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Zeta functions of groups and rings

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
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Zeta functions of groups and rings

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The representation growth of a $\mathcal{T}$-group is polynomial. We study the rate of polynomial growth and the spectrum of possible growth, showing that any rational number α can be realized as the rate of polynomial growth of a class 2 nilpotent $\mathcal{T}$-group. This is in stark contrast to the related subject of subgroup growth of $\mathcal{T}$-groups where it has been shown that the set of possible growth rates is discrete in $\mathbb{Q}$.

We derive a formula for almost all of the local representation zeta functions of a T_2-group with centre of Hirsch length 2. A consequence of this formula shows that the representation zeta function of such a group is finitely uniform. In contrast, we explicitly derive the representation zeta function of a specific T_2-group with centre of Hirsch length 3 whose representation zeta function is not finitely uniform.

We give formulae, in terms of Igusa's local zeta function, for the subring, left-, right- and two-sided ideal zeta function of a 2-dimensional ring. We use these formulae to compute a number of examples. In particular, we compute the subring zeta function of the ring of integers in a quadratic number field.

Contents

Author's declaration

I, Robert Snocken, declare that the thesis entitled *Zeta functions of groups and rings* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;

Signed..

Date...

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Chapter 1

Introduction

The study of the representation growth of finitely generated torsion-free nilpotent groups (or $\mathcal{T}$-groups, for short) was introduced in [20] and studied implicitly, for a single group, in [33]. These papers were motivated by the analogy with the concept of subgroup growth and by the study of representation varieties, respectively.

We briefly outline these topics, but for a more comprehensive survey one should consult [30] and [29] respectively. We survey what is known about the representation growth of $\mathcal{T}$-groups and outline the content and layout of the thesis.

1.1 Representation varieties of finitely generated groups

Let G be a finitely generated group and k be an algebraically closed field of characteristic zero. We are interested in the representation theory of G over k, that is, the homomorphisms from G to $\mathrm{GL}_n(k)$ for some $n \in \mathbb{N}$.

If G is finite then the theory is well-studied. Every representation is semisimple, that is, a sum of irreducible representations, and there are only finitely many irreducible representations up to isomorphism. The isomorphism class of a representation is determined by its character, and so, the representations of G are completely known once the character table is computed.

If G is infinite then none of the assertions made in the previous paragraph for finite groups hold in general. The analogue of character theory is the parameterisation of representations by affine varieties. The details for the discussion below can be found in [29].

Let $\langle x_1, \dots, x_d \mid \mathscr{R} \rangle$ be a presentation for G and $n \in \mathbb{N}$. A d-tuple $(A_1, \dots, A_d) \in \mathrm{GL}_n(k)^d$ determines a representation $\rho : G \to \mathrm{GL}_n(k)$ if and only if the matrices $A_1, \dots, A_d$ satisfy the relations $\mathscr{R}$. We can translate the group relations $\mathscr{R}$ into polynomial conditions on the entries of the A_i, and so the set of representations of G over k of dimension n has the structure of an affine variety. Note that we do not require that G is finitely presented. If G were not finitely presented we would have infinitely many polynomial conditions, but these conditions would be equivalent to some finite set of polynomials. We denote this affine variety by $\mathrm{Rep}_n(G)$. The variety $\mathrm{Rep}_n(G)$ depends on the field k, but we suppress this in the notation. Importantly, the isomorphism class of the variety $\mathrm{Rep}_n(G)$ does not depend on the presentation of G.

The orbits of the natural action of $\mathrm{GL}_n(k)$ on $\mathrm{Rep}_n(G)$ are not necessarily Zariski-closed, and so the orbit space $\mathrm{Rep}_n(G)/\mathrm{GL}_n(k)$ need not have the structure of an affine variety. Let $\mathrm{Rep}_n(G)/\!/\mathrm{GL}_n(k)$ denote the space of closed orbits. This does have the structure of an affine variety and is parameterised by the isomorphism classes of semisimple representations. For details on geometric invariant theory and the construction and properties of the quotient variety the reader is referred to [32, Chapter 3]. We denote $\mathrm{Rep}_n(G)/\!/\mathrm{GL}_n(k)$ by $\mathrm{SS}_n(G)$ and call it the variety of semisimple representations. Furthermore, the set of isomorphism classes of irreducible n-dimensional representations, which we denote by $\mathrm{Irr}_n(G)$, is an open sub-variety of $\mathrm{SS}_n(G)$. Alternatively, we can think of $\mathrm{SS}_n(G)$ and $\mathrm{Irr}_n(G)$ as being the spaces of n-dimensional semisimple and irreducible characters of G, respectively. A geometric description of these varieties can be viewed as the analogue of determining the characters of a finite group.

For a $\mathcal{T}$-group G, the varieties $\mathrm{Irr}_n(G)$ and $\mathrm{SS}_n(G)$ are well understood qualitatively, see [29, Section 6]. To give such a description we first need a very important definition. Note that $\mathrm{Rep}_1(G) = \mathrm{SS}_1(G) = \mathrm{Irr}_1(G)$ is a group under the tensor product.

Definition 1.1. Let G be a group. Two representations ρ_1, ρ_2 are *twist-equivalent* if there exists a 1-dimensional representation χ of G such that $\rho_1 = \chi \otimes \rho_2$.

Note that the twisting action descends to give an action on $\mathrm{SS}_n(G)$ and $\mathrm{Irr}_n(G)$. As $\mathrm{Irr}_1(G)$ is a group, one can easily check that twist-equivalence is an equivalence relation. The orbits of $\mathrm{SS}_n(G)$ and $\mathrm{Irr}_n(G)$ under the action of $\mathrm{Irr}_1(G)$ are called *twist-isoclasses.*

Note that, in general, the operations of matrix conjugation and twisting by 1-dimensional representations commute, so that two representations ρ_1, ρ_2 are in the same twist-isoclass if ρ_1 is isomorphic to a representation that differs from ρ_2 by a 1-dimensional representation.

We now describe qualitatively the varieties $\mathrm{SS}_n(G)$ and $\mathrm{Irr}_n(G)$ in the case where G is a finitely generated nilpotent group. We denote the abelianisation of G by G^{ab} and the Hirsch length of a group G by $\mathrm{h}(G)$.

Theorem 1.2. [29, Section 6, (I) and (II)] *Let G be a finitely generated nilpotent group and let $n \in \mathbb{N}$.*

1. *G has finitely many isomorphism classes of semisimple representations $\sigma_{n,1}, \ldots, \sigma_{n,s(n)}$ such that $\mathrm{SS}_n(G)$ is equal to the disjoint union of the twist-isoclasses of $\sigma_{n,1}, \ldots, \sigma_{n,s(n)}$. Furthermore, this is a partition of $\mathrm{SS}_n(G)$ into open and closed subvarieties.*

2. $\mathrm{Irr}_n(G)$ *is non-singular. Each of its irreducible components has dimension* $h(G^{ab})$ *and consists of a single twist-isoclass.*

1.2 Subgroup growth of $\mathcal{T}$-groups

Let G be a finitely generated group. Let $a_n(G)$ denote the number of subgroups of index n in G. Since G is finitely generated, $a_n(G)$ is finite for all $n \in \mathbb{N}$. Loosely speaking 'subgroup growth' is the study of the properties of the sequence $(a_n(G))$ and how these properties relate to the structure of G. An outstanding result in this area is the polynomial subgroup growth theorem. We say that a group G has polynomial subgroup growth if there exist $c, d \in \mathbb{R}_{\geq 0}$ such that $a_n(G) \leq cn^d$ for all $n \in \mathbb{N}$.

Theorem 1.3. [30, Theorem 5.1] *Let G be a finitely generated residually finite group. Then G has polynomial subgroup growth if and only if G is virtually soluble of finite rank.*

In the landmark paper [16], Grunewald, Segal and Smith introduced zeta functions to study the subgroup growth of a $\mathcal{T}$-group. Given a $\mathcal{T}$-group G, its *subgroup zeta function* is defined as

$$\zeta_G(s) := \sum_{n=1}^{\infty} a_n(G) n^{-s}. \tag{1.1}$$

The overarching questions in the theory of subgroup growth for $\mathcal{T}$-groups concern connections between a group's algebraic structure, the arithmetic properties of its subgroup growth and the analytic properties of the subgroup zeta function. In [10] du Sautoy and Grunewald show that the subgroup zeta function of a $\mathcal{T}$-groups satisfies certain analytic properties.

Theorem 1.4. *Let G be a $\mathcal{T}$-group with subgroup zeta function $\zeta_G(s)$. There exists $\alpha \in \mathbb{Q}$ such that $\zeta_G(s)$ converges for $\Re(s) > \alpha$. The zeta function $\zeta_G(s)$ admits a meromorphic continuation beyond its abscissa of convergence* *and furthermore the continued function has only a single pole on the line* $\Re(s) = \alpha$ *located at* $s = \alpha$.

As a corollary to this theorem one can deduce a precise asymptotic statement about the subgroup growth of G. The proof of this theorem is deep and requires an application of resolution of singularities, see [10] for the details. In the proof of the result the growth rate α is determined in terms of the numerical data associated to a resolution of singularities. However, while this data is, in principal, available, in practice it is very difficult to ascertain. Furthermore, it is not clear what structural properties of G are reflected therein.

1.3 Representation growth of $\mathcal{T}$-groups

The first topic of this thesis is the representation growth of $\mathcal{T}$-groups. In this section we state the basic definitions and record the main results in the area.

Let G be a $\mathcal{T}$-group. In general, G has uncountably many irreducible representations in infinitely many dimensions. However we have seen that the number of twist-isoclasses of a given dimension is finite. This is because the number of twist-isoclasses is equal to the number irreducible components of an affine variety. We introduce the main invariants to be investigated. For $n \in \mathbb{N}$, set

$$\widetilde{\mathrm{r}}_n(G) := \#\{\text{twist-isoclasses of irreducible complex representations of } G \text{ of dimension } n\}.$$

In the sequel, by *twist-isoclasses* we mean twist-isoclasses of irreducible representations unless explicitly stated otherwise. We also require notation for the sequence of partial sums. For $n \in \mathbb{N}$, set

$$\widetilde{R}_n(G) := \sum_{i=1}^{n} \widetilde{\mathrm{r}}_i(G)$$

Being a non-decreasing function, $\widetilde{R}_n(G)$ has the potential to 'smoothen out' some of the variation in the values $\widetilde{\mathrm{r}}_n(G)$. The sequence $(\widetilde{R}_n(G))$ is called the representation growth of G. A sequence (a_n) of natural numbers has polynomial growth if there exist $c, d \in \mathbb{R}_{\geq 0}$ such that $a_n \leq cn^d$ for all n. By Lemma 4.4, a $\mathcal{T}$-group has polynomial representation growth, that is, the sequence $(\widetilde{R}_n(G))$ has polynomial growth. We are interested in the rate of polynomial growth

$$\alpha^{\widetilde{\mathrm{irr}}}(G) := \inf\{d \in \mathbb{R} \mid \exists c \in \mathbb{R}, \widetilde{R}_n(G) \leq cn^d, \forall n \in \mathbb{N}\}. \tag{1.2}$$

We introduce an important tool in the study of representation growth, the representation zeta function. For a complex variable s, set

$$\zeta_G^{\widetilde{\mathrm{irr}}}(s) := \sum_{n=1}^{\infty} \widetilde{\mathrm{r}}_n(G) n^{-s}. \tag{1.3}$$

This Dirichlet series converges precisely on the right half plane $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > \alpha^{\widetilde{\mathrm{irr}}}(G)\}$ (cf. [26, Chapter VIII]).

Example 1.5. The discrete Heisenberg group H is the set of 3×3 upper uni-triangular matrices over $\mathbb{Z}$. In [33] Magid and Nunley studied the representation varieties of H. As stated in Section 1.1 the number of irreducible components of $\widetilde{\mathrm{Irr}}_n(H)$ is equal to number of twist-isoclasses of dimension n. They computed that $\widetilde{\mathrm{r}}_n(H) = \phi(n)$, the Euler totient function. The Dirichlet series associated with the Euler totient function is well-known (see [1, Section 11.4]), namely,

$$\zeta_H^{\widetilde{\mathrm{irr}}}(s) = \sum_{n=1}^{\infty} \phi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)},$$

where $\zeta(s)$ denotes the Riemann zeta function. Therefore, $\alpha^{\widetilde{\mathrm{irr}}}(H) = 2$.

We present a couple of theorems regarding the representation theory of $\mathcal{T}$-groups that will be needed.

Theorem 1.6. [29, Proposition 1] *Let G be a $\mathcal{T}$-group. For all $n \in \mathbb{N}$ there exists a finite quotient $G(n)$ such that all irreducible n-dimensional representations of G are twist-equivalent to a representation that factors through $G(n)$.*

Theorem 1.7. [5, Theorem 11.3] *$\mathcal{T}$-groups are monomial, that is every irreducible representation of a $\mathcal{T}$-group G is induced from a 1-dimensional representation of a finite index subgroup of G.*

A priori, we do not know which finite quotients are candidates for $G(n)$. In Section 2.2 we show how the Kirillov orbit method can be applied to find all irreducible representations by inducing 1-dimensional representations from finite-index subgroups. The method produces an almost canonical set of subgroups and 1-dimensional representations. This set is only almost canonical, because, as we shall see, there is some choice.

The focus of the fledgling subject of representation growth of $\mathcal{T}$-groups has been to explore the properties of the representation zeta function and to calculate specific examples.

Theorem 1.8. [12, Theorem 1.1] *Let K be a quadratic number field with ring of integers $\mathcal{O}_K$. Let $H(\mathcal{O}_K)$ be the Heisenberg group over the ring of integers $\mathcal{O}_K$, that is the 3×3 upper uni-triangular matrices over $\mathcal{O}_K$. Then,*

$$\widetilde{\zeta^{\mathrm{irr}}_{H(\mathcal{O}_K)}}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)}, \tag{1.4}$$

where $\zeta_K(s)$ is the Dedekind zeta function of the number field K.

Ezzat's calculations are very constructive: for a specific generating set, he gives explicit matrices for representatives of each twist-class. In [12] Ezzat conjectures that the formula (1.4) should generalise to arbitrary number fields. In [36] Stasinski and Voll define three infinite families of class-2-nilpotent groups. Each of these families generalises the Heisenberg group. Ezzat's conjecture follows from [36, Theorem B]. It should be noted that

Stasinski and Voll's results are not as constructive, specifically they do not give explicit matrices for the images of a generating set for a representative of each twist-isoclass.

1.4 Localisation

In this section we describe how the process of calculating the representation zeta function of a $\mathcal{T}$-group can be 'localized'. Let G be a $\mathcal{T}$-group. For a prime p we define the p-local representation zeta function

$$\zeta^{\widetilde{\text{irr}}}_{G,p}(s) := \sum_{i=0}^{\infty} \widetilde{\text{r}}_{p^i}(G)(p^{-s})^i. \tag{1.5}$$

The p-local zeta function is the Dirichlet generating function that encodes the numbers of twist-isoclasses of p-power dimension. In Section 4.1.2 we will see that the global zeta function satisfies the Euler factorisation

$$\zeta^{\widetilde{\text{irr}}}_{G}(s) = \prod_{p} \zeta^{\widetilde{\text{irr}}}_{G,p}(s), \tag{1.6}$$

where the product is taken over all primes. This result is key. In view of this factorisation, the p-local representation zeta functions are often referred to simply as the p-local factors. The remainder of this section collects some of what is known about these local factors.

Theorem 1.9. [20, Theorem 8.4] *Let G be a $\mathcal{T}$-group. For all primes p, the local representation zeta function $\zeta^{\widetilde{\text{irr}}}_{G,p}(s)$ is a rational function of p^{-s}.*

A consequence of this theorem is that the sequence $(\widetilde{\text{r}}_{p^i}(G))$ satisfies a linear recurrence relation. The proof of Theorem 1.9 requires a deep application of a model-theoretic result, for the details see [20].

The p-local zeta function of G can be seen as the global zeta function of a local object, namely the pro-p completion of G. Let $\widehat{G}_p$ denote the pro-p completion of G. We consider only continuous representations, those of finite image. A representation of $\widehat{G}_p$ is called *p-admissible* if it factors through a finite p-group. Two representations are p-twist-equivalent if they are twist-equivalent by a p-admissible 1-dimensional representation. We define $\widetilde{\mathrm{r}}_n^{(p)}(\widehat{G}_p)$ to be the number of p-twist-isoclasses of p-admissible irreducible complex representation. Note that $\widetilde{\mathrm{r}}_n^{(p)}(\widehat{G}_p)$ is zero unless n is a p-power as irreducible representations of finite p-groups have p-power dimension. The representation zeta function of $\widehat{G}_p$ is as defined as follows:

$$\zeta^{\widetilde{\mathrm{irr}}}_{\widehat{G}_p}(s) = \sum_{i=0}^{\infty} \widetilde{\mathrm{r}}_{p^i}^{(p)}(\widehat{G}_p)(p^{-s})^i. \tag{1.7}$$

Proposition 1.10. [20, Lemma 8.5] *Let G be a $\mathcal{T}$-group. For all primes p,*

$$\zeta^{\widetilde{\mathrm{irr}}}_{G,p}(s) = \zeta^{\widetilde{\mathrm{irr}}}_{\widehat{G}_p}(s).$$

Actually, Hrushovski and Martin show that there is a canonical bijection between the set of twist-isoclasses of p^i-dimensional representations of G and the p-twist-isoclasses of representations of $\widehat{G}_p$ of dimension p^i.

We finish this section by recalling two results, due to Voll, about the local factors.

Theorem 1.11. [38] *Let G be a $\mathcal{T}$-group. There exist rational functions $W_1(X,Y)$, …, $W_k(X,Y)$ over $\mathbb{Q}$ and smooth projective varieties $V_1, \ldots, V_k$ over $\mathbb{Q}$ such that, for almost all primes p,*

$$\zeta^{\widetilde{\mathrm{irr}}}_{G,p}(s) = \sum_{i=1}^{k} |\overline{V}_i(\mathbb{F}_p)| W(p, p^{-s}),$$

where $|\overline{V}_i(\mathbb{F}_p)|$ denotes the number of $\mathbb{F}_p$-rational points of $\overline{V}_i$, the reduction of V_i modulo p.

The proof is non-constructive. In general, the specific rational functions and smooth projectives varieties are unknown. The proof requires an application of a deep result from algebraic geometry: principalisation of ideals, a generalisation of the concept of resolution of singularities. Voll uses this formula to show that the local factors satisfy a local functional equation.

Theorem 1.12. [38, Theorem D] *Let G be a $\mathcal{T}$-group with derived group of Hirsch length d. For almost all primes p,*

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s)|_{p\to p^{-1}} = p^d \widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s).$$

For details on local functional equations, see [38, Section 1]. Conceptually, this result has an interesting consequence. The representation growth of a $\mathcal{T}$-group G 'knows' the Hirsch length of the derived group of G. This is one of the first results demonstrating the connection between the structure of the group and the properties of the sequence $(\widetilde{r}_n(G))$.

1.5 Summary of results

The main results of this thesis concern the representation growth of $\mathcal{T}_2$-groups and the subring and ideal growth a 2-dimensional rings. In this section we outline the layout of the thesis and summarize the main results.

Chapter 2 describes the main technical machinery necessary to enumerate twist-isoclasses of representations, namely, the Kirillov orbit method and elementary divisors. We show that the computation of (almost all) local representation zeta functions of a $\mathcal{T}$-group G can be reduced to enumerating the elementary divisors of a matrix associated to the structure of G.

Chapter 3 introduces Igusa's local zeta function and describes its basic properties and presents a series of tools for explicit computation.

In Chapter 4 we investigate the abscissa of convergence and its relation to the structure of the group. We establish a number of elementary properties and determine the abscissa of convergence for direct products and certain classes of central products. The latter result allows use to deduce the first major result of the thesis, that every positive rational number is realised as the abscissa of convergence of a $\mathcal{T}_2$-group. This is particularly interesting because this is not the case for subgroup growth. Finally, we give bounds on the abscissa of convergence in terms of the structure of the group. This last result is due to joint work with Shannon Ezzat.

Chapter 5 concerns uniformity. Loosely speaking, a representation zeta function satisfies uniformity properties if its local factors are similar, see Chapter 5 for a precise definition. The main result of this chapter states that if the center of $\mathcal{T}_2$-group has rank 2 then the representation zeta function satisfies a uniformity property. Furthermore, a T_2-group with center of rank 3 is exhibited and its representation zeta function is shown not to be uniform.

In Chapter 6 we use the tools from Chapter 2 to compute a number of representation zeta functions.

Chapter 7 concerns subring and ideal growth in 2-dimensional rings. The main results of this section shows that the subring and ideal zeta functions of a 2-dimensional ring R can be expressed in terms of Igusa's local zeta function associated to ideal which depends on the structure of R. As a corollary to these main results, a classification of the possible growth rates is given. Finally, the formulae are used to compute a number of specific examples. These results are similar to results obtained in [22], where the authors focus on the rings $\mathbb{Z}^n$ equipped with component-wise multiplication.

Chapter 2

Enumerating twist-isoclasses

In this chapter we present the main techniques that will be used in the investigation of representation zeta functions of $\mathcal{T}$-groups throughout this thesis, namely, the Kirillov orbit method and elementary divisors. For details on the Kirillov orbit method for $\mathcal{T}$-groups consult [19] and [36]. An alternative method for enumerating twist-isoclasses is developed in [12].

2.1 Lie rings

Definition 2.1. A *Lie ring* L is an abelian group (written additively) with an operation $[\cdot, \cdot]$, called the *Lie bracket*, satisfying

- Bilinearity: $[x + y, z] = [x, z] + [y, z]$, for all $x, y, z \in L$
- The Jacobi Identity: $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$, for all $x, y, z \in L$
- For all $x \in L$, $[x, x] = 0$.

In particular, $[\cdot, \cdot]$ is antisymmetric. We assume further that $(L, +)$ is finitely generated and torsion-free, that is, isomorphic to $\mathbb{Z}^n$. If $A, B \subseteq L$ then $[A, B]$ denotes the Lie subring spanned by all Lie brackets of elements from A with elements from B.

The lower central series $\{\gamma_i(L)\}_{i\in\mathbb{N}}$ of a Lie ring L is defined inductively by $\gamma_1(L) = L$ and $\gamma_{i+1}(L) = [L, \gamma_i(L)]$. A Lie ring is *class-c-nilpotent* if $\gamma_{c+1}(L) = 0$, but $\gamma_c(L) \neq 0$. The derived subalgebra $\gamma_2(L)$ is denoted by L'. The *centre* $Z(L)$ of Lie ring L is defined as $Z(L) = \{y \in L \mid [y, x] = 0, \forall x \in L\}$. We say that a Lie subring A of L is *saturated* in L if, for $n \in \mathbb{N}$, $nx \in A$ implies that $x \in A$.

If L is a class-2-nilpotent Lie ring then $L' \subseteq Z(L)$. For a class-2-nilpotent Lie ring L let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be an additive basis such that $\langle y_1, \ldots, y_n\rangle = Z(L)$. Clearly such a basis exists because $Z(L)$ is saturated in L. For all $i, j \in [1, m]$, $i < j$ the Lie bracket $[x_i, x_j]$ is an element of the centre. Thus $[x_i, x_j] = \sum_{k=1}^{n} \lambda_{ij}^k y_k$, for some $\lambda_{ij}^k \in \mathbb{Z}$. The λ_{ij}^k are called the structure constants of L with respect to the chosen basis.

Conversely, let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be an additive basis of $\mathbb{Z}^{m+n}$. We can define a Lie bracket on $\mathbb{Z}^{m+n}$ by choosing arbitrary structure constants $\lambda_{ij}^k \in \mathbb{Z}$ for $1 \leq k \leq m$ and $1 \leq i < j \leq m$. Set $[x_i, x_j] = \sum_{k=1}^{n} \lambda_{ij}^k y_k$ and prescribe each y_k to be central. The Lie bracket can be extended to the whole of $\mathbb{Z}^{m+n}$ by anti-symmetry and bilinearity. The Jacobi identity will then be trivially satisfied as any nested bracket will be equal to 0.

A presentation of a class-2-nilpotent Lie ring L consists of an additive basis $x_1, \ldots, x_m$, $y_1, \ldots, y_n$ and for each $i, j \in [1, m]$, $i < j$ a linear form in $y_1, \ldots, y_n$. By convention, Lie brackets that do not follow from those presented are assumed to be trivial.

We write $\mathcal{T}_2$ to denote the class of class-2-nilpotent $\mathcal{T}$-groups. We now show how to construct a class-2-nilpotent Lie ring from a $\mathcal{T}_2$-group and conversely.

Let G be a $\mathcal{T}_2$-group such that $h(G/Z(G)) = m$ and $h(Z(G)) = n$. A Mal'cev basis for G is a set of generators $x_1, \ldots, x_m, y_1, \ldots, y_n$, where the images of $x_1, \ldots, x_m$ are a basis of the abelian quotient $G/Z(G)$ and $y_1, \ldots y_n$ are a basis for $Z(G)$. For details on Mal'cev bases see [15, Sections 1 & 5].

Since G is class-2-nilpotent each group commutator $[x_i, x_j]$ is central and there exist $\lambda_{ij}^k \in \mathbb{Z}$ such that G has a group presentation

$$G = \left\langle \begin{array}{c} x_1, \dots x_n, \\ y_1, \dots y_d \end{array} \middle| \, [x_i, x_j] = \prod_{k=1}^{d} {y_k}^{\lambda_{ij}^k}, \ 1 \leq i, j \leq n \right\rangle , \qquad (2.1)$$

where all other commutators that do not follow from those presented are trivial. The λ_{ij}^k are called the group structure constants and depend on the chosen Mal'cev basis. It follows that $g \in G$ can be written uniquely as $g = {x_1}^{e_1} \dots {x_m}^{e_m} {y_1}^{f_1} \dots {y_n}^{f_n}$, where $e_i, f_j \in \mathbb{Z}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

The abelian group $G/Z(G) \oplus Z(G)$ has a natural Lie ring structure induced from the group commutators of G. Fix a Mal'cev basis for G. Identify $x_1, \dots, x_m$ with their images in $G/Z(G)$. Then $x_1, \dots, x_m, y_1, \dots, y_n$ are a generating set for $G/Z(G) \oplus Z(G) \cong \mathbb{Z}^{m+n}$, which is written as an additive group. For $i \neq j$ we define the Lie bracket of x_i and x_j to be $[x_i, x_j] := \sum_{k=1}^{d} \lambda_{ij}^k y_k$, where the λ_{ij}^k are those appearing in the presentation of G. Of course, $[x_i, x_i] := 0$ and for $1 \leq i \leq n$ the Lie bracket of y_i with any other element is trivial. The Lie bracket is extended to $G/Z(G) \oplus Z(G)$ by anti-symmetry and bi-linearity.

The additive group $G/Z(G) \oplus Z(G)$ together with the Lie bracket defined above is called the Lie ring associated with G and is denoted $L(G)$. Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$ then the element $a_1 x_1 + \dots + a_m x_m + b_1 y_1 + \dots + b_n y_n \in L(G)$ will be denoted $\mathbf{x}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}}$. If $\mathbf{x}^{\mathbf{a}'} \mathbf{y}^{\mathbf{b}'}$ is another element of $L(G)$, then $\mathbf{x}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}} + \mathbf{x}^{\mathbf{a}'} \mathbf{y}^{\mathbf{b}'} = \mathbf{x}^{\mathbf{a}+\mathbf{a}'} \mathbf{y}^{\mathbf{b}+\mathbf{b}'}$.

The group structure can be reconstructed from the Lie structure. First, note that the structure constants λ_{ij}^k are determined by the Lie bracket on $L(G)$. We define a binary operation $\star$ on $L(G)$ as follows. Declare the monomials $\mathbf{y}^{\mathbf{b}}$ to be central and, for $1 \leq i < j \leq m$ define

$$x_i^{a_i} \star x_j^{a_j} = x_i^{a_i} x_j^{a_j} {y_1}^{a_i a_j \lambda_{ij}^1} \dots {y_m}^{a_i a_j \lambda_{ij}^m}$$

The definition of the binary operation $\star$ is then extended to $L(G)$ in the obvious way and we have that $(L(G),\star)$ is isomorphic to G. For further details on this construction see [36, Section 2].

While we can think of having one set that has both the structure of a Lie ring and a group, we prefer to think of them as separate objects. We denote the map from $L(G)$ to G determined by the above construction by λ^{-1} and its inverse by λ. The map λ is used to define the adjoint action of G on $L(G)$. Let $g \in G$ then

$$\begin{aligned} \mathrm{Ad}_g : L(G) &\to L(G) \\ x &\mapsto \lambda(g^{-1}) \star x \star \lambda(g) \ . \end{aligned}$$

The adjoint action is given by

$$\begin{aligned} G \times L(G) &\to L(G) \\ (g,x) &\mapsto \mathrm{Ad}_g(x) \ . \end{aligned} \tag{2.2}$$

The dual group $\widehat{L(G)} := \mathrm{Hom}_{\mathbb{Z}}(L(G),\mathbb{C}^\star)$ is the set of homomorphisms from the underlying abelian group of $L(G)$ to $\mathbb{C}^\star$. For $g \in G, x \in L(G)$ and $\psi \in \widehat{L(G)}$ the coadjoint action of G on $\widehat{L(G)}$ is given by

$$\mathrm{Ad}_g^\star \psi(x) = \psi(\mathrm{Ad}_g(x)) \ . \tag{2.3}$$

2.2 Kirillov orbit method

The Kirillov orbit method is an umbrella term for a collection of related results in the representation theory of several classes of groups. It was first developed in [23] by Kirillov to study the unitary representations of nilpotent Lie groups. The method has been adapted to other classes of groups. In [19], Howe gives a treatment for $\mathcal{T}$-groups.

In this section we recall the Kirillov orbit method for the specialised case of $\mathcal{T}_2$-groups, as described in [36, Section 2.4].

Let $\psi \in \widehat{L(G)}$ and define a binary form on $L(G)$. Namely,

$$B_\psi : L(G) \times L(G) \to \mathbb{C}^\star$$
$$(x, y) \mapsto \psi([x, y]) \ .$$

A subalgebra P of $L(G)$ is called a *polarising subalgebra* for ψ if $B_\psi|_{P\times P} \equiv 1$ and P is maximal with respect to that property.

For a subalgebra A of $L(G)$, $\psi \in \widehat{L(G)}$ is called *rational* on A if $\psi|_A$ is a torsion element, that is there exists $n \in \mathbb{N}$ such that $\psi(A)^n \equiv 1$. Let $\psi \in \widehat{L(G)}$ be rational on $L(G)'$. By [36, Lemma 2.14], finite-index polarising subalgebras for ψ exist.

Let $\psi \in \widehat{L(G)}$ be rational on $L(G)'$ and suppose that P is a polarising subalgebra for ψ. For a $\mathcal{T}_2$-group G the maps λ and λ^{-1} establish an index preserving bijection between the finite index subgroups of G and the finite index subalgebras of $L(G)$. Furthermore, under the aforementioned correspondence, normal subgroups are paired with ideals. Let $\Pi = \lambda^{-1}P$, then Π is a finite index normal subgroup of G.

