Galois-Module Theory for Wildly Ramified Covers of Curves over Finite Fields

by

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Abstract. Given a Galois cover of curves over $\mathbb{F}_p$, we relate ε-constants appearing in functional equations of Artin L-functions to an equivariant Euler characteristic. Our main theorem generalises a result of Chinburg from the tamely to the weakly ramified case. We furthermore apply Chinburg's result to obtain a 'weak' relation in the general case.

Introduction

Relating global invariants to invariants that are created from local data is a fundamental topic in number theory. In this paper, we will study this local-global principle in the context of curves over finite fields. More precisely, our goal is to relate ε-constants appearing in functional equations of Artin L-functions to an equivariant Euler characteristic of the underlying curve.

Let X be an irreducible smooth projective curve over $\mathbb{F}_p$, let k denote the algebraic closure of $\mathbb{F}_p$ in the function field of X and let G be a finite subgroup of $\mathrm{Aut}(X/k)$. We consider the equivariant Euler characteristic

$$\chi(G, \bar{X}, \mathcal{O}_{\bar{X}}) := \left[H^0(\bar{X}, \mathcal{O}_{\bar{X}})\right] - \left[H^1(\bar{X}, \mathcal{O}_{\bar{X}})\right]$$

of $\bar{X} := X \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ as an element of the Grothendieck group $K_0(G, \bar{\mathbb{F}}_p)$ of all finitely generated modules over the group ring $\bar{\mathbb{F}}_p[G]$. We recall, if $k = \mathbb{F}_p$, then $H^0(\bar{X}, \mathcal{O}_{\bar{X}})$ is the trivial representation $\bar{\mathbb{F}}_p$ and $H^1(\bar{X}.\mathcal{O}_{\bar{X}})$ is isomorphic to the dual $H^0(\bar{X}, \Omega_{\bar{X}})^*$ of the canonical representation of G on the space $H^0(\bar{X}, \Omega_{\bar{X}})$ of global holomorphic differentials on $\bar{X}$.

On the other hand, for any finite-dimensional complex representation V of G, we consider the ε-constant $\varepsilon(V)$ appearing in the functional equation of the Artin L-function associated with V and the action of G on X. It is known that $\varepsilon(V) \in \bar{\mathbb{Q}}$. We denote the standard p-adic valuation on $\bar{\mathbb{Q}}_P^\times$ by v_p, we fix a field embedding $j_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, we identify the classical ring $K_0(\mathbb{C}[G])$ of virtual complex representations with $K_0(\bar{\mathbb{Q}}[G])$ and we let j_p denote also the isomorphism $K_0(\bar{\mathbb{Q}}[G]) \ \tilde{\to} \ K_0(\bar{\mathbb{Q}}_p[G])$ induced by j_p. We then define the element $E(G,X) \in K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}} := K_0(\bar{\mathbb{Q}}_p[G]) \otimes \mathbb{Q}$ by the equations

$$\langle E(G,X), j_p(V)\rangle = -v_p(j_p(\varepsilon(V^*))), \qquad \text{for } V \text{ as above},$$

where $\langle \quad , \quad \rangle$ denotes the classical (character) pairing on $K_0(\bar{\mathbb{Q}}_p[G])$. It may be worth pointing out that the element $E(G,X)$ does not depend on j_p; furthermore, it follows from Frobenius reciprocity and the definition of Artin L-functions and ε-constants that, when restricted to a sugroup H of G, the element $E(G,X)$ becomes $E(H,X)$.

In Section 5 (which from the logical point of view does not depend on the earlier sections), we will prove the following general relation between the global invariant $\chi(G,\bar{X},\mathcal{O}_{\bar{X}})$ and the invariant $E(G,X)$ created from local data. Let $d : K_0(\bar{\mathbb{Q}}_p[G]) \to K_0(G,\bar{\mathbb{F}}_p)$ denote the (surjective) decomposition map from classical modular representation theory.

Theorem ('Weak' Formula)**.** *We have*

$$d(E(G,X)) = \chi(G,\bar{X},\mathcal{O}_{\bar{X}}) \quad \textit{in} \quad K_0(G,\bar{\mathbb{F}}_p)_{\mathbb{Q}}.$$

While we do not assume any condition on the type of ramification of the corresponding projection

$$\pi : X \to X/G =: Y$$

for this formula, it is only a 'weak' formula in the sense that it does not describe $E(G,X)$ itself, but only the image of $E(G,X)$ in $K_0(G,\bar{\mathbb{F}}_p)$, which for instance, if the order of G is a power of p, captures only the rank of $E(G,X)$. The weakness of this formula may also be explained by recalling that two $\bar{\mathbb{F}}_p[G]$-modules whose classes are equal in $K_0(G,\bar{\mathbb{F}}_p)$ are not necessarily isomorphic but only have the same composition factors.

The main object of this paper is to establish a 'strong' formula. For this, we assume that π is weakly ramified, i.e. that the second ramification group $G_{\mathfrak{p},2}$ vanishes for all $\mathfrak{p} \in X$. We recall that, by the Deuring-Shafarevic formula, this condition is always satisfied if X is ordinary, i. e. that this condition holds in a sense generically. Let $\bar{X}^{\mathrm{w}}$ denote the set of all points P in $\bar{X}$ such that $\bar{\pi}$ is wildly ramified at P. We furthermore define the subset $\bar{Y}^{\mathrm{w}} := \bar{\pi}(\bar{X}^{\mathrm{w}})$ of $\bar{Y} := Y \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ and the divisor $\bar{D}^{\mathrm{w}} := -\sum_{P \in \bar{X}^{\mathrm{w}}} [P]$ on $\bar{X}$. By [Kö], the equivariant Euler characteristic $\chi(G, \bar{X}, \mathcal{O}_{\bar{X}}(\bar{D}^{\mathrm{w}}))$ of $\bar{X}$ with values in the invertible G-sheaf $\mathcal{O}_{\bar{X}}(\bar{D}^{\mathrm{w}})$ is then equal to the image $c(\psi(G, \bar{X}))$ of a unique element $\psi(G, \bar{X})$ in the Grothendieck group $K_0(\bar{\mathbb{F}}_p[G])$ of all finitely generated *projective* $\bar{\mathbb{F}}_p[G]$-modules under the Cartan homomorphism $c : K_0(\bar{\mathbb{F}}_p[G]) \to K_0(G, \bar{\mathbb{F}}_p)$. Furthermore, for every $Q \in \bar{Y}$, we fix a point $\tilde{Q}$ in the fibre $\bar{\pi}^{-1}(Q)$, we denote the decomposition group of $\bar{\pi}$ at $\tilde{Q}$ by $G_{\tilde{Q}}$ and we denote the trivial representation of rank 1 by $\mathbf{1}$. Finally, let $e : K_0(\bar{\mathbb{F}}_p[G]) \to K_0(\bar{\mathbb{Q}}_p[G])$ denote the third homomorphism from the classical *cde*-triangle. The following relation between $\psi(G, \bar{X})$ and $E(G, X)$ is the main result of Section 4 and of this paper.

Theorem ('Strong' formula)**.** *If π is weakly ramified, we have*

$$(1) \qquad E(G, X) = e(\psi(G, \bar{X})) + \sum_{Q \in \bar{Y}^{\mathrm{w}}} \left[\mathrm{Ind}_{G_{\tilde{Q}}}^{G}(\mathbf{1})\right] \quad \textit{in} \quad K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}.$$

In particular, $E(G, X)$ belongs to the integral part $K_0(\bar{\mathbb{Q}}_p[G])$ of $K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}$.

While the 'strong' formula is a generalisation of the (first) main theorem in Chinburg's seminal paper [Ch] (applied to curves), the 'weak' formula is basically a corollary of it. More precisely, using Artin's induction theorem for modular representation theory, one quickly sees that it suffices to prove the 'weak' formula only in the case when G is cyclic and p does not divide the order of G. That case is even more restrictive than the tamely ramified case considered in [Ch].

In order to prove the 'strong' formula, we follow an approach different to that used in [Ch]. The idea and some of the steps of this alternative approach for tamely ramified covers of curves have been sketched in Erez's beautiful survey article [Er], but the preprint [CEPT5] cited there and authored by Chinburg, Erez, Pappas and Taylor seems to have not been published.

We now give an overview of our proof of the 'strong' formula (1). As already explained earlier, the left-hand side of (1) is compatible with restriction to any subgroup of G. Using Mackey's double coset formula, we will see that the added induced representations on the right-hand side ensure that also the right-hand side is compatible with restriction, in the obvious sense.

We need to show that both sides of (1) are equal after pairing them with $j_p(V)$ as above. As usual, the classical Artin induction theorem implies that it suffices to assume that G is cyclic and V corresponds to a multiplicative character χ. In that case (and in fact also in the slightly more general case when G is abelian), both sides can be explicitly computed as follows.

Let r denote the degree of k over $\mathbb{F}_p$ and let g_{Y_k} denote the genus of the geometrically irreducible curve Y/k. Furthermore, let Y^{t} (respectively Y^{w}) denote the set of all points $\mathfrak{q} \in Y$ such that π is tamely ramified (respectively wildly ramified) at one (and then all) point(s) $\mathfrak{p} \in X$ above $\mathfrak{q}$. For $\mathfrak{q} \in Y^t$, we restrict the character χ to the inertia subgroup $I_{\tilde{\mathfrak{q}}}$ for some $\tilde{\mathfrak{q}} \in \pi^{-1}(\mathfrak{q})$ and compose it with the norm residue homomorphism from local class field theory to obtain a multiplicative character $\chi_{k(\mathfrak{q})}$ on the residue field $k(\mathfrak{q})$ and to obtain the Gauss sum $\tau(\chi_{k(\mathfrak{q})}) \in \bar{\mathbb{Q}}$. In Section 3, we will prove the following explicit formula which essentially computes the right-hand side of (1). We will actually prove a more general formula that applies not only to D^{w} but to all divisors $D = \sum_{\mathfrak{p} \in X} n_{\mathfrak{p}}[\mathfrak{p}]$ on X for which, for all $\mathfrak{p} \in X$, the coefficient $n_{\mathfrak{p}}$ is congruent to -1 modulo the order of the (first) ramification group $G_{\mathfrak{p},1}$.

Theorem (Equivariant Euler characteristic formula). *We have*

$$\langle e(\psi(G,\bar{X})), j_p(\chi)\rangle = r(1-g_{Y_k}) - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} v_p(j_p(\tau(\chi_{k(\mathfrak{q})}))) - \sum_{\mathfrak{q} \in Y^{\mathrm{w}}} \deg(\mathfrak{q}). \tag{2}$$

The main ingredient in the proof of this theorem is the explicit description of $\psi(G,\bar{X})$ given in [Kö]. After plugging that explicit description into the left-hand side of (2), it then takes somewhat lengthy calculations to arrive at the right-hand side of (2). At the end of these calculations we use a variant of Stickelberger's formula for the valuation of Gauss sums that will be developed in Section 1 using local class field theory, particularly Lubin-Tate theory.

On the other hand, we will prove the following formula for $\varepsilon(\chi)$ in Section 2. We will actually prove a more general formula which applies to the general case when no condition on the type of ramification of π is assumed.

Theorem (ε-constant formula). *Up to a multiplicative root of unity we have*

$$\varepsilon(\chi^{-1}) = |k|^{g_{Y_k}-1} \cdot \prod \tau(\chi_{k(\mathfrak{q})}) \cdot \prod |k(\mathfrak{q})| \tag{3}$$

where the first product runs over all $\mathfrak{q} \in Y$ such that χ is tamely ramified (but not unramified) at $\mathfrak{q}$ and the second product runs over all $\mathfrak{q} \in Y$ such that χ is wildly ramified at $\mathfrak{q}$.

The main ingredient in the proof of (3) is the Deligne-Langlands description of $\varepsilon(\chi)$ as a product of local ε-constants, see [De]. To be able to apply the Deligne-Langlands formula we need to compute the Tamagawa measure on the ring $\mathbb{A}_{K(Y)}$ of adeles of the function field $K(Y)$ of Y and we need to construct an additive character on $\mathbb{A}_{K(Y)}$ that vanishes on $K(Y)$. The former is accomplished by a Mittag-Leffler type argument and the latter by using the theory of residues and in particular the residue theorem. To complete the proof of (3) we compute some local p-adic integrals.

The Euler characteristic formula (2) and the ε-constant formula (3) finally imply the 'strong' formula (1) after observing that the difference between the set Y^{w} and the set of all points $\mathfrak{q} \in Y$ such that χ is wildly ramified at $\mathfrak{q}$ is accounted for by the sum $\sum_{Q \in \bar{Y}^{\mathrm{w}}} \left[\mathrm{Ind}_{G_{\bar{Q}}}^{G}(\mathbf{1})\right]$.

Notations. Let p be a prime and let $\mathbb{Q}_p$ denote the p-adic completion of the field $\mathbb{Q}$ of rational numbers. We fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and denote the residue field of $\bar{\mathbb{Q}}_p$ by $\bar{\mathbb{F}}_p$, an algebraic closure of the field $\mathbb{F}_p$ with p elements. In particular, we have a well-defined reduction map from the ring of integers of $\bar{\mathbb{Q}}_p$ to $\bar{\mathbb{F}}_p$ which we denote by η_p. If q is a power of p, the unique subfield of $\bar{\mathbb{F}}_p$ with q elements is denoted by $\mathbb{F}_q$. Furthermore, let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ inside the field $\mathbb{C}$ of complex numbers. Throughout this paper, we fix a field embedding $j_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. The unique extension of the standard p-adic valuation on $\mathbb{Q}$ to $\bar{\mathbb{Q}}_p$ will be denoted by v_p and takes values in $\mathbb{Q}$.

