A Coalgebraic Approach to Linear-Time Logics

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Abstract. We extend recent work on defining linear-time behaviour for state-based systems with branching, and propose modal and fixpoint logics for specifying linear-time temporal properties of states in such systems. We model systems with branching as coalgebras whose type arises as the composition of a branching monad and a polynomial endofunctor on the category of sets, and employ a set of truth values induced canonically by the branching monad. This yields logics for reasoning about quantitative aspects of linear-time behaviour. Examples include reasoning about the probability of a linear-time behaviour being exhibited by a system with probabilistic branching, or about the minimal cost of a linear-time behaviour being exhibited by a system with weighted branching. In the case of non-deterministic branching, our logic supports reasoning about the *possibility* of exhibiting a given linear-time behaviour, and therefore resembles an existential version of the logic LTL.

1 Introduction

Linear-time temporal logics such as LTL interpreted over non-deterministic transition systems and its probabilistic interpretation over Markov chains (see e.g. [1]) have been used successfully as specification logics in model checking. These logics share the same notion of linear-time behaviour, and employ a set of truth values which depends on the type of branching: a two-valued logic is used for non-deterministic transition systems, whereas elements of the unit interval are the possible truth values in the case of Markov chains. Despite such commonalities, a general and uniform account of linear-time logics is still missing.

The present paper fills this gap by building on recent work on defining linear-time behaviour for states in coalgebras with branching [2]. We model systems as coalgebras whose type incorporates branching, and define modal and fixpoint logics that are *parametric* in both the branching type and the transition type. The branching type canonically induces a set of truth values, whereas the transition type canonically induces the notion of observable linear behaviour. In addition to non-deterministic and probabilistic branching, our approach also instantiates to weighted branching. Our approach can be summarised as follows:

- We model systems as coalgebras of an endofunctor obtained as the composition of a *branching monad* $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ with a *polynomial endofunctor* $F : \mathsf{Set} \to \mathsf{Set}$. The elements of the final F-coalgebra provide the observable linear-time behaviours, whereas the set $\mathsf{T}1$ (with 1 a one-element set) is taken as domain of truth values.

- Fundamental to our approach is a (partial) semiring structure on the set $\mathsf{T}1$, studied in [8, 3, 2]. On the one hand, its (partial) addition operation induces an order on $\mathsf{T}1$ which is used to generalise the notion of predicate typically employed in the semantics of modal logics, by considering predicates valued in $\mathsf{T}1$. This subsequently supports the interpretation of fixpoint formulas. On the other hand, the multiplication operation on $\mathsf{T}1$ is used to canonically associate a set of *predicate liftings* to a polynomial endofunctor on Set.
- We employ two kinds of liftings of endofunctors on Set to the category of generalised predicates: one is inspired by coalgebraic modal logic (see e.g. [9]) and is used to provide semantics to individual modalities of a linear-time logic, while another is used to abstract away branching.
- We define modal and fixpoint linear-time logics for coalgebras with branching, and provide an alternative relational semantics for these logics that is amenable to model checking. The relational semantics relies on a generalised notion of relation lifting studied in [2], and currently applies to fixpoint formulas with only one type of fixpoints (either least or greatest ones).

While our approach builds on [2], the study of generalised predicate liftings and the definition of linear-time modal and fixpoint logics are new. Our results apply to systems with probabilistic or weighted branching, and yield linear-time logics for reasoning about the probability or the minimal cost of exhibiting a given linear-time behaviour. Our relational semantics provides a *global approach to model-checking linear-time logics*, whereby the truth values of all sub-formulas of a given formula are computed simultaneously. In this approach, computing the truth values of desirable (undesirable) system properties formalised using least (respectively greatest) fixpoints is done through a sequence of approximations, and this computation can be stopped once a satisfactory threshold is reached.

The remainder of this paper is structured as follows. Section 2 introduces basic definitions (Section 2.1) and gives a summary of our previous work on linear-time behaviour (Sections 2.2 and 2.3). Section 3 defines generalised predicate liftings, which are used in Section 4 to define multi-valued, linear-time modal logics for coalgebras with branching. Fixpoint extensions of these logics are considered in Section 5, where an outline of a relational approach to model checking such logics is also given. Section 6 describes ongoing and future work.

2 Preliminaries

2.1 Monads and Semirings

In what follows, we use monads (T, η, μ) on Set (where $\eta : \mathsf{Id} \Rightarrow \mathsf{T}$ and $\mu : \mathsf{T} \circ \mathsf{T} \Rightarrow \mathsf{T}$ are the *unit* and *multiplication* of T) to capture branching in coalgebraic types. Moreover, we assume that these monads are *strong* and *commutative*. A *strong monad* is equipped with a *strength map* $\mathsf{st}_{X,Y} : X \times \mathsf{T}Y \to \mathsf{T}(X \times Y)$, natural in X and Y and subject to coherence conditions w.r.t. η and μ. For such a monad, one can also define a *swapped strength map* $\mathsf{st}'_{X,Y} : \mathsf{T}X \times Y \to \mathsf{T}(X \times Y)$ by:

$$\mathsf{T}X \times Y \xrightarrow{\mathsf{tw}_{\mathsf{T}X,Y}} Y \times \mathsf{T}X \xrightarrow{\mathsf{st}_{Y,X}} \mathsf{T}(Y \times X) \xrightarrow{\mathsf{T}\mathsf{tw}_{Y,X}} \mathsf{T}(X \times Y)$$

where $\mathsf{tw}_{X,Y} : X \times Y \to Y \times X$ is the *twist map* taking $(x, y) \in X \times Y$ to (y, x). *Commutative monads* are strong monads where the maps $\mu_{X,Y} \circ \mathsf{T}\mathsf{st}'_{X,Y} \circ \mathsf{st}_{\mathsf{T}X,Y} : \mathsf{T}X \times \mathsf{T}Y \to \mathsf{T}(X \times Y)$ and $\mu_{X,Y} \circ \mathsf{T}\mathsf{st}_{X,Y} \circ \mathsf{st}'_{X,\mathsf{T}Y} : \mathsf{T}X \times \mathsf{T}Y \to \mathsf{T}(X \times Y)$ coincide, yielding a *double strength map* $\mathsf{dst}_{X,Y} : \mathsf{T}X \times \mathsf{T}Y \to \mathsf{T}(X \times Y)$ for each choice of sets X, Y.