In general, the map $\psi \circ \lambda : G \to \mathbb{C}^\star$ is not a homomorphism. However, the restriction to Π is a homomorphism. This follows from the fact that for $g_1, g_2 \in \Pi$, $\psi([g_1, g_2]) = 1$. That is, $\psi \circ \lambda : G \to \mathbb{C}^\star$ restricts to a 1-dimenstional representation of Π. Write $\pi(\psi)$ for the induction of this 1-dimensional representation from Π to G. Thus $\pi(\psi)$ is a $|G : \Pi|$-dimensional representation of G.

In the context of representation growth the dimension of the induced representation, equivalently the index of the normal subgroup Π, is our focus. Let $\psi \in \widehat{L(G)}$ be rational, we know that a polarising subalgebra P for ψ exists, but in general, polarising subalgebras are not unique. However, the index of the polarising subalgebras for ψ is an invariant. The radical Rad_ψ of ψ is $\{x \in L \mid \psi([x, L]) = 1\}$.

Lemma 2.2. [36, cf. Lemma 2.13] *Let $\psi \in \widehat{L(G)}$ be rational on $L(G)'$ and* *let P be a polarising subalgebra for ψ. Then $|L(G) : \mathrm{Rad}_\psi| = |L(G) : P|^2$.*

For $\psi \in \widehat{L(G)}$, let $\Omega(\psi)$ denote the orbit of ψ under the coadjoint action of G on $\widehat{L(G)}$.

Theorem 2.3. [36, Propostion 2.16] *Let G be a $\mathcal{T}_2$-group. For every rational $\psi \in \widehat{L(G)}$ the representation $\pi(\psi)$ is irreducible and of dimension $|\Omega(\psi)|^{1/2}$. Furthermore, every irreducible representation of G arises in this manner. For $\psi, \phi \in \widehat{L(G)}$ the representations $\pi(\psi)$, $\pi(\phi)$ are isomorphic if and only $\Omega(\psi) = \Omega(\phi)$. They are twist-equivalent if and only if $\psi|_{L(G)'} = \phi|_{L(G)'}$.*

Let Ω be a finite orbit of the coadjoint action of G on $\widehat{L(G)}$ and let $\psi \in \Omega$, write $\mathrm{Stab}_G(\psi)$ for the stabiliser of ψ. By the Orbit-Stabiliser Theorem, $|\Omega| = |G : \mathrm{Stab}_G(\psi)|$ and therefore $\dim \pi(\psi) = |G : \mathrm{Stab}_G(\psi)|^{1/2}$. Under the index-preserving bijection between finite-index subgroups of G and the finite-index subalgebras of $L(G)$ the stabiliser $\mathrm{Stab}_G(\psi)$ is associated with the radical Rad_ψ. Therefore, $\dim \pi(\psi) = |L(G) : \mathrm{Rad}_\psi|^{1/2}$.

See [12] for a different method for computing the representations of a $\mathcal{T}$-group.

2.3 Elementary divisors

In this section we define the elementary divisor type of a matrix with entries in $\mathbb{Z}/p^N$. These definitions can be extended to much more general situations. We restrict our definitions to the context we need. We then prove some simple facts about elementary divisor types that will be required throughout.

Let $A \in \mathrm{Mat}_d(\mathbb{Z}/p^N)$, the set of $d \times d$ matrices over $\mathbb{Z}/p^N$. By the Elementary Divisor Theorem [27, Theorem 7.8], there exist $\alpha, \beta \in GL_d(\mathbb{Z}/p^N)$

and $m_1, \ldots, m_d \in [0, N]$ such that

$$\alpha A \beta = \begin{pmatrix} p^{m_1} & & & 0 \\ & p^{m_2} & & \\ & & \ddots & \\ 0 & & & p^{m_d} \end{pmatrix}, \tag{2.4}$$

where $m_1 \leq m_2 \leq \cdots \leq m_d$. Note that α, β are not necessarily unique. We say that A has elementary divisor type $(m_1, \ldots, m_d)$ and write $\nu(A) = (m_1, \ldots, m_d)$.

The elementary divisor type of a matrix A is simpler to determine if A has a large unit minor.

Lemma 2.4. *Let A be matrix with unit i-minor. Then $m_1 = \cdots = m_i = 0$.*

Proof. First note that a matrix with unit determinant must have elementary divisor type (0,. . . ,0). Now if A has a unit i-minor we use row and column operations to move the corresponding $i \times i$ submatrix of A to the upper left corner. We use further operations to transform the upper left corner into the $i \times i$ identity matrix. After the described transformations the matrix A now has the following form:

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & (i) \\ & & 1 & \\ \hline & (ii) & & (iii) \\ & & & \end{array}\right). \tag{2.5}$$

The areas (i) and (ii) can be cleared of non-zero entries easily. We can now preform row and column operations on area (iii) that will not affect the upper left section or areas (i) and (ii). Hence, the elementary divisor type of A is $(0, \ldots, 0, m_{i+1}, \ldots, m_d)$, for some $m_{i+1}, \ldots, m_d \in \mathbb{N}_0$. □

More generally, the elementary divisors are determined by minors of A. Let $\boldsymbol{\sigma}_i$ denote the set of i-minors of A.

Lemma 2.5. *Suppose a d-dimensional matrix $A \in \mathrm{Mat}_d(\mathbb{Z}/p^N)$ has elementary divisor type $(m_1, \dots, m_d)$, where $0 \le m_1 \le \cdots \le m_d \le N$, then $m_1 + \cdots + m_i = \min\{v_p(\sigma) \mid \sigma \in \boldsymbol{\sigma}_i\}$.*

This follows from the fact that row and column operations do not affect the p-adic valuations of minors. Lemma 2.5 gives us another approach to calculating the elementary divisor type of an arbitrary matrix. However, it is very computationally intensive.

The elementary divisor types of antisymmetric matrices is vital to the computation of local representation zeta functions. We now present a few key facts concerning the elementary divisor types of anti-symmetric matrices.

Lemma 2.6. *Let A be a $d \times d$ anti-symmetric matrix over $\mathbb{Z}/p^N$, for some $N \in \mathbb{N}$. If d is even then there exists $\alpha \in \mathrm{GL}_d(\mathbb{Z}/p^N)$ such that*

$$\alpha A \alpha^T = \begin{pmatrix} 0 & p^{m_1} & & & & \\ -p^{m_1} & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & p^{m_{d/2}} \\ & & & & -p^{m_{d/2}} & 0 \end{pmatrix},$$

where $m_1 \le \cdots \le m_{\lfloor n/2 \rfloor} \le N$. If d is odd, then there exists $\alpha \in \mathrm{GL}_d(\mathbb{Z}/p^N)$ such that

$$\alpha A \alpha^T = \begin{pmatrix} 0 & p^{m_1} & & & & & \\ -p^{m_1} & 0 & & & & & \\ & & \ddots & & & & \\ & & & & 0 & p^{m_{\lfloor d/2 \rfloor}} & \\ & & & & -p^{m_{\lfloor d/2 \rfloor}} & 0 & \\ & & & & & & 0 \end{pmatrix},$$

In particular, if d is even then $\nu(A) = (m_1, m_1, \ldots, m_{d/2}, m_{d/2})$ and if d is odd then $\nu(A) = (m_1, m_1, \ldots, m_{\lfloor d/2 \rfloor}, m_{\lfloor d/2 \rfloor}, N)$.

Proof. We can construct such an α implicitly by describing simultaneous row and column operations that have the desired affect. Let a be an entry of A with minimum p-adic valuation amongst the entries of A. Use simultaneous row and column operations to bring a to the (1,2) position. By anti-symmetry the (2,1) position is now occupied by $-a$. If $v_p(a) = m_1$, then $a = up^{m_1}$ for some unit $u \in \mathbb{Z}/p^N$. Multiply rows and columns 1 and 2 by u^{-1}. After the operations described above the matrix A has been transformed into a matrix of the following form.

$$\left(\begin{array}{cc|c} 0 & p^{m_1} & (i) \\ -p^{m_1} & 0 & \\ \hline & & \\ (ii) & & (iii) \\ & & \end{array}\right). \tag{2.6}$$

Now, any non-zero entries in areas (i) and (ii) have p-adic valuation greater than or equal to m_1. We can therefore use simulataneous row and column operations to turn these entries to 0. Now, area (iii) is an antisymmetric matrix. The result then follows by induction. □

We already know that the elementary divisor type of A is determined by the minors of A. In the case where A is anti-symmetric we can say more. Recall that a submatrix of A is determined by the intersection of a subset I of its rows with a subset J of its rows. A minor is called *principal* if it is the determinant of a submatrix determined by subsets I and J with $I = J$. Let $\boldsymbol{\sigma}_i^{\text{prin}}$ denote the set of principal i-minors of A. Let $\nu(A) = (m_1, m_1, m_2, m_2, \ldots)$.

Lemma 2.7. [18, Section 3] *Let A be a non-zero $n \times n$ anti-symetric matrix over $\mathbb{Z}/p^N$. Then $2m_1 + 2m_2 + \ldots 2m_i = \min\{v_p(\sigma) \mid \sigma \in \boldsymbol{\sigma}_{2i}\} = \min\{v_p(\sigma) \mid$*

$\sigma \in \boldsymbol{\sigma}_{2i}^{\mathrm{prin}}\}$.

In words, the elementary divisor type of an anti-symmetric matrix A is determined by its principal minors. In practice, Lemma 2.7 can ease computations significantly.

2.4 Formulae for local zeta functions

In [38, Section 3.4] Voll uses Howe's description [19, Section II] of the finite-dimensional representations of a $\mathcal{T}$-group G to, for almost all primes p, give several formulae for the p-local representation zeta function $\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s)$. Later in [36, Section 2.4.2] Stasinski and Voll gave a description in the special case of $\mathcal{T}_2$-groups.

This section presents these results. In order to compute the global representation zeta of a group G it is sufficient to calculate all local factors. The first formula is derived from the results of the Kirillov orbit method. It first appeared in [38, Corollary 3.1], but is proved for all primes as [36, Corollary 2.17].

Theorem 2.8. *Let G be a $\mathcal{T}_2$-group and p be any prime. Then*

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \sum_{\substack{\psi \in \widehat{L(G)'} \\ \text{rational of} \\ p\text{-power period}}} |L(G) : \mathrm{Rad}_\psi|^{-s/2} . \tag{2.7}$$

We say that ψ has p-power period if the image of $L(G)'$ under ψ is contained in the p-power roots of unity in $\mathbb{C}^\star$.

A subalgebra A of an algebra B is saturated in B if $nx \in A$ implies that $x \in A$. If the derived ring $L(G)'$ of $L(G)$ is saturated in the center $Z(L(G))$, then the results of this section are valid for all primes. In the case that $L(G)'$ is not saturated in $Z(L(G))$ there is a finite index subgroup Λ of G such that $L(\Lambda)^{prime}$ is saturated in the center $Z(L(\Lambda))$.

Let $L(G)$ have presentation

$$\left\langle \begin{matrix} x_1, \dots x_n, \\ y_1, \dots y_d \end{matrix} \;\middle|\; [x_i, x_j] = \sum_{k=1}^{d} \lambda_{ij}^k y_k,\ 1 \leq i, j \leq n \right\rangle ,$$

where for $1 \leq k \leq d$, y_k is central. Write $\mathbf{y} = (y_1, \dots, y_d)$, the structure matrix $\mathcal{R}_{L(G)}(\mathbf{y})$ of the Lie ring $L(G)$ is the $n \times n$ matrix of linear forms where $(\mathcal{R}_{L(G)}(\mathbf{y}))_{ij} = \sum_{k=1}^{d} \lambda_{ij}^k y_k$. The structure matrix $\mathcal{R}_{L(G)}(\mathbf{y})$ depends on the set of generators chosen for $L(G)$. However, the properties of the structure matrix that feature in the sequel are invariants of the Lie ring $L(G)$. To ease notation, $\mathcal{R}_{L(G)}(\mathbf{y})$ is refered to as *the* structure matrix of $L(G)$.

Let $\psi \in \widehat{L(G)'}$ be of p-power period. Then ψ is determined by $\{\psi(y_k) \mid 1 \leq k \leq d\}$ and for $1 \leq k \leq d$, $\psi(y_k)$ is a p-power root of unity. Let $N \in \mathbb{N}$ be the maximum natural number such that $\{\psi(y_k) \mid 1 \leq k \leq d\}$ contains a primitive p^Nth root of unity, but does not contain a p^{N+1}th root. Identify ψ with $(\psi(y_1), \dots, \psi(y_k)) \in (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d$. Via this identification, there is a bijection between the non-trivial p-power elements of $\widehat{L(G)'}$ and $\bigcup_{N=1}^{\infty} (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d$.

The structure matrix, $\mathcal{R}_{L(G)}(\mathbf{y})$, can be considered as a matrix of linear forms. Let $\mathbf{Y} = (Y_1, \dots, Y_d)$ be a d-tuple of variables. Then $\mathcal{R}_{L(G)}(\mathbf{Y})$ is $d \times d$ matrix over $\mathbb{Z}[\mathbf{Y}]$. Suppose that $\psi \in \widehat{L(G)'}$ is identified with $\mathbf{a} \in (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d$. Then the dimension of the representation $\pi(\psi)$ of G associated with ψ is determined by the elementary divisor type of $\mathcal{R}_{L(G)}(\mathbf{a})$. Now, $\mathcal{R}_{L(G)}(\mathbf{a})$ is a matrix with entries in the finite ring $\mathbb{Z}/p^N\mathbb{Z}$. There exist $\alpha, \beta \in \mathrm{GL}_n(\mathbb{Z}/p^N\mathbb{Z})$ such that

$$\alpha \mathcal{R}_{L(G)}(\mathbf{a}) \beta = \begin{pmatrix} p^{m_1} & & \\ & \ddots & \\ & & p^{m_n} \end{pmatrix},$$

where $m_1 \leq \cdots \leq m_n \leq N$ and noting that $p^N = 0$ in $\mathbb{Z}/p^N\mathbb{Z}$. That is, $\mathcal{R}_{L(G)}(\mathbf{a})$ has elementary divisor type $\mathbf{m} = (m_1, \dots, m_n)$. If $\pi(\psi)$ is the representation of G associated with $\mathbf{a} \in (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d$, then $\dim \pi(\psi) = p^{\frac{1}{2}\sum_{i=1}^{n}(N-m_i)}$, see the proof of [38, Proposition 3.1] for the details. The second formula expresses the p-local representation zeta function in terms of elementary divisors. Let $Z = Z(G)$ denote the centre of G.

Theorem 2.9. [36, cf. Proposition 2.18] *Let G be a $\mathcal{T}_2$-group such that $h(G/Z) = n$ and $h(G') = d$. Then for almost all primes p,*

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{N=1}^{\infty} \sum_{\mathbf{m}\in\mathbb{N}_0^n} \mathcal{N}_{N,\mathbf{m}} p^{-\frac{s}{2}\sum_{i=1}^{n}(N-m_i)}, \tag{2.8}$$

where $\mathcal{N}_{N,\mathbf{m}} = \#\{\mathbf{a} \in (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d \mid \nu(\mathcal{R}_{L(G)}(\mathbf{a})) = \mathbf{m}\}$.

A p-local factor is called *exceptional* if equation (2.8) is not valid for p. The formula (2.8) is valid for all primes if $L(G)'$ is saturated in $Z(L(G))$ and each entry of the structure matrix is either equal to $\pm y_k$ for some $k \in [1, d]$ or 0. These conditions are sufficient, but they are not necessary.

The final formulation presented in this section expresses the local factor $\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s)$ in terms of a p-adic integral. For $1 \leq j \leq n$, let $\boldsymbol{\sigma}_j$ denote the set of j-minors of $\mathcal{R}_{L(G)}(\mathbf{y})$. Note that $\boldsymbol{\sigma}_0 = 1$ and let $f := \max\{j \in [0, n] \mid (\boldsymbol{\sigma}_j) \neq (0)\}$.

Theorem 2.10. [38, Section 2.2] *Let G be a $\mathcal{T}_2$-group such that $h(G/Z) = n$ and $h(Z) = d$. Then for almost all primes p,*

$$\begin{aligned}&\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s)\\ &= 1 + \frac{1}{(1-p^{-1})}\prod_{i=1}^{d}\frac{1}{1-p^{-i}}\int_{\substack{p\mathbb{Z}_p\times\mathbb{Z}_p^d\setminus p\mathbb{Z}_p^d\\(x,\mathbf{y})}} |x|^{fs-d-1}\prod_{j=1}^{f}\frac{||\boldsymbol{\sigma}_j \cup x\boldsymbol{\sigma}_{j-1}||^s}{||\boldsymbol{\sigma}_{j-1}||^s}d\mu,\end{aligned} \tag{2.9}$$

where $|\cdot|$ denotes the p-adic absolute value and for a set A, $||A|| := max_{a\in A}\{|a|\}$.

A detailed exposition on p-adic integrals such as the one in equation (2.9) in given in Chapter 3.

Chapter 3

Igusa's local zeta function

Igusa's local zeta function is a key tool in the study of zeta functions associated to $\mathcal{T}$-groups, both in the proof of general results and in specific computations. We begin this chapter by giving a brief outline of the history of Igusa's local zeta function. In Section 3.2 we highlight some general results of the theory and Section 3.3 consists of a 'toolbox' of methods for calculating specific examples.

3.1 History

For p be prime, let X be a smooth projective variety defined over $\mathbb{F}_p$. We define $N_{p^m}(X)$ to be the number of $\mathbb{F}_{p^m}$-rational points of X. The variety X is defined by homogeneous equations and the numbers $N_{p^m}(X)$ are simply the number of solutions of the equations over $\mathbb{F}_{p^m}$. The sequence $(N_{p^m}(X))_{m\in\mathbb{N}}$ encodes a lot of arithmetical information about the variety X. To study the sequence we introduce the p-local zeta function of X at the prime p:

$$Z_X(p,t) = \exp\left(\sum_{m=1}^{\infty} N_{p^m}(X)\frac{t^m}{m}\right) \tag{3.1}$$

In [40] Weil made a number of highly influential conjectures. Although they are now theorems they are still known as the Weil conjectures.

Theorem 3.1. [17, Appendix C] *Let X be a smooth projective variety defined over $\mathbb{F}_p$ of dimension n.*

1. *(Rationality.) $Z_X(p,t)$ is a rational function of t.*

2. *(Functional equation.) There exists $E \in \mathbb{Z}$ such that*

$$Z_X(p, p^{-n}t^{-1}) = \pm p^{nE/2} t^E Z_X(p,t).$$

3. *(Analogue of the Riemann Hypothesis.) It is possible to write*

$$Z_X(p,t) = \frac{P_1(t)P_2(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)},$$

where $P_0(t) = 1-t$, $P_{2n}(t) = 1-p^n t$ and for each $1 \le t \le 2n-1$, $P_i(t)$ is a polynomial with integer coeffients. Further, $P_i(t)$ can be written as a finite product

$$P_i(t) = \prod_j (1 - \alpha_{ij} t),$$

where the α_{ij} are algebraic integers such that $|\alpha_{ij}| = p^{i/2}$.

Weil established these conjectures in the case of curves in 1948; cf. [39]. The rationality and functional equation for higher-dimensional varieties was first established by Dwork in [11]. The analog of the Riemann Hypothesis was finally established in 1974 by Deligne in [6]. The proofs are deep and require significant machinery. For a more detailed discussion of the history of the Weil conjectures see [17, Appendix C].

The Weil conjectures have many deep consequences. In particular, rationality implies that the sequence $(N_{p^m}(X))$ is determined by a finite number of the $N_{p^m}(X)$.

Now let X be a smooth projective variety defined over $\mathbb{Q}$. For each prime p we can consider the p-local zeta function associated with $\overline{X}$, the reduction modulo p of X.

The L-function of a smooth projective variety X defined over $\mathbb{Q}$ is defined, loosely speaking, by multiplying all of the p-local zeta functions of

X together. One hopes that the L-function contains information about the global structure of the variety.

For example, in the case where X is an elliptic curve the L-function converges on $\mathfrak{Re}(s) > 3/2$ and has meromorphic contiuation to the entire complex plane. It is well known that an elliptic curve over $\mathbb{Q}$ carries the structure of an abelian group. The Birch Swinnerton-Dyer Conjecture states that the rank of this abelian group is equal to the multiplicity of the zero at $s = 1$. For details on L-functions associated with elliptic curves see [35, Appendix B, Section 16].

The success of studying varieties defined over $\mathbb{Q}$ by considering the numbers of solutions in the tower of finite fields $\mathbb{F}_{p^m}$ lead number theorists to consider another 'tower'. The field of p elements can be viewed as the first level in a different system, namely the rings $\mathbb{Z}/p^m\mathbb{Z}$. Let $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ and let $N_m(F)$ denote the number of roots of F over $(\mathbb{Z}/p^m\mathbb{Z})^n$. Motivated by the success of the Weil conjectures, we study these numbers simultaneously by encoding them in a single power series $\mathrm{P}_F(t)$, which is called the Poincaré series associated with the polynomial F. We set

$$\mathrm{P}_F(t) = \sum_{m=0}^{\infty} N_m(F)(p^{-n}t)^m. \tag{3.2}$$

Analogously to towers of finite fields, it was conjectured (see [4, Section 5, Problem 9]) that the number of solutions modulo p^m should be determined by a finite number of levels or, more precisely, that $\mathrm{P}_F(t)$ is a rational function in t. Igusa showed that this problem can be phrased in terms of certain p-adic integrals, which we now introduce.

3.2 Igusa's local zeta function

The p-adic integers $\mathbb{Z}_p$ are compact. By the Haar Theorem [28, Section 29] there exists a unique Haar measure μ on the additive group of $\mathbb{Z}_p$ such that $\mu(\mathbb{Z}_p) = 1$. Then the Haar measure on $\mathbb{Z}_p^n$ is given as the product measure and is also denoted by μ. We use this measure to define integration over $\mathbb{Z}_p^n$. Denote, for $z \in \mathbb{Q}_p$, by $v_p(z)$ the p-adic valuation and by $|z| = p^{-v_p(z)}$ the p-adic absolute value.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $F(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$. For a complex variable s, we define Igusa's local zeta function associated to the polynomial F by setting

$$\mathrm{Z}_F(s) := \int_{\mathbb{Z}_p^n} |F(\mathbf{x})|^s \mathrm{d}\mu. \tag{3.3}$$

For a more comprehensive introduction to Igusa's local zeta function see [7]. We now explain the connection between Igusa's local zeta function $\mathrm{Z}_F(s)$ and the Poincare series $\mathrm{P}_F(t)$. The Haar measure μ on the p-adic integers $\mathbb{Z}_p$ satisfies the following property:

$(*)$ $\mu(S) = \mu(a + S)$ for any measurable subset $S \subset \mathbb{Z}_p$ and $a \in \mathbb{Z}_p$.

We normalise the Haar measure so that $\mu(\mathbb{Z}_p) = 1$. The property $(*)$ implies that $\mu(p^m\mathbb{Z}_p) = \mu(a + p^m\mathbb{Z}_p) = p^{-m}$. The additive cosets $a + p^m\mathbb{Z}_p$ form a basis for the topology of $\mathbb{Z}_p$, thus we can calculate the Haar measure of any open subset. The Haar measure on $\mathbb{Z}_p^n$ is given as the product measure, therefore, $\mu(p^m\mathbb{Z}_p^n) = \mu(p^m\mathbb{Z}_p)^n = p^{-nm}$. We rewrite the integral $\mathrm{Z}_F(s)$ over subsets where the integrand is constant. We clearly have

$$\mathrm{Z}_F(s) = \sum_{m=0}^{\infty} \int_{V(m)} |F(\mathbf{x})|^s d\mu,$$

where

$$V(m) := \{\mathbf{x} \in \mathbb{Z}_p^n \mid v_p(F(\mathbf{x})) = m\}.$$

However, since the integrand is constant on $V(m)$ we see that

$$\mathrm{Z}_F(s) = \sum_{m=0}^{\infty} \mu(V(m)) p^{-ms}.$$

Now we make the connection between the measure of the subsets $V(m)$ and the numbers $N_m(F)$. If $W(m) := \{\mathbf{x} \in \mathbb{Z}_p^n \mid v_p(F(\mathbf{x})) \geq m\} = \{\mathbf{x} \in \mathbb{Z}_p^n \mid$

$F(\mathbf{x}) \equiv 0 \bmod p^m\}$, then $V(m) = W(m) \smallsetminus W(m+1)$. $W(m)$ is the union of $N_m(F)$ cosets of $p^m\mathbb{Z}_p^n$ each of measure p^{-mn} and so,

$$\mu(V(m)) = N_m(F)p^{-nm} - N_{m+1}(F)p^{-n(m+1)}.$$

Therefore, by (3.2),

$$\begin{aligned} \mathrm{Z}_F(s) &= \sum_{m=0}^{\infty}(N_m(F)p^{-nm} - N_{m+1}(F)p^{-n(m+1)})p^{-ms} \\ &= \sum_{m=0}^{\infty} N_m(F)(p^{-n-s})^m - p^s \sum_{m=0}^{\infty} N_{m+1}(F)(p^{-n-s})^{m+1} \\ &= \mathrm{P}_F(p^{-s}) - p^s(\mathrm{P}_F(p^{-s}) - 1). \end{aligned} \tag{3.4}$$

Rearranging, we see that

$$\mathrm{P}_F(p^{-s}) = \frac{1 - p^{-s}\,\mathrm{Z}_F(s)}{1 - p^{-s}}.$$

In [21] Igusa showed that $\mathrm{Z}_F(s)$ is a rational function of p^{-s}, which in turn proved that the Poincare series $\mathrm{P}_F(t)$ associated to the polynomial $F(\mathbf{x})$ is a rational function in t.

3.3 Toolbox

The rest of this chapter is devoted to explaining various techniques for calculating Igusa's local zeta function for particular classes of polynomials that are required for our computations.

Example 3.2. Consider

$$\mathrm{Z}_x(s) = \int_{\mathbb{Z}_p} |x|^s d\mu.$$

Since $N_m(x) = 1$ for all m, we have

$$\mathrm{P}_x(p^{-s}) = \sum_{m=0}^{\infty}(p^{-1-s})^m = \frac{1}{1 - p^{-1-s}}.$$

Therefore, by (3.4),

$$\mathrm{Z}_x(s) = \int_{\mathbb{Z}_p} |x|^s d\mu = \frac{1-p^{-1}}{1-p^{-1-s}}.$$

Alternatively, we can calculate the integral directly. Indeed,

$$\mathrm{Z}_x(s) = \sum_{m=0}^{\infty} \mu(\{x \in \mathbb{Z}_p \mid v_p(x) = m\})p^{-ms}.$$

Let $x \in \mathbb{Z}_p$ then $x = \sum_{i=0}^{\infty} x_i p^i$, where $0 \leq x_i \leq p-1$. One has $v_p(x) = m$ if and only if $x_1 = \cdots = x_m = 0$ and $x_{m+1} \neq 0$, thus $\mu(\{x \in \mathbb{Z}_p \mid v_p(x) = m\}) = (1-p^{-1})p^{-m}$, so

$$\mathrm{Z}_x(s) = \sum_{m=0}^{\infty} (1-p^{-1})p^{-m}p^{-ms} = \frac{1-p^{-1}}{1-p^{-1-s}}.$$

3.3.1 Fubini's theorem

In the general theory of integration Fubini's theorem [3, Theorem 10.10] gives a criterion for when the order of integration can be reversed in an iterated integral. We require the use of a standard corollary to the theorem, which we present specialised to the case of Igusa's local zeta function.

Proposition 3.3. *Let $F(x_1, \ldots, x_d)$ be a polynomial with coefficients in $\mathbb{Z}_p$. If $F(x_1, \ldots, x_d) = \prod_{i=1}^{d} f_i(x_i)$ for polynomials $f_i(x) \in \mathbb{Z}_p[x], i \in [1, d]$ then*

$$\int_{\mathbb{Z}_p^d} |F(x_1, \ldots, x_d)|^s d\mu = \prod_{i=1}^{d} \int_{\mathbb{Z}_p} |f_i(x_i)|^s d\mu.$$

Example 3.4. Let $F(\mathbf{x}) = \prod_{i=1}^{d} x_i^{e_i}$ where each $e_i \in \mathbb{N}$. By Proposition 3.3 and Example 3.2,

$$\mathrm{Z}_F(s) = \int_{\mathbb{Z}_p^d} |x_1^{e_1} x_2^{e_2} \ldots x_d^{e_d}|^s d\mu = \prod_{i=1}^{d} \int_{\mathbb{Z}_p} |x_i^{e_i}|^s d\mu = \prod_{i=1}^{d} \frac{1-p^{-1}}{1-p^{-1-e_i s}}.$$

This example shows that we can easily calculate Igusa's local zeta function associated with any monomial. A technique in calculating more complicated examples is to reduce the problem to calculating Igusa's local zeta function for monomials. A lot of the techniques that follow do just that.

3.3.2 Change of variables

When evaluating an integral it is often convenient to change the coordinate system of the domain of integration by a differentiable map. In doing so, the Jacobian keeps track of the change in measure, cf. [13, Section 235]

Let $F(\mathbf{x}) \in \mathbb{Z}_p[x_1, \ldots, x_d]$ and $\Phi : U \to V$ be a bijection between subsets U, V of $\mathbb{Z}_p^d$ given by d polynomials $y_1(x_1, \ldots, x_d), \ldots, y_d(x_1, \ldots, x_d)$ defined over $\mathbb{Z}_p$. The Jacobian of this transformation is

$$J_\Phi = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_d}{\partial x_1} & \cdots & \frac{\partial y_d}{\partial x_d} \end{pmatrix}$$

and

$$\mathrm{Z}_F(s) = \int_U |F(\mathbf{x})|^s d\mu = \int_V |(F \circ \Phi)(\mathbf{x})|^s | \det(J_\Phi)| d\mu,$$

Example 3.5. Let $F(x, y) = x^2 + xy$. Consider the change of variables given by the map $(x, y) \mapsto (x, y - x)$. The determinant of the Jacobian of this transformation is equal to 1. Therefore, using Example 3.4, we have

$$\begin{aligned} \mathrm{Z}_F(s) &= \int_{\mathbb{Z}_p^2} |x^2 + xy|^s d\mu = \int_{\mathbb{Z}_p^2} |x^2 + x(y - x)|^s |1| d\mu \\ &= \int_{\mathbb{Z}_p^2} |xy|^s d\mu = \int_{\mathbb{Z}_p} |x|^s d\mu \int_{\mathbb{Z}_p} |y|^s d\mu = \left(\frac{1 - p^{-1}}{1 - p^{-1-s}} \right)^2. \end{aligned}$$

We can also use a change of coordinates to calculate integrals over certain domains of integration different from $\mathbb{Z}_p^d$.