For any finite group G and field F, the Grothendieck group of all finitely generated projective modules over the group ring $F[G]$ will be denoted by $K_0(F[G])$ and the Grothendieck group of all finitely generated $F[G]$-modules by $K_0(G, F)$. The isomorphism $K_0(G, \bar{\mathbb{Q}}) \to K_0(G, \bar{\mathbb{Q}}_p)$ induced by the embedding j_p (and other homomorphisms induced by j_p) will be denoted by j_p as well. We have a canonical ismomorphism between $K_0(G, \bar{\mathbb{Q}})$ and $K_0(G, \mathbb{C})$ and identify these two groups with the classical ring of virtual characters of G. The group of n^{th} roots of unity in F will be denoted by $\mu_n(F)$. If furthermore H/F is a Galois extension, the corresponding trace map is denoted by $\mathrm{Tr}_{H/F}$ or just by Tr.

For any $r \in \mathbb{R}$, the integral part $\lfloor r \rfloor$ and fractional part $\{r\}$ are related by $\lfloor r \rfloor = r - \{r\}$.

A point in a scheme will always mean a closed point.

1 Some Local Class Field Theory

Let L/K be a finite *abelian* Galois extension of local fields with Galois group G. The multiplicative group $k^\times$ of the residue field of K and the 'tame subquotient' G_0/G_1 of G are related in two natural ways: on the one hand, the norm residue homomorphism from class field theory induces a natural epimorphism $\gamma_{L/K}$ from $k^\times$ to G_0/G_1; on the other hand, the natural representation of the inertia subgroup G_0 of G on the 'cotangent space of L' induces a monomorphism $\chi_{L/K}$ from G_0/G_1 to $k^\times$. In Proposition 1.1 below we compute the endomorphism $\chi_{L/K} \circ \gamma_{L/K}$ of the cyclic group $k^\times$. We use this later in Corollary 1.3 to reformulate a well-known formula for the p-adic valuation of the Gauss sum $\tau(\chi)$ associated with a multiplicative character χ of G_0/G_1.

We denote the valuation rings of L and K by $\mathcal{O}_L$ and $\mathcal{O}_K$, the maximal ideals by $\mathfrak{m}_L$ and $\mathfrak{m}_K$ and the residue fields by l and k, respectively. The characteristic of k and l is denoted by $p > 0$ and the cardinality of k is denoted by $q = p^r$. We write $G_i = G_i(L/K)$ and $G^i = G^i(L/K)$ for the i^{th} higher ramification group of L/K in lower and upper numbering, respectively. Furthermore, let $e^{\text{t}} = e^{\text{t}}_{L/K} = \text{ord}(G_0)/\text{ord}(G_1)$ be the tame part of the ramification index $e = e_{L/K} = \text{ord}(G_0)$ of L/K, and let $e^{\text{w}} = e^{\text{w}}_{L/K} = \text{ord}(G_1)$ denote the wild part of e. In other words, e^{w} is the p-part of e and e^{t} is the non-p-part of e.

We re-normalise the norm residue homomorphism $(\quad, L/K) : K^\times \to G$ defined in Chapters IV and V of [Ne] by composing it with the homomorphism that takes every element to its inverse and denote the resulting composition by

$$\gamma_{L/K} : K^\times \to G.$$

By definition, the map $\gamma_{L/K}$ is surjective and maps every prime element of $\mathcal{O}_K$ to a pre-image of the inverse of the Frobenius automorphism $x \mapsto x^q$ under the canonical epimorphism $G \to G/G_0 \cong \text{Gal}(l/k)$. Furthermore, by Theorem V(6.2) in [Ne], it maps the group $\mathcal{O}_K^\times$ of units of $\mathcal{O}_K$ onto the subgroup G_0 of G and the subgroup $1 + \mathfrak{m}_K$ of $\mathcal{O}_K^\times$ onto the subgroup G^1 of G_1. Thus, the norm residue homomorphism $\gamma_{L/K}$ induces the epimorphism

$$\gamma_{L/K} : k^\times = \mathcal{O}_K^\times/(1 + \mathfrak{m}_K) \twoheadrightarrow G_0/G^1$$

(denoted $\gamma_{L/K}$ again). Hence $G^1 = G_1$, the tame part e^{t} of e divides $|k^\times| = q - 1$ and the cyclic group $k^\times$ contains all $(e^{\text{t}})^{\text{th}}$ roots of unity.

The one-dimensional l-representation $\mathfrak{m}_L/\mathfrak{m}_L^2$ of G_0 defines a multiplicative character from G_0 to $l^\times$ which we may view as a homomorphism

$$\chi_{L/K} : G_0/G_1 \to k^\times$$

because G_1 is a p-group and because $k^\times$ contains all $(e^{\mathrm{t}})^{\mathrm{th}}$ roots of unity; in fact $\chi_{L/K}$ is injective by Proposition IV.2.7 in [Se1].

Proposition 1.1. *The composition $\chi_{L/K} \circ \gamma_{L/K}$ raises every element of $k^\times$ to the power $\frac{q-1}{e^{\mathrm{t}}} \cdot \frac{1}{e^{\mathrm{w}}}$.*

Here, exponentiating with $\frac{1}{e^{\mathrm{w}}}$ just means the inverse map of exponentiating with e^{w}, as usual. This is the identity map if e^{w} is a power of q which in turn holds true for instance if $q = p$ or if $e^{\mathrm{w}} = 1$.

Proof. Let π be a prime element of $\mathcal{O}_K$ and let

$$K \subseteq L_1 \subseteq L_2 \subseteq \ldots$$

denote the corresponding Lubin-Tate extensions. We recall (see §V.5 in [Ne]) that the finite extension L_n/K is obtained from K by adjoining the π^n-division points $F(n)$ associated with a Lubin-Tate module F over $\mathcal{O}_K$.
By Korollar V(5.7) in [Ne] we may embed L into the compositum $\tilde{K}L_n$ of a finite unramified Galois extension $\tilde{K}$ and a Lubin-Tate extension L_n of K. We now consider the diagram

$$\begin{array}{ccccc}
k^\times & \xrightarrow{\gamma_{\tilde{K}L_n/K}} & G_0(\tilde{K}L_n/K)/G_1(\tilde{K}L_n/K) & \xrightarrow{\chi_{\tilde{K}L_n/K}} & k^\times \\
\| & & \downarrow & & \downarrow{\scriptstyle e_{\tilde{K}L_n/L}} \\
k^\times & \xrightarrow{\gamma_{L/K}} & G_0(L/K)/G_1(L/K) & \xrightarrow{\chi_{L/K}} & k^\times .
\end{array}$$

The left-hand square of this diagram commutes by functoriality of the norm residue homomorphism (see Satz IV(5.8) in [Ne]). From the definition of $\chi_{\tilde{K}L_n/K}$ and $\chi_{L/K}$ we easily obtain that the right-hand square commutes as well. It therefore suffices to prove Proposition 1.1 for $\tilde{K}L_n$ over K. Replacing in the lower row of the above diagram the extension L/K with the extension L_n/K we obtain similarly that it suffices to show Proposition 1.1 for $L = L_n$ over K.

In order to prove Proposition 1.1 for $L = L_n$ we need to show that the composition $\chi_{L_n/K} \circ \gamma_{L_n/K}$ is the identity map because $e^{\mathrm{t}}_{L_n/K} = q - 1$ and

$e^{\mathrm{w}}_{L_n/K} = q^{n-1}$ by [Ne, Theorem V(5.4)]. By *loc. cit.*, every $\lambda \in F(n)\backslash F(n-1)$ is a prime element of L_n and, by [Ne, Theorem V(5.5)], we have

$$\left(\gamma_{L_n/K}(u)\right)(\lambda) = [u]_F(\lambda)$$

for every unit $u \in \mathcal{O}_K^\times$, where $[u]_F$ denotes the endomorphism of $F(n)$ associated with u via the Lubin-Tate module structure of $\mathcal{O}_K$ on $F(n)$. Furthermore we have

$$[u]_F(\lambda) \equiv u\lambda \mod (\lambda^2)$$

by definition of a formal $\mathcal{O}_K$-module [Ne, Definition V(4.4)]. This shows that the composition $\chi_{L_n/K} \circ \gamma_{L_n/K} : k^\times \to k^\times$ is the identity map, as was to be shown. □

For any $d \in \mathbb{Z}$ we define

$$S_{L/K}(d) := \left\{ \left\{ \frac{dp^i}{e^{\mathrm{t}}} \right\} : i = 0, \dots, r-1 \right\},$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of any $x \in \mathbb{R}$. Although we have used set brackets in the definition of $S_{L/K}(d)$, we rather consider $S_{L/K}(d)$ as an unordered tuple, i.e., multiple entries of the same rational number are allowed. As e^{t} divides $q-1 = p^r - 1$, we have

$$S_{L/K}(dp^N) = S_{L/K}(d) \quad \text{for any } N \in \mathbb{N}.$$

Let now $\bar{\chi} : G_0/G_1 \to \bar{\mathbb{F}}_p^\times$ be a multiplicative character. (We will later define the character $\chi : G_0/G_1 \to \bar{\mathbb{Q}}^\times$ corresponding to $\bar{\chi}$.) Furthermore we fix a field embedding $\beta : k \hookrightarrow \bar{\mathbb{F}}_p$ of k into the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$. As $\chi_{L/K}$ is injective, there exists a unique integer $d(\bar{\chi}) \in \{0, \dots, e^{\mathrm{t}} - 1\}$ such that $\bar{\chi}$ is the $d(\bar{\chi})^{\mathrm{th}}$ power of the composition

$$G_0/G_1 \xhookrightarrow{\chi_{L/K}} k^\times \xhookrightarrow{\beta} \bar{\mathbb{F}}_p^\times .$$

While $d(\bar{\chi})$ depends on the embedding β, the unordered tuple $S_{L/K}(d(\bar{\chi}))$ does not. Indeed, any other such embedding is equal to $\beta \circ F^N = \beta^{(p^N)}$ for some $N \in \mathbb{N}$, where $F : k \to k$, $x \mapsto x^p$, denotes the Frobenius homomorphisms. Thus, choosing a different embedding amounts to multiplying $d(\bar{\chi})$ with a power of p and therefore does not change $S_{L/K}(d(\bar{\chi}))$.

Furthermore, there exists a unique integer $c(\bar{\chi}) \in \{0, \ldots, q-2\}$ such that the composition

$$k^\times \xrightarrow{\gamma_{L/K}} G_0/G_1 \xrightarrow{\bar{\chi}} \bar{\mathbb{F}}_p^\times$$

is the $c(\bar{\chi})^{\mathrm{th}}$ power of β.

Corollary 1.2. *We have the following equality of unordered tuples:*

$$S_{L/K}(d(\bar{\chi})) = \left\{ \left\{ \frac{c(\bar{\chi})p^i}{q-1} \right\} : i = 0, \ldots, r-1 \right\}.$$

Proof. By definition of $c(\bar{\chi})$ and of $d(\bar{\chi})$ and by Proposition 1.1 we have

$$\beta^{c(\bar{\chi})} = \bar{\chi} \circ \gamma_{L/K} = (\beta \circ \chi_{L/K})^{d(\bar{\chi})} \circ \gamma_{L/K} = \beta^{d(\bar{\chi}) \cdot \frac{q-1}{e^{\mathrm{t}}} \cdot \frac{1}{e^{\mathrm{w}}}}.$$

Hence

$$c(\bar{\chi}) \equiv \frac{d(\bar{\chi})(q-1)}{e^{\mathrm{t}}} \cdot \frac{q^N}{e^{\mathrm{w}}} \mod q-1$$

with N chosen big enough so e^{w} divides q^N. This implies

$$\left\{ \frac{c(\bar{\chi})}{q-1} \right\} = \left\{ \frac{d(\bar{\chi})}{e^{\mathrm{t}}} \cdot \frac{q^N}{e^{\mathrm{w}}} \right\}$$

and finally

$$S_{L/K}(d(\bar{\chi})) = S_{L/K}\left(d(\bar{\chi}) \frac{q^N}{e^{\mathrm{w}}} \right) = \left\{ \left\{ \frac{c(\bar{\chi})p^i}{q-1} \right\} : i = 0, \ldots, r-1 \right\},$$

as was to be shown. □

We recall that $\eta_p : \mu_{e^t}(\bar{\mathbb{Q}}_p) \xrightarrow{\sim} \mu_{e^t}(\bar{\mathbb{F}}_p^\times)$ denotes the reduction map modulo p and that we have fixed an emebedding $j_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. There obviously exists a unique character $\chi : G_0/G_1 \to \bar{\mathbb{Q}}^\times$ such that $\eta_p j_p \chi = \bar{\chi}$. Composing with the norm residue homomorphism defines the multiplicative character

$$\chi_k := \chi \circ \gamma_{L/K} : k^\times \to \bar{\mathbb{Q}}^\times.$$

Let $\zeta_p := e^{\frac{2\pi i}{p}} \in \bar{\mathbb{Q}} \subset \mathbb{C}$. We define the additive character

$$\psi_k : k \to \bar{\mathbb{Q}}^\times, \quad x \mapsto \zeta_p^{\mathrm{Tr}(x)} = \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{k/\mathbb{F}_p}(x) \right),$$

where $\mathrm{Tr} : k \to \mathbb{F}_p$ denotes the trace map. Furthermore we define the Gauss sum

$$\tau(\chi) := \sum_{x \in k^\times} \chi_k(x)^{-1} \cdot \psi_k(x) \in \bar{\mathbb{Q}}.$$

The p-adic valuation of $\tau(\chi)$ is usually described using fractions with denominator $q-1 = \mathrm{ord}(k^\times)$, see the proof of Corollary 1.3 below. By definition, our χ_k factorises through G_0/G_1; this allows us to give a formula using fractions with denominator e^{t}, as follows.