Example 1. As examples of monads, we consider:

1. the *powerset monad* $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, given by $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$, modelling non-deterministic computations, with unit given by singletons and multiplication given by unions. Its strength and double strength are given by
$$\mathsf{st}_{X,Y}(x, V) = \{x\} \times V \qquad\qquad \mathsf{dst}_{X,Y}(U, V) = U \times V$$
for $x \in X$, $U \in \mathcal{P}X$ and $V \in \mathcal{P}Y$.
2. the *sub-probability distribution monad* $\mathcal{S} : \mathsf{Set} \to \mathsf{Set}$, given by
$$\mathcal{S}(X) = \{\varphi : X \to [0, 1] \mid \sum_{x \in \mathsf{supp}(\varphi)} \varphi(x) \leq 1\}$$
and modelling probabilistic computations. Here, $\mathsf{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ is called the *support* of φ. The unit of $\mathcal{S}$ is given by the Dirac distributions (i.e. $\eta_X(x) = (x \mapsto 1)$), and its multiplication is given by $\mu_X(\Phi) = \sum_{\varphi \in \mathsf{supp}(\Phi)} \sum_{x \in \mathsf{supp}(\varphi)} \Phi(\varphi) * \varphi(x)$, with $*$ denoting multiplication on $[0, 1]$. Its strength and double strength are given by
$$\mathsf{st}_{X,Y}(x, \psi)(z, y) = \begin{cases} \psi(y) & \text{if } z = x \\ 0 & \text{otherwise} \end{cases} \qquad \mathsf{dst}_{X,Y}(\varphi, \psi)(z, y) = \varphi(z) * \psi(y)$$
for $x \in X$, $\varphi \in \mathcal{S}(X)$, $\psi \in \mathcal{S}(Y)$, $z \in X$ and $y \in Y$.
3. the *semiring monad* $\mathsf{T}_S : \mathsf{Set} \to \mathsf{Set}$ with $(S, +, 0, \bullet, 1)$ a commutative semiring, given by
$$\mathsf{T}_S(X) = \{f : X \to S \mid \mathsf{supp}(f) \text{ is finite}\}$$
Its unit, multiplication, strength and double strength are defined similarly to the sub-probability distribution monad (see [2] for details). As a concrete example we consider the semiring $W = (\mathbb{N}^\infty, \min, \infty, +, 0)$ (sometimes called the *tropical semiring*), and use T_W to model weighted computations.

We restrict attention to commutative, *partially additive* monads [2], as these have been shown in loc. cit. to induce partial commutative semirings, whose carriers will serve as our domains of truth values. To define partial additivity, we begin by observing that any monad $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ with $\mathsf{T}\emptyset = 1$ is such that, for any X, $\mathsf{T}X$ has a *zero element* $0 \in \mathsf{T}X$, obtained as $(\mathsf{T}!_X)(*)$, where $*$ denotes the unique element of 1. This yields a *zero map* $0 : Y \to \mathsf{T}X$ for any X, Y, given by the composition
$$Y \xrightarrow{!_Y} \mathsf{T}\emptyset \xrightarrow{T!_X} \mathsf{T}X$$

with the maps $!_Y : Y \to \mathsf{T}\emptyset$ and $!_X : \emptyset \to X$ arising by finality and initiality, respectively. Partial additivity is then defined using the following map:

$$T(X+Y) \xrightarrow{\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle} \mathsf{T}X \times \mathsf{T}Y \tag{1}$$

where $p_1 = [\eta_X, 0] : X + Y \to \mathsf{T}X$ and $p_2 = [0, \eta_Y] : X + Y \to \mathsf{T}Y$.

Definition 1. *A monad* $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ *is called* partially additive *if* $\mathsf{T}\emptyset = 1$ *and the map in (1) is a monomorphism.*

Remark 1. When the map in (1) is an isomorphism, then T is called *additive*. Additive monads were studied in [8, 3].

A (partially) additive monad T induces a (partial) addition operation $+$ on the set $\mathsf{T}X$, given by $\mathsf{T}[1_X, 1_X] \circ q_{X,X}$:

$$\mathsf{T}X \xleftarrow{\mathsf{T}[1_X,1_X]} \mathsf{T}(X+X) \underset{q_{X,X}}{\overset{\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle}{\rightleftarrows}} \mathsf{T}X \times \mathsf{T}X, \qquad + : \mathsf{T}X \times \mathsf{T}X \to \mathsf{T}X$$

where $q_{X,X} : \mathsf{T}X \times \mathsf{T}X \to \mathsf{T}(X+X)$ is the (partial) left inverse of the map $\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle$. That is, $a + b$ is defined if and only if $(a, b) \in \mathsf{Im}(\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle)$. Hence, when T is additive, $+$ is a total operation.

The next result relates commutative, partially additive monads to *partial commutative semirings*. The latter are given by a set S carrying a partial commutative monoid structure $(S, +, 0)$, as well as a commutative monoid structure $(S, \bullet, 1)$, with $\bullet$ distributing over $+$. Specifically, for all $s, t, u \in S$, $s \bullet 0 = 0$, and whenever $t + u$ is defined, so is $s \bullet t + s \bullet u$ and moreover $s \bullet (t + u) = s \bullet t + s \bullet u$.

Proposition 1 ([2]). *Let* T *be a commutative, (partially) additive monad. Then* $(\mathsf{T}1, 0, +, \bullet, \eta_1(*))$ *is a (partial) commutative semiring.*

Example 2. For the monads in Example 1, one obtains the commutative semirings $(\{\bot, \top\}, \vee, \bot, \wedge, \top)$ when $\mathsf{T} = \mathcal{P}$ and S when $\mathsf{T} = \mathsf{T}_S$, and the partial commutative semiring $([0, 1], +, 0, *, 1)$ when $\mathsf{T} = \mathcal{S}$ (where in the latter case $a + b$ is defined if and only if $a + b \leq 1$).

2.2 Generalised Relations and Relation Lifting

Throughout this section we fix a partial commutative semiring $(S, +, 0, \bullet, 1)$ and, following [2], define a preorder relation $\sqsubseteq$ on S by

$$x \sqsubseteq y \quad \text{if and only if} \quad \text{there exists } z \in S \text{ such that } x + z = y \tag{2}$$

for $x, y \in S$. It follows immediately from the axioms of a partial commutative semiring (see [2]) that $\sqsubseteq$ has $0 \in S$ as bottom element and is preserved by $\bullet$ in each argument.

We now let Rel denote the category[1] with objects given by triples (X, Y, R), where $R : X \times Y \to S$ is a function defining a *multi-valued relation* (or S-*relation*), and with arrows from (X, Y, R) to (X', Y', R') given by pairs of functions (f, g) as below, such that $R \sqsubseteq R' \circ (f \times g)$:

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & X' \times Y' \\ {\scriptstyle R}\downarrow & \sqsubseteq & \downarrow{\scriptstyle R'} \\ S & = & S \end{array}$$

Here, the order $\sqsubseteq$ on S has been extended pointwise to S-relations with the same carrier. We write $q : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$ for the functor taking (X, Y, R) to (X, Y) and (f, g) to itself. It follows easily that q is a fibration, with reindexing functors $(f, g)^* : \mathsf{Rel}_{X',Y'} \to \mathsf{Rel}_{X,Y}$ taking $R' : X' \times Y' \to S$ to $R' \circ (f \times g) : X \times Y \to S$. We also write $\mathsf{Rel}_{X,Y}$ for the *fibre over* (X, Y), i.e. the subcategory of Rel with objects given by S-relations over $X \times Y$ and arrows given by $(1_X, 1_Y)$.

