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Q-Q Plots with Confidence

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES

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Q–Q PLOTS WITH CONFIDENCE

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There are two types of procedures for assessing whether a population has a normal distribution based on a random sample: graphical tests (e.g., Q-Q plots) and non-graphical tests (e.g., the Anderson-Darling and Shapiro-Wilk tests). The graphical tests are more intuitive and more easily interpretable than non-graphical ones. However, their disadvantage is that different people can make different interpretations of the plots. Thus, graphical tests are usually regarded as informal techniques because the conclusions arrived at may be influenced by the subjectivity of users.

Among graphical tests, probability plots are the most commonly used, particularly the normal Q-Q plot that compares the empirical quantiles of sample data $(Y_1, \ldots, Y_n)$, i.e. the ordered values $Y_{[1]} < \ldots < Y_{[n]}$, with the corresponding quantiles of a theoretical distribution, i.e. a normal distribution. If the plotted points are close to a straight line, that indicates the observations come from a normal distribution. Within this context, two existing statistics are examined, which are the Kolmogorov-Smirnov D and Michael's (1983) D_{sp}, and three new statistics D_e, D_{be} and D_{bi} are proposed. Each statistic gives a set of simultaneous $1 - \alpha$ probability intervals for $Y_{[k]}, k = 1, \ldots, n$ in the normal Q-Q plots, which are used to detect non-normality in a graphical manner.

Therefore, simultaneous probability intervals for each of the order statistics are constructed to avoid the subjectivity of the users and to ensure the objectivity of the conclusions, regardless of the users. The idea behind the construction of the D_e, D_{be} and D_{bi} statistics is similar to that of the D_{sp} statistic: make all the n intervals have similar marginal probabilities of containing the corresponding $Y_{[k]}$ under H_0. The interval derived from the D_{be} test has some additional properties.

Normal Q-Q plots are also routinely used to check whether the random errors are independently and normally distributed based on the residuals from a linear re-

gression model. However, the statistical textbooks and statistical software often do not treat the residuals and the independently and identically distributed observations differently in producing and interpreting a normal Q-Q plot. Thus, the major concern is whether these two situations should be treated differently. In this thesis, the five graphical statistics D, D_{sp}, D_e, D_{be} and D_{bi} are proposed to provide a set of simultaneous probability intervals for the order statistics of the residuals in the normal Q-Q plot.

Furthermore, the graphical tests for the normal distribution are extended in this thesis to test whether a dataset follows Weibull and exponential distributions. Specifically, we obtain the simultaneous probability intervals suitable for Q-Q plots of the Weibull and exponential distributions.

Throughout this thesis, simulation studies have been carried out to compare the powers among the graphical tests and some non-graphical tests. In power comparison for testing normality of an i.i.d. sample, the popular non-graphical Anderson-Darling and Shapiro-Wilk statistics are used. For residuals from a linear regression model, the non-graphical T_n test of Hušková and Meintanis (2007) is used. Finally, for testing Weibull and exponential distributions, the non-graphical Anderson-Darling and Craḿer-von-Mises statistics are selected for power comparisons.

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Declaration of Authorship

I, WANPEN CHANTARANGSI, declare that the thesis entitled

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CHAPTER 1 Introduction

1.1 Background

The importance of graphical tests can never be overstated in statistics. In particular, plots present a pattern and detail which are not available in any non–graphical tests. Even though non-graphical tests provide an objective judgement of normality (i.e., significance or non–significance at some α levels), they do not specify the reason, or reasons, why a hypothesis is rejected, nor do they have the ability to show the hidden effects within the data which may bring about acceptance of a null hypothesis. From the normal probability plot or normal Q-Q plot, if and only if the n points fall close to a straight line, the population is claimed to be normally distributed. In addition, other non-normal features of the population distribution, such as skewness, and long or short tails, can be easily identified from the normal probability plot. Thus the normal probability plot is not only useful in determining whether the population distribution may be normal but also in diagnosing what kinds of deviations may be present. The hypothesis tests of normality, such as the Anderson–Darling and Shapiro–Wilk tests, do not have this diagnostic feature and cannot be represented graphically on a probability plot. The patterns of nonlinearity in a normal probability plot indicate the feature of deviation from normality.

The normal probability plot can also be used to check whether the random errors in a linear regression model are independently and normally distributed based on the residuals. Furthermore, the Q-Q plot is widely used for testing not only normality, but also Weibull and exponential distributions.

Although the Q-Q plot has many advantages, as discussed above, it is regarded as an informal technique because the conclusions arrived at may be influenced by the subjectivity of the user. Some users may judge that the n points fall close to a straight line on a Q-Q plot but others may not. Therefore, throughout this thesis the researcher has tried to set up means on Q-Q plots that allow an objective judgement as to whether the n points fall close to a straight line.

The objective judgement of a Q-Q plot is proposed in this thesis by providing an interval for each ordered value from a sample containing n observations which will fall into the corresponding intervals simultaneously with probability $1-\alpha$. Each of these n intervals can be depicted in a Q-Q plot as a vertical interval. Therefore, if at least one point of ordered values does not fall within the corresponding interval, one can claim, with a $(1-\alpha)100\%$ confidence level, that a sample does not follow the underlying distribution.

To date, only a few existing statistics can be presented graphically on the plots e.g., the Kolmogorov-Smirnov (1933) statistic D and the Michael's (1983) statistic D_{sp}. Therefore, the three new graphical statistics D_e, D_{be} and D_{bi} are proposed in this thesis in order to augment the choices of graphical tests.

1.2 Objectives

The main objectives of this thesis are:

1. To construct the five different sets of simultaneous intervals associated with the Kolmogorov-Smirnov's D, Michael's (1983) D_{sp}, D_e, D_{be}, and D_{bi} tests

2. To provide a set of simultaneous probability intervals on Q-Q plots, in order to make an objective judgement on "whether the points are close to a straight line"

3. To extend the idea of the graphical tests for normality in the case of a simple random sample to the vector of residuals from a linear regression model

4. To generalise the idea of the graphical tests for normality to Weibull and exponential distributions

5. To compare the powers of these graphical tests with some non-graphical tests.

1.3 The structure of the thesis

The thesis is organised as follows:

In Chapter 1, the motivation, the main objectives and the structure of this thesis are explained.

In Chapter 2, the researcher constructs the five graphical tests for normality based on a simple random sample and presents the simultaneous probability intervals of the graphical tests on the plots. Also, the two non-graphical tests, Anderson-Darling and Shapiro-Wilk, are considered and powers of the tests are compared.

In Chapter 3, the focus of this chapter is the vector of random errors in a linear regression model. Since the errors are unobservable, the residuals are utilised as replacements of errors to test normality. The graphical tests from Chapter 2 are applied, except for the D_{be} test for reasons that are addressed later. In addition, the T_n test of Hušková and Meintanis (2007), which is a selected non-graphical test, is compared with the graphical tests in power. The simultaneous probability intervals of the graphical tests on the plots are used to judge whether the errors follow a normal distribution.

In Chapter 4, the author generalised the five graphical tests for normality from Chapter 2 to obtain the graphical tests for Weibull distribution. The log-transformation is used to change the three–parameter Weibull distribution to the smallest extreme value distribution, which is a member of a location–scale family. The three estimators of the location and scale parameters are scrutinised in terms of power.

In Chapter 5, the five graphical tests for normal distribution described in Chapter 2, which are used to obtain the graphical tests for exponentiality, are extended. The three estimators of the location and scale parameters, as described in Chapter 4, are again investigated in order to have good power.

Finally, in Chapter 6, the conclusions are drawn and future work is suggested.

CHAPTER 2 Normality tests for a simple random sample

The predominance of the normal distribution is attributed to the fact that it has a number of advantages, such as the facilitation of mathematical handling and the superiority of derived results, and many currently developed statistical methods are based on the assumption of normality. Therefore, when carrying out a statistical analysis using normal distribution, validating the assumption of normality is of fundamental concern for the analyst.

Since the early 1900s, many normality tests have been developed by different authors. First, Pearson (1900) suggested the Chi-square test of normality which inspired other statisticians to also produce normality tests. The two most popular empirical distribution function (edf) based tests are a) the one introduced by Kolmogorov (1933), which proposed the comparison of the edf with the cumulative distribution function (cdf) of the normal distribution through the maximum difference and b) the test by Anderson and Darling (1954), which involves a combination of all the differences. In the meantime, significant numbers of new normality tests have been proposed over time. There are nearly 40 tests of normality available in the statistical literature; see for example, Dufour et al. (1998). Mage (1982) reviewed the formulae for plotting probability and techniques for subjectively drawing lines on probability plots, and he presented a method for plotting data and drawing an objective line on the probability plot to obtain a test of the distributional assumption. One year later, Michael (1983) proposed the stabilised probability plot that enhances its interpretability. This prompted the definition of a new and powerful goodness-of-fit statistic D_{sp}, which is analogous to the standard Kolmogorov-Smirnov Statistic D, and is defined to be the maximum deviation of the plotted points from their theoretical values. By using either D or D_{sp}, it is shown how to construct acceptance regions for the Q-Q and P-P plots and the new plots. Acceptance regions can help remove much of the subjectivity from the interpretation of these probability plots. Bera and Ng (1995) presented a graphical alternative to the Q-Q plot for detecting departures from normality using the score function. They concluded that the estimated score function is informative in performing exploratory data analysis.

Scrucca (2000) proposed a graphical method based on the characterisation of the multivariate normal distribution in terms of univariate normality of all linear combinations of the variables in the set. Moreover, he reviewed some methods for choosing directions to look for departure from the hypothesis of normality, and proposed an interactive dynamic graphic approach for checking the joint distribution. Breton et al. (2008) reported on a systematic comparison over a rich family of alternative distributions that span a broad range of of skewness and kurtosis combinations. The approach constituted an easily implemented tool for the practitioner to exercise in deciding which test and which implementation to employ.

Recently, Razali and Yap (2011) focused on comparing the power of four normality tests - Shapiro-Wilk, Kolmogorov-Smirnov, Lilliefors and Anderson-Darling - via the Monte Carlo simulation of sample data generated from alternative distributions that follow symmetrical and asymmetrical distributions. Results showed that the Shapiro-Wilk test was the most powerful normality test, followed by the Anderson-Darling, Lilliefors and Kolmogorov-Smirnov tests, respectively. However, the power of all four tests was still low for small sample sizes. Noughabi and Arghami (2011) considered seven normality tests and compared them with each other, by using Monte Carlo simulations, under classified alternatives. The choices of the seven tests were based on popularity (e.g., the Kolmogorov-Smirnov and Anderson-Darling tests) and powerfulness (e.g., the Shapiro-Wilk test). This article revealed that no single test procedure is uniformly more powerful than others; that is, some tests are more powerful than other tests for some alternatives and some are better for other alternatives.

Broadly speaking, there are two types of procedures in assessing whether a population has a normal distribution based on a random sample: graphical tests (e.g., Q-Q plots) and non–graphical tests (e.g., the Anderson-Darling and Shapilo-Wilk tests). The normal Q–Q plot or the normal probability plot is the most commonly used diagnostic tool for assessing whether a random sample is drawn from a normally distributed population.

Normally, the graphical tests visualise the distribution of a random variable or difference between an empirical distribution and a theoretical distribution (e.g., the standard normal distribution). Suppose that $Y_1, ..., Y_n$ are n independent observations from a continuous distribution. The corresponding order statistics are $Y_{[1]} \leq Y_{[2]} \leq ... \leq Y_{[n]}$. Let $\overline{Y} = \frac{1}{n}\Sigma_{k=1}^{n} Y_k$ and $\hat{\sigma}_Y = \sqrt{\frac{\Sigma_{k=1}^{n}(Y_k - \overline{Y})^2}{n-1}}$ be the sample mean and sample standard deviation, respectively. Besides, $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution and $\Phi^{-1}(\cdot)$ denotes the inverse function of $\Phi(\cdot)$. Let $z_1 < ... < z_n$ be a set of n reference values. A normal probability plot consists of the n points $(z_k, Y_{[k]})$, $k = 1, ..., n$. There are several ways to choose the reference values z_k. One is to use the normal probability scores of a set of given probabilities $0 < p_1 < ... < p_n < 1$, that is, $\Phi(z_k) = p_k$ for $k = 1, ..., n$. The most commonly used one is $p_k = (k - 0.5)/n$ for $k = 1, .., n$ given by Hazen (1914), which is also used in the software packages R (when $n > 10$) and Matlab.

When providing an objective mean to judge whether the points $(z_k, Y_{[k]})$ fall close to a straight line, one can augment the normal probability plot by providing an interval for each $Y_{[k]}$ $k = 1, ..., n$ which will fall into the corresponding intervals simultaneously with probability $1 - \alpha$. Each of these n intervals can be depicted in a normal probability plot as a vertical interval at the corresponding z_k. Therefore, if at least one point $(z_k, Y_{[k]})$ does not fall within the corresponding interval, one can claim, with a confidence level $1 - \alpha$, that a sample is not normally distributed.

Next, each graphical test which provides a set of simultaneous intervals for $Y_{[k]}$ $(k = 1, ..., n)$ in the normal probability plot is described.

2.1 Graphical tests for normality

In this section, we introduce two existing graphical tests, namely the Kolmogorov-Smirnov test and the test based on Michael's (1983). Throughout this thesis, the Kolmogorov-Smirnov test is called the“ D test” and the test based on Michael (1983) is represented by the “D_{sp} test”.

2.1.1 The Kolmogorov-Smirnov test (The D test)

The idea of the test is to compare the empirical distribution function (edf) which is estimated based on the data with the cumulative distribution function (cdf) of normal distribution to see if there is a good agreement between them. One of the most commonly used tests was developed by Kolmogorov in 1933. To construct the simultaneous intervals, the Kolmogorov-Smirnov statistic is defined by

$$D = \max_{1\le k\le n}\left|\Phi\big((Y_{[k]}-\overline{Y})/\hat{\sigma}_Y\big)-(k-0.5)/n\right|. \tag{2.1.1}$$

Let c_D be a critical constant so that $P\{D < c_D\} = 1-\alpha$ when a sample follows a normal distribution.