Example 3.6. Consider the integral $\int_{p\mathbb{Z}_p} |x|^s d\mu$. The map $\phi : \mathbb{Z}_p \to p\mathbb{Z}_p$ given by $x \mapsto px$ has Jacobian (p). Therefore

$$\int_{p\mathbb{Z}_p} |x|^s d\mu = \int_{\mathbb{Z}_p} |px|^s |p| d\mu = p^{-1-s} \int_{\mathbb{Z}_p} |x|^s d\mu = \frac{(1 - p^{-1}) p^{-1-s}}{1 - p^{-1-s}},$$

by Example 3.2.

Let $F(x_1, \ldots, x_d)$ be a homogeneous polynomial of degree n. Then the transformation $\phi : \mathbb{Z}_p^d \to p\mathbb{Z}_p^d$ given by $(x_1, \ldots, x_d) \mapsto (px_1, \ldots, px_d)$ yields

$$\int_{p\mathbb{Z}_p^d} |F(\mathbf{x})|^s d\mu = \int_{\mathbb{Z}_p} |p^n F(\mathbf{x})|^s |p^d| d\mu = p^{-d-ns} \int_{\mathbb{Z}_p^d} |F(\mathbf{x})|^s d\mu \qquad (3.5)$$

3.3.3 Coset decompostion

One general approach to calculating an integral is to divide the domain of integration into pieces on which the integrand takes a simpler form. In the case of p-adic integration, the arithmetic suggests a very useful decomposition $\mathbb{Z}_p$ into cosets modulo p^m.

Example 3.7. Let $F(x) = (x-1)(x-2)(x-3) \in \mathbb{Z}[x]$. In this case a transformation of coordinates will not allow us to obtain a monomial. For simplicity, assume that $p > 3$. We decompose the integral into cosets modulo p.

$$\int_{\mathbb{Z}_p} |(x-1)(x-2)(x-3)|^s d\mu = \sum_{a \in \mathbb{F}_p} \int_{\left\{ \substack{x \in \mathbb{Z}_p \mid \\ x \equiv a \bmod p} \right\}} |(x-1)(x-2)(x-3)|^s d\mu$$

Here the sum is not over $\mathbb{F}_p$, but rather the Teichmüller representative of the cosets of $p\mathbb{Z}_p$. The Teichmüller representatives are the p solutions of $x^p - x = 0$ in $\mathbb{Z}_p$. We denote them by $\mathbb{F}_p$ because there is a map from $\mathbb{F}_p$ to the Teichmüller representatives which preserves the multiplicative structure. It does not, of course, preserve the additive structure. We do not make use of these facts. We simply use $\mathbb{F}_p$ as notation for the Teichmüller representatives.

Each summand of the right hand side is now easy to calculate. If $x \equiv 1 \bmod p$ then $v_p(x-2) = v_p(x-3) = 0$ so that

$$\int_{\left\{ \substack{x \in \mathbb{Z}_p \mid \\ x \equiv 1 \bmod p} \right\}} |(x-1)(x-2)(x-3)|^s d\mu = \int_{\left\{ \substack{x \in \mathbb{Z}_p \mid \\ x \equiv 1 \bmod p} \right\}} |(x-1)|^s d\mu,$$

and by the change of variables $x \mapsto x + 1$, which has Jacobian of unit determinant,

$$\int_{\left\{\substack{x\in\mathbb{Z}_p \mid \\ x\equiv 1 \bmod p}\right\}} |(x-1)|^s d\mu = \int_{p\mathbb{Z}_p} |x|^s d\mu = \frac{(1-p^{-1})p^{-1-s}}{1-p^{-1-s}}$$

by Example 3.6. A similar arguement holds for the cases where $x \equiv 2, 3 \mod p$. If $x \not\equiv 1, 2, 3 \mod p$ then $v_p((x-1)(x-2)(x-3)) = 0$ and so the integral is equal to the measure of the domain of integration, namely $\mu(a + p\mathbb{Z}_p) = \mu(p\mathbb{Z}_p) = p^{-1}$. Therefore

$$\begin{aligned}\int_{\mathbb{Z}_p} |(x-1)(x-2)(x-3)|^s d\mu &= 3\frac{(1-p^{-1})p^{-1-s}}{1-p^{-1-s}} + (p-3)p^{-1} \\ &= \frac{1-3p^{-1}+2p^{-1-s}}{1-p^{-1-s}}.\end{aligned}$$

In the example above we decompose $\mathbb{Z}_p$ into its cosets modulo p. This works because, in some sense, the arithmetic of the polynomial we are integrating is determined modulo p. The integral in Example 3.7 can be calculated in the cases $p = 2$ or $p = 3$ by decomposing the integral into cosets modulo p^2. As discussed earlier the number of solutions of any polynomial in the finite rings $\mathbb{Z}/p^m\mathbb{Z}$ is determined by only finitely many levels. In the example above it is determined in the first (non-trivial) level. This property is fairly general.

3.3.4 Hensel's lemma

We begin by stating the simplest form of Hensel's lemma.

Theorem 3.8. [34, Section 2.2] *Let $F(x) \in \mathbb{Z}[x]$ and p a prime. Suppose that there exists $\alpha_0 \in \mathbb{Z}_p$ such that*

$$F(\alpha_0) \equiv 0 \bmod p,$$

but

$$F'(\alpha_0) \not\equiv 0 \bmod p.$$

Then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $\alpha \equiv \alpha_0 \bmod p$ and $F(\alpha) = 0$.

The strength of Hensel's lemma is that it essentially allows us to work over the finite field $\mathbb{F}_p$.

Example 3.9. Let $F(x) = x^2 - k$ where k is a square-free integer. We exclude the cases $p = 2$ and $p|k$. We use Hensel's lemma to calculate Igusa's local zeta function associated with F in the remaining cases. The remaining primes are divided into two classes, depending on whether or not k is a quadratic residue modulo p. Suppose that k is not a quadratic residue modulo p. This implies that for all $\alpha \in \mathbb{Z}_p$ it is the case that $\alpha^2 - k \not\equiv 0 \mod p$ and thus $v_p(\alpha^2 - k) = 0$ for all $\alpha \in \mathbb{Z}_p$. Thus

$$\int_{\mathbb{Z}_p} |x^2 - k|^s d\mu = 1.$$

On the other hand, if k is a quadratic residue modulo p then, by definition, there exists $\alpha_0 \in [1, p-1]$ such that $\alpha_0^2 - k \equiv 0 \mod p$. We have that $F'(x) = 2x$ so that $F'(\alpha_0) \not\equiv 0 \mod p$, since we are assuming that $p \neq 2$. By Hensel's lemma α_0 lifts to a unique $\alpha \in \mathbb{Z}_p$ such that $F(\alpha) = 0$. Therefore, in $\mathbb{Z}_p$, $x^2 - k = (x + \alpha)(x - \alpha)$ and

$$\int_{\mathbb{Z}_p} |x^2 - k|^s d\mu = \int_{\mathbb{Z}_p} |(x + \alpha)(x - \alpha)|^s d\mu.$$

Now, since $p \neq 2$ we know that α is not congruent to $-\alpha$ modulo p. We can then decompose the domain of integration into the cosets modulo p and similar to Example 3.7 we calculate

$$\int_{\mathbb{Z}_p} |x^2 - k|^s d\mu = \frac{1 - 2p^{-1} + p^{-1-s}}{1 - p^{-1-s}}.$$

We can systemise the process illustrated in the example into the following corollary.

Corollary 3.10. *Let $F(x) \in \mathbb{Z}[x]$ and p a prime. If $a \in \mathbb{Z}_p$ is such that $F'(a) \not\equiv 0 \mod p$, then*

$$\int_{a+p\mathbb{Z}_p} |F(x)|^s d\mu = \begin{cases} p^{-1} & \text{if } F(a) \not\equiv 0 \mod p, \\ \frac{(1-p^{-1})p^{-1-s}}{1-p^{-1-s}} & \text{if } F(a) \equiv 0 \mod p. \end{cases}$$

Hensel's lemma can be generalised in a number of directions. We now discuss one such generalisation and give an example to illustrate its usefulness. We explore the multivariate case.

Let $F(x_1, \ldots, x_d)$ be a polynomial in d variables with integer coefficients and p be a prime. Now suppose that there exists $a = (a_1, \ldots a_d) \in \mathbb{Z}_p^d$ such that $F(a) \equiv 0 \mod p$ and $\frac{\partial F}{\partial x_i}(a) \not\equiv 0 \mod p$ for some $i \in [1, d]$. Then the polynomial $F^\star(x_i) = F(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_d)$ is a polynomial in one variable, such that $F^\star(a_i) \equiv 0 \mod p$ and $\partial F^\star / \partial x_i(a_i) \not\equiv 0 \mod p$ so that, by the single variate Hensel's lemma, there exists $\alpha \in Z_p$ such that $\alpha \equiv a_i \mod p$ and $F^\star(\alpha) = 0$. In [8] Denef and Hoornaert use the above observation to prove the following proposition.

Proposition 3.11. [8, Proposition 3.1]
Let $F(x) = F(x_1, \ldots, x_d) \in \mathbb{Z}_p[x_1, \ldots, x_d]$ be a polynomial and let $a \in \mathbb{Z}_p^n$. Suppose that the set of congruences

$$\begin{cases} F(x) \equiv 0 \mod p, \\ \frac{\partial F}{\partial x_i}(x) \equiv 0 \mod p, \quad i \in [1, d] \end{cases}$$

has no simultaneous solution in the coset $a + (p\mathbb{Z}_p)^d$. Then for a complex variable s, we have, for $\mathfrak{Re}(s) > 0$,

$$\int_{a+(p\mathbb{Z}_p)^d} |F(x)|^s d\mu = \begin{cases} p^{-d} & \textit{if } F(a) \not\equiv 0 \mod p, \\ \frac{(1-p^{-1})p^{-d-s}}{1-p^{-1-s}} & \textit{if } F(a) \equiv 0 \mod p. \end{cases}$$

If a polynomial $F(x_1, \ldots, x_d)$ satisfies the hypothesis of the proposition for all cosets modulo p, then, using coset decomposition, the calculation of the associated integral is reduced to counting the number of solutions of F over the finite field $\mathbb{F}_p$.

Example 3.12. Let $F(x) = F(x_1, \ldots, x_d)$ be a polynomial which satisfies the hypothesis of Proposition 3.11 for all cosets modulo p and let $N_1(F)$ be the number of solutions of F over the finite field $\mathbb{F}_p$. Then

$$\begin{aligned}\int_{\mathbb{Z}_p^d} |F(x)|^s d\mu &= \sum_{a\in\mathbb{F}_p^d} \int_{a+(p\mathbb{Z}_p)^d} |F(x)|^s d\mu \\ &= (p^d - N_1(F))p^{-d} + N_1(F)\left(\frac{(1-p^{-1})p^{-d-s}}{1-p^{-1-s}}\right) \\ &= 1 + N_1(F)\frac{(p^{-s}-1)p^{-d}}{1-p^{-1-s}}.\end{aligned}$$

Here we are using $\mathbb{F}_p$ to denote the Teichmüller representatives.

3.3.5 Homogeneous ideals

In this section we define Igusa's local zeta function associated to an ideal. Then we explore the case of a homogenous ideal.

Let p be a prime, $\mathbf{x} = (x_1, \ldots, x_d)$ and let $\mathbf{f} = (f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})) \lhd \mathbb{Z}_p[\mathbf{x}]$ be an ideal. For $\mathbf{a} \in \mathbb{Z}_p^n$, define $||f_1(\mathbf{a}), \ldots, f_l(\mathbf{a})|| := \max\{|f_1(\mathbf{a})|, \ldots, |f_l(\mathbf{a})|\} = p^{-\min\{v_p(f_1(\mathbf{a})),\ldots,v_p(f_l(\mathbf{a}))\}}$. It follows from the definition of the p-adic absolute value that for all $f \in \mathbf{f}$, we have $|f(\mathbf{a})| \leq ||f_1(\mathbf{a}), \ldots, f_l(\mathbf{a})||$.

Conversely, let $\mathbf{f}$ denote an ideal of $\mathbb{Z}_p[\mathbf{x}]$. Since $\mathbb{Z}_p[\mathbf{x}]$ is Noetherian, there exist $f_1(\mathbf{x}), \ldots, f_l(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ such that $\mathbf{f} = (f_1(\mathbf{x}), \ldots, f_l(\mathbf{x}))$. We define Igusa's local zeta function $\mathrm{Z}_{\mathbf{f}}(s)$ associated to the ideal $\mathbf{f} = (f_1(\mathbf{x}), \ldots, f_l(\mathbf{x}))$ as

$$\mathrm{Z}_{\mathbf{f}}(s) := \int_{\mathbb{Z}_p^d} ||f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})||^s \mathrm{d}\mu. \tag{3.6}$$

Of course, the integral is independent of the choice of generating set. Also note that Igusa's local zeta function associated with a single polynomial $F \in \mathbb{Z}_p[\mathbf{x}]$ is equal to Igusa's local zeta function associated with the principal ideal $(F) \lhd \mathbb{Z}_p[\mathbf{x}]$ generated by F.

We have seen that Igusa's local zeta associated with a single polynomial $F(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$ is related to the Poincare series encoding the number of solutions of F modulo p^m by formula (3.4). We establish the analogous formula for Igusa's local zeta function associated with an ideal. For $\mathbf{a} \in \mathbb{Z}_p^n$, set $v_p(\mathbf{f}(\mathbf{a})) := \min\{v_p(f_1(\mathbf{a})), \ldots, v_p(f_l(\mathbf{a}))\}$. For $m \in \mathbb{N}$, we have $v_p(\mathbf{f}(\mathbf{a})) = m$ if and only if $f(\mathbf{a}) \equiv 0$ modulo p^m for all $f \in \mathbf{f}$. Set $N_m := \#\{\mathbf{a} \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid \forall f \in \mathbf{f}, f(\mathbf{a}) = 0\}$.

The Poincaré series associated with $\mathbf{f}$ is given by

$$\mathrm{P}_{\mathbf{f}}(t) := \sum_{m=0}^{\infty} N_m (p^{-d}t)^m. \tag{3.7}$$

Using very similar arguments to those carried out in Section 3.2 and by setting $t = p^{-s}$, we arrive at the formula

$$\mathrm{P}_{\mathbf{f}}(s) = \frac{1 - t\,\mathrm{Z}_{\mathbf{f}}(s)}{1-t}. \tag{3.8}$$

This generalises formula (3.4).

Definition 3.13. For $n \in \mathbb{N}_0$, an ideal $\mathbf{f} \lhd \mathbb{Z}[\mathbf{x}]$ is *homogeneous of degree* n if there exist $f_1(\mathbf{x}), \ldots, f_l(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ such that $\mathbf{f} = (f_1(\mathbf{x}), \ldots, f_l(\mathbf{x}))$ such that, for $1 \leq i \leq l$, each $f_i(\mathbf{x})$ is homogeneous of degree n.

We introduce Poincaré series and Igusa's local zeta function in the 'projective' case. We generalise [25, Lemma 2.1]. The proof is almost identical and is included because we need the intermediate identities in Chapters 5 and 7.

We introduce notation for Igusa's local zeta function with integration over $\mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d$.

$$\mathrm{Z}^{\star}_{\mathbf{f}}(s) := \int_{\mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d} ||f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})||^s \mathrm{d}\mu.$$

We have already seen, in Section 3.3.3 that Igusa's local zeta function can be decomposed into a sum of integrals over the cosets modulo p^k, $k \in \mathbb{N}$. We now show that if $\mathbf{f}$ is a homogeneous ideal then the integral over the coset $p\mathbb{Z}_p$ is determined by the integrals over the remaining cosets. Let $\mathbf{f} = (f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})) \in \mathbb{Z}[\mathbf{x}]$ be a homogeneous ideal of degree n. By using the change of variables $(x_1, \ldots, x_d) \mapsto (px_1, \ldots, px_d)$, we have

$$\int_{(p\mathbb{Z}_p)^d} ||f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})||^s d\mu = p^{-d-ns} \int_{\mathbb{Z}_p^d} ||f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})||^s d\mu.$$

Now,

$$\begin{aligned} \mathrm{Z}_{\mathbf{f}}(s) &= \mathrm{Z}^{\star}_{\mathbf{f}}(s) + \int_{(p\mathbb{Z}_p)^d} ||f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})||^s d\mu \\ &= \mathrm{Z}^{\star}_{\mathbf{f}}(s) + p^{-n-ds} \int_{\mathbb{Z}_p^d} ||f_1(\mathbf{x}), \ldots, f_l(\mathbf{x})||^s d\mu. \end{aligned}$$

Thus,

$$\mathrm{Z}_{\mathbf{f}}(s) = \frac{1}{1 - p^{-n-ds}} \mathrm{Z}^{\star}_{\mathbf{f}}(s). \tag{3.9}$$

For $m \in \mathbb{N}$ we set

$$N^{\star}_m := \#\{\mathbf{a} \in (\mathbb{Z}/p^m)^d \setminus p(\mathbb{Z}/p^m)^d \mid \forall f \in \mathbf{f}, f(a) = 0\}$$

and define a Poincare series

$$\mathrm{P}^{\star}_{\mathbf{f}}(t) = \sum_{m=0}^{\infty} N^{\star}_m (p^{-d}t)^m. \tag{3.10}$$

Setting $\mu^{\star}_m := \mu(\{\mathbf{a} \in \mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d \mid v_p(\mathbf{f}(\mathbf{a})) = m\})$, we have

$$\mathrm{Z}^{\star}_{\mathbf{f}}(s) = \sum_{m=0}^{\infty} \mu^{\star}_m p^{-ms}. \tag{3.11}$$

Now

$$\mu^{\star}_m = \mu(\{\mathbf{a} \in \mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d \mid v_p(\mathbf{f}(\mathbf{a})) \geq m\}) - \mu(\{\mathbf{a} \in \mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d \mid v_p(\mathbf{f}(\mathbf{a})) \geq m+1\}),$$

and

$$\mu(\{\mathbf{a} \in \mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d \mid v_p(\mathbf{f}(\mathbf{a})) \geq m\}) = \begin{cases} N^{\star}_m p^{-dm} & \text{if } m \geq 1, \\ \mu(\{\mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d\}) = 1 - p^{-d} & \text{if } m = 0. \end{cases}$$

We have

$$\mu^{\star}_m = \frac{N^{\star}_m}{p^{dm}} - \frac{N^{\star}_{m+1}}{p^{d(m+1)}} - \delta_{m,0} p^{-d}. \tag{3.12}$$

By substituting (3.12) into (3.11) we have

$$\begin{aligned} \mathrm{Z}^{\star}_{\mathbf{f}}(s) &= \sum_{m=0}^{\infty} \left(\frac{N^{\star}_m}{p^{dm}} - \frac{N^{\star}_{m+1}}{p^{d(m+1)}} - \delta_{m,0} p^{-d} \right) p^{-ms} & (3.13) \\ &= \sum_{m=0}^{\infty} \frac{N^{\star}_m}{p^{dm}} p^{-ms} - \sum_{m=0}^{\infty} \frac{N^{\star}_{m+1}}{p^{d(m+1)}} p^{-ms} - p^{-d} \\ &= \mathrm{P}^{\star}_{\mathbf{f}}(t) - t^{-1}(\mathrm{P}^{\star}_{\mathbf{f}}(t) - 1) - p^{-d}. & (3.14) \end{aligned}$$

We rearrange equality (3.14) to arrive at the statement of [25, Lemma 2.1] generalised to case of a homogeneous ideal in $\mathbb{Z}_p[x_1, \dots, x_d]]$,

$$\mathrm{P}^{\star}_{\mathbf{f}}(t) = \frac{1 - p^{-d}t - t\,\mathrm{Z}^{\star}_{\mathbf{f}}(s)}{1 - t}. \tag{3.15}$$

Example 3.14. Consider the ideal $(x_1, \dots, x_d) \lhd \mathbb{Z}[x_1, \dots, x_d]$. It is homogeneous of degree 1. Igusa's local zeta function associated to this ideal can easily be computed using (3.9).

$$\int_{\mathbb{Z}_p^d} |x_1, \dots, x_d|^s d\mu = \frac{1}{1 - p^{-d-s}} \int_{\mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d} ||x_1, \dots, x_d||^s d\mu = \frac{1 - p^{-d}}{1 - p^{-d-s}}.$$

Chapter 4

The abscissa of convergence

We would like to describe relationships between the algebraic structure of a $\mathcal{T}$-group and the abscissa of convergence of its representation zeta function. An ambitious aim is to provide a closed formula for the abscissa of convergence in terms of well-known group invariants.

In this chapter we investigate what can be said about the abscissa of convergence in certain circumstances. In the first section we record proofs of some results that might be considered 'folklore' in as much as the results are known to the experts, but often no proof currently exists in the literature. We also show that the abscissa of convergence is a commensurability invariant, a new result.

In the second section of this chapter we investigate central products and use our results to show that any positive rational number can be realised as the abscissa of convergence of the representation zeta function of some $\mathcal{T}_2$-group. In the final section we present joint work with Shannon Ezzat. We give bounds on the p-local abscissa of convergence.

4.1 Basic properties

Throughout this section G denotes an arbitrary $\mathcal{T}$-group. We need some basic lemmas concerning the twist-isoclasses of such a group and those of a finite-index subgroup. Let $\mathrm{Irr}_n(G)$, $\widetilde{\mathrm{Irr}}_n(G)$ denote the set of isomorphism classes of n-dimensional irreducible representations of G and the set of twist-isoclasses of n-dimensional irreducible representations of G respectively.

Let $H \leq G$ be a subgroup. For $\sigma_1, \sigma_2 \in \mathrm{Irr}_n(H)$, we say that σ_1 is G-twist-equivalent to σ_2 if there exists $\chi \in \mathrm{Irr}_1(G)$ such that $\sigma_1 = \chi|_H \sigma_2$. We write $\sigma_1 \sim_G \sigma_2$. We denote by $\widetilde{\mathrm{Irr}}_n^G(H)$ the set of G-twist-isoclasses of degree-n characters of H.

Lemma 4.1. *For $H \leq G$ of finite index we have*

$$|\widetilde{\mathrm{Irr}}_n^G(H)| \leq |G' \cap H : H'||\widetilde{\mathrm{Irr}}_n(H)|.$$

Proof. Let N be the image of the restriction map $\mathrm{Irr}_1(G) \to \mathrm{Irr}_1(H)$ and let $\chi_1, \dots, \chi_k$ be the coset representatives of N in $\mathrm{Irr}_1(H)$. If there exist characters σ_1, σ_2 of H and $\chi \in \mathrm{Irr}_1(H)$ such that $\sigma_2 = \chi\sigma_1$, then σ_2 is G-twist-equivalent to at least one of $\chi_1\sigma_1, \dots, \chi_k\sigma_1$.

Consider the restriction map $\gamma : \mathrm{Irr}_1(H) \to \mathrm{Irr}_1(G' \cap H)$. The image of this map is isomorphic to $\mathrm{Irr}_1((G' \cap H)/H')$. This is clear as any element of $\mathrm{Irr}_1(H)$ is trivial on H'.

We show N is the kernel of γ. It is clear that N is contained within the kernel. Now, let $\tau \in \ker(\gamma)$ and let $\overline{\tau}$ denote the corresponding element of $\mathrm{Irr}_1((G' \cap H)/H')$. By the second isomorphism theorem $(G' \cap H)/H' \cong G'H/G'$. So we may consider $\overline{\tau}$ as an element of $\mathrm{Irr}_1(G'H/G')$. By extending $\overline{\tau}$ to G we obtain a 1-dimensional representation that restricts to τ and so $\tau \in N$. Therefore, $k = |\mathrm{Irr}_1(H) : N| = |(G' \cap H)/H'|$.

□

Corollary 4.2. *For $H \leq G$ of finite index we have*

$$|\widetilde{\mathrm{Irr}}_1^G(H)| = |G' \cap H : H'|$$

Proof. All 1-dimensional representations of a $\mathcal{T}$-group are twist-equivalent. Therefore $|\widetilde{\mathrm{Irr}}_1(H)| = 1$. By Lemma 4.1, $|\widetilde{\mathrm{Irr}}_1^G(H)| \leq |G' \cap H : H'|$. To prove equality we construct $|G' \cap H : H'|$ 1-dimensional representations of that are not G-twist-equivalent. The group $(G' \cap H)/H'$ is finite abelian and has precisely $|G' \cap H : H'|$ 1-dimensional representations. These induce $|G' \cap H : H'|$ distinct 1-dimensional representations of $G' \cap H$ which we extend to H. Clearly, these 1-dimensional representations of H are not G-twist-equivalent. □

Lemma 4.3. *Let H be a finite index subgroup of G. Two 1-dimensional representations τ_1, τ_2 of H induce representations of G that are in the same twist-isoclass if $\tau_1 \sim_G \tau_2$.*

Proof. This follows from Frobenius reciprocity. In particular, for a 1-dimensional representation τ of H and a 1-dimensional representation χ of G we have

$$\chi \otimes \mathrm{Ind}_H^G(\tau) = \mathrm{Ind}_H^G(\mathrm{Res}_H^G(\chi) \otimes \tau).$$

□

Before we can embark on an investigation of the abscissa of convergence we must first know that the representation zeta function of a $\mathcal{T}$-group converges on some half plane. That is, we must show that the representation growth is polynomial. This is established in [36]:

Lemma 4.4. [36, Lemma 2.1] *For a $\mathcal{T}$-group G, the sequence $(\widetilde{r}_n(G))$ is bounded by a polynomial. Equivalently, the representation zeta function $\zeta_G^{\widetilde{\mathrm{irr}}}(s)$ converges on some complex half plane.*

4.1.1 Direct products

Let $G \cong H_1 \times H_2$ be a direct product of $\mathcal{T}$-groups. In many contexts, the study of the group G can be reduced in some way to the study of the two components H_1, H_2. So far, this does not appear to be the case in subgroup growth. If we happen to know the subgroup zeta functions $\zeta_{H_1}(s)$, $\zeta_{H_2}(s)$ this does not necessarily tell us anything about the subgroup zeta function of G. However, in the case of representation zeta functions the study of G can be reduced to the study of its direct factors.

Let G, H be groups. If $\tilde{\rho} \in \widetilde{\operatorname{Irr}}(G)$ and $\tilde{\sigma} \in \widetilde{\operatorname{Irr}}(H)$, then we denote by $\tilde{\rho} \otimes \tilde{\sigma} := \{\rho_1 \otimes \sigma_1 \mid \rho_1 \in \tilde{\rho}, \sigma_1 \in \tilde{\sigma}\}$ the set of representations of $G \times H$ that are constructed as tensor products of representations from the twist-isoclasses $\tilde{\rho}$ of G and $\tilde{\sigma}$ of H.

Lemma 4.5. *Let $G \cong H_1 \times \cdots \times H_k$. Then,*

$$\widetilde{\operatorname{Irr}}_n(G) = \bigcup_{d_1 d_2 \ldots d_k = n} \{\tilde{\rho_1} \otimes \cdots \otimes \tilde{\rho_k} \mid \forall j \in [1, k], \tilde{\rho_j} \in \widetilde{\operatorname{Irr}}_{d_j}(H_j)\}$$

Proof. It is sufficient to prove the statement for $k = 2$. Suppose that $G \cong H_1 \times H_2$. We describe the twist-isoclass of G in terms of the twist-isoclasses H_1 and H_2. It is well known that all irreducible representations of G can be constructed as the tensor product of irreducible representations of H_1 and H_2. That is, for all $\rho \in \operatorname{Irr}(G)$ there exist $\tau_1 \in \operatorname{Irr}(H_1)$ and $\tau_2 \in \operatorname{Irr}(H_2)$ such that $\rho = \tau_1 \otimes \tau_2$. Additionally, we must have $\dim(\rho) = \dim(\tau_1)\dim(\tau_2)$. For $\sigma_1, \sigma_2 \in \operatorname{Irr}(H_1)$ and $\delta_1, \delta_2 \in \operatorname{Irr}(H_2)$ let $\rho_1 = \sigma_1 \otimes \delta_1$ and $\rho_2 = \sigma_2 \otimes \delta_2$. We show that $\rho_1 \sim_G \rho_2$ if and only if $\sigma_1 \sim_{H_1} \sigma_2$ and $\delta_1 \sim_{H_2} \delta_2$.

Suppose that $\rho_1 \sim_G \rho_2$. There exists $\chi \in \operatorname{Irr}_1(G)$ such that $\rho_1 = \chi \otimes \rho_2$. There exist $\chi_1 \in \operatorname{Irr}_1(H_1)$ and $\chi_2 \in \operatorname{Irr}_1(H_2)$ such that $\chi = \chi_1 \otimes \chi_2$. By associativity of the tensor product and because 1-dimensional representations are central we have

$$\sigma_1 \otimes \delta_1 = \rho_1 = \chi \otimes \rho_2 = (\chi_1 \otimes \chi_2) \otimes (\sigma_2 \otimes \delta_2) = (\chi_1 \otimes \sigma_2) \otimes (\chi_2 \otimes \delta_2).$$

Thus $\sigma_1 \sim_{H_1} \sigma_2$ and $\delta_1 \sim_{H_2} \delta_2$. The proof of the converse is very similar.

Therefore the twist-isoclasses of G are of the form $\tilde{\rho} = \tilde{\sigma} \otimes \tilde{\delta}$ and the result follows from the fact that $\dim(\tilde{\rho}) = \dim(\tilde{\sigma})\dim(\tilde{\delta})$. □

Corollary 4.6. *Let $G \cong H_1 \times \cdots \times H_k$. Then*

$$\widetilde{\mathrm{r}}_n(G) = \sum_{d_1 d_2 \dots d_k = n} \widetilde{\mathrm{r}}_{d_1}(H_1) \dots \widetilde{\mathrm{r}}_{d_k}(H_k). \tag{4.1}$$

Proof. Clearly, we have

$$|\{\tilde{\rho}_1 \otimes \cdots \otimes \tilde{\rho}_k \mid \tilde{\rho}_j \in \widetilde{\mathrm{Irr}}_{d_j}(H_j), \forall j \in [1,k]\}| = \widetilde{\mathrm{r}}_{d_1}(H_1) \dots \widetilde{\mathrm{r}}_{d_k}(H_k).$$

□

Corollary 4.7. *Let $G \cong H_1 \times \cdots \times H_k$. Then*

$$\zeta_G^{\widetilde{\mathrm{irr}}}(s) = \zeta_{H_1}^{\widetilde{\mathrm{irr}}}(s)\zeta_{H_2}^{\widetilde{\mathrm{irr}}}(s) \dots \zeta_{H_k}^{\widetilde{\mathrm{irr}}}(s).$$

Proof. The right-hand side of equation (4.1) is precisely the Dirichlet convolution product formula for the nth coefficient of the Dirichlet series $\zeta_{H_1}^{\widetilde{\mathrm{irr}}}(s)\zeta_{H_2}^{\widetilde{\mathrm{irr}}}(s) \dots \zeta_{H_k}^{\widetilde{\mathrm{irr}}}(s)$. Obviously, the left-side of (4.1) is the nth coefficient of the Dirichlet series $\zeta_G^{\widetilde{\mathrm{irr}}}(s)$. Two series are equal precisely when they have the same coefficients. See [1, Section 2.6] for details regarding Dirichlet convolution. □

Corollary 4.8. *Let $G \cong H_1 \times \cdots \times H_k$. Then*

$$\alpha^{\widetilde{\mathrm{irr}}}(G) = \max\{\alpha^{\widetilde{\mathrm{irr}}}(H_1), \dots, \alpha^{\widetilde{\mathrm{irr}}}(H_k)\}.$$

Proof. It is a standard fact that the abscissa of convergence of a finite product of Dirichlet series is equal to the maximum abscissa of convergence of the factors. The result follows. □

4.1.2 Euler product decomposition

In this subsection we prove that the representation zeta function of a $\mathcal{T}$-group satisfies an Euler product decomposition into a product of p-local zeta function. This result is well-known, but no complete proof has been recorded.