[2] shows how to canonically lift *polynomial* endofunctors on Set (that is, endofunctors constructed from identity and constant functors using *finite* products and set-indexed coproducts) to the category of generalised relations. To define such liftings, an additional assumption that the unit 1 of $\bullet$ is a top element for $\sqsubseteq$ is made. The *relation lifting of* $F : \mathsf{Set} \to \mathsf{Set}$ is an endofunctor $\mathsf{Rel}(F) : \mathsf{Rel} \to \mathsf{Rel}$ making the following diagram commute:

$$\begin{array}{ccc} \mathsf{Rel} & \xrightarrow{\mathsf{Rel}(F)} & \mathsf{Rel} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{Set} \times \mathsf{Set} & \xrightarrow[F \times F]{} & \mathsf{Set} \times \mathsf{Set} \end{array}$$

The definition of $\mathsf{Rel}(F)$ is by induction on the structure of F, and makes use of the $\bullet$ operation in the case of products of polynomial functors. The reader is referred to [2] for details.

A special relation lifting, called *extension lifting* and induced canonically by a commutative, partially additive monad T, is also defined in [2]. This time, relations are valued into the partial commutative semiring induced by T (i.e. $S = \mathsf{T}1$), and the extension lifting $\mathsf{E_T} : \mathsf{Rel} \to \mathsf{Rel}$ lifts the endofunctor $\mathsf{T} \times \mathsf{Id}$ to Rel

$$\begin{array}{ccc} \mathsf{Rel} & \xrightarrow{\mathsf{E_T}} & \mathsf{Rel} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{Set} \times \mathsf{Set} & \xrightarrow[\mathsf{T} \times \mathsf{Id}]{} & \mathsf{Set} \times \mathsf{Set} \end{array}$$

and takes $R : X \times Y \to \mathsf{T}1$ to the relation $\mathsf{E_T}(R) : \mathsf{T}X \times Y \to \mathsf{T}1$ given by

$$\mathsf{T}X \times Y \xrightarrow{\mathsf{st}'_{X,Y}} \mathsf{T}(X \times Y) \xrightarrow{\mathsf{T}(R)} \mathsf{T}^2 1 \xrightarrow{\mu_1} \mathsf{T}1$$

[1] To keep notation simple, the dependency on S is left implicit.

(The actual definition of the extension lifting in [2] is given in terms of unique 1-linear extensions of relations of type $X \times Y \to \mathsf{T}1$ to relations of type $\mathsf{T}X \times Y \to \mathsf{T}1$. However, as shown in loc. cit., the above is an equivalent characterisation.)

2.3 Linear-Time Behaviour via Relation Lifting

This section summarises the definition of the linear-time behaviour of a state in a coalgebra with branching, as proposed in [2]. The approach in loc. cit. applies to coalgebras of functors obtained as arbitrary compositions of a single branching monad and a finite number of polynomial endofunctors on Set. Here we restrict attention to compositions of type $\mathsf{T} \circ F$. Thus, we model systems with branching as $\mathsf{T} \circ F$-coalgebras on Set, where the partially additive, commutative monad $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ specifies the type of branching, and the polynomial endofunctor $F : \mathsf{Set} \to \mathsf{Set}$ specifies the structure of individual transitions.

Given an arbitrary endofunctor $F : \mathsf{Set} \to \mathsf{Set}$, an F*-coalgebra* is given by a pair (C, γ) with C a set (of states), and $\gamma : C \to FC$ a function describing the one-step evolution of the states. The notion of *coalgebraic bisimulation* provides a canonical and uniform observational equivalence relation between states of F-coalgebras. One of the many (and under mild assumptions, equivalent) definitions of bisimulation involves lifting the endofunctor F to the category of *standard relations* (obtained in our setting by taking $S = (\{\bot, \top\}, \vee, \bot, \wedge, \top)$). A similar approach is taken in [2] to define the extent to which a state in a coalgebra with branching can exhibit a given linear-time behaviour. The definition in loc. cit. differs from the relational definition of bisimulation (for which we refer the reader to [6]) in two ways: (i) generalised relations are used in place of standard relations, and (ii) the relation lifting employed also involves the extension lifting E_T defined earlier, as the goal is to relate branching-time and linear-time behaviours, as opposed to behaviours of the same coalgebraic type.

Having fixed the branching type T and the transition type F, the final F-coalgebra (Z, ζ) provides a natural choice as domain of observable linear-time behaviours (which we will also refer to as *maximal traces*), whereas the (partial) commutative semiring $(\mathsf{T}1, +, 0, \bullet, 1)$ induced by T (see Proposition 1) provides, as argued in [2], a natural choice as set of truth values. Throughout this section, and in the remainder of the paper, we assume that the preorder $\sqsubseteq$ induced by this semiring (defined in (2)) is an ω^{op}-chain complete partial order, and has the unit 1 of $\bullet$ as top element. This assumption is satisfied by the preorders associated to the semirings in Example 2, namely $\leq$ on $\{\bot, \top\}$ for $\mathsf{T} = \mathcal{P}$, $\leq$ on $[0, 1]$ for $\mathsf{T} = \mathcal{S}$, and $\geq$ on $\mathbb{N}^\infty$ for $\mathsf{T} = \mathsf{T}_W$.

The next definition provides a canonical notion of linear-time behaviour of states in coalgebras with branching. It is inspired by a characterisation of coalgebraic bisimilarity (i.e. the largest bisimulation) between states of coalgebras of the same type as the greatest fixpoint of a monotone operator on the category of standard relations (see e.g. [2][Section 2.2] for a summary). It also resembles partition refinement algorithms for computing largest bisimulations on labelled transition systems with finite state spaces [7].

Definition 2 ([2]). *The* linear-time behaviour *of a state in a* $\mathsf{T} \circ F$*-coalgebra* (C, γ) *is the greatest fixpoint of the operator* O *on* $\mathsf{Rel}_{C,Z}$ *given by the composition*

$$\mathsf{Rel}_{C,Z} \xrightarrow{\mathsf{Rel}(F)} \mathsf{Rel}_{FC,FZ} \xrightarrow{\mathsf{E}_\mathsf{T}} \mathsf{Rel}_{\mathsf{T}(FC),FZ} \xrightarrow{(\gamma\times\zeta)^*} \mathsf{Rel}_{C,Z} \qquad (3)$$

Monotonicity of the operator O is an immediate consequence of the functoriality of $\mathsf{Rel}(F)$, E_T and $(\gamma \times \delta)^*$. The existence of a greatest fixpoint for O is then guaranteed by the following standard result on the existence of fixpoints in chain-complete partial orders, applied to the *dual* of the order $\sqsubseteq$.