The required critical constant c_D can be evaluated straightforwardly as the $(1-\alpha)$-quantile of D by using the following simulations:

1. Simulate one D by generating i.i.d. $Y_1, ..., Y_n$ from $N(0,1)$.
2. Calculate $\overline{Y}, \hat{\sigma}_Y$ and $(Y_{[1]}, ..., Y_{[n]})$, and compute D using formula (2.1.1).
3. Simulate a large number R of independent copies of $D: D_1, ..., D_R$.
4. Use the $(1-\alpha)$ sample quantile of $D_1, ..., D_R$ as an approximation of the critical constant c_D.

For the other tests, the critical constants can be evaluated by using the same algorithms as the D test. Thus, this probability statement can be rewritten as

$$\begin{aligned}
1-\alpha &= P\{D < c_D\}\\
&= P\Big\{\max_{1\le k\le n}\left|\Phi\big((Y_{[k]}-\overline{Y})/\hat{\sigma}_Y\big)-(k-0.5)/n\right| \le c_D\Big\}\\
&= P\Big\{-c_D \le \Phi\big((Y_{[k]}-\overline{Y})/\hat{\sigma}_Y\big)-(k-0.5)/n \le c_D \ \text{ for } k=1,\ldots,n\Big\}\\
&= P\Big\{Y_{[k]} \in \overline{Y}+\hat{\sigma}_Y\Phi^{-1}\big((k-0.5)/n \pm c_D\big) \ \text{ for } k=1,\ldots,n\Big\}. \qquad (2.1.2)
\end{aligned}$$

Note that the expression (2.1.1) has nothing to do with the unknown parameters μ and σ^2 since $\dfrac{Y_1-\mu}{\sigma}, ..., \dfrac{Y_n-\mu}{\sigma} \overset{i.i.d.}{\sim} N(0,1)$. Let $Z_k = \dfrac{Y_k-\mu}{\sigma}$ and $Z_{[k]} = \dfrac{Y_{[k]}-\mu}{\sigma}$ for $k = 1, ..., n$. Then $Y_{[k]} = \mu + \sigma Z_{[k]}$ for $k = 1, ..., n$ and

$$\begin{aligned}
\frac{Y_{[k]}-\overline{Y}}{\hat{\sigma}_Y} &= \frac{(\mu+\sigma Z_{[k]})-(\mu+\sigma\bar{Z})}{\sigma\hat{\sigma}_Z}\\
&= \frac{Z_{[k]}-\bar{Z}}{\hat{\sigma}_Z}.
\end{aligned}$$

Therefore, $\dfrac{Y_{[k]}-\overline{Y}}{\hat{\sigma}_Y}$ has nothing to do with μ and σ^2 and we can generate the random sample $Y_1, ..., Y_n$ from $N(0,1)$ instead of $N(\mu, \sigma^2)$.

From (2.1.2), $Y_{[k]}$ should fall in the corresponding interval

$$\overline{Y}+\hat{\sigma}_Y\Phi^{-1}\big((k-0.5)/n \pm c_D\big) \text{ for } k=1,\ldots,n \tag{2.1.3}$$

with a simultaneous probability $1-\alpha$ if the sample data follow a normal distribution.

The Kolmogorov-Smirnov test, however, is appropriate in the case that the parameters of the null hypothesis are completely specified. If the unknown parameters of the null distribution need to be estimated from the sample data, the standard tables used for the Kolmogorov-Smirnov test are no longer valid. Lilliefors (1967) proposed the Lilliefors test for a situation in which the parameters are estimated. The Lilliefors statistic is the same as (2.1.1). Even though the estimated parameters for μ and σ in the Kolmogorov-Smirnov test are used here, this does not affect the validity of the test because the critical values are evaluated from the generated samples, not from the standard tables.

2.1.2 The D_{sp} test

According to Michael (1983), the statistic D is less efficient for detecting normality since $\Phi\big((Y_{[k]} - \overline{Y})/\hat{\sigma}_Y\big)$ has a smaller variance for a k close to either 1 or n compared to a k close to $n/2$, and hence $Y_{[k]}$ has a small chance of falling outside its corresponding interval for a k close to either 1 or n compared to a k close to $n/2$. The statistic D_{sp} is defined as

$$D_{sp} = \max_{1\le k\le n} \left| (2/\pi)\arcsin\sqrt{\Phi\big((Y_{[k]} - \overline{Y})/\hat{\sigma}_Y\big)} - (2/\pi)\arcsin\sqrt{(k-0.5)/n} \right|, \tag{2.1.4}$$

and the methodology behind this statistic is discussed below.

Order statistics are useful for probability plots because of the transformation

$$U_{[k]} = F_X(X_{[k]}),$$

where $F_X(\cdot)$ is the cdf of X_k, produces a random variable which is the k^{th} order statistic of a random sample of size n from a standard uniform distribution $U(0,1)$. This result is called the *probability integral transformation*; see, Mukhopadhyay (2006).

Theorem 2.1.1. *Let the random variable X have cdf F_X. If F_X is strictly monotonic and continuous then $Y = F_X(X)$ has a $U(0,1)$ distribution.*

Proof. We have

$$\begin{aligned} G_Y(y) &= P(Y \le y) \\ &= P(F_X(X) \le y) \\ &= P(F_X^{-1}(F_X(X)) \le F_X^{-1}(y)) \\ &= P(X \le F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \\ &= y, \qquad 0 < y < 1, \end{aligned}$$

and hence $Y = F_X(X) \sim U(0,1)$. This completes the proof. □

Immediately, we have that if $X_{[1]}, ..., X_{[n]}$ are the order statistics of $X_1, ..., X_n \overset{i.i.d.}{\sim} F_X(\cdot)$, then $F_X(X_{[1]}) < ... < F_X(X_{[n]})$ are the order statistics of a size n random sample from the continuous uniform distribution on $(0,1)$ regardless of the original distribution F_X as long as it is continuous.

Now, let $U_1, ..., U_n$ be independent and identically distributed random variables with uniform distribution on $(0,1)$ and $U_{[1]} \le ... \le U_{[n]}$ be the order statistics. We show that $U_{[k]}$ for $k = 1, \ldots, n$ has the distribution $beta(k, n-k+1)$. Note that the pdf and cdf of U_k are $f(u) = 1$ and $F(u) = u$, respectively. So, the pdf of $U_{[k]}$ for $k = 1, \ldots, n$ is

$$\begin{aligned} f_{U_{[k]}}(u) &= \frac{n!}{(n-k)!(k-1)!}\big(F(u)\big)^{k-1}\big(1-F(u)\big)^{n-k} f(u) \\ &= \frac{n!}{(n-k)!(k-1)!}(u)^{k-1}(1-u)^{n-k} \\ &= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)}(u)^{k-1}(1-u)^{n-k} \ , \ 0 < u < 1. \end{aligned}$$

Therefore, $U_{[k]} \sim beta(k, n-k+1)$ with

$$\begin{aligned} \mathrm{E}(U_{[k]}) &= \frac{k}{n+1}, \\ \mathrm{Var}(U_{[k]}) &= \frac{k(n+1-k)}{(n+1)^2(n+2)}. \end{aligned}$$

The following result can be found in Arnold et al. (2008).

Theorem 2.1.2. *For $0 < p < 1$, assume that $\frac{k}{n} \to p$ as $n \to \infty$, then we have*

$$\sqrt{n}\Big(U_{[k]} - p\Big) \xrightarrow{d} N\big(0, p(1-p)\big).$$

Proof. We know that $U_{[k]} \sim beta(k, n-k+1)$, and so

$$\mathrm{E}(U_{[k]}) = \frac{k}{n+1} \to p, \tag{2.1.5}$$

$$\mathrm{Var}(U_{[k]}) = \frac{k(n-k+1)}{(n+1)^2(n+2)} \to \frac{p(1-p)}{n}. \tag{2.1.6}$$

Note that $U_{[k]}$, a $beta(k, n-k+1)$ random variable, can be written as

$$U_{[k]} = \frac{\sum_{i=1}^{k} V_i}{\sum_{i=1}^{k} V_i + \sum_{i=k+1}^{n+1} V_i} = \frac{A}{A+B},$$

where $A = \sum_{i=1}^{k} V_i$, $B = \sum_{i=k+1}^{n+1} V_i$, and the $V_i's$ are i.i.d. $Exp(0,1)$ random variables with $\mathrm{E}(V_i) = 1$ and $\mathrm{Var}(V_i) = 1$. From the central limit theorem, we have

$$\frac{A-k}{\sqrt{k}} \xrightarrow{d} N\big(0\,,1\big).$$

Recall that $\frac{k}{n} \to p$ and so we get

$$\frac{A-k}{\sqrt{n}} \xrightarrow{d} N\big(0\,,p\big).$$

Similarly, we also have $\frac{B-(n-k+1)}{\sqrt{n-k+1}} \xrightarrow{d} N\big(0\,,1\big)$ and so $\frac{B-(n-k+1)}{\sqrt{n}} \xrightarrow{d} N\big(0\,,1-p\big)$. Observe that

$$\begin{aligned}
\sqrt{n}\Big(U_{[k]} - p\Big) &= \sqrt{n}\Big(\frac{A}{A+B} - p\Big) \\
&= \frac{\sqrt{n}}{A+B}\Big((1-p)A - pB\Big) \\
&= \frac{1}{\sqrt{n}}\Big[\frac{(1-p)(A-k) - p(B-(n-k+1)) + (1-p)k - p(n-k+1)}{\frac{A+B}{n}}\Big].
\end{aligned}$$

We can see that $\frac{1}{\sqrt{n}}[(1-p)k - p(n-k+1)] \to 0$ as $n \to \infty$. Since A and B are independent for all n, then

$$(1-p)\Big(\frac{A-k}{\sqrt{n}}\Big) - p\Big(\frac{B-(n-k+1)}{\sqrt{n}}\Big) \xrightarrow{d} N(0, p(1-p)).$$

By using the weak law of large numbers, we obtain

$$\frac{A+B}{n} = \frac{\sum_{i=1}^{n+1} V_i}{n} \xrightarrow{P} 1.$$

Hence, by Slutsky's Theorem,

$$\sqrt{n}\Big(U_{[k]} - p\Big) \xrightarrow{d} N\big(0\,, p(1-p)\big).$$

This completes the proof. □

From the expressions (2.1.5) and (2.1.6), we also have

$$\begin{aligned} \mathrm{E}(U_{[k]}) &\approx p, \\ \mathrm{Var}(U_{[k]}) &\approx \frac{p(1-p)}{n}. \end{aligned}$$

Hence the asymptotic normal distribution of $U_{[k]}$ is the same as the asymptotic normal distribution of the sample mean of i.i.d. Bernoulli(p) random variables.

Variance Stabilising Transformation

Consider a sequence of real-valued statistics $\{T_n; n \geq 1\}$ such that

$$\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0, \sigma^2) \text{ as } n \longrightarrow \infty. \tag{2.1.7}$$

Then, the Delta method gives

$$\sqrt{n}\big(g(T_n) - g(\theta)\big) \xrightarrow{d} N\big(0, [\sigma g'(\theta)]^2\big) \tag{2.1.8}$$

if $g(\cdot)$ is a continuous real-valued function and $g'(\theta)$ is finite and nonzero. The following example can be found in Mukhopadhyay (2006).

Example Suppose that $X_1, ..., X_n$ are i.i.d. Bernoulli(p) random variables. Let $\hat{p}_n = \bar{X}$ be the sample proportion of successes in n independent trials with

$$\mathrm{E}_p[\hat{p}_n] = p \text{ and } \mathrm{Var}_p[\hat{p}_n] = \frac{p(1-p)}{n} = \frac{\sigma^2}{n}.$$

Thus,

$$\sqrt{n}(\hat{p}_n - p) \xrightarrow{d} N\big(0, p(1-p)\big) \text{ as } n \longrightarrow \infty.$$

We seek a suitable function $g(\cdot)$ such that the asymptotic variance of

$$\sqrt{n}\big(g(\hat{p}_n) - g(p)\big) \text{ does not involve } p.$$

We must have $\sigma g'(p) = k$ (constant), and so

$$\begin{aligned} g(p) &= \int g'(p)dp \\ &= k\int \frac{dp}{\sigma} \\ &= k\int \frac{dp}{\sqrt{p(1-p)}}. \end{aligned}$$

Next, we substitute $p = \sin^2\theta$ to write $\theta = \arcsin\sqrt{p}$. That is

$$\begin{aligned} g(p) &= k\int \frac{dp}{\sqrt{p(1-p)}} \\ &= k\int \frac{d\sin^2\theta}{\sqrt{\sin^2\theta\cos^2\theta}} \\ &= k\int \frac{2\sin\theta\cos\theta\, d\theta}{\sin\theta\cos\theta} \\ &= 2k\theta \\ &= 2k\arcsin\sqrt{p}. \end{aligned}$$

Thus, $g'(p) = \dfrac{d}{dp}\arcsin\sqrt{p} = \dfrac{1}{\sqrt{1-p}}\dfrac{d}{dp}\sqrt{p} = \dfrac{1}{2\sqrt{p(1-p)}}$ and so

$$[\sigma g'(p)]^2 = \Big(\sqrt{p(1-p)}\frac{1}{2\sqrt{p(1-p)}}\Big)^2 = \frac{1}{4}.$$

So, we apply the variance stabilising transformation to obtain

$$\arcsin\sqrt{U_{[k]}} \sim N\big(\arcsin\sqrt{p}\,, \frac{1}{4n}\big),$$

i.e.,

$$\big(\arcsin\sqrt{U_{[k]}} - \arcsin\sqrt{p}\,\big) \ \xrightarrow{d}\ N\big(0, \frac{1}{4n}\big). \qquad (2.1.9)$$

The arcsine square root transformation can be used to stabilise the variance of a uniform order statistic just as it does for a binomial random variable.

