the points at which it presents local extrema.

The students are expected to observe the local extrema of the graph and to meditate on ways of locating them. For this purpose they can use at first the tool of *Line* which changes in parallel to the $x'x$ axis and locates its points of intersection with the graph.

In the same file press the button *Line y=k*, so that the horizontal line appears, which you can shift in parallel, with the help of parameter k. This shift in parallel may help you in locating the local extrema.

Q1: At which points does the function present local extrema?
Which of these points are interior points of the interval that we study?

We expect students to locate the local extrema of the function, to characterize them (local maximum or minimum) and to distinguish which of these are interior points of the interval.

Q2: Which property in relation to the curve do you think that the horizontal line $y = k$ has, when it passes through a local extreme?

We expect students to notice that, when the line passes through a local extreme, then locally, in the extreme point's neighborhood, it has a unique common point with the curve.

Press the button *Magnification*, to use the tool in a neighborhood of a local maximum.

Q3: Which additional property related to the curve do you think that the line $y = k$ has, when it passes through an interior local extreme?

The previous conclusion of E2, combined with the property of the local straightness of the curve (that is, magnified in a region of the local extreme the curve seems to coincide with the line $y = k$) can lead to the connection of the problem with the notion of the tangent.

The examination of the graph follows, with the help of the tool Tangent and the study of its slope, when the point of contact moves on the curve.

Press the button *Tangent*, so that the tangent of the function at a point will appear, as well as the counter of the value for the corresponding slope (rate of change).

Afterwards, you can observe the value of the slope of the tangent for various positions on the graph of the function, by altering the abscissa of the point of contact.

Q4: What do you observe concerning the slope of the tangent at local extrema?

Students experiment by shifting the tangent of the graph through the abscissa of the point of contact and at the same time they observe the counter of the slope. We expect them to observe that at the interior points it becomes zero and to be lead to a meditation regarding the observed condition. The goal is the formulation of a conjecture for Fermat theorem as well as the checking of the endpoints of the interval. Here the teacher can spend some time for a discussion in the class, so that the students will be able to express their opinions.

The tangent as limit of secants can lead the discussion to the slope of the chord AM and to the sign of the ratio of change $\frac{f(x) - f(x_0)}{x - x_0}$.

The function f presents local maximum at the interior point x_0, at which it is differentiable.

In the program press the button *Secant*, in order to make appear a chord of the graph with endpoints at the maximum $\mathrm{M}\left(x_0, f(x_0)\right)$ and a random point of $\mathrm{A}\left(x, f(x)\right)$ with $x \neq x_0$, as well as the counter of the slope of the section AM.
Shift, with the help of its abscissa, the random point x on the graph of the function, keeping it close enough to x_0.

Q5: When *x* approaches x_0 from the left (smaller values) but without coinciding, what do you observe concerning the sign of the slopes for the variable line segment AM;

It is expected to observe that the slopes remain constantly positive, when x approaches very close to x_0 from smaller values. Possibly the teacher should suggest to the students in which way they can change decimal points of different orders in the abscissa x_{M} of point M.

Q6: Can you find an algebraic formula expressing the slope of the line segment AM?

We expect students, with or without teacher's help, to give the ratio of change $\frac{f(x) - f(x_0)}{x - x_0}$.

Q7: With the help of the above formula and the relative signs, could you justify the result, observed in E5, concerning the sign of the slopes of AM?

It is expected that students will take under consideration the definition of the local maximum, namely that $f(x) \leq f(x_0)$ for every x within a zone with centre x_0, as well as that $x < x_0$, in order to prove that the ratio of change is non negative.

Q8: What could you deduce in relation to the limit of the slopes of AM, as *x* approaches x_0 from smaller values?

We expect students to reach the conclusion that the limit is a non negative number. Here, reference could be made to the relative theorem from the chapter of limits:

«If for a real function g, $g(x) \geq 0$ in a neighborhood of x_0 and the limit, $\lim_{x \to x_0} g(x)$ exists and is a real number, then $\lim_{x \to x_0} g(x) \geq 0$».

Q9: With the help of the previous theorem and the answer you gave in question E8, what can you deduce for the limit $\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$ **?**

It is expected that the students will reach the conclusion that $\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$.

Q10: Respectively with the previous reasonings, what can you deduce for the limit $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$ **?**

It is expected from the students to reach the conclusion or/and to prove that $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$.

Q11: What do you conclude for the derivative of f **at point** x_0 **?**

It is expected from the students to conclude that $f'(x_0) = 0$.

Q12: If at point x_1 f **presents local minimum what do you deduce for the derivative of** f **at point** x_1 **?**

We expect that with the help of the software the students will have observed that the derivative becomes zero at all the interior local extrema. Thus, the following question can be posed in a general way:

Q13: If at a local extreme the derivative exists, will it be necessarily equal to zero?

On the same graph students can notice that the conclusion of E11 is not necessarily valid, if the extreme is at the endpoint of the interval. So the previous question is restated:

Q14: If a local extreme of a function is an interior point of its domain and the function is differentiable at this point, which will its derivative be?

Q15: How could you formulate with the help of mathematical terms and symbols the conclusion reached at E14?

It is expected students to formulate Fermat theorem.

Q16: Could you give a complete mathematical proof of *Fermat theorem* which you stated before?

Q17: Can a function have a derivative equal to zero at a point, without this point being an extreme?

Function $y = x^3$ at $x_0 = 0$ could be a counter-example.

Open *4.3.2.activity.en.euc* where the graph of function $y = x^3$ is given and try to answer question E17.

Q18: What information does Fermat theorem give us in relation to the local extrema?

At this point the teacher can trigger a discussion as to what the meaning of necessary and sufficient condition is. The goal is that students can comprehend that the theorem provides a necessary but not sufficient condition for the local extrema. Namely, the result of the theorem is that an interior point of the domain of the function with zero derivative is a possible point of local extreme.

Q19: Can a function present a local extreme at an interior point of interval of its domain, without being differentiable at this point?

We expect students to consider the interior angular points. The function $y = |x|$ is a simple characteristic example.

Q20: If the domain of a differentiable function is an interval, at which points of the interval would you seek possible local or global extrema?

We expect students to state the interior points of the interval at which the derivative is zero, the points at which the function is not differentiable as well as the endpoints at which the interval is closed (if they exist).

Q21: Can you estimate the possible local extrema of the function $f(x) = -0{,}0451\left(\frac{1}{4}\cdot x^4 - \frac{14}{3}\cdot x^3 + \frac{53}{2}\cdot x^2 - 40x\right) + 5{,}4$ **with** $0 \le x \le 10$ **?**
Examine if f **presents global extrema and which.**

In order to check your results you can consult the graph.

Students should calculate the derivative of the function and should find its roots. This can be done either by factorization or by using the Horner form. The endpoints of the domain of definition should also be mentioned as possible extrema. During the various stages they can refer to the graph of the function in order to check their conclusions. Afterwards with the help of Weierstrass theorem (maximum and minimum value) the existence of points of global maximum and minimum is ensured. These points are those possible extrema at which the function has the maximum and minimum value respectively.

4.4 Activity: The Mean Value Theorem of Differential Calculus

Content of the activity

This activity introduces the Mean Value theorem (M.V.T) of differential calculus without its proof.

Goals of the activity

Through this activity we aim students:

- To be driven through the exploration of a specific case and the possibility of its expansion, to the conjecture of the theorem of Mean Value.
- To examine the necessary presuppositions of the M.V.T. and to comprehend that the point ξ resulting from the theorem is not unique.
- To be able to formulate the theorem

The rationale of the activity

The logical structure of the activity is the following:

At first we consider the relationship between the average and the instant velocity in a motion problem and through the graphical interpretation of this relationship the slope of the chord is related to the slope of the tangent at a certain point. Next, we examine the prerequisites of the generalization of this connection as well as if the resulting point is unique. The above leads to the typical formulation of the theorem. The proof of the theorem is not included since it is not usually taught in the secondary education.

Activity and curriculum

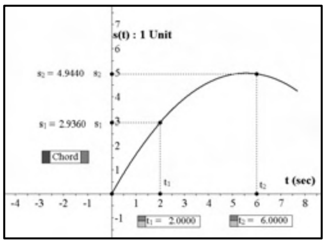
This activity introduces the Mean Value Theorem without its proof and it could be used for instruction in the secondary education.

Estimated time needed one teaching hour.

4.4.1 Worksheet Analysis
Slope of chord and derivative: The Mean Value Theorem

PROBLEM

The motion of a train is described by the function y = s(t), whose graph is given in the next figure. The independent variable t represents the time of the motion of the train and the dependent variable s(t) the distance covered by the train up to time t.



Open *4.4.1.activity.en.euc* file.

Q1: Can you estimate the average velocity of the train during the time interval $[t_1, t_2]$, where $t_1 = 2$ *sec* and $t_2 = 6$ *sec* ?
Which is the geometrical representation of the estimated average velocity in the above figure?

Here we expect from the students to relate the average velocity with the slope of the secant.

Q2: Which is the principal mathematical notion underlying all following expressions: *Instantaneous velocity, Instant (or limiting) rate of change, Slope of the tangent of the graph of a function at a point?*

It is desirable for the students to be driven, with teacher's help if necessary, to the notion of the derivative number $f'(x_0)$.

Q3: What is the graphical meaning of the measure of instant velocity of the train when $t_0 = 4$ *sec* ?

The desirable answer is the slope of the tangent of the graph at the point $(t_0, s(t_0))$.

Q4: Do you think that during the motion of the train from $t_1 = 2$ *sec* *to* $t_2 = 6$ *sec* there is a t_0 , when the measure of the instant velocity is equal to the average velocity which you estimated above for the time interval $[t_1, t_2]$?

An intuitive answer from the students would be satisfactory.

Q5: Is the conclusion of E4 valid for any moments t_1 and t_2 ? Can you express your answer with the help of symbols?

The aim is that students are driven to the conclusion that in any interval $[t_1, t_2]$ there is a moment t_0, such as $s'(t_0) = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ or otherwise that $v(t_0) = v_\mu$, where $v(t_0)$ the instant velocity at t_0 and v_μ the average velocity in the interval $[t_1, t_2]$.

Q6: Try to give a geometrical interpretation of the answer in E5.

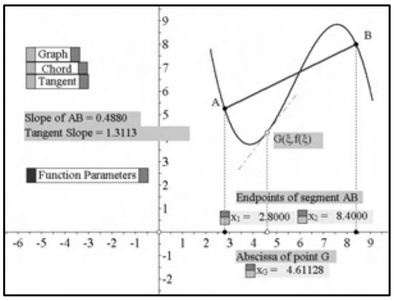
Here, with the help of the previous questions but also with the help from the teacher, we expect students to be lead to the geometrical interpretation of the M.V.T. using the terminology of the problem (average velocity, instant velocity-slope of the secant, slope of the tangent).

Using as starting points the two different notions of average and instant velocity, the teacher could, possibly by using some of the intuitive answers of the students, contribute to the formulation of a conjecture for M.V.T. This could be the motivation for students for further analysis and exploration.

Q7: Could you generalize the conclusion of E5 for a function f defined at an interval $[x_1, x_2]$?
Which is the corresponding formulation?

Open *4.4.2.activity.en.euc* file and press the appearance buttons in order to see the environment:

With the help of the relative buttons you can appear the graph of a function, the chord AB and a random point $G(\xi, f(\xi))$ of the graph as well as the tangent at that point. By changing the value of the abscissa x_G you can move the point G on the graph and make observation concerning the slopes of the tangent of the graph and of the chord AB.



Q8: By moving the point of contact G between points A and B, can you check, if there is a point in the domain of the function, which satisfies the conjecture of E6 ?

Students experiment with the graph of the specific function, trying to find out if there is a point between A and B at which the tangent will be parallel to the chord AB. It is expected from the students to ascertain graphically the M.V.T. and this will help with the rest of the activity. We note, that in order to achieve equality between the two slopes, several decimal points of different orders will have to be changed in the abscissa x_G of point G.
The teacher must help students in this technical part by explaining in what way they can change decimal points of different orders.

Afterwards students can alter the shape of the function by interfering in some of its parameters or even at the endpoints x_1, x_2, in order to find out that the slopes become equal in all these different cases.

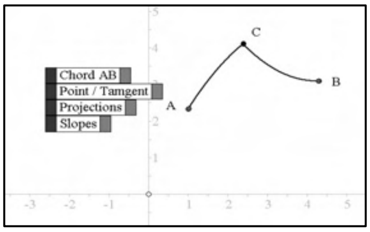
Q9: Is the point resulting from M.V.T. the unique with this specific property?

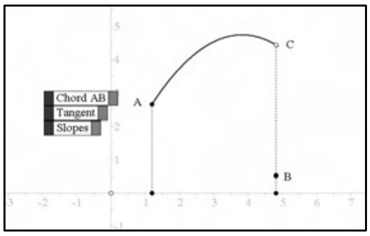
The teacher should emphasize the fact that the existence of an object does not imply uniqueness. In addition, the possible existence of a second point with the M.V.T. property has been shown from the graph of the previous file *4.4.1.activity.en.euc*. All these could help students clarify the use of the expression “at least one point”, which is used in the formal definition.

Q10: Which properties do you think that function f should have, in order for the above conjecture to be valid?

Here a general formulation of M.V.T. for the function *f* is desired with the help of the geometrical interpretation. We aim at a discussion concerning its presuppositions and its final formulation with the help of the following questions. Through the following examples emphasis is given to the presuppositions of the theorem: The differentiability in the interior of the interval and the continuity at the ending points are absolutely necessary.

Q11: Is there a real number ξ in the interior of the relative interval satisfying the conjecture of E6, for each of the following graphs?

A) 

B) 

Open the corresponding files *4.4.3.activity.en.euc* and *4.4.4.activity.en.euc*, so that with the help of the counters you can find out if the conjecture of question E6 is valid.

Students are asked with the help of the counters of the program to ascertain that the slope of AB in any case, is far out of the range of change of the slope of the tangent and therefore it is impossible for these two slopes to become equal.
Here, the teacher should make a comment about the computing deficiency which provides a slope even to an angular point, while this is not correct.