Corollary 1.3. *We have:*

$$v_p\left(j_p\left(\tau(\chi)\right)\right) = \sum_{i=0}^{r-1} \left\{ \frac{d(\bar{\chi})p^i}{e^{\mathrm{t}}} \right\}. \tag{4}$$

Proof. By Corollary 1.2, the right-hand side of (4) is equal to

$$\sum_{i=0}^{r-1} \left\{ \frac{c(\bar{\chi})p^i}{q-1} \right\}.$$

Let $s(c(\bar{\chi}))$ denote the sum of digits of the p-adic expansion of $c(\bar{\chi})$. Then the previous term is equal to

$$\frac{s(c(\bar{\chi}))}{p-1}$$

by the first two lines of the proof of the Lemma on page 96 in [La, IV, §4]. This in turn is equal to the left-hand side of (4) by Theorem 9 in [La, IV, §3] or by Theorem 27 in [Fr]. (Note that our $\tau(\chi)$ is equal to $\tau(\chi^{-1})$ in [La], that the distinguished character χ_φ in [La] corresponds to our β^{-1} via composing with $\eta_p j_p$ and that the p-adic valuation of the number $\omega - 1$ in [La] is $(p-1)^{-1}$. Similar remarks apply when Theorem 27 in [Fr] is applied.) $\square$

2 Computing ε-Constants

Let X be a smooth projective curve over some finite field k of characteristic p. We assume that X is geometrically irreducible over k, i.e. that k is algebraically closed in the function field $K(X)$ of X. Furthermore, let G be a finite subgroup of $\mathrm{Aut}(X/k)$ of order n. For any $\bar{\mathbb{Q}}$-representation V of G, let $\varepsilon(V)$ denote the ε-constant associated with X, G and V, as defined in

Sections 3.11 and 5.11 of [De] (and equal to $\varepsilon(V^*)$ defined in the introduction, see Section 4). The goal of this section is to explicitly describe $\varepsilon(V)$ up to a multiplicative root of unity when G is abelian and the representation V corresponds to a multiplicative character $\chi : G \to \bar{\mathbb{Q}}^\times$.

Let

$$\pi : X \to X/G =: Y$$

be the canonical projection, let $K := K(Y)$ denote the function field of Y and let $\mathbb{A}_K$ denote the ring of adeles of K. We start by explicitly describing additive characters $\psi : \mathbb{A}_K \to \bar{\mathbb{Q}}^\times$ and by computing the Tamagawa measure on $\mathbb{A}_K$.

For any $\mathfrak{q} \in Y$ let $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$ denote the completion of the local ring $\mathcal{O}_{Y,\mathfrak{q}}$ at $\mathfrak{q}$, let $k(\mathfrak{q})$ denote its residue field and let $K_\mathfrak{q}$ denote the field of fractions of $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$, i.e. $K_\mathfrak{q}$ is the completion of the function field $K := K(Y)$ of Y at $\mathfrak{q}$. The ring $\mathbb{A}_K$ of adeles of K is then the restricted product, over all $\mathfrak{q} \in Y$, of the fields $K_\mathfrak{q}$ relative to the subrings $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$. We embed K into $\mathbb{A}_K$ diagonally and endow $\mathbb{A}_K$ with its usual topology.

In the next few paragraphs we construct a non-trivial continuous additive character $\psi : \mathbb{A}_K \to \bar{\mathbb{Q}}^\times$ that is trivial on K. (The construction outlined in Exercises 5 and 6 on pp. 299-300 in [RV] seems to contain some flaws.) By Proposition 7-15 on p. 270 in [RV], any other such character $\tilde{\psi}$ is then given by $\tilde{\psi}(x) = \psi(ax)$ for some unique $a \in K$. While there is a so-called standard character if K is a number field (e.g. see Exercise 4 on p. 299 in [RV]), it seems not to be possible to single out a standard character if K is a function field. There is however a natural way to parameterize all characters ψ as above with meromorphic differentials (rather than with field elements as in the number field case).

Let Ω_K denote the module of differentials of K over k, a vector space of dimension 1 over K. For each $\mathfrak{q} \in Y$ let $\Omega_{K_\mathfrak{q}}$ denote $K_\mathfrak{q} \otimes_K \Omega_K$, a vector space over $K_\mathfrak{q}$ of dimension 1. This module may be viewed as the 'universally finite' module of differentials of $K_\mathfrak{q}$ over k in the sense of §11 in [Ku]. For each $\mathfrak{q} \in Y$ let

$$\mathrm{res}_\mathfrak{q} : \Omega_\mathfrak{q} \to k$$

denote the residue map at $\mathfrak{q}$ defined e.g. in [Ta], see also Theorem 7.14.1 on p. 247 in [Ha]. It can be computed as follows. Let $\pi_\mathfrak{q} \in \mathcal{O}_{Y,\mathfrak{q}}$ be a local parameter and let $x\, d\pi_\mathfrak{q} \in \Omega_{K_\mathfrak{q}}$. We write $x = \sum_{k=-\infty}^{\infty} \tilde{a}_k \pi_\mathfrak{q}^k$ with 'digits' $\tilde{a}_k \in \mathcal{O}_{Y,\mathfrak{q}}$ representing $a_k \in k(\mathfrak{q})$. Then we have

$$\mathrm{res}_\mathfrak{q}(x\, d\pi_\mathfrak{q}) = \mathrm{Tr}_{k(\mathfrak{q})/k}(a_{-1}) \quad \text{in} \quad k.$$

For each meromorphic differential $\omega \in \Omega_K$ we now define the additive character

$$\psi_\omega : \mathbb{A}_K \to \bar{\mathbb{Q}}^\times, \quad (x_\mathfrak{q})_{\mathfrak{q}\in Y} \mapsto \exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{k/\mathbb{F}_p}\left(\sum_{\mathfrak{q}\in Y}\mathrm{res}_\mathfrak{q}(x_\mathfrak{q}\omega)\right)\right).$$

Note that the sum on the right-hand side is finite because $x_\mathfrak{q} \in \hat{\mathcal{O}}_{Y,\mathfrak{q}}$ for almost all $\mathfrak{q}$ and because ω has at most finitely many poles.

Proposition 2.1. *For each $\omega \in \Omega_K$ the additive character ψ_ω is trivial on K.*

Proof. This follows from the residue theorem, see Corollary on p. 155 in [Ta] or Theorem 7.14.2 on p. 248 in [Ha]. □

The quotient $\mathbb{A}_K/K$ is compact by Theorem 5-11 on p. 192 in [RV]. The following proposition computes its volume. Let g_Y denote the genus of Y.

Proposition 2.2. *For each $\mathfrak{q} \in Y$ let $dx_\mathfrak{q}$ be that Haar measure on the additive group $K_\mathfrak{q}$ for which the volume of $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$ is equal to 1. Then the volume of $\mathbb{A}_K/K$ with respect to the measure $\prod_{\mathfrak{q}\in Y} \mathrm{d}x_\mathfrak{q}$ on $\mathbb{A}_K$ is equal to $|k|^{g_Y-1}$.*

Proof. Let $\mathcal{M}_Y$ denote the sheaf of meromorphic functions on Y, i.e. $\mathcal{M}_Y$ is the constant sheaf associated with K. The canonical short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{M}_Y \to \mathcal{M}/\mathcal{O}_Y \to 0$$

induces the long exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{M}_Y) \to H^0(Y, \mathcal{M}_Y/\mathcal{O}_Y) \\ \to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{M}_Y) \to \dots \end{aligned}$$

Since Y is geometrically irreducible, the canonical inclusion $k \hookrightarrow H^0(Y, \mathcal{O}_Y)$ is bijective and Serre duality (see Remark 7.12.2 on p. 246 in [Ha]) shows that $\dim_k H^1(Y, \mathcal{O}_Y) = g_Y$. Furthermore the sheaf $\mathcal{M}_Y$ is constant as a presheaf because Y is noetherian, so $H^0(Y, \mathcal{M}_Y) \cong K$ and $H^1(Y, \mathcal{M}_Y) = 0$. Finally, the canonical map from the sheaf $\mathcal{M}_Y/\mathcal{O}_Y$ to the direct product of sky scraper sheaves associated with $K/\mathcal{O}_{Y,\mathfrak{q}}$, $\mathfrak{q} \in Y$, has its image within the direct sum and then obviously becomes an isomorphism, hence

$$H^0(Y, \mathcal{M}_Y/\mathcal{O}_Y) \cong \bigoplus_{\mathfrak{q}\in Y} K/\mathcal{O}_{Y,\mathfrak{q}}.$$

Thus the above long exact sequence shows that we have a natural exact sequence

$$0 \to k \to K \to \bigoplus_{\mathfrak{q}\in Y} K/\mathcal{O}_{Y,\mathfrak{q}} \to H^1(Y, \mathcal{O}_Y) \to 0.$$

Using the canonical epimorphism

$$\mathbb{A}_K \twoheadrightarrow \bigoplus_{\mathfrak{q}\in Y} K/\mathcal{O}_{Y,\mathfrak{q}}$$

we therefore obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k & \longrightarrow & \prod_{\mathfrak{q}\in Y} \mathcal{O}_{Y,\mathfrak{q}} & \longrightarrow & \left(\prod_{\mathfrak{q}\in Y} \mathcal{O}_{Y,\mathfrak{q}}\right) \Big/ k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \mathbb{A}_K & \longrightarrow & \mathbb{A}_K/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K/k & \longrightarrow & \bigoplus_{\mathfrak{q}\in Y} K/\mathcal{O}_{Y,\mathfrak{q}} & \longrightarrow & H^1(Y, \mathcal{O}_Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose rows and columns are exact. Hence $\mathbb{A}_K/K$ is the disjoint union of the $|k|^{g_Y}$ cosets of $\left(\prod_{\mathfrak{q}\in Y} \mathcal{O}_{Y,\mathfrak{q}}\right)/k$ in $\mathbb{A}_K/K$, parameterized by $H^1(Y, \mathcal{O}_Y)$, and we obtain

$$\mathrm{vol}(\mathbb{A}_K/K) = |k|^{g_Y} \cdot \mathrm{vol}\left(\left(\prod_{\mathfrak{q}\in Y} \mathcal{O}_{Y,\mathfrak{q}}\right) \Big/ k\right).$$

We now fix a point $\mathfrak{q}_0 \in Y$ and a complement V of k in the vector space $k(\mathfrak{q}_0)$ over k and let $\mathfrak{n}_{\mathfrak{q}_0,V}$ denote the set of elements x in $\mathcal{O}_{Y,\mathfrak{q}_0}$ whose residue class is contained in V. Then the product $\left(\prod_{\mathfrak{q}\in Y\setminus\{\mathfrak{q}_0\}} \mathcal{O}_{Y,\mathfrak{q}}\right) \times \mathfrak{n}_{\mathfrak{q}_0,V}$ is obviously a fundamental domain for $\left(\prod_{\mathfrak{q}\in Y} \mathcal{O}_{Y,\mathfrak{q}}\right) \Big/ k$ in $\prod_{\mathfrak{q}\in Y} \mathcal{O}_{Y,\mathfrak{q}}$ and its volume is equal to

$$\left(\prod_{\mathfrak{q}\in Y\setminus\{\mathfrak{q}_0\}} \mathrm{vol}(\mathcal{O}_{Y,\mathfrak{q}})\right)\cdot \mathrm{vol}(\mathfrak{n}_{\mathfrak{q}_0,V}) = \mathrm{vol}(\mathfrak{n}_{\mathfrak{q}_0,V}) = |k|^{-1}.$$

Therefore we have

$$\mathrm{vol}(\mathbb{A}_K/K) = |k|^{g_Y}\cdot|k|^{-1} = |k|^{g_Y-1},$$

as was to be shown. □

Recall that $\pi : X \to Y$ is a non-constant finite morphism between geometrically irreducible smooth projective curves over k such that the corresponding extension $K(X)/K$ of function fields is a Galois extension with Galois group G of order n. Henceforth, we assume that G is abelian, we fix a multiplicative character

$$\chi : G \to \bar{\mathbb{Q}}^\times$$

and we denote the corresponding ε-constant by $\varepsilon(\chi)$.

As above, for each $\mathfrak{q}\in Y$, let $\mathrm{d}x_\mathfrak{q}$ be that Haar measure on the additive group $K_\mathfrak{q}$ for which the volume of $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$ is equal to 1. Furthermore we define an additive character $\psi_\mathfrak{q}$ on $K_\mathfrak{q}$ that is trivial on $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$ but not anymore on $\mathfrak{m}_\mathfrak{q}^{-1}$ where $\mathfrak{m}_\mathfrak{q}$ denotes the maximal ideal of $\hat{\mathcal{O}}_{Y,\mathfrak{q}}$. To this end we fix a generator $\pi_\mathfrak{q}\in\mathcal{O}_{Y,\mathfrak{q}}$ of the ideal $\mathfrak{m}_\mathfrak{q}$ and, given any $x\in K_\mathfrak{q}$, we write $x=\sum_{k=-\infty}^{\infty}\tilde{a}_k\pi_\mathfrak{q}^k$ with 'digits' $\tilde{a}_k\in\hat{\mathcal{O}}_{Y,\mathfrak{q}}$ representing $a_k\in k(\mathfrak{q})$; we then define

$$\psi_\mathfrak{q} : K_\mathfrak{q}\to\bar{\mathbb{Q}}^\times,\quad x\mapsto\psi_\mathfrak{q}(x) := \exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{k(\mathfrak{q})/\mathbb{F}_p}(a_{-1})\right).$$

The restriction of the character χ to the decomposition group $G_{\tilde{\mathfrak{q}}}$ of some $\tilde{q}\in X$ lying above $\mathfrak{q}$ is denoted by $\chi_{\tilde{\mathfrak{q}}}$. Let $\varepsilon(\chi_{\tilde{\mathfrak{q}}},\psi_\mathfrak{q},\mathrm{d}x_\mathfrak{q})$ denote the local ε-constant associated with $\chi_{\tilde{\mathfrak{q}}}$, $\psi_\mathfrak{q}$ and with $\mathrm{d}x_\mathfrak{q}$, as defined in §4 of [De].

For any two complex numbers w,z we write $w\sim z$ if there exists a root of unity ζ (i.e. $\zeta\in\exp(2\pi i\mathbb{Q})$) such that $w=\zeta z$. Note that this equivalence relation is finer than the equivalence relation $\sim$ defined in the Appendix of [De].