Let $U_{[k]} = \Phi(\frac{Y_{[k]}-\bar{Y}}{\hat{\sigma}_Y})$. Thus, the expression (2.1.9) suggests that

$$\arcsin\sqrt{\Phi\big((Y_{[k]}-\bar{Y})/\hat{\sigma}_Y\big)} - \arcsin\sqrt{(k-0.5)/n} \ \xrightarrow{d}\ N\big(0, \frac{1}{4n}\big). \qquad (2.1.10)$$

From (2.1.10), Michael (1983) proposed that the intervals are constructed based on the statistic:

$$D_{sp} = \max_{1\le k\le n}\left|(2/\pi)\arcsin\sqrt{\Phi\big((Y_{[k]}-\overline{Y})/\hat{\sigma}_Y\big)} - (2/\pi)\arcsin\sqrt{(k-0.5)/n}\right|.$$

Let c_{sp} be a critical constant so that $P\{D_{sp} < c_{sp}\} = 1-\alpha$ when the population has a normal distribution. The critical constant c_{sp} can be computed by simulation in a similar way as c_D can. The probability statement

$$\begin{aligned} 1-\alpha &= P\{D_{sp} < c_{sp}\} \\ &= P\Big\{\max_{1\le k\le n}\left|(2/\pi)\arcsin\sqrt{\Phi\big((Y_{[k]}-\overline{Y})/\hat{\sigma}_Y\big)} - (2/\pi)\arcsin\sqrt{(k-0.5)/n}\right| \le c_{sp}\Big\} \\ &= P\Big\{Y_{[k]} \in \overline{Y} + \hat{\sigma}_Y\Phi^{-1}\big(\sin^2[\arcsin\sqrt{(k-0.5)/n} \pm \frac{\pi}{2}c_{sp}]\big) \text{ for } k = 1,\ldots,n\Big\} \end{aligned}$$

provides the simultaneous intervals for $Y_{[k]}$ for $k = 1, ..., n$.

Furthermore, we have developed three new tests, namely the D_e, D_{be}, and D_{bi} tests for testing normality.

2.1.3 The D_e test

We proposed another test based on which a set of simultaneous $1-\alpha$ level intervals for $Y_{[k]}$ $(k=1,\dots,n)$ can be constructed and the statistic corresponding to the individual intervals have similar variances. It is shown that this set of simultaneous intervals for $Y_{[k]}$ is close to the set of simultaneous intervals proposed by Michael in 1983.

Let $Z_1,...,Z_n$ be a simple random sample drawn from the standard normal distribution $N(0,1)$ and let $Z_{[1]}\le ... \le Z_{[n]}$ be their ordered values. It is straightforward to compute numerically, for $1\le k\le n$,

$$\mu_k = \mathrm{E}(Z_{[k]}) = \int_{-\infty}^{\infty} z f_k(z)dz, \quad (2.1.11)$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \int_{-\infty}^{\infty} z^2 f_k(z)dz - \mu_k^2, \quad (2.1.12)$$

where $f_k(z)$ is the probability density function of $Z_{[k]}$ and is given by

$$f_k(z) = \frac{n!}{(k-1)!(n-k)!}\,\Phi(z)^{k-1}\phi(z)\big(1-\Phi(z)\big)^{n-k}\ ,\ -\infty\le z\le\infty. \quad (2.1.13)$$

Since the Y_k's are taken from the distribution $N(\mu,\sigma^2)$, it is clear that $(Y_{[1]},...,Y_{[n]})$ have the same joint distribution as $(\mu+\sigma Z_{[1]},...,\mu+\sigma Z_{[n]})$. In particular, we have

$$\mathrm{E}(Y_{[k]}) = \mu+\sigma\mu_k \ \text{ and } \ \mathrm{Var}(Y_{[k]}) = \sigma^2\sigma_k^2 \quad (2.1.14)$$

for $k=1,...,n$. Specially, we construct the simultaneous intervals

$$Y_{[k]} \in [\overline{Y}+\hat{\sigma}_Y\mu_k - c_e\hat{\sigma}_Y\sigma_k, \overline{Y}+\hat{\sigma}_Y\mu_k + c_e\hat{\sigma}_Y\sigma_k] \ \text{ for } k=1,...,n \quad (2.1.15)$$

where c_e is a critical constant chosen so that all the $Y_{[k]}$'s are contained in the corresponding intervals simultaneously with probability $1-\alpha$.
Note that

$$\begin{aligned} & P\{Y_{[k]} \in [\overline{Y}+\hat{\sigma}_Y\mu_k - c_e\hat{\sigma}_Y\sigma_k, \overline{Y}+\hat{\sigma}_Y\mu_k + c_e\hat{\sigma}_Y\sigma_k] \ \text{ for } k=1,...,n\} \\ = \ & P\Big\{\max_{1\le k\le n}\frac{|Y_{[k]}-(\overline{Y}+\hat{\sigma}_Y\mu_k)|}{\hat{\sigma}_Y\sigma_k}\le c_e\Big\} \quad (2.1.16) \\ = \ & P\Big\{\max_{1\le k\le n}\frac{|(Y_{[k]}-\mu)/\sigma - \big[(\overline{Y}-\mu)/\sigma + (\hat{\sigma}_Y/\sigma)\mu_k\big]|}{(\hat{\sigma}_Y/\sigma)\mu_k}\le c_e\Big\} \\ = \ & P\Big\{\max_{1\le k\le n}\frac{|Z_{[k]}-(\overline{Z}+\hat{\sigma}_Z\mu_k)|}{\hat{\sigma}_Z\sigma_k}\le c_e\Big\}, \quad (2.1.17) \end{aligned}$$

where $\bar{Z}$ and $\hat{\sigma}_Z$ are the sample mean and sample standard deviation, respectively, of $Z_1,...,Z_n$. It is clear from (2.1.17), (2.1.11) and (2.1.12) that the simultaneous coverage probability depends only on the sample size and the critical constant, and has nothing to do with the unknown parameters μ and σ^2. So, the simultaneous intervals in (2.1.15) are based on the statistic

$$D_e = \max_{1\le k\le n} \frac{|Y_{[k]} - (\overline{Y} + \hat{\sigma}_Y \mu_k)|}{\hat{\sigma}_Y \sigma_k}. \tag{2.1.18}$$

where the subscript "e" indicates the expectation used in (2.1.11).

2.1.4 The D_{be} test

We introduce the new test which is constructed by the following steps:

Step 1. Since $U_{[k]} = \Phi(\frac{Y_{[k]}-\mu}{\sigma}) \sim beta(k, n+1-k)$, we construct p^* level highest-density probability interval for $U_{[k]}$. That is, $[L(p^*, k, n), U(p^*, k, n)]$ is the shortest probability interval for $U_{[k]}$ among all the p^* level probability intervals for $U_{[k]}$.

Step 2. Construct simultaneous probability intervals for $Y_{[1]} \le ... \le Y_{[n]}$ based on

$$L(p^*, k, n) \le \Phi\Big(\frac{Y_{[k]} - \overline{Y}}{\hat{\sigma}_Y}\Big) \le U(p^*, k, n) \text{ for } k = 1, ..., n,$$

where p^* is chosen so that

$$K(p^*) \equiv P\Big\{\Phi^{-1}\Big(L(p^*, k, n)\Big) \le \frac{Y_{[k]} - \overline{Y}}{\hat{\sigma}_Y} \le \Phi^{-1}\Big(U(p^*, k, n)\Big) \text{ for } k = 1, ..., n\Big\} = 1-\alpha.$$

Step 3. Such a p^* can be found by simulation :

(i) for each p^*, find $K(p^*)$;

(ii) search over $p^* \in (1 - \frac{\alpha}{n}, 1-\alpha)$ so that $K(p^*) = 1-\alpha$.

Therefore, the simultaneous probability intervals for $Y_{[1]} \le ... \le Y_{[n]}$ can be expressed as

$$\overline{Y} + \hat{\sigma}_Y \Phi^{-1}\Big(L(p^*, k, n)\Big) \le Y_{[k]} \le \overline{Y} + \hat{\sigma}_Y \Phi^{-1}\Big(U(p^*, k, n)\Big) \text{ for } k = 1, \ldots, n.$$

Note that the subscript "be" in D_{be} indicates the beta distribution used in the construction.

2.1.5 The D_{bi} test

For $0 < p < 1$, assume that $\frac{k}{n} \to p$ as $n \to \infty$. From Theorem 2.1.2, we have

$$\begin{aligned} U_{[k]} &\sim N\Big(p, \frac{p(1-p)}{n}\Big), \\ \mathrm{E}(U_{[k]}) &\approx p, \\ \text{and} \quad \mathrm{Var}(U_{[k]}) &\approx \frac{p(1-p)}{n}. \end{aligned}$$

From

$$\frac{U_{[k]} - \frac{k}{n}}{\sqrt{\frac{\frac{k}{n}(1-\frac{k}{n})}{n}}} \overset{approx.}{\sim} N(0,1),$$

we define the statistic

$$D_{bi} = \max_{1\le k\le n} \frac{\left|\Phi\big((Y_{[k]} - \overline{Y})/\hat{\sigma}_Y\big) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} \qquad (2.1.19)$$

where the subscript "*bi*" indicates that the construction is based on the asymptotic normal distribution related to a binomial random variable.
Let c_{bi} be a critical constant so that $P\Big\{D_{bi} < c_{bi}\Big\} = 1-\alpha$; c_{bi} can be determined by simulation as before. The probability statement for $Y_{[1]}, \dots, Y_{[n]}$ is given by

$$\begin{aligned} 1-\alpha &= P\Big\{D_{bi} < c_{bi}\Big\} \\ &= P\Big\{\max_{1\le k\le n} \frac{\left|\Phi\big((Y_{[k]} - \overline{Y})/\hat{\sigma}_Y\big) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi}\Big\} \\ &= P\Big\{Y_{[k]} \in \overline{Y} + \hat{\sigma}_Y \Phi^{-1}\Big((k-0.5)/n \pm c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\Big) \\ &\qquad\qquad \text{for } k = 1, \dots, n\Big\} \end{aligned}$$

which provides the simultaneous intervals for $Y_{[1]}, \dots, Y_{[n]}$.

2.2 Non–graphical tests for normality

Our focus is on the graphical tests, which provide simultaneous probability intervals for $Y_{[k]}$. These intervals can be used in the normal probability plot to judge whether the points in the normal probability plot fall close to a straight line. We want to not only compare the power among the graphical tests but also concentrate on the two non-graphical tests; that is, the Anderson–Darling and Shapiro–Wilk tests. However, it needs to be emphasised that there are no simultaneous intervals associated with the Anderson-Darling and Shapiro–Wilk tests.

2.2.1 The Anderson–Darling test

The Anderson–Darling test (1954) is chosen because it is one of the most practical non–graphical tests.

The Anderson–Darling test statistic (AD) is

$$AD = -\sum_{k=1}^{n} \Big[\frac{(2k-1)\{log(Z_k) + log(1 - Z_{n+1-k})\}}{n}\Big] - n, \qquad (2.2.1)$$

where $Z_k = \Phi\big((Y_{[k]} - \overline{Y})/\hat{\sigma}_Y\big)$. The critical constant c_{AD} which satisfies $P\{AD < c_{AD}\} = 1 - \alpha$ can be determined by simulation as the critical constant c_D for the D test.

2.2.2 The Shapiro–Wilk test

The Shapiro–Wilk statistic (1965) for normality is defined as

$$SW = \frac{[\sum_{k=1}^{n} a_k Y_{[k]}]^2}{\sum_{k=1}^{n}(Y_{[k]} - \overline{Y})^2} \quad (2.2.2)$$

$$\text{with } \boldsymbol{a}' = (a_1, \ldots, a_n)' = \frac{\boldsymbol{m}'\mathbf{V}^{-1}}{\big(\boldsymbol{m}'\mathbf{V}^{-1}\mathbf{V}^{-1}\boldsymbol{m}\big)^{1/2}}. \quad (2.2.3)$$

where $\boldsymbol{m}' = (m_1, ..., m_n)$ is the vector of the expected values of the standard normal order statistics and $\mathbf{V} = (v_{kj})$ is the corresponding $n \times n$ covariance matrix of these order statistics. To obtain $\boldsymbol{a}$, there are two different approaches. The first approach for the coefficient vector $\boldsymbol{a}$ is to compute $\boldsymbol{m}$ and $\boldsymbol{V}$ straightforwardly. In 1965, when Shapiro and Wilk introduced this statistic, the exact values for the variances and covariances of normal order statistics were approximated up to sample of size 20. Then, Parrish (1992a, 1992b) extended the approximation of variances and covariances of normal order statistics for a sample size up to 50. Due to the difficulty in handling double integrals, the computation of $\boldsymbol{V}$ is more complex than the computation of $\boldsymbol{m}$. The second approach is to approximate $\boldsymbol{a}$ directly. One method to approximate $\boldsymbol{a}$ was proposed by Royston (1992). The idea of this computation results from the conducting of a polynomial regression analysis for estimating the value of $\boldsymbol{a}$. Based on the regression equation, the predicted value $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ can be employed as an approximation for $\boldsymbol{a}$ if $4 \le n \le 1000$. The coefficient $\boldsymbol{a}$ of Royston (1992) differs from the Weisberg and Bingham statistic (1975) mainly in the first two terms (also in the last two, but $a_k = -a_{n-k+1}$).