Theorem 1 ([4, Theorem 8.22]). *Let* P *be a complete partial order and let* $\mathsf{O} : P \to P$ *be order-preserving. Then* O *has a least fixpoint.*

Example 3. For $T = \mathcal{P}$, the greatest fixpoint of O relates a state c in a $\mathcal{P} \circ F$-coalgebra (C, γ) with a state z of the final F-coalgebra if and only if there exists a sequence of choices in the unfolding of γ starting from c, that results in an F-behaviour bisimilar to z. For $T = T_\mathcal{S}$, the greatest fixpoint of O yields, for each state in a $\mathcal{S} \circ F$-coalgebra and each maximal trace, the accumulated probability of this trace being exhibited (across all branches). In particular, for *infinite* traces, the associated probability is often 0. The logics defined later provide the ability to also reason about the probability of exhibiting *finite prefixes* of infinite traces. For $\mathsf{T} = \mathsf{T}_W$, the greatest fixpoint of O maps a pair (c, z), with c a state in a $\mathsf{T}_W \circ F$-coalgebra and z a maximal trace, to the minimal cost of exhibiting that trace. Intuitively, this is computed by adding the weights of individual transitions along the same branch, and minimising this sum across the various branches.

Remark 2. We recall from [2] that a relation between states of two $\mathsf{T} \circ F$-coalgebras (C, γ) and (D, δ) can also be defined in a similar way, namely as the greatest fixpoint of the operator $\mathsf{O}' : \mathsf{Rel}_{C,D} \to \mathsf{Rel}_{C,D}$ given by the composition

$$\mathsf{Rel}_{C,D} \xrightarrow{\mathsf{Rel}(F)} \mathsf{Rel}_{FC,FD} \xrightarrow{\mathsf{E}'_\mathsf{T}} \mathsf{Rel}_{\mathsf{T}(FC),\mathsf{T}(FD)} \xrightarrow{(\gamma\times\zeta)^*} \mathsf{Rel}_{C,D}$$

where $\mathsf{E}'_\mathsf{T} : \mathsf{Rel} \to \mathsf{Rel}$ is the lifting of $\mathsf{T} \times \mathsf{T}$ to Rel:

$$\begin{array}{ccc} \mathsf{Rel} & \xrightarrow{\mathsf{E}_\mathsf{T}} & \mathsf{Rel} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{Set} \times \mathsf{Set} & \xrightarrow[\mathsf{T}\times\mathsf{T}]{} & \mathsf{Set} \times \mathsf{Set} \end{array}$$

taking $R : X \times Y \to \mathsf{T}1$ to the relation $\mathsf{E}'_\mathsf{T}(R) : \mathsf{T}X \times \mathsf{T}Y \to \mathsf{T}1$ given by

$$\mathsf{T}X \times \mathsf{T}Y \xrightarrow{\mathsf{dst}_{X,Y}} \mathsf{T}(X \times Y) \xrightarrow{\mathsf{T}(R)} \mathsf{T}^2 1 \xrightarrow{\mu_1} \mathsf{T}1$$

Example 4. For non-deterministic systems ($\mathsf{T} = \mathcal{P}$), the greatest fixpoint of O' relates two states if and only if they admit a common maximal trace (element of the final F-coalgebra). For probabilistic systems ($\mathsf{T} = \mathcal{S}$), the greatest fixpoint measures the probability of two states exhibiting the same maximal trace (*any* maximal trace), whereas for weighted systems ($\mathsf{T} = \mathsf{T}_W$), the greatest fixpoint measures the *joint* minimal cost of two states exhibiting the same maximal trace. In the latter case, $\mathsf{E}'_W : \mathsf{Rel} \to \mathsf{Rel}$ takes a relation $R : X \times Y \to W$ to the relation $\mathsf{E}'_W(R) : \mathsf{T}_W X \times \mathsf{T}_W Y \to W$ given by

$$\mathsf{E}'_W(R)(f, g) = \min_{x \in \mathsf{supp}(f), y \in \mathsf{supp}(g)} (f(x) + g(y) + R(x, y))$$

The modal and fixpoint logics we introduce later have a similar flavour to the previous example. In particular, for non-deterministic (respectively probabilistic) systems, the resulting logics will support reasoning about the possibility (respectively likelihood) of a state satisfying a certain linear-time property.

3 Generalised Predicates and Predicate Lifting

The standard approach to defining the semantics of modal logics involves interpreting formulas as predicates over the state space of the system of interest. In the coalgebraic approach to modal logic, individual modal operators are interpreted using so called *predicate liftings* [9]. In order to follow the same approach for linear-time logics, we introduce *generalised predicates*, i.e. predicates valued in a partial commutative semiring $(S, +, 0, \bullet, 1)$ with induced order $\sqsubseteq$.

We let Pred denote the category with objects given by pairs (X, P) with $P : X \to S$ a function defining a *multi-valued predicate* (or *S-predicate*), and arrows from (X, P) to (X', P') given by functions $f : X \to X'$ such that $P \sqsubseteq P' \circ f$:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle P}\downarrow & \sqsubseteq & \downarrow{\scriptstyle P'} \\ S & = & S \end{array}$$

As in the case of generalised relations, we obtain a fibration $p : \mathsf{Pred} \to \mathsf{Set}$, with p taking (X, P) to X. The fibre over X is denoted Pred_X, and the reindexing functor $f^* : \mathsf{Pred}_{X'} \to \mathsf{Pred}_X$ takes $P' : X' \to S$ to $P' \circ f : X \to S$.

The next definition generalises predicate liftings as used in the semantics of coalgebraic modal logics [9].

Definition 3. *A* predicate lifting of arity $n \in \omega$ *for an endofunctor* $F : \mathsf{Set} \to \mathsf{Set}$ *is a functor* $\mathsf{L} : \mathsf{Pred}^n \to \mathsf{Pred}$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathsf{Pred}^n & \xrightarrow{\mathsf{L}} & \mathsf{Pred} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathsf{Set} & \xrightarrow[F]{} & \mathsf{Set} \end{array}$$

where the category Pred^n *has objects given by tuples* $(X, P_1, \ldots, P_n)$ *with* $P_i : X \to S$ *for* $i \in \{1, \ldots, n\}$*, and arrows from* $(X, P_1, \ldots, P_n)$ *to* $(X', P'_1, \ldots, P'_n)$ *given by functions* $f : X \to X'$ *such that* $P_i \sqsubseteq P'_i \circ f$ *for all* $i \in \{1, \ldots, n\}$*.*

We now restrict attention to polynomial functors $F : \mathsf{Set} \to \mathsf{Set}$, and show how to define a canonical *set* of predicate liftings for F by induction on its structure. Since in Set finite products distribute over arbitrary coproducts, any polynomial endofunctor is naturally isomorphic to a coproduct of finite (including empty) products of identity functors. The next definition exploits this observation.