Lemma 4.9. *Let G be a finite nilpotent group, p be any prime and P be the (possibly trivial) Sylow p-subgroup of G. Then, for all $e \in \mathbb{N}$,*

$$\widetilde{\mathrm{r}}_{p^e}(G) = \widetilde{\mathrm{r}}_{p^e}(P).$$

Proof. It is well known that a finite nilpotent group is isomorphic to the direct product of its Sylow subgroups. In particular, there exists a finite nilpotent group H such that $p \nmid |H|$ and $G \cong P \times H$. By Corollary 4.6 we have

$$\widetilde{\mathrm{r}}_{p^e}(G) = \sum_{e_1+e_2=e} \widetilde{\mathrm{r}}_{p^{e_1}}(P)\widetilde{\mathrm{r}}_{p^{e_2}}(H).$$

However, $\widetilde{\mathrm{r}}_{p^{e_2}}(H) \neq 0$ if and only if $e_2 = 0$, because the dimension of an irreducible representation of a finite group must divide the order of the group, and in this case $\widetilde{\mathrm{r}}_1(H) = 1$ and the result follows. □

Lemma 4.10. *Let G be a finite nilpotent group and n be a natural number with prime factorisation $n = p_1^{e_1} \dots p_k^{e_k}$. Then, we have*

$$\widetilde{\mathrm{r}}_n(G) = \widetilde{\mathrm{r}}_{p_1^{e_1}}(G) \dots \widetilde{\mathrm{r}}_{p_k^{e_k}}(G).$$

Proof. For $i \in [1, k]$, let P_i denote the (possibly trivial) Sylow p_i-subgroup of G. There exists a finite nilpotent group H such that n is coprime to $|H|$ and $G \cong P_1 \times \dots \times P_k \times H$. By Corollary 4.6,

$$\widetilde{\mathrm{r}}_n(G) = \sum_{d_1 d_2 \dots d_{k+1}=n} \widetilde{\mathrm{r}}_{d_1}(P_1) \dots \widetilde{\mathrm{r}}_{d_k}(P_k)\widetilde{\mathrm{r}}_{d_{k+1}}(H).$$

First note that for $d_{k+1} \mid n$, $\widetilde{\mathrm{r}}_{d_{k+1}}(H) \neq 0$ if and only if $d_{k+1} = 1$. Further, each $\widetilde{\mathrm{r}}_{d_i}(P_i)$ is non-zero if and only if d_i is a p_i power. We must have $d_1 \dots d_k = n$ and so $d_i = p_i^{e_i}$. The result then follows from Lemma 4.9. □

Proposition 4.11. *Let G be a $\mathcal{T}$-group and n be a natural number with prime factorisation $n = p_1^{e_1} \dots p_k^{e_k}$. Then, we have*

$$\widetilde{\mathrm{r}}_n(G) = \widetilde{\mathrm{r}}_{p_1^{e_1}}(G) \dots \widetilde{\mathrm{r}}_{p_k^{e_k}}(G).$$

In other words, the arithmetic function $\widetilde{\mathrm{r}}_n(G)$ is multiplicative.

Proof. By Theorem 1.6 there exists a finite quotient $G(n)$ such that every n-dimensional irreducible representation of G is twist-equivalent to a representation that factors through $G(n)$. Denote the kernel of natural map $G \to G(n)$ by $N(n)$. The finite quotients $\{G(n) \mid n \in \mathbb{N}\}$ are not uniquely determined by this property and without loss we may assume that the normal subgroups $\{N(n) \mid n \in \mathbb{N}\}$ form a chain. In particular, if $m \leq n$ then $G(n)$ maps onto $G(m)$.

We claim that $\widetilde{\mathrm{r}}_{p_i^{e_i}}(G(n)) = \widetilde{\mathrm{r}}_{p_i^{e_i}}(G(p_i^{e_i}))$ and then by applying Theorem 1.6 and Lemma 4.10 we have

$$\begin{aligned} \widetilde{\mathrm{r}}_n(G) = \widetilde{\mathrm{r}}_n(G(n)) &= \widetilde{\mathrm{r}}_{p_1^{e_1}}(G(n)) \dots \widetilde{\mathrm{r}}_{p_k^{e_k}}(G(n)) \\ &= \widetilde{\mathrm{r}}_{p_1^{e_1}}(G(p_1^{e_1})) \dots \widetilde{\mathrm{r}}_{p_k^{e_k}}(G(p_k^{e_k})) = \widetilde{\mathrm{r}}_{p_1^{e_1}}(G) \dots \widetilde{\mathrm{r}}_{p_k^{e_k}}(G). \end{aligned}$$

To prove the claim, first note that any $p_i^{e_i}$-dimensional representation of $G(p_i^{e_i})$ induces a representation of $G(n)$ via the canonical map. Conversely if ρ if a $p_i^{e_i}$-dimensional representation of $G(n)$ then ρ must be twist-equivalent to a representation that factors through $G(p_i^{e_i})$ by the defining property of $G(p_i^{e_i})$. □

Corollary 4.12. *Let G be a $\mathcal{T}$-group. The representation zeta function $\zeta_G^{\widetilde{\mathrm{irr}}}(s)$ has an Euler factorisation*

$$\zeta_G^{\widetilde{\mathrm{irr}}}(s) = \prod_p \zeta_{G,p}^{\widetilde{\mathrm{irr}}}(s).$$

4.1.3 Commensurability

The main result of this section is to show that the rate of representation growth $\alpha^{\widetilde{\mathrm{irr}}}(G)$ is a commensurability invariant. It suffices to show that if H is a finite index subgroup of G then $\alpha^{\widetilde{\mathrm{irr}}}(H) = \alpha^{\widetilde{\mathrm{irr}}}(G)$.

Proposition 4.13. *Let $H \leq G$ be a subgroup of finite index in G. Let $k_1 = |G : H|$ and $k_2 = |G' \cap H : H'|$ then, for all $n \in \mathbb{N}$,*

$$\tilde{\mathcal{R}}_n(H) \leq k_1 \tilde{\mathcal{R}}_{k_1 n}(G), \tag{4.2}$$

$$\tilde{\mathcal{R}}_n(G) \leq k_1 k_2 \tilde{\mathcal{R}}_n(H). \tag{4.3}$$

Proof. We define a map $\Psi : \cup_{i=1}^{n} \widetilde{\mathrm{Irr}}_i(H) \to \cup_{i=1}^{kn} \widetilde{\mathrm{Irr}}_i(G)$ as follows. Let $\tilde{\sigma} \in \cup_{i=1}^{n} \widetilde{\mathrm{Irr}}(H)$. We pick any representative σ of the twist-isoclass $\tilde{\sigma}$ and induce to G. Let ρ be any irreducible component of $\mathrm{Ind}_H^G \sigma$ and define $\Psi(\tilde{\sigma}) = \tilde{\rho}$. We repeat this process for each element of $\cup_{i=1}^{n} \widetilde{\mathrm{Irr}}_i(H)$. (There are many possible choices for Ψ.)

Suppose that we have $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m \in \cup_{i=1}^{n} \widetilde{\mathrm{Irr}}_i(H)$ such that $\tilde{\sigma}_i \neq \tilde{\sigma}_j$ for $i \neq j$ but $\Psi(\tilde{\sigma}_1) = \cdots = \Psi(\tilde{\sigma}_m) = \tilde{\rho}$ for some $\tilde{\rho} \in \cup_{i=1}^{kn} \widetilde{\mathrm{Irr}}(G)$. The construction of Ψ implies that for $i \in \{1, \ldots, m\}$ there exist $\sigma_i \in \tilde{\sigma}_i$ and $\rho_i \in \tilde{\rho}$ such that $\rho_i \in \mathrm{Ind}_H^G \sigma_i$. By Frobenius reciprocity, $\sigma_i \in \mathrm{Res}_H^G \rho_i$. We also have that the $\mathrm{Res}_H^G \rho_i$ are all twist-equivalent. This implies that for each σ_i there exists χ_i such that $\chi_i \sigma_i \in \mathrm{Res}_H^G \rho_1$.

The σ_i are in separate twist-isoclasses. This implies that $\mathrm{Res}_H^G \rho_1 = \chi_1 \sigma_1 \oplus \cdots \oplus \chi_m \sigma_m \oplus \rho'$ for some representation ρ' of H. By relabeling we may assume that σ_1 has mininum dimension among the σ_i. Noting that $\dim \rho_1 = \dim \mathrm{Res}_H^G \rho_1$ we have that $\dim \rho_1 \geq m \dim \sigma_1$. We also know that $\dim \rho_1 \leq k \dim \sigma_1$ therefore the map Ψ is at most k-to-1. This establishes (4.2).

We define a map $\Phi : \cup_{i=1}^{n} \widetilde{\mathrm{Irr}}_i(G) \to \cup_{i=1}^{n} \widetilde{\mathrm{Irr}}_i^G(H)$ as follows. Let $\tilde{\rho} \in \cup_{i=1}^{n} \widetilde{\mathrm{Irr}}_i(G)$, pick any representative ρ of the twist-isoclass $\tilde{\rho}$. Let σ be any irreducible component of Res_H^G and define $\Phi(\tilde{\rho}) = \tilde{\sigma}^G$, the G-twist-isoclass of σ.

Suppose that we have $\tilde{\rho}_1, \ldots, \tilde{\rho}_m \in \cup_{i=1}^n \widetilde{\mathrm{Irr}}_i(G)$ such that $\tilde{\rho}_i \neq \tilde{\rho}_j$ for $i \neq j$ but $\Psi(\tilde{\rho}_1) = \cdots = \Psi(\tilde{\rho}_m) = \tilde{\sigma}^G$ for some $\tilde{\rho} \in \cup_{i=1}^n \widetilde{\mathrm{Irr}}^G(H)$. The construction of Φ implies that for $i \in \{1, \ldots, m\}$ there exist $\rho_i \in \tilde{\rho}_i$ and $\sigma_i \in \tilde{\sigma}^G$ such that $\sigma_i \in \mathrm{Res}_H^G \rho_i$. By Frobenius reciprocity, $\rho_i \in \mathrm{Ind}_H^G \sigma_i$. The $\mathrm{Ind}_H^G \sigma_i$ are all twist-equivalent. This follows from the fact that the σ_i are G-twist equivalent on H. This implies that for each ρ_i there exists a 1-dimensional representation τ_i of G such that $\tau_i \rho_i \in \mathrm{Ind}_H^G \sigma_1$.

The ρ_i are not twist-equivalent. It follows that $\mathrm{Ind}_H^G \sigma_1 = \tau_1\rho_1 \oplus \cdots \oplus \tau_m\rho_m \oplus \sigma'$, for some representation σ' of G. By relabeling we may assume that ρ_1 has minimum dimension among the ρ_i. Thus $\dim \mathrm{Ind}_H^G \sigma_1 \geq m \dim \rho_1$. We also have $\dim \rho_1 \geq \dim \sigma_1$ and so $k_1 \dim \rho_1 \geq \dim \mathrm{Ind}_H^G \sigma_1 \geq m \dim \rho_1$. Thus $m \leq k_1$.

We have established that $\tilde{\mathcal{R}}_n(G) \leq k_1 \tilde{\mathcal{R}}_n^G(H)$. Inequality (4.3) then follows from Corollary 4.2. □

Corollary 4.14. *Let $H \leq G$ be a subgroup of finite index in G. Then $\alpha^{\widetilde{\mathrm{irr}}}(G) = \alpha^{\widetilde{\mathrm{irr}}}(H)$.*

Proof. Let $\alpha \in \mathbb{R}$ and suppose that there exists γ such that

$$\tilde{\mathcal{R}}_n(G) < \gamma n^\alpha$$

for all $n \in \mathbb{N}$. Then by Proposition 4.13 we have

$$\tilde{\mathcal{R}}_n(H) \leq k_1 \tilde{\mathcal{R}}_{k_1 n}(G) < k_1 \gamma (k_1 n)^\alpha = k_1^{\alpha+1} \gamma n^\alpha$$

for all $n \in N$. Thus, $\alpha^{\widetilde{\mathrm{irr}}}(H) \leq \alpha^{\widetilde{\mathrm{irr}}}(G)$. Conversely, for $\alpha \in \mathbb{R}$ suppose that there exists γ such that

$$\tilde{\mathcal{R}}_n(H) < \gamma n^\alpha$$

for all $n \in \mathbb{N}$. Then by Proposition 4.13 we have

$$\tilde{\mathcal{R}}_n(G) \leq k_1 k_2 \tilde{\mathcal{R}}_n(H) < k_1 k_2 \gamma (n)^\alpha = k_1 k_2 \gamma n^\alpha$$

for all $n \in \mathbb{N}$. Thus, $\alpha^{\widetilde{\mathrm{irr}}}(G) \leq \alpha^{\widetilde{\mathrm{irr}}}(H)$ and the proof is complete. □

Corollary 4.15. *The polynomial rate of representation growth of a $\mathcal{T}$-group is a commensurability invariant.*

Proposition 4.16. *Let G be a $\mathcal{T}$-group and H be a subgroup of finite index in G. Then $\alpha_p^{\widetilde{\mathrm{irr}}}(G) = \alpha_p^{\widetilde{\mathrm{irr}}}(H)$.*

Proof. It suffices to show that $\alpha^{\widetilde{\mathrm{irr}}}(\widehat{G}_p) = \alpha^{\widetilde{\mathrm{irr}}}(\widehat{H}_p)$. In this case a slight adjustment to the proof of Proposition 4.13 is sufficient. □

4.2 Central products

In this section we investigate the representation growth of central products of $\mathcal{T}$-groups. This leads to a construction of a family of groups that realise every positive rational number as an abscissa of convergence.

We begin by recalling the definition of central products of $\mathcal{T}$-groups and a few known results on the representation theory of such groups.

Theorem 4.17. [14, Theorem 5.3] *Let H, K, M be groups with $M \subseteq Z(H)$ and suppose there is an isomorphism θ of M into $Z(K)$. Then if we identify M with its image $\theta(M)$, there exists a group of the form $G = HK$ with $M = H \cap K \subseteq Z(G)$ such that H centralizes K.*

The group G in Theorem 4.17 is called the central product of H and K with respect to θ. If $M = 1$ then the central product is just the direct product $H \times K$. The central product G is isomorphic to $(H \times K)/N$, where $N = \{(h, k) \mid h \in H, k \in K, \theta(h) = k\}$.

Lemma 4.18. *Let G, H be $\mathcal{T}$-groups and suppose that there exists a normal subgroup $N \lhd G$ of G such that $G/N \cong H$. Then $\alpha^{\widetilde{\mathrm{irr}}}(H) \leq \alpha^{\widetilde{\mathrm{irr}}}(G)$.*

Proof. Let $\phi : G \to H$ be the canonical homomorphism. Any representation of H can be realised as a representation of G by factoring via ϕ.

We need to check that representations that are not H-twist-equivalent do not induce G-twist-equivalent representations on factoring via ϕ. Let τ_1, τ_2 be irreducible n-dimensional representations of H which are not H-twist-equivalent. Now suppose that the irreducible n-dimensional representations $\tau_1 \circ \phi, \tau_2 \circ \phi$ of G are G-twist-equivalent. That is, there exists $\chi \in \mathrm{Irr}_1(G)$ such that

$$\chi \otimes (\tau_1 \circ \phi) = \tau_2 \circ \phi. \tag{4.4}$$

Let $n \in N$. Now $\tau_1 \circ \phi(n) = \tau_2 \circ \phi(n) = 1$ because N is the kernal of ϕ. Further, $(\chi \otimes (\tau_1 \circ \phi))(n) = \chi(n)$. By comparison with the right-hand side of (4.4) we must have $\chi(n) = 1$ and N is contained in the kernal of χ.

This means that χ factors through H and there exists a 1-dimensional representation σ of H such that $\chi = \sigma \circ \phi$. Finally note that $(\sigma \circ \phi) \otimes (\tau_1 \circ \phi) = (\sigma \otimes \tau_1) \circ \phi$. This implies that τ_1 and τ_2 are H-twist-equivalent. This is a contradiction. □

Corollary 4.19. *Let G be a $\mathcal{T}$-group that is a central product of two $\mathcal{T}$-groups H and K. Then $\widetilde{\alpha^{\mathrm{irr}}}(G) \leq \max\{\widetilde{\alpha^{\mathrm{irr}}}(H), \widetilde{\alpha^{\mathrm{irr}}}(K)\}$.*

Proof. Corollary 4.8 states that $\widetilde{\alpha^{\mathrm{irr}}}(H \times K) = \max\{\widetilde{\alpha^{\mathrm{irr}}}(H), \widetilde{\alpha^{\mathrm{irr}}}(K)\}$. Now, any central product of H and K is isomorphic to a quotient of the direct product. Lemma 4.18 states that the abscissa of convergence of a quotient is less than or equal to the abscissa of convergence of the original group. □

We now examine a special class of central products. Let G_1, G_2 be $\mathcal{T}_2$-groups with isomorphic centres. If we have presentations

$$G_1 = \left\langle \begin{array}{c} x_{11}, \dots x_{1n_1} \\ y_1, \dots y_d \end{array} \middle| \mathscr{R}_1 \right\rangle, G_2 = \left\langle \begin{array}{c} x_{21}, \dots x_{2n_2} \\ y_1, \dots y_d \end{array} \middle| \mathscr{R}_2 \right\rangle,$$

then the *canonical central product*, denoted $G_1 \times_Z G_2$ has presentation

$$G_1 \times_Z G_2 = \left\langle \begin{array}{c} x_{11}, \dots x_{1n_1}, x_{21}, \dots x_{2n_2} \\ y_1, \dots y_d \end{array} \middle| \mathscr{R}_1, \mathscr{R}_2 \right\rangle.$$

It is also very easy to write down the structure matrix of the Lie ring $L(G_1 \times_Z G_2)$. It is simply the diagonal sum of the structure matrices of the Lie rings $L(G_1)$ and $L(G_2)$:

$$\mathcal{R}_{L(G_1 \times_Z G_2)}(\mathbf{Y}) = \left(\begin{array}{c|c} \mathcal{R}_{L(G_2)}(\mathbf{Y}) & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{R}_{L(G_1)}(\mathbf{Y}) \end{array} \right).$$

4.2.1 The k-fold canonical central product

The k-fold canonical central product of a $\mathcal{T}_2$-group G with itself is defined as

$$\times_Z^k G := \underbrace{G \times_Z \cdots \times_Z G}_{k \text{ copies}}$$

Example 4.20. Let H denote the discrete Heisenberg group it has presentation

$$H = \langle x_1, x_2, y \mid [x_1, x_2] = y \rangle$$

and $L(H)$ has structure matrix

$$\mathcal{R}_{L(H)}(Y) = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}.$$

Therefore, the k-fold canonical central product of the Heisenberg group has presentation

$$\times_Z^k H = \left\langle \begin{array}{cc|c} x_{11}, x_{12}, & & [x_{11}, x_{12}] = y \\ \dots & y & \dots \\ x_{k1}, x_{k2}, & & [x_{k1}, x_{k2}] = y \end{array} \right\rangle.$$

and the associated Lie ring $L(\times_Z^k H)$ has the following structure matrix:

$$\mathcal{R}_{L(\times_Z^k H)}(Y) = \begin{pmatrix} 0 & Y & & & & & \\ -Y & 0 & & & & & \\ & & 0 & Y & & & \\ & & -Y & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & Y \\ & & & & & -Y & 0 \end{pmatrix}.$$

Proposition 4.21. *Let G be a $\mathcal{T}_2$-group such that $h(G/Z) = n$ and $h(Z) = d$. For almost all primes p we have the following formula for the local representation zeta function of the k-fold canonical central product of G.*

$$\widetilde{\zeta^{\mathrm{irr}}_{\times_Z^k G,p}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{G,p}}(ks).$$

Proof. By Theorem 2.9, for almost all primes p the p-local representation zeta function is given by the formula.

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{\substack{N=1 \\ \mathbf{m}\in\mathbb{N}_0^n}}^{\infty} \mathcal{N}_{N,\mathbf{m}} p^{-\frac{s}{2}\sum_{i=1}^{n}(N-m_i)}, \tag{4.5}$$

where $\mathcal{N}_{N,\mathbf{m}} = \#\{\mathbf{a} \in (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d \mid \nu(\mathcal{R}_{L(G)}(\mathbf{a})) = \mathbf{m}\}$. Since $\mathcal{R}_{L(\times_Z^k G)}(\mathbf{Y})$ is the diagonal sum of k copies of $\mathcal{R}_{L(G)}(\mathbf{Y})$ if $\nu(\mathcal{R}_{L(G)}(\mathbf{a})) = (m_1, m_2, \ldots, m_n)$, then

$$\mathcal{R}_{L(\times_Z^k G)}(\mathbf{a}) = (\underbrace{m_1, \ldots, m_1}_{k}, \ldots, \underbrace{m_n, \ldots, m_n}_{k}).$$

Therefore,

$$\begin{aligned}\widetilde{\zeta^{\mathrm{irr}}_{\times_Z^k G,p}}(s) &= 1 + \sum_{\substack{N=1 \\ \mathbf{m}\in\mathbb{N}_0^n}}^{\infty} \mathcal{N}_{N,\mathbf{m}} p^{-\frac{s}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}(N-m_i)} \\ &= 1 + \sum_{\substack{N=1 \\ \mathbf{m}\in\mathbb{N}_0^n}}^{\infty} \mathcal{N}_{N,\mathbf{m}} p^{-\frac{ks}{2}\sum_{i=1}^{n}(N-m_i)} = \widetilde{\zeta^{\mathrm{irr}}_{G,p}}(ks).\end{aligned}$$

□

Proposition 4.21 is key in the proof of the main theorem of this section.

Theorem 4.22. *Let α be a positive rational number. Then there exists a $\mathcal{T}_2$-group G with abscissa of convergence α. That is, $\widetilde{\alpha^{\mathrm{irr}}}(G) = \alpha$.*

Proof. Grenham's groups, denoted by G_n, are in introduced in Example 6.2, where it is shown that G_n has representation zeta function

$$\widetilde{\zeta^{\mathrm{irr}}_{G_n}}(s) = \frac{\zeta(s-n+1)}{\zeta(s)},$$

which has abscissa of convergence $\widetilde{\alpha^{\mathrm{irr}}}(G_n) = n$. In the case of G_n, formula (2.8) is valid for all primes. Therefore the k-fold canonical central product of G_n has representation zeta function

$$\widetilde{\zeta^{\mathrm{irr}}_{\times_Z^k G_n}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{G_n}}(ks) = \frac{\zeta(ks-n)}{\zeta(ks)},$$

which has abscissa of convergence $\widetilde{\alpha^{\mathrm{irr}}}(\times_Z^k G_n) = (n+1)/k$. It is clear that for any positive rational number α there are infinitely many choices for n and k such that $\widetilde{\alpha^{\mathrm{irr}}}(\times_Z^k G_n) = \alpha$. □

4.3 Bounds

This section consists of joint work with Shannon Ezzat. In this section we prove some bounds for the local abscissa of convergence $\widetilde{\alpha_p^{\mathrm{irr}}}(G)$. First we need to discuss and reformulate formula (2.8) slightly. Theorem 2.9 states that if G is a $\mathcal{T}_2$-group such that $h(G/Z) = n$ and $h(Z) = d$, then for almost all primes p,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{N=1}^{\infty} \sum_{\mathbf{m} \in \mathbb{N}_0^n} \mathcal{N}_{N,\mathbf{m}} p^{-\frac{s}{2}\sum_{i=1}^{n}(N-m_i)},$$

where $\mathcal{N}_{N,\mathbf{m}} = \#\{\mathbf{a} \in (\mathbb{Z}/p^N\mathbb{Z})^d \smallsetminus p(\mathbb{Z}/p^N\mathbb{Z})^d \mid \nu(\mathcal{R}_{L(G)}(\mathbf{a})) = \mathbf{m}\}$.

In Section 2.4 we outline how this formula is derived. A representative of each twist-isoclass of p-power-dimensional irreducible representations of

G is constructed from $\phi \in \widehat{L(G)'}$ of p-power period. There is a bijection between the non-trivial elements of $\widehat{L(G)'}$ and $\cup_{N\in\mathbb{N}}(\mathbb{Z}/p^N)^d \smallsetminus p(\mathbb{Z}/p^N)^d$. The irreducible representation associated with ϕ is denoted $\pi(\phi)$. Associated with ϕ is some $\mathbf{a} \in \cup_{N\in\mathbb{N}}(\mathbb{Z}/p^N)^d \smallsetminus p(\mathbb{Z}/p^N)^d$. We also denote the representation by $\pi(\mathbf{a})$. We can rewrite formula (2.8) as

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{N\in\mathbb{N}} \sum_{\mathbf{a}\in(\mathbb{Z}/p^N)^d \smallsetminus p(\mathbb{Z}/p^N)^d} \dim(\pi(\mathbf{a}))^{-s}. \tag{4.6}$$

We use elementary divisors to calculate $\dim(\pi(\mathbf{a}))$. If $\nu(\mathcal{R}_{L(G)}(\mathbf{a})) = (m_1, m_1, \dots, m_n, m_n)$ or $(m_1, m_1, \dots, m_n, m_n, 0)$ then $\dim(\pi(\mathbf{a})) = p^{\sum_{i=1}^{n}(N-m_i)}$, cf. Section 2.4.

Before we establish the main result of the section we require a technical lemma. For a Dirichlet series $Z(s)$ we denote the abscissa of convergence by $\alpha(Z(s))$. Note that for a Dirichlet series with non-negative coefficients the abscissa of absolute convergence coincides with the abscissa of conditional convergence, this follows from [26, Chapter VIII, Theorem 1].

Lemma 4.23. *Let A be a countable set and suppose that $\mu : A \to \mathbb{N}$ is a map such that the Dirichlet series $\sum_{a\in A}\mu(a)^{-s}$ converges on some half plane. Further, suppose that there exist functions* upp $: A \to \mathbb{N}$ *and* low $: A \to \mathbb{N}$ *such that*

$$\mathrm{low}(a) \le \mu(a) \le \mathrm{upp}(a)$$

for all $a \in A$. Then,

$$\alpha\left(\sum_{a\in A}\mathrm{upp}(a)^{-s}\right) \le \alpha\left(\sum_{a\in A}\mu(a)^{-s}\right) \le \alpha\left(\sum_{a\in A}\mathrm{low}(a)^{-s}\right).$$

Proof. First we can rewrite $\sum_{a\in A}\mu(a)^{-s}$ as $\sum_{n=1}^{\infty}\bar{\mu}(n)n^{-s}$, where $\bar{\mu}(n) = |\{a \in A \mid \mu(a) = n\}|$. The abscissa of convergence $\alpha\left(\sum_{a\in A}\mu(a)^{-s}\right)$ is defined in terms of the sequence of partial sums of the $\bar{\mu}(n)$: $\alpha := \alpha\left(\sum_{a\in A}\mu(a)^{-s}\right)$ is the smallest value such that

$$\sum_{n=1}^{N}\bar{\mu}(n) < O(N^{\alpha+\epsilon}),$$

for every $\epsilon \in \mathbb{R}_{>0}$.

If we define $\overline{\text{upp}}(n) := |\{a \in A \mid \text{upp}(a) = n\}|$ and $\overline{\text{low}}(n) := |\{a \in A \mid \text{low}(a) = n\}|$. It follows that

$$\sum_{n=1}^{N} \overline{\text{upp}}(n) < \sum_{n=1}^{N} \bar{\mu}(n) < \sum_{n=1}^{N} \overline{\text{low}}(n)$$

and the result follows. □

We are ready to prove the main result of this section. This result is similar to [2, Theorem 1.1].

Theorem 4.24. *Let G be a $\mathcal{T}_2$-group such that $h(G/Z) = n$ and $h(G') = d$. Then for almost all primes p*

$$\frac{d}{\lfloor \frac{n}{2} \rfloor} \leq \alpha_p^{\widetilde{\text{irr}}}(G) \leq d.$$

Proof. We apply Lemma 4.23. Recall formula (4.6):

$$\zeta_{G,p}^{\widetilde{\text{irr}}}(s) = 1 + \sum_{N \in \mathbb{N}} \sum_{\mathbf{a} \in (\mathbb{Z}/p^N)^d \setminus p(\mathbb{Z}/p^N)^d} \dim(\pi(\mathbf{a}))^{-s}.$$

The summation is over $W := \cup_{N=1}^{\infty} (\mathbb{Z}/p^N)^d \setminus p(\mathbb{Z}/p^N)^d$. Consider the map $W \to \mathbb{N}$, $\mathbf{a} \mapsto \dim(\pi(\mathbf{a}))$. In order to apply Lemma 4.23 it is necessary to find upper and lower bounds for $\dim(\pi(\mathbf{a}))$. Now, using the notation from the discussion preceding Lemma 4.23, we know that $\dim(\pi(\mathbf{a})) = p^{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(N-m_i)}$. Clearly, $p^{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(N)}$ is an upper bound for $\dim(\pi(\mathbf{a}))$. We obtain the lower bound p^N by noting that, by Lemma 2.4, the first elementary divisor m_1 must be zero and taking the remaining elementary divisors to be equal to N.