Let $\tilde{m}_k = \Phi^{-1}\big(\frac{k-0.375}{n+0.25}\big)$, $x = n^{-1/2}$ and $\boldsymbol{c} = \frac{\boldsymbol{m}'}{(\boldsymbol{m}'\boldsymbol{m})^{1/2}} = (c_1, ..., c_n)$. Therefore, we can estimate the Shapiro–Wilk statistic by the approximate $\tilde{\boldsymbol{a}}$ for $\boldsymbol{a}$:

1. For $n \le 5$,

$$\tilde{a}_n = c_n + 0.221157x - 0.147981x^2 - 2.071190x^3 + 4.434687x^4 - 2.706056x^5$$

$$\phi = \frac{1 - 2\tilde{a}_n^2}{\tilde{\boldsymbol{m}}'\tilde{\boldsymbol{m}} - 2\tilde{m}_n^2}$$

$$\text{Also, } \tilde{a}_k = \phi^{1/2}\tilde{m}_k \quad \text{for} \quad k = 2, \ldots, n-1.$$

2. For $n > 5$,

$$\begin{aligned}
\tilde{a}_n &= c_n + 0.221157x - 0.147981x^2 - 2.071190x^3 + 4.434687x^4 - 2.706056x^5 \\
\tilde{a}_{n-1} &= c_{n-1} + 0.042981x - 0.293762x^2 - 1.752461x^3 + 5.682633x^4 - 3.582663x^5 \\
\phi &= \frac{1 - 2\tilde{a}_n^2 - 2\tilde{a}_{n-1}^2}{\tilde{\boldsymbol{m}}'\tilde{\boldsymbol{m}} - 2\tilde{m}_n^2 - 2\tilde{m}_{n-1}^2} \\
\text{Also, } \tilde{a}_k &= \phi^{1/2}\tilde{m}_k \quad \text{for} \quad k = 3, \ldots, n-2.
\end{aligned}$$

Note that with the estimates $\tilde{a}_n$ and $\tilde{a}_{n-1}$ we always have approximations of $\tilde{a}_1 = -\tilde{a}_n$ and $\tilde{a}_2 = -\tilde{a}_{n-1}$, respectively.

The Shapiro–Wilk test is a single-tailed test by which the null hypothesis will be rejected if $SW \leq c_{SW}$ where c_{SW} is the critical constant for the test statistic at a given sample size n and a significance level α. The value of SW lies between zero and one. Small values of SW lead to the rejection of normality whereas a value of one indicates normality of the data. The critical constant c_{SW} which satisfies $P\{SW < c_{SW}\} = \alpha$ can be determined by simulation as above.

Investigated Tests for Normality

For our empirical studies we included tests for normality presented in this thesis. A short summary is given in Table 2.1 where the notation of the test statistic is repeated.

Table 2.1: Abbreviations for the tests of normality in the empirical power studies

Test Symbol	Test Name
D	Kolmogorov–Smirnov test
D_{sp}	Michael's test (1983)
D_e	New test
D_{be}	New test
D_{bi}	New test
AD	Anderson–Darling test
SW	Shapiro–Wilk test

2.3 Simultaneous intervals

In Table 2.2, we construct the simultaneous intervals corresponding to the standardised samples $\dfrac{Y_{[k]}-\overline{Y}}{\hat{\sigma}_Y}$ for $k=1,\ldots,n$ for the graphical tests D, D_{sp}, D_e, D_{be} and D_{bi}.

Table 2.2: Simultaneous intervals for the standardised observations from a simple random sample based on the five graphical tests

Graphical tests	Simultaneous intervals of $\frac{Y_{[k]}-\overline{Y}}{\hat{\sigma}_Y}$ for $k=1,\ldots,n$
D	$\Phi^{-1}\big((k-0.5)/n \pm c_D\big)$
D_{sp}	$\Phi^{-1}\big(\sin^2[\arcsin\sqrt{(k-0.5)/n} \pm (\pi/2)c_{sp}]\big)$
D_e	$\mu_k \pm c_e \sigma_k$
D_{be}	$(\Phi^{-1}(L), \Phi^{-1}(U))$
D_{bi}	$\Phi^{-1}\Big((k-0.5)/n \pm c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\Big)$

These simultaneous intervals have nothing to do with the alternative distributions because they depend only on $k=1,\ldots,n$ and their critical constants of each test. In order to construct the simultaneous intervals of the above tests, we have to calculate the critical constants. Now, 500,000 samples with $n=5(5)30$, 50 and 100 were generated from the standard normal distribution. The critical constants were estimated based on the test statistics calculated. Table 2.3 shows the critical values for the D, D_{sp}, D_e, D_{bi}, AD and SW tests at $\alpha=0.01, 0.05$ and 0.1, respectively. Some studies calculated their own critical values for the tests although critical constants have already been available from other works.

A typical standardised sample of size n from a given distribution $F(\cdot)$ can be calculated by using simulation as follows, in which N is a large number (e.g., 500,000).

1. Simulate the i^{th} sample from $F(\cdot)$: $y^*_{i1}, y^*_{i2}, \ldots, y^*_{in}$ for $i=1,\ldots,N$.

2. Sort each sample in ascending order : $y^*_{i[1]} \le y^*_{i[2]} \le \ldots \le y^*_{i[n]}$ and compute $\bar{y}^*_i = \frac{1}{n}\sum_{k=1}^n y^*_{ik}$ and $\hat{\sigma}_{y^*_i} = \sqrt{\frac{1}{n-1}\sum_{k=1}^n \left(y^*_{ik}-\bar{y}^*_i\right)^2}$ for $i=1,\ldots,N$.

3. Compute $y^*_{[k]} = \frac{1}{N}\sum_{i=1}^N y^*_{i[k]}$, $\bar{y}^* = \frac{1}{N}\sum_{i=1}^N \bar{y}^*_i$ and $\hat{\sigma}_{y^*} = \frac{1}{N}\sum_{i=1}^N \hat{\sigma}_{y^*_i}$ for $k=1,\ldots,n$.

4. The typical standardised sample is given by $\dfrac{y^*_{[k]}-\bar{y}^*}{\hat{\sigma}_{y^*}}$, $\quad k=1,\ldots,n$.

Figures 2.1, 2.2, and 2.3 pinpoint the simultaneous intervals of the five graphical tests generated from the typical samples for the sample size $n = 30$ at $\alpha = 0.05$. For Figure 2.1, some points of the typical sample from the Chi-square distribution with 1 degree of freedom fall outside the intervals; therefore, the tests should have a good chance of detecting the non–normality.

As seen in Figure 2.2, roughly speaking, the non-normality has a good chance to be detected by the D_{sp}, D_{be} and D_{bi} tests; with a smaller chance to be detected by the D test and D_e test. On the other hand, in Figure 2.3 all the observations of the typical standardised sample taken from the t distribution with 6 degrees of freedom lie within the intervals, so all the tests should have a small chance of detecting non-normality. This is not surprising since the t distribution with 6 degrees of freedom is also a bell-shaped distribution.

Table 2.3: Critical constants of the D, D_{sp} D_e, D_{bi}, Anderson-Darling and Shapilo-Wilk tests for a simple random sample at the significance levels $\alpha = 0.01, 0.05$ and 0.1

(a) $\alpha = 0.01$

n	D	D_{sp}	D_e	D_{bi}	AD	SW
5	0.2966	0.1929	1.3283	1.4489	0.7960	0.7006
10	0.2542	0.1719	1.7104	2.1135	0.9414	0.7840
15	0.2217	0.1535	2.0068	2.5349	0.9785	0.8346
20	0.1978	0.1403	2.2242	2.8381	0.9897	0.8669
25	0.1819	0.1310	2.3661	3.0950	1.0058	0.8880
30	0.1677	0.1231	2.5020	3.2894	1.0077	0.9032
50	0.1351	0.1043	2.7924	3.8983	1.0179	0.9364
100	0.0988	0.0816	3.1313	4.6033	0.1096	0.9655

(b) $\alpha = 0.05$

n	D	D_{sp}	D_e	D_{bi}	AD	SW
5	0.2432	0.1597	1.1394	1.1742	0.5992	0.7774
10	0.2122	0.1442	1.4734	1.6867	0.6871	0.8450
15	0.1856	0.1298	1.6772	1.9868	0.7109	0.8818
20	0.1667	0.1194	1.8070	2.1968	0.7209	0.9045
25	0.1530	0.1113	1.9034	2.3660	0.7262	0.9193
30	0.1422	0.1050	1.9766	2.5016	0.7325	0.9300
50	0.1143	0.0883	2.1764	2.8739	0.7389	0.9540
100	0.0838	0.0687	2.4182	3.3388	0.7455	0.9747

(c) $\alpha = 0.1$

n	D	D_{sp}	D_e	D_{bi}	AD	SW
5	0.2191	0.1438	1.0322	1.0415	0.5158	0.8128
10	0.1911	0.1306	1.3517	1.4796	0.5795	0.8705
15	0.1678	0.1183	1.5222	1.7310	0.5973	0.9015
20	0.1514	0.1091	1.6284	1.9067	0.6056	0.9199
25	0.1389	0.1018	1.7133	2.0274	0.6105	0.9326
30	0.1293	0.0960	1.7738	2.1411	0.6137	0.9415
50	0.1044	0.0809	1.9350	2.4351	0.6235	0.9612
100	0.0766	0.0627	2.1306	2.793	0.6221	0.9786

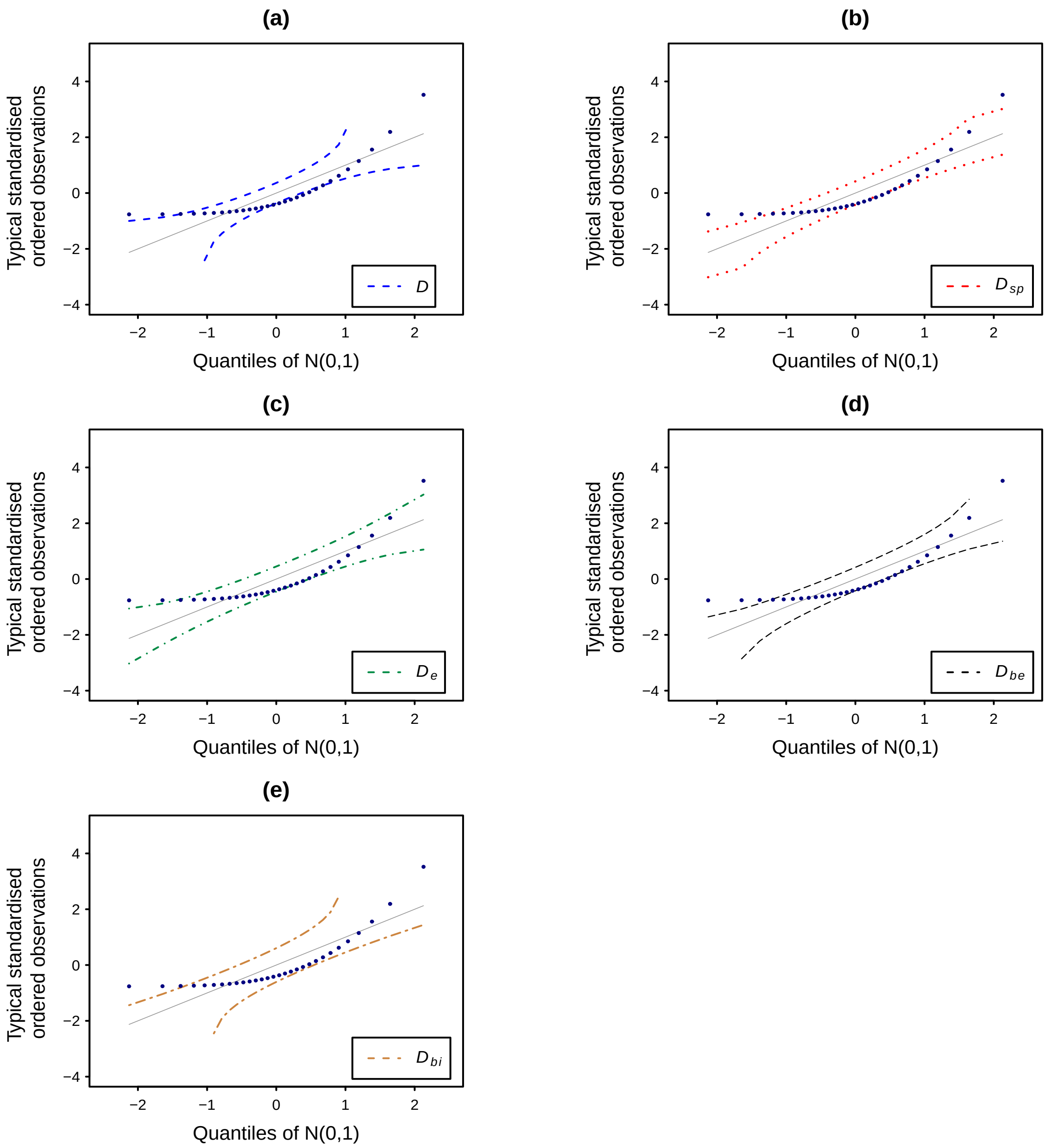

Figure 2.1: Simultaneous intervals in detecting non–normality of the typical standardised ordered observations simulated from Chi-square distribution with 1 degree of freedom based on the five graphical tests $n = 30$ at $\alpha = 0.05$