Definition 4. *Let $F = \coprod_{i \in I} \mathsf{Id}^{j_i}$, with $j_i \in \omega$ for $i \in I$. The set of predicate liftings $\Lambda = \{L_i \mid i \in I\}$ has elements $L_i : \mathsf{Pred}^{j_i} \to \mathsf{Pred}$ with $i \in I$ given by:*

$$(L_i)_X(P_1, \ldots, P_{j_i})(f) \;=\; \begin{cases} P_1(x_1) \bullet \ldots \bullet P_{j_i}(x_{j_i}) & \textit{if } f = (x_1, \ldots, x_{j_i}) \in \iota_i(\mathsf{Id}^{j_i}) \\ 0 & \textit{otherwise} \end{cases}$$

The functoriality of this definition follows from the preservation of $\sqsubseteq$ by $\bullet$.

Example 5. For $F = 1 + A \times \mathsf{Id} \times \mathsf{Id} \;\simeq\; 1 + \coprod_{a \in A} \mathsf{Id} \times \mathsf{Id}$, F-coalgebras are binary trees with internal nodes labelled by elements of A. Definition 4 yields a nullary predicate lifting L_0 and an A-indexed set of binary predicate liftings $(L_a)_{a \in A}$:

$$L_0(f) \;=\; \begin{cases} 1 & \text{if } f = \iota_1(*) \\ 0 & \text{otherwise} \end{cases}$$

$$(L_a)_X(P_1, P_2)(f) \;=\; \begin{cases} P_1(x_1) \bullet P_2(x_2) & \text{if } f = \iota_a(x_1, x_2) \\ 0 & \text{otherwise} \end{cases}$$

Remark 3. One can also define a single, unary predicate lifting $\mathsf{Pred}(F)$ for each polynomial functor $F : \mathsf{Set} \to \mathsf{Set}$, again by induction on the structure of F:

- If $F = \mathsf{Id}$, $\mathsf{Pred}(F)$ takes an S-predicate to itself.
- If $F = 1$, $\mathsf{Pred}(F)$ takes an S-predicate to the predicate $* \mapsto 1$.
- If $F = F_1 \times F_2$, $\mathsf{Pred}(F)(P) : F_1X \times F_2X \to S$ is given by

$$\mathsf{Pred}(F)(P)(f_1, f_2) = \mathsf{Pred}(F_1)(P)(f_1) \bullet \mathsf{Pred}(F_2)(P)(f_2), \quad \text{for } P : X \to S.$$

- if $F = \coprod_{i \in I} F_i$, $\mathsf{Pred}(F)(P) : \coprod_{i \in I} F_iX \to S$ is given by

$$\mathsf{Pred}(F)(P)(\iota_i(f_i)) = \mathsf{Pred}(F_i)(P)(f_i) \quad \text{for } P : X \to S,\ i \in I \text{ and } f_i \in F_iX.$$

Indeed, this is the approach taken in [5]. However, this predicate lifting turns out to yield a modal logic with limited expressive power. We show later how $\mathsf{Pred}(F)$ yields a coinductive interpretation of truth in a system with branching.

Example 6. Let $F : \mathsf{Set} \to \mathsf{Set}$ be as in Example 5. Then $\mathsf{Pred}(F)$ is given by

$$\mathsf{Pred}(F)(P)(\iota_1(*)) = 1 \qquad \mathsf{Pred}(F)(P)(\iota_a(x_1, x_2)) = P(x_1) \bullet P(x_2)$$

As can be seen, the resulting unary modality requires the same property (P) to hold on both the left- and the right subtree.

As we are interested in linear-time logics, a special *extension lifting*, akin to the extension lifting of Section 2.2, will be used to abstract away branching.

Definition 5. *Let* $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ *be a commutative, partially additive monad. The* extension lifting $\mathsf{P_T} : \mathsf{Pred} \to \mathsf{Pred}$ *is the lifting of* $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ *to* Pred

$$\begin{array}{ccc} \mathsf{Pred} & \xrightarrow{\mathsf{P_T}} & \mathsf{Pred} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathsf{Set} & \xrightarrow[\mathsf{T}]{} & \mathsf{Set} \end{array}$$

which takes $P : X \to \mathsf{T}1$ *to the predicate* $\mathsf{P_T}(P) : \mathsf{T}X \to \mathsf{T}1$ *given by* $\mu_1(\mathsf{T}(P))$.

Remark 4. As in the case of $\mathsf{E_T}$, $\mathsf{P_T}(P)$ can alternatively be defined as the unique extension of $P : X \to \mathsf{T}1$ to a T-algebra homomorphism $(\mathsf{T}X, \mu_X) \to (\mathsf{T}1, \mu_1)$.

4 Linear-Time Modal Logics

We are now ready to define linear-time logics for coalgebras of type $\mathsf{T} \circ F$, where the partially additive monad $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ and the polynomial functor $F : \mathsf{Set} \to \mathsf{Set}$ are used as in Section 2.3. Our logics will be valued into the partial semiring $(\mathsf{T}1, +, 0, \bullet, 1)$ induced by the monad T (see Section 2.1).

We begin by fixing a set Λ of modal operators with associated predicate liftings $(\mathsf{P}_\lambda)_{\lambda \in \Lambda}$ for F. A canonical choice for Λ is given by the set of predicate liftings in Definition 4. The next definition adapts the definition of coalgebraic modal logics [9] in order to provide reasoning about linear-time behaviours.

Definition 6. *The logic* $\mathcal{L}_\Lambda$ *has syntax given by*

$$\varphi ::= \top \mid [\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)})$$

with $\lambda \in \Lambda$ *of arity* $\mathsf{ar}(\lambda)$*, and semantics* $[\![\_]\!]_\gamma : \mathcal{L}_\Lambda \to \mathsf{Pred}_C$ *(where* (C, γ) *is a* $\mathsf{T} \circ F$*-coalgebra) defined inductively on the structure of formulas by*

- $[\![\top]\!]_\gamma(c) = \top$
- $[\![[\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)})]\!]_\gamma = \gamma^*(\mathsf{P_T}(\mathsf{P}_\lambda([\![\varphi_1]\!]_\gamma, \ldots, [\![\varphi_n]\!]_\gamma)))$

where $\gamma^* : \mathsf{Pred}_C \to \mathsf{Pred}_{\mathsf{T}FC}$ *performs reindexing of predicates along* γ.

The semantics of $\mathcal{L}_\Lambda$ resembles that of coalgebraic modal logics (see e.g. [9]), with two differences: (i) the interpretation of a formula is a generalised predicate over the state space as opposed to a subset of the state space, and (ii) the extension lifting $\mathsf{P_T} : \mathsf{Pred} \to \mathsf{Pred}$ of Definition 5 is used to abstract away branching. In particular, the use of $\mathsf{P_T}$ is what makes $\mathcal{L}_\Lambda$ a *linear-time* logic.