Q12: For what reason do you think the conjecture of E6 is not valid in each of the above cases?

The absolute necessity of M.V.T.'s presuppositions should be stressed by the teacher who could add questions such as:
Which are the "problematic" points of the graphs at each case? For what reason? etc.

Q13: How could you express with the help of mathematical terms and symbols, the conjecture formulated in the previous questions?

In this stage the teacher can support the students in formulating the M.V.T. in typical mathematical language, and could also give the name of the theorem. In this way he could return to some questions such as:
Do you think that M.V.T. could be applied in any function?
What kind of properties should a function have so that the theorem could be applied? (Emphasis on the presuppositions of the theorem).

4.5 Activity: Definitions and theorem of Monotonicity of a function

Content of the activity

This activity deals with the notion of the monotonicity of a function and then it introduces the theorem of monotonicity for a differentiable function on an interval as well as its proof. Finally, the activity ends with an application of this theorem on the study of a function as to the monotonicity and the extrema.

Goals of the activity

Through this activity we aim that students:

- Relate the monotonicity of a function to the signs of the slopes of chords whose endpoints are on it.
- Approach intuitively the notion of monotonicity of a function and following to be lead to the formal definitions.
- Comprehend the need for studying the subintervals of the domain of a function, on which it is monotonic.
- Be able to handle in combination symbolic and graphical representations in order to be lead to a conjecture, understanding, formulation of the theorem of monotonicity and finally to its proof.
- Understand the necessity of the presuppositions of the theorem as well as the impossibility of its rational reversion.
- Use the theorem for the study of the monotonicity and the local extrema of a function.
- Complete and interpret the table of changes of a function.

The rational of the activity

The logical structure of the activity is the following:

In the first part we try with the help of the software to relate intuitively the sign of the slope of a segment AB with the relative positions of its endpoints A and B on the graph of strictly monotonic function. The study of the intervals on the left and on the right of an extreme leads to the definitions of the monotonicity of a function. Its kind is related graphically and algebraically to the sign of the slopes of the chords with endpoints on the graph or respectively of the ratios of change.

Afterwards through the graph the sign of the derivative is related intuitively with the monotonicity and the students are lead step by step to the conjecture of the theorem of monotonicity, its formal formulation and its formal proof.

In the last part, the students, via the graphs, are asked to relate information for the function and its derivative. Then they should use the previous theorems, in order to be lead to the algebraic calculation necessary for the completion of the classic table of changes.

The students collect information by combining algebraic, numeric and graphical representations which are provided by the software.

The algebraic process and the completion of the table of changes are the final result after the conceptual understanding of the preceded notions and are not the main or /and exclusive didactical aim.

Activity and curriculum

This activity can be utilised for the following subjects:

A) The introduction of the notion of monotonicity. This part, since it does not use the notion of the derivative, could be used in an elementary Algebra course of a lower class (for instance in A class of Lyceum for Greece), in the introduction to functions.

B) The introduction, the proof and the use of the theorem of Monotonicity in the study of function, at the secondary education level or in an introductory lesson of differential calculus.

Depending on the partial didactical aims this activity can be given as a whole or by omitting some parts, as for example those that contain proofs. Estimated time is 3 instructive hours.

4.5.1 Worksheet Analysis
Monotonicity of a function

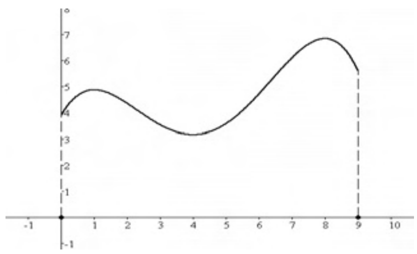
PROBLEM

The foreseen population y of a herd of deer in a forest (in hundreds) is given approximately by a function $y = f(x)$ with $0 \leq x \leq 10$, where x are the years during the time period from 1/1/2000 to 1/1/2009.
An environmentalist bureau in interested to know the periods during which the herd population increases and the periods during which it decreases.

Open the file *4.5.1.activity.en.euc*, in which the graph of the function f has been sketched. This function expresses the number of deer during the period 2000-2008 as a function of time x.
With the help of the graph of the function and the tool *Coordinates of a Point* which presents and shifts a point M on it, try to answer the following questions:

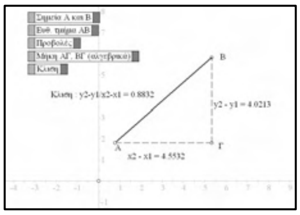
Q1: In which time periods do you estimate that the number of deer in the herd increases?
What do you observe concerning the shape of the graph in these periods?

In this, as well as in the following question E2 the desired aim is to enable the students to relate the increase or decrease of the values of the function with the shape of its graph (it *ascends* or *descends* respectively from the left to the right).



Q2: In which time periods do you estimate that the number of deer in the herd decreases?
What do you observe concerning the shape of the graph in these periods?

The aim of the following file as well as of question E3 is to familiarize students with the notion of the slope of a segment as well as to become conscious of the fact that the sign of the slope depends on the relative positions of its endpoints.



Open the file *4.5.2.activity.gr.euc*, in order to see the two points A and B, as well as the counter of the slope[2] of the segment AB. You can shift any of the two points A or B and observe the signs of the differences $y_2 - y_1$ and $x_2 - x_1$, as well as the sign of the slope of the segment AB with the help of the counter.

The shift of the points is achieved with Ctrl+1 and then by pressing the mouse on the point and dragging with the left button pressed.

Q3: After your observations by shifting at will the two points, which should the relative positions of A and B be, so that the slope of the segment AB to be:

a. Positive?
b. Negative?
c. Zero?

The teacher could probably add questions such as:
If B is at the right of A, do we always have a positive slope?
With B above A do we always have a positive slope? etc.

The special case where $x_1 = x_2$ obviously cannot be handled by the software and for this reason a comment should be needed. It concerns the examination of vertical lines for which a slope is not defined[3] .

After having closed the file *4.5.2,* in the previous file 4.5.1 press the buttons *Local Maximum* which presents a local maximum of the function and *Chord AB*, in order for the two points $\mathrm{A}(x_1, f(x_1))$ and $\mathrm{B}(x_2, f(x_2))$, the corresponding chord AB and the *counter* of its slope, to appear on the graph of the function. By pressing Ctrl+2 and then with the mouse you can shift at will A and B on the graph of the function.

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[2] Here it would be useful that the teacher reminds the notion of the slope of a line on the Cartesian plane, so that it is clarified for the rest of the activity.

[3] More generally it might be worth while on several occasions, to refer and emphasize the innate deficiency of the calculator when the denominator of a fraction is zero, in order to avoid students' misunderstandings.

Q4: By shifting points A and B on the graph of the function, on the left of point $\mathrm{M}\left(x_0, f(x_0)\right)$, what to you observe in relation to the sign of the slope of the variable chord AB?

Students are asked to observe that at the part of the graph, where the population increases, the slopes of the chords remain positive.

Q5: Which algebraic formula expresses the slope of the chord AB?

We expect the formula $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Q6: Considering $x_1 < x_2$, what can you conclude for the relation of $f(x_1)$ and $f(x_2)$?
Can you justify your answer with the help of the ratio of change?

We expect students to answer that if $x_1 < x_2$ then $f(x_1) < f(x_2)$ (1). This can be observed directly from the graph and afterwards they can be lead to the reasoning that, since the ratio of change is positive and $x_1 < x_2$, then the conclusion (1) is valid.

A function satisfying the above condition for all pairs x_1, x_2 on an interval Δ of its domain will be called strictly increasing on Δ.

Q7: With the help of mathematical terms and symbols try to express when a function is called strictly increasing on an interval Δ of its domain.

We expect students to be able to formulate the formal definition of a strictly increasing function, while the teacher should contribute to the institutionalisation and interpretation of the universal quantifier[4] $\forall x_1, x_2 \in \Delta$. This definition, because of the impossibility of its practical utilisation, will become the motive for the study of monotonicity of a function, with the help of various tools, like the sign of its derivative.

[4] The quantifier problem is multiple: conceptual, symbolical etc. Here it is used with the notion of universality and this makes difficult the direct proofs.

Q8: By shifting points A and B on the graph of the function, on the right of point $M(x_0, f(x_0))$, what to you observe in relation to the sign of the slope of the variable chord AB?

Students are asked to observe that on the part of the graph where the population decreases, the slopes of the chords remain negative.

Q9: Considering $x_1 < x_2$, what can you conclude for the relation of $f(x_1)$ and $f(x_2)$?
Can you justify your answer with the help of the ratio of change?

We expect students to answer that if $x_1 < x_2$ then $f(x_1) > f(x_2)$. This can be observed directly from the graph and afterwards they can conclude that since the ratio of change is negative and $x_1 < x_2$, then the conclusion is valid.

A function satisfying the above condition for all pairs x_1, x_2 on an interval Δ of its domain will be called strictly decreasing on Δ.

Q10: With the help of mathematical terms and symbols try to express when a function is called strictly decreasing on an interval Δ of its domain.

We expect students to give the formal definition for a strictly decreasing function. The previous remark concerning the quantifier *«for every»* is valid. If any students have difficulties with the typical statements, the formal definitions will be given after oral explanations such as:«As the values of x increase, the corresponding values of the function increase or decrease etc.».

A function f which is strictly increasing or strictly decreasing on an interval Δ will be called strictly monotonic on Δ.

The aim of the two following questions is to summarize all the previous, strengthening the relation between the monotonicity of a function on an interval of its domain or on whole $\mathbb{R}$, with the preservation of a constant sign for the slopes of all possible chords with endpoints on the graph.

Let f be a function strictly monotonic on an interval Δ and two random points $x_1, x_2 \in \Delta$ with $x_1 \neq x_2$.

Q11: i) If f is strictly increasing on Δ, what do you conclude for the sign of the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$?

ii) If f is strictly decreasing on Δ, what do you conclude for the sign of the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$?

The justification for the signs can be graphical or algebraic.

Here we attempt to relate the kind of monotonicity of a function with the sign of the ratio of changes for any two distinct points of the domain of the function. For the rest of the activity we are interested in the inverse relation, namely if the constant sign of the ratios of change for any two points of the interval Δ can ensure the kind of monotonicity of the function.

Inversely: Suppose a function f defined on an interval Δ and two points $x_1, x_2 \in \Delta$ with $x_1 \neq x_2$.

Q12: i) If for every $x_1, x_2 \in \Delta$ the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ is positive, what can you conclude for the monotonicity of the function f on Δ?

ii) If for every $x_1, x_2 \in \Delta$ the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ is negative, what can you conclude for the monotonicity of the function f on Δ?

Here finally the teacher should, in relation to the previous questions, help the students reach the natural conclusion that the kind of monotonicity of a function on an interval is being transferred to the sign of the slope (ratio of change) for any two distinct points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ of the interval in question (namely that the following equivalence is valid: Function $f:\Re \to \Re$ is strictly increasing on Δ, if and only if , is valid that $\frac{f(x_2)-f(x_1)}{x_2-x_1} > 0 \;\; \forall x_1, x_2 \in \Delta$ με $x_1 \neq x_2$). The whole procedure, at this stage, relates the graphic representation (image) to the algebraic approach (signs, operations etc).

At this point a discussion could take place, relating the monotonicity with the problem of locating the local extrema. Fermat theorem provides us with information as to what happens when we have local extrema on interior points of an interval, where the function is differentiable (and consequently a possible way of looking for local extrema on the whole). Nevertheless it does not answer the question of which of the roots of the derivative are local extrema and whether there are other which are not roots of the derivative. Therefore the problem of determining the local extrema of a function remains open. The following questions aim to lead to the possibility of finding the local extrema of a function with the help of its monotonicity intervals.

Q13: Suppose a function f defined on an interval Δ and x_0 interior point of Δ.

Which conditions near x_0 are sufficient for f to satisfy, so that it presents:

A) Local maximum?

B) Local minimum?

Students are expected to observe that a function presents, for example, local maximum at a point if it is strictly increasing at its left and strictly decreasing at its right. Afterwards the teacher can formalize the answer with the following statement: Let *f* be a function defined on an interval Δ and x_0 an interior point of Δ. If there exist intervals of the form $(\alpha, x_0]$ and $[x_0, \beta)$ at the left and the right of the point x_0 on which the function has a different kind of monotonicity then it presents a local extreme at x_0.

Also reference could be made to the case where the function presents different kind of monotonicity on intervals of the form (α, x_0) and (x_0, β). In this case the continuity at x_0 is necessary in order for this point to be a local extreme. A few straightforward examples of graphs of functions could be given, showing that, if f is not continuous at x_0, the change of the monotonicity at the left and the right of the point does not ensure always the existence of an extreme. Finally, in the context of further exploration students could be asked to open the file *4.5.3.activity.gr.euc* where there is the graph of the function $f(x) = \begin{cases} \left| x \cdot \eta\mu \dfrac{1}{x} \right| & , x \neq 0 \\ 0 & , x = 0 \end{cases}$, which, although continuous at $x_0 = 0$, where it presents a global minimum, nevertheless there exists no interval of the form $(\alpha, 0]$ or $[0, \beta)$, on which it is strictly monotonic. This counterexample shows that the previous condition with the monotonicity intervals is sufficient for the existence of interior extrema but not necessary.

Q14: What can help us in determining the local extrema of a function?

As a rational consequence of the answer to the previous question Q14 we expect students to refer to the intervals of the domain, where the function is strictly monotonic.

4.5.2 Worksheet Analysis
Connection of monotonicity and sign of the derivative

In many cases it is not easy to check the monotonicity of a function by using the definition (for example how could we prove with the use of the definition that the function $f(x) = 2^x$, $x \in \mathbb{R}$ is strictly increasing?). Using this difficulty as a motive a discussion could take place in the class, leading to the necessity of finding tools that will help in the study of the monotonicity of a function. The connection of the monotonicity with the preservation of a constant sign of the slopes of the chords, which was studied in Worksheet 4.5.1 will lead through the Mean Value Theorem to the sign of the derivative.