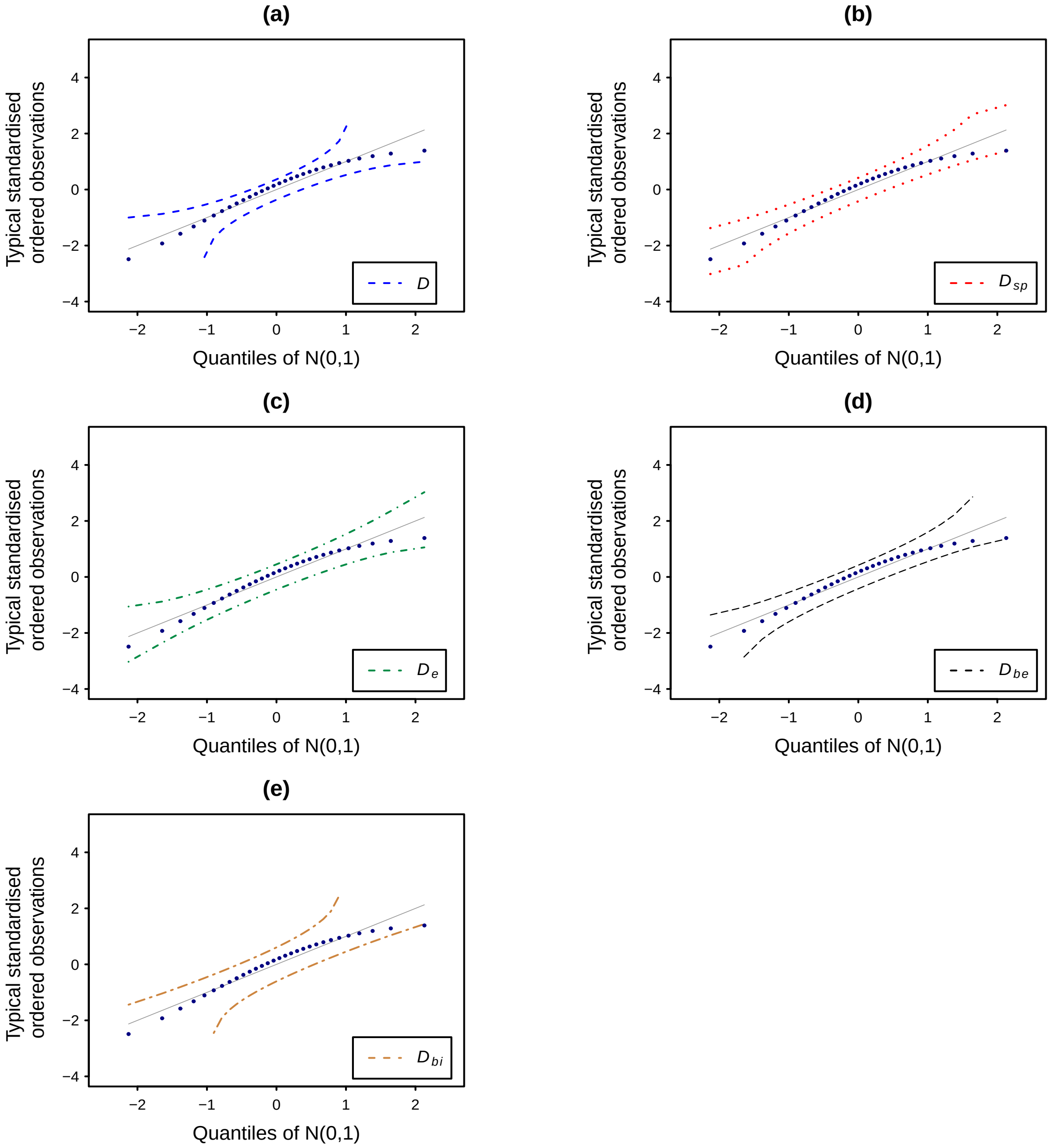

Figure 2.2: Simultaneous intervals in detecting non–normality of the typical standardised ordered observations simulated from $beta(5, 1.5)$ based on the five graphical tests $n = 30$ at $\alpha = 0.05$

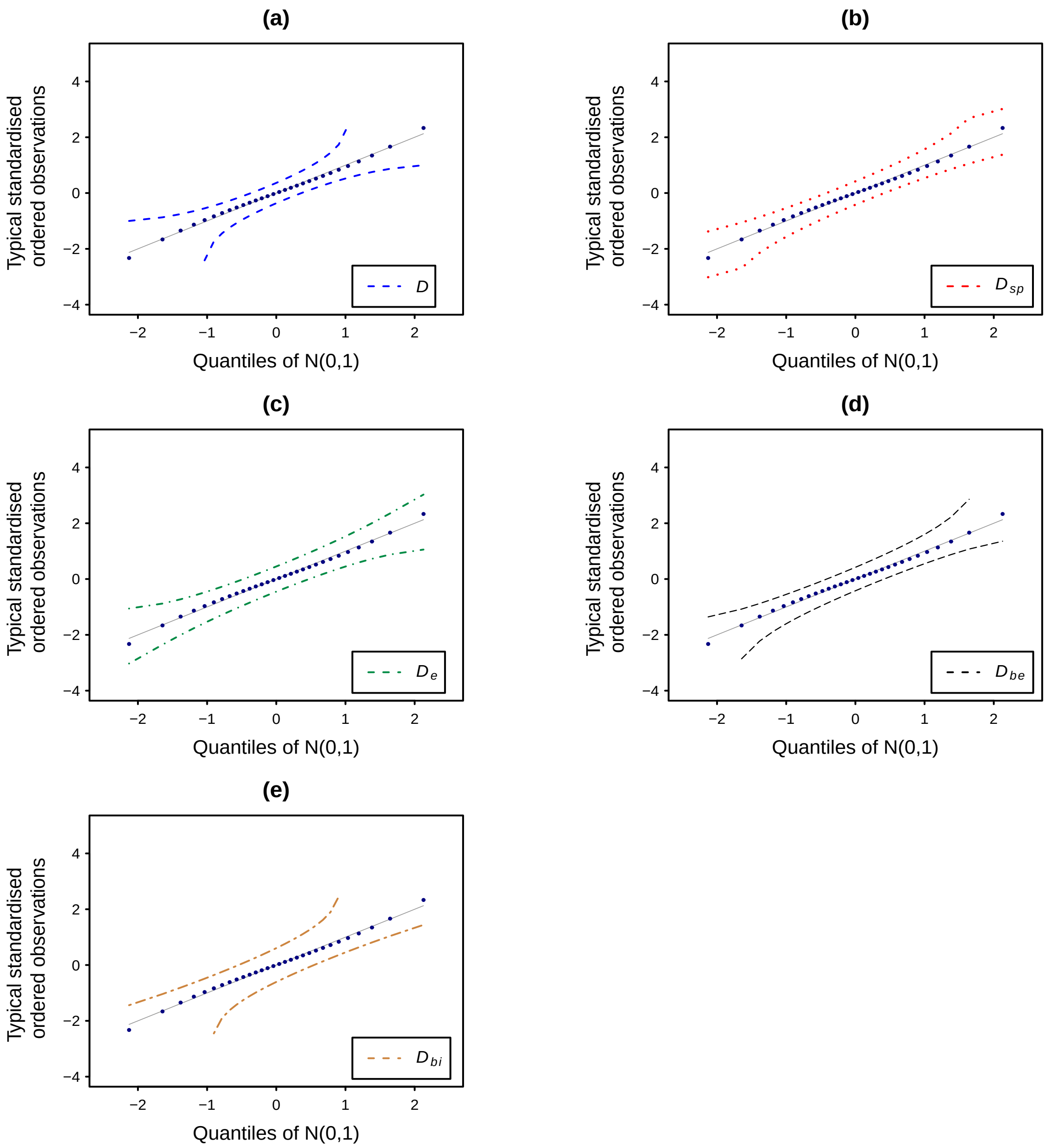

Figure 2.3: Simultaneous intervals in detecting non–normality of the typical standardised ordered observations simulated from t distribution with 6 degrees of freedom based on the five graphical tests $n = 30$ at $\alpha = 0.05$

Example 1

The application of the graphical tests to ERBAT data

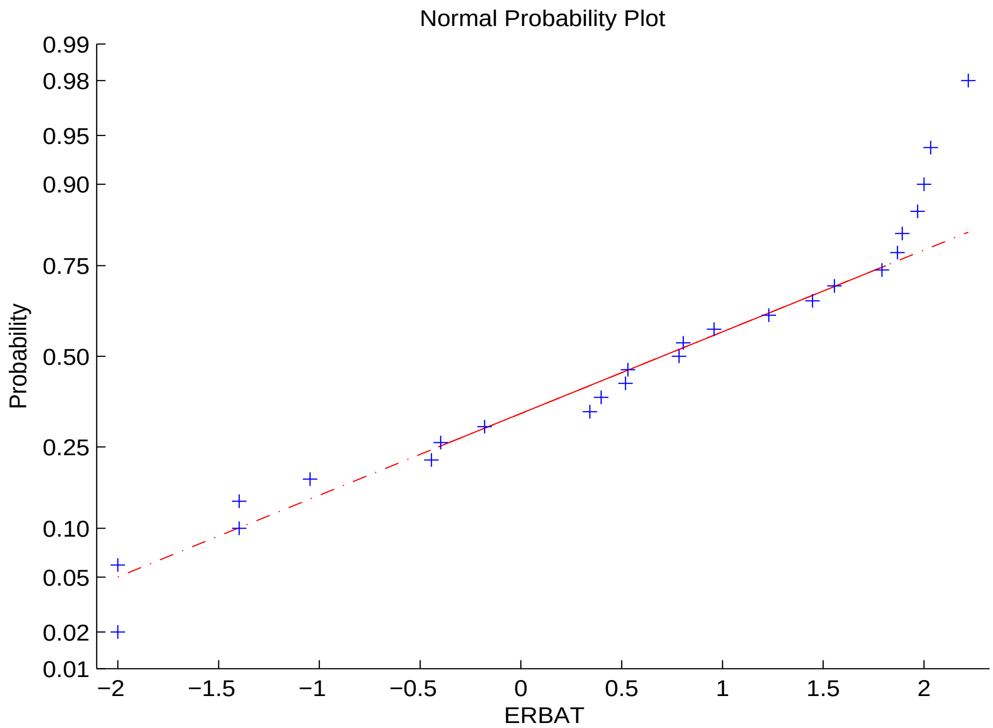

Figure 2.4: Normal probability plot of ERBAT data

In order to clarify how to apply the graphical tests based on a real dataset, we exemplify the case study of the estrogen receptor–binding affinity of tripheni-lacrylonitriles (ERBAT); see, for example, Mukherjee et al. (2005). Twenty five observations on ERBAT data are −1.046, 1.556, 0.342, 0.519, 1.792, 1.869, 0.785, 2.22, 1.447, 0.398, 1.968, 1.892, 0.959, −0.18, 1.23, −0.444, 0.806, −2, 0.531, 2.033, −0.398, −2, −1.398, 2, and −1.398. Bolboaca and Jantschi (2009) pointed out that the investigated experimental ERBAT data are not quite normally distributed if we judge it by a classic graphical test (the normal probability plot) as shown in Figure 2.4. Similarly, the Shapiro–Wilk test identifies that ERBAT data are not normally distributed. Nevertheless, when the Kolmogorov–Smirnov and Anderson-Darling tests are applied to the data set, they draw the conclusion that such data are normally distributed.

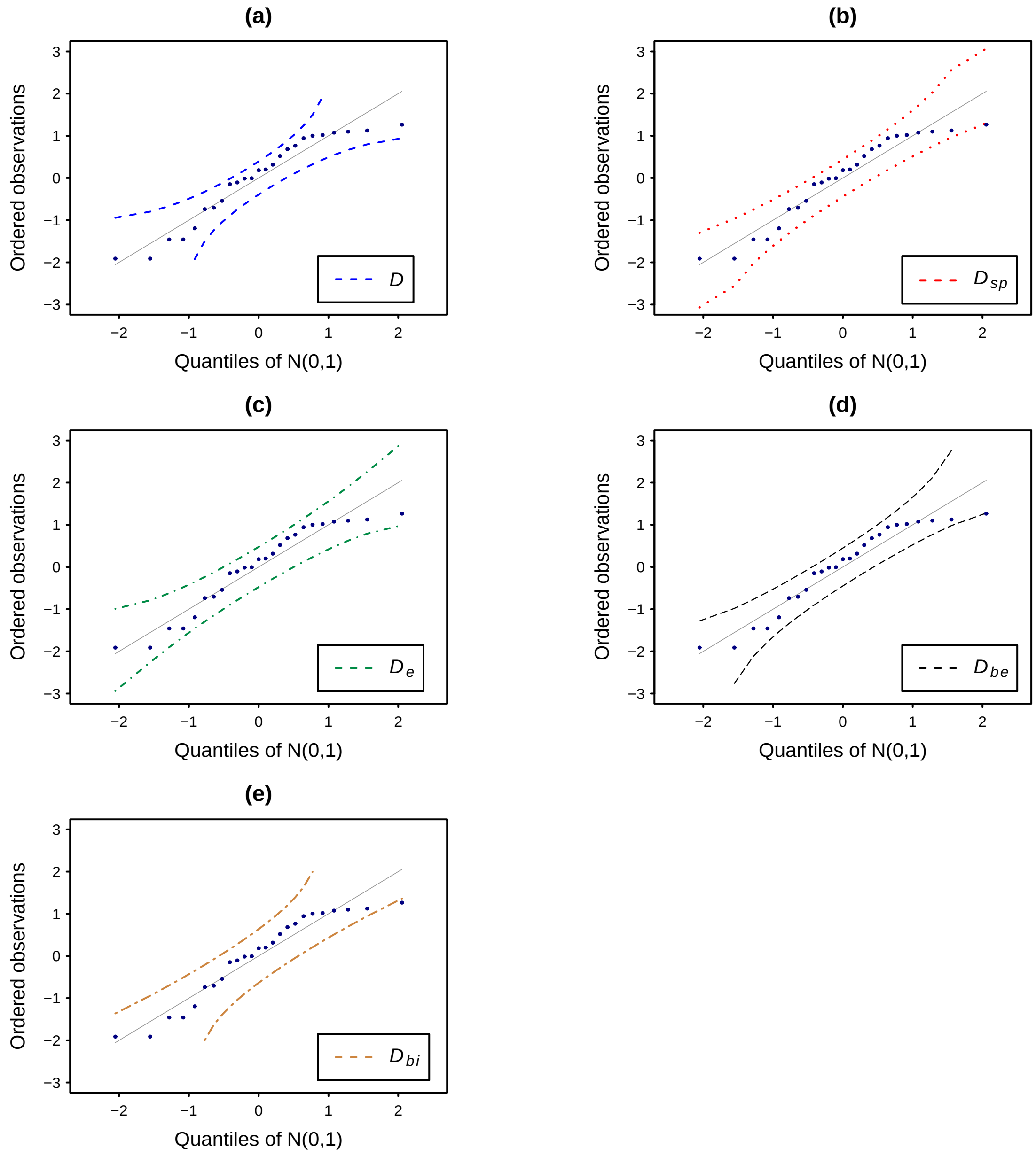

Figure 2.5: Simultaneous intervals of the 25 observations from ERBAT data for testing normality based on the five graphical tests (a) D test, (b) D_{sp} test, (c) D_e test, (d) D_{be} test and (e) D_{bi} test at $\alpha = 0.05$