It turns out that an equivalent definition of the semantics of $\mathcal{L}_\Lambda$ can be given in terms of relation lifting. To show this, we let $\mathsf{L}_\Lambda = \sum_{\lambda \in \Lambda} \mathsf{Id}^{\mathsf{ar}(\lambda)}$, and note that $\mathsf{L}_\Lambda(L) \simeq \{[\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)}) \mid \lambda \in \Lambda,\ \varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)} \in L\}$. We now consider the lifting $\mathsf{D} : \mathsf{Rel} \to \mathsf{Rel}$ of the functor $F \times \mathsf{L}_\Lambda : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}$ defined through case analysis by

$$\mathsf{D}(R)(f, [\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)})) = \mathsf{P}_\lambda(R^\sharp(\varphi_1), \ldots, R^\sharp(\varphi_{\mathsf{ar}(\lambda)}))(f)$$

for $R : C \times L \to \mathsf{T}1$, $f \in FC$ and $\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)} \in L$, where $R^\sharp : L \to \mathsf{Pred}_C$ is obtained from R by currying.

Lemma 1. $\mathsf{D} : \mathsf{Rel} \to \mathsf{Rel}$ *is a functor making the following diagram commute:*

$$\begin{array}{ccc} \mathsf{Rel} & \xrightarrow{\mathsf{D}} & \mathsf{Rel} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathsf{Set} \times \mathsf{Set} & \xrightarrow[F \times \mathsf{L}_\Lambda]{} & \mathsf{Set} \times \mathsf{Set} \end{array}$$

Proof (Sketch). Functoriality of D follows from the functoriality of P_λ for $\lambda \in \Lambda$.

An alternative definition of the semantics of $\mathcal{L}_\Lambda$ can now be given by turning the initial $(\{\top\} + \mathsf{L}_\Lambda)$-algebra $(\mathcal{L}_\Lambda, \alpha)$ into a $(\{\top\} + \mathsf{L}_\Lambda)$-coalgebra $(\mathcal{L}_\Lambda, \alpha^{-1})$.

Proposition 2. *Consider the operator* S *on* $\mathsf{Rel}_{C,\mathcal{L}_\Lambda}$ *given by the composition:*

$$\mathsf{Rel}_{C,\mathcal{L}_\Lambda} \xrightarrow{\mathsf{D}} \mathsf{Rel}_{FC,\mathsf{L}_\Lambda \mathcal{L}_\Lambda} \xrightarrow{\mathsf{E}_\mathsf{T}} \mathsf{Rel}_{\mathsf{T}FC,\mathsf{L}_\Lambda \mathcal{L}_\Lambda} \xrightarrow{\mathsf{X}} \mathsf{Rel}_{\mathsf{T}FC,\{\top\}+\mathsf{L}_\Lambda \mathcal{L}_\Lambda} \xrightarrow{(\gamma \times \alpha^{-1})^*} \mathsf{Rel}_{C,\mathcal{L}_\Lambda}$$

where E_T *is the extension lifting of Section 2.2, and where the lifting* $\mathsf{X} : \mathsf{Rel} \to \mathsf{Rel}$ *of* $\mathsf{Id} \times (\{\top\} + \mathsf{Id}) : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set} \times \mathsf{Set}$ *takes* $R : C \times L \to \mathsf{T}1$ *to the relation* $\mathsf{X}(R) : C \times (\{\top\} + L) \to \mathsf{T}1$ *given by*

$$\mathsf{X}(R)(c, \iota_1(\top)) = 1, \qquad \mathsf{X}(R)(c, \iota_2(l)) = R(c, l) \quad \textit{for } c \in C \textit{ and } l \in L.$$

Then, the least and greatest fixpoints of S *coincide, and the semantics of* $\mathcal{L}_\Lambda$ *is obtained via currying from this unique fixpoint* $\mathsf{fix}(\mathsf{S}) \in \mathsf{Rel}_{C,\mathcal{L}_\Lambda}$.

Proof (Sketch). Let $\mathsf{fix}(\mathsf{S})^\sharp : \mathcal{L}_\Lambda \to \mathsf{Pred}_C$ be obtained from $\mathsf{fix}(\mathsf{S}) : C \times \mathcal{L}_\Lambda \to \mathsf{T}1$ by currying. It follows by induction on the modal depth of a formula φ (degree of nesting of the modalities) that for φ of depth n, $\mathsf{fix}(\mathsf{S})(c, \varphi)$ can be computed in n steps for any $c \in C$ and moreover, $\mathsf{fix}(\mathsf{S})^\sharp(\varphi) = [\![\varphi]\!]_\gamma$. The proof of the inductive step exploits the close relationships between D and $(\mathsf{P}_\lambda)_{\lambda \in \Lambda}$ on the one hand, and between the extension liftings E_T and P_T on the other.

In Section 5, a fixpoint extension $\mu\mathcal{L}_\Lambda$ of $\mathcal{L}_\Lambda$ is defined and a similar result is proved for a fragment of $\mu\mathcal{L}_\Lambda$.

We note the absence of conjunction and disjunction from $\mathcal{L}_\Lambda$. Restricted versions of these operators can be incorporated into the modal operators, as illustrated by the next example, and this appears to be sufficient in practice. On the other hand, if the domain of truth values carries a lattice structure (which is the case for all three branching monads considered here), then canonical interpretations for conjunction and disjunction exist. The addition of such operators in the general case, as well as the expressiveness of $\mathcal{L}_\Lambda$, are left as future work.

Example 7. Let $F = 1 + A \times \mathsf{Id} \simeq 1 + \coprod_{a \in A} \mathsf{Id}$, and let the nullary modality $*$, the unary modality $\langle a \rangle$ and the *binary* modality $[a]$ be defined using the predicate liftings $\mathsf{P}_* : 1 \to \mathsf{Pred}$, $\mathsf{P}_{\langle a \rangle} : \mathsf{Pred} \to \mathsf{Pred}$ and $\mathsf{P}_{[a]} : \mathsf{Pred} \times \mathsf{Pred} \to \mathsf{Pred}$ for F, given by

$$\begin{aligned}
&\mathsf{P}_*(\iota_1(*)) = 1 && \mathsf{P}_*(\iota_a(x)) = \mathsf{P}_*(\iota_{a'}(x)) = 0\\
&P_{\langle a\rangle}(P)(\iota_a(x)) = P(x) && \mathsf{P}_{\langle a\rangle}(P)(\iota_1(*)) = \mathsf{P}_{\langle a\rangle}(P)(\iota_{a'}(x)) = 0\\
&\mathsf{P}_{[a]}(P_1,P_2)(\iota_a(x)) = P_1(x) && \mathsf{P}_{[a]}(P_1,P_2)(\iota_{a'}(x)) = P_2(x)\\
&\mathsf{P}_{[a]}(P_1,P_2)(\iota_1(*)) = 0
\end{aligned}$$

where $a' \in A \setminus \{a\}$ in the above. Then, the formula $\langle a\rangle\top$ measures the extent to which the output a is observed in the next step. Also, the formula $[a](\top,*)$ measures the extent to which either the output a is observed in the next step, or an output $a' \neq a$ is observed and following that, the computation terminates.