Theorem 2.3. *We have*

$$\varepsilon(\chi)\sim|k|^{g_Y-1}\prod_{\mathfrak{q}\in Y}\varepsilon(\chi_{\tilde{\mathfrak{q}}},\psi_\mathfrak{q},\mathrm{d}x_\mathfrak{q}).$$

Proof. We fix a non-zero meromorphic differential $\omega \in \Omega_K$. By Proposition 2.1, the differential ω determines a non-trivial continuous additive character

$$\psi_\omega : \mathbb{A}_K \to \bar{\mathbb{Q}}^\times$$

that is trivial on K. By Proposition 2.2, the measure $|k|^{1-g_Y} \cdot \prod_{\mathfrak{q}\in Y} \mathrm{d}x_\mathfrak{q}$ is then a Tamagawa measure on $\mathbb{A}_K$, i.e. the volume of $\mathbb{A}_K/K$ is equal to 1. According to (5.11.2) and (5.3) in [De], the ε-constant $\varepsilon(\chi)$ can be decomposed as a product of local ε-constants as follows:

$$\varepsilon(\chi) = |k|^{1-g_Y} \prod_{\mathfrak{q}\in Y} \varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\omega,\mathfrak{q}}, \mathrm{d}x_\mathfrak{q});$$

here the local additive character $\psi_{\omega,\mathfrak{q}}$ is given by

$$\psi_{\omega,\mathfrak{q}} : K_\mathfrak{q} \to \bar{\mathbb{Q}}^\times, \quad x \mapsto \exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{k/\mathbb{F}_p}(\mathrm{res}_\mathfrak{q}(x\omega))\right).$$

We now write $\omega = y_\mathfrak{q}\mathrm{d}\pi_\mathfrak{q}$ with some $y_\mathfrak{q} \in K^\times \subset K_\mathfrak{q}^\times$. Then we obviously have

$$\psi_\mathfrak{q}(x) = \psi_{\omega,\mathfrak{q}}(y_\mathfrak{q}^{-1}x) \quad \text{for } x \in K_\mathfrak{q}.$$

Recall that the surjective norm-residue homomorphism

$$\gamma_{\tilde{\mathfrak{q}}} : K_\mathfrak{q}^\times \twoheadrightarrow G_{\tilde{\mathfrak{q}}}$$

(see Section 1) maps the group $\hat{\mathcal{O}}_{Y,\mathfrak{q}}^\times$ of units onto the inertia subgroup $I_{\tilde{\mathfrak{q}}} = G_{\tilde{q},0}$ of $G_{\tilde{\mathfrak{q}}}$. Let now

$$\underline{\chi}_{\tilde{\mathfrak{q}}} : K_\mathfrak{q}^\times \xrightarrow{\gamma_{\tilde{\mathfrak{q}}}} G_{\tilde{\mathfrak{q}}} \xrightarrow{\chi_{\tilde{\mathfrak{q}}}} \bar{\mathbb{Q}}^\times$$

denote the composition of $\chi_{\tilde{\mathfrak{q}}}$ with the norm-residue homomorphism and let $n_\mathfrak{q}$ denote the valuation of $y_\mathfrak{q}$ at $\mathfrak{q}$. By Formula (5.4) in [De] we then have

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\omega,\mathfrak{q}}, \mathrm{d}x_\mathfrak{q}) = \underline{\chi}_{\tilde{\mathfrak{q}}}(y_\mathfrak{q})\ |k(\mathfrak{q})|^{n_\mathfrak{q}}\ \varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_\mathfrak{q}, \mathrm{d}x_\mathfrak{q}) \sim |k(\mathfrak{q})|^{n_\mathfrak{q}}\ \varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_\mathfrak{q}, \mathrm{d}x_\mathfrak{q}).$$

In fact we have equality here whenever π is unramified at $\tilde{\mathfrak{q}}$ and ω does not have a pole or zero at $\mathfrak{q}$; in particular we have equality for almost all $\mathfrak{q} \in Y$. Moreover we have

$$\sum_{\mathfrak{q}\in Y} \deg(\mathfrak{q}) n_\mathfrak{q} = \deg(\omega) = 2g_Y - 2$$

by Example 1.3.3 on p. 296 in [Ha]. Thus we obtain

$$\varepsilon(\chi) \sim |k|^{g_Y-1} \prod_{\mathfrak{q}\in Y} \varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}),$$

as was to be shown. □

In the next three propositions we are going to compute $\varepsilon(\chi_{\tilde{q}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}})$. We distinguish three cases. Recall that the character χ is said to be *unramified at* $\mathfrak{q}$ if χ is trivial on the inertia subgroup $I_{\tilde{\mathfrak{q}}} = G_{\tilde{\mathfrak{q}},0}$, it is *tamely ramified at* $\mathfrak{q}$ if χ is trivial on the ramification subgroup $G_{\tilde{\mathfrak{q}},1}$ and it is *wildly ramified at* $\mathfrak{q}$ if χ is not tamely ramified at $\mathfrak{q}$.

The first proposition concerns the unramified case and in particular tells us that the product in Theorem 2.3 is indeed a finite product.

Proposition 2.4. *If χ is unramified at $\mathfrak{q}$, then*

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}) = 1.$$

Proof. This is stated in Section 5.9 of [De]. □

We now assume that χ is tamely ramified at $\mathfrak{q}$. Then the character $\underline{\chi}_{\tilde{\mathfrak{q}}} : K_{\mathfrak{q}}^{\times} \to \bar{\mathbb{Q}}^{\times}$ (defined in the proof of Theorem 2.3) is trivial on $1 + \mathfrak{m}_{\mathfrak{q}}$ by Theorem V(6.2) in [Ne] and hence induces a multiplicative character

$$\chi_{k(\mathfrak{q})} : k(\mathfrak{q})^{\times} \to \bar{\mathbb{Q}}^{\times}$$

on the finite field $k(\mathfrak{q})$. Furthermore, as in Section 1, we introduce the standard additive character

$$\psi_{k(\mathfrak{q})} : k(\mathfrak{q}) \to \bar{\mathbb{Q}}^{\times}, \quad x \mapsto \exp\left(\frac{2\pi i}{p}\mathrm{Tr}_{k(\mathfrak{q})/\mathbb{F}_p}(x)\right).$$

and the Gauss sum

$$\tau(\chi_{k(\mathfrak{q})}) := \sum_{x\in k(\mathfrak{q})^{\times}} \chi_{k(\mathfrak{q})}^{-1}(x)\psi_{k(\mathfrak{q})}(x)$$

associated with $\chi_{k(\mathfrak{q})}$ and $\psi_{k(\mathfrak{q})}$.

Proposition 2.5. *If χ is tamely ramified at $\mathfrak{q}$, then*

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}) \sim \tau(\chi_{k(\mathfrak{q})}).$$

Proof. By the previous proposition we may assume that χ is not unramified at $\mathfrak{q}$. Let the character $\psi'_{\mathfrak{q}}$ be defined by

$$\psi'_{\mathfrak{q}}(x) = \psi_{\mathfrak{q}}(\pi_{\mathfrak{q}}^{-1}x) \quad \text{for } x \in K_{\mathfrak{q}}.$$

Furthermore let $\overline{\mathrm{d}x_{\mathfrak{q}}}$ denote that Haar measure on $K_{\mathfrak{q}}$ for which the volume of $\mathfrak{m}_{\mathfrak{q}}$ is equal to 1, i.e. $\overline{\mathrm{d}x_{\mathfrak{q}}} = |k(\mathfrak{q})| \, \mathrm{d}x_{\mathfrak{q}}$. By Formulas (5.3) and (5.4) and Section 5.10 in [De] we then have

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}) = \underline{\chi}_{\tilde{\mathfrak{q}}}(\pi_{\mathfrak{q}})\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi'_{\mathfrak{q}}, \overline{\mathrm{d}x_{\mathfrak{q}}}) \sim \varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi'_{\mathfrak{q}}, \overline{\mathrm{d}x_{\mathfrak{q}}}) = \tau(\chi_{k(\mathfrak{q})})$$

because $\psi'_{\mathfrak{q}}$ obviously induces $\psi_{k(\mathfrak{q})}$ and the character $\chi_{k(\mathfrak{q})}$ is non-trivial. □

In the final proposition we treat the wildly ramified case. If χ is ramified at $\mathfrak{q}$, recall that the *conductor* $\mathrm{cd}_{\mathfrak{q}} = \mathrm{cd}_{\mathfrak{q}}(\chi)$ of χ at $\mathfrak{q}$ is the smallest number m such that $\underline{\chi}_{\tilde{\mathfrak{q}}} : K_{\mathfrak{q}}^{\times} \to \bar{\mathbb{Q}}^{\times}$ is trivial on $1 + \mathfrak{m}_{\mathfrak{q}}^{m}$.

Proposition 2.6. *If χ is widly ramified at $\mathfrak{q}$ of conductor $\mathrm{cd}_{\mathfrak{q}}$, then*

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}) \sim |k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}-1}.$$

Proof. As χ is not tamely ramified at $\mathfrak{q}$, we have $\mathrm{cd}_{\mathfrak{q}} \geq 2$. The isomorphisms

$$\mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}-1}/\mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}} \ \tilde{\to}\ (1+\mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}-1})/(1+\mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}}), \quad \overline{a} \mapsto \overline{1+a},$$

and

$$\mathfrak{m}_{\mathfrak{q}}^{-\mathrm{cd}_{\mathfrak{q}}}/\mathfrak{m}_{\mathfrak{q}}^{-\mathrm{cd}_{\mathfrak{q}}+1} \ \tilde{\to}\ \mathrm{Hom}\left(\mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}-1}/\mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}}, \bar{\mathbb{Q}}^{\times}\right), \quad \overline{\gamma} \mapsto (\overline{a} \mapsto \psi_{\mathfrak{q}}(\overline{\gamma a})),$$

show that there exists a $\gamma \in \mathfrak{m}_{\mathfrak{q}}^{-\mathrm{cd}_{\mathfrak{q}}} \backslash \mathfrak{m}_{\mathfrak{q}}^{-\mathrm{cd}_{\mathfrak{q}}+1}$, unique modulo $\mathfrak{m}_{\mathfrak{q}}^{-\mathrm{cd}_{\mathfrak{q}}+1}$, such that

$$\chi_{\tilde{\mathfrak{q}}}(1+a) = \psi_{\mathfrak{q}}(\gamma a) \quad \text{for all } a \in \mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}-1}.$$

By Formula (3.4.3.2) in [De] we have

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}) = \int_{\gamma \hat{\mathcal{O}}_{Y,\mathfrak{q}}^{\times}} \chi_{\tilde{\mathfrak{q}}}^{-1}(x)\psi_{\mathfrak{q}}(x)\mathrm{d}x_{\mathfrak{q}} = \sum_{b \in k(\mathfrak{q})^{\times}} \int_{\gamma \tilde{b}(1+\mathfrak{m}_{\mathfrak{q}})} \chi_{\tilde{\mathfrak{q}}}^{-1}(x)\psi_{\mathfrak{q}}(x)\mathrm{d}x_{\mathfrak{q}}$$

where $\tilde{b} \in \hat{\mathcal{O}}_{Y,\mathfrak{q}}^{\times}$ represents $b \in k(\mathfrak{q})^{\times}$. For all $b \in k(\mathfrak{q})^{\times}$ and $a \in \mathfrak{m}_{\mathfrak{q}}^{\mathrm{cd}_{\mathfrak{q}}-1}$ we have

$$\chi_{\tilde{\mathfrak{q}}}^{-1}(\gamma\tilde{b}(1+a))\psi_{\mathfrak{q}}(\gamma\tilde{b}(1+a))) = \chi_{\tilde{\mathfrak{q}}}^{-1}(\gamma\tilde{b})\psi_{\mathfrak{q}}(\gamma\tilde{b})\psi_{\mathfrak{q}}(\gamma(\tilde{b}-1)a).$$

If $\tilde{b} = 1$, the function $\chi_{\tilde{\mathfrak{q}}}^{-1}(x)\psi_{\mathfrak{q}}(x)$ is therefore constant on $\gamma\tilde{b}(1+\mathfrak{m}_{\mathfrak{q}})$ and we hence obtain

$$\int_{\gamma\tilde{b}(1+\mathfrak{m}_{\mathfrak{q}})} \chi_{\tilde{\mathfrak{q}}}^{-1}(x)\psi_{\mathfrak{q}}(x)\mathrm{d}x_{\mathfrak{q}} = \chi_{\tilde{\mathfrak{q}}}^{-1}(\gamma)\psi_{\mathfrak{q}}(\gamma)\mathrm{vol}(\gamma\tilde{b} + \gamma\mathfrak{m}_{\mathfrak{q}})$$
$$= \chi_{\tilde{\mathfrak{q}}}^{-1}(\gamma)\psi_{\mathfrak{q}}(\gamma)|k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}-1}.$$

If $b \neq 1$, the substitution $x = \gamma\tilde{b}(1+a)$ gives $\mathrm{d}x_{\mathfrak{q}} = |k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}}\mathrm{d}a$ and hence

$$\int_{\gamma\tilde{b}(1+\mathfrak{m}_{\mathfrak{q}})} \chi_{\tilde{\mathfrak{q}}}^{-1}(x)\psi_{\mathfrak{q}}(x)\mathrm{d}x_{\mathfrak{q}} = \chi_{\tilde{\mathfrak{q}}}^{-1}(\gamma\tilde{b})\psi_{\mathfrak{q}}(\gamma\tilde{b})|k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}} \int_{\mathfrak{m}_{\mathfrak{q}}} \psi_{\mathfrak{q}}(\gamma(\tilde{b}-1)a)\mathrm{d}a = 0$$

because $\psi_{\mathfrak{q}}$ is a non-trivial additive character on $\mathfrak{m}_{\mathfrak{q}}^{-1}/\hat{\mathcal{O}}_{Y,\mathfrak{q}}$. Hence

$$\varepsilon(\chi_{\tilde{\mathfrak{q}}}, \psi_{\mathfrak{q}}, \mathrm{d}x_{\mathfrak{q}}) = \chi_{\tilde{\mathfrak{q}}}^{-1}(\gamma)\psi_{\mathfrak{q}}(\gamma)|k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}-1} \sim |k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}-1},$$

as was to be shown. □

Corollary 2.7. *We have*

$$\varepsilon(\chi) \sim |k|^{g_Y-1} \cdot \prod \tau(\chi_{k(\mathfrak{q})}) \cdot \prod |k(\mathfrak{q})|^{\mathrm{cd}_{\mathfrak{q}}(\chi)-1}$$

where the first product runs over all $\mathfrak{q} \in Y$ such that χ is tamely ramified (but not unramified) at $\mathfrak{q}$ and the second product runs over all $\mathfrak{q} \in Y$ such that χ is wildly ramified at $\mathfrak{q}$.