It follows from Lemma 4.23 that

$$\alpha\left(1 + \sum_{\mathbf{a} \in W} (p^{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(N)})^{-s}\right) \leq \alpha_p^{\widetilde{\text{irr}}}(G) \leq \alpha\left(1 + \sum_{\mathbf{a} \in W} (p^N)^{-s}\right).$$

By noting that $|(\mathbb{Z}/p^N)^d \setminus p(\mathbb{Z}/p^N)^d| = (1-p^{-d})p^{dN}$ we have

$$\alpha\left(1 + \sum_{N=1}^{\infty} (1-p^{-d})p^{dN}(p^{\lfloor \frac{n}{2} \rfloor N})^{-s}\right) \leq \alpha_p^{\widetilde{\text{irr}}}(G) \leq \alpha\left(1 + \sum_{N=1}^{\infty} (1-p^{-d})p^{dN}(p^N)^{-s}\right)$$

and therefore

$$\alpha\left(\frac{1-p^{-\lfloor\frac{n}{2}\rfloor s}}{1-p^{d-\lfloor\frac{n}{2}\rfloor s}}\right)\leq\widetilde{\alpha_p^{\mathrm{irr}}}(G)\leq\alpha\left(\frac{1-p^{-s}}{1-p^{d-s}}\right).$$

The result follows after reacalling that $\frac{1}{1-p^{-s}}$ converges for $\Re(s)>0$ $\square$

Chapter 5

$\mathcal{T}$-groups with small derived group

Definition 5.1. The representation zeta function $\widetilde{\zeta^{\mathrm{irr}}_G}(s)$ of a $\mathcal{T}$-group G is *finitely uniform* if there exist rational functions $W_1(X,Y),\dots,W_k(X,Y) \in \mathbb{Q}[X,Y]$ and a function $f : \mathcal{P} \to [1,k]$ from the set of all primes $\mathcal{P}$ to $[1,k]$ such that for all primes p,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = W_{f(p)}(p, p^{-s}).$$

This chapter contains results concerning the uniformity of the representation zeta functions of $\mathcal{T}_2$-groups. It is shown that the representation zeta function of a $\mathcal{T}_2$-group with centre of Hirsch length at most 2 is finitely uniform. This result is obtained using a classification of such groups up to commensurability, which is recalled in Section 5.4.

We also calculate the representation zeta function a particular $\mathcal{T}_2$-group with centre of Hirsch length 3; it is not finitely uniform. This result rely on a formula for the local factors of the representation zeta for $\mathcal{T}_2$-groups, whose associated Lie ring has a structure matrix of a specific shape.

5.1 Simple elementary divisors

This section constructs a formula for almost all the local factors of the representation zeta function of a $\mathcal{T}_2$-group G whose associated Lie ring $L(G)$ has structure matrix with a particular 'shape' to its elementary divisor type.

Definition 5.2. Let $f \in \mathbb{N}$ and G be a $\mathcal{T}_2$-group with $h(G') = d$, say. Then G is said to have *simple elementary divisor type of length f* if for all $N \in \mathbb{N}$ and $\mathbf{b} \in (\mathbb{Z}/p^N)^d \setminus p(\mathbb{Z}/p^N)^d$ there exists $m \in [N]$ such that

$$\nu(\mathcal{R}_{L(G)}(\mathbf{b})) = (\underbrace{0, \ldots, 0, m, m}_{2f}, N, \ldots, N).$$

Theorem 5.3. *Let G be a $\mathcal{T}_2$-group that has simple elementary divisor type of length f. Let $h(G') = d$, $h(G/Z) = n$ and let $\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{Y}) \subseteq \mathbb{Z}[Y_1, \ldots, Y_d]$ denote the ideal generated by the principal $2f$-minors of $\mathcal{R}_G(\mathbf{Y})$. Then for almost all primes p,*

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{1 - t^{f-1}}{1 - p^d t^{f-1}} + \frac{(t-1)p^d t^{f-1}}{(1 - p^d t^{f-1})(1 - p^d t^f)} Z^{\star}_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d),$$

where $Z^{\star}_{\mathbf{F}}(s)$ is the variant of Igusa's local zeta function defined in Section 3.3.5.

Proof. Theorem 2.9 states, for almost all primes p:

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \sum_{\substack{N \in \mathbb{N}_0 \\ \mathbf{m} \in \mathbb{N}^n}} \mathcal{N}_{N,\mathbf{m}} p^{-\frac{s}{2} \sum_{i=1}^{n} (N - m_i)}. \tag{5.1}$$

Now suppose that G has simple elementary type of length f. In this case $m_1 = \cdots = m_{2f-2} = 0$ and $m_{2f+1} = \cdots = m_n = N$. We write $m = m_{2f-1} = m_{2f}$. Substituting this information into (5.1) and writing $t = p^{-s}$ we obtain

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \sum_{N=0}^{\infty} \sum_{m=0}^{N} \mathcal{N}^{\star}_{N,m} t^{fN-m}, \tag{5.2}$$

where $\mathcal{N}^{\star}_{N,m} := \#\{\mathbf{b} \in (\mathbb{Z}/p^N\mathbb{Z})^d \setminus (\mathbb{Z}/p^N\mathbb{Z})^d \mid \nu(\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{b})) = 2m\}$ and $\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{Y})$ denotes the ideal generated by the $2f$-minors of $\mathcal{R}_{L(G)}(\mathbf{Y})$. Recall that $\nu(\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{b}))$ denotes the minimum p-adic valuation of any element of the ideal $\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{Y}) \subset \mathbb{Z}[Y_1, \ldots, Y_d]$ when evaluated at $\mathbf{b}$. This minimum is always realised by one of the elements of any generating set of $\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{Y})$. In particular, the minimum is always realised by at least one of the $2f$-minors of $\mathcal{R}_{L(G)}(\mathbf{Y})$. Furthermore, by Proposition 2.7 this minimum is attained by at least one of the principal $2f$-minors. Therefore, $\mathcal{N}^{\star}_{N,m} = \#\{\mathbf{b} \in (\mathbb{Z}/p^N\mathbb{Z})^d \setminus (\mathbb{Z}/p^N\mathbb{Z})^d \mid \nu(\tilde{\boldsymbol{\sigma}}_{2f}(\mathbf{b})) = m\}$. We now use the identities explored in Section 3.3.5 to express $\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s)$ in terms of Igusa's local zeta function associated to the ideal $\tilde{\boldsymbol{\sigma}}_{2f}$.

This procedure is similar to one employed in the proof of [25, Theorem 1.1]. Rearranging the summation of (5.2) and then comparing the definition $\mathcal{N}^{\star}_{N,m}$ with the definitions of $\mu^{\star}_m$ and $\mathcal{N}^{\star}_m$, given in Section 3.3.5 we have

$$\begin{aligned}\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) &= \sum_{N=1}^{\infty}\sum_{m=0}^{N-1} \mathcal{N}^{\star}_{N,m} t^{fN-m} + \sum_{N=0}^{\infty} \mathcal{N}^{\star}_{N,N} t^{fN-N} \\ &= \sum_{N=1}^{\infty}\sum_{m=0}^{N-1} \mu^{\star}_m p^{dN} t^{fN-m} + \sum_{N=0}^{\infty} \mathcal{N}^{\star}_{N} (t^{f-1})^N. \qquad (5.3)\end{aligned}$$

Equation (3.10) implies that $\sum_{N=0}^{\infty} \mathcal{N}^{\star}_N (t^{f-1})^N = \mathrm{P}^{\star}_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1})$. Using this fact and (3.12) we rewrite (5.3) as follows:

$$\begin{aligned}\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) &= \sum_{N=1}^{\infty}\sum_{m=0}^{N-1} \left(\frac{\mathcal{N}^{\star}_m}{p^{dm}} - \frac{\mathcal{N}^{\star}_{m+1}}{p^{d(m+1)}} - \delta_{0,m} p^{-d}\right) p^{dN} t^{fN-m} + \mathrm{P}^{\star}_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1}) \\ &= \sum_{N=1}^{\infty} \left(\sum_{m=0}^{N-1} \left(\frac{\mathcal{N}^{\star}_m}{p^{dm}} - \frac{\mathcal{N}^{\star}_{m+1}}{p^{d(m+1)}}\right) p^{dN} t^{fN-m} - p^{-d} p^{dN} t^{fN}\right) + \mathrm{P}^{\star}_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1}) \\ &= \left(\sum_{N=1}^{\infty}\sum_{m=0}^{N-1} \left(\frac{\mathcal{N}^{\star}_m}{p^{dm}} - \frac{\mathcal{N}^{\star}_{m+1}}{p^{d(m+1)}}\right) p^{dN} t^{fN-m}\right) - \sum_{N=1}^{\infty} p^{d(N-1)} t^{fN} + \mathrm{P}^{\star}_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1})\end{aligned}$$

$$= \sum_{m=0}^{\infty} \left(\frac{\mathcal{N}_m^\star}{p^{dm}} - \frac{\mathcal{N}_{m+1}^\star}{p^{d(m+1)}} \right) t^{-m} \sum_{N=m+1}^{\infty} p^{dN} t^{fN} \qquad (5.4)$$

$$- \sum_{N=1}^{\infty} p^{d(N-1)} t^{fN} + \mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1})$$

$$= \sum_{m=0}^{\infty} \left(\frac{\mathcal{N}_m^\star}{p^{dm}} - \frac{\mathcal{N}_{m+1}^\star}{p^{d(m+1)}} \right) t^{-m} \frac{p^{d(m+1)} t^{f(m+1)}}{1 - p^d t^f} - \frac{t^f}{1 - p^d t^f} + \mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1})$$

$$= \frac{p^d t^f}{1 - p^d t^f} \underbrace{\sum_{m=0}^{\infty} \left(\frac{\mathcal{N}_m^\star}{p^{dm}} - \frac{\mathcal{N}_{m+1}^\star}{p^{d(m+1)}} \right) p^{dm} t^{(f-1)m}}_{=:(A)} - \frac{t^f}{1 - p^d t^f} + \mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1}). \qquad (5.5)$$

We use (3.13) to express (A) in terms of $\mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(s)$:

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{p^d t^f}{1 - p^d t^f} (\mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d) + p^{-d}) - \frac{t^f}{1 - p^d t^f} + \mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1})$$

$$= \frac{p^d t^f}{1 - p^d t^f} (\mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d)) + \frac{t^f}{1 - p^d t^f} - \frac{t^f}{1 - p^d t^f} + \mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1}).$$

The identity (3.15) is used to express $\mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(t)$ in terms of Igusa's local zeta function and complete the proof of the theorem.

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{p^d t^f}{1 - p^d t^f} (\mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d)) + \mathrm{P}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}(p^d t^{f-1})$$

$$= \frac{p^d t^f}{1 - p^d t^f} (\mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d)) + \frac{1 - t^{f-1} - p^d t^{f-1} \, \mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d)}{1 - p^d t^{f-1}}$$

$$= \frac{1 - t^{f-1}}{1 - p^d t^f} + \frac{(t-1) p^d t^{f-1}}{(1 - p^d t^{f-1})(1 - p^d t^f)} \mathrm{Z}^\star_{\tilde{\boldsymbol{\sigma}}_{2f}}((f-1)s - d).$$

□

5.2 Du Sautoy's elliptic curve example

In this section we calculate the representation zeta function of the $\mathcal{T}_2$-group known as du Sautoy's elliptic curve example. Du Sautoy constructed the group in [9] and used it to demonstrate the subgroup zeta function a $\mathcal{T}$-group is not necessarily finitely uniform, answering a question [16, pp. 188] of Grunewald, Segal and Smith in the negative. The group is given by the presentation

$$G = \left\langle \begin{array}{c} x_1, \ldots, x_6 \\ y_1, y_2, y_3 \end{array} \;\middle|\; \begin{array}{c} [x_1, x_4] = y_3, [x_1, x_5] = y_1, [x_1, x_6] = y_2 \\ [x_2, x_4] = y_1, [x_2, x_5] = y_3 \\ [x_3, x_4] = y_2, [x_3, x_6] = y_1 \end{array} \right\rangle \tag{5.6}$$

and has associated structure matrix

$$\mathcal{R}_{L(G)}(\mathbf{Y}) = \begin{pmatrix} 0 & 0 & 0 & Y_3 & Y_1 & Y_2 \\ 0 & 0 & 0 & Y_1 & Y_3 & 0 \\ 0 & 0 & 0 & Y_2 & 0 & Y_1 \\ -Y_3 & -Y_1 & -Y_2 & 0 & 0 & 0 \\ -Y_1 & -Y_3 & 0 & 0 & 0 & 0 \\ -Y_2 & 0 & -Y_1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

Note that the Pfaffian $E(\mathbf{Y}) := \mathrm{Pf}(\mathcal{R}_G(\mathbf{Y})) = Y_1 {Y_3}^2 - {Y_1}^3 + {Y_2}^2 Y_3$ of the structure matrix determines an elliptic curve E. In [9] du Sautoy showed that the local factors of the subgroup zeta function of G are closely linked to the number of $\mathbb{F}_p$-rational points of the elliptic curve described by $E(\mathbf{Y})$. As a consequence of this fact he deduced that the subgroup zeta function of G is not finitely uniform. Later, in [37], Voll gave an explicit formula for almost all of the local factors of the normal subgroup zeta function of G. In this section we compute $\widetilde{\zeta^{\mathrm{irr}}_G}(s)$, including all local factors.

Theorem 5.4. *Let G be as above. Then*

$$\widetilde{\zeta^{\mathrm{irr}}_G}(s) = \prod_p \left(\frac{1-t^3}{1-p^3t^3} + E(\mathbb{F}_p) \cdot \frac{(p-1)(t-1)t}{(1-p^2t^2)(1-p^3t^3)} \right),$$

where $E(\mathbb{F}_p) := \#\{\mathbf{b} \in \mathbb{P}^2(\mathbb{F}_p) \mid E(\mathbf{b}) = 0\}$ and $t := p^{-s}$.

Proof. First we show that G has simple elementary divisor type of length 6. It is sufficient to show that, for all $N \in \mathbb{N}$ and $\mathbf{b} \in (\mathbb{Z}/p^N)^3 \setminus p(\mathbb{Z}/p^N)^3$, the set of 4-minors of $\mathcal{R}_L(G)(\mathbf{b})$ contains a p-adic unit. This is established in [37, Section 1]. Clearly, G satisfies the conditions established in Section 2.4. Therefore, the formula presented in Theorem 5.3 is valid for all primes p. That is,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{1-t^2}{1-p^3t^2} + \frac{(t-1)p^3t^3}{(1-p^3t^2)(1-p^3t^3)} Z^{\star}_{E(\mathbf{Y})}(2s-3). \tag{5.8}$$

It remains to calculate Igusa local zeta function associated with the Pfaffian $E(\mathbf{Y})$.

$$Z^{\star}_{E(\mathbf{Y})}(s) = \int_{\mathbb{Z}_p^3 \smallsetminus p\mathbb{Z}_p^3} |Y_1{Y_3}^2 - {Y_1}^3 + {Y_2}^2Y_3|^s d\mu.$$

Suppose that $p \neq 2$. Then $E(\mathbf{Y}) = Y_1{Y_3}^2 - {Y_1}^3 + {Y_2}^2Y_3$ satisfies the hypothesis of Proposition 3.11. Therefore,

$$Z^{\star}_{E(\mathbf{Y})}(s) = (p^3 - 1 - |E(\mathbb{F}_p)|)p^{-3} + |E(\mathbb{F}_p)|\frac{(1-p^{-1})p^{-3-s}}{1-p^{-1-s}}.$$

By substituting this expression into (5.8) and performing some algebraic manipulation the formula for the local zeta function is

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{1-t^3}{1-p^3t^3} + E(\mathbb{F}_p)\frac{(p-1)(t-1)t}{(1-p^2t^2)(1-p^3t^3)}.$$

For $p=2$, Proposition 3.11 is not applicable to the coset $(1,0,1)+2\mathbb{Z}_2^3$. In essence, Proposition 3.11 says that the solutions of $E(\mathbf{Y})$ modulo p lift uniformly to solutions modulo p^k, for $k \geq 1$. In the case $p=2$, a generalisation of Hensel's Lemma [34, Section 2.2] can be applied to show that solutions of $E(\mathbf{Y})$ modulo 2^3 lift uniformly. The analysis in this case must be done modulo 2^3.

$$\int_{(1,0,1)+2\mathbb{Z}_2^3} |E(\mathbf{Y})|^s d\mu = \sum_{\substack{\mathbf{b}\in\mathbb{Z}^3/2^3 \\ \mathbf{b}\equiv(1,0,1) \mod 2}} \int_{\mathbf{b}+8\mathbb{Z}_2} |E(\mathbf{Y})|^s d\mu.$$

The sum is really over a set a representatives for the cosets of $2^3\mathbb{Z}_2$, but we choose these to be the Teichmüller representatives and identify them with elements of $\mathbb{Z}^3/2^3$. If $E(\mathbf{b}) \not\equiv 0$ modulo 8, then the summand corresponding to $\mathbf{b}$ has constant valuation and thus

$$\int_{\mathbf{b}+8\mathbb{Z}_2} |E(\mathbf{Y})|^s d\mu = \mu(\mathbf{b}+8\mathbb{Z}_2)p^{-v_p(E(\mathbf{b}))}.$$

If $E(\mathbf{b}) \equiv 0$ modulo 8, then using [34, Lemma 1, Section 2.2] the solutions lift uniformly. In this specific example, it happens that the solutions of $E(\mathbf{b}) \equiv 0$ modulo 2 lift uniformly, but this is not a direct consequence of Hensel's Lemma or one of its generalisations. The formula is therefore the same as in the case $p \neq 2$. □

Corollary 5.5. *The representation zeta function $\widetilde{\zeta_G^{\mathrm{irr}}}(s)$ of du Sautoy's elliptic curve example G is not finitely uniform.*

Proof. It is shown in [9, Section 1] that if almost all the local factors of a subgroup zeta function enumerating subgroups have the form

$$\zeta_{G,p}(s) = W_1(p, p^{-s}) + |E(\mathbb{F}_p)|W_2(p, p^{-s}),$$

where $W_1(X,Y), W_2(X,Y) \in \mathbb{Z}[X,Y]$ and $W_2(X,Y) \neq 0$. Then the subgroup zeta function is not finitely uniform. As the local factors of the representation zeta function of du Sautoy's Elliptic curve example have this form we are done. □

5.3 Non-principal ideal example

In this section we compute the representation zeta function of a family of $\mathcal{T}_2$-groups that are of simple elementary divisor type. However, the structure matrices of the members of this family are not of maximal rank and so the elementary divisor types depend on a non-principal ideal.

For $1 \leq r < d$ let $G(r,d)$ be the $\mathcal{T}_2$-group given by the presentation

$$G(r,d) = \left\langle \begin{array}{c} x_1, \dots, x_{d+1} \\ y_1, \dots, y_d \end{array} \middle| \begin{array}{cc} [x_1, x_i] = y_{i-1} & 2 \leq i \leq r+1 \\ [x_2, x_j] = y_j & r+1 \leq j \leq d \end{array} \right\rangle,$$

with associated structure matrix

$$\mathcal{R}_{L(G(r,d))}(\mathbf{Y}) = \begin{pmatrix} 0 & Y_1 & \dots & Y_r & 0 & \dots & 0 \\ -Y_1 & 0 & \dots & 0 & Y_{r+1} & \dots & Y_d \\ \vdots & \vdots & & & & & \\ -Y_r & 0 & & & & & \\ 0 & -Y_{r+1} & & & & & \\ \vdots & \vdots & & & & & \\ 0 & -Y_d & & & & & \end{pmatrix}.$$

The group $G(r,d)$ has centre of Hirsch length d. For all $N \in \mathbb{N}$ and $\mathbf{b} \in (Z/p^N)^d \setminus p(Z/p^N)^d$ the matrix $\mathcal{R}_{G(r,d)}(\mathbf{b})$ has rank 4. Therefore, $\nu(\mathcal{R}_{G(r,d)}(\mathbf{b})) = (0,0,m,m,N,\dots,N)$ and so $G(r,d)$ has simple elementary type of length 2. Let $\boldsymbol{\sigma}_4(\mathbf{Y})$ denote the set of 4-minors of $\mathcal{R}_{G(r,d)}(\mathbf{Y})$. For almost all primes p, it follows from Theorem 5.3 that

$$\widetilde{\zeta^{\mathrm{irr}}_{G(r,d),p}}(s) = \frac{1-t}{1-p^d t} + \frac{(t-1)p^d t}{(1-p^d t)(1-p^d t^2)} \cdot \mathrm{Z}^{\star}_{\boldsymbol{\sigma}_4}(s-d). \tag{5.9}$$

The computation of $\widetilde{\zeta^{\mathrm{irr}}_{G(r,d),p}}(s)$ has been reduced to calculating $\mathrm{Z}^{\star}_{\boldsymbol{\sigma}_4}(s)$. By Proposition 2.7, $|\boldsymbol{\sigma}_4(\mathbf{b})| = |\tilde{\boldsymbol{\sigma}}_4(\mathbf{b})|$, where $\tilde{\boldsymbol{\sigma}}_4(\mathbf{Y})$ denotes the set of principal 4-minors of $\mathcal{R}_{G(r,d)}(\mathbf{Y})$. Clearly, $\tilde{\boldsymbol{\sigma}}_4(\mathbf{Y}) = \{Y_i^2 Y_j^2 \mid i \in [2,r], j \in [r+1,d]\} \cup \{0\}$. Therefore, using Fubini's Theorem and Example 3.14,

$$\begin{aligned}
\mathrm{Z}^{\star}_{\boldsymbol{\sigma}_4}(s) &= \int_{\mathbb{Z}_p^d \setminus p\mathbb{Z}_p^d} ||\ \{Y_i Y_j \mid i \in [2,r], j \in [r+1,d]\}\ ||^s d\mu \\
&= (1-p^{-d-2s}) \int_{\mathbb{Z}_p^d} ||\ \{Y_i Y_j \mid i \in [2,r], j \in [r+1,d]\}\ ||^s d\mu \\
&= (1-p^{-d-2s}) \int_{\mathbb{Z}_p^d} ||Y_2,\dots,Y_r||^s ||Y_{r+1},\dots,Y_d||^s d\mu \\
&= (1-p^{-d-2s}) \int_{\mathbb{Z}_p} d\mu \int_{\mathbb{Z}_p^{r-1}} ||Y_2,\dots,Y_r||^s d\mu \int_{\mathbb{Z}_p^{d-r}} ||Y_{r+1},\dots,Y_d||^s d\mu \\
&= (1-p^{-d-2s}) \cdot \frac{1-p^{1-r}}{1-p^{1-r-s}} \cdot \frac{1-p^{r-d}}{1-p^{r-d-s}}.
\end{aligned}$$

The third equality follows from the fact that for two ideals $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{Z}[Y_1,\dots,Y_d]$ and $\mathbf{b} \in \mathbb{Z}_p^d$ we have $|(\mathbf{F}_1\mathbf{F}_2)(\mathbf{b})| = |\mathbf{F}_1(\mathbf{b})||\mathbf{F}_2(\mathbf{b})|$. The final expression for

$\zeta^{\widetilde{\mathrm{irr}}}_{G(r,d),p}(s)$ is achieved by substitution of the above expression for $\mathrm{Z}^{\star}_{\boldsymbol{\sigma}_4}(s)$ into equation (5.9). The p-local and global representation zeta functions are

$$\zeta^{\widetilde{\mathrm{irr}}}_{G(r,d),p}(s) = \frac{(1-t)(1-pt)}{(1-p^{d-r+1}t)(1-p^r t)}$$

and

$$\zeta^{\widetilde{\mathrm{irr}}}_{G(r,d)}(s) = \frac{\zeta(s-r)\zeta(s-d+r-1)}{\zeta(s)\zeta(s-1)}.$$

Therefore, $\alpha^{\widetilde{\mathrm{irr}}}(G(r,d)) = \max\{r+1, d-r+2\}$.

5.4 $D^{\star}$-groups

In this section we study the representation zeta function of $\mathcal{T}_2$-group with centre of Hirsch length 2 The main result of this section of the following.

Theorem 5.6. *Let G be a $\mathcal{T}_2$-group with centre of Hirsch length 2. Then $\zeta^{\widetilde{\mathrm{irr}}}_{G}(s)$ is finitely uniform.*

In general, a group H is called *radicable* if for every $x \in H$ and $m \in \mathbb{N}$ there exists $y \in H$ with $y^m = x$. For a $\mathcal{T}$-group G, we say that G, is radicable if $G^{\mathbb{Q}}$, the set of $\mathbb{Q}$-points of G is radicable. For a precise definition of $G^{\mathbb{Q}}$ see [15, Section 1].

Definition 5.7. A $D^{\star}$-group is a radicable $\mathcal{T}_2$-group with centre of Hirsch length 2.

In [15] Grunewald and Segal gave a classification of $\mathcal{T}_2$-groups with centre of Hirsch length 2 up to commensurability. They showed that every such group is commensurable to a $D^{\star}$-group. In order to prove Theorem 5.6 it is sufficient to show that the representation zeta function of $D^{\star}$-group is finitely uniform.

First the definition and classification of $D^{\star}$-groups are recalled. The classification shows that a $D^{\star}$-group can be broken up into indecomposable constituents. The representation zeta functions of indecomposable $D^{\star}$-groups are computed next. Finally, an expression for a general $D^\star$-group is derived and it is shown that the representation zeta function of a $\mathcal{T}_2$-group with centre of Hirsch length 2 is finitely uniform.

Definition 5.8. Let G be a $D^\star$-group. A central decomposition of G is a family $\{\Lambda_1, \dots, \Lambda_k\}$ of subgroups of G such that:

(1) $Z(\Lambda_i) = Z(G)$ for each $i \in [1, k]$,

(2) $G/Z(G)$ is the direct product the subgroups $\Lambda_i/Z(G)$,

(3) For $i, j \in [1, k]$, $[\Lambda_i, \Lambda_j] = 1$ whenever $i \neq j$.

The group G is called indecomposable if the only such decomposition is $\{G\}$.

Theorem 5.9. [15, Theorem 6.2] *Every $D^\star$-group has a central decomposition into indecomposable constituents, and the decomposition is unique, up to automorphisms of G. In particular the constituents are unique up to isomorphism.*

Theorem 5.10. [15, Theorem 6.3] *Let G be an indecomposable $D^\star$-group of Hirsch length $n+2$. Then there exists a Mal'cev basis $\{x_1, \dots, x_n, y_1, y_2\}$ for G such that the structure matrix of G has the following form.*

$$\mathcal{R}_{L(G)}(\mathbf{Y}) = \begin{pmatrix} 0 & B(\mathbf{Y}) \\ -B(\mathbf{Y})^T & \end{pmatrix},$$

where, if $n = 2m+1$,

$$B(\mathbf{Y}) = \begin{pmatrix} Y_1 & Y_2 & & & \\ & Y_1 & Y_2 & & \\ & & \ddots & \ddots & \\ & & & Y_1 & Y_2 \end{pmatrix} \in \mathrm{Mat}_{m,m+1}(\mathbb{Z}[Y_1, Y_2])$$

and if $n = 2m$*,*

$$B(\mathbf{Y}) = \begin{pmatrix} Y_1 & Y_2 & & & \\ & Y_1 & Y_2 & & \\ & & \ddots & \ddots & \\ & & & Y_1 & Y_2 \\ a_m Y_2 & a_{m-1} Y_2 & \dots & a_2 Y_2 & Y_1 + a_1 Y_2 \end{pmatrix} \in \mathrm{Mat}_m(\mathbb{Z}[Y_1, Y_2]),$$

where $\det(B(Y_1, 1)) = Y_1^m - a_1 Y_1^{m-1} - \cdots - a_{m-1} Y_1 - a_m$ *is a power of an irreducible polynomial over* $\mathbb{Q}$*.*

Furthermore, if G *is any* $D^\star$*-group then* $\mathcal{R}_{L(G)}(\mathbf{Y})$ *is a diagonal sum of matrices representing indecomposable constituents.*

Proposition 5.11. *Let* G *be an indecomposable* $D^\star$*-group of Hirsch length* $n+2$*, where* $n = 2m+1$ *with* $m > 1$*. Then for all primes* p*,*

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{1 - p^{-ms}}{1 - p^{2-ms}}.$$

Proof. The dimension of $\mathcal{R}_G(\mathbf{Y})$ is n. For all $N \in \mathbb{N}$ and $\mathbf{b} \in (\mathbb{Z}/p^N)^2 \smallsetminus p(\mathbb{Z}/p^N)^2$ the matrix $\mathcal{R}_G(\mathbf{b})$ has a $2m$-minor which is a unit. Consider the $2m$-minors obtained from $\mathcal{R}_G(\mathbf{b})$ by deleting the $m+1$ row and column and the $2m+1$ row and column. These minors are b_1^{2m} and b_2^{2m} respectively. Since one of b_1 and b_2 is a unit it follows that

$$\nu(\mathcal{R}_G(\mathbf{b})) = (\underbrace{0, \dots, 0}_{2m}, N).$$

Therefore, by Theorem 2.9,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{N=1}^{\infty} (1 - p^{-2}) p^{2N} p^{-mNs} = 1 + \frac{(1 - p^{-2}) p^{2-ms}}{1 - p^{2-ms}} = \frac{1 - p^{-ms}}{1 - p^{2-ms}}.$$

□

Proposition 5.12. *Let* G *be an indecomposable* $D^\star$*-group of Hirsch length* $n+2$*, where* $n = 2m$ *with* $m > 1$*. Let* $F(Y_1, Y_2)$ *be the Pffafian of* $\mathcal{R}_{L(G)}(\mathbf{Y})$*.*

There exists $e \in \mathbb{N}$ and an irreducible polynomial $f(Y)$ such that $F(Y_1, 1) = f(Y_1)^e$. Then, for almost all primes p,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{1-t^m}{1-p^2t^m} + \ell \cdot \frac{(p-1)(t-1)(1-p^{2e}t^{(m-1)e})}{(1-p^2t^{m-1})(1-p^2t^m)(1-p^{2e-1}t^{(m-1e)})},$$

where ℓ is the number of roots of $\bar{f}(Y)$, the reduction modulo p of $f(Y)$.