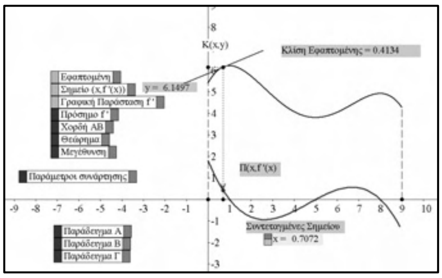
PROBLEM

How could we determine the time periods where the number of deer of the herd increases or decreases?

The use of the derivative and its contribution to the study of the function has been already established in the activity 4.3 with Fermat theorem. Here another discussion could take place for its connection with the monotonicity. Both those notions could be correlated through the slopes of the chords in the following way: As far as monotonicity is concerned, it is already related to the sign of the slopes of the chords in 4.5.1, and the derivative refers to the tangents as limit of secants. Consequently the slopes of the chords is expected to be posed as a subject for discussion before question Q7 which follows.

Open the file *4.5.4.activity.gr.euc*, where the graph of a function f appears. This function expresses the population of a herd of deer related to time. Press the button *Tangent*, in order to make appear the tangent of f at a random point $\mathrm{K}\left(x_{\mathrm{K}}, f(x_{\mathrm{K}})\right)$. The slope of the tangent at point K is shown by the counter and it can be sketched with the pen when point K shifts on the graph of the function. This can be done by shifting the abscissa x_{K} of the point, while at the same time we hold the button F7 pressed. You can observe the sign of the slope of the tangent on the counter and at the same time the graph of the derivative function $y = f'(x)$ which is drawn on the same system of axes. This graph is the curve which is sketched by the point $\Sigma\left(x_{\mathrm{K}}, f'(x_{\mathrm{K}})\right)$ as the abscissa x_{K} shifts. With Ctrl+z you can refute (in the reverse order) all the last actions you operated on the program.

Q1: On which intervals is the derivative of f posititive, on which negative and on which points is it equal to zero?



We expect students to have a first contact with the connection of the graph to the sign of the derivative. They can shift point K on the graph with the help of its abscissa, in order to ascertain, either visually or with the help of the counter, the intervals on which the slope of the tangent is positive or negative as well as the points at which its equals zero (roots of the derivative).
Students experiment with the software and we expect them to relate the numeric information with the image provided by the graph. Furthermore they can have on the same screen the graphs of the function as well as of its derivative, from where they can deduce the sign and the roots of the derivative and connect them to the monotonicity and the extrema of the function.
The graph of f' can be refuted with Ctrl+2, so that the students can experiment.

Q2: What do you deduce by observing the graph of the derivative f' in relation to the one of the original function f ?

The aim of this question is students to connect the monotonicity with the sign of the derivative.

Q3: If a function f is strictly increasing and differentiable on an interval Δ, what do you deduce for the sign of its derivative on this interval?
Try to prove this deduction.

The aim is that students first notice from the graph the connection of monotonicity to the sign of the derivative and then reach to the algebraic proof.

Since f is strictly increasing with $x \neq x_0$ it will be $\dfrac{f(x)-f(x_0)}{x-x_0} > 0$ and therefore, since it is differentiable, it is valid that $\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0} \geq 0$ for every x_0 of the interval.

The case for which we have $\dfrac{f(x)-f(x_0)}{x-x_0}>0$ for every $x \in$ □ με $x \neq x_0$ but $f'(x_0)=0$, should be noted. As an example, the function $f(x)=x^3$ at point $x_0=0$ could be referred.

Q4: If a function f is strictly decreasing and differentiable on an interval, what do you deduce for the sign of its derivative on this interval?
Try to prove this deduction.

The corresponding remarks of the previous question are valid. Also, the students could be asked to prove the deduction of E4 using the deduction of Q3 and the fact that a function f is strictly decreasing if and only if $-f$ is strictly increasing. Also we could ask students to give a example similar to the one of the previous question as for example the function $y=-x^3$.

Q5: Do you think that the sign of the derivative of a function on an interval can determine its monotonicity?
If yes, formulate the conjecture which you think is valid.

The aim is students to be lead to the conjecture that the maintenance of the sign of the derivative on an interval ensures its strict monotonicity. For this purpose we could try other functions. At this point the intuitive answers are sufficient.
The following questions aim to lead students to the proof of the conjecture.

Let Δ be an interval and two points $x_1, x_2 \in \Delta$ with $x_1 \neq x_2$.
On the screen press the button *Chord AB*, in order to make appear the chord AB with endpoints $\mathrm{A}\left(x_1, f(x_1)\right)$ and $\mathrm{B}\left(x_2, f(x_2)\right)$ on the graph of the function f. Shift (Ctrl+2) with the mouse the points inside the interval $(6,8)$, on which the function is strictly increasing.

Q6: In what way is the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ represented geometrically on the figure?

The desired answer is the slope of the chord.

Q7: How can the slope of the chord AB be connected to the derivative?

The aim of the question is students to use M.V.T. Those who have difficulty can use the software pressing the button *Theorem*, in order to see the point at which the tangent is parallel to the chord and thus to be lead to the M.V.T.

Q8: Reformulate, if you think it is necessary, and prove the conjecture you made in E5.

The aim of this question is students to formulate and prove the following theorem:

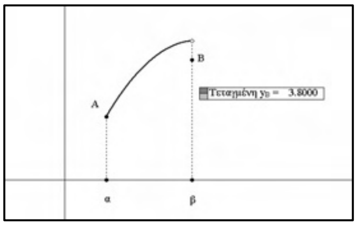
Let $f:[\alpha,\beta]\to\mathbb{R}$ be a function, continuous on $[\alpha,\beta]$ and differentiable on (α,β). Then if $f'(x)>0$ for every $x\in(\alpha,\beta)$, f is strictly increasing and if $f'(x)<0$ for every $x\in(\alpha,\beta)$, f is strictly decreasing.

The presuppositions of the theorem will arise from the use of M.V.T.

The following example shows the necessity of continuity at the endpoints of the interval.

Open the file*4.5.5.activity.gr.euc*. In this you can shift point B in parallel to $y'y$ axe with the help of its ordinate y_B and make your observations.

E9: Can you check from the graph of the example if the function satisfies the presuppositions of the above theorem?
Is the monotonicity of the function affected by the position of point B?
If yes, in what way?



The aim of this question is to show the necessity of the presupposition of continuity at the endpoints of the interval in order for the deduction of the theorem to be valid. At its original position the function satisfies the presuppositions of the theorem. The possibility of vertical shift of B and the examination of the monotonicity of the function for the various positions of B shows that the discontinuity of the function at one of the endpoints might lead to a function which is not strictly monotonic.

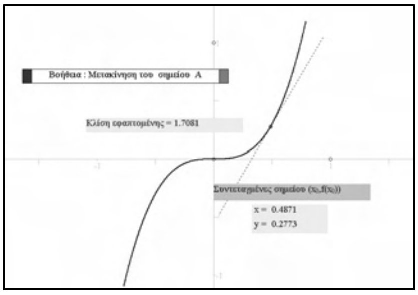
Q10: Try to formulate the reverse of the theorem of monotonicity of question E8.

The possibility of reversing a theorem in mathematics (but also in general of any rational deduction from real life) is not so clear to the students and if might be worth spending some time commenting it.

Q11: Examine if the reverse of the theorem of monotonicity is valid.

Through this question we should try to make the students understand that a counterexample is needed in order to justify that the statement is not valid.
Students are encouraged to formulate conjectures in reference to the above question.
The following actions can help them reply to E11.

Open the file *4.5.6.activity.gr.euc* so that the graph of the function $f(x) = x^3$ will appear. You can shift with the mouse point A, in order to see the changes of the slope of the tangent as it covers the graph of the function.



Now, try to answer question Q11.

4.5.3 Worksheet Analysis
Applications of the theorem of Monotonicity

In this worksheet the formula of a function expressing the population of a deer herd is given and students are asked to find the intervals of monotonicity as well as the local maxima and minima.

PROBLEM

The population of a deer herd related to time is given by the function

$$f(x) = -0,0451\left(\frac{1}{4}\cdot x^4 - \frac{14}{3}\cdot x^3 + \frac{53}{2}\cdot x^2 - 40x\right) + 5,4 \text{ with } 0 \le x \le 10.$$

How can we calculate:

A) The time periods during which it increases or decreases?

B) The moments at which it becomes momentary maximum or minimum?

C) The moments at which it has the maximum or minimum value of the whole time period that we are examining?

Q1: Can you determine the intervals of monotonicity and the extrema for the function f ?

The application of the theorem of Monotonicity should lead to the differentiation of the function and then to the finding of the roots of the derivative (by factorization or using Horner method) and the solving of the relative inequality 3rd degree. If some students have difficulties with the calculus, the intervention of the teacher or /and the separation of the students in groups can accelerate the procedure.

Q2: For each interval that you found in question E1 fill in the blanks of the following table of changes with the sign of the derivative (+ or -) as well as the symbols or for strictly increasing or strictly decreasing respectively.

x				
f΄(x)				
f(x)				

Q3: On which intervals is the preceding function strictly increasing and on which strictly decreasing?

Q4: For which values of the variable x does the function present local extrema?

Q5: Does the function f present global extrema?
If yes, for which values of the variable x?

Q6: Can you now answer the questions (A), (B) and (C) of the original problem for the population of the herd?

Q7: Can you sketch the graph of a function f, whose derivative satisfies the conditions of the table:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,3)$	3	$(3,+\infty)$
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$

The reverse possibility, namely to extract from the sign some characteristics for the shape of the graph of the function and the roots of the derivative, can enrich this connection for the students.

5. INTRODUCTION TO THE DEFINITE INTEGRAL

5.1 Activity: Introduction to the definite integral

Content of the activity:

This activity introduces the students to the notion of *Riemann integral* through the calculation of a parabolic area.

Goals of the activity

This activity intends:

- To introduce the students to the calculation of a non-linear plane.
- To understand intuitively the approximation process of the area in question by using the Riemann sums.
- To manipulate arithmetically, symbolically and geometrically the approximation process.

Rationale of the activity

Research findings show that many students, though capable to calculate definite integrals, they have not comprehended the concept itself. With this activity, which concerns the calculation of a plane area which can not be determined by the traditional methods of area calculation, an attempt is made so that students can approximate intuitively the concept of the definite integral of a continuous function.

The Riemann sums are introduced to achieve that goal. The activity consists of two parts. In the first part the dynamic environment supports the treatment of the problem of calculating the area. Dynamic tools as the control parameters for the number of covering rectangles and the magnification tool are inevitably used so as to help students understand the meaning of the questions. In the second part the students are asked to calculate an easy to compute area. In other words students can calculate the Riemann sums and determine their limits. In this way they carry out themselves the process they did during the previous phase with the aid of the program.

In conclusion, the principal goal of the activity is to form a teaching “milieu” which, on the whole contributes to the development of a mathematical meaning attached to the concept of definite integral through the formation and control of conjectures.

Activity and curriculum

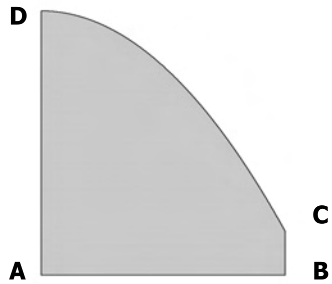
The activity can be used for the introduction of the concept of non-rectilinear plane area and the Riemann integral. The design has taken into account the students’ previous knowledge regarding the calculation of areas of rectilinear shapes and the calculation of the limit of a rational function. Its estimated execution time is 1-2 instruction hours.

5.1.1. Worksheet Analysis
Area calculation of a parabolic plane region I

Finding the area between the graph of a function and the x-axis has a surprising number of applications. For example it is well known from physics that the area enclosed by a velocity-time curve, the x΄x axis and the lines $x=t_1$, $x=t_2$ equals the distance travelled over the given time interval.

PROBLEM

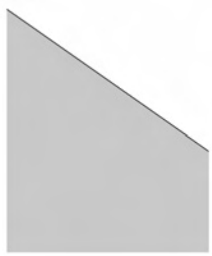
We are looking to find a way to calculate the area of the semi-parabolic region ABCD, which is bounded by three line segments AB, AD, BC, and the parabolic segment CD.



Q1: How can we think about simpler problems of the same type?

If for example instead of the parabolic function we had a linear function, how would you try to calculate the following areas?

i. $y = c$ or ii. $y = \alpha x + \beta$



Q2: i. Which plane rectilinear geometrical figures you know to calculate their area?

ii. Give the formulas for the areas of the figures mentioned before.

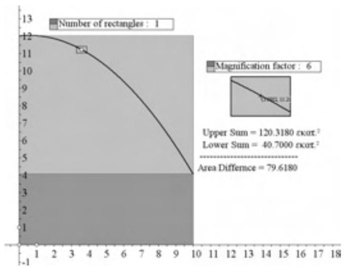
Q3: Could you use these shapes and the corresponding formulas to calculate the parabolic area asked? Why?

Q4: Could you find a rectangle that has an area smaller than the parabolic one and another rectangle with the same base having a bigger area?

The goal of the question is to prompt the students to consider the rectangles with the same base. Another formulation for question Q4, that invites students to search for the appropriate geometrical shape could be :

"How could you connect the area asked and the areas of simple geometrical figures that can be easily calculated?"

Open the *5.1.1.activity.en.euc*. file. Change (if needed) the parameter *n* which controls the number of the covering rectangles by setting it equal to 1. You could use the area counters on the screen to get a first estimation (write down a double inequality) for the area in question.



Q5: Fill in the blanks with the numbers you found before:

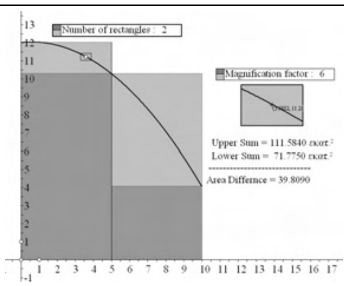
i. ....< E < ....

ii. The difference between the upper sum and the lower sum is:

$$D_1 = S_1 - s_1 = ...$$

Q6: How could you get a better approximation for area in question Q5?