Proof. This immediately follows from Theorem 2.3 and Propositions 2.4, 2.5 and 2.6. □

3 Computing Equivariant Euler Characteristics

Let X be a geometrically irreducible smooth projective curve over $\mathbb{F}_q$, let k denote the algebraic closure of $\mathbb{F}_q$ in the function field $K(X)$ of X and let G be a finite *abelian* subgroup of $\mathrm{Aut}(X/k)$ of order n. In addition, we assume in this section that the canonical projection

$$\pi : X \to X/G =: Y$$

is at most weakly ramified, i.e. that the second ramification group $G_{\mathfrak{p},2}$ vanishes for all $\mathfrak{p} \in X$. The object of this section to find the multiplicity of each irreducible character of G in the 'characteristic-zero versions' of the equivariant Euler characteristics associated with certain G-invariant divisors on X.

For each $\mathfrak{p} \in X$ let $e_\mathfrak{p}^{\mathrm{w}}$ and $e_\mathfrak{p}^{\mathrm{t}}$ denote the wild part (i.e., p-part) and tame part (i.e., non-p-part) of the ramification index $e_\mathfrak{p}$ of π at $\mathfrak{p}$, respectively. Furthermore, let $G_\mathfrak{p} := \{\sigma \in G : \sigma(\mathfrak{p}) = \mathfrak{p}\}$, $I_\mathfrak{p} := \ker(G_\mathfrak{p} \to \mathrm{Aut}(k(\mathfrak{p})))$ and $G_{\mathfrak{p},1}$ denote the decomposition group, inertia group and first ramification group at $\mathfrak{p}$, respectively. We recall that the exponent of $G_{\mathfrak{p},1}$ is p and that $e_\mathfrak{p} = \mathrm{ord}(I_\mathfrak{p})$, $e_\mathfrak{p}^{\mathrm{w}} = \mathrm{ord}(G_{\mathfrak{p},1})$ and $e_\mathfrak{p}^{\mathrm{t}} = \mathrm{ord}(I_\mathfrak{p}/G_{\mathfrak{p},1})$. We call $\mathfrak{p}$ *tamely ramified*, if $e_\mathfrak{p}^{\mathrm{w}} = 1$ and we call $\mathfrak{p}$ *wildly ramified* if $e_\mathfrak{p}^{\mathrm{w}} \neq 1$.

Our assumptions imply the following somewhat unexpected statement.

Lemma 3.1. *For each $\mathfrak{p} \in X$ we have $e_\mathfrak{p}^{\mathrm{t}} = 1$ or $e_\mathfrak{p}^{\mathrm{w}} = 1$.*

Proof. As π is weakly ramified, the natural action of $I_\mathfrak{p}/G_{\mathfrak{p},1}$ on $G_{\mathfrak{p},1}\backslash\{\mathrm{id}\}$ is free by Proposition 9 in §2, Ch. IV of [Se1]. As G is abelian, this can only happen if $e_\mathfrak{p}^{\mathrm{t}} = 1$ or $e_\mathfrak{p}^{\mathrm{w}} = 1$. □

Let $D = \sum_{\mathfrak{p} \in X} n_\mathfrak{p}[\mathfrak{p}]$ be a G-invariant divisor on X such that $n_\mathfrak{p} \equiv -1 \mod e_\mathfrak{p}^{\mathrm{w}}$ for all $\mathfrak{p} \in X$. Let

$$\bar{\pi} : \bar{X} := X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_p \to Y \times_{\mathbb{F}_q} \bar{\mathbb{F}}_p =: \bar{Y}$$

denote the morphism induced by π. For any $P \in \bar{X}$ lying above $\mathfrak{p} \in X$ we write n_P for $n_\mathfrak{p}$. Furthermore we put $\bar{D} := \sum_{P \in \bar{X}} n_P[P]$.

By Theorem 2.1(a) in [Kö], applied to each of the irreducible components of $\bar{X}$, the equivariant Euler characteristic

$$\chi(G, \bar{X}, \mathcal{O}_{\bar{X}}(\bar{D})) := \left[H^0(\bar{X}, \mathcal{O}_{\bar{X}}(\bar{D}))\right] - \left[H^1(\bar{X}, \mathcal{O}_{\bar{X}}(\bar{D}))\right] \in K_0(G, \bar{\mathbb{F}}_p)$$

of $\bar{X}$ with values in the G-equivariant invertible $\mathcal{O}_{\bar{X}}$-module $\mathcal{O}_{\bar{X}}(\bar{D})$ lies in the image of the (injective) Cartan homorphism

$$c : K_0(\bar{\mathbb{F}}_p[G]) \to K_0(G, \bar{\mathbb{F}}_p).$$

We define $\psi(G, \bar{X}, \bar{D}) \in K_0(\bar{\mathbb{F}}_p[G])$ by

$$c(\psi(G, \bar{X}, \bar{D})) = \chi(G, \bar{X}, \mathcal{O}_{\bar{X}}(\bar{D})).$$

Let

$$\begin{array}{ccc} & K_0(G,\bar{\mathbb{Q}}_p) & \\ {\scriptstyle e}\nearrow & & \searrow{\scriptstyle d} \\ K_0(\bar{\mathbb{F}}_p[G]) & \xrightarrow{\quad c \quad} & K_0(G,\bar{\mathbb{F}}_p) \end{array}$$

be the so-called *cde*-triangle and let

$$\langle \quad , \quad \rangle : K_0(G,\bar{\mathbb{Q}}_p) \times K_0(G,\bar{\mathbb{Q}}_p) \to \mathbb{Z}$$

denote the classical character pairing (given by

$$\langle [M],[N] \rangle = \dim_{\bar{\mathbb{Q}}_p} \mathrm{Hom}_{\bar{\mathbb{Q}}_p[G]}(M,N)$$

for any finitely generated $\bar{\mathbb{Q}}_p[G]$-modules M, N).

Furthermore, let $\chi : G \to \bar{\mathbb{Q}}^\times$ be a multiplicative character of G, which we also consider as a map to the group $\mu_n(\bar{\mathbb{Q}})$ of n^{th} roots of unity in $\bar{\mathbb{Q}}$ and also as an element of $K_0(G,\bar{\mathbb{Q}})$.

The main theorem of this section, see Theorem 3.2 below, will give a formula for $\langle e(\psi(G,\bar{X},\bar{D})), j_p\chi\rangle$, that is the multiplicity of the character $j_p\chi$ in the virtual $\bar{\mathbb{Q}}_p$-representation $e(\psi(G,\bar{X},\bar{D}))$ of G. To state the theorem we need to introduce further notations.

Let r denote the degree of k over $\mathbb{F}_q$. When viewed as curve over k, the curve Y becomes geometrically irreducible curve and we denote the genus of that curve by g_{Y_k}. For each $\mathfrak{p} \in X$ let

$$\chi_{\mathfrak{p}} : I_{\mathfrak{p}} \to k(\mathfrak{p})^\times$$

denote the multiplicative character corresponding to the obvious representation of $I_{\mathfrak{p}}$ on the one-dimensional cotangent space of X at $\mathfrak{p}$. Note that, in contrast to the previous section, the restriction of χ to the decomposition group $G_{\mathfrak{p}}$ is not denoted by $\chi_{\mathfrak{p}}$ but by $\mathrm{Res}^G_{G_{\mathfrak{p}}}(\chi)$ in this and the next sections. Furthermore we write

$$n_{\mathfrak{p}} = (e^{\mathrm{w}}_{\mathfrak{p}} - 1) + (l_{\mathfrak{p}} + m_{\mathfrak{p}} e^{\mathrm{t}}_{\mathfrak{p}}) e^{\mathrm{w}}_{\mathfrak{p}}$$

with $l_{\mathfrak{p}} \in \{0,\ldots,e^{\mathrm{t}}_{\mathfrak{p}} - 1\}$ and $m_{\mathfrak{p}} \in \mathbb{Z}$. Note that

$$\begin{array}{ll} l_{\mathfrak{p}} = 0 \text{ and } m_{\mathfrak{p}} = n_{\mathfrak{p}} & \text{if } \pi \text{ is unramified at } \mathfrak{p} \\ l_{\mathfrak{p}} + m_{\mathfrak{p}} e_{\mathfrak{p}} = n_{\mathfrak{p}} & \text{if } \pi \text{ is tamely ramified at } \mathfrak{p} \\ l_{\mathfrak{p}} = 0 \text{ and } m_{\mathfrak{p}} = \dfrac{n_{\mathfrak{p}} - e_{\mathfrak{p}} + 1}{e_{\mathfrak{p}}} & \text{if } \pi \text{ is wildly ramified at } \mathfrak{p} \end{array}$$

by Lemma 3.1. As above, for any $P \in \bar{X}$ lying above $\mathfrak{p} \in X$ we write $m_P := m_{\mathfrak{p}}$ and $l_P := l_{\mathfrak{p}}$.

For each $\mathfrak{q} \in Y$, we let $\deg(\mathfrak{q}) := [k(\mathfrak{q}) : \mathbb{F}_q]$ denote the degree of $\mathfrak{q}$, we fix a point $\tilde{\mathfrak{q}} \in \pi^{-1}(\mathfrak{q}) \subset X$ and a field embedding $\alpha_{\tilde{\mathfrak{q}}} : k(\tilde{\mathfrak{q}}) \hookrightarrow \bar{\mathbb{F}}_p$ and we write $e_{\mathfrak{q}} := e_{\tilde{\mathfrak{q}}}, e^{\mathrm{t}}_{\mathfrak{q}} := e^{\mathrm{t}}_{\tilde{\mathfrak{q}}}, m_{\mathfrak{q}} := m_{\tilde{\mathfrak{q}}}$, etc. If π is tamely ramified at $\tilde{\mathfrak{q}}$, the character $\chi_{\tilde{\mathfrak{q}}}$ is injective by Proposition IV.2.7 in [Se1] and we define $d_{\mathfrak{q}}(\chi)$ to be the unique integer $d \in \{0, \ldots, e_{\mathfrak{q}} - 1\}$ such that the composition

$$\mathrm{Res}^G_{I_{\tilde{\mathfrak{q}}}}(\eta_p j_p \chi) : \; I_{\tilde{\mathfrak{q}}} \hookrightarrow G \xrightarrow{\chi} \mu_n(\bar{\mathbb{Q}}) \xrightarrow{j_p} \mu_n(\bar{\mathbb{Q}}_p) \xrightarrow{\eta} \mu_n(\bar{\mathbb{F}}_p)$$

is the d^{th} power of the composition

$$I_{\tilde{\mathfrak{q}}} \xrightarrow{\chi_{\tilde{\mathfrak{q}}}} k(\tilde{\mathfrak{q}})^\times \xrightarrow{\alpha_{\tilde{\mathfrak{q}}}} \bar{\mathbb{F}}_p.$$

Let Y^{t} denote the set of $\mathfrak{q} \in Y$ such that π is tamely but not unramified at $\tilde{\mathfrak{q}}$.

Theorem 3.2. *We have*

(5)
$$\langle e(\psi(G, \bar{X}, \bar{D})), j_p\chi \rangle = r(1 - g_{Y_k}) + \sum_{\mathfrak{q} \in Y} \deg(\mathfrak{q})\, m_{\mathfrak{q}} - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \sum_{i=0}^{\deg(\mathfrak{q})-1} g_i(l_{\mathfrak{q}}, \chi)$$

where $g_i(l_{\mathfrak{q}}, \chi)$ denotes the unique rational number in the interval $[-\frac{l_{\mathfrak{q}}}{e_{\mathfrak{q}}}, 1 - \frac{l_{\mathfrak{q}}}{e_{\mathfrak{q}}}[$ that is congruent to $\frac{d_{\mathfrak{q}}(\chi) q^i}{e_{\mathfrak{q}}}$ modulo $\mathbb{Z}$.

Proof. We have a well-defined pairing

$$\langle \quad , \quad \rangle : K_0(\bar{\mathbb{F}}_p[G]) \times K_0(G, \bar{\mathbb{F}}_p) \to \mathbb{Z}$$

given by

$$\langle [P], [M] \rangle = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\bar{\mathbb{F}}_p[G]}(P, M)$$

for any finitely generated projective $\bar{\mathbb{F}}_p[G]$-module P and any finitely generated $\bar{\mathbb{F}}_p[G]$-module M. By [Se2, 15.4b)] we have

$$\langle e(x), y \rangle = \langle x, d(y) \rangle$$

for all $x \in K_0(\bar{\mathbb{F}}_p[G])$ and $y \in K_0(G, \bar{\mathbb{Q}}_p)$.

Hence the left-hand side of (5) is equal to

(6)
$$\langle \psi(G, \bar{X}, \bar{D}), d(j_p\chi) \rangle.$$

For any point P of $\bar{X}$ let I_P denote the inertia group of $\bar{\pi}$ at P. Recall that I_P is equal to the decomposition group $G_P = \{\sigma \in G : \sigma(P) = P\}$ and that $e_P = \mathrm{ord}(I_P)$ is the ramification index. The multiplicative character $\chi_P : I_P \to \bar{\mathbb{F}}_P^\times$ afforded by the representation of I_P on the 1-dimensional cotangent space at P is given by

$$\chi_P(\sigma)t_P \equiv \sigma(t_P) \mod (t_P^2)$$

where $\sigma \in I_P$ and t_P is a local parameter at P. For any $d \in \mathbb{Z}$, the d^{th} (tensor) power of χ_P is denoted by χ_P^d.