Proof. The structure matrix $\mathcal{R}_{L(G)}(\mathbf{Y})$ has Pfaffian $\mathrm{Pf}(\mathcal{R}_{L(G)}(\mathbf{Y})) = Y_1^m + c_2Y_1^{m-1}Y_2 + \cdots + c_mY_m =: F(\mathbf{Y})$. The $2m-2$-minors given by removing the $(m-1, m+1)$ rows and columns and by removing the $(m-1, 2m)$ rows and columns are Y_1^{2m-2} and Y_2^{2m-2} respectively. Therefore, G has simple elementary divisor type of length m and by Theorem 5.3,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{1-t^{m-1}}{1-p^2t^{m-1}} + \frac{(t-1)p^2t^{m-1}}{(1-p^2t^{m-1})(1-p^2t^m)} Z_F^\star((m-1)s-2).$$

The computation of $\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s)$ has been reduced to computing $Z_F^\star(s)$. Now consider the coset decomposition

$$Z_F^\star(s) = \sum_{\mathbf{a}\in\mathbb{F}_p^2\smallsetminus\{\mathbf{0}\}} \int_{\mathbf{a}+p\mathbb{Z}_p^2} |F(Y_1,Y_2)|^s d\mu. \tag{5.10}$$

Here, strictly speaking, the summation is not over $\mathbf{a} = (a_1, a_2) \in \mathbb{F}_p^2 \smallsetminus \{\mathbf{0}\}$, but rather over the representatives of the non-zero cosets of $p\mathbb{Z}_p^2$. If $a_2 \equiv 0$ modulo p then $a_1 \not\equiv 0$ modulo p and $v_p(F(a_1, a_2)) = 0$. In this case

$$\int_{\mathbf{a}+p\mathbb{Z}_p^2} |F(Y_1,Y_2)|^s d\mu = \mu(\mathbf{a} + p\mathbb{Z}_p^2) = p^{-2}.$$

In all other cases we have $a_2 \not\equiv 0$ modulo p. Now,

$$\int_{\substack{\mathbf{a}+p\mathbb{Z}_p^2\\ a_2\not\equiv 0 \mod p}} |F(Y_1,Y_2)|^s d\mu = \int_{(a_1a_2^{-1},1)+p\mathbb{Z}_p^2} |F(Y_1,Y_2)|^s d\mu,$$

by a change of variables. Therefore,

$$\sum_{\substack{\mathbf{a}\in\mathbb{F}_p^2\\ a_2\not\equiv 0 \mod p}} \int_{\mathbf{a}+p\mathbb{Z}_p^2} |F(Y_1,Y_2)|^s d\mu = (p-1)\sum_{a_1\in\mathbb{F}_p} \int_{(a_1,1)+p\mathbb{Z}_p^2} |F(Y_1,Y_2)|^s d\mu$$

$$= (p-1)p^{-1}\sum_{a_1\in\mathbb{F}_p} \int_{a_1+p\mathbb{Z}_p} |F(Y_1,1)|^s d\mu.$$

By assumption, $F(Y_1, 1) = f(Y_1)^e$, where f is irreducible over $\mathbb{Q}$, and so,

$$\sum_{a_1 \in \mathbb{F}_p} \int_{a_1 + p\mathbb{Z}_p} |F(Y_1, 1)|^s d\mu = \int_{\mathbb{Z}_p} |f(Y_1)|^{es} d\mu$$

$$= Z_f(es).$$

For almost all primes p, Igusa's local zeta function associated with an irreducible polynomial f has a form which only depends on the splitting behaviour of the prime p in the ring of integers in the splitting field of f. Let K denote the splitting field of f and let $\mathcal{O}$ denote the ring of integers in K.

Suppose that the prime p is unramified in $\mathcal{O}$ (which is the case for all but finitely many primes) and that $p\mathcal{O} = \mathcal{P}_1 \dots \mathcal{P}_k$, where $\mathcal{P}_1, \dots, \mathcal{P}_k$ are prime ideals of $\mathcal{O}$. Then, the reduction modulo p of f has the form $\bar{f} = \bar{f}_1 \dots \bar{f}_k$ where $\bar{f}_1, \dots, \bar{f}_k$ are irreducible polynomials over $\mathbb{F}_p$. For each $\bar{f}_i$ we have either $\bar{f}_i(a) \not\equiv 0$ modulo p for all $a \in \mathbb{F}_p$ or $\bar{f}_i(Y_1) = Y_1 - \alpha_i$ for some $\alpha_i \in \mathbb{F}_p$.

Furthermore, the roots of $\bar{f}$ in $\mathbb{F}_p$ are distinct and each lifts, via Hensel's Lemma, to a unique root of f in $\mathbb{Z}_p$. Let ℓ denote the number of solutions of $f(Y_1) \equiv 0$ modulo p. We have,

$$Z_f(s) = \int_{\mathbb{Z}_p} |f(Y_1)|^s d\mu$$

$$= \int_{\mathbb{Z}_p} |(Y_1 - \alpha_1) \dots (Y_1 - \alpha_\ell)|^s d\mu,$$

where the α_i are distinct, and so,

$$Z_f(s) = \ell \cdot \frac{(1 - p^{-1})p^{-1-s}}{1 - p^{-1-s}} + (p - \ell)p^{-1}.$$

All the summands of equation (5.10) are now understood. By summing and performing some algebraic reductions the result is achieved. $\square$

Theorem 5.13. *Let G be an arbitrary $D^\star$-group. There exists $\ell \in \mathbb{N}$ and irreducible polynomials $F_1(Y), \dots, F_\ell(Y) \in \mathbb{Q}[Y]$ and rational functions $W_0(Y_1, Y_2), \dots, W_\ell(Y_1, Y_2)$ such that for almost all primes p*

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = W_0(p, p^{-s}) + \sum_{i=1}^{\ell} F_i(\mathbb{F}_p) W_i(p, p^{-s}),$$

where

$$F_i(\mathbb{F}_p) := \#\{x \in \mathbb{F}_p \mid F_i(x) \equiv 0 \mod p\}.$$

Proof. Let $\{\Lambda_1, \ldots, \Lambda_{m+n}\}$ be a central decomposition of G where $\Lambda_1, \ldots, \Lambda_m$ are indecomposable $D^\star$-groups of even Hirsch length and $\Lambda_{m+1}, \ldots, \Lambda_{m+n}$ indecomposable $D^\star$-groups of odd Hirsch length. The structure matrix $\mathcal{R}_G(Y_1, Y_2)$ is the diagonal sum of the structure matrices $\mathcal{R}_{\Lambda_i}(Y_1, Y_2)$.

Denote the Hirsch length of Λ_i by $k_i + 2$ so that the Hirsch length of G is equal to $k_1 + \cdots + k_{m+n} + 2$. Let $\boldsymbol{\sigma}_{2r}$ denote the set of $2r$-minors of $\mathcal{R}_G(Y_1, Y_2)$. For $\mathbf{b} \in (\mathbb{Z}/p^N)^2 \smallsetminus p(\mathbb{Z}/p^N)^2$, the matrices $\mathcal{R}_{\Lambda_i}(b_1, b_2)$ of the indecomposable constituents of G have large unit minors. Precisely, for $i \in [1, m]$ each $\mathcal{R}_{\Lambda_i}(b_1, b_2)$ has a unit $(k_i - 2)$-minor and for $i \in [m+1, m+n]$ each $\mathcal{R}_{\Lambda_i}(b_1, b_2)$ has a unit $(k_i - 1)$-minor. Further, for $i \in [m+1, m+n]$ each $\mathcal{R}_{\Lambda_i}(b_1, b_2)$ has rank $k_i - 1$.

These observations imply that $\mathcal{R}_{\Lambda_i}(b_1, b_2)$ has an elementary divisor type of a very specific 'shape'.

$$\nu(\mathcal{R}_{\Lambda_i}(b_1, b_2)) = (\underbrace{0, \ldots, 0}_{(\sum k_i) - 2m - n}, \kappa_1, \kappa_1, \ldots, \kappa_m, \kappa_m, \underbrace{N, \ldots, N}_{n}),$$

Let $f_i(Y) \in \mathbb{Q}[Y]$ be the irreducible polynomial determined by the indecomposable component Λ_{m+i}. That is, if $\mathrm{Pf}_i(Y_1, Y_2)$ denotes the Pfaffian of $\mathcal{R}_{\Lambda_i}(Y_1, Y_2)$. Then, for $i \in [1, n]$ there exists an irreducible polynomial f_i and $e_i \in \mathbb{N}$ such that $\mathrm{Pf}_i(Y, 1) = f_i(Y)^{e_i}$.

Let $\boldsymbol{\sigma}_{2r}$ denote the set of Pfaffians of the principal $2r$-minors of $\mathcal{R}_G(Y_1, Y_2)$ and $k := \frac{1}{2}(\sum k_i - n)$. By Theorem 2.10,

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \frac{1}{(1-p^{-1})^2(1-p^{-2})} \underbrace{\int_{\substack{p\mathbb{Z}_p \times \mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2 \\ (X, \mathbf{Y})}} |X|^{\tilde{s}} \prod_{\kappa \in [1,k]} \frac{|\boldsymbol{\sigma}_{2\kappa} \cup X\boldsymbol{\sigma}_{2\kappa-2}|^s}{|\boldsymbol{\sigma}_{2\kappa}|^s} d\mu}_{=: Z(s, \tilde{s})},$$

where $\tilde{s} = (\sum k_i)s + 1$. Decompose the integral $Z(s, \tilde{s})$ with respect to cosets modulo p and gather the cosets that are equivalent up to multiplication by a unit in $\mathbb{Z}_p$.

$$\begin{aligned}Z(s,\tilde{s}) &= \sum_{\mathbf{a}\in\mathbb{F}_p^2\setminus\mathbf{0}} \int_{p\mathbb{Z}_p\times(\mathbf{a}+p\mathbb{Z}_p^2)} |X|^{\tilde{s}} \prod_{\kappa\in[1,k]} \frac{|\boldsymbol{\sigma}_{2\kappa}\cup X\boldsymbol{\sigma}_{2\kappa-2}|^s}{|\boldsymbol{\sigma}_{2\kappa}|^s} d\mu \\ &= (p-1)\sum_{\mathbf{a}\in\mathbb{P}^1(\mathbb{F}_p)} \int_{p\mathbb{Z}_p\times(\mathbf{a}+p\mathbb{Z}_p^2)} |X|^{\tilde{s}} \prod_{\kappa\in[1,k]} \frac{|\boldsymbol{\sigma}_{2\kappa}\cup X\boldsymbol{\sigma}_{2\kappa-2}|^s}{|\boldsymbol{\sigma}_{2\kappa}|^s} d\mu \\ &= (p-1)\sum_{\mathbf{a}\in\mathbb{P}^1(\mathbb{F}_p)} \int_{p\mathbb{Z}_p\times(\mathbf{a}+p\mathbb{Z}_p^2)} |X|^{\tilde{s}} \prod_{i\in[1,m]} |\operatorname{Pf}_i(Y_1,Y_2),X|^s d\mu.\end{aligned}$$

Here we identify $\mathbb{P}^1(\mathbb{F}_p)$ with the following set of representatives of the homothety classes: $\{(a_1,1)\mid a_1\in\mathbb{F}_p\}\cup\{(1,0)\}$ and denote by $\mathbf{a}+p\mathbb{Z}_p^2$ the coset of $p\mathbb{Z}_p^2$ in which all elements are congruent to $\mathbf{a}$ modulo p.

$$\begin{aligned}Z(s,\tilde{s}) = (p-1)\Bigg(&\sum_{\mathbf{a}=(a_1:1)\in\mathbb{P}^1(\mathbb{F}_p)} \int_{p\mathbb{Z}_p\times(a_1+p\mathbb{Z}_p)} |X|^{\tilde{s}} \prod_{i\in[1,m]} |f_i(Y_1)^{e_i},X|^s d\mu \\ &+ \int_{p\mathbb{Z}_p\times((0,1)+p\mathbb{Z}_p^2)} |X|^{\tilde{s}} \prod_{i\in[1,m]} |f_i(Y_1,Y_2),X|^s d\mu\Bigg)\end{aligned}$$

$$\begin{aligned}&= (p-1)\Bigg(\sum_{\mathbf{a}=(a_1:1)\in\mathbb{P}^1(\mathbb{F}_p)} \int_{p\mathbb{Z}_p\times(a_1+p\mathbb{Z}_p)} |X|^{\tilde{s}} \prod_{i\in[1,m]} |f_i(Y_1)^{e_i},X|^s d\mu \\ &+ p^{-2}\frac{(1-p^{-1})p^{-1-\tilde{s}}}{1-p^{-1-\tilde{s}}}\Bigg)\end{aligned} \tag{5.11}$$

For p a sufficiently large prime, two distinct irreducible polynomials g_1 and g_2 over $\mathbb{Q}$ have distinct roots modulo p. Indeed, for polynomials g_1, g_2 the resultant $\operatorname{Res}(g_1,g_2)$ is defined as

$$\operatorname{Res}(g_1,g_2) = \prod_{\substack{\alpha_1,\alpha_2\in\overline{\mathbb{Q}} \\ g_1(\alpha_1)=0,\, g_2(\alpha_2)=0}} (\alpha_1-\alpha_2).$$

Note that if g_1, g_2 are irreducible polynomials then $\operatorname{Res}(g_1,g_2)$ is a rational number. Therefore, if g_1, g_2 have a simultaneous root modulo p then the p-adic valuation $v_p(\operatorname{Res}(g_1,g_2))$ of the resultant $\operatorname{Res}(g_1,g_2)$ is non-zero, but this can only happen for finitely many primes.

Suppose that $\{f_1(Y), \ldots, f_m(Y)\} = \{F_1(Y), \ldots, F_\ell(Y)\}$, where the $F_i(Y)$ are distinct. For $a_1 \in \mathbb{F}_p$ and for p sufficiently large a_1 can only be a root modulo p of one of the ℓ distinct irreducibles polynomials. Suppose that a_1 is a root of F_j for $j \in [1, \ell]$ and let $S_j := \{i \in [1, m] \mid f_i(Y) = F_j(Y)\}$ then,

$$\begin{aligned}\int_{p\mathbb{Z}_p \times (a_1 + p\mathbb{Z}_p)} |X|^t \prod_{i \in [1,m]} |f_i(Y_1)^{e_i}, X|^s d\mu &= \int_{p\mathbb{Z}_p \times (a_1 + p\mathbb{Z}_p)} |X|^t \prod_{i \in S_j} |F_j(Y_1)^{e_i}, X|^s d\mu \\ &= \int_{p\mathbb{Z}_p \times p\mathbb{Z}_p} |X|^t \prod_{i \in S_j} |Y_1^{e_i}, X|^s d\mu.\end{aligned}$$

The second equality follows from the fact that a is a simple root of F_j. If $a \in \mathbb{F}_p$ is not a root of any of the F_j then

$$\int_{p\mathbb{Z}_p \times (a_1 + p\mathbb{Z}_p)} |X|^t \prod_{i \in [1,m]} |f_i(Y_1)^{e_i}, X|^s d\mu = p^{-1} \int_{p\mathbb{Z}_p} |X|^t d\mu$$

In either case, the integral is a rational function in p, p^{-s} whose coefficients do not depend on p. The summands of expression (5.11) are now known and the multplicity of each summand is given by $F_i(\mathbb{F}_p)$. □

Corollary 5.14. *The representation zeta function of a $\mathcal{T}_2$-group G with centre of Hirsch length 2 is finitely uniform.*

Proof. For any $\mathcal{T}$-group G and finite-index subgroup H and for almost all primes p we have $\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{H,p}}(s)$. A $\mathcal{T}_2$-group with derived group of Hirsch length 2 is commensurable to a $D^\star$-group. Therefore, it suffices to prove the statement for a $D^\star$-group. Almost all the local factors of the representation zeta function of G are given by the formula given in the theorem. The numbers $F_i(\mathbb{F}_p)$ are determined by the splitting behaviour of the ideal (p) in the ring of integers of the splitting field of F_i. Since there are only finitely many splitting behaviours possible, the result follows. □

Chapter 6

Computations

This chapter presents computations of the representation zeta function of several (families of) $\mathcal{T}_2$-groups. In each computation we give a presentation of a $\mathcal{T}_2$-group G and, after implicitely choosing a basis for $L(G)$, the structure matrix $\mathcal{R}_{L(G)}(\mathbf{Y})$. All examples in this chapter satisfy the conditions in Section 2.4 that imply the Theorem 2.9 is valid for all primes p.

Example 6.1. *The Heisenberg group.* Let H denote the Heisenberg group, which has presentation $\langle x_1, x_2, y_1 \mid [x_1, x_2] = y_1 \rangle$. The fact that $\widetilde{\mathrm{r}}_n(H) = \phi(n)$ was first shown in [33, Theorem 5] by direct computation of the twist-isoclasses. It was later noted in [20] that this implies that

$$\zeta_H^{\widetilde{\mathrm{irr}}}(s) = \frac{\zeta(s-1)}{\zeta(s)}. \tag{6.1}$$

The representation zeta function of H is also calculated using the method of elementary divisors in [24, Chapter 3, Example 3.4]. The computation is recorded here for completeness. The structure matrix $\mathcal{R}_{L(H)}(Y)$ of the associated Lie ring $L(H)$ is as follows:

$$\mathcal{R}_{L(H)}(Y) = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}.$$

Theorem 2.9 implies that

$$\widetilde{\zeta^{\mathrm{irr}}_{G_n,p}}(s) = \sum_{\substack{N\in\mathbb{N}\\ m\in\mathbb{N}}} \mathcal{N}_{N,m} p^{-s(N-m)},$$

where $\mathcal{N}_{N,m} = \#\{b \in (\mathbb{Z}/p^N) \setminus p(\mathbb{Z}/p^N) \mid \nu(\mathcal{R}(Y)) = (m,m)\}$. It then follows that

$$\mathcal{N}_{N,m} = \begin{cases} 1 & \text{if } N = m = 0, \\ (1-p^{-1})p^N & \text{if } N > 0, m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \widetilde{\zeta^{\mathrm{irr}}_{G_n,p}}(s) &= 1 + \sum_{N\in\mathbb{N}} (1-p^{-1}) p^{N(1-s)} \\ &= 1 + (1-p^{-1}) \frac{p^{1-s}}{1-p^{1-s}} = \frac{1-p^{-s}}{1-p^{1-s}}. \end{aligned}$$

and after taking the Euler product of the local factors, formula (6.1) is recovered.

Example 6.2. *Grenham's groups.* Let G_n denote Grenham's group given by the presentation

$$G_n = \left\langle \begin{matrix} x_0, \dots, x_{n-1} \\ y_1, \dots, y_{n-1} \end{matrix} \;\middle|\; [x_0, x_i] = y_i,\ 1 \le i \le n \right\rangle.$$

The associated structure matrix for $L(G_n)$ is

$$\mathcal{R}_{L(G_n)}(\mathbf{Y}) = \begin{pmatrix} 0 & Y_1 & \dots & Y_{n-1} \\ -Y_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -Y_{-1} & 0 & \dots & 0 \end{pmatrix}.$$

For any $\mathbf{b} = (b_1, \dots, b_{n-1}) \in (\mathbb{Z}/p^N)^{n-1} \setminus (\mathbb{Z}/p^N)^{n-1}$ the matrix $\mathcal{R}_{L(G_n)}(\mathbf{b})$ has rank 2. Thus, $\nu(\mathcal{R}_{L(G_n)}(\mathbf{b})) = (m_1, m_1, N, \dots, N)$ and by Lemma 2.4,

$m_1 = 0$ and

$$\widetilde{\zeta^{\mathrm{irr}}_{G_n,p}}(s) = 1 + \sum_{N\in\mathbb{N}} (1 - p^{-(n-1)}) p^{(n-1)N} p^{-Ns}$$
$$= 1 + (1 - p^{-n+1}) \frac{p^{(n-1)-s}}{1 - p^{(n-1)-s}}$$
$$= \frac{1 - p^{-s}}{1 - p^{(n-1)-s}}$$

By taking the Euler product of the local factors the global zeta function is

$$\widetilde{\zeta^{\mathrm{irr}}_{G_n}}(s) = \frac{\zeta(s-n+1)}{\zeta(s)}.$$

The representation zeta function $\widetilde{\zeta^{\mathrm{irr}}_{G_n}}(s)$ converges for $\Re(s) > n$ and has meromorphic continuation to the whole complex plane.

Example 6.3. Let B_n denote the $\mathcal{T}_2$-group given by the following presentation:

$$B_n = \langle x_1, \dots x_n, y_1 \mid [x_i, x_{i+1}] = y_1,\ 1 \le i \le n-1 \rangle.$$

It has associated structure matrix

$$\mathcal{R}_{L(B_n)}(Y) = \begin{pmatrix} 0 & Y & 0 & & & \\ -Y & 0 & Y & & & \\ 0 & -Y & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & Y \\ & & & & -Y & 0 \end{pmatrix}.$$

For $b \in \mathbb{Z}/p^N \setminus p(\mathbb{Z}/p^N)$ and $n = 2f$ or $2f+1$ the matrix $\mathcal{R}_{L(B_n)}(b)$ has rank $2f$. If $n = 2f$ then $\nu(\mathcal{R}_{L(B_n)}(b)) = (\underbrace{0, \dots, 0}_{2f})$ and if $n = 2f+1$ then $\nu(\mathcal{R}_{L(B_n)}(b)) = (\underbrace{0, \dots, 0}_{2f}, N)$. In either case, the local factors of the representation zeta function are the same.

$$\widetilde{\zeta^{\mathrm{irr}}_{B_n,p}}(s) = 1 + \sum_{N\in\mathbb{N}} (1 - p^{-1}) p^N p^{-fNs}$$
$$= 1 + (1 - p^{-1}) \frac{p^{1-fs}}{1 - p^{1-fs}} = \frac{1 - p^{-fs}}{1 - p^{1-fs}}.$$

By taking the Euler product of the local factors the global zeta function is

$$\widetilde{\zeta^{\mathrm{irr}}_{B_n}}(s) = \frac{\zeta(fs-1)}{\zeta(fs)}.$$

The representation zeta function $\widetilde{\zeta^{\mathrm{irr}}_{B_n}}(s)$ converges for $\Re(s) > 2/f$ and has meromorphic continuation to the whole complex plane.

Lemma 6.4. *For $m, n, a, b \in \mathbb{N}$ and writing $t = p^{-s}$,*

$$\begin{aligned} A(s) &:= \sum_{\substack{N_1 \in \mathbb{N} \\ N_2 \in \mathbb{N}}} p^{mN_1 + nN_2} p^{-s(aN_1 + bN_2 - \min\{N_1, N_2\})} \\ &= \frac{p^{m+n} t^{a+b-1}(1 - p^{m+n} t^{a+b})}{(1 - p^m t^a)(1 - p^n t^b)(1 - p^{m+n} t^{a+b-1})}. \end{aligned} \tag{6.2}$$

Proof. In order to resolve the minimum on the left hand side of equation (6.2) divide the domain of summation into three parts:

(i) $N_1 > N_2$,

(ii) $N_1 = N_2$,

(iii) $N_1 < N_2$.

Case (i): For $D \in \mathbb{N}$ substitute $N_1 = N_2 + D$ into the left hand side of equation (6.2).

$$A(s)|_{(i)} = \sum_{\substack{N_2 \in \mathbb{N} \\ D \in \mathbb{N}}} p^{(m+n)N_2 + mD} p^{-s((a+b-1)N_2 + aD)} = \frac{p^{m+n} t^{a+b-1}}{1 - p^{m+n} t^{a+b-1}} \cdot \frac{p^m t^a}{1 - p^m t^a}.$$

Case (ii): If $N_1 = N_2$ then

$$A(s)|_{(ii)} = \sum_{N \in \mathbb{N}} p^{(m+n)N} p^{-s(a+b-1)N} = \frac{p^{m+n} t^{a+b-1}}{1 - p^{m+n} t^{a+b-1}}.$$

Case (iii): By symmetry with case (i),

$$A(s)|_{(iii)} = \frac{p^{m+n} t^{a+b-1}}{1 - p^{m+n} t^{a+b-1}} \cdot \frac{p^n t^b}{1 - p^n t^b}.$$

The result is obtained by summing the three cases and some algebraic manipulation. □

Example 6.5. Let G be the $\mathcal{T}_2$-group with the presentation

$$G = \left\langle \begin{array}{c} x_1, \dots x_4 \\ y_1, y_2 \end{array} \middle| \begin{array}{c} [x_1, x_2] = y_1, \ [x_1, x_4] = y_2 \\ [x_2, x_3] = y_2 \end{array} \right\rangle$$

and associated structure matrix

$$\mathcal{R}_{L(G)}(\mathbf{Y}) = \begin{pmatrix} 0 & Y_1 & 0 & Y_2 \\ -Y_1 & 0 & Y_2 & 0 \\ 0 & -Y_2 & 0 & 0 \\ -Y_2 & 0 & 0 & 0 \end{pmatrix}$$

For $\mathbf{b} \in (\mathbb{Z}/p^N)^2 \backslash p(\mathbb{Z}/p^N)^2$ the elementary divisors of $\mathcal{R}_{L(G)}(\mathbf{b})$ depend only on $v_p(b_2)$. The strategy employed is to divide the domain of summation into pieces where the elementary divisors of $\mathcal{R}_{L(G)}(\mathbf{b})$ are constant. By formula (2.8)

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{\substack{N \in \mathbb{N} \\ 0 \leq m \leq N}} \mathcal{N}_{N,m} p^{-s(2N-m)}, \tag{6.3}$$

where $\mathcal{N}_{N,m} = \#\{\mathbf{b} \in (\mathbb{Z}/p^N)^2 \setminus p(\mathbb{Z}/p^N)^2 \mid \nu(\mathcal{R}_{L(G)}(\mathbf{b})) = (0,0,m,m)\}$. The determinant $\det(\mathcal{R}_{L(G)}(\mathbf{Y})) = Y_2^4$ and so $\nu(\mathcal{R}_{L(G)}(\mathbf{b})) = (0,0,m,m)$, where $m = \min\{2v_p(b_2), N\}$. Writing $N_2 := v_p(b_2)$ equation (6.3) becomes

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = 1 + \sum_{\substack{N \in \mathbb{N} \\ 0 \leq N_2 \leq N}} \mathcal{N}_{N,N_2} p^{-s(2N-\min\{2N_2,N\})}, \tag{6.4}$$

where $\mathcal{N}_{N,N_2} = \#\{\mathbf{b} \in (\mathbb{Z}/p^N)^2 \setminus p(\mathbb{Z}/p^N)^2 \mid v_p(b_2) = N_2\}$. The summand of equation (6.4) is split into three cases in order to resolve the minimum.

(i) $N_2 = 0$,

(ii) $0 < N_2 < N$,

(iii) $N_2 = N$.

Case(i): For $N \in \mathbb{N}$ and $N_2 = 0$, $\mathcal{N}_{N,0} = (1-p^{-1})p^{2N}$ and so

$$\sum_{\substack{N\in\mathbb{N}\\N_2=0}} \mathcal{N}_{N,N_2} p^{-s(2N-\min\{2N_2,N\})} = \sum_{N\in\mathbb{N}} (1-p^{-1})p^{2N}p^{-2Ns}$$
$$= (1-p^{-1})\frac{p^{2-2s}}{1-p^{2-2s}}.$$

Case (ii): Write $N = N_2 + D$, where $D \in \mathbb{N}$. In this case $\mathcal{N}_{N,N_2} = (1-p^{-1})^2p^{2N-N_2}$ and

$$\sum_{\substack{N\in\mathbb{N}\\0<N_2<N}} \mathcal{N}_{N,N_2} p^{-s(2N-\min\{2N_2,N\})} = \sum_{\substack{N_2\in\mathbb{N}\\D\in\mathbb{N}}} (1-p^{-1})^2 p^{N_2+2D}p^{-s(N_2+2D-\min\{N_2,D\})}$$
$$= (1-p^{-1})^2\frac{p^3t^2(1-p^3t^3)}{(1-pt)(1-p^2t^2)(1-p^3t^2)},$$

by Lemma 6.4.

Case (iii): If $N_2 = N$, then $\mathcal{N}_{N,N_2} = (1-p^{-1})p^N$ and

$$\sum_{\substack{N\in\mathbb{N}\\N_2=N}} \mathcal{N}_{N,N_2} p^{-s(2N-\min\{2N_2,N\})} = \sum_{N\in\mathbb{N}} (1-p^{-1})p^Np^{-Ns}$$
$$= (1-p^{-1})\frac{pt}{(1-pt)}.$$

Therfore, the local factor of the representation zeta

$$\widetilde{\zeta^{\mathrm{irr}}_{G,p}}(s) = \frac{(1-t)(1-p^2t^2)}{(1-pt)(1-p^3t^2)}$$

and the global zeta function

$$\widetilde{\zeta^{\mathrm{irr}}_{G}}(s) = \frac{\zeta(s-1)\zeta(2s-3)}{\zeta(s)\zeta(2s-2)}.$$

The zeta function $\widetilde{\zeta^{\mathrm{irr}}_{G}}(s)$ converges for $\Re(s) > 2$ and has meromorphic continuation to the whole of $\mathbb{C}$. The continued function has a double pole at $s = 2$.

Example 6.6. *Class-2-nilpotent quotients of $U_d(\mathbb{Z})$.* Let T_n denote the maximal class-2-nilpotent quotient of the (upper) unitriangular $n\times n$ matrix group over $\mathbb{Z}$. The $\mathcal{T}_2$-group T_n has presentation

$$T_n = \langle x_1, \ldots x_n, y_1, \ldots, y_{n-1} \mid [x_i, x_{i+1}] = y_i,\ 1 \le i \le n-1\rangle$$

and associated structure matrix

$$\mathcal{R}_{L(T_n)}(\mathbf{Y}) = \begin{pmatrix} 0 & Y_1 & 0 & & & \\ -Y_1 & 0 & Y_2 & & & \\ 0 & -Y_2 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & Y_{n-1} \\ & & & & -Y_{n-1} & 0 \end{pmatrix}.$$

Note that, $T_2 = H$, the Heisenberg group and $T_3 \cong G_3$, Grenham's group. The first group of the form T_n that is not treated above is T_4. The determinant $\det(\mathcal{R}_{L(T_4)}(\mathbf{Y})) = Y_1^2 Y_3^2$ implies that for $\mathbf{b} \in (\mathbb{Z}/p^N)^3 \setminus p(\mathbb{Z}/p^N)^3$ the elementary divisor type $\nu(\mathcal{R}_{L(T_4)}(\mathbf{Y})) = (0, 0, m, m)$ only depends the p-adic valuations of b_1 and b_3. Writing $N_1 = v_p(b_1)$ and $N_3 = v_p(b_3)$, we obtain

$$\widetilde{\zeta^{\mathrm{irr}}_{T_4,p}}(s) = 1 + \sum_{\substack{N \in \mathbb{N} \\ 0 \le N_1 \le N,\ 0 \le N_3 \le N}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N - \min\{N_1+N_3,N\})}, \qquad (6.5)$$

where $\mathcal{N}_{N,(N_1,N_3)} := \#\{\mathbf{b} \in (\mathbb{Z}/p^N)^3 \setminus p(\mathbb{Z}/p^N)^3 \mid v_p(b_1) = N_1,\ v_p(b_3) = N_3\}$. The minimum in equation (6.5) can be resolved by breaking the sum up with respect to the values of N_1 and N_3. Initially, the sum is expanded as 9 cases. However, due to the symmetry in expression (6.5) the 9 cases can immediately be reduced to 6 cases. These cases are summarised in the table below.

	$N_1 = 0$	$0 < N_1 < N$	$N_1 = N$
$N_3 = 0$	(a)	(b)	(c)
$0 < N_3 < N$	(b)	(d)	(e)
$N_3 = N$	(c)	(e)	(f)

In all cases except (d), the minimum is resolved and the computation of the summation is reduced to calculating $\mathcal{N}_{N,(N_1,N_3)}$ and expressing the resulting sum in terms of geometric series. In case (d) a further case distinction is required.