Another wording could be: In what way could you proceed, so that the difference between the upper and the lower area bounds is getting less?



Parameter n controls the number of the covering rectangles, both above and under the curve, which are constructed by the software. Give to this parameter the value n=2 to construct 2 such rectangles respectively.

The area counters S_n and s_n on the screen provide the sum of the rectangles' areas above the curve and under the curve respectively. These are called Riemann sums.

Q7: Could you explain the exact way by which the values S_2 and s_2 are calculated by the program?

Q8: How could you obtain an even better approximation for the area in question? Use the area counters on the screen to get the upper and the lower Riemann sums.
Fill in the blanks with the numbers you found before:

i. .... < E < ....

ii. The difference between the upper sum and the lower sum is:

$$D_{....} = S_{....} - s_{....} = ...$$

Q9: Could you find a better approximation for the area in question?

Another wording could be: In which way can you proceed in the same fashion in order to decrease even more the difference between the upper and the lower bound for the area in question?

With the help of the same parameter n used before, change the number of rectangles having equal bases which cover exactly the interval [0,10] to n=5.

Q10: Could you find a third (and even better) approximation for the area in question? Use the area counters on the screen to get the upper and the lower Riemann sums.
Fill in the blanks with the numbers you found before:

i. .... < E < ....

ii. The difference between the upper sum and the lower sum is:

$$D_{....} = S_{....} - s_{....} = ...$$

Q11: For which number n of rectangles could you obtain an approximation of number E to the digit of integer units?
Fill in the blanks with the numbers you found before:
i. .... < E < ....
ii. The difference between the upper sum and the lower sum is:

$$D_{...} = S_{...} - s_{...} = ...$$

By pressing the *Magnification* button, you can make the magnification tool appear. Use this tool to get some sense for the accuracy of covering the parabolic area with rectangles. Change this magnification factor at will.

Q12: What do you observe concerning the differences between the areas of upper and lower rectangles appearing in the magnification window, as you alter the number n of rectangles?

Q13: Could you find an approximation to the first decimal digit for the area E? How?
Fill in the blanks with the numbers you found before:
i. .... < E < ....
ii. The difference between the upper sum and the lower sum is: $D_{....} = S_{....} - s_{....} = ...$

Q14: Fill in the blanks of the following table with the numbers you found just before.

n	**S_n**	**s_n**	**Difference : $D_n = S_n - s_n$**
1	**120.3180**	**40.7000**	**79.6180**
2			
5			
115			
809			

2516			
11096			
281068			

As you alter the number of covering rectangles:

Q15: What kind of changes can you observe regarding the values of the upper and the lower Riemann sums included in the previous table?

Q16: How is the difference $D_n = S_n - s_n$ modified?

Q17: Do you think that this process can lead to the calculation of the area in question with absolute accuracy?

Alternatively the teacher could pose the question: Is it always possible to find a value of n so that the upper and lower Riemann sums turn equal?

Q18: Do you think that this process will be completed?

Q19: Which number do you think the sum area difference $D_n = S_n - s_n$ is approaching?

Q20: How close to zero do you think that this difference D_n can approach?

It is expected that the students answer the difference can approach 0 as much we want by increasing the number of the covering rectangles.

Q21: How close to the area in question do you think that we can reach through this process?

A similar answer to the previous question is expected.

Q22: Do you think that the whole process can constitute a new way of measuring the unknown area E?

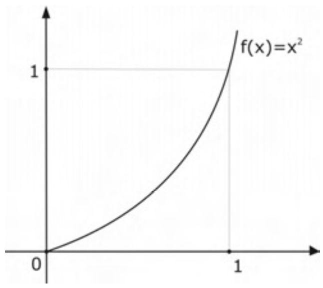
The students are expected comprehend that the above mentioned process constitutes a way of measuring only in the cases that the straightforward absolute measurement of a quantity is not possible.

5.1.2 Worksheet analysis
Area calculation of a parabolic plane region II

This activity aims at familiarizing the student with the notion of area approximation using the ancient method of bounds. The sequence of numerical intervals produced and calculated, is realized through geometrical construction-numerical calculation-algebraic processing. At this stage, it is important for the student to realize two points in particular:

- The construction which leads to the upper and lower Riemann sums, between of which the area in question lies. This entails understanding the way the upper and lower sum of rectangles are produced by the software that were used in the previous activity as well.
- The process of finding a quantity through an arbitrarily close aproximation of it.

We willl determine the area between the graph of the function $f(x) = x^2$ and the lines x=1 and y=0.



1. Divide the interval [0,1] into two equal parts

The length of each subinterval [0,1/2] and [1/2,1] is ……………

The maximum value of f on interval [0,1/2] is …......... and the minimum value is ……….

The area of the rectangle with base the interval [0,1/2] and height the maximum value of f on this interval is ………………………

The area of the rectangle with base the interval [0,1/2] and height the minimum value of f on this interval is ………………………

The maximum value of f on interval [1/2,1] is …......... and the minimum value is ……….

The area of the rectangle with base the interval [1/2,1] and height the maximum value of f on this interval is ………………………
The area of the rectangle with base the interval [1/2,1] and height the minimum value of f on this interval is ………………………

The sum of rectangles area produced by taking the maximum value of f at each interval equals to $S_2 =$ ……………..
The sum of rectangles area produced by taking the minimum value of f at each interval equals to $s_2 =$ ……………..
The difference between the area sums (Upper – Lower) is: $S_2 - s_2 =$ …………

2. Divide the interval [0,1] into three equal parts
The subintervals in this case are [,] [,] [,]
The length of each subinterval is ……….

The maximum value of f on [0,1/3] is ….......... and the minimum value is ……….
The maximum value of f on [1/3,2/3] is ….......... and the minimum value is ……….
The maximum value of f on [2/3,1] is ….......... and the minimum value is ……….

The area of the big rectangle on the interval [0,1/3] is……….. and the area of the small one is …..
The area of the big rectangle on the interval [1/3,2/3] is……….. and the area of the small one is …..
The area of the big rectangle on the interval [2/3,1] is……….. and the area of the small one is …..

The sum of the areas of the three big rectangles equals to $S_3 =$ ……………..
The sum of the areas of the three small rectangles equals to $s_3 =$ ……………..
The difference of these two is $S_3 - s_3 =$ …………

3. Divide the interval [0,1] into n equal parts; the subintervals constructed are :
[,] [,] [,] … [,]
Each one of them having length ……
The maximum value of f on the 1st subinterval is ……….. and the minimum is …….

The maximum value of f on the 2^{nd} subinterval is ……….. and the minimum is …….

………… ………… ………… ………… ………… ………… ………… ……….

The maximum value of f on the n^{th} subinterval is ……….. and the minimum is …….

The big rectangle on the 1^{st} subinterval has an area equal to ………. and the small one has an area equal to ………… .

The big r rectangle on the 2^{nd} subinterval has an area equal to ………. and the small one has an area equal to ………… …………

The big r rectangle on the 3^{rd} subinterval has an area equal to ………. and the small one has an area equal to ………… …………

The Sum of the *n* big rectangles is S_v =…………
and the sum of the *n* small rectangles is s_v = ……………

The difference of the area sums is $S_n - s_n =$ ………………..

Which number is the difference $S_n - s_n$ approaching to as *n* is increasing?

Is it possible through this process to determine the area of the region bounded by the curve $y = x^2$ and the lines x=0 and x=1?

APPENDIX
WORKSHEETS

1.1.1 Worksheet
Introduction to Infinite Processes

PROBLEM
How can we calculate the area of the circle having radius R=1?

Q1. What does it mean that the area of a triangle equals 4.5?

Q2: Find geometrical figures whose area can be measured with the previous method.

Q3: Can we divide the circle into a certain number of figures with measurable area?

Q4: In which way is it possible to link the area under question with polygons areas?

Draw two squares: One inscribed in the circle and the other one circumscribed around it.

Try to answer the question with the use of *1.1.activity.en.euc* file.

The environment provides us with the following:

We can see the circle.

The *sides* button controls the number n of the sides of the regular inscribed and circumscribed polygons.

When the *circumscribed* button is pressed, the circumscribed polygon appears. Another click results to the polygon disappearance.

The *inscribed* button acts in a similar way.

The areas of the polygons and their difference are also displayed.

The *magnification* button displays a window around a predetermined circle point. By enlarging *magnification scale* we achieve a more delicate focus.

Q5: What is the relation between the circle area E and the areas of these two squares?

Q6: What is the difference of these squares areas?

Q7: Through which process is it possible to obtain a better approximation of E?

Q8: Complete the following table:

n	Inscribed n-gon area	Circumscribed n-gon area	Areas difference lees or equal to
4			
5			
6			
10			
12			
			0.09
	3.1…	3.1…	
			0.009
	3.14…	3.14….	
			0.0009
	3.141…	3.141…	
			0.00009

Q9: Is there a step in this process in which the inscribed or the circumscribed polygon has the same area with that of the circle?

To answer the question you can use the *zoom-in* tool and focus on a circle point. By enlarging the *zooming scale* we can achieve better focus

Q10: Will this process come to an end?

Q11: Which is the number approached by the area difference?

Q12: How close to this number can the area difference ever get?

Q13: How close to the circle area can we approach?

1.2.1. Worksheet
Introduction to Sequence Limit I

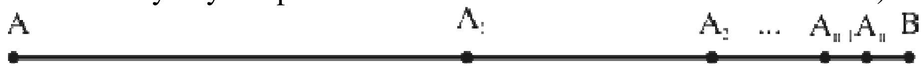
We start with a line segment AB having a length equal to 1. A point moves from A towards B in the following way:

During the first day the point covers the interval AA_1 equal to the half of the interval AB.

During the second day the point covers the interval A_1A_2 equal to the half of the interval A_1B.

Keeping on this way, during the n[th] day it covers the interval $A_{n-1}A_n$ equal to the half of the interval $A_{n-1}B$.

(That is every day the point covers half of the distance left to reach B).



Q1: Will this point arrive at B?

Q2: Calculate the length of the intervals A_nB for *n*=1,2,…

$A_1B = \frac{1}{2}$

$A_2B = \frac{1}{4}$

...

$A_nB = \frac{1}{2^n}$

Q3: Let C_1 be a point of AB so that $C_1B = 10^{-6}$.
Will the moving point go beyond C_1?

Q4: Consider the same question for points C_2, C_3
such that $C_2B = 10^{-100}$, $C_3B = 10^{-1000}$

Q5: Let C be a random point between A and B. Will the moving point get beyond C?

Q6: Can you describe the result you reached in question C5?

Q7: Complete your answer to the question Q6 in such a manner to include the information that if some day the point surpasses C then the same will hold for all the next days .

1.2.2 Worksheet
Introduction to Sequence Limit II

Let ABCD be a square having side equal to 1 and K the intersection of the two diagonals.

Let A_1, B_1, C_1 and D_1 be the midpoints of KA, KB, KC and KD respectively.

We construct the square $A_1B_1C_1D_1$.

Let A_2, B_2, C_2 and D_2 be the midpoints of KA_1, KB_1, KC_1, and KD_1 respectively.

We construct the square $A_2B_2C_2D_2$.

In general if A_n, B_n, C_n, D_n are the midpoints of KA_{n-1}, KB_{n-1}, KC_{n-1}, and KD_{n-1} respectively, we construct the square

$A_nB_nC_nD_n$, for n = 2, 3, …

This means that each square has its vertices on the midpoints of the line segments that connect K with the vertices of the square in the previous step .

Q1: Are there any other common points except K in the interior of all constructed squares ?

Open *1.2.1.activity.en.euc* file and experiment.

The number labeled "area" represents the area E_n of the square $A_nB_nC_nD_n$.

If needed, you can use the magnification tool.

Q2: Calculate the square area E_n of $A_nB_nC_nD_n$, n = 1, 2, 3, …

Q3: Is there any of these squares with area less than 10^{-60}?

Q4: Consider the same question for the number $10^{-1.000.000}$.

Q5: Let $\varepsilon > 0$. Is there a square, whose area is less than ε?

Q6: Can you find a description for the result you reached on Q5 ?

Q7: Supplement your answer on Q6 taking into account the information that if at some step the square area is less than ε, then the same holds for all the remaining steps.

Further Exploration

1. Let the sequence $\alpha_n = (-1)^n \frac{1}{n}$, n = 1, 2, …

(i) Complete the following table:

n	1	2	3	10^3	11^6	10^{100}
α_n						

(ii) Is there any real number λ being approached by the terms of α_n as n increases?

(iii) Is there a term of the sequence, so that the distance of λ from it and its successors is less than 10^{-6}?

Consider the same question for numbers 10^{-100} and 10^{-1000} respectively.

(iv) Let ε > 0. Is there a term of the sequence, so that the distance of λ from it and its successors is less than ε?

(v) Could you describe the conclusion you reached in question (iv) in a formal way?

2. Consider the same questions for the sequence $\beta_n = \dfrac{n+1}{n+2}$, n = 1, 2,…

2.1.1 Worksheet
Introduction to function limit at a point

PROBLEM
A camera has recorded a 100m race.
How could the camera's recording assist in calculating a runner's velocity at T=6sec?

- Open *activity2.1.euc EucliDraw* file. In this environment we can get the camera's recordings.
- When changing the values of t, the values of s(t), that represent the distance the runner has covered up to t, also change.
- t can approach T from less and greater values.
- Display the average velocity.

The yellow box displays the average velocity $\frac{s(T)-s(t)}{T-t}$ in the interval defined by t and T.

Q1: Fill the empty cells in the following table.

t	$\frac{s(T)-s(t)}{T-t}$	t	$\frac{s(T)-s(t)}{T-t}$
4		8	
5		7	
5.5		6.5	
5.8		6.3	
5.9		6.1	
5.93		6.07	
5.95		6.03	
5.99		6.01	
5.995		6.005	
5.999		6.001	
5.9999		6.0001	
5.99999		6.00001	

Q1: Which number does the average velocity approach as t approaches T=6sec?