By Theorems 4.3 and 4.5 in [Kö], applied to each of the r irreducible components of $\bar{X}$, we have

$$\begin{aligned}(7)\qquad &\psi(G, \bar{X}, \bar{D}) \\ &= -\frac{1}{n} \sum_{P \in \bar{X}} \sum_{d=1}^{e_P^{\mathrm{t}}-1} e_P^{\mathrm{w}}\, d \left[\mathrm{Ind}_{I_P}^G \left(\mathrm{Cov}(\chi_P^d)\right)\right] \\ &\quad + \sum_{Q \in \bar{Y}} \sum_{d=1}^{l_{\tilde{Q}}} \left[\mathrm{Ind}_{I_{\tilde{Q}}}^G \left(\mathrm{Cov}(\chi_{\tilde{Q}}^{-d})\right)\right] \\ &\quad + \left(r(1-g_{Y_k}) + \sum_{Q \in \bar{Y}} m_{\tilde{Q}}\right) [\bar{\mathbb{F}}_p[G]] \quad \text{in} \quad K_0(\bar{\mathbb{F}}_p[G]);\end{aligned}$$

here, $\tilde{Q} \in \bar{X}$ denotes a fixed preimage of $Q \in \bar{Y}$ under $\bar{\pi}$, and "Cov" means "projective cover of"; note that, despite the fraction $\frac{1}{n}$, the first of the three summands above is an element of $K_0(\bar{\mathbb{F}}_p[G])$, see Theorem 4.3 in [Kö].

By definition of the decomposition map d, the element $d(j_P\chi)$ is equal to the class of the 1-dimensional $\bar{\mathbb{F}}_p$-representation of G given by the composition

$$G \xrightarrow{\chi} \mu_n(\bar{\mathbb{Q}}) \xrightarrow{j_p} \mu_n(\bar{\mathbb{Q}}_p) \xrightarrow{\eta_p} \mu_n(\bar{\mathbb{F}}_p^\times).$$

Hence (6) is equal to

$$
\begin{aligned}
(8)\qquad & -\frac{1}{n}\sum_{P\in\bar{X}}\sum_{d=1}^{e_P^{\mathrm{t}}-1} e_P^{\mathrm{w}}\, d\left\langle \mathrm{Ind}_{I_P}^G\left(\mathrm{Cov}(\chi_P^d)\right), \eta_p j_p \chi\right\rangle \\
& + \sum_{Q\in\bar{Y}}\sum_{d=1}^{l_{\tilde{Q}}} \left\langle \mathrm{Ind}_{I_{\tilde{Q}}}^G\left(\mathrm{Cov}(\chi_{\tilde{Q}}^{-d})\right), \eta_p j_p \chi\right\rangle \\
& + \left(r(1-g_{Y_k}) + \sum_{Q\in\bar{Y}} m_{\tilde{Q}}\right)\left\langle \bar{\mathbb{F}}_p[G], \eta_p j_p \chi\right\rangle .
\end{aligned}
$$

If H is a subgroup of G, P a finitely generated projective $\bar{\mathbb{F}}_p[H]$-module and M a finitely generated $\bar{\mathbb{F}}_p[G]$-module, then we have

$$
\begin{aligned}
\langle \mathrm{Ind}_H^G(P), M\rangle &= \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\bar{\mathbb{F}}_p[G]}\left(\bar{\mathbb{F}}_p[G]\otimes_{\bar{\mathbb{F}}_p[H]} P, M\right) \\
&= \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\bar{\mathbb{F}}_p[H]}\left(P, \mathrm{Res}_H^G(M)\right) = \langle P, \mathrm{Res}_H^G(M)\rangle,
\end{aligned}
$$

i.e., the map $\mathrm{Ind}_H^G : K_0(\bar{\mathbb{F}}_p[H]) \to K_0(\bar{\mathbb{F}}_p[G])$ is left adjoint to the map $\mathrm{Res}_H^G : K_0(G, \bar{\mathbb{F}}_p) \to K_0(H, \bar{\mathbb{F}}_p)$ with respect to the pairings $\langle\ \ ,\ \ \rangle$. Hence (8) is equal to

$$
\begin{aligned}
(9)\qquad & -\frac{1}{n}\sum_{P\in\bar{X}}\sum_{d=1}^{e_P^{\mathrm{t}}-1} e_P^{\mathrm{w}}\, d\left\langle \mathrm{Cov}(\chi_P^d), \mathrm{Res}_{I_P}^G(\eta_p j_p \chi)\right\rangle \\
& + \sum_{Q\in\bar{Y}}\sum_{d=1}^{l_{\tilde{Q}}} \left\langle \mathrm{Cov}(\chi_{\tilde{Q}}^{-d}), \mathrm{Res}_{I_{\tilde{Q}}}^G(\eta_p j_p \chi)\right\rangle \\
& + r(1-g_{Y_k}) + \sum_{Q\in\bar{Y}} m_{\tilde{Q}}.
\end{aligned}
$$

It is well-known that, if R is the group ring of a finite group over a field and if J denotes the Jacobson radical of R, then $JM = 0$ and $\mathrm{Cov}(M)/J\mathrm{Cov}(M) \cong M$ for any simple R-module M. As the characters $\mathrm{Res}_{I_P}^G(\eta_p j_p \chi)$ and χ_P^d are simple, we hence obtain

$$\begin{aligned}
&\big\langle \mathrm{Cov}(\chi_P^d), \mathrm{Res}^G_{I_P}(\eta_p j_p \chi)\big\rangle \\
&\quad = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\bar{\mathbb{F}}_p[I_P]} \big(\mathrm{Cov}(\chi_P^d), \mathrm{Res}^G_{I_P}(\eta_p j_p \chi)\big) \\
&\quad = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{\bar{\mathbb{F}}_p[I_P]} \big(\chi_P^d, \mathrm{Res}^G_{I_P}(\eta_p j_p \chi)\big) \\
&\quad = \begin{cases} 1 & \text{if} \quad \chi_P^d = \mathrm{Res}^G_{I_P}(\eta_p j_p \chi) \\ 0 & \text{else} \end{cases}
\end{aligned}$$

for all $P \in \bar{X}$ and $d \in \mathbb{Z}$.

For each $P \in \bar{X}$, let $G_{P,1}$ denote the first ramification group of $\bar{\pi}$ at P. As $G_{P,1}$ is a p-group, every character from I_P to $\bar{\mathbb{F}}_p^\times$ factorises modulo $G_{P,1}$. The character $\bar{\chi}_P : I_P/G_{P,1} \to \bar{\mathbb{F}}_p^\times$ induced by χ_P is injective by Proposition IV.2.7 in [Se1] and hence generates the group $\mathrm{Hom}(I_P/G_{P,1}, \bar{\mathbb{F}}_p^\times)$. In particular, there is unique integer $d(\chi_P, \chi) \in \{0, \ldots, e_P^{\mathrm{t}} - 1\}$ such that

$$\chi_P^{d(\chi_P, \chi)} = \mathrm{Res}^G_{I_P}(\eta_p j_p \chi).$$

For instance, $d(\chi_P, \chi) = 0$ if $e_P^{\mathrm{t}} = 1$.

For each $Q \in Y$ we now write $e_Q := e_{\tilde{Q}}$, $m_Q := m_{\tilde{Q}}$, etc.

All this allows us to rewrite (9) in the following way:

$$\begin{aligned}
&-\frac{1}{n} \sum_{P \in \bar{X}} e_P^{\mathrm{w}}\, d(\chi_P, \chi) \\
&+ \big|\big\{Q \in \bar{Y} : d(\chi_{\tilde{Q}}, \chi) \geq e_Q^{\mathrm{t}} - l_Q\big\}\big| \qquad (10) \\
&+ r(1 - g_{Y_k}) + \sum_{Q \in \bar{Y}} m_Q.
\end{aligned}$$

For each point $\mathfrak{p} \in X$, we have a canonical bijection between the set

$$\mathrm{Hom}_{\mathbb{F}_q}(k(\mathfrak{p}), \bar{\mathbb{F}}_p)$$

of $\mathbb{F}_q$-embeddings of the residue field $k(\mathfrak{p})$ into $\bar{\mathbb{F}}_p$ and the fibre in $\bar{X}$ above $\mathfrak{p}$. Then, if $P \in \bar{X}$ lies above $\mathfrak{p} \in X$ and $\alpha : k(\mathfrak{p}) \to \bar{\mathbb{F}}_p$ corresponds to P, we have $I_P = I_{\mathfrak{p}}$ and the character χ_P is equal to the composition

$$I_{\mathfrak{p}} \xrightarrow{\chi_{\mathfrak{p}}} k(\mathfrak{p})^\times \xrightarrow{\alpha} \bar{\mathbb{F}}_p^\times .$$

Hence (10) is equal to

$$
(11)\qquad
\begin{aligned}
&-\frac{1}{n}\sum_{\mathfrak{p}\in X}\ \sum_{\alpha\in\mathrm{Hom}_{\mathbb{F}_q}(k(\mathfrak{p}),\bar{\mathbb{F}}_p)} e^{\mathrm{w}}_{\mathfrak{p}}\, d(\alpha\circ\chi_{\mathfrak{p}},\chi)\\
&+\left|\left\{Q\in\bar{Y} : d(\chi_{\tilde{Q}},\chi)\geq e^{\mathrm{t}}_Q-l_Q\right\}\right|\\
&+r(1-g_{Y_k})+\sum_{\mathfrak{q}\in Y}\deg(\mathfrak{q})\, m_{\mathfrak{q}}.
\end{aligned}
$$

For each $\sigma\in G$ and $\mathfrak{p}\in X$, the character $\chi_{\sigma(\mathfrak{p})}$ is equal to the composition

$$
I_{\sigma(\mathfrak{p})}\xrightarrow{\ \sim\ } I_{\mathfrak{p}}\xrightarrow{\ \chi_{\mathfrak{p}}\ } k(\mathfrak{p})^{\times}\xrightarrow{\ \sigma\ } k(\sigma(\mathfrak{p}))^{\times}
$$

where the first map is given by conjugation with σ and the last map is the inverse of the isomorphism induced by the local homomorphism $\sigma^{\#}:\mathcal{O}_{X,\sigma(\mathfrak{p})}\to\mathcal{O}_{X,\mathfrak{p}}$. It is also easy to check that the character $\mathrm{Res}^G_{I_{\sigma(\mathfrak{p})}}(\eta_p j_p\chi)$ is equal to the composition

$$
I_{\sigma(\mathfrak{p})}\xrightarrow{\ \sim\ } I_{\mathfrak{p}}\xrightarrow{\ \mathrm{Res}^G_{I_{\mathfrak{p}}}(\eta_p j_p\chi)\ }\bar{\mathbb{F}}_p^{\times}
$$

where again the first map is given by conjugation with σ. In particular we have

$$
d(\alpha'\circ\chi_{\sigma(\mathfrak{p})},\chi)=d(\alpha'\circ\sigma\circ\chi_{\mathfrak{p}},\chi)
$$

for every $\alpha'\in\mathrm{Hom}_{\mathbb{F}_q}(k(\sigma(\mathfrak{p})),\bar{\mathbb{F}}_p)$. Furthermore, if α' runs through the set $\mathrm{Hom}_{\mathbb{F}_q}(k(\sigma(\mathfrak{p})),\bar{\mathbb{F}}_p)$, then $\alpha=\alpha'\circ\sigma$ runs through $\mathrm{Hom}_{\mathbb{F}_q}(k(\mathfrak{p}),\bar{\mathbb{F}}_p)$.

Recall that $\tilde{\mathfrak{q}}\in X$ denotes a (fixed) point above $\mathfrak{q}\in Y$. We have $\pi^{-1}(\mathfrak{q})=\{\sigma(\tilde{\mathfrak{q}}):\sigma\in G/G_{\tilde{\mathfrak{q}}}\}$ and hence $|\pi^{-1}(\mathfrak{q})|=\frac{n}{e_{\mathfrak{q}}f_{\mathfrak{q}}}$ where $f_{\mathfrak{q}}=f_{\tilde{\mathfrak{q}}}=[k(\tilde{\mathfrak{q}}):k(\mathfrak{q})]$ denotes the inertia degree of π at $\tilde{\mathfrak{q}}$. Furthermore, we recall that $d(\alpha\circ\chi_{\tilde{q}},\chi)=0$ if $\mathfrak{q}\notin Y^{\mathrm{t}}$ (by Lemma 3.1). Thus (11) is equal to

$$
(12)\qquad
\begin{aligned}
&-\sum_{\mathfrak{q}\in Y^{\mathrm{t}}}\ \sum_{\alpha\in\mathrm{Hom}_{\mathbb{F}_q}(k(\tilde{\mathfrak{q}}),\bar{\mathbb{F}}_p)}\frac{1}{e_{\mathfrak{q}}f_{\mathfrak{q}}}\, d(\alpha\circ\chi_{\tilde{\mathfrak{q}}},\chi)\\
&+\sum_{\mathfrak{q}\in Y^{\mathrm{t}}}\frac{1}{f_{\mathfrak{q}}}\ \left|\left\{\alpha\in\mathrm{Hom}_{\mathbb{F}_q}(k(\tilde{\mathfrak{q}}),\bar{\mathbb{F}}_p): d(\alpha\circ\chi_{\tilde{\mathfrak{q}}},\chi)\geq e_{\mathfrak{q}}-l_{\mathfrak{q}}\right\}\right|\\
&+r(1-g_{Y_k})+\sum_{\mathfrak{q}\in Y}\deg(\mathfrak{q})\, m_{\mathfrak{q}}.
\end{aligned}
$$

