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Learning congruency-based proofs in geometry via a web-based learning system

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Congruence, and triangle congruence in particular, is generally taken to be a key topic in school geometry. This is because the three conditions of congruent triangles are very useful in proving geometrical theorems and also because triangle congruency leads on to the idea of mathematical similarity via similar triangles. Despite the centrality of congruence in general, and of congruent triangles in particular, there appears to be little research on the topic. In this paper, we use evidence from an on-going research project to illustrate how a web-based learning system for geometrical proof might help to develop Year 9 pupils' capability with congruent triangles. Using the notion of 'conceptions of congruency' as our framework, we first characterise our web-based learning system in terms of four different 'conceptions' of congruency by comparing the online tasks with activities from a Year 9 textbook. We then discuss how the web-based learning system would aid pupils when they are tackling congruency-based proofs in geometry.

Keywords: geometry, congruency, proof, web-based learning system

Introduction

Congruency constitutes a key topic in school geometry. The reasons for this are two-fold. First, the three conditions for triangle congruency (side-angle-side, SAS; side-side-side, SSS; side, angle, angle, SAA) can be used to prove many more propositions, as in the books of Euclid. Second, triangle congruency links to the even more powerful mathematical topic of similarity (via the idea of similar triangles). Despite congruence in general, and congruent triangles in particular, holding a key position in the geometry curriculum, there is, as far as we have been able to ascertain, little research on the teaching of congruency; at least, little published in English.

An exception to the general lack of research on the teaching of congruence is a paper by González and Herbst (2009, 115) in which they identify different "conceptions of congruency" that teaching might develop. In this paper, we use evidence from an on-going research project to illustrate how the use of a web-based learning system for geometrical proof (originated in Japan; see Miyazaki et al. 2011) can be utilised to develop Year 9 pupils' capability with congruent triangles. Using the notion of 'conceptions of congruency' as our framework, we characterise the web-based learning system in terms of the different 'conceptions' of congruency by comparing the online tasks with activities from a Grade 8 (equivalent to Year 9) textbook in common use in Japan. We then discuss how our web-based learning system would aid pupils when they are tackling congruency-based proofs in geometry. We begin by explaining how triangle congruency is specified in the mathematics curriculum in Japan.

Triangle congruency in school mathematics in Japan

The specification of the mathematics curriculum for Japan is given in the 'Course of Study' (MEXT 2008). Mathematical content is divided into 'Numbers and Algebraic Expressions', 'Functions', 'Geometrical Figures' and 'Making Use of Data'. Our focus is 'Geometrical Figures' for Grade 8 (pupils aged 13-14); Table 1 provides the details of this topic.

(1) Through activities like observation, manipulation and experimentation, to be able to find out the properties of basic plane figures and verify them based on the properties of parallel lines. (a) To understand the properties of parallel lines and angles and basing on it, to verify and explain the properties of geometrical figures. (b) To know how to find out the properties of angles of polygons based on the properties of parallel lines and angles of triangle. (2) To understand the congruence of geometrical figures and deepen the way of viewing geometrical figures, to verify the properties of geometrical figures based on the facts like the conditions for congruence of triangles, and to foster the ability to think and represent logically. (a) To understand the meaning of congruence of plane figures and the conditions for congruence of triangles. (b) To understand the necessity, meaning and methods of proof. (c) To verify logically the basic properties of triangles and parallelograms based on the facts like the conditions for congruence of triangles, and to find out new properties by interpreting proofs of the properties of geometrical figures.

Table 1 'Geometrical figures' in course of study for Grade 8 in Japan

From Table 1 we see that understanding congruent figures is one of the main objectives. Also we can see that the conditions of congruent triangles are expected to be used in verifying properties of geometrical figures. Our interest is what characterizes the approach to congruency presented in tasks that are tackled by pupils as these mediate between the 'intended curriculum' (as laid out in the 'Course of Study') and the 'attained curriculum' that is learnt by students.

Web-based proof learning system in geometry

Considering the fact that congruency is the main topic in Japanese geometry teaching in lower secondary schools, in an on-going research project we are developing a web-based learning support system (available in Japanese, English and Chinese) designed for pupils who are just starting to tackle congruency-based proofs in geometry; see Miyazaki et al. 2011, and www.schoolmath.jp/flowchart_en/home.html.

When using this learning system, pupils can tackle geometric problems by dragging sides, angles and triangles to on-screen cells. As this happens, the system automatically translates the figural elements to their symbolic form. Pupils also select from a choice of congruency conditions. From each set of actions, feedback is provided from the system. Thus the system offers opportunities for students to learn proofs in a way that is different from traditional textbook-based learning. As such, we are interested in how the tasks in our system can be characterised in terms of the conception of congruency, and whether we might be able to identify similarities and differences between tasks in the textbook and our system.