Modalities of this kind can be defined for an arbitrary polynomial endofunctor, but space limitations prevent us from including the general case here.

We conclude this section with a brief discussion on the expressiveness of $\mathcal{L}_\Lambda$. We immediately note that $\mathcal{L}_\Lambda$ is intended as a specification logic, and therefore finding a semantically-defined relation that captures the indistinguishability of states by formulas is not the primary concern. This paper does not provide a definitive answer on the expressiveness of $\mathcal{L}_\Lambda$. However, it does provide an answer in the case when the predicate liftings in Λ are the canonical ones from Definition 4. In this case, $\mathcal{L}_\Lambda$ is (isomorphic to) the initial $(\{\top\} + F)$-algebra, whose elements can be thought of as *finite trace prefixes*, and formulas of $\mathcal{L}_\Lambda$ measure the extent to which finite-trace prefixes are exhibited by states of $\mathsf{T}\circ F$-coalgebras. Thus, two states are indistinguishable by formulas if and only if the extent to which they can exhibit each *finite* linear-time behaviour is the same.

Example 8. For $F = 1 + A \times \mathsf{Id} \simeq 1 + \coprod_{a\in A} \mathsf{Id}$, finite trace prefixes are in one-to-one correspondence with finite sequences of one of the forms $a_1 \ldots a_n\top$ or $a_1 \ldots a_n*$ with $n \in \omega$ and $a_1,\ldots,a_n \in A$, where the latter sequence is also a maximal trace. For $F = 1 + A \times \mathsf{Id} \times \mathsf{Id} \simeq 1 + \coprod_{a\in A} \mathsf{Id} \times \mathsf{Id}$, finite trace prefixes are given by finite binary trees with internal nodes labelled by elements of A and with leafs labelled by either $*$ or $\top$.

5 Linear-Time Fixpoint Logics

We now extend the logic $\mathcal{L}_\Lambda$ with fixpoints, and describe an approach to model checking a fragment of the resulting logic, whose formulas do not contain both least and greatest fixpoints. In order to interpret both greatest and least fixpoints, we additionally assume that the order $\sqsubseteq$ induced by the (partial) semiring of Proposition 1 is ω-chain complete. This assumption holds in all our examples.

Definition 7. *Let $\mathcal{V}$ be a set of variables. The logic $\mu\mathcal{L}_\Lambda$ has syntax given by*

$$\varphi ::= x \mid \top \mid [\lambda](\varphi_1,\ldots,\varphi_{\mathsf{ar}(\lambda)}) \mid \mu x.\varphi \mid \nu x.\varphi$$

with $x \in \mathcal{V}$ and $\lambda \in \Lambda$, and semantics $[\![\_]\!]^V_\gamma : \mu\mathcal{L}_\Lambda \to \mathsf{Pred}_C$ (where (C,γ) is a $\mathsf{T}\circ F$-coalgebra and $V : \mathcal{V} \to \mathsf{Pred}_C$ is a valuation*) defined inductively on the structure of formulas by*

- $[\![x]\!]_\gamma^V = V(x)$,
- $[\![\mu x.\varphi]\!]_\gamma^{V\setminus\{x\}}$ ($[\![\nu x.\varphi]\!]_\gamma^{V\setminus\{x\}}$) *is the least (respectively greatest) fixpoint of the operator on* Pred_C *defined by* $P \longmapsto [\![\varphi]\!]_\gamma^{V[P/x]}$, *where the valuation* $V[P/x] : \mathcal{V} \to \mathsf{Pred}_C$ *is given by* $V[P/x](x) = P$ *and* $V[P/x](y) = V(y)$ *for* $y \in \mathcal{V}\setminus\{x\}$

and clauses for $\top$ *and* $[\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)})$ *similar to Definition 6.*

The fact that the operator used to interpret fixpoint formulas is order-preserving follows from the functoriality of predicate liftings. Existence of the required least and greatest fixpoints then follows by Theorem 1.

Example 9. For $\mathsf{T} = \mathcal{P}$, predicate liftings for F are as used in the semantics of coalgebraic modal logic [9], and $\mu\mathcal{L}_\Lambda$-formulas can be interpreted on F-coalgebras. In this case, the logic $\mu\mathcal{L}_\Lambda$ can be viewed as an *existential* version of the logic LTL, wherein a linear-time formula holds in a state whenever a trace satisfying the formula can be exhibited from that state. Our logic is however more general, as it applies to transition structures defined by an arbitrary polynomial functor F. For $\mathsf{T} = \mathcal{S}$ or $\mathsf{T} = \mathsf{T}_W$, $\mu\mathcal{L}_\Lambda$-formulas measure the likelihood, respectively minimal cost, of satisfying a certain linear-time property.

Example 10. Using the modalities $[a]$ and $\langle a \rangle$ of Example 7, the extent to which $a \in A$ appears (i) eventually, (ii) always and (iii) infinitely many times in the unfolding of a state in a $\mathsf{T} \circ F$-coalgebra is measured by the formulas $\mu x.[a](\top, x)$, $\nu x.\langle a \rangle x$, and respectively $\nu x.\mu y.[a](x, y)$.

Remark 5. The formula $\nu x.\circ x$, with $\circ$ the predicate lifting defined in Remark 3, can be viewed as providing a coinductive interpretation of truth. When $\mathsf{T} = \mathcal{P}$, $\nu x.\circ x$ holds in a state precisely when there exists a maximal trace from that state, arising from a sequence of choices in the branching behaviour. (Such a path will not exist from a state that offers no choices for proceeding.) For $\mathsf{T} = \mathsf{T}_W$, the truth value associated to $\nu x.\circ x$ in a particular state is the minimum accumulated weight that can be achieved along any maximal trace from that state.

Ongoing work concerns the formulation of a result similar to Proposition 2 for the logic $\mu\mathcal{L}_\Lambda$, and its exploitation for model checking $\mu\mathcal{L}_\Lambda$-formulas. Here we only present a restricted version of such a result, which concerns the fragment of $\mu\mathcal{L}_\Lambda$ whose formulas do not contain both least and greatest fixpoints.

The following definitions are standard in the fixpoint logic literature.

Definition 8. *A formula* $\varphi \in \mu\mathcal{L}_\Lambda$ *is* clean *if every variable is bound at most once in* φ, *and* guarded *if every occurrence of a bound variable appears within the scope of a modal operator. A set* $C \subseteq \mu\mathcal{L}_\Lambda$ *of formulas is* closed *if*

- $\varphi \in C$ *whenever* $[\lambda]\varphi \in C$, *for* $\lambda \in \Lambda$,
- $\varphi[\eta x.\varphi/x] \in C$ *whenever* $\eta x.\varphi \in C$, *for* $\eta \in \{\mu, \nu\}$.

The closure $\mathsf{Cl}(\varphi)$ *of a* $\mu\mathcal{L}_\Lambda$*-formula* φ *is the smallest closed set containing* φ.