Now let $\mathfrak{q} \in Y^{\mathrm{t}}$. The norm residue homomorphism induces a surjective homomorphism

$$\gamma_{\mathfrak{q}} : k(\mathfrak{q})^{\times} \to I_{\tilde{\mathfrak{q}}},$$

see Section 1. We recall that this implies that $e_{\mathfrak{q}}$ divides $|k(\mathfrak{q})| - 1$ and that the multiplicative group $k(\mathfrak{q})^{\times}$ contains all $e_{\mathfrak{q}}^{\mathrm{th}}$ roots of unity. As in Section 1, we now consider the character $\chi_{\tilde{\mathfrak{q}}} : I_{\tilde{\mathfrak{q}}} \to k(\tilde{\mathfrak{q}})^{\times}$ as a map

$$\chi_{\tilde{\mathfrak{q}}} : I_{\tilde{\mathfrak{q}}} \to k(\mathfrak{q})^{\times}$$

from $I_{\tilde{\mathfrak{q}}}$ to the subgroup $k(\mathfrak{q})^{\times}$ of $k(\tilde{\mathfrak{q}})^{\times}$. All $\mathbb{F}_q$-embeddings $\alpha : k(\tilde{\mathfrak{q}}) \hookrightarrow \bar{\mathbb{F}}_p$ which extend a given $\mathbb{F}_q$-embedding $\beta : k(\mathfrak{q}) \hookrightarrow \bar{\mathbb{F}}_p$ therefore yield the same composition with $\chi_{\tilde{\mathfrak{q}}}$. For each β there are $f_{\mathfrak{q}}$ such extensions α. Therefore (12) is equal to

$$\begin{aligned}
& - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \frac{1}{e_{\mathfrak{q}}} \sum_{\beta \in \mathrm{Hom}_{\mathbb{F}_q}(k(\mathfrak{q}), \bar{\mathbb{F}}_p)} d(\beta \circ \chi_{\tilde{\mathfrak{q}}}, \chi) \\
(13) \qquad & + \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \left| \left\{ \beta \in \mathrm{Hom}_{\mathbb{F}_q}(k(\mathfrak{q}), \bar{\mathbb{F}}_p) : d(\beta \circ \chi_{\tilde{\mathfrak{q}}}, \chi) \geq e_{\mathfrak{q}} - l_{\mathfrak{q}} \right\} \right| \\
& + r(1 - g_{Y_k}) + \sum_{\mathfrak{q} \in Y} \deg(\mathfrak{q})\, m_{\tilde{\mathfrak{q}}}.
\end{aligned}$$

For each $\mathfrak{q} \in Y^{\mathrm{t}}$ let $\beta_{\mathfrak{q}}$ denote the embedding

$$\beta_{\mathfrak{q}} : \; k(\mathfrak{q}) \hookrightarrow k(\tilde{\mathfrak{q}}) \xrightarrow{\alpha_{\tilde{\mathfrak{q}}}} \bar{\mathbb{F}}_p \, .$$

All other embeddings of $k(\mathfrak{q})$ into $\bar{\mathbb{F}}_p$ are then given by

$$k(\mathfrak{q}) \xrightarrow{F^i} k(\mathfrak{q}) \xrightarrow{\beta_{\mathfrak{q}}} \bar{\mathbb{F}}_p \, , \quad i = 1, \ldots, \deg(\mathfrak{q}),$$

where $F : k(\mathfrak{q}) \to k(\mathfrak{q})$, $x \mapsto x^q$, denotes the Frobenius automorphism of $k(\mathfrak{q})$ over $\mathbb{F}_q$. Thus we can rewrite (13) as

$$\begin{aligned}
& - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \frac{1}{e_{\mathfrak{q}}} \sum_{i=1}^{\deg(\mathfrak{q})} d((\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}})^{q^i}, \chi) \\
(14) \qquad & + \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \left| \{ i = 1, \ldots, \deg(\mathfrak{q}) : d((\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}})^{q^i}, \chi) \geq e_{\mathfrak{q}} - l_{\mathfrak{q}} \} \right| \\
& + r(1 - g_{Y_k}) + \sum_{\mathfrak{q} \in Y} \deg(\mathfrak{q})\, m_{\mathfrak{q}}.
\end{aligned}$$

For a moment let $\chi_0 = \beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}}$, $\chi_1 = \mathrm{Res}^G_{I_{\tilde{\mathfrak{q}}}}(\eta_p j_p \chi)$ and $d = d(\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}}, \chi)$. The defining identity $\chi_0^d = \chi_1$ implies

$$(\chi_0^{q^i})^{dq^{\deg(\mathfrak{q})-i}} = \chi_1^{q^{\deg(\mathfrak{q})}} = \chi_1^{|k(\mathfrak{q})|} = \chi_1$$

because $e_{\mathfrak{q}}$ divides $|k(\mathfrak{q})| - 1$. Thus we obtain

$$d((\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}})^{q^i}, \chi) \equiv d(\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}}, \chi) q^{\deg(\mathfrak{q})-i} \mod e_{\mathfrak{q}}.$$

When i runs from 1 to $\deg(\mathfrak{q})$, then $\deg(\mathfrak{q}) - i$ runs from $\deg(\mathfrak{q}) - 1$ to 0. Finally, if a and m are any positive integers, then the residue class of a modulo m in $\{0, \dots, m-1\}$ is given by $m\{\frac{a}{m}\}$. Therefore (14) is equal to

$$\begin{aligned} & - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \sum_{i=0}^{\deg(\mathfrak{q})-1} \left\{ \frac{d(\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}}, \chi) q^i}{e_{\mathfrak{q}}} \right\} \\ & + \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \left| \left\{ i = 0, \dots, \deg(\mathfrak{q}) - 1 : \left\{ \frac{d(\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}}, \chi) q^i}{e_{\mathfrak{q}}} \right\} \geq 1 - \frac{l_{\mathfrak{q}}}{e_{\mathfrak{q}}} \right\} \right| \\ & + r(1 - g_{Y_k}) + \sum_{\mathfrak{q} \in Y} \deg(\mathfrak{q})\, m_{\mathfrak{q}}. \end{aligned} \tag{15}$$

As obviously $d(\beta_{\mathfrak{q}} \circ \chi_{\tilde{\mathfrak{q}}}, \chi) = d_{\mathfrak{q}}(\chi)$ for every $\mathfrak{q} \in Y^{\mathrm{t}}$, the term (15) is equal to the right-hand side of (5) and the proof of Theorem 5 is finished. □

For $\mathfrak{q} \in Y^{\mathrm{t}}$, let the Gauss sum $\tau(\chi_{k(\mathfrak{q})}) \in \bar{\mathbb{Q}}$ be defined as in the previous section.

Corollary 3.3. *If $q = p$, we have*

$$\begin{aligned} & \langle e(\psi(G, \bar{X}, \bar{D})), j_p \chi \rangle \\ & \quad = r(1 - g_{Y_k}) + \sum_{\mathfrak{q} \in Y} \deg(\mathfrak{q})\, m_{\mathfrak{q}} - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} v_p(j_p(\tau(\chi_{k(\mathfrak{q})}))) \\ & \qquad + \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} \left| \left\{ i = 0, \dots, \deg(\mathfrak{q}) - 1 : \left\{ \frac{d_{\mathfrak{q}}(\chi) p^i}{e_{\mathfrak{q}}} \right\} \geq 1 - \frac{l_{\mathfrak{q}}}{e_{\mathfrak{q}}} \right\} \right|. \end{aligned}$$

Proof. This follows immediately from Theorem 3.2 (see also (15)) and Corollary 1.3. □

Let now X^{w} denote the set of $\mathfrak{p} \in X$ such that π is is wildly ramified at $\mathfrak{p}$, let $Y^{\mathrm{w}} := \pi(X^{\mathrm{w}})$, let D^{w} denote the divisor

$$D^{\mathrm{w}} := -\sum_{\mathfrak{p} \in X^{\mathrm{w}}} [\mathfrak{p}]$$

and let

$$\psi(G, \bar{X}) := \psi(G, \bar{X}, \bar{D}^{\mathrm{w}}).$$

Corollary 3.4. *If $q = p$, we have*

$$\langle e(\psi(G, \bar{X})), j_p\chi\rangle = r(1 - g_{Y_k}) - \sum_{\mathfrak{q} \in Y^{\mathrm{w}}} \deg(\mathfrak{q}) - \sum_{\mathfrak{q} \in Y^{\mathrm{t}}} v_p(j_p(\tau(\chi_{k(\mathfrak{q})}))).$$

Proof. For the divisor D^{w} we have $l_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in X$, $m_{\mathfrak{q}} = -1$ for all $\mathfrak{q} \in Y^{\mathrm{w}}$ and $m_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in Y \backslash Y^{\mathrm{w}}$. This corollary is therefore a special case of Corollary 3.3 □

4 The Main Theorem

Let X be an irreducible smooth projective curve over $\mathbb{F}_p$, let k denote the algebraic closure of $\mathbb{F}_p$ within the function field $K(X)$ of X and let G be a finite subgroup of $\mathrm{Aut}(X/k)$. As in the previous section, we assume that the canonical projection

$$\pi : X \to X/G =: Y$$

is at most weakly ramified. The object of this section is to prove our 'strong' formula stated in in the introduction and in Theorem 4.2 below and relating the ε-constants associated with X and finite-dimensional complex representations of G to an equivariant Euler characteristic of $\bar{X} := X \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$.

Remark 4.1. Note that, in this section, our base field is $\mathbb{F}_p$ rather than $\mathbb{F}_q$ or k. Euler characteristics of the geometrically irreducible curve X/k (or of $X \times_k \bar{\mathbb{F}}_p/\bar{\mathbb{F}}_p$) are finer invariants than the corresponding Euler characteristics of $X/\mathbb{F}_p$ and, as already explained in Remark 5.3 of [Ch], it seems not to be possible to relate these finer invariants to ε-constants associated with X and G. In this paper, this becomes apparent in the transition from Theorem 3.2 to Corollaries 3.3 and 3.4. More concretely, if for instance $\mathfrak{q} \in Y^{\mathrm{t}}$ is a k-rational point, the sum $\sum_{i=0}^{\deg(\mathfrak{q})-1} \left\{ \frac{d_{\mathfrak{q}}(\chi) q^i}{e_{\mathfrak{q}}} \right\}$ occuring in (15) is reduced to $\frac{d_{\mathfrak{q}}(\chi)}{e_{\mathfrak{q}}}$ and seems not to be related to the p-adic valuation of the corresponding Gauss sum.

We use notations similar to those introduced in the previous chapter; most notably, the element $\psi(G,\bar{X}) \in K_0(\bar{\mathbb{F}}_p[G])$ is defined by the equation

$$c(\psi(G,\bar{X})) = [H^0(\bar{X}, \mathcal{O}_{\bar{X}}(\bar{D}^{\mathrm{w}}))] - [H^1(\bar{X}, \mathcal{O}_{\bar{X}}(\bar{D}^{\mathrm{w}}))] \text{ in } K_0(G, \bar{\mathbb{F}}_p)$$

where $\bar{D}^{\mathrm{w}}$ is the divisor

$$\bar{D}^{\mathrm{w}} = \bar{D}^{\mathrm{w}}(G,X) = -\sum_{P \in \bar{X}^{\mathrm{w}}} [P]$$

and $\bar{X}^{\mathrm{w}}$ denotes the set of all points $P \in \bar{X} := X \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ such that the induced projection

$$\bar{\pi} : \bar{X} \to \bar{Y} := Y \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$$

is wildly ramified at P. We put $\bar{Y}^{\mathrm{w}} := \bar{\pi}(\bar{X}^{\mathrm{w}})$. For every representation V of G and for every subgroup H of G, let V^H denote the subspace of V fixed by H.

The Grothendieck L-function associated with X, G and a finite-dimensional $\bar{\mathbb{Q}}$-representation V of G is

$$L(V,t) := \prod_{\mathfrak{q} \in Y} \det\left(1 - \mathrm{Frob}(\tilde{\mathfrak{q}}) t^{\deg(\mathfrak{q})} | V^{I_{\tilde{\mathfrak{q}}}}\right)^{-1},$$

where $\mathrm{Frob}(\tilde{\mathfrak{q}}) \in G$ denotes a geometric Frobenius automorphism at $\bar{\mathfrak{q}}$, i.e. $\mathrm{Frob}(\tilde{\mathfrak{q}})$ induces the inverse of the usual Frobenius automorphism of the field extension $k(\tilde{\mathfrak{q}})/k(\mathfrak{q})$. We recall that replacing the geometric with the arithmetic Frobenius automorphism in this definition amounts to defining the Artin L-function which in turn is equal to the Grothendieck L-function $L(V^*,t)$ corresponding to the contragredient representation V^* of V, see [Mi, Exercise V.2.21(h)]. By [De, Théorème 7.11(iii)], the Grothendieck L-function satisfies the functional equation

$$L(V,t) = \varepsilon(V)\ t^a\ L(V^*, \frac{1}{pt})$$

with some $a \in \mathbb{N}$ and with $\varepsilon(V)$ denoting the ε-constant considered in the previous sections. By [Mi, Theorem VI.13.3] (or by Corollary 1.3 and Artin's induction theorem), we have $\varepsilon(V) \in \bar{\mathbb{Q}}$. After applying $j_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ to $\varepsilon(V)$, we may consider its p-adic valuation $v_p(j_p(\varepsilon(V)))$. The following theorem computes this p-adic valuation in terms of the element $\psi(G,\bar{X})$ introduced above.

Theorem 4.2. *For every finite-dimensional $\bar{\mathbb{Q}}$-representation V of G we have*

$$-v_p(j_p(\varepsilon(V))) = \langle e(\psi(G,\bar{X})), j_pV\rangle + \sum_{Q\in\bar{Y}^{\mathrm{w}}} \dim_{\bar{\mathbb{Q}}}(V^{G_{\tilde{Q}}}). \tag{16}$$

In particular, the p-adic valuation of $j_p(\varepsilon(V))$ is an integer.

Remark 4.3. Using the notation $E(G,X)$ introduced in the next section, Theorem 4.2 can obviously be reformulated in the following succinct way.

We have:

$$E(G,X) = e(\psi(G,\bar{X})) + \sum_{Q\in\bar{Y}^{\mathrm{w}}} \left[\mathrm{Ind}_{G_{\tilde{Q}}}^{G}(\mathbf{1})\right] \quad \textit{in} \quad K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}},$$

where $\mathbf{1}$ means the trivial representation of rank 1. In particular, $E(G,X)$ belongs to the integral part $K_0(\bar{\mathbb{Q}}_p[G])$ of $K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}$.