Q2: What is the runner's velocity at T=6sec?

- Display the Average velocity Function U(t) in *EucliDraw* and confirm your findings graphically.
- Display the ε-zone in the *EuclidDraw* file. The points in the ε-zone have ordinate that is bigger than L-ε and less than L+ε.
- Move t so that (t,U(t)) lies inside the epsilon zone, and observe the values of the average velocity.

Q3: For which values of t is the point (t,U(t)) inside the ε=0.8 zone?

You may have some assistance on answering this question by displaying the delta zone. Points inside the δ-zone have abscissa bigger than T-δ and smaller than T+δ. Points simultaneously inside epsilon and delta zones are coloured in green. Points outside the epsilon zone are coloured in red.

Q4: Try to find a δ such that no points of the graph lie in the red area.

Q5: Decrease ε to 0.5 and find a δ such that the points (t,U(t)) do not lie inside the red area.

Q6: If ε=0.05 can you find such a δ?

You can display the magnification window. It can assist you in viewing inside a small area around (T,L).

Q7: If ε gets less and less, will we be always able to find a suitable δ with the abovementioned property?

Q8: Fill in the blank with a suitable colour in the following statement in order to express the conclusion of Q7.

"For every ε>0 we can find a δ>0 such that the function does not lie in the ……………. area."

Q9: Fill in the blanks so that the following statement bears the same conclusion as Q7

The ………………..………………………. can be arbitrarily close to …………..……………. as long as the …………………….are close enough to …………………….. and different than …..………..

Q10: Try to formulate the conclusion of Q7 using mathematical symbols

3.1 Worksheet
Introduction to continuity of a function at a point

FIRST STEP

A chemical and health care corporation is about to produce a new antibiotic pill which will be able to cure a certain disease.

It is known that the pill should contain 3gr in order to provide the patient with the right dose of medicine.

The function $f(x)=\sqrt{x+1}-1$ gives the amount $f(x)$ of the antibiotic which is detected in the blood, when a patient gets a pill with x mgr of the substance.

According to current research results, if the antibiotic detected in blood is equal or less than 0.8gr, there will be no effect on the patient's health and if it is more than 1.2gr the patient is in danger due to overdose.

Q1: Which amount of medicine is presumably detected in the patient's blood?

Q2: Which is the allowed error ε of divergence of the detected amount of medicine from the ideal value, such that the pills stay safe and effective?

The number ε=0.2 sets a boundary $(1-0.2, 1+0.2)=(0.8, 1.2)$ on the accepted quantities of antibiotic around f(3)=1. We will call ε the allowed error.

The machine available to the corporation that produces pills of t=3gr has an accuracy level adjusted to $\delta=1.1$

This means that although the machine is programmed to produce 3gr pills, the pills are not always 3gr but their weights vary between 3 – 1.1gr and 3 + 1.1gr.

Q3: Is the machine suitably adjusted to produce safe and effective pills?

- Open *EucliDraw* file 3.1.1a.activity.gr.euc and try to answer question E3 using it.
- In this environment we get the graph of f(x). By changing ε we can alter the allowed error, and by changing δ we can alter machine's accuracy.
- The magnification window allows us to focus in a neighbourhood of point (3,1).

The machine can be adjusted to another accuracy level.

Q4: Can the machine be adjusted in order to produce pills within the allowed error?

The results of a new research indicated that the error level should be reduced to $\varepsilon=0.1$.

Q5: Is there any problem with this change in pill production? Does the accuracy level have to be adjusted anew?

SECOND STEP

Q1: If the results of another research suggest that ε should be lessen more, will the corporation be always able to adjust suitably the machine?

Open *activity3.1.1.euc* file and check whether we can always find an adequate $\delta>0$, as ε gets smaller and smaller. Experiment graphically.

Display the Red/Green region and describe what it means for the function graph to lay in the green, red or white region.

If necessary, use the magnification window.

In this question, the student plays an ε-δ game giving less and less epsilons. When ε lessens enough the zones are indistinguishable, so the interest moves to the magnification window.

Q2: In each of the following sentences, fill in the blanks with the correct colours:

a. Whenever we are given an ε, we can find a δ,
such that the function does not lie in the .....red........ region.

b. For every ε we can find a δ such that for every x in the accuracy limits of the machine, (x,f(x)) lies in the ....green........... region.

Q3: Write sentence b, replacing the colours with algebraic relations.

THIRD STEP

Another research shows that the formula given by the function above which gives the quantity of drug traced in blood works well for values less than 3mgr. When values of x exceed or equal 3 it shows 0.06mgr less than the real amount detected in blood.

Q1. Find out the formula of the new function that gives the real quantity of drug detected in blood, taking into account the results of the last research.

$$g(x) = \begin{cases} \sqrt{x+1} - 1 & , \ x < 3 \\ \ldots\ldots\ldots\ldots\ldots\ldots & , \ x \geq 3 \end{cases}$$

Q2: Can the machine be adjusted properly to produce effective and safe pills?
Which δ should work for ε=0.1?

Open *activity3.1.2.euc* file

Give your answer by giving suitable values δ.

If needed use the magnification window.

Q3: What will happen if ε is reduced to 0.06? Can you find an adequate δ?

Q4: What causes this failure?

Q5. In each of the following sentences, fill in the blanks with the correct colours:

a. For a given an ε, no δ could prevent the function from lying in the .... region.

b. There is an ε, such that for every δ, there is an x in the accuracy field of the machine such that (x,g(x)) lies in the .... region.

Q6. Write sentence Q5b using algebraic relationships instead of colours.

4.1.1 Worksheet

Introduction to the notion of derivative

FIRST STEP "The circle tangent"

Euclid in «Elements» states that if we have a circle and its tangent at one of his points A, there exists no half-line Ax which lies between the tangent and the circle.
Let's investigate the validity of this proposition.

As a motivation of the students in this activity we used a simplified property of

In a new *EucliDraw* file, sketch a circle with centre *O*, a point of it *A* and a line *l* through *A* and perpendicular to the radius *OA,* that is the tangent of the circle at *A*.

Q1: Check if there is a line *xx´* through point *A*, different than *l*, such as at least one the half-lines *Ax* or *Ax´* to be between the line *l* and the circle.

(Hint: Draw a line *xx´* through *A* and, if it is needed, magnify the region of your figure around *A* using the tool of magnification to check if your drawn line has this property. Try different positions of the line *xx´* and check them in the magnification window.)

Q2: How does the circle look in the magnification window?

Q3: If the line *xx´* does not coincide with *l* how many are the common points of *xx´* and circle?

Supposing that *xx´* is a line different than *l*, passing through point *A*, name *B* the other common point of this line and the circle. Move point B so as to approach point A.

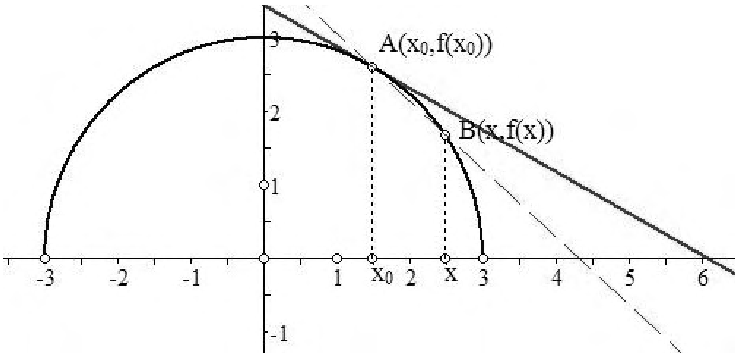
Q4: What could you say about the line AB if the point B gets closer and closer to point A?

Q5: Can you write another definition of the circle tangent line to a point of it *A*?

WHAT COULD YOU SAY IF INSTEAD OF A CIRCLE, THE CURVE WAS A FUNCTION GRAPH?

SECOND STEP "Tangent line of a graph: Derivative"

In the following figure we have drawn the graph of the function $f(x) = \sqrt{9 - x^2}$, $x \in [-3, 3]$ which corresponds to a semicircle of radius 3 and center the origin of the axes. The tangent of the semicircle at a point *A* and a random secant *AB* are also drawn.

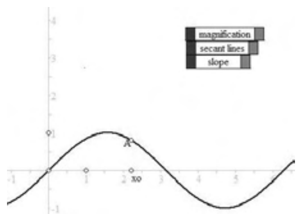


Try to answer the following questions:

Q6: Which is the slope of the line *AB*?

Q6: Which is the slope of the tangent at *A*?

Open *EucliDraw* file *4.1.1.b.activity.en.euc* in which the graph of the function $f(x)$ = sinx, has been drawn, as it can be seen in the next picture. Press the red square of the *magnification* button. On the graph you will see the points $B(x_0+h, f(x_0+h))$ and $C(x_0-h, f(x_0-h))$. You can change the h to move these points. As h decreases the magnification factor increases. Decrease h to move points *B* and *C* closer to *A*



and observe what is changing in the construction. Keep notes of your observations.

Q7: What do you observe with respect to the behaviour of the curve in the interval [x_0-*h*, x_0+*h*] as *h* becomes smaller and smaller?

Press the red square of the button: *secant lines* to display the secants *AB* and *AC* of the points *B*(x_0-*h*, *f*(x_0-*h*)) and *C*(x_0+*h*, f(x_0+*h*)) of the curve. Decrease *h* (absolute value) and observe what is happening with these lines.

Q8: What do you notice about the behaviour of the lines *AB* and *AC* as the absolute value of *h* becomes smaller and smaller?

Press the red square of the button *slope* in order to appear the slopes of the lines *AB* and *AC*. Decrease *h* and observe what happens with the slopes of the lines *AB* and *AC*. In the following table write the slopes of the lines *AB* and *AC* which correspond to the given values of *h*:

h	**Slope of *AB***	**Slope of *AC***
1		
0.1000		
0.0100		
0.0010		
0.0001		

Q9: What do you observe with respect to the slopes of *AB* and *AC* as *h* becomes smaller and smaller?

Let f be a function and $A\ (x, f(x))$ a point of its graph

Q10: Can you define the tangent of the graph of the function at point *A*?

Q11: Can you write a formula for the calculation of the slope of this line?

Q12: Can you write the equation of this line?

CAN YOU ALWAYS FIND A LINE WITH THE ABOVE PROPERTY AT ANY POINT OF THE GRAPH OF ANY FUNCTION?

THIRD STEP "Non-differentiable function"

At the previous file of *EucliDraw 4.1.1.b.activity.en.euc* change the type of the function to $f(x)$= abs(sin(x)).

(Hint: With *right click* on the graph select *Parameters*, the window for the handling of functions will appear. In this you will be able to define the new function after having changed the type to abs(sin(x)) from sin(x) and then select the button *Redefine Function*.)

Q13: Move point *A* to different places of the graph. Do you think that there is a tangent line at any place of the point *A*?

Let's examine what happens when point *A* is at the origin of the axes *O*(0,0). Move point *A* at the origin of the axes *O*. Decrease the absolute value of *h* and write down your observations concerning :

iii. the secants *AB* and *AC*

iv. the behaviour of the graph in a small area of *A*.

Q14: What do you notice about the limiting values of the slopes of the secants?

Q15: Is there a tangent of the graph of the function *f*(*x*)=abs(sin(x)) at point *O*? Justify your answer.

4.1.2 Worksheet
Non-differentiability / differentiability and continuity

Suppose a function with formula:

$$f(x)=\begin{cases} x^2-5 & , \quad x\le a \\ cx^2+a^2-b-ca^2, & x>a \end{cases}$$

Where *α, b* and *c* are real numbers.

Q1: Find the proper values of the parameters *b* and *c* in order for the function *f* to be differentiable at *x=a,* for every value of the real number *a*.

Open the *EucliDraw* file *4.1.2.activity.en.euc* in which the above function is sketched. Check the validity of your results by changing the values of the parameters. Afterwards write down your observations.

c. **The function is continuous at *x=a,* for every value of the real number *a*, when *b*=…**5**… and *c* =…**any real number**…**

d. **The function is differentiable at *x=α*, for every value of the real number *α*, when b=…**5**… and c =…**1**…**

Suppose the function with the formula:

$$f(x)=\begin{cases} x^2-5 & , \quad x\le a \\ cx^2+a^2-5-ca^2, & x>a \end{cases}$$

where α **and** c **are real numbers with** $c\neq 1$.

Q2: In the environment of the software examine if there exists a value of α **for which the function f is differentiable regardless the value of** c.

Q3: Can you prove the above result?

4.1.3 Worksheet
More about the tangent line I

We suppose a function with the formula $f(x) = ax^2 + bx + c$, where *α, b* and *c* are real numbers. Let $A\left(x_0, f(x_0)\right)$ be a point of the graph of the above function and *L* a line passing through *A* with slope *s*.

Write the equation of the line *L*: *L*(*x*) = … $y = s(x - x_0) + f(x_0)$ …

Show that $\lim_{h \to 0} \left(f(x) - L(x)\right) = 0$

Is line *L* the tangent line?

If YES, why?

If NOT, why?

Can you calculate the correct formula of the tangent line :

***K*(*x*)=**……

Open the *EucliDraw* file *4.1.3.activity.en.euc* in which the graph of the above function *f* is sketched. You may change the slope *s* of the line *L* and the software will calculate the differences and the ratios of the differences at every case. Try different values of the magnification factor and write down your observations.

4.1.4 Worksheet
More about the tangent line II

We suppose functions *f* and *h* with formulas: $f(x) = x^2$ **and** $h(x) = |x|$**, for** $x \in \mathbb{R}$ **. Open the *EucliDraw* file *4.1.4.activity.en.euc* in which the above functions *f* and *h* are sketched. Move the point *A* closer to the origin *O*.**

Q1: What do you observe regarding the slopes of the half-lines *OB, OC* and *OD, OE*?

Q2: What do you observe about the derivative of *f* and of h at *x*=0?