Table 2.4: The ERBAT data and the corresponding probability intervals for each ordered observation

		D		D_{sp}		D_e		D_{be}		D_{bi}	
observation	standardized value	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
1	-1.911	NaN	-0.9437	-3.0659	-1.2988	-2.9391	-0.9915	$-\infty$	-1.2787	NaN	-1.362
2	-1.911	NaN	-0.7973	-2.5579	-0.9621	-2.2581	-0.7905	-2.7589	-0.9815	NaN	-0.8425
3	-1.4579	NaN	-0.6662	-2.0251	-0.7488	-1.9021	-0.6234	-2.1185	-0.7677	NaN	-0.6976
4	-1.4579	NaN	-0.5457	-1.7179	-0.5809	-1.6535	-0.4801	-1.7769	-0.5962	NaN	-0.5099
5	-1.193	-1.9213	-0.4326	-1.4928	-0.4372	-1.4579	-0.3522	-1.5349	-0.4492	NaN	-0.3511
6	-0.74	-1.4958	-0.3248	-1.3104	-0.3085	-1.0934	-0.2347	-1.3424	-0.3176	-1.9962	-0.2095
7	-0.7054	-1.2407	-0.2207	-1.154	-0.1896	-1.1492	-0.1246	-1.1791	-0.1963	-1.6318	-0.0787
8	-0.5413	-1.0479	-0.119	-1.0148	-0.0775	-1.0191	-0.0196	-1.0348	-0.082	-1.3916	0.0451
9	-0.1485	-0.8877	-0.0184	-0.8876	0.03	-0.8991	0.0819	-0.9036	0.0276	-1.202	0.1645
10	-0.1064	-0.7476	0.0819	-0.7689	0.1347	-0.7865	0.1812	-0.7818	0.1342	-1.0402	0.2816
11	-0.0153	-0.6208	0.1831	-0.6565	0.2377	-0.6794	0.2793	-0.6666	0.2392	-0.8956	0.3982
12	-0.0063	-0.5034	0.2862	-0.5485	0.3403	-0.5763	0.3772	-0.5562	0.3437	-0.7623	0.516
13	0.1849	-0.3925	0.3925	-0.4435	0.4435	-0.4759	0.4759	-0.449	0.449	-0.6367	0.6367
14	0.2007	-0.2862	0.5034	-0.3403	0.5485	-0.3772	0.5763	-0.3437	0.5562	-0.516	0.7623
15	0.3158	-0.1831	0.6208	-0.2377	0.6565	-0.2793	0.6794	-0.2392	0.6666	-0.3982	0.8956
16	0.5198	-0.0819	0.7476	-0.1347	0.7689	-0.1812	0.7865	-0.1342	0.7818	-0.2816	1.0402
17	0.6831	0.0184	0.8877	-0.03	0.8876	-0.0819	0.8991	-0.0276	0.9036	-0.1645	1.202
18	0.7651	0.119	1.0479	0.0775	1.0148	0.0196	1.0191	0.082	1.0348	-0.0451	1.3916
19	0.9427	0.2207	1.2407	0.1896	1.154	0.1246	1.1492	0.1963	1.1791	0.0787	1.6318
20	1.0007	0.3248	1.4958	0.3085	1.3104	0.2347	1.2934	0.3176	1.3424	0.2095	1.9962
21	1.018	0.4326	1.9213	0.4372	1.4928	0.3522	1.4579	0.4492	1.5349	0.3511	NaN
22	1.0752	0.5457	NaN	0.5809	1.7179	0.4801	1.6535	0.5962	1.7769	0.5099	NaN
23	1.0992	0.6662	NaN	0.7488	2.0251	0.6234	1.9021	0.7677	2.1186	0.6976	NaN
24	1.1241	0.7973	NaN	0.9621	2.5579	0.7905	2.2581	0.9815	2.7589	0.9425	NaN
25	1.2648	0.9437	NaN	1.2988	3.0659	0.9915	2.9391	1.2787	∞	1.362	NaN

NaN stands for Not a Number representing an undefined value.

Table 2.4 shows the standardised order observations of the ERBAT data and the numerical results of the simultaneous intervals from the D, D_{sp}, D_e, D_{be} and D_{bi} tests. In case we cannot distinguish from a plot whether a point falls inside the corresponding interval, the numerical results will be applied for comparison.

In Figure 2.5, the graphical tests based on a simple random sample are applied to the ERBAT data. The null (H_0) and alternative (H_1) hypotheses are

$$H_0 \ : \ \text{The data follow the normal distribution}$$

$$H_1 \ : \ \text{The data do not follow the normal distribution.}$$

The null hypothesis is rejected if there is at least one point falling outside the corresponding interval on the plot. Figures 2.5(a) and 2.5(c) reveal that all standardised observations from the ERBAT data fall inside the corresponding simultaneous intervals. Therefore, the inference is that the ERBAT data follow the normal distribution based on the D and D_e tests.

When Figure 2.5(e) is considered, it is clear that the 25^{th} standardised ordered observation lies outside its corresponding interval. Thus, the ERBAT data do not follow the normal distribution based on the D_{bi} test.

As seen in Figures 2.5(b) and 2.5(d), the 25^{th} standardised ordered observations of both figures are so close to their corresponding lower bounds that we cannot assess whether they fall inside the corresponding intervals. For this reason, the numerical results are applied. For Figure 2.5(b), the simultaneous interval of the 25^{th} ordered observation is $[1.2988, 3.0659]$. Comparing the 25^{th} ordered value, 1.2648, with its interval, we can observe that the corresponding interval does not cover the value. Therefore, the ERBAT data do not follow the normal distribution based on the D_{sp} test. Likewise, the simultaneous interval of the 25^{th} standardised ordered observation is $[1.362, NaN]$ from Figure 2.5(d). This leads to the conclusion that the ERBAT data are not taken from the normal population based on the D_{be} test.
Table 2.5 summarises the conclusion drawn from when the seven tests are applied to assess whether data from Table 2.4 follow a normal distribution.

Table 2.5: Test results for Example 1

Test	Inference
D	Does not reject H_0
D_{sp}	Rejects H_0
D_e	Does not reject H_0
D_{be}	Rejects H_0
D_{bi}	Rejects H_0
AD	Does not reject H_0
SW	Rejects H_0

2.4 Power comparison

Power is the ability to detect when a sample comes from a non-normal distribution and it is the most frequently used measure of the value of a test for normality. When presenting a new test, the term "power" must be referred in order to judge which tests are better than others. Thode (2002) recommended that there was no common standard for the design of the power comparisons. Generally, different studies employed different sample sizes and significance levels. The different numbers of replications used in each study result in reliability of the estimated power.

Having introduced a variety of different tests for normality, the purpose is to compare all the tests directly with one another. Simulations based on the Monte Carlo experiments have to be conducted. This is the major concern of this chapter.

Definition 2.4.1. The test decision of not rejecting the null hypothesis H_0 when in fact the alternative hypothesis H_1 is true, is called a **Type II error**. Let the probability of making such an error be denoted with β. The **power** $1-\beta$ of a statistical test is then the probability that it will not make a Type II error. This is equivalent to the probability of rejecting a false null hypothesis.

In the scope of testing for normality, the power is the ability to detect when a sample comes from a non-normal distribution. Accordingly, a sample is taken from a distribution which we already know is not normally distributed. Tests with higher power should be more likely to reject H_0 than tests with lower power. This is the underlying idea when determining the power with an empirical simulation study. The principle of an empirical power analysis is simple to describe. The general approach is given in the following definition.

Definition 2.4.2. Let R be the number of randomly generated independent samples of sizes n where all R samples follow the same distribution that is not a normal distribution. The **empirical power** $1-\hat{\beta}$ of a given test for normality for a given significance level α is given by

$$1-\hat{\beta}=\frac{r}{R}$$

where $r \leq R$ is the number of the R tests that reject the null hypothesis of a normally distributed sample at a significance level α.

2.4.1 Alternative non-normal distributions

Here the alternative non-normal distributions are briefly presented. We only give the probability density functions of the distributions and their theoretical means and variances. All the detail of the following distributions were taken from Krishnamoorthy (2006).

- Y is the chi-square distribution with ν degree(s) of freedom, denoted by $\chi^2(\nu)$, if its probability density function (pdf) is given by

$$f(y) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{(\nu/2)-1} e^{-y/2}, \qquad y > 0, \nu > 0$$

  with mean ν and variance 2ν.

- Y is the exponential distribution with parameter $-\infty < \mu < \infty$ and $\lambda > 0$, denoted by $Exp(\mu, \sigma)$, if its pdf is given by

$$f(y) = \frac{1}{\sigma} \exp\Big(- \frac{y-\mu}{\sigma}\Big) \quad y > \mu,\, \sigma > 0.$$

  with mean $\mu + \sigma$ and variance σ^2.

- Y is the log-normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, denoted by $LogN(\mu, \sigma)$, if its pdf is given by

$$f(y) = \frac{1}{\sigma y\sqrt{2\pi}} \exp\Big(- \frac{(\log y - \mu)^2}{2\sigma^2}\Big), \qquad y > 0,\, \sigma > 0, -\infty < \mu < \infty$$

  with mean $\exp(\mu + \frac{\sigma^2}{2})$ and variance $\exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$.

- Y is the uniform distribution over an interval $[a, b]$ denoted by $U(a, b)$, if its probability density function (pdf) is given by

$$f(y) = \frac{1}{b-a}, \qquad a \le y \le b$$

  with mean $\frac{b+a}{2}$ and variance $\frac{(b-a)^2}{12}$.

- Y is the beta distribution with parameter a and b denoted by $beta(a, b)$, if its probability density function (pdf) is given by

$$f(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}, \qquad 0 \le y \le 1, a > 0, b > 0$$

  with mean $\frac{a}{a+b}$ and variance $\frac{ab}{(a+b)^2(a+b+1)}$.

- Y is the Laplace distribution with parameters a and b denoted by $Laplace(a, b)$, if its probability density function (pdf) is given by

$$f(y) = \frac{1}{2b} \exp\Big(-\frac{|y-a|}{b}\Big), \qquad -\infty < y < \infty, -\infty < a < \infty, b > 0$$

with mean a and variance $2b^2$.

- Y is the Student's t distribution with parameter ν denoted by $t(\nu)$, if its probability density function (pdf) is given by

$$f(y) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}}(1+\frac{y^2}{\nu})^{-\frac{(\nu+1)}{2}}, \qquad -\infty < y < \infty,\ \nu \geq 1$$

with mean 0 for $\nu > 1$; undefined for $\nu = 1$ and variance $\dfrac{\nu}{\nu-2}, \quad \nu > 2.$

We categorised the 18 non-normal distributions into three groups based on the support and shape of their densities. Generating the simple random samples corresponding to the alternative distributions above, we utilised the corresponding functions in Matlab.

Group I : Support = $(0, \infty)$; asymmetric

1. $\chi^2(1)$
2. $LogN(0, 1)$
3. $Exp(0, 1)$
4. $\chi^2(3)$
5. $\chi^2(4)$
6. $\chi^2(10)$

Group II : Support = $(0, 1)$

1. $U(0, 1)$
2. $beta(2, 2)$
3. $beta(2, 5)$
4. $beta(5, 1.5)$

5. $beta(0.5, 0.5)$
6. $beta(0.5, 3)$
7. $beta(1, 2)$

Group III : Support= $(-\infty, \infty)$; symmetric

1. $Laplace(0, 1)$
2. $t(1)$
3. $t(3)$
4. $t(4)$
5. $t(6)$

In order to obtain the estimated power at $\alpha = 0.05, 0.01$ and 0.1, for each sample size, $300,000$ samples were generated from each of the 18 different non-normal distributions with different sample sizes n. The appropriated critical values were obtained for each test for sample size $n = 5(5)30$, 50 and 100 based on $500,000$ simulated samples from a standard normal distribution. These alternatives were used by Michael (1983), and Noughabi and Arghami (2011) in their study of power comparisons. The alternative distributions were classified into three groups depending on the support and shape of their densities as shown previously.

2.4.2 Results on powers

In this part of the simulation study, the percentage powers of the tests are shown in Tables 2.6−2.14 against 18 alternative distributions and different sample sizes at $\alpha = 0.01, 0.05$ and 0.1.