Overall, our interest is to investigate how and why our system can be an effective tool to promote students' proof learning experience. So far, evidence from our pilot studies (e.g. Miyazaki et al. 2011; Fujita, Jones, and Miyazaki 2011) suggests that learners' proving processes can be enriched when learners used our proof system. In this paper, we explore further the features of our system by characterising the tasks it contains.

Analytic framework and method

In the analysis we present in this paper we follow the approach of González and Herbst (2009, 154) in taking a 'conception' as being "the interaction between the cognizant subject and the milieu – those features of the environment that relate to the knowledge at stake". In this approach, a conception comprises the following quadruplet (P, R, L, ∑): P: a set of problems or tasks in which the conception is operational; R: a set of operations that the agent could use to solve problems in that set; L: a representation system within which those problems are posed and their solution expressed; ∑: a control structure (for example, a set of statements accepted as true). In their paper, González and Herbst (2009, 155-156) propose the following four conceptions of congruency:

- The perceptual conception of congruency (PERC) "relies on visual perception to control the correctness of a solution to the problem of determining if two objects (or more) are congruent".
- The measure-preserving conception of congruency (MeaP) "describes the sphere of practice in which a student establishes that two objects (e.g. segments or angles) are congruent by way of checking that they have the same measure (as attested by a measurement instrument)".
- The correspondence conception of congruency (CORR) is such that "two objects (segments or angles) are congruent if they are corresponding parts in two triangles that are known to be congruent".
- The transformation conception of congruency (TRANS) "establishes that two objects are congruent if there is a geometric transformation, mapping one to the other, which preserves metric invariants".

By using the above ideas as our analytic framework, we have analysed tasks which can be found in a commonly-used Grade 8 textbook in Japan; see Jones and Fujita (2013). What we found, in brief, is that the Japanese textbook contained a lesson progression from PERC or MeaP to CORR. Nevertheless, National Survey data from Japan has indicated that Japanese Grade 8 students struggle to solve geometrical problems. For example, a recent national survey in Japan reported that the proportion of Grade 9 students who could identify the pair of equal angles known to be equal by the SAS condition in a given proof was 48.8% (National Institute for Educational Policy Research 2010). This indicates that many students in Japan have not fully developed their CORR conception of congruency despite studying congruent triangles and related proofs during Grade 8.

With our proof learning system, learners can select and drag the sides and angles of various shapes, and also select from a choice of congruency conditions. From each set of actions, feedback is provided from the system. This is likely to influence learners' subsequent actions. Thus the system offers opportunities for students to learn proofs in a way that is different from traditional textbook-based learning. As such, we are interested in how the tasks in our system can be characterised in terms of the conception of congruency, and whether we might be able to identify similarities and differences between tasks in the textbook and our system.

From our analysis of a commonly-used Japanese Grade 8 textbook (Jones and Fujita, 2013), we know that the Japanese textbook includes many tasks which are related to congruent triangles. Some of the tasks entail identifying congruent figures, while others focus on proving properties of geometrical figures using congruency-based arguments. Because our web-based learning support system focuses especially on proof-related tasks, we chose the tasks shown in Table 2 as our sample for analysis. These tasks are similar to each other at a first glance. Our intention is to see if different intended conceptions might be available with our system because of the technology that underpins it.

Following the approach of Gon le and Herbst, we undertook an a priori analysis of the tasks in following way:

- we used the quadruplet (Problems; Operations; Representation system; Control structure) to characterise the sample tasks selected from the Grade 8 textbook and from our geometry proof system;
- we used the information from our analysis to characterise the approach to triangle congruency utilised in the sampled tasks.

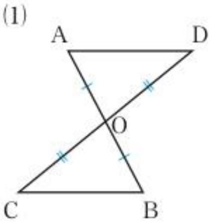
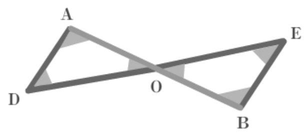
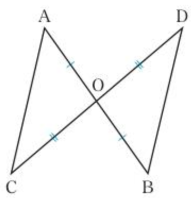
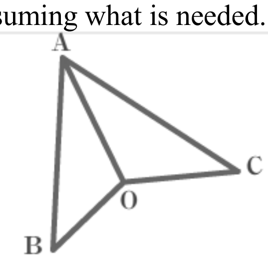
<table>
<tr><th>Tasks from the Grade 8 textbook</th><th>Tasks from the geometry proof system</th></tr>
<tr><td>A task taken from lesson 10
In the diagram below, identify a pair of congruent triangles and name them using the / sign. Also, name the congruence condition used. The sides and angles labelled with the same marks in each diagram may be considered equal.
(1)
</td><td>Lesson II-1
In the diagram below, prove triangles ADO and BEO are congruent by assuming what is needed (AO=BO assumed)

(Students can construct more than one proof in this problem situation.)</td></tr>
<tr><td>A task from lesson 13
In the diagram below, if O is the mid-point of line segments AB and CD, then prove that angles OAC=OBD.
</td><td>Lesson III-2
In the diagram below, prove that angles ABO=ACO by using triangle congruence and by assuming what is needed.