Press the red square of *Ratios* in order to see how the ratios $\left|\frac{f(x)}{x}\right|$ and $\left|\frac{h(x)}{x}\right|$ change. The red and green segments correspond to the values of *f*(*x*) and *h*(*x*), respectively. Move the point *A* closer to the origin *O*.

What do you observe with respect to:

a. the ratios?

b. the values of *f*(*x*) and *h*(*x*)?

4.1.5 Worksheet
Vertical tangent line

We consider the function with formula: $f(x)=\sqrt{|x|}$, where x is a real number.

Q1: Check whether f is differentiable at x=0.

Q2: If O(0,0) and B(h, $f(h)$), h>0, what happens to the line OB as h approaches zero ?

Open *EucliDraw* file *4.1.5.activity.en.euc* in which the graph of f is sketched. Check the correctness of your results by choosing small absolute values of h and by changing the magnification factor. What do you observe?

4.1.6 Worksheet
Geometric interpretation of the derivation of the inverse function

We consider the function with formula:

$f(x) = \tan\left(\frac{x}{3}\right),\ x \in \left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

Q1: Prove that the inverse f^{-1} exists.

(Hint: Check if *f* is 1-1 in its domain).

In a new *EucliDraw* file, sketch the graphs of f and f^{-1}.

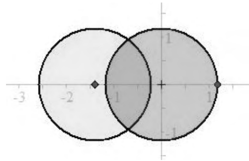
(Hint: For the construction of the graph of f^{-1} plot the line *y*=*x* and Reflection of the graph of *f* on the line *y*=*x*. If we you have any difficulty with the construction you may use the already constructed file: *4.1.6.activity.en.euc*.)

Plot the tangents of C_f and $C_{f^{-1}}$ at points *A*(*x, f*(*x*)) and *B*(*f*(*x*), *x*), respectively (or press the red square of *tangent line*).

Q2: What do you observe about the slopes of the two curves? Justify your answers.

Further exploration

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ function given by $f(x) = x^n \sin\left(\frac{1}{x}\right), x \neq 0$, $f(0) = 0$ and n natural number. Does the tangent line of f at point $A(0, f(0))$ for different values of n exist?



2. Let C_x be a circle with centre $(x, 0)$ and radius 1, for all $x \in R$. Calculate the area of the intersection of the two circles C_x and C_0. How does this area change for the different values of x?

4.2.1 Worksheet
Use of the graph for the introduction of the notions of global and local extrema

PROBLEM

The foreseen population y (in hundreds) of a herd of deer in a forest is described approximately by a function $y = P(x)$ with $0 \leq x \leq 10$, where x represents the years during the period from 1/1/2000 to 1/1/2010.
An environmental agency is interested to know at which moment of the period we are studying the herd had the maximum number of deer and at which moment it had the minimum.

Open the *4.2.1.activity.en.euc* file and on its screen press the button *Graph*, in order to see the graph of the function $y = P(x)$, where 0 of the $x'x$ axe corresponds to the year 2000.

Using the button *Point - Coordinates* a point M with its coordinates can appear on the graph of the function. You can change the abscissa of the point x_M, in order to shift its place on the graph and observe the corresponding ordinate y_M in various positions. Also, with the help of the parameter k *Line* $y = k$ you can shift in parallel the line $y = k$. The points of intersection of the above line with the graph, when they exist, are marked.

Q1: Is there a moment at which the herd has the maximum quantity of deer?
If yes, when does this occur and how many deer are there in the herd at that moment?

Q2: We designate x_0 the moment which resulted from E1. Let $x \in [0,16]$. In what way are $P(x)$ and $P(x_0)$ related?

At the point x_0 we say that the function $P(x)$ presents *global maximum*.

Q3: Try to complete the following definition:

Definition: Let f be a function with domain A. f presents global maximum at point x_0 of A the value $f(x_0)$, if…………

Q4: Is there a moment at which the herd has the minimum population?

If yes, when does this happen and how many deer do you estimate that exist at that moment in the herd?

Q5: Can you, respectively to E3, give a definition for the global minimum?

Q6: During the time period 2000-2002 is there a moment x_0 at which the deer population becomes maximum?
How many deer do you estimate that exist in the herd at that moment?

Q7: In what way are $P(x)$ and $P(x_0)$ related for $x \in (0, 2)$;

At the point x_0 we say that the function $P(x)$ presents *local maximum*.

Q8: Try to complete the following definition:
Definition: Let f be a function with domain A. f presents local maximum at point x_0 of A the value $f(x_0)$, if………..

Q9: During the time period 2002-2005 is there a moment at which the population of the herd becomes minimum?
How many deer do you estimate that exist at that moment in the herd?

Q10: Can you, respectively with E8, give a definition for the local minimum?

Q11: In the year 2009 is there a moment in which the population of the herd becomes maximum?
How many deer do you estimate that exist at that moment in the herd?

Q12: Do you think that the previous definitions which you gave for the local extrema also include the case where x_0 is the endpoint of the interval in which the function is defined?

Q13: Do you think that a local maximum or minimum, when it exists, is necessarily unique at a function?

Q14: Can you deduce, if there are other local extrema, which you did not notice before, by observing the graph of the function $y = P(x)$?

Q15: Can you write down all the local and global extrema of the function $y = P(x)$ that you found?

x						
P(x)						
Type of extreme (LM/LM, GM/GM)						

Q16: Do you think that the values which you found with the help of the software are absolutely exact? Why?

4.2.2 Worksheet
Further exploration of local and global extrema

Open the *4.2.2.activity.en.euc* file. After having opened the existing tools and the graphics that they contain, you may alter at will the parameters and make observations concerning the local or global extrema of the appearing graphs.

From the graphs of the functions which you can construct try to locate in each case the local and global extrema, if they exist.

With the help of the parameters you can alter the previous graphs. After making your observations you could answer the following:

Q1: Do you think that a local maximum is always greater than a local minimum (or that a local minimum is always less than a local maximum)?
You can make a construction with the help of the program or you can draw a graph that supports your claim.

Q2: Do you think that a function has always a global maximum or minimum?
When this exists, is it unique for a function?

Q3: Do you think that, if a function has a unique local maximum, this is always a global maximum as well?

See also the original graph (File 4.2.1) combined with the previous file 4.2.2, in order to answer the following questions.

Q4: If a function has many local maxima, then the greater of these is also a global maximum?
For the previous population function is something like this valid?
Do you think that this is always valid? Under which conditions?

Q5: If a function has many local minima then the least of these is also a global minimum?
For the previous population function is something like this valid?
Do you think that this is always valid? Under which conditions?

4.3.1 Worksheet
Fermat theorem

PROBLEM
How could we find some general conditions related to the local extrema of a function, which could also help us in determining them?

Open the file *4.3.1.activity.en.euc*. In this appears the graph of the function:

$$f(x) = -0,0451\left(\frac{1}{4}\cdot x^4 - \frac{14}{3}\cdot x^3 + \frac{53}{2}\cdot x^2 - 40x\right) + 5,4 \text{ with } 0 \le x \le 10$$

which expresses the population of a deer herd related to time? Observe its shape and the points at which it presents local extrema.

In the same file press the button *Line y=k*, so that the horizontal line appears, which you can shift in parallel, with the help of parameter k. This shift in parallel may help you in locating the local extrema.

Q1: At which points does the function present local extrema?
Which of these points are interior points of the interval that we study?

Q2: Which property in relation to the curve do you think that the horizontal line $y = k$ has, when it passes through a local extreme?

Press the button *Magnification*, to use the tool in a neighbourhood of a local maximum.

Q3: Which additional property related to the curve do you think that the line $y = k$ has, when it passes through an interior local extreme?

Press the button *Tangent*, so that the tangent of the function at a point will appear, as well as the counter of the value for the corresponding slope (rate of change).
Afterwards, you can observe the value of the slope of the tangent for various positions on the graph of the function, by altering the abscissa of the point of contact.

Q4: What do you observe concerning the slope of the tangent at local extrema?

The function f presents local maximum at the interior point x_0, at which it is differentiable.
In the program press the button *Secant*, in order to make appear a chord of the graph with endpoints at the maximum $\mathrm{M}\left(x_0, f(x_0)\right)$ and a random point of $\mathrm{A}\left(x, f(x)\right)$ with $x \neq x_0$, as well as the counter of the slope of the section AM.
Shift, with the help of its abscissa, the random point x on the graph of the function, keeping it close enough to x_0.

Q5: When *x* approaches x_0 from the left (smaller values) but without coinciding, what do you observe concerning the sign of the slopes for the variable line segment AM;

Q6: Can you find an algebraic formula expressing the slope of the line segment AM?

Q7: With the help of the above formula and the relative signs, could you justify the result, observed in E5, concerning the sign of the slopes of AM?

Q8: What could you deduce in relation to the limit of the slopes of AM, as x approaches x_0 from smaller values?

Q9: With the help of the previous theorem and the answer you gave in question E8, what can you deduce for the limit

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} ?$$

Q10: Respectively with the previous reasonings, what can you deduce for the limit

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} ?$$

Q11: What do you conclude for the derivative of f at point x_0?

Q12: If at point x_1 f presents local minimum what do you deduce for the derivative of f at point x_1?

Q13: If at a local extreme the derivative exists, will it be necessarily equal to zero?

Q14: If a local extreme of a function is an interior point of its domain and the function is differentiable at this point, which will its derivative be?

Q15: How could you formulate with the help of mathematical terms and symbols the conclusion reached at E14?

Q16: Could you give a complete mathematical proof of *Fermat theorem* which you stated before?

Q17: Can a function have a derivative equal to zero at a point, without this point being an extreme?

Open *4.3.2.activity.en.euc* where the graph of function $y = x^3$ is given and try to answer question E17.

Q18: What information does Fermat theorem give us in relation to the local extrema?

Q19: Can a function present a local extreme at an interior point of interval of its domain, without being differentiable at this point?

Q20: If the domain of a differentiable function is an interval, at which points of the interval would you seek possible local or global extrema?

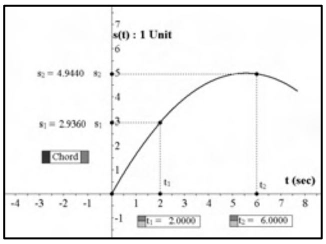
Q21: Can you estimate the possible local extrema of the function $f(x) = -0,0451\left(\frac{1}{4}\cdot x^4 - \frac{14}{3}\cdot x^3 + \frac{53}{2}\cdot x^2 - 40x\right) + 5,4$ **with** $0 \le x \le 10$ **?**
Examine if f **presents global extrema and which.**

In order to check your results you can consult the graph.

4.4.1 Worksheet
Slope of chord and derivative: The Mean Value Theorem

PROBLEM

The motion of a train is described by the function y = s(t), whose graph is given in the next figure. The independent variable t represents the time of the motion of the train and the dependent variable s(t) the distance covered by the train up to time t.



Open *4.4.1.activity.en.euc* file.

Q1: Can you estimate the average velocity of the train during the time interval $[t_1, t_2]$, where $t_1 = 2$ *sec* and $t_2 = 6$ *sec* ?
Which is the geometrical representation of the estimated average velocity in the above figure?

Q2: Which is the principal mathematical notion underlying all following expressions: *Instantaneous velocity, Instant (or limiting) rate of change, Slope of the tangent of the graph of a function at a point?*

Q3: What is the graphical meaning of the measure of instant velocity of the train when $t_0 = 4\ sec$?

Q4: Do you think that during the motion of the train from $t_1 = 2\ sec$ to $t_2 = 6\ sec$ there is a t_0 , when the measure of the instant velocity is equal to the average velocity which you estimated above for the time interval $[t_1 , t_2]$?

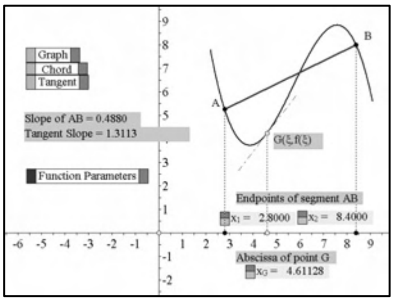
Q5: Is the conclusion of E4 valid for any moments t_1 and t_2 ? Can you express your answer with the help of symbols?

Q6: Try to give a geometrical interpretation of the answer in E5.

Q7: Could you generalize the conclusion of E5 for a function f defined at an interval $[x_1, x_2]$?
Which is the corresponding formulation?

Open *4.4.2.activity.en.euc* file and press the appearance buttons in order to see the environment:

With the help of the relative buttons you can appear the graph of a function, the chord AB and a random point $G(\xi, f(\xi))$ of the graph as well as the tangent at that point. By changing the value of the abscissa x_G you can move the point G on the graph and make observation concerning the slopes of the tangent of the graph and of the chord AB.

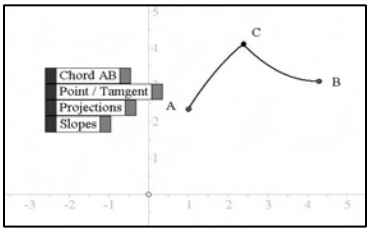


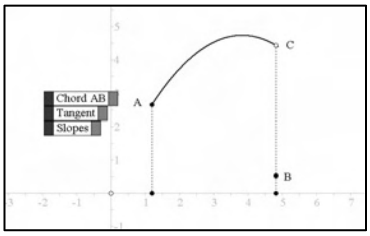
Q8: By moving the point of contact G between points A and B, can you check, if there is a point in the domain of the function, which satisfies the conjecture of E6 ?

Q9: Is the point resulting from M.V.T. the unique with this specific property?

Q10: Which properties do you think that function f should have, in order for the above conjecture to be valid?

Q11: Is there a real number ξ in the interior of the relative interval satisfying the conjecture of E6, for each of the following graphs?

B) 

B) 

Open the corresponding files *4.4.3.activity.en.euc* and *4.4.4.activity.en.euc*, so that with the help of the counters you can find out if the conjecture of question E6 is valid.

Q12: For what reason do you think the conjecture of E6 is not valid in each of the above cases?