Proof. By Artin's induction theorem [Se2, Corollaire of Théorème 17], every element of $K_0(\bar{\mathbb{Q}}[G])_{\mathbb{Q}}$ can be written as a rational linear combination of representations induced from multiplicative characters of cyclic subgroups of G. It therefore suffices to prove the following three statements.

(a) When applied to the direct sum $V_1 \oplus V_2$ of two finite-dimensional $\bar{\mathbb{Q}}$-representations V_1 and V_2 of G, each side of equation (16) is equal to the sum of the corresponding values for V_1 and V_2.

(b) Each side of equation (16) is invariant under induction with respect to V.

(c) Equation (16) is true if G is a cyclic group and V corresponds to a multiplicative character χ of G.

Statement (a) is obvious for the right-hand side of (16). For the left-hand side, this follows from (5.2) and (5.11.2) in [De].

It is well-known that the L-fucntion $L(V,t)$ is invariant under induction with respect to V. This immediately implies that the left-hand side of (16) is invariant under induction. We now prove that the right-hand side of (16) is invariant under induction as well. Let H be a subgroup of G and let $\bar{\pi}_H : \bar{X} \to \bar{X}/H$ denote the corresponding projection. Furthermore, let W

be a finite-dimensional $\bar{\mathbb{Q}}$-representation of H. Then the right-hand side of (16) applied to $V = \mathrm{Ind}_H^G(W)$ is equal to

$$\langle e(\psi(G,\bar{X})), j_p \mathrm{Ind}_H^G(W)\rangle + \sum_{Q \in \bar{Y}^{\mathrm{w}}} \langle \mathbf{1}, \mathrm{Res}^G_{G_{\tilde{Q}}}(\mathrm{Ind}_H^G(W))\rangle \tag{17}$$

where $\mathbf{1}$ means the trivial representation of dimension 1. By Frobenius reciprocity [Se2, Théorème 13], this is equal to

$$\langle e(\mathrm{Res}^G_H(\psi(G,\bar{X}))), j_p W\rangle + \sum_{Q \in \bar{Y}^{\mathrm{w}}} \langle \mathrm{Res}^G_H(\mathrm{Ind}^G_{\tilde{Q}}(\mathbf{1})), W\rangle. \tag{18}$$

For each $Q \in \bar{Y}$ we have a canonical bijection

$$\{H\sigma G_{\tilde{Q}} : \sigma \in G\} \to \bar{\pi}_H(\bar{\pi}^{-1}(Q)), \quad H\sigma G_{\tilde{Q}} \mapsto \bar{\pi}_H(\sigma(\tilde{Q})).$$

Therefore, Mackey's double coset formula [CR1, Theorem 44.2] implies that (18) is equal to

$$\begin{aligned} &\langle e(\mathrm{Res}^G_H(\psi(G,\bar{X})), j_p W\rangle + \sum_{R \in \bar{\pi}_H(\bar{X}^{\mathrm{w}})} \langle \mathrm{Ind}^H_{H_{\tilde{R}}}(\mathbf{1}), W\rangle \\ &\quad = \langle e(\mathrm{Res}^G_H(\psi(G,\bar{X})), j_p W\rangle + \sum_{R \in \bar{\pi}_H(\bar{X}^{\mathrm{w}})} \dim_{\bar{\mathbb{Q}}}(W^{H_{\tilde{R}}}), \end{aligned} \tag{19}$$

where, of course, $\tilde{R}$ is a point in $\bar{\pi}_H^{-1}(R)$ and $H_{\tilde{R}} := G_{\tilde{R}} \cap H$. Let $\mathcal{S}$ denote the set of all points $R \in \bar{X}/H$ such that $\bar{\pi} : \bar{X} \to \bar{X}/G = Y$ is wildly ramified at $\tilde{R}$, but $\bar{\pi}_H : \bar{X} \to \bar{X}/H$ is tamely ramified at $\tilde{R}$. Then for each $R \in \mathcal{S}$, the coefficient of the divisor $D^{\mathrm{w}}(G,\bar{X})$ at $\tilde{R}$ is $-1 = (e_R(H) - 1) + (-1)e_R(H)$, where $e_R(H)$ denotes the ramification index of $\bar{\pi}_H$ at $\tilde{R}$. Using [Kö, Theorem 4.5] (see also equation (7)), we now obtain:

$$\begin{aligned} \mathrm{Res}^G_H(\psi(G,\bar{X})) &= \psi(H,\bar{X},D^{\mathrm{w}}(G,\bar{X})) \\ &= \psi(H,\bar{X},D^{\mathrm{w}}(H,\bar{X})) + \sum_{R\in\mathcal{S}} \left(\sum_{d=1}^{e_R(H)-1} \left[\mathrm{Ind}^H_{H_{\tilde{R}}}(\chi_{\tilde{R}}^{-d})\right] - \left[\bar{\mathbb{F}}_p[H]\right] \right) \\ &= \psi(H,\bar{X}) - \sum_{R\in\mathcal{S}} \left[\mathrm{Ind}^H_{H_{\tilde{R}}}(\mathbf{1})\right], \end{aligned}$$

where $\chi_{\tilde{R}}$ also denotes the restriction of $\chi_{\tilde{R}} : G_{\tilde{R}} \to \bar{\mathbb{F}}_p$ to $H_{\tilde{R}}$. Therefore (19) is equal to

$$\begin{aligned}(20)\qquad &\langle e(\psi(H,\bar{X})), j_p W\rangle - \sum_{R\in\mathcal{S}} \dim_{\bar{\mathbb{Q}}}(W^{H_{\tilde{R}}}) + \sum_{R\in\bar{\pi}_H(\bar{X}^{\mathrm{w}})} \dim_{\bar{\mathbb{Q}}}(W^{H_{\tilde{R}}}) \\ &= \langle e(\psi(H,\bar{X})), j_P W\rangle + \sum_{R\in(\bar{X}/H)^{\mathrm{w}}} \dim_{\bar{\mathbb{Q}}}(W^{H_{\tilde{R}}}),\end{aligned}$$

where $(\bar{X}/H)^{\mathrm{w}}$ denotes the set of all $R \in \bar{X}/H$ such that $\bar{\pi}_H$ is not tamely ramified at $\tilde{R}$. This finishes the proof of statement (b).

We finally prove statement (c). Let G be cyclic and let χ be a multiplicative character of G. Let r denote the degree of k over $\mathbb{F}_p$. By [Ch, Remark 5.4], the ε-constant of $X/\mathbb{F}_p$ is the same as the ε-constant of X considered as a scheme over k. Hence, by Corollary 2.7 we have

$$(21)\qquad -v_p(j_p(\varepsilon(\chi))) = r(1-g_{Y_k}) - \sum v_p(j_p(\tau(\chi_{k(\mathfrak{q})}))) - \sum \deg(\mathfrak{q}).$$

Here the first sum runs over all $\mathfrak{q} \in Y$ such that χ is tamely ramified at $\mathfrak{q}$ and the second sum runs over all $\mathfrak{q} \in \bar{Y}$ such that χ is not tamely ramified at $\tilde{q}$. These two sets differ from Y^{t} and Y^{w} by the set of those $\mathfrak{q} \in \bar{Y}^{\mathrm{w}}$ such that χ vanishes on $G_{\tilde{\mathfrak{q}},1}$. For such $\mathfrak{q}$ we have $G_{\tilde{\mathfrak{q}}} = G_{\tilde{\mathfrak{q}},1}$ by Lemma 3.1 and the corresponding Gauss sum is trivial. Hence (21) is equal to

$$(22)\quad r(1-g_{Y_k}) - \sum_{\mathfrak{q}\in Y^{\mathrm{t}}} v_p(j_p(\tau(\chi_{\mathfrak{q}}))) - \sum_{\mathfrak{q}\in Y^{\mathrm{w}}} \deg(\mathfrak{q}) + \sum_{\mathfrak{q}\in Y^{\mathrm{w}}:\mathrm{Res}^G_{G_{\tilde{\mathfrak{q}}}}(\chi)=\mathbf{1}} \deg(\mathfrak{q})$$

which in turn is equal to

$$(23)\qquad \langle e(\psi(G,\bar{X})), j_p\chi\rangle + \sum_{Q\in\bar{Y}^{\mathrm{w}}} \langle \mathbf{1}, \mathrm{Res}^G_{G_{\tilde{Q}}}(\chi)\rangle$$

by Corollary 3.4, as was to be shown. □

5 A Weak, but General Formula

As in the previous section, let X be an irreducible smooth projective curve over $\mathbb{F}_p$, let k denote the algebraic closure of $\mathbb{F}_p$ in $K(X)$ and let G be a finite subgroup of $\mathrm{Aut}(X/k)$. Without assuming any condition on the type of ramification of the associated projection $\pi : X \to X/G =: Y$ we give, in this section, a 'weak' relation between the equivariant Euler characteristic $\chi(G, \bar{X}, \mathcal{O}_{\bar{X}})$ of $\bar{X} := X \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ and ε-constants associated with X and finite-dimensional complex representations of G.

By [De, (5.2)], associating with every finite-dimensional $\bar{\mathbb{Q}}$-representation V of G the p-adic valuation $v_p(j_p(\varepsilon(V)))$ of the ε-constant $j_p(\varepsilon(V))$ defines a homomorphism from $K_0(\bar{\mathbb{Q}}[G])_{\mathbb{Q}} := K_0(\bar{\mathbb{Q}}[G]) \otimes \mathbb{Q}$ to $\mathbb{Q}$. As the classical character pairing $\langle \quad , \quad \rangle : K_0(\bar{\mathbb{Q}}_p[G]) \times K_0(\bar{\mathbb{Q}}_p[G]) \to \mathbb{Z}$ is non-degenerate, there is a unique element $E(G, X) \in K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}$ such that

$$\langle E(G, X), j_p(V) \rangle = -v_p(j_p(\varepsilon(V)))$$

for all finite-dimensional $\bar{\mathbb{Q}}$-representations V of G. It follows for instance from the definition of $L(V, t)$ that, for every $\alpha \in \mathrm{Aut}(\bar{\mathbb{Q}})$, we have $\varepsilon(\alpha(V)) = \alpha(\varepsilon(V))$ and that therefore $E(G, X)$ does not depend on the embedding j_p. Recall that

$$d : K_0(\bar{\mathbb{Q}}_p[G]) \to K_0(G, \mathbb{F}_p)$$

denotes the decomposition map.

Theorem 5.1. *We have*

$$d(E(G, X)) = \chi(G, \bar{X}, \mathcal{O}_{\bar{X}}) \quad \text{in} \quad K_0(G, \bar{\mathbb{F}}_p)_{\mathbb{Q}}. \tag{24}$$

In particular, $d(E(G, X))$ *lies in the integral part* $K_0(G, \bar{\mathbb{F}}_p)$ *of* $K_0(G, \bar{\mathbb{F}}_p)_{\mathbb{Q}}$.

Proof. As the canonical pairing

$$\langle \quad , \quad \rangle : K_0(\bar{\mathbb{F}}_p[G]) \times K_0(G, \bar{\mathbb{F}}_p) \to \mathbb{Z}$$

(see the beginning of the proof of Theorem 4.2) is non-degenerate as well [Se2, §14.5(b)], it suffices to show that

$$\langle P, d(E(G, X)) \rangle = \langle P, \chi(G, \bar{X}, \mathcal{O}_{\bar{X}}) \rangle \tag{25}$$

for all finitely generated projective $\bar{\mathbb{F}}_p[G]$-modules P. By Artin's induction theorem for modular representation theory [Se2, Théorème 40], every element in $K_0(\bar{\mathbb{F}}_p[G])_{\mathbb{Q}}$ can be written as a rational linear combination of representations induced from one-dimensional projective representations of cyclic subgroups of G. Furthermore, by Frobenius reciprocity and the fact that ε-constants are invariant under induction, both sides of (25) are invariant under induction with respect to P. As in the proof of Theorem 4.2, we may therefore assume that G is cyclic and that P corresponds to a character $\chi : G \to \bar{\mathbb{F}}_p^\times$. The fact that P is projective moreover implies that p does not divide the order of G. In particular, the projection π is tamely ramified and we conclude from Theorem 4.2 (or actually already from Theorem 5.2 in [Ch]) that $E(G, X) = e(\psi(G, \bar{X}))$. We therefore have

$$d(E(G,X)) = d(e(\psi(G,\bar{X}))) = c(\psi(G,\bar{X})) = \chi(G,\bar{X},\mathcal{O}_{\bar{X}}),$$

as was to be shown. □

Remark 5.2. If π is weakly ramified, Theorem 5.1 can be derived from Theorem 4.2 also in the following way:

$$\begin{aligned} d(E(G,X)) &= d\left(e(\psi(G,\bar{X})) + \sum_{Q\in\bar{Y}^{\mathrm{w}}} \mathrm{Ind}_{G_{\tilde{Q}}}^{G}(\mathbf{1})\right) \\ &= c(\psi(G,\bar{X})) + \sum_{Q\in\bar{Y}^{\mathrm{w}}} d(\mathrm{Ind}_{G_{\tilde{Q}}}^{G}(\mathbf{1})) \\ &= \chi(G,\bar{X},\mathcal{O}_{\bar{X}}(\bar{D}^{\mathrm{w}})) + \sum_{Q\in\bar{Y}^{\mathrm{w}}} \mathrm{Ind}_{G_{\tilde{Q}}}^{G}(\mathbf{1}) \\ &= \chi(G,\bar{X},\mathcal{O}_{\bar{X}}). \end{aligned}$$

Here, the first equality follows from Thoerem 4.2, and the last equality follows from Theorem 3.1 in [Kö] or from the simpler formula [Bo, Théorème 4.10].

We end with the following problem.

Problem 5.3. Describe $E(G, X)$ within $K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}$ in terms of global geometric invariants of $\bar{X}$ in a way that generalises Theorem 4.2 from the weakly ramified situation to the general situation considered in this section and that proves the conjecture that $E(G, X)$ belongs to the integral part $K_0(\bar{\mathbb{Q}}_p[G])$ of $K_0(\bar{\mathbb{Q}}_p[G])_{\mathbb{Q}}$.

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