(Students can construct more than one proof in this problem situation.)</td></tr>
<tr><td>A task from Lesson 17
In triangle ABC, prove that if AB = AC, then angle B (ABD) = angle C (ACD).</td><td>Lesson V-1
If AB=AC and angle BAD = angle CAD, then prove that angle ABD = angle ACD.</td></tr>
</table>

Table 2: tasks selected from the textbook and from the web-based learning support system

Finding and discussion

Table 3 summarises the result of our analysis of Lesson 10 (textbook) and task II-1 (proof system). In terms of the four conceptions of congruency, both tasks can be characterised as being the correspondence conception of congruency (CORR) as both tasks require learners to identify corresponding parts to deduce congruent triangles. Similar characteristics were identified for other tasks we analysed.

Despite both tasks in Table 2 being characterised as CORR, the table suggests striking differences between the way in which the same intended conception is realised in the tasks in the textbook and in our proof system. In particular, whereas both tasks provide similar problems (P), learners would face quite different learning experience in terms of operation (R), representations (L), and control structure (∑) thanks to the technology in our system.

	Tasks from G8 textbook	Tasks from the proof system
P	10Pa: To identify two congruent triangles. 10Pb: To identify the conditions of congruent triangles. 10Pc: To use symbols correctly.	II-1 Pa: To prove triangles ADO and BEO are congruent.
R	10Ra: To find pairs of congruent sides and angles. 10Rb: To identify equal sides/angles including not symbolised ones. 10Rc: To apply the conditions of congruent triangles. 10Rd: To apply already known facts	II-1 Ra: To identify what assumptions and conclusions are. II-1 Rb: To drag and drop sides, and angles. II-1 Rc: To choose statements (conditions of congruent) II-1 Rd: To check answers by clicking a button II-1 Re: To review already completed answers by clicking stars
L	10La: The diagram is the medium for the presentation of the problem. 10Lb The symbols are the registers of equal sides and angles. 10Lc: Already known facts such as vertically opposite angles or the conditions of congruent triangles mediate for the solution and reasoning.	II-1 La: The diagram on the computer screen is the medium for the presentation of the problem. II-1 Lb: Dragged sides/angles/triangles are the registers of equal sides/angles/triangles. II-1 Lc: The structure of the proof is visualised by the flow-chart format. II-1 Ld: Tabs are the medium of right statements to be chosen.
∑	10∑a: If we can find three components of triangles (SSS, ASA, SAS). 10∑b: If one of the conditions of congruent triangles is applied to two triangles.	II-1 ∑a: If one of the conditions of congruent triangles is applied to two triangles. II-1 ∑b: If the system gives feedback 'your proof is correct'.

Table 3: analysis of tasks selected from the textbook and from the web-based system

In the textbook task, learners have to correspond figural elements to symbolic ones by themselves, but this can be quite hard for many learners who are just developing their CORR conceptions. The system supports this process by enabling learners to drag and drop figural elements to cells connected with the equal sign (=) or congruent sign (≡), and, as a result, learners can concentrate on formulating logical relationships in their proof. Also, the system does not have any measurement or superposition tools and these restrictions might help make learners aware that it is possible to study geometry theoretically as well as practically. To complete a proof, learners have to specify which congruency condition should be applied. The system supports learners as the known facts to be used are shown in the tabs and provides various forms of feedback in accordance with learners' actions.

Concluding comment

Given the sparse research on the topic of congruency, we have analysed selected congruency-related tasks in a well-used textbooks and our web-based learning system. We did this through an analysis utilising the four conceptions of congruency proposed by González and Herbst (2009). In our analysis of tasks in a Japanese textbook (see Jones and Fujita 2013) we showed that the textbook is based on a learning progression from PERC or MeaP to CORR, i.e. from a practical conception of congruency to a correspondence conception. Our analysis in this paper shows that our web-based proof system might usefully be used during the introductory stage of proof learning because the tasks provided in the web-based system are similarly designed to help learners to bridge between PERC or MeaP and CORR. One reason for developing our web-based system is that national survey data from Japan shows this progression might not be straightforward for many learners and that it might be necessary to support many more of them to develop CORR in their learning of proofs in geometry.

In addition to aiming to support the development of students' CORR, we aim, with our web-based proof system, to support students' learning in various other ways, including mediating figural and symbolic elements of geometrical proofs, scaffolding the students' use of known facts, and supporting their control structure by providing relevant and timely feedback. We argue that such learning experience should be useful as students proceed to more complex and formal learning in geometry and proving, and that is why the learning with our system can be located in the introductory stage of proof learning.

Our next task is to characterise actual students' conceptions when they interact with various congruent triangle problems. In this way we aim to examine more systematically how our web-based proof learning system would contribute to supporting the development of students' correspondence conception of congruency.

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