Table 2.6: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group I for several sample size n at $\alpha = 0.01$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$\chi^2(1)$	5	10.72	10.72	10.88	10.64	11.05	13.83	**14.61**
	10	30.31	43.08	35.28	36.71	**59.52**	47.88	50.53
	15	51.48	82.40	48.87	76.96	**90.07**	75.18	78.33
	20	70.15	96.74	60.28	95.15	**98.33**	90.19	92.68
	25	83.79	99.57	70.03	99.25	**99.78**	96.56	97.96
	30	92.22	99.96	78.95	99.93	**99.98**	98.90	99.50
	50	99.86	**100**	98.09	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	8.96	8.96	8.92	8.78	9.03	10.65	**10.92**
	10	26.64	32.41	31.99	29.16	**42.74**	39.03	41.02
	15	44.78	63.10	47.21	58.03	**72.34**	63.20	65.90
	20	60.41	83.82	58.91	80.09	**88.54**	79.79	83.07
	25	72.93	93.83	67.94	91.86	**95.86**	89.42	92.40
	30	82.33	97.89	75.41	97.13	**98.63**	94.91	96.66
	50	97.61	99.99	92.76	99.98	**99.99**	99.82	99.94
	100	**100**	**100**	99.96	**100**	**100**	**100**	**100**
$Exp(0,1)$	5	4.78	4.78	4.77	4.70	4.81	5.66	**5.81**
	10	13.04	16.32	16.59	14.08	**25.36**	21.35	23.35
	15	23.12	41.54	25.54	35.41	**54.42**	40.01	43.77
	20	33.74	67.79	32.76	61.04	**76.71**	57.41	63.56
	25	44.36	85.09	39.05	80.32	**89.98**	71.08	77.98
	30	54.91	94.07	45.26	91.94	**96.19**	81.46	87.57
	50	85.24	99.95	67.41	99.92	**99.97**	98.13	99.42
	100	99.87	**100**	98.63	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	3.15	3.15	3.14	3.09	3.15	3.51	**3.54**
	10	8.07	9.25	10.64	8.41	13.81	12.77	**13.94**
	15	14.04	22.45	17.13	18.82	**31.48**	24.34	27.08
	20	20.22	40.54	22.52	34.33	**49.99**	36.60	42.22
	25	26.80	58.40	27.27	51.61	**66.73**	48.03	56.31
	30	33.99	73.42	31.90	68.20	**79.51**	58.94	67.56
	50	60.72	97.80	49.28	96.86	**98.45**	87.98	93.97
	100	97.79	**100**	82.61	**100**	**100**	99.84	**100**
$\chi^2(4)$	5	2.51	2.51	2.50	2.44	2.50	2.67	**2.69**
	10	5.89	6.51	7.73	6.12	9.19	8.85	**9.82**
	15	9.89	14.42	12.96	12.28	**20.25**	16.85	19.11
	20	14.07	25.87	17.06	21.73	**33.08**	25.54	30.31
	25	18.67	38.92	21.07	33.19	**46.50**	34.08	41.47
	30	23.76	52.19	24.73	46.99	**59.28**	43.03	51.57
	50	44.39	88.18	38.82	85.32	**90.44**	73.64	83.63
	100	83.15	99.96	67.07	99.92	**99.97**	98.45	99.73
$\chi^2(10)$	5	1.48	1.48	1.48	1.49	1.48	1.50	**1.61**
	10	2.70	2.83	3.42	2.77	3.40	3.57	**3.82**
	15	3.91	4.72	5.68	4.25	5.89	5.83	**6.81**
	20	5.07	7.02	7.43	6.15	8.47	8.17	**10.29**
	25	6.33	9.76	9.06	8.41	11.60	10.63	**14.12**
	30	7.82	12.98	10.82	11.66	14.99	13.37	**17.67**
	50	14.34	28.16	17.04	25.84	29.65	26.55	**35.79**
	100	34.22	66.43	31.08	64.88	65.28	60.30	**75.08**

The bold number is the highest power among the seven tests for each sample size.

Table 2.7: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group II for several sample size n at $\alpha = 0.01$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$U(0,1)$	5	0.94	0.94	0.98	1.03	0.94	1.08	**1.17**
	10	1.18	0.91	1.20	0.83	0.82	**1.26**	1.07
	15	1.58	1.04	0.66	0.94	1.63	**2.22**	1.69
	20	1.97	1.83	0.38	1.38	3.32	**3.95**	3.30
	25	2.55	3.30	0.28	2.26	5.63	**6.19**	5.67
	30	3.41	5.85	0.19	4.25	9.33	**9.59**	8.87
	50	6.97	26.09	0.26	19.23	33.29	27.27	**35.33**
	100	24.98	93.82	0.61	88.99	95.82	79.05	**94.42**
$beta(2,2)$	5	0.84	0.84	**0.86**	0.84	0.84	0.84	**0.86**
	10	**0.79**	0.62	0.67	0.59	0.53	0.68	0.53
	15	**0.88**	0.54	0.26	0.51	0.60	0.78	0.45
	20	0.94	0.59	0.12	0.49	0.76	**0.99**	0.57
	25	1.01	0.68	0.07	0.54	0.97	**1.15**	0.71
	30	1.14	0.85	0.04	0.69	1.26	**1.39**	0.77
	50	1.59	1.96	0.02	1.50	2.77	**12.97**	2.22
	100	3.51	10.16	0.01	7.38	13.15	10.45	**13.32**
$beta(2,5)$	5	1.21	1.21	1.21	1.16	1.21	1.25	**1.33**
	10	1.74	1.60	1.80	1.61	1.87	1.96	**2.01**
	15	2.47	2.43	2.00	2.10	**3.54**	3.05	2.92
	20	3.12	3.91	2.20	3.11	**5.87**	4.34	4.48
	25	3.88	6.19	2.26	4.59	**8.97**	5.74	6.39
	30	4.90	9.27	2.25	7.33	**2.79**	7.36	8.22
	50	9.18	29.06	2.65	24.06	**34.18**	16.94	20.76
	100	24.67	83.55	3.70	78.93	**86.40**	49.62	65.64
$beta(5,1.5)$	5	1.64	1.64	1.64	1.60	1.64	1.72	**1.79**
	10	2.85	2.81	3.24	2.70	3.90	3.76	**4.18**
	15	4.56	5.66	4.26	4.59	**9.30**	6.88	7.43
	20	6.37	11.26	5.01	8.50	**16.83**	10.72	12.42
	25	8.36	19.52	5.50	14.77	**26.94**	14.81	18.00
	30	10.77	29.92	5.97	24.46	**38.24**	19.79	23.97
	50	21.74	73.14	8.30	63.35	**78.12**	43.85	55.59
	100	54.53	99.74	15.31	99.56	**99.84**	88.23	96.81
$beta(0.5,0.5)$	5	2.37	2.37	2.66	2.38	2.43	2.99	**3.27**
	10	4.01	3.83	5.31	2.98	5.95	**7.52**	7.28
	15	6.92	9.47	4.51	6.80	15.38	**17.53**	17.48
	20	10.62	20.48	4.42	15.28	29.41	31.55	**34.98**
	25	15.27	36.06	4.55	28.22	47.52	46.30	**53.87**
	30	20.97	54.50	5.18	46.23	66.34	60.69	**70.26**
	50	48.13	97.61	10.54	95.60	**98.71**	94.14	98.59
	100	93.87	**100**	43.54	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	7.28	7.28	7.43	7.12	7.50	9.41	**9.96**
	10	19.28	27.54	22.20	22.21	**43.61**	32.45	34.49
	15	34.11	66.88	29.24	58.97	**79.57**	57.64	61.16
	20	49.36	90.62	35.55	86.68	**94.76**	76.99	81.72
	25	64.25	98.05	42.12	96.84	**98.96**	88.75	92.82
	30	76.87	99.69	49.62	99.49	**99.85**	95.03	97.36
	50	98.22	**100**	84.87	**100**	**100**	99.92	99.99
	100	99.99	**100**	99.99	**100**	**100**	**100**	**100**
$beta(1,2)$	5	1.43	1.43	1.56	1.49	1.54	**1.69**	**1.69**
	10	2.40	2.08	2.39	1.98	**3.27**	3.07	3.14
	15	3.58	4.44	2.00	3.54	**8.65**	5.62	5.21
	20	5.40	10.48	1.67	7.44	**17.16**	9.15	8.82
	25	6.79	20.34	1.49	14.57	**28.94**	13.2	13.67
	30	10.02	33.31	1.34	26.17	**42.46**	18.24	19.34
	50	20.20	80.51	1.43	75.33	**85.33**	43.84	52.52
	100	54.51	99.95	3.75	99.93	**99.98**	90.65	97.64

The bold number is the highest power among the seven tests for each sample size.

Table 2.8: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group III for several sample size n at $\alpha = 0.01$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$Laplace(0,1)$	5	**2.30**	**2.30**	2.28	2.28	**2.30**	**2.30**	2.29
	10	4.81	5.67	6.00	5.62	5.59	6.13	**6.24**
	15	7.10	8.67	**10.25**	9.06	6.30	9.72	9.42
	20	9.05	10.85	13.31	11.82	5.95	13.19	**13.42**
	25	11.02	12.55	15.93	13.88	5.51	16.43	**16.96**
	30	13.15	13.79	18.31	16.24	4.95	19.76	**20.03**
	50	21.91	18.09	27.34	22.79	3.13	**34.16**	33.88
	100	46.17	28.64	44.37	38.33	1.27	**65.90**	63.70
$t(1)$	5	17.25	17.25	17.16	17.07	17.31	**17.91**	17.78
	10	44.00	46.31	46.31	46.58	45.15	**48.70**	47.14
	15	62.46	65.10	64.00	66.18	58.89	**68.58**	66.38
	20	75.03	77.16	75.28	78.21	67.68	**81.15**	79.68
	25	83.60	84.75	83.04	86.00	74.21	**88.71**	87.56
	30	89.42	89.90	88.22	91.09	79.02	**93.37**	92.53
	50	98.33	98.04	97.52	98.52	90.74	**99.31**	99.14
	100	99.99	99.98	99.96	99.98	99.02	**100**	**100**
$t(3)$	5	2.80	2.80	2.78	2.70	2.80	**2.90**	2.79
	10	6.98	8.23	9.22	8.42	8.40	9.21	**9.72**
	15	10.60	13.17	**16.69**	13.58	11.36	14.84	15.90
	20	13.82	17.35	**22.62**	18.29	12.86	20.06	22.17
	25	16.81	20.75	27.71	21.94	13.87	24.65	**27.88**
	30	19.76	23.56	32.44	25.86	14.43	29.01	**32.94**
	50	30.95	33.10	48.14	37.15	15.56	45.36	**50.93**
	100	55.62	50.92	72.45	58.40	17.52	73.97	**79.50**
$t(4)$	5	1.94	1.94	1.93	1.92	1.95	**2.02**	2.01
	10	4.23	5.06	5.78	5.21	5.30	5.67	6.03
	15	6.13	7.98	**10.85**	8.17	6.93	8.92	10.06
	20	7.58	10.23	**14.79**	10.70	7.39	11.75	13.90
	25	8.98	11.98	**18.23**	12.70	7.83	14.29	17.34
	30	10.39	13.40	**21.58**	14.98	7.86	16.74	20.63
	50	15.80	17.94	32.88	21.08	7.47	26.51	**33.11**
	100	29.46	26.38	53.43	33.14	6.31	48.18	**58.29**
$t(6)$	5	1.47	1.47	1.46	1.38	1.47	1.49	**1.54**
	10	2.40	2.83	3.24	2.97	3.01	3.09	**3.40**
	15	3.08	4.04	**6.00**	4.24	3.62	4.48	5.37
	20	3.59	5.08	**8.14**	5.26	3.73	5.72	7.04
	25	3.91	5.60	**9.88**	5.97	3.68	6.53	8.76
	30	4.40	6.08	**11.60**	6.96	3.62	7.45	10.59
	50	5.92	7.19	**17.92**	8.85	2.92	11.09	16.62
	100	9.70	8.44	30.19	12.05	1.78	19.85	**30.62**

The bold number is the highest power among the seven tests for each sample size.

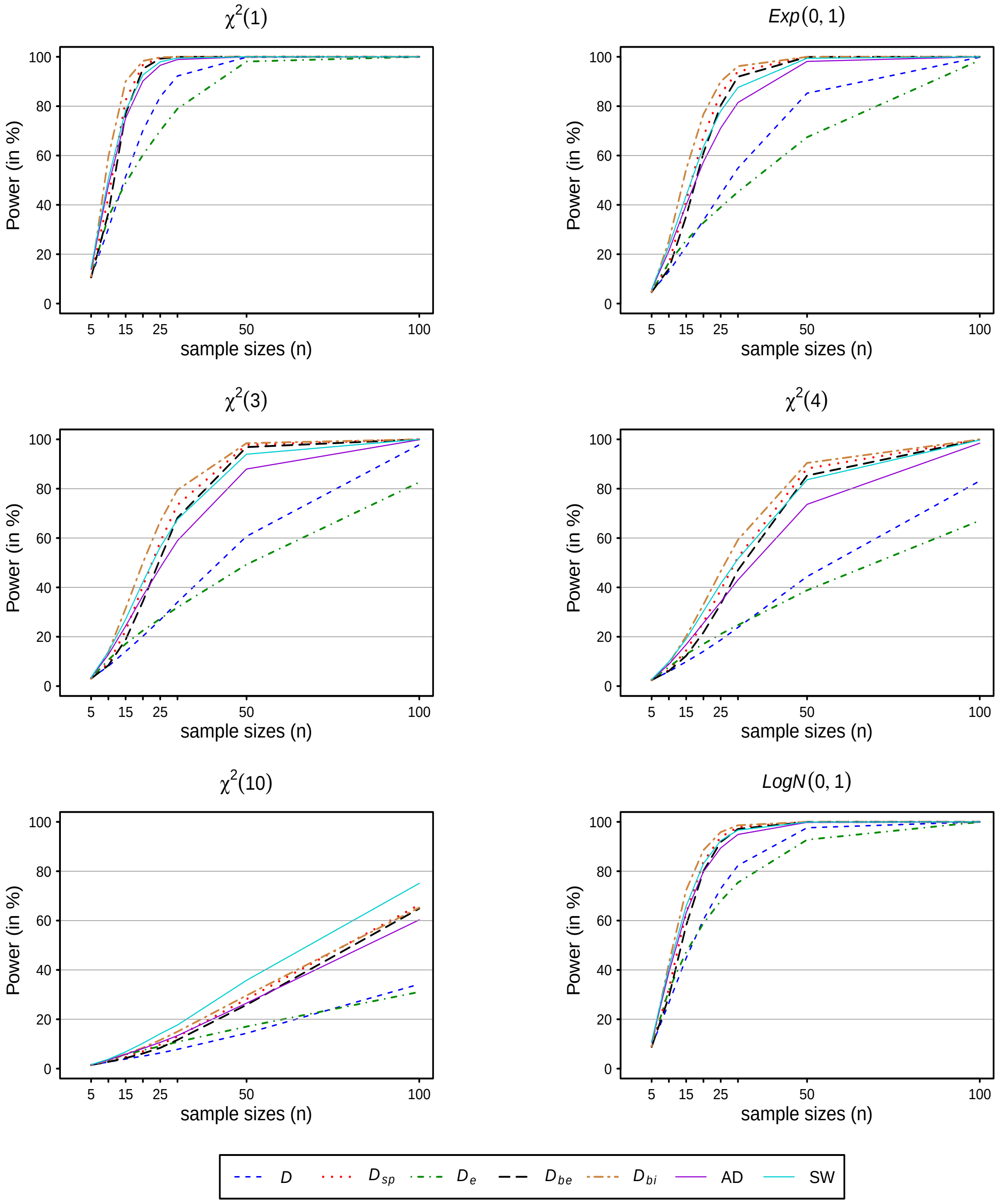

Figure 2.6: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group I for several sample size n at $\alpha = 0.01$