Case (a): If $N_1 = N_3 = 0$, then $\mathcal{N}_{N,(N_1,N_3)} = (1-p^{-1})^2 p^{3N}$ and

$$\sum_{\substack{N\in\mathbb{N}\\ N_1=N_3=0}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N-\min\{N_1+N_3,N\})} = \sum_{N\in\mathbb{N}} (1-p^{-1})^2 p^{3N} p^{-2Ns}$$
$$= (1-p^{-1})^2 \frac{p^3t^2}{1-p^3t^2}.$$

Case (b): If $0 < N_1 < N$ and $N_3 = 0$, then $\mathcal{N}_{N,(N_1,N_3)} = (1-p^{-1})^2 p^{3N-N_1}$. Writing $N = N_1 + D$,

$$\sum_{\substack{N\in\mathbb{N}\\ 0<N_1<N,\ N_3=0}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N-\min\{N_1+N_3,N\})} = \sum_{N_1,D\in\mathbb{N}} (1-p^{-1})^2 p^{2N_1+3D} p^{-s(N_3+2D)}$$
$$= (1-p^{-1})^2 \frac{p^2t}{1-p^2t} \cdot \frac{p^3t^2}{1-p^3t^2}.$$

Case (c): If $N_1 = N$ and $N_3 = 0$, then $\mathcal{N}_{N,(N_1,N_3)} = (1-p^{-1})p^{2N}$ and

$$\sum_{\substack{N\in\mathbb{N}\\ N_1=N,\ N_3=0}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N-\min\{N_1+N_3,N\})} = \sum_{N\in\mathbb{N}} (1-p^{-1}) p^{2N} p^{-Ns}$$
$$= (1-p^{-1}) \frac{p^2t}{1-p^2t}.$$

Case (d): If $0 < N_1, N_3 < N$, then $\mathcal{N}_{N,(N_1,N_3)} = (1-p^{-1})^3 p^{3N-N_1-N_3}$,

$$\sum_{\substack{N\in\mathbb{N}\\ 0<N_1<N_3<N}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N-\min\{N_1+N_3,N\})}$$
$$= \sum_{\substack{N\in\mathbb{N}\\ 0<N_1<N_3<N}} (1-p^{-1})^3 p^{3N-N_1-N_3} p^{-s(2N-\min\{N_1+N_3,N\})}$$

and the minimum is not resolved. Case (d) is further subdivided into the following three cases:

(i) $N_1 < N_3$,

(ii) $N_1 = N_3$,

(iii) $N_1 > N_3$.

Due to the symmetry of (6.5) the summations of cases (i) and (iii) is equal. Case (i): Write $N_3 = N_1 + D$ and $N = N_1 + D + D'$, where $D, D' \in \mathbb{N}$, then

$$\begin{aligned}\sum_{\substack{N\in\mathbb{N}\\0<N_1<N_3<N}} &(1-p^{-1})^3 p^{3N-N_1-N_3} p^{-s(2N-\min\{N_1+N_3,N\})}\\ &= \sum_{N,D,D'\in\mathbb{N}} (1-p^{-1})^3 p^{N_1+2D+3D'} p^{-s(N_1+D+2D'-\min\{N_3,D'\})}\\ &= (1-p^{-1})^3 \frac{p^2t}{1-p^2t} \sum_{N,D'\in\mathbb{N}} p^{N_1+3D'} p^{-s(N_1+2D'-\min\{N_3,D'\})}.\end{aligned} \tag{6.6}$$

The minimum is still unresolved, but the final sum of expression (6.6) is of the form described in Lemma 6.4. By applying this lemma the final formula for case (i) is obtained as

$$\frac{(1-p^{-1})^3 p^6 t^3 (1-p^4t^3)}{(1-pt)(1-p^2t)(1-p^3t^2)(1-p^4t^2)}.$$

Case (ii): Write $N = N_1 + D$. Then by utilising Lemma 6.4

$$\begin{aligned}\sum_{\substack{N\in\mathbb{N}\\0<N_1=N_3<N}} &(1-p^{-1})^3 p^{3N-N_1-N_3} p^{-s(2N-\min\{N_1+N_3,N\})}\\ &= \sum_{N_1,D\in\mathbb{N}} (1-p^{-1})^3 p^{N_1+3D} p^{-s(N_1+2D-\min\{N_1,D\})}\\ &= \frac{(1-p^{-1})^3 p^4 t^2 (1-p^4t^3)}{(1-pt)(1-p^3t^2)(1-p^4t^2)}.\end{aligned}$$

Case (e): If $N_1 = N$ and $0 < N_3 < N$, then $\mathcal{N}_{N,(N_1,N_3)} = (1 -$

$p^{-1})^2p^{2N-N_3}$. Write $N = N_3 + D$, where $D \in \mathbb{N}$, then

$$\sum_{\substack{N\in\mathbb{N}\\ N_1=N,\ 0<N_3<0}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N-\min\{N_1+N_3,N\})}$$

$$= \sum_{N,D\in\mathbb{N}} (1-p^{-1})^2 p^{N_3+2D} p^{-s(N_3+D)}$$

$$= \sum_{N\in\mathbb{N}} (1-p^{-1}) p^{2N-N_3} p^{-Ns}$$

$$= (1-p^{-1})^2 \frac{p^3t^2}{(1-pt)(1-p^2t)}.$$

Case (f): If $N_1 = N_3 = N$, then $\mathcal{N}_{N,(N_1,N_3)} = (1-p^{-1})p^N$ and

$$\sum_{\substack{N\in\mathbb{N}\\ N_1=N_3=N}} \mathcal{N}_{N,(N_1,N_3)} p^{-s(2N-\min\{N_1+N_3,N\})} = \sum_{N\in\mathbb{N}} (1-p^{-1}) p^n p^{-Ns}$$

$$= (1-p^{-1}) \frac{pt}{1-pt}.$$

The local factor of the representation zeta function for T_4 is obtained by summing all the cases with appropriate multiplicity.

$$\widetilde{\zeta^{\mathrm{irr}}_{T_4,p}}(s) = \frac{(1-t)(1-pt)}{(1-p^2t)^2}.$$

The global representation zeta function therefore

$$\widetilde{\zeta^{\mathrm{irr}}_{T_4}}(s) = \frac{\zeta(s-2)^2}{\zeta(s)\zeta(s-1)},$$

converges for $\Re(s) > 3$ and has meromorphic continuation to the whole complex plane. The continued function has a double pole at $s = 3$.

The author has computed the representation zeta function $\widetilde{\zeta^{\mathrm{irr}}_{T_5}}(s)$ for T_5. The computation is very long and is similar in flavour to the calculation for T_4. The formula is

$$\widetilde{\zeta^{\mathrm{irr}}_{T_5}}(s) = \frac{\zeta(s-2)^3}{\zeta(s)\zeta(s-1)^2}.$$

Remark 6.7. For $n \in [2, 5]$,

$$\widetilde{\zeta^{\mathrm{irr}}_{T_n}}(s) = \frac{\zeta(s-1)\zeta(s-2)^{n-2}}{\zeta(s)\zeta(s-1)^{n-2}}.$$

It is interesting to ask whether this formula is valid for all $n \geq 2$.

Chapter 7

Zeta functions of 2-dimensional rings

This chapter contains ideas similar, but different, to the material in the rest of this thesis. We discuss the zeta functions enumerating subrings and ideals of 2-dimensional rings.

For the purposes of this chapter a ring R is always additively isomorphic to $\mathbb{Z}^d$, for some $d \in \mathbb{N}$, and equipped by a bi-additive product. Note that this product is not necessarily associative. If R is additively isomorphic to $\mathbb{Z}^d$ then R is said to be d-dimensional.

Definition 7.1. For Ξ a class of sub-objects of a ring R, the Ξ zeta function of R is the Dirichlet generating function

$$\zeta_R^{\Xi}(s) = \sum_{H \in \Xi} |R : H|^{-s},$$

where the sum ranges over all finite index subobjects in the class Ξ and s is a complex variable.

We explore the enumeration of four particular classes of sub-objects. Namely, subrings, left-, right- and two-sided ideals.

Remark 7.2. In general these classes yield four Dirichlet generating functions. If R is (anti-)commutative then the three types of ideal coincide.

Let R be a ring that is additively isomorphic to $\mathbb{Z}^d$. Fix a $\mathbb{Z}$-basis $\mathbf{y} = (y_1, \ldots, y_d)$ of R. Each product $y_i y_j$ is a linear combination of the basis elements. That is, for all $i, j, k \in [1, d]$, there exist $\lambda_{ij}^k \in \mathbb{Z}$ such that $y_i y_j = \sum_{k=1}^d \lambda_{ij}^k y_k$.

Let $\mathscr{R}_R(\mathbf{Y}) \in \mathrm{Mat}_d(\mathbb{Z}[Y_1, \ldots, Y_d])$ be the matrix of linear forms whose ij-entry is $\sum_{k=1}^d \lambda_{ij}^k Y_k$. Then $\mathscr{R}_R(\mathbf{Y})$ is the structure matrix of R with respect to the chosen basis. Once a basis is chosen we refer to $\mathscr{R}_R(\mathbf{Y})$ as the structure matrix of R. The structure matrix is determined, with respect to the chosen basis, by the multiplication of R. Conversely, if we write down any $d \times d$ matrix of linear forms of $\mathbb{Z}[Y_1, \ldots, Y_d]$ this determines a ring R by extending the multiplication linearly.

7.1 Localisation

In this section we localise the problem of counting subobjects and give criteria for an additive subgroup of finite index to be a ring or ideal.

The ring R is a $\mathbb{Z}$-algebra and so, for any prime p, we can consider the tensor product $R_p := R \otimes \mathbb{Z}_p$. Then R_p is a $\mathbb{Z}_p$-algebra whose multiplication is obtained by linearly extending that of R. We define the Ξ zeta function of R_p to be the Dirichlet generating function $\zeta_{R_p}^{\Xi}(s)$ enumerating the finite index sub-objects in the class Ξ.

Proposition 7.3. [16, Section 3] *Let R be a ring. If Ξ is the class of subrings, left-, right- or two-sided ideals, then $\zeta_R^{\Xi}(s)$ has the following Euler product.*

$$\zeta_R^{\Xi}(s) = \prod_p \zeta_{R_p}^{\Xi}(s), \tag{7.1}$$

where the product is taken over all primes.

The zeta function $\zeta^{\Xi}_{R_p}(s)$ is the *p-local factor* of the the global zeta function $\zeta^{\Xi}_{R}(s)$. We now present an enumeration of all finite-index subgroups of R_p, which is additively isomorphic to $\mathbb{Z}_p^d$.

Remark 7.4. In this chapter we consider global rings R and their localisation R_p, but all the results concerning R_p are valid for general $\mathbb{Z}_p$-algebras, not only those that occur as the localisation of a $\mathbb{Z}$-algebra.

We follow [38, Section 3]. Every subring or ideal is a subgroup of the underlying abelian group. Once we have provided an enumeration of all finite-index subgroups we present criteria for a subgroup to be a subring, left-, right- or two-sided ideal.

Write $R_p = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_d$. Let $M \in \mathrm{GL}_d(\mathbb{Q}_p) \cap \mathrm{Mat}_d(\mathbb{Z}_p)$, the set of $d \times d$ matrices of over $\mathbb{Z}_p$ with non-zero determinant. We identify M with the sublattice of R_p generated by the elements whose coordinates with respect to $(e_1, \ldots, e_d)$ are encoded in the rows of M. The index of the subgroup is equal to the determinant of M. The left-action of $\Gamma := \mathrm{GL}_d(\mathbb{Z}_p)$ on $\mathrm{Mat}_d(\mathbb{Z}_p)$, by left multiplication, corresponds to row operations, which means that any element of the coset ΓM corresponds to the same subgroup of R_p. Conversely, if we fix a finite-index subgroup then every matrix with which it identifies is an element of the coset ΓM.

We have established a one-to-one correspondence between the finite-index subgroups of R_p and the right cosets of Γ. By the elementary divisor theorem each coset ΓM contains a representative of the form $D\alpha^{-1}$, where $\alpha \in \Gamma$ and $D = \mathrm{diag}(p^{r_0 + \cdots + r_{d-1}}, \ldots, p^{r_0}) = \mathrm{diag}(D_1, \ldots, D_d)$ is a diagonal matrix with $r_i \in \mathbb{N}_0$ for $i \in [1, d]$.

Proposition 7.5. [38, Section 3] *Let R_p be a ring with structure matrix $\mathscr{R}_R(\mathbf{y})$. A sublattice of R_p corresponding to a right coset $\Gamma D\alpha^{-1}$ is a subalgebra of R_p if and only if the following congruences hold:*

$$\forall i \in [1, d] : D\alpha^{-1}\mathscr{R}_R(\alpha[i])(\alpha^{-1})^T D \equiv 0 \bmod D_i, \qquad \text{(SUB)}$$

where $\alpha[i]$ denotes the i^{th} column of α.

We generalise Proposition 7.5 to right-, left- and two-sided ideals.

Proposition 7.6. *Let R_p be a ring with structure matrix $\mathscr{R}_R(\mathbf{y})$. A sublattice of R_p corresponding to a right coset $\Gamma D\alpha^{-1}$ is a right-ideal of R_p if and only if the following congruences hold:*

$$\forall i \in [1,d] : D\alpha^{-1}\mathscr{R}_R(\alpha[i]) \equiv 0 \bmod D_i. \qquad \text{(R-IDEAL)}$$

Proof. Let Λ be a finite-index subgroup of R_p. Then Λ is a right ideal if and only if $\Lambda R_p \subseteq \Lambda$. Let $C_{(j)} : R \to R, x \mapsto xe_j$ be the matrix corresponding to right multiplication by the j^{th} basis element with respect to the chosen basis. Consider elements of R_p as row vectors. Recall $M = D\alpha^{-1}$ and denote by M_i the i^{th} row of the matrix M. M_i corresponds to a generator of the subgroup corresponding to M. Then M is a right-ideal if and only if

$$\begin{aligned} &\forall i,j \in [1,d] : M_iC_{(j)} \in \mathbb{Z}_p^d M \\ \Leftrightarrow\ &\forall i,j \in [1,d] : M_iC_{(j)}\alpha \in \mathbb{Z}_p^d D \\ \Leftrightarrow\ &\forall i,j,k \in [1,d] : D_k | (M_iC_{(j)}\alpha)[k] \end{aligned}$$

For all $k \in [1,d]$ we have $(C_{(1)}[k]) \dots C_{(d)}[k]) = \mathscr{R}_R(\alpha[k])$. By writing M instead of M_i and collecting the $C_{(j)}$ using the previous identity we arrive at the result. □

Proposition 7.7. *Let R_p be a ring with structure matrix $\mathscr{R}_R(\mathbf{y})$. A sublattice of R_p corresponding to a right coset $\Gamma D\alpha^{-1}$ is a left-ideal of R_p if and only if the following congruences hold:*

$$\forall i \in [1,d] : \mathscr{R}_R(\alpha[i])(\alpha^{-1})^T D \equiv 0 \bmod D_i. \qquad \text{(L-IDEAL)}$$

The proof of Proposition 7.7 is very similar to the proof of Proposition 7.6.

Proposition 7.8. *Let R_p be a ring with structure matrix $\mathscr{R}_R(\mathbf{y})$. A sublattice of R_p corresponding a right coset $\Gamma D\alpha^{-1}$ is a two-sided ideal of R_p if and only if the following congruences hold:*

$$\forall i \in [1,d] : \begin{cases} D\alpha^{-1}\mathscr{R}_R(\alpha[i]) \equiv 0 \bmod D_i, \\ \mathscr{R}_R(\alpha[i])(\alpha^{-1})^T D \equiv 0 \bmod D_i. \end{cases} \tag{IDEAL}$$

Proof. By definition, an ideal is two-sided if it is simultaneously a right- and left-ideal. The congruences in (IDEAL) are simply the conjunction of the congruences in (R-IDEAL) and (L-IDEAL). □

7.2 Formulae for local zeta functions

We want to determine whether or not a given finite-index subgroup is a subring or an ideal. In general, given a finite-index subgroup, the criteria given in Propositions 7.5-7.8 reduce this question to verifying a number of congruences.

In [25] Klopsch and Voll consider the subrings of 3-dimensional $\mathbb{Z}_p$-Lie algebras. In this case the combination of the small dimension and the anti-commutivity of the multiplication of a $\mathbb{Z}_p$-Lie algebra mean that all but one of the congruences of (SUB) are satisfied for all finite-index subgroups.

They deduce a formula for the subring zeta function in terms of Igusa's local zeta function associated with a single ternary quadratic form. We use similar methods but focus on 2-dimensional $\mathbb{Z}_p$-algebras (with no restriction on the multiplication).

We write $\zeta^{\leq}_{R_p}(s)$ for the zeta function enumerating all finite-index subrings of R_p and recall our convention that $t = p^{-s}$.

Theorem 7.9. *Let R be a 2-dimensional ring with structure constants λ^k_{ij} for $i,j,k \in \{1,2\}$ with respect to a chosen $\mathbb{Z}$-basis. Then*

$$\zeta^{\leq}_{R_p}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)}\,\mathrm{Z}_F(s-1),$$

where $F(x_1, x_2) \in \mathbb{Z}[x_1, x_2]$ *is a cubic binary form given by*

$$F(x_1, x_2) = \lambda_{22}^1 x_1^3 + (\lambda_{22}^2 - \lambda_{12}^1 - \lambda_{21}^1) x_1^2 x_2 + (\lambda_{11}^1 - \lambda_{12}^2 - \lambda_{21}^2) x_1 x_2^2 + \lambda_{22}^1 x_2^3.$$

Proof. For $D = \mathrm{diag}(p^{r_1+r_0}, p^{r_0})$ and $\alpha \in \mathrm{GL}_2(\mathbb{Z}_p)$, consider the coset $\Gamma D \alpha^{-1}$ of $\Gamma = \mathrm{GL}_2(\mathbb{Z}_p)$. We want to check if the finite-index subgroup Λ corresponding to $\Gamma D \alpha^{-1}$ is a subring. In the case of 2-dimensional rings the set of congruences labelled (SUB) in Proposition 7.5 consists of eight congruences. By writing the congruences out, one notices that seven of the eight are automatically satisfied for all finite-index subgroups. In fact, (SUB) reduces to verifying the single congruence

$$p^{r_0} F(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1},$$

where $F(x_1, x_2)$ is as in the statement of the proposition.

The matrix α is determined only up to right multiplication by elements of the stabilizer $\mathrm{Stab}_\Gamma(\Gamma D)$. For a fixed r_1 we can describe $\mathrm{Stab}_\Gamma(\Gamma D)$ precisely.

$$\mathrm{Stab}_\Gamma(\Gamma D) = \begin{pmatrix} \mathbb{Z}_p^\star & \mathbb{Z}_p \\ p^{r_1}\mathbb{Z}_p & \mathbb{Z}_p^\star \end{pmatrix}.$$

This means the first column of α is determined modulo p^{r_1} by Λ. In addition, α must have non-zero determinant and so the pairs $(\alpha_{11}, \alpha_{21})$ that correspond to unique sublattices are in bijection with the points of the finite projective space $\mathbb{P}^1(\mathbb{Z}/p^{r_1})$. Therefore,

$$\zeta_{R_p}^{\leq}(s) = \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} M_{r_0,r_1}^{\#} t^{2r_0+r_1},$$

where $M_{r_0,r_1}^{\#} := \#\{\mathbf{x} \in \mathbb{P}^1(\mathbb{Z}/p^{r_1}) \mid v_p(F_{r_0})(\mathbf{x}) \geq r_1\}$ and $F_{r_0}(\mathbf{x}) = p^{r_0} F(x_1, x_2)$. In the case $r_1 > 0$, define $M_{r_0,r_1}^{\star} := \#\{\mathbf{x} \in (\mathbb{Z}/p^{r_1})^2 \setminus p(\mathbb{Z}/p^{r_1})^2 \mid v_p(F_{r_0})(\mathbf{x}) = 0\}$. It is clear that

$$M_{r_0,r_1}^{\#} = \begin{cases} (1-p^{-1})^{-1} p^{-r_1} M_{r_0,r_1}^{\star} & \text{if } r_1 \neq 0, \\ 1 & \text{if } r_1 = 0. \end{cases}$$

Thus

$$\zeta^{\leqq}_{R_p}(s) = \sum_{r_0=0}^{\infty}\left(t^{2r_0} + \sum_{r_1=1}^{\infty}\frac{M^{\star}_{r_0,r_1}}{(1-p^{-1})p^{r_1}}t^{2r_0+r_1}\right)$$

by direct substitution. Using identities (3.10) and (3.15) we obtain:

$$\begin{aligned}
\zeta^{\leqq}_{R_p}(s) &= \sum_{r_0=0}^{\infty}\left(t^{2r_0} + \frac{t^{2r_0}}{1-p^{-1}}\left(\mathrm{P}^{\star}_{F_{r_0}}(pt)-1\right)\right)\\
&= \sum_{r_0=0}^{\infty}\left(t^{2r_0} + \frac{t^{2r_0}}{1-p^{-1}}\left(\frac{1-p^{-1}t-pt\,\mathrm{Z}^{\star}_{F_{r_0}}(s-1)}{1-pt}-1\right)\right)\\
&= \frac{1}{1-t^2} + \frac{pt-p^{-1}t}{(1-p^{-1})(1-t^2)(1-pt)}\\
&\quad - \frac{pt}{(1-p^{-1})(1-pt)}\sum_{r_0=1}^{\infty} t^{2r_0}\,\mathrm{Z}^{\star}_{F_{r_0}}(s-1)
\end{aligned}$$

For the final step, we use identity (3.9) and note that $\mathrm{Z}_{p^{r_0}F}(s) = t^{r_0}\,\mathrm{Z}_F(s)$.

$$\zeta^{\leqq}_{R_p}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)}\,\mathrm{Z}_F(s-1),$$

□

Now let Ξ be the class of all finite-index right ideals. We define the right-ideal zeta function as

$$\zeta^{\triangleleft_r}_{R_p}(s) := \sum_{H\in\Xi}|R:H|^{-s}.$$

Theorem 7.10. *Let R be a 2-dimensional ring with structure constants λ^k_{ij} for $i,j,k\in\{1,2\}$ with respect to a chosen $\mathbb{Z}$-basis. Then*

$$\zeta^{\triangleleft_r}_{R_p}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)}\,\mathrm{Z}_{\mathbf{F}}(s-1),$$

where $\mathbf{F}\triangleleft\mathbb{Z}[x_1,x_2]$ is the ideal generated by the polynomials $f_1(x_1,x_2)$, $f_2(x_1,x_2)\in\mathbb{Z}[x_1,x_2]$ given by

$$f_1(x_1,x_2) = \lambda^1_{21}x_1^2 + (\lambda^2_{21}-\lambda^1_{11})x_1x_2 - \lambda^2_{11}x_2^2,$$

$$f_2(x_1,x_2) = \lambda^1_{22}x_1^2 + (\lambda^2_{22}-\lambda^1_{12})x_1x_2 - \lambda^2_{12}x_2^2.$$

Proof. By Proposition 7.6 a finite-index subgroup Λ is a right ideal if the congruences (R-IDEAL) are satisfied by the matrix $D\alpha^{-1}$ corresponding to Λ. Since R is 2-dimensional, we see that there are eight congruences in (R-IDEAL) and that six of them are satisfied automatically. Suppose that a finite-index subgroup corresponds to the matrix $D\alpha^{-1}$. Then Λ is a right ideal if the following two congruences hold:

$$f_1(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1},$$
$$f_2(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1}.$$

Similarly to the proof of Theorem 7.9 it follows that

$$\zeta_{R_p}^{\triangleleft_r}(s) = \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} M_{r_1}^{\#} t^{2r_0+r_1},$$

where $M_{r_1}^{\#} := \#\{\mathbf{x} \in \mathbb{P}(\mathbb{Z}/p^{r_1}) \mid \min\{v_p(f_1(\mathbf{x}), v_p(f_2(\mathbf{x})\} \geq r_1\}$. Since $M_{r_1}^{\#}$ does not depend on r_0 we can separate the sums and obtain

$$\zeta_{R_p}^{\triangleleft_r}(s) = \frac{1}{1-t^2} \sum_{r_1=0}^{\infty} M_{r_1}^{\#} t^{r_1}.$$

For $r_1 > 0$ we define $M_{r_1}^{\star} := \#\{\mathbf{x} \in (\mathbb{Z}/p^{r_1})^2 \setminus p(\mathbb{Z}/p^{r_1})^2 \mid f_1(\mathbf{x}) = f_2(\mathbf{x}) = 0\}$. We have

$$M_{r_1}^{\#} = \begin{cases} (1-p^{-1})^{-1} p^{-r_1} M_{r_1}^{\star} & \text{if } r_1 \neq 0, \\ 1 & \text{if } r_1 = 0. \end{cases}$$

Therefore

$$\zeta_{R_p}^{\triangleleft_r}(s) = \frac{1}{1-t^2} \left(1 + \frac{1}{1-p^{-1}} \sum_{r_1=1}^{\infty} M_{r_1}^{\star} (p^{-1}t)^{r_1} \right).$$

The final formula results from using of identities (3.10), (3.15), (3.9).

$$\begin{aligned}
\zeta_{R_p}^{\triangleleft_r}(s) &= \frac{1}{1-t^2}\left(1+\frac{1}{1-p^{-1}}(\mathrm{P}_{\mathbf{F}}^{\star} -1)\right) \\
&= \frac{1}{1-t^2}\left(1+\frac{1}{1-p^{-1}}\left(\mathrm{P}_{\mathbf{F}}^{\star}(pt)-1\right)\right) \\
&= \frac{1}{1-t^2}\left(1+\frac{1}{1-p^{-1}}\left(\frac{1-p^{-1}t-pt\,\mathrm{Z}_{\mathbf{F}}^{\star}(s-1)}{1-pt}-1\right)\right) \\
&= \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)(1-t^2)}\,\mathrm{Z}_{\mathbf{F}}^{\star}(s-1) \\
&= \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)}\,\mathrm{Z}_{\mathbf{F}}(s-1).
\end{aligned}$$

□

Now let Ξ be the class of all finite-index left ideals. We define the left-ideal zeta function as

$$\zeta_{R_p}^{\triangleleft_\ell}(s) := \sum_{H\in\Xi} |R:H|^{-s}.$$

Theorem 7.11. *Let R be a 2-dimensional ring, with structure constants λ_{ij}^k for $i,j,k\in\{1,2\}$ with respect to a chosen $\mathbb{Z}$-basis. Then*

$$\zeta_{R_p}^{\triangleleft_\ell}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)}\,\mathrm{Z}_{\mathbf{G}}(s-1),$$

where $\mathbf{G}\triangleleft\mathbb{Z}[x_1,x_2]$ is the ideal generated by the polynomials $g_1(x_1,x_2)$, $g_2(x_1,x_2)\in\mathbb{Z}[x_1,x_2]$ given by

$$\begin{aligned}
g_1(x_1,x_2) &= \lambda_{12}^1 x_1^2 + (\lambda_{12}^2-\lambda_{11}^1)x_1x_2 - \lambda_{11}^2 x_2^2, \\
g_2(x_1,x_2) &= \lambda_{22}^1 x_1^2 + (\lambda_{22}^2-\lambda_{21}^1)x_1x_2 - \lambda_{21}^2 x_2^2.
\end{aligned}$$

The proof of Theorem 7.11 is very similar to the proof of Theorem 7.10. Finally, we write $\zeta_{R_p}^{\triangleleft}(s)$ for the zeta function enumerating two-sided ideals.

Theorem 7.12. *Let R be a 2-dimensional ring, by fixing a basis we determine a structure matrix $\mathscr{R}_R(\mathbf{y})$ and structure constants λ_{ij}^k for $i,j,k\in\{1,2\}$. Then*

$$\zeta_{R_p}^{\triangleleft}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)}\,\mathrm{Z}_{\mathbf{F}+\mathbf{G}}(s-1),$$

where $\mathbf{F}, \mathbf{G} \lhd \mathbb{Z}[x_1, x_2]$ *are the ideals defined in Theorems 7.10 and 7.11 respectively.*

Proof. By Proposition 7.8 a finite-index subgroup is a two-sided ideal if the corresponding matrix $D\alpha^{-1}$ satisfies (IDEAL), which for a 2-dimensional ring consists of sixteen congruences. However, we notice that twelve of the congruences are satisfied automatically. The remaining four congruences are

$$f_1(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1},$$
$$f_2(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1},$$
$$g_1(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1},$$
$$g_2(\alpha_{11}, \alpha_{21}) \equiv 0 \bmod p^{r_1},$$

where f_1, f_2, g_1, g_2 are the polynomials defined in Theorems 7.10 and 7.11. The remainder of the proof is very similar to the proof of Theorem 7.10. □

In the case where the ring R is (anti-)commutative the left-, right- and two-sided ideal zeta function are equal. In this case we denote the single ideal zeta function by $\zeta^{\lhd}_{R_p}(s)$.

7.3 Pole spectra

In Section 7.2 we showed that the p-local subring and ideal zeta functions of a 2-dimensional ring can be expressed in terms of Igusa's local zeta function associated with a polynomial or ideal given by the structure constants. In this section we use the formulae presented in Theorems (7.9)-(7.12) to examine the local and global pole spectra for the various zeta functions associated with a 2-dimensional ring.

We are interested in the poles of zeta functions because they give us information about the growth of the sequence encoded in the Dirichlet series. In particular, the pole with largest real part controls the rate of polynomial growth.

The *pole spectrum* of a class of rings $\mathfrak{X}$ is the set S of real numbers such that if $R \in \mathfrak{X}$ and $\zeta_R^{\equiv}(s)$ has leading pole on the real line at $s = \alpha$, then $s \in S$. Conversely, if $\alpha \in S$, then there exists $R \in \mathfrak{X}$ such $\zeta_R^{\equiv}(s)$ has leading pole at $s = \alpha$. Recall that, by Theorem [10, Theorem 1.1], the zeta functions of rings all have meromorphic continuation beyond the abscissa of convergence, so that the pole spectrum of any class of rings is non-empty. The *local pole spectrum* is defined analogously.

In practice we may not be able to calculate the subring (or ideal) zeta function of given ring R. However, we might be able to provide a superset for its pole spectrum. Of course, the pole spectrum for the class of all rings is a natural superset for the pole spectrum of a given ring. In this section we give supersets for the subring and idea pole spectrum of 2-dimensional rings.

Before we can describe the pole spectra we need to introduce some notation and recall some well known facts. For a prime p the p-local factor of the Riemann zeta function $\zeta_p(s)$ is defined as

$$\zeta_p(s) = \frac{1}{1-t}.$$

The p-local factor of the Riemann zeta function converges for $\mathfrak{Re}(s) > 0$, has meromorphic continuation to the whole complex and the extended function has a simple pole at $s = 0$. The Riemann zeta function $\zeta(s)$ is the Euler product of the p-local factors:

$$\zeta(s) = \prod_p \zeta_p(s).$$

The Riemann zeta function converges for $\mathfrak{Re}(s) > 1$, has meromorphic continuation to the whole complex plane and the extended function has a simple pole at $s = 1$.

The main result of [16] implies that the local factors of the subring, left-, right- and two-sided zeta functions are rational functions. Furthermore, in [10] it is shown that they have the following form.