Q13: How could you express with the help of mathematical terms and symbols, the conjecture formulated in the previous questions?

4.5.1 Worksheet
Monotonicity of a function

PROBLEM

The foreseen population y of a herd of deer in a forest (in hundreds) is given approximately by a function $y = f(x)$ with $0 \leq x \leq 10$, where x are the years during the time period from 1/1/2000 to 1/1/2009.
An environmentalist bureau in interested to know the periods during which the herd population increases and the periods during which it decreases.

Open the file *4.5.1.activity.en.euc*, in which the graph of the function f has been sketched. This function expresses the number of deer during the period 2000-2008 as a function of time x.
With the help of the graph of the function and the tool *Coordinates of a Point* which presents and shifts a point M on it, try to answer the following questions:

Q1: In which time periods do you estimate that the number of deer in the herd increases?
What do you observe concerning the shape of the graph in these periods?

Q2: In which time periods do you estimate that the number of deer in the herd decreases?
What do you observe concerning the shape of the graph in these periods?

Open the file *4.5.2.activity.gr.euc*, in order to see the two points A and B, as well as the counter of the slope of the segment AB. You can shift any of the two points A or B and observe the signs of the differences $y_2 - y_1$ and $x_2 - x_1$, as well as the sign of the slope of the segment AB with the help of the counter.

The shift of the points is achieved with Ctrl+1 and then by pressing the mouse on the point and dragging with the left button pressed.

Q3: After your observations by shifting at will the two points, which should the relative positions of A and B be, so that the slope of the segment AB to be:

a. Positive?
b. Negative?
c. Zero?

After having closed the file *4.5.2,* in the previous file 4.5.1 press the buttons *Local Maximum* which presents a local maximum of the function and *Chord AB*, in order for the two points $\mathrm{A}\left(x_1, f(x_1)\right)$ and $\mathrm{B}\left(x_2, f(x_2)\right)$, the corresponding chord AB and the *counter* of its slope, to appear on the graph of the function. By pressing Ctrl+2 and then with the mouse you can shift at will A and B on the graph of the function.

Q4: By shifting points A and B on the graph of the function, on the left of point $\mathrm{M}\left(x_0, f(x_0)\right)$, what to you observe in relation to the sign of the slope of the variable chord AB?

Q5: Which algebraic formula expresses the slope of the chord AB?

Q6: Considering $x_1 < x_2$, what can you conclude about the relation of $f(x_1)$ and $f(x_2)$?
Can you justify your answer with the help of the ratio of change?

A function satisfying the above condition for all pairs x_1, x_2 on an interval Δ of its domain will be called strictly increasing on Δ.

Q7: With the help of mathematical terms and symbols try to express when a function is called strictly increasing on an interval Δ of its domain.

Q8: By shifting points A and B on the graph of the function, on the right of point $M(x_0, f(x_0))$, what to you observe in relation to the sign of the slope of the variable chord AB?

Q9: Considering $x_1 < x_2$, what can you conclude for the relation of $f(x_1)$ and $f(x_2)$?
Can you justify your answer with the help of the ratio of change?

A function satisfying the above condition for all pairs x_1, x_2 on an interval Δ of its domain will be called strictly decreasing on Δ.

Q10: With the help of mathematical terms and symbols try to express when a function is called strictly decreasing on an interval Δ of its domain.

A function f which is strictly increasing or strictly decreasing on an interval Δ will be called strictly monotonic on Δ.

Let f be a function strictly monotonic on an interval Δ and two random points $x_1, x_2 \in \Delta$ with $x_1 \neq x_2$.

Q11: i) If f is strictly increasing on Δ, what do you conclude for the sign of the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$?
ii) If f is strictly decreasing on Δ, what do you conclude for the sign of the ratio of change $\frac{f(x_2)-f(x_1)}{x_2-x_1}$?

Inversely: Suppose a function f defined on an interval Δ and two points $x_1, x_2 \in \Delta$ with $x_1 \neq x_2$.

Q12: i) If for every $x_1, x_2 \in \Delta$ the ratio of change $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is positive, what can you conclude for the monotonicity of the function f on Δ?

ii) If for every $x_1, x_2 \in \Delta$ the ratio of change $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is negative, what can you conclude for the monotonicity of the function f on Δ?

Q13: Suppose a function f defined on an interval Δ and x_0 interior point of Δ.

Which conditions near x_0 are sufficient for f to satisfy, so that it presents:

A) Local maximum?

B) Local minimum?

Q14: What can help us in determining the local extrema of a function?

4.5.2 Worksheet
Connection of monotonicity and sign of the derivative

PROBLEM
How could we determine the time periods where the number of deer of the herd increases or decreases?

Open the file *4.5.4.activity.gr.euc*, where the graph of a function f appears. This function expresses the population of a herd of deer related to time. Press the button *Tangent*, in order to make appear the tangent of f at a random point $\mathrm{K}\left(x_{\mathrm{K}}, f(x_{\mathrm{K}})\right)$. The slope of the tangent at point K is shown by the counter and it can be sketched with the pen when point K shifts on the graph of the function. This can be done by shifting the abscissa x_{K} of the point, while at the same time we hold the button F7 pressed. You can observe the sign of the slope of the tangent on the counter and at the same time the graph of the derivative function $y = f'(x)$ which is drawn on the same system of axes. This graph is the curve which is sketched by the point $\Sigma\left(x_{\mathrm{K}}, f'(x_{\mathrm{K}})\right)$ as the abscissa x_{K} shifts. With Ctrl+z you can refute (in the reverse order) all the last actions you operated on the program.

Q1: For which intervals is the derivative of f positive, for which negative and for which points is it equal to zero?

Q2: What do you deduce by observing the graph of the derivative f' in relation to the one of the original function f ?

Q3: If a function f is strictly increasing and differentiable on an interval Δ, what do you deduce for the sign of its derivative on this interval?
Try to prove this deduction.

Q4: If a function f is strictly decreasing and differentiable on an interval, what do you deduce for the sign of its derivative on this interval?

Try to prove this deduction.

Q5: Do you think that the sign of the derivative of a function on an interval can determine its monotonicity?

If yes, formulate the conjecture which you think is valid.

Let Δ be an interval and two points $x_1, x_2 \in \Delta$ with $x_1 \neq x_2$.
On the screen press the button *Chord AB*, in order to make appear the chord AB with endpoints $\mathrm{A}\left(x_1, f(x_1)\right)$ and $\mathrm{B}\left(x_2, f(x_2)\right)$ on the graph of the function f. Shift (Ctrl+2) with the mouse the points inside the interval $(6,8)$, on which the function is strictly increasing.

Q6: In what way is the ratio of change $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ represented geometrically on the figure?

Q7: How can the slope of the chord AB be connected to the derivative?

Q8: Reformulate, if you think it is necessary, and prove the conjecture you made in E5.

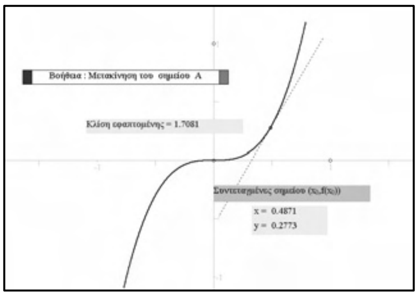
Open the file*4.5.5.activity.gr.euc*. In this you can shift point B in parallel to $y'y$ axe with the help of its ordinate y_B and make your observations.

E9: Can you check from the graph of the exemple if the function satisfies the presuppositions of the above theorem?
Is the monotonicity of the function affected by the position of point B?
If yes, in what way?

Q10: Try to formulate the reverse of the theorem of monotonicity of question E8.

Q11: Examine if the reverse of the theorem of monotonicity is valid.

Open the file *4.5.6.activity.gr.euc* so that the graph of the function $f(x) = x^3$ will appear. You can shift with the mouse point A, in order to see the changes of the slope of the tangent as it covers the graph of the function.



Now, try to answer question Q11.

4.5.3 Worksheet
Applications of the theorem of Monotonicity for the study of a function

PROBLEM

The population of a deer herd related to time is given by the function

$$f(x) = -0,0451\left(\frac{1}{4}\cdot x^4 - \frac{14}{3}\cdot x^3 + \frac{53}{2}\cdot x^2 - 40x\right) + 5,4 \text{ with } 0 \le x \le 10.$$

How can we calculate:

A) The time periods during which it increases or decreases?

B) The moments at which it becomes momentary maximum or minimum?

C) The moments at which it has the maximum or minimum value of the whole time period that we are examining?

Q1: Can you determine the intervals of monotonicity and the extrema for the function f ?

Q2: For each interval that you found in question E1 fill in the blanks of the following table of changes with the sign of the derivative (+ or -) as well as the symbols □ or □ for strictly increasing or strictly decreasing respectively.

x				
f′(x)				
f(x)				

Q3: On which intervals is the preceding function strictly increasing and on which strictly decreasing?

Q4: For which values of the variable x does the function present local extrema?

Q5: Does the function f present global extrema?
If yes, for which values of the variable x?

Q6: Can you now answer the questions (A), (B) and (C) of the original problem for the population of the herd?

Q7: Can you sketch the graph of a function f, whose derivative satisfies the conditions of the table:

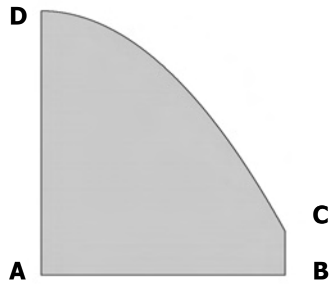
x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,3)$	3	$(3,+\infty)$
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$

5.1.1. Worksheet
Area calculation of a parabolic plane region I

Finding the area between the graph of a function and the x-axis has a surprising number of applications. For example it is well known from physics that the area enclosed by a velocity-time curve, the x΄x axis and the lines $x=t_1$, $x=t_2$ equals the distance travelled over the given time interval.

PROBLEM

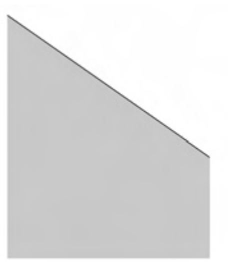
We are looking to find a way to calculate the area of the semi-parabolic region ABCD, which is bounded by three line segments AB, AD, BC, and the parabolic segment CD.



Q1: How can we think about simpler problems of the same type?

If for example instead of the parabolic function we had a linear function, how would you try to calculate the following areas?

i. $y = c$ or ii. $y = \alpha x + \beta$

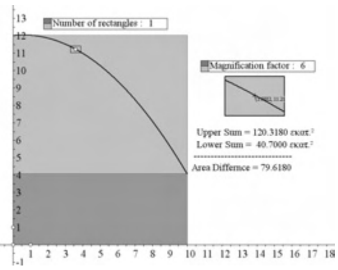


Q2: i. Which plane rectilinear geometrical figures you know to calculate their area?
ii. Give the formulas for the areas of the figures mentioned before.

Q3: Could you use these shapes and the corresponding formulas to calculate the parabolic area asked? Why?

Q4: Could you find a rectangle that has an area smaller than the parabolic one and another rectangle with the same base having a bigger area?

Open the *5.1.1.activity.en.euc*. file. Change (if needed) the parameter *n* which controls the number of the covering rectangles by setting it equal to 1. You could use the area counters on the screen to get a first estimation (write down a double inequality) for the area in question.



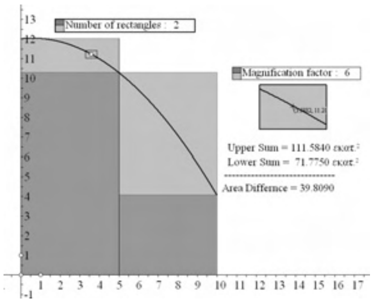
Q5: Fill in the blanks with the numbers you found before:

i. $.................. < E < ....$

ii. The difference between the upper sum and the lower sum is:

$$D_1 = S_1 - s_1 = ...$$

Q6: How could you get a better approximation for area in question Q5?



Parameter n controls the number of the covering rectangles, both above and under the curve, which are constructed by the software. Give to this parameter the value n=2 to construct 2 such rectangles respectively.

The area counters S_n and s_n on the screen provide the sum of the rectangles' areas above the curve and under the curve respectively. These are called Riemann sums.

Q7: Could you explain the exact way by which the values S_2 and s_2 are calculated by the program?

Q8: How could you obtain an even better approximation for the area in question? Use the area counters on the screen to get the upper and the lower Riemann sums.
Fill in the blanks with the numbers you found before:

i. $.................. < E < ....$

ii. The difference between the upper sum and the lower sum is:

$$D_{....} = S_{....} - s_{....} = ...$$

Q9: Could you find a better approximation for the area in question?

With the help of the same parameter n used before, change the number of rectangles having equal bases which cover exactly the interval [0,10] to n=5.

Q10: Could you find a third (and even better) approximation for the area in question? Use the area counters on the screen to get the upper and the lower Riemann sums.
Fill in the blanks with the numbers you found before:
i. .... < E < ....
ii. The difference between the upper sum and the lower sum is:

$$D_{....} = S_{....} - s_{....} = ...$$

Q11: For which number n of rectangles could you obtain an approximation of number E to the digit of integer units?
Fill in the blanks with the numbers you found before:
i. .... < E < ....
ii. The difference between the upper sum and the lower sum is:

$$D_{...} = S_{...} - s_{...} = ...$$

By pressing the *Magnification* button, you can make the magnification tool appear. Use this tool to get some sense for the accuracy of covering the parabolic area with rectangles. Change this magnification factor at will.

Q12: What do you observe concerning the differences between the areas of upper and lower rectangles appearing in the magnification window, as you alter the number n of rectangles?

Q13: Could you find an approximation to the first decimal digit for the area E? How?

Fill in the blanks with the numbers you found before:

i. .... < E < ....

ii. The difference between the upper sum and the lower sum is: $D_{....} = S_{....} - s_{....} =$ ....