We now proceed by observing that the set $\mathcal{F} := \mathsf{Cl}(\varphi)$ carries a $\{\top\} + \mathsf{L}_\Lambda + \mathsf{Id}$-coalgebra structure $\alpha : \mathcal{F} \to \{\top\} + \mathsf{L}_\Lambda\mathcal{F} + \mathcal{F}$, defined by:

- $\alpha(\top) = \iota_1(\top)$,
- $\alpha([\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)})) = \iota_2(\iota_\lambda(\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)}))$,
- $\alpha(\eta x.\varphi) = \iota_3(\varphi[\eta x.\varphi/x])$ for $\eta \in \{\mu, \nu\}$.

Our goal is to characterise the semantics of $\mu\mathcal{L}_\Lambda$ using relation lifting, as this was done for the logic $\mathcal{L}_\Lambda$. The slight difficulty here is that an unfolding of the $\mathsf{T}\circ F$-coalgebra γ (performed along an unfolding of the $\{\top\}+\mathsf{L}_\Lambda+\mathsf{Id}$-coalgebra α) is only required in the case of formulas whose outer-most operator is a modality. For formulas whose outer-most operator is a fixpoint operator, only an unfolding of the respective fixpoint formula should be performed (by unfolding α). This explains the somewhat involved next definition, which, in particular, replaces the coalgebra γ as used in Proposition 2 by the coalgebra $\langle \gamma, \mathsf{id}_C\rangle : C \to \mathsf{T}FC \times C$.

Definition 9. *The operator* $\mathsf{S}_\mu : \mathsf{Rel}_{C,\mathcal{F}} \to \mathsf{Rel}_{C,\mathcal{F}}$ *is defined by the composition*

$$\mathsf{Rel}_{C,\mathcal{F}} \xrightarrow{\mathsf{F}} \mathsf{Rel}_{\mathsf{T}FC\times C,\mathsf{L}_\Lambda\mathcal{F}+\mathcal{F}} \xrightarrow{\mathsf{X}} \mathsf{Rel}_{\mathsf{T}FC\times C,\{\top\}+\mathsf{L}_\Lambda\mathcal{F}+\mathcal{F}} \xrightarrow{(\langle\gamma,\mathsf{id}_C\rangle\times\alpha)^*} \mathsf{Rel}_{C,\mathcal{F}}$$

where the lifting $\mathsf{F} : \mathsf{Rel} \to \mathsf{Rel}$ *of* $((\mathsf{T}\circ F)\times \mathsf{Id})\times(\mathsf{L}_\lambda + \mathsf{Id}) : \mathsf{Set}\times\mathsf{Set} \to \mathsf{Set}\times\mathsf{Set}$ *takes* $R : C \times L \to \mathsf{T}1$ *to the relation* $\mathsf{F}(R) : (\mathsf{T}FC \times C) \times (\mathsf{L}_\Lambda L + L) \to \mathsf{T}1$ *given by*

$$\mathsf{F}(R)((u,c), \iota_1(\iota_\lambda(\varphi_1,\ldots,\varphi_{\mathsf{ar}(\lambda)}))) = \mathsf{E}_\mathsf{T}(\mathsf{D}(R))(u, \iota_\lambda(\varphi_1,\ldots,\varphi_{\mathsf{ar}(\lambda)}))$$
$$\mathsf{F}(R)((u,c), \iota_2(\varphi)) = R(c,\varphi)$$

The lifting F of Definition 9 plays a rôle similar to that of $\mathsf{E}_\mathsf{T}\circ\mathsf{D}$ in Proposition 2, only its definition is more involved for the reasons identified above.

Theorem 2. *Let* $\varphi \in \mu\mathcal{L}_\Lambda$ *be a clean, guarded formula containing no free variables, and only least (or only greatest) fixpoint operators. Let* $\mathcal{F} := \mathsf{Cl}(\varphi)$*, and let* (C,γ) *be a* $\mathsf{T}\circ F$*-coalgebra. Then,* $[\![\varphi]\!]_\gamma \in \mathsf{Pred}_C$ *can be obtained as* $\mathsf{fix}(\mathsf{S}_\mu)^\sharp(\varphi)$*, where* $\mathsf{fix}(\mathsf{S}_\mu) : C \times \mathcal{F} \to \mathsf{T}1$ *is the least (respectively greatest) fixpoint of the operator* S_μ *of Definition 9, and* $(\_)^\sharp$ *denotes currying.*

Proof (Sketch). The statement follows by induction on the nesting depth of fixpoint operators. Once the equivalence in Proposition 2 is taken into account, the only difference between the two characterisations of the fixpoint semantics is that in the relational semantics, the approximations of outer fixpoints used in the computation of the inner fixpoints are updated *while* the computation of the inner fixpoints is taking place. Given that all the fixpoints are of the same nature (either least or greatest), this is not a problem. The proof of the inductive step uses the observation that the above difference only impacts on *how quickly* the fixpoint is reached, and not on the truth value of the outer fixpoint formula. For example, in the case of the formula $\mu x.\mu y.[a](x,y)$, the only effect of updating the truth value of x (with a *more accurate* approximation) during the computation of the inner fixpoint is that the outer fixpoint is potentially reached earlier.

We conclude by describing the relevance of Theorem 2 to model checking $\mu\mathcal{L}_\Lambda$-formulas. We believe the value of this result stands in providing (so far only for a fragment of $\mu\mathcal{L}_\Lambda$), a global approximation procedure that does not require inner fixpoints to be fully computed before the computation of the outer fixpoints can resume. With this procedure, assuming a finite state space, and since the closure of a formula is itself finite, one obtains increasingly accurate approximations of the truth value of a formula in finite time, and can choose to stop computing these approximations as soon as a satisfactory threshold is reached. This methodology can be applied to desirable properties captured by formulas only involving least fixpoints, as well as to undesirable properties captured by formulas only involving greatest fixpoints. In the latter case, as the approximations decrease the truth values of formulas, computing them can be stopped as soon as the truth value of the property of interest is sufficiently small in the initial state(s) of the system.

6 Conclusions and Future Work

We have described a uniform approach to defining linear-time fixpoint logics for a large class of state-based systems, modelled as coalgebras whose type incorporates branching. In our view, employing a universe of truth values derived from the type of branching yields more natural logics which may in time prove easier to model-check. In particular, our results apply to systems with weighted branching, for which temporal logics and associated model checking techniques have hardly been studied. Such systems can be used to model resources, including time, memory or computational power.

Ongoing work concerns extending Theorem 2 to arbitrary $\mu\mathcal{L}_\Lambda$-formulas. Such an extension will provide the necessary support for model checking algorithms based on the relational semantics. Future work will investigate similar logics for coalgebras of even more general types, including arbitrary compositions of a single branching monad with several polynomial endofunctors, as considered in [2]. The expressiveness of the proposed logics also deserves further study.

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