For a ring R, a prime p and $\star \in \{\leq, \lhd_r, \lhd_\ell, \lhd\}$ we have

$$\zeta^\star_{R_p}(s) = \frac{\Psi_{p.1}(p,t)}{\Psi_{p,2}(p,t)},$$

where $\Psi_{p,1}(X,Y), \Psi_{p,2}(X,Y) \in \mathbb{Z}[X,Y]$ are polynomials and

$$\Psi_{p,2}(X,Y) = \prod_{i \in I_p} (1 - p^{A_{p,i}} t^{B_{p,i}}),$$

where $A_{p,i}, B_{p,i} \in \mathbb{N}_0$ and I_p is a finite indexing set. It follows that the location of the real part of any poles in an element of the set $\{A_{p,i}/B_{p,i} \mid i \in I\}$. It is also shown in [10] that the location of the leading pole is $\max_{\substack{p \text{ prime} \\ i \in I_p}} \{\frac{A_{p,i}+1}{B_{p,i}}\}$.

Therefore, from knowledge of all possible pairs $(A_{p,i}, B_{p,i})$ we know the possible local and global pole spectra. The set of all such pairs is called the pole location data.

Theorem 7.13. *For the class of all 2-dimensional rings:*
(1) The pole spectra of the local subring zeta function is $\{0, 1/3, 1/2, 2/3, 1\}$. *The pole spectra for the leading pole of the global subring zeta functions is* $\{2/3, 1, 2\}$.
(2) The pole spectra of either the local left-, right- or two-sided ideal zeta function is $\{0, 1/2, 1\}$. *The pole spectra for the leading pole of the either the global left-, right- or two-sided ideal zeta functions is* $\{1/2, 1, 2\}$.

Proof. For all primes p and $\star \in \{\leq, \lhd_r, \lhd_\ell, \lhd\}$ and any 2-dimensional ring R, there exist an ideal $\mathbf{F} \lhd \mathbb{Z}[x_1, x_2]$ such that

$$\zeta^\star_{R_p}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)} \mathrm{Z}_\mathbf{F}(s-1)$$

The ideal $\mathbf{F}$ depends on $\star$ and R and is described explicitly in Theorems 7.9-7.12, but it is generated by a single cubic polynomial if $\star = \leq$ or a set of quadratic polynomials if $\star \in \{\lhd_r, \lhd_\ell, \lhd\}$. In either case, we see immediately that $(0,1)$ and $(1,1)$ are part of the pole location data.

Our analysis is complete if we understand the pole location data for $\mathrm{Z}_{\mathbf{F}}(s-1)$. Note that if (A, B) is part of the pole for $\mathrm{Z}_{\mathbf{F}}(s)$, then $(A+B, B)$ is part of the pole data for $\mathrm{Z}_{\mathbf{F}}(s-1)$. We complete our analysis by determining the pole data for $\mathrm{Z}_{\mathbf{F}}(s)$ when $\mathbf{F} \lhd \mathbb{Z}[x_1, x_2]$ is a homogeneous ideal of degree $n = 2$ for the ideal case and degree $n = 3$ for the subring case. By identity (3.9)

$$\mathrm{Z}_{\mathbf{F}}(s) = \frac{1}{1 - p^{-2}t^n} \mathrm{Z}_{\mathbf{F}}^{\star}(s).$$

Therefore, (-2, n) is in the pole location data for $\mathrm{Z}_{\mathbf{F}}(s)$ which implies $(1, 3)$ is part of the pole location data for $\zeta_{R_p}^{\leq}(s)$ and $(0, 2)$ is part of the pole location data for $\zeta_{R_p}^{\star}(s)$, where $\star \in \{\lhd_r, \lhd_\ell, \lhd\}$. Now, $\mathrm{Z}_{\mathbf{F}}^{\star}(s)$ has a pole if and only if every element of $\mathbf{F}$ has a simultaneous root in $\mathbb{Z}_p^2 \smallsetminus p\mathbb{Z}_p^2$. The elements of $\mathbf{F}$ can have only finitely many distinct simultaneous solutions. Pick $m \in \mathbb{N}$ sufficiently large such that the distinct solutions are not equal modulo p^m and consider the coset decomposition

$$\mathrm{Z}_{\mathbf{F}}^{\star}(s) = \sum_{\mathbf{a} \in (\mathbb{Z}/p^m)^2 \smallsetminus p(\mathbb{Z}/p^m)^2} \underbrace{\int_{\substack{\mathbf{x} \in (\mathbb{Z}_p^2 \smallsetminus p(\mathbb{Z}_p)^2 \\ \mathbf{x} \equiv \mathbf{a} \bmod p^m}} ||\mathbf{F}(\mathbf{x})||^s d\mu}_{=: Z_{\mathbf{F},\mathbf{a}}(s)}.$$

If $\mathbf{a} = (a_1, a_2)$ is not congruent to one of the solutions then $Z_{\mathbf{F},\mathbf{a}}(s)$ is a polynomial in t and therefore cannot contribute a pole. Now suppose that $\mathbf{a}$ is congruent to one of the roots. The denominator of $Z_{\mathbf{F},\mathbf{a}}(s)$ only depends on the order of the root. Denote the order of the root by k. The denominator of $Z_{\mathbf{F},\mathbf{a}}(s)$ is $1 - p^{-1-ks}$ which means that $(-1, k)$ for $1 \leq k \leq n$ are part of the possible pole data for $\mathrm{Z}_{\mathbf{F}}(s)$ which implies that $(0, 1), (1, 2)$ and $(2, 3)$ are added to the pole location data for $\zeta_{R_p}^{\leq}(s)$ and $(0, 1), (1, 2)$ are added to the pole location data for $\zeta_{R_p}^{\star}(s)$, where $\star \in \{\lhd_r, \lhd_\ell, \lhd\}$.

We have provided a superset for the the pole spectra for the class of 2-dimensional rings. The proof is completed by showing that each of the possible poles is actually exhibited by a zeta function associated to a 2-dimensional ring. Each possible pole location data is exhibited in the examples computed in Section 7.4. $\square$

7.4 Examples

In this section we use the results of Section 7.2 to compute the subring and ideal zeta functions for several rings. In each case the calculation is reduced to computing Igusa's local zeta function associated to an ideal that is generated by polynomials whose coefficients are given in terms of the structure matrix of the ring.

To calculate Igusa's local zeta function we use the tools and examples presented in Section 3.3. Throughout this section let $F(x_1, x_2)$ be as defined in Theorem 7.9, $\mathbf{F}, f_1(x_1, x_2), f_2(x_1, x_2)$ be as defined in Theorem 7.10 and $\mathbf{G}, g_1(x_1, x_2), g_2(x_1, x_2)$ be as defined in Theorem 7.11. Recall that $\zeta(s)$ denotes the Riemann zeta function.

Example 7.14. Let R be additively isomorphic to $\mathbb{Z}^2$ and equipped with component-wise multiplication. If we choose the basis $\{(1,0),(0,1)\}$, then

$$\mathcal{R}_R(\mathbf{Y}) = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}.$$

The ring R is commutative, and so has only one ideal zeta function associated with it. In this case $F(x_1, x_2) = x_1 x_2 (x_1 + x_2)$ and $\mathbf{F} = \langle f_1(x_1, x_2), f_2(x_1, x_2) \rangle = \langle -x_1 x_2, x_1 x_2 \rangle$.

Proposition 7.15. *For $R = \mathbb{Z}^2$, equipped with component-wise multiplication, we have*

$$\zeta_R^{\leq}(s) = \frac{\zeta(s)^3 \zeta(3s-1)}{\zeta(2s)^2} \tag{7.2}$$

$$\zeta_R^{\triangleleft}(s) = \zeta(s)^2. \tag{7.3}$$

Proof. We first calculate the subring zeta function. By Theorem 7.9

$$\zeta_{R_p}^{\leq}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)} \mathrm{Z}_F(s-1),$$

where $F(x_1, x_2) = x_1x_2(x_1 + x_2)$. It suffices to compute

$$\mathrm{Z}_F(s) = \int_{\mathbb{Z}_p^2} |x_1x_2(x_1 + x_2)|^s d\mu.$$

First note that $\mathrm{Z}_F(s) = \frac{1}{1-p^{-2-3s}} \mathrm{Z}_F^\star(s)$. We compute $\mathrm{Z}_F^\star(s)$ by first decomposing the integral over the $p^2 - 1$ cosets of $p(\mathbb{Z}_p)^2$.

$$\mathrm{Z}_F^\star(s) = \sum_{\mathbf{a}\in\mathbb{F}_p^2\setminus\{\mathbf{0}\}} \underbrace{\int_{\substack{\mathbf{x}\in\mathbb{Z}_p^2\setminus p(\mathbb{Z}_p^2)\\ \mathbf{x}\equiv\mathbf{a} \bmod p}} |x_1x_2(x_1 + x_2)|^s d\mu}_{=:I_{\mathbf{a}}(s)}$$

We compute the integrals $I_{\mathbf{a}}(s)$ in cases. If $v_p(a_1) \neq 0$, then $v_p(a_2) = 0$ and

$$I_{\mathbf{a}}(s) = \int_{\substack{\mathbf{x}\in\mathbb{Z}_p^2\setminus p(\mathbb{Z}_p^2)\\ \mathbf{x}\equiv(0,a_2) \bmod p}} |x_1x_2(x_1 + x_2)|^s d\mu = \int_{x_2\in p\mathbb{Z}_p} |x_2|^s d\mu = \frac{(1-p^{-1})p^{-1-s}}{1-p^{-1-s}}.$$

If $v_p(a_2) \neq 0$, then $v_p(a_1) = 0$ and

$$I_{\mathbf{a}}(s) = \int_{\substack{\mathbf{x}\in\mathbb{Z}_p^2\setminus p(\mathbb{Z}_p^2)\\ \mathbf{x}\equiv(a_1,0) \bmod p}} |x_1x_2(x_1 + x_2)|^s d\mu = \int_{x_1\in p\mathbb{Z}_p} |x_1|^s d\mu = \frac{(1-p^{-1})p^{-1-s}}{1-p^{-1-s}}.$$

If $v_p(a_1) = v_p(a_2) = 0$, then

$$I_{\mathbf{a}}(s) = \int_{\substack{\mathbf{x}\in\mathbb{Z}_p^2\setminus p(\mathbb{Z}_p^2)\\ \mathbf{x}\equiv(a_1,a_2) \bmod p}} |x_1 + x_2|^s d\mu.$$

Further, if $a_1 \not\equiv -a_2 \bmod p$, then $|a_1 + a_2| = 1$ and $I_{\mathbf{a}}(s) = p^{-2}$. However, if $a_1 \equiv -a_2 \bmod p$, then, by performing a linear change of variables, we have

$$I_{\mathbf{a}}(s) = \int_{\substack{\mathbf{x}\in\mathbb{Z}_p^2\setminus p(\mathbb{Z}_p^2)\\ \mathbf{x}\equiv(a_1,-a_1) \bmod p}} |x_1 + x_2|^s d\mu = \int_{x_1\in p\mathbb{Z}_p} |x_1|^s d\mu = \frac{(1-p^{-1})p^{-1-s}}{1-p^{-1-s}}.$$

By gathering the summands and performing some algebraic reduction, we have

$$\zeta_{R_p}^{\leq}(s) = \frac{(1-p^{-2s})^2}{(1-p^{-s})^3(1-p^{1-3s})}.$$

Formula (7.2) is achieved by taking the Euler product.

Now we prove Formula (7.3). The ring R is commutative, therefore, by Theorem 7.10

$$\zeta_{R_p}^{\triangleleft}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)} \mathrm{Z}_{\mathbf{F}}(s-1), \qquad (7.4)$$

where $\mathbf{F} \lhd \mathbb{Z}[x_1, x_2]$ is the ideal generated by the polynomials $f_1(x_1, x_2) = -x_1 x_2$ and $f_2(x_1, x_2) = x_1 x_2$. It suffices to compute the integral

$$\mathrm{Z}_{\mathbf{F}}(s) = \int_{\mathbb{Z}_p^2} |x_1 x_2|^s d\mu = \int_{\mathbb{Z}_p} |x_1|^s d\mu \int_{\mathbb{Z}_p} |x_2|^s d\mu = \left(\frac{1 - p^{-1}}{1 - p^{-1-s}} \right)^2.$$

Here we have used Fubini's Theorem and Example (3.2). By substituting the expression for $\mathrm{Z}_{\mathbf{F}}(s)$ into (7.4), we have

$$\zeta^{\lhd}_{R_p}(s) = \frac{1}{(1 - p^{-s})^2}.$$

Formula (7.3) is achieved by taking the Euler product. □

Recall that, if $A(n), B(n)$ are arithmetic functions, then we write $A(n) \sim B(n)$ if $\lim_{n\to\infty} \frac{A(n)}{B(n)} = 1$.

Corollary 7.16. *Consider $\mathbb{Z}^2$ as a ring with componentwise multiplication. Then*

$$\sum_{i=1}^{n} a_n^{\leq}(\mathbb{Z}^2) \sim \frac{3}{\pi} n (\log n)^2$$

and

$$\sum_{i=1}^{n} a_n^{\lhd}(\mathbb{Z}^2) \sim n(\log n).$$

Furthermore, we have $a_n^{\lhd}(R) = d(n)$, where $d(n)$ denotes the divisor function.

Proof. The two asymptotic formulae follow at once from [10, Theorem 4.20]. For the final statement, it is well-known, see for example [1, Section 11.4, Example 5], that the Dirichlet series associated with the divisor function $d(n)$ is $\zeta(s)^2$. □

Corollary 7.17. *For R additively isomorphic to $\mathbb{Z}^2$ and equipped with component-wise multiplication,*

$$\zeta_R^{\leq}(s) = \frac{\zeta(s)\zeta(3s-1)\zeta_R^{\lhd}(s)}{\zeta_R^{\lhd}(2s)}.$$

Proof. This follows immediately from Proposition 7.15. □

Example 7.18. Let K be a quadratic number field. It is well-known that $K = \mathbb{Q}(\sqrt{d})$ for some square-free integer d. Let R be the ring of integers of K. Then R is a 2-dimensional ring whose structure depends on the residue class of d modulo 4. Precisely,

$$R = \begin{cases} \mathbb{Z}\left[1, \frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \bmod 4, \\ \mathbb{Z}[1, \sqrt{d}] & \text{if } d \equiv 2, 3 \bmod 4. \end{cases}$$

For a number field K with ring of integers R, recall that the Dedekind zeta function $\zeta_K(s)$ of K is defined as $\zeta_K(s) = \sum_{I \lhd R} N(I)^{-s}$, where $N(I)$ denotes the norm of the ideal I.

The ideal zeta function $\zeta_R^{\lhd}(s)$ is precisely the Dedekind zeta function $\zeta_K(s)$. The p-local factors of the Dedekind zeta function $\zeta_K(s)$ are well-known and depend only on the splitting behaviour of the prime ideal (p) in the ring of integers, see [26, Chapter VIII, Section 2]. We provide a new proof in the case where $K = \mathbb{Q}(\sqrt{d})$. For details on the splitting of prime ideals see [26, Chapter 1, Section 7].

Proposition 7.19. *Let R be the ring of integers in a quadratic number field K.*

$$\zeta_{R,p}^{\lhd}(s) = \begin{cases} \frac{1}{1-p^{-s}} & \textit{if } (p) \textit{ is ramified in } R, \\ \frac{1}{1-p^{-2s}} & \textit{if } (p) \textit{ is inert in } R, \\ \frac{1}{(1-p^{-s})^2} & \textit{if } (p) \textit{ is split in } R. \end{cases}$$

Proof. Let R be the rings of integers in the number field $K = \mathbb{Q}(\sqrt{d})$. We prove the statement in the case $d \equiv 3 \bmod 4$. The proofs of the other cases are very similar.

The ring R is generated by $\{1, \sqrt{d}\}$ and with respect to that basis has structure matrix

$$\mathcal{R}_R(\mathbf{Y}) = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & dY_1 \end{pmatrix}.$$

The ring R is commutative and therefore, By Theorem 7.11,

$$\zeta_{R,p}^{\lhd}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)} \mathrm{Z}_{\mathbf{F}}(s-1), \tag{7.5}$$

where $\mathbf{F} = \langle 0, dx_1^2 - x_2^2 \rangle$. It suffices to compute

$$Z_{\mathbf{F}}^{\star}(s) = \int_{\mathbb{Z}_p^2 \smallsetminus p\mathbb{Z}_p^2} |dx_1^2 - x_2^2|^s d\mu.$$

In a quadratic number field a prime ideal (p) ramifies if $p = 2$ or $p|d$. We treat these two cases first. Let $p = 2$. We decompose $Z_{\mathbf{F}}^{\star}(s)$ over the cosets of $2\mathbb{Z}_2^2$.

$$Z_{\mathbf{F}}^{\star}(s) = \sum_{(a_1,a_2) \in \mathbb{F}_2^2 \smallsetminus \{\mathbf{0}\}} \int_{\substack{(x_1,x_2) \in \mathbb{Z}_2^2 \smallsetminus 2\mathbb{Z}_2^2 \\ (x_1,x_2) \equiv (a_1,a_2) \bmod 2}} |dx_1^2 - x_2^2|^s d\mu.$$

Notice that on the cosets $(0,1)+2\mathbb{Z}_2^2$ and $(1,0)+2\mathbb{Z}_2^2$, the valuation $v_p(dx_1^2 - x_2^2)$ is identically 0. If we break the coset $(1,1) + 2\mathbb{Z}_2^2$ up modulo 4 we see that the valuation $v_p(dx_1^2 - x_2^2)$ is identically 1. Therefore,

$$Z_{\mathbf{F}}^{\star}(s) = 2\mu(2\mathbb{Z}_2^2) + 2^{-s}\mu(2\mathbb{Z}_2^2) = 2^{-1} + 2^{-2-s}.$$

By substituting $Z_{\mathbf{F}}(s) = \frac{1}{1-2^{-2-2s}} Z_{\mathbf{F}}^{\star}(s)$ into equation (7.5) and performing some algebraic manipulation the result is acquired.

$$\zeta_{R,2}^{\triangleleft}(s) = \frac{1}{1-2^{-s}}.$$

Now suppose that $p|d$. Once again decompose $Z_{\mathbf{F}}^{\star}(s)$ over the cosets of $p\mathbb{Z}_p^2$. Since d is square-free, p divides d exactly once. It is easy to show that

$$v_p(dx_1^2 - x_2^2) \equiv \begin{cases} 0 & \text{if } v_p(x_2) = 0, \\ 1 & \text{if } v_p(x_2) \geq 1. \end{cases}$$

Therefore, to compute $Z_{\mathbf{F}}^{\star}(s)$ we need only count the multiplicities of each case:

$$Z_{\mathbf{F}}^{\star}(s) = (p^2 - p)p^{-2} + (p-1)p^{-2-s} = (1 - p^{-1})(1 + p^{-1-s}).$$

By substituting $Z_{\mathbf{F}}(s) = \frac{1}{1-2^{-2-2s}} Z_{\mathbf{F}}^{\star}(s)$ into equation (7.5) and preforming some algebraic manipulation the result is acquired.

$$\zeta_{R,p}^{\triangleleft}(s) = \frac{1}{1-p^{-s}}.$$

Now suppose that $p \neq 2$ and $p \nmid d$. Additionally suppose that d is a quadratic residue modulo p. There is $\tilde{r} \in \mathbb{F}_p$ such that $\tilde{r}^2 - d \equiv 0 \bmod p$. As $p \neq 2$, by Hensel's Lemma, $\tilde{r}$ lifts to $r \in \mathbb{Z}_p$ such that $r^2 = d$. Therefore, $dx_1^2 - x_2^2 = (rx_1 - x_2)(rx_1 + x_2)$ and by performing a change of variables

$$\mathrm{Z}_{\mathbf{F}}(s) = \int_{\mathbb{Z}_p^2} |(rx_1 - x_2)(rx_1 + x_2)|^s d\mu = \int_{\mathbb{Z}_p^2} |x_1 x_2|^s d\mu = \left(\frac{1 - p^{-1}}{1 - p^{-1-s}}\right)^2.$$

By substituting $\mathrm{Z}^{\star}_{\mathbf{F}}(s-1)$ into (7.5), we obtain

$$\zeta^{\triangleleft}_{R,p}(s) = \frac{1}{(1 - p^{-s})^2}.$$

Finally, suppose that $p \neq 2$, $p \nmid d$ and d is not a quadratic residue modulo p. For $(x_1, x_2) \in \mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2$ one of the valuations $v_p(dx_1^2), v_p(x_2^2)$ must be zero, additionally dx_1^2 is not a quadratic residue modulo p and, of course, x_2^2 is a quadratic residue modulo p. Therefore, $v_p(dx_1^2 - x_2^2) = 0$ and

$$\mathrm{Z}^{\star}_{\mathbf{F}}(s) = \mu(\mathbb{Z}_p^2 \setminus p\mathbb{Z}_p^2) = 1 - p^{-2}.$$

By substituting $\mathrm{Z}^{\star}_{\mathbf{F}}(s-1)$ into (7.5),

$$\zeta^{\triangleleft}_{R,p}(s) = \frac{1}{1 - p^{-2s}}.$$

□

Proposition 7.19 is well-known from the classical theory of Dedekind zeta functions. However, the subring zeta function of the ring of integers is a new object of study, although similar Dirichlet series have been studied, see for example [31].

Proposition 7.20. *Let R be the ring of integers in a quadratic number field K.*

$$\zeta^{\leq}_{R,p}(s) = \begin{cases} \frac{1-p^{-2s}}{(1-p^{-s})^2(1-p^{1-3s})} & \text{if } (p) \text{ is ramified in } R, \\ \frac{1-p^{-4s}}{(1-p^{-s})(1-p^{-2s})(1-p^{1-3s})} & \text{if } (p) \text{ is inert in } R, \\ \frac{1-p^{-2s}}{(1-p^{-s})^3(1-p^{1-3s})} & \text{if } (p) \text{ is split in } R. \end{cases} \qquad (7.6)$$

Proof. Let R be the rings of integers in the number field $K = \mathbb{Q}(\sqrt{d})$. We prove the statement in the case $d \equiv 3 \bmod 4$. The proofs of the other cases are very similar.

By Theorem 7.9 the p-local subring zeta function

$$\zeta_{R,p}^{\leq}(s) = \frac{1}{(1-t)(1-pt)} - \frac{pt}{(1-p^{-1})(1-pt)} \, \mathrm{Z}_F(s-1), \tag{7.7}$$

where $F(x_1, x_2) = x_1(dx_1^2 - x_2^2)$. It suffices to compute

$$\mathrm{Z}_F^{\star}(s) = \int_{\mathbb{Z}_p^2 \smallsetminus p\mathbb{Z}_p^2} |x_1(dx_1^2 - x_2^2)|^s d\mu.$$

First suppose that $p = 2$. We break up $\mathrm{Z}_F^{\star}(s)$ over the cosets modulo $p\mathbb{Z}_p^2$. From the proof of Proposition 7.5 we know that

$$v_2(dx_1^2 - x_2^2) = \begin{cases} 0 & \text{if } (x_1, x_2) \equiv (0,1) \text{ or } (x_1, x_2) \equiv (1,0) \bmod 2, \\ 1 & \text{if } (x_1, x_2) \equiv (1,1) \bmod 2. \end{cases}$$

Therefore,

$$\mathrm{Z}_F^{\star}(s) = \int_{2\mathbb{Z}_2^2} |x_1|^s d\mu + 2^{-2} + 2^{-2-s} = \frac{(1-2^{-1})2^{-2-s}}{1-2^{-1-s}} + 2^{-2} + 2^{-2-s}.$$

By substituting $\mathrm{Z}_F(s-1) = \frac{1}{1-2^{1-3s}} \mathrm{Z}_F^{\star}(s-1)$ into formula (7.7) and performing some algebraic simplification,

$$\zeta_{R,2}^{\leq}(s) = \frac{1-2^{-2s}}{(1-2^{-s})^2(1-2^{1-3s})}.$$

Now suppose that $p|d$. We decompose the integral $\mathrm{Z}_F^{\star}(s)$ into cosets modulo p. The integral over a coset $(a_1, a_2) + p(p\mathbb{Z}_p)^2$ only depends on whether a_1, a_2 are units or non-units. We gather the cosets into three sets:

$$\begin{aligned} \mathrm{Z}_F^{\star}(s) &= (p-1)\int_{p(\mathbb{Z}_p)^2} |x_1|^s d\mu + (p-1)^2 p^{-2} + (p-1)p^{-2-s} \\ &= \frac{(p-1)(1-p^{-1})p^{-2-s}}{1-p^{-1-s}} + (p-1)^2 p^{-2} + (p-1)p^{-2-s}. \end{aligned}$$

By substituting $\mathrm{Z}_F(s-1) = \frac{1}{1-p^{1-3s}} \mathrm{Z}_F^{\star}(s)$ into formula (7.7),

$$\zeta_{R,p}^{\leq}(s) = \frac{1-p^{-2s}}{(1-p^{-s})^2(1-p^{1-3s})}.$$

Now suppose that $p \neq 2$ and $p \nmid d$. Additionally suppose that d is a quadratic residue modulo p. There is $\tilde{r} \in \mathbb{F}_p$ such that $\tilde{r}^2 - d \equiv 0 \bmod p$. As $p \neq 2$, by Hensel's Lemma, $\tilde{r}$ lifts to $r \in \mathbb{Z}_p$ such that $r^2 = d$. Therefore, $x_1(dx_1^2 - x_2^2) = x_1(rx_1 - x_2)(rx_1 + x_2)$ and

$$\begin{aligned}
\mathrm{Z}_F^\star(s) &= \sum_{\mathbf{a} \in \mathbb{F}_p \smallsetminus \{\mathbf{0}\}} \int_{\substack{\mathbf{x} \in \mathbb{Z}_p^2 \smallsetminus p\mathbb{Z}_p^2 \\ \mathbf{x} \equiv \mathbf{a} \bmod p}} |x_1(rx_1 - x_2)(rx_1 + x_2)|^s d\mu \\
&= 3(p-1) \int_{(p\mathbb{Z}_p)^2} |x_1|^s d\mu + (p-1)(p-2)p^{-2} \\
&= 3(p-1) \frac{(1-p^{-1})p^{-2-s}}{1-p^{-1-s}} + (p-1)(p-2)p^{-2}.
\end{aligned}$$

By substituting $\mathrm{Z}_F(s-1) = \frac{1}{1-p^{1-3s}} \mathrm{Z}_F^\star(s)$ into formula (7.7),

$$\zeta_{R,p}^{\leq}(s) = \frac{1-p^{-2s}}{(1-p^{-s})^3(1-p^{1-3s})}.$$

Finally, suppose that $p \neq 2$, $p \nmid d$ and that d is not a quadratic residue modulo p. For $(x_1, x_2) \in \mathbb{Z}_p^2 \smallsetminus p\mathbb{Z}_p^2$, the valuation $v_p(dx_1^2 - x_2^2)$ is identically zero. Therefore,

$$\begin{aligned}
\mathrm{Z}_F^\star(s) = \int_{\mathbb{Z}_p^2 \smallsetminus p\mathbb{Z}_p^2} |x_1|^s d\mu &= (p-1) \int_{(p\mathbb{Z}_p)^2} |x_1|^s d\mu + (p^2-p)p^{-2} \\
&= (p-1) \frac{(1-p^{-1})p^{-2-s}}{1-p^{-1-s}} + (p^2-p)p^{-2}.
\end{aligned}$$

By substituting $\mathrm{Z}_F(s-1) = \frac{1}{1-p^{1-3s}} \mathrm{Z}_F^\star(s)$ into formula (7.7),

$$\zeta_{R,p}^{\leq}(s) = \frac{1-p^{-4s}}{(1-p^{-s})(1-p^{-2s})(1-p^{1-3s})}.$$

□

Theorem 7.21. *Let R be the ring of integers in a quadratic number field K,*

$$\zeta_R^{\leq}(s) = \frac{\zeta(s)\zeta(3s-1)\zeta_K(s)}{\zeta_K(2s)}.$$

Proof. This follows from inspection of Propositions 7.19 and 7.20. □

Corollary 7.22. *Let R be the ring of integers in a quadratic number field K. There exists $\gamma \in \mathbb{R}$ such that*

$$\sum_{i=1}^{n} a_n^{\leq}(R) \sim \gamma n \log n.$$

Proof. The zeta function $\zeta_R^{\leq}(s)$ is a quotient of Dedekind zeta functions; it has meromorphic continuation to the whole complex plane. It satisfies the hypothesis of Theorem [10, Theorem 4.20] and the result follows. $\square$

In the remainder of this section we present a number of examples of 2-dimensional rings whose subring and ideal zeta functions are calculated using Theorems 7.9-7.12. The computations are similar to those in Propositions 7.15, 7.19 and 7.20 and are omitted.

Example 7.23. Let R be the 2-dimensional ring with trivial multiplication. For any basis it has structure matrix

$$\mathcal{R}_R(\mathbf{Y}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The zeta functions are equal to the subgroup zeta function of $\mathbb{Z}^2$.

$$\zeta_R^{\leq}(s) = \zeta_R^{\triangleleft}(s) = \zeta(s)\zeta(s-1).$$

Example 7.24. Let C_2 denote the group of order 2. Let $R = \mathbb{Z}[C_2]$, the integral group ring of C_2. The ring R is 2-dimensional. For the basis $\{1e, 1a\}$, where e denotes the identity element and a denotes the non-trivial element of C_2, the ring R has structure matrix

$$\mathcal{R}_R(\mathbf{Y}) = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & Y_1 \end{pmatrix}.$$

The subring and ideal zeta functions are

$$\zeta_R^{\leq}(s) = \frac{\zeta(s)^3\zeta(3s-1)}{\zeta(2s)^2},$$

$$\zeta_R^{\triangleleft}(s) = \zeta(s)^2.$$

Note that the subring and ideal zeta functions of $\mathbb{Z}[C_2]$ are equal to those of $\mathbb{Z}^2$. However, these rings are clearly not isomorphic. This shows that the subring and ideal zeta functions do not determine the isomorphism class of a ring.

Example 7.25. Let R be the 2-dimensional soluble Lie ring given by the presentation $\langle x, y \mid [x, y] = x \rangle$, with respect to the basis $\{x, y\}$ it has structure matrix

$$\mathcal{R}_R(\mathbf{Y}) = \begin{pmatrix} 0 & Y_1 \\ -Y_1 & 0 \end{pmatrix}$$

and subring and ideal zeta functions

$$\zeta_R^{\leq}(s) = \zeta(s)\zeta(s-1),$$
$$\zeta_R^{\triangleleft}(s) = \zeta(s)\zeta(2s-1).$$

Example 7.26. Let R be a 2-dimensional ring with structure matrix

$$\mathcal{R}_R(\mathbf{Y}) = \begin{pmatrix} Y_1 & 0 \\ Y_2 & Y_1 \end{pmatrix}$$

The ring R is not commutative. We have,

$$\zeta_R^{\leq}(s) = \frac{\zeta(s)\zeta(2s-1)\zeta(2s-3)}{\zeta(4s-2)},$$
$$\zeta_R^{\triangleleft_\ell}(s) = \zeta(s)\zeta(2s-1),$$
$$\zeta_R^{\triangleleft_\ell}(s) = \zeta_R^{\triangleleft}(s) = \zeta(2s).$$

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