Q14: Fill in the blanks of the following table with the numbers you found just before.

n	**S_n**	**s_n**	**Difference : $D_n = S_n - s_n$**
1	**120.3180**	**40.7000**	**79.6180**
2			
5			
115			
809			
2516			
11096			
281068			

As you alter the number of covering rectangles:

Q15: What kind of changes can you observe regarding the values of the upper and the lower Riemann sums included in the previous table?

Q16: How is the difference $D_n = S_n - s_n$ modified?

Q17: Do you think that this process can lead to the calculation of the area in question with absolute accuracy?

Q18: Do you think that this process will be completed?

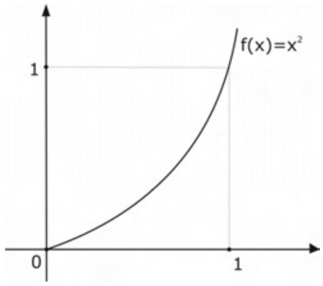
Q19: Which number do you think the sum area difference $D_n = S_n - s_n$ is approaching?

Q20: How close to zero do you think that this difference D_n can approach?

Q21: How close to the area in question do you think that we can reach through this process?

Q22: Do you think that the whole process can constitute a new way of measuring the unknown area E?

5.1.2 Worksheet
Area calculation of a parabolic plane region II



1. Divide the interval [0,1] into two equal parts

The length of each subinterval [0,1/2] and [1/2,1] is ……………

The maximum value of f on interval [0,1/2] is …......... and the minimum value is ……….

The area of the rectangle with base the interval [0,1/2] and height the maximum value of f on this interval is ………………………

The area of the rectangle with base the interval [0,1/2] and height the minimum value of f on this interval is ………………………

The maximum value of f on interval [1/2,1] is …......... and the minimum value is ……….

The area of the rectangle with base the interval [1/2,1] and height the maximum value of f on this interval is ………………………

The area of the rectangle with base the interval [1/2,1] and height the minimum value of f on this interval is ………………………

The sum of rectangles area produced by taking the maximum value of f at each interval equals to $S_2 =$ ……………..

The sum of rectangles area produced by taking the minimum value of f at each interval equals to $s_2 =$ ……………..

The difference between the area sums (Upper – Lower) is: $S_2 - s_2 =$ …………

2. Divide the interval [0,1] into three equal parts

The subintervals in this case are [,] [,] [,]

The length of each subinterval is ……….

The maximum value of f on [0,1/3] is …......... and the minimum value is ……….

The maximum value of f on [1/3,2/3] is …......... and the minimum value is ……….

The maximum value of f on [2/3,1] is …......... and the minimum value is ……….

The area of the big rectangle on the interval [0,1/3] is……….. and the area of the small one is …..

The area of the big rectangle on the interval [1/3,2/3] is……….. and the area of the small one is …..

The area of the big rectangle on the interval [2/3,1] is……….. and the area of the small one is …..

The sum of the areas of the three big rectangles equals to $S_3 =$ ……………..

The sum of the areas of the three small rectangles equals to $s_3 =$ ……………..

The difference of these two is $S_3 - s_3 =$ …………

3. Divide the interval [0,1] into n equal parts

The subintervals constructed are :

[,] [,] [,] … [,]

Each one of them having length ……

The maximum value of f on the 1st subinterval is ……….. and the minimum is …….

The maximum value of f on the 2nd subinterval is ……….. and the minimum is …….

………… ………… ………… ………… ………… ………… ………… ……….

The maximum value of f on the nth subinterval is ……….. and the minimum is …….

The big rectangle on the 1st subinterval has an area equal to ………. and the small one has an area equal to ………… .

The big r rectangle on the 2nd subinterval has an area equal to ………. and the small one has an area equal to ………… …………

The big r rectangle on the 3rd subinterval has an area equal to ………. and the small one has an area equal to ………… …………

The Sum of the n big rectangles is S_v =…………
and the sum of the n small rectangles is s_v = ……………

The difference of the area sums is $S_n - s_n =$ ………………..

Which number is the difference $S_n - s_n$ approaching to as n is increasing?

Is it possible through this process to determine the area of the region bounded by the curve $y = x^2$ and the lines x=0 and x=1?

BIBLIOGRAPHY

Alcock, L.J., & Simpson, A.P. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics, 22*(2), 28–34.

Alcock, L.J., & Simpson, A.P. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *57*(1), 1-32.

Arcavi A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215–241.

Artigue, M. (1991). Analysis. In D.O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 166–198). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Artigue, M. (1997). Teaching and learning elementary Analysis: What can we learn from didactical research and curriculum evolution? In G. Makridis (Ed.), *Proceedings of 1st Mediterranean Conference on Mathematics* (pp. 207-219). Nicosia, Cyprus.

Artigue, M. (2000). Teaching and learning Calculus: What can be learned from education research and curricular changes in France? In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education* (pp. 1–15). Providence, RI: American Mathematical Society

Artigue, M. (2003). Learning and teaching analysis: What can we learn from the past in order to think about the future? In D. Coray, F. Furinghetti, H. Gispert, B.R. Hodgson & G. Schubring (Eds.). *One Hundred Years of L'Enseignement Mathématique. Moments of Mathematics Education in the Twentieth Century.* (pp. 213–223). Genève: L'Enseignement Mathématique

Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. K. Lester, Jr (Ed.), *Second*

Handbook on Mathematics Teaching and Learning (pp. 1011-1049). NCTM

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In: E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. 2, pp. 1–32). Providence, RI: American Mathematical Society.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior, 16*(3), 399-431.

Berry, J., & Nyman, M. (2003). Promoting students' graphical understanding of the calculus. *Journal of Mathematical Behaviour*, *22*, 481-497.

Biza, I., Christou, C., & Zachariades, T. (2008). Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education, 10*(1), 53-70.

Borgen, K.L., & Manu, S.S. (2002). What do students really understand? *Journal of Mathematical Behavior, 21*(2), 151-165.

Bremigan, E.G. (2005). An analysis of diagram modification and construction in students' solutions to applied Calculus problems. *Journal for Research in Mathematics Education, 36*(3), 248-277.

Carlson, M., Jacobs, S., Coe, T., Larsen, S., & Hsu, E. (2002) Applying co-variational reasoning while modelling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures: Un example concret, celui de la tangente. *Recherche en didactique des mathématiques, 15*(1), 7-47.

Christou, C., Pitta-Pantazi, D., Souyoul, A., & Zachariades, T. (2005). The embodied, proceptual and formal worlds in the context of functions. *Canadian Journal of Science, Mathematics and Technology Education, 5*(2), 241-252.

Cipra, B.A. (1988). Calculus: crisis looms in mathematics' future. *Science*, *239*, 1491-1492.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Damarin, S. K., & White, C.M. (1986). Examining a model for courseware development, *Journal of Computers in Mathematics and Science Teaching, 6*(1) 38-43.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5*(3), 281-303.

Dreyfus, T., & Halevi, T. (1990/91). QuadFun: A case study of pupil-computer interaction. *Journal of Computers in Mathematics and Science Teaching*, *10*(2) 43-48.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*(3), 271-300.

Dubinsky, E., Weller, K., Mcdonald, M.A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An APOS analysis: Part 2. *Educational Studies in Mathematics, 60*(2), 253-266.

Ferrara, F., Pratt, D., & Robutti, O. (2006). The role and uses of technologies for the teaching of Algebra and Calculus. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 237-273). Rotterdam, The Netherlands: Sense Publishers.

Ferrini-Mundy, J., & Graham K.G., (1991). An overview of the calculus curriculum reform effort: Issues for Learning, Teaching, and Curriculum Development. *American Mathematical Monthly*, *98*(7), 627-635.

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics, 3*(2), 9–24.

Fischbein, E. (1987). *Intuition in Science and Mathematics.* Dordrecht, The Netherlands: Reidel.

Fischbein, E. (2001). Tacit models and infinity. *Educational Studies in Mathematics, 48*(2-3), 309-329.

Giaquinto, M. (2007). *Visual Thinking in Mathematics.* New York: Oxford University Press.

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. Some PME Perspectives. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 147–172). Rotterdam, The Netherlands: Sense Publishers.

Hershkowitz, R., & Kieran, C. (2001). Algorithmic and meaningful ways of joining together representatives within the same mathematical activity: An experience with graphic calculators. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th PME International Conference* (Vol. 1, pp. 96-107). Utrecht, The Netherlands.

Hanna, G., & Sidoli, N. (2007). Visualisation and proof: A brief survey of philosophical perspectives. *ZDM International Journal on Mathematics Education*, *39*, 73-78.

Heid, K. M. (1988). Re-sequencing skills and concepts in applied calculus using the computer as a tool, *Journal for Research in Mathematics Education*, *19*(1), 3-25.

Hsaio, F. S., (1984/85). A new CAI approach to teaching calculus. *Computers in Mathematics and Science Teaching*, *4*(2), 29-36.

Janvier C. (Ed.). (1987). *Problems of Representations in the Teaching and Learning of Mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Jones, K. (2000). The student experience of mathematical proof at university level. *International Journal of Mathematical Education*, *31*(1), 53-60.

Jones, K. (2004). Celebrating 20 years of computers in mathematics education: a research bibliography, *MicroMath*, *20*(1), 29-30.

Judson, & Nishimori, (2005). Concepts and skills in high school Calculus: An examination of a special case in Japan and the United States. *Journal for Research in Mathematics Education*, *36*(1), 24-43.

Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Kaput, J. (1994). Democratizing access to calculus: New roots to old roots. In A. Schoenfeld (Ed.), *Mathematical Thinking and Problem Solving* (pp. 77-156). Hillsdale, NJ, USA: Lawrence Erlbaum.

Kaput, J. J., & Thompson, P. W., (1994). Technology in mathematics education research: The first 25 years in the JRME. *Journal for Research in Mathematics Education*, *25*(6), 676-684.

Kleiner, I. (2001). History of the infinitely small and the infinitely large in calculus. *Educational Studies in Mathematics, 48*(2-3), 137-174.

Lauten, A. D, Graham, K., & Ferrini-Mundy, J. (1994). Student understanding of basic Calculus concepts: Interaction with the graphics calculator. *Journal of Mathematical Behavior, 13*, 225-237.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics, 52*(1), 29–55.

Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *The Journal of Mathematical Behavior, 23*(4), 405-427.

Little, C. (2008), A first lesson on differentiation with *Geometer's Sketchpad*. *Mathematics in School,* vol 37

Little, C. (2008), Teaching calculus: can *GeoGebra* bridge the algebra-geometry divide? *Mathematics in School,* vol 37

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal; A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics, 48*(2)*,* 259–288.

Mancosu, P., Jorgensen, K. F., & Pedersen, S. A. (Eds.). (2005). *Visualization, Explanation and Reasoning Styles in Mathematics*. Dordrecht, The Netherlands: Springer.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: an empirical approach. *Educational Studies in Mathematics, 56*(2), 255-286.

Monaghan, J. (1991). Problems with the language of limit. *For the Learning of Mathematics, 11*(3), 20–24.

Monaghan, J. (2001). Young people's ideas of infinity. *Educational Studies in Mathematics, 48*(2/3), 239–257.

Moreira, L. (2002). Mathematics education and critical consciousness. In A. D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26th PME In-*

ternational Conference (Vol. 3, pp. 369-376). Norwich, United Kingdom.

Moschkovich, J.N. (2004). Appropriating mathematical practices: A case of learning to use and explore functions through interaction with a tutor. *Educational Studies in Mathematics, 55*(1-3), 49-80.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14,* 1-18.

Orton, A., (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14,* 235-250.

Rozier, S., & Viennot, L. (1991). Students' reasoning in thermodynamics. *International Journal of Science Education*, *13*, 159-170.

Sajka, M. (2003). A secondary school student's understanding of the concept of function: A case study. *Educational Studies in Mathematics,53*(3), 229-254.

Schoenfeld, A. (Ed.). (1994). *Mathematical Thinking and Problem Solving.* Hillsdale, NJ: Lawrence Erlbaum.

Selden, A. (2005). New developments and trends in tertiary mathematics education: Or, more of the same? *International Journal of Mathematical Education in Science and Technology, 36*(2/3), 131–147.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18*(4), 371–397.

Steen, L. (1988). *Calculus for a New Century*. Washington, DC: Mathematical Association of America.

Tall, D. (1987). Constructing the concept image of a tangent. In J. C. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th PME International Conference* (Vol. 3, pp. 69-75). Montréal, Canada.

Tall, D.O. (Ed.). (1991). *Advanced Mathematical Thinking*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Tall, D.O., & Pinto, M. (1999). Student constructions of formal theory: giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME International Conference* (Vol. 4, pp. 65–73). Haifa, Israel.

Tall, D. O., & Schwarzenberger R. L. (1978). Conflicts in the learning of real numbers and limits, *Mathematics Teaching*, *83*, 44-49.

Tall, D.O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Tsamir, P., & Dreyfus, T. (2005). How fragile is consolidated knowledge? Ben's comparisons of infinite sets. *The Journal of Mathematical Behavior, 24*(1), 15-38.

Vinner, S. (1991). The role of definitions in the teaching and learning mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65–81). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of a function, *Journal for Research in Mathematics Education*, *20*(4), 356-366.

Weber, K. (2003). A procedural route toward understanding of the concept of proof. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th PME International Conference* (Vol. 4, pp. 395–401). Honolulu, Hawaii.

Weber, K. (2004). A framework for describing the processes that undergraduates use to construct proofs. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference* (Vol. 4, pp. 425–432). Bergen, Norway.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior 23,* 115-133.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory Calculus, *Journal for Research in Mathematics Education*, *27*(1), 79-95.

Williams, S. (1991). Models of limit held by college calculus students. *Journal of Research in Mathematics Education, 22*(3), 219-236.

Zazkis, R., Liljedahl, P., & Gadowsky, K. (2003). Students' conceptions of function translation: Obstacles, intuitions and rerouting. *Journal of Mathematical Behavior, 22,* 437-450.

Zimmermann, W., & Cunningham, S. (Eds.). (1991). *Visualization in Teaching and Learning Mathematics.* Washington, DC: Mathematical Association of America.

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