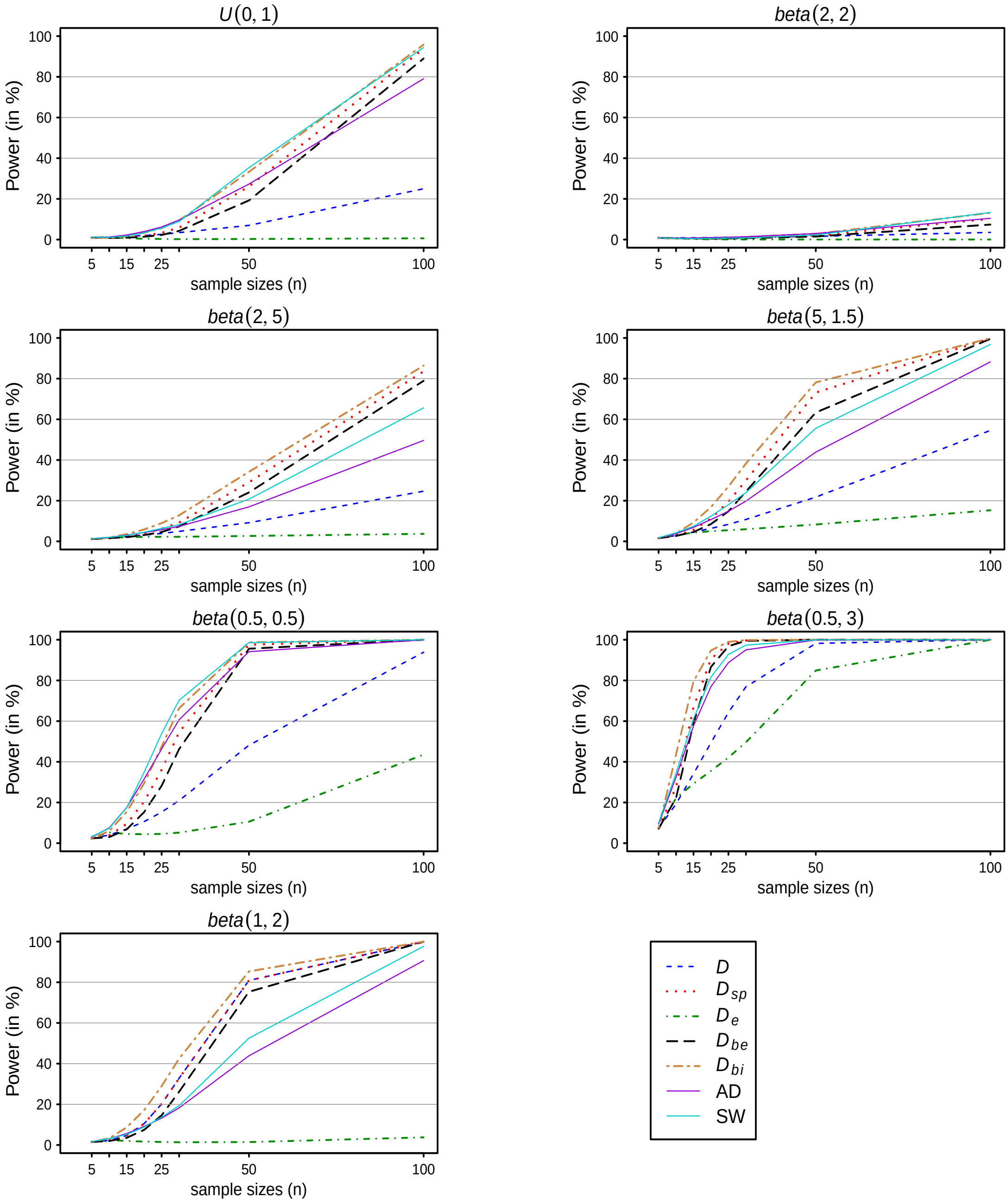

Figure 2.7: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group II for several sample size n at $\alpha = 0.01$

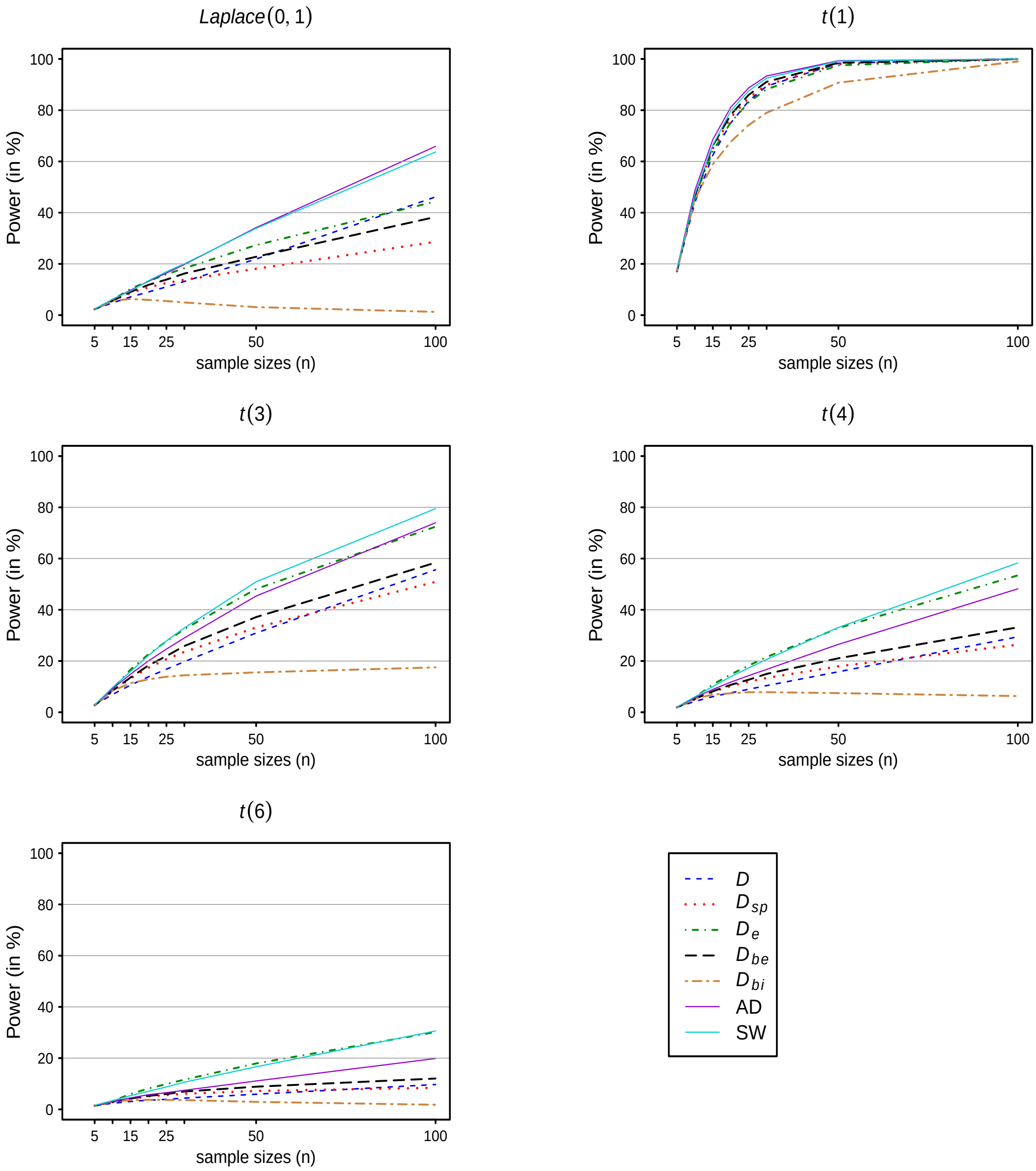

Figure 2.8: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group III for several sample size n at $\alpha = 0.01$

Table 2.9: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group I for several sample size n at $\alpha = 0.05$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$\chi^2(1)$	5	24.05	23.91	25.47	22.03	25.55	30.12	**31.04**
	10	53.30	66.65	56.89	64.93	**77.59**	69.59	73.57
	15	75.02	93.25	75.96	92.11	**96.45**	89.44	92.51
	20	88.33	99.18	88.38	98.88	**99.64**	96.96	98.39
	25	99.22	99.91	95.23	99.88	**99.97**	99.21	99.70
	30	98.24	**99.99**	98.68	99.25	**99.99**	99.82	99.95
	50	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	20.07	19.96	20.01	19.92	21.56	24.13	**24.26**
	10	46.12	52.42	50.04	51.59	**61.35**	57.91	60.68
	15	65.38	79.63	68.08	77.94	**85.53**	79.17	82.53
	20	79.14	93.26	80.25	91.84	**95.54**	90.58	93.26
	25	87.90	97.74	87.25	97.31	**98.72**	95.96	97.56
	30	93.25	99.43	93.13	99.23	**99.68**	98.41	99.24
	50	99.53	99.99	99.54	**100**	99.99	99.97	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	5	13.88	13.62	13.09	13.43	14.58	16.65	**16.82**
	10	30.07	35.75	35.05	34.72	**45.63**	41.22	44.76
	15	44.81	64.81	48.41	62.09	**74.72**	62.43	68.11
	20	58.20	85.32	59.88	82.44	**90.45**	77.66	83.69
	25	69.50	94.44	69.11	93.33	**96.97**	87.57	92.46
	30	78.30	98.39	78.64	97.78	**99.16**	93.43	96.74
	50	96.15	**99.99**	96.94	**99.99**	**99.99**	99.67	99.45
	100	99.99	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	10.76	10.42	10.12	10.05	11.01	**12.21**	12.18
	10	21.13	23.57	24.25	23.16	30.12	28.65	**31.17**
	15	31.47	43.74	34.88	41.35	**53.26**	44.64	49.80
	20	41.54	64.53	44.59	60.2	**72.14**	58.76	66.04
	25	50.86	78.29	50.75	75.59	**85.23**	70.57	78.40
	30	59.03	88.90	59.77	86.37	**92.63**	79.65	86.75
	50	82.81	99.51	82.29	99.36	**99.76**	96.32	98.79
	100	99.14	**100**	99.65	**100**	**100**	99.98	**100**
$\chi^2(4)$	5	8.99	8.93	8.73	9.05	9.47	**10.08**	10.04
	10	16.83	18.06	18.96	17.92	22.60	22.13	**24.11**
	15	24.56	32.34	27.55	30.51	**39.67**	34.46	38.86
	20	32.36	49.10	35.79	44.85	**56.31**	46.27	53.08
	25	39.82	61.78	41.13	58.74	**70.06**	56.88	65.24
	30	46.64	74.49	49.60	70.74	**80.34**	66.21	75.10
	50	69.83	96.25	68.86	95.35	**97.76**	89.13	95.08
	100	95.31	**99.99**	96.35	**99.99**	**99.99**	99.74	99.98
$\chi^2(10)$	5	6.57	6.38	6.08	6.06	6.68	6.83	**6.85**
	10	9.53	9.56	10.34	9.77	11.04	11.36	**11.99**
	15	12.41	14.08	14.73	13.52	16.33	15.99	**17.95**
	20	15.34	20.15	18.65	18.07	21.94	20.75	**24.41**
	25	18.53	24.20	21.64	22.84	27.90	25.59	**30.50**
	30	21.21	30.91	26.15	28.12	33.56	30.32	**36.48**
	50	33.15	51.10	36.70	48.93	54.54	48.47	**58.89**
	100	59.72	86.05	60.29	85.01	86.78	80.38	**90.24**

The bold number is the highest power among the seven tests for each sample size.

Table 2.10: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group II for several sample size n at $\alpha = 0.05$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$U(0,1)$	5	5.31	5.34	6.09	4.62	4.78	6.09	**6.25**
	10	6.20	5.45	6.39	4.33	5.14	7.95	**8.49**
	15	7.93	7.82	6.59	5.83	8.97	11.91	**13.23**
	20	9.91	12.44	6.74	8.69	15.13	17.20	**20.35**
	25	12.15	16.65	6.63	13.22	23.54	23.38	**28.74**
	30	14.39	25.02	7.43	19.30	33.16	30.13	**38.34**
	50	25.87	61.10	9.81	54.40	74.53	57.71	**75.31**
	100	58.71	99.48	26.95	98.99	**99.86**	94.96	99.62
$beta(2,2)$	5	4.44	4.52	**4.88**	4.19	4.21	4.65	4.71
	10	**4.48**	3.94	4.25	3.39	3.40	4.42	4.31
	15	4.72	3.89	3.20	3.12	3.77	**4.91**	4.56
	20	5.25	4.28	2.67	3.41	4.83	**5.84**	5.42
	25	5.58	4.83	2.34	3.78	6.10	**6.79**	6.34
	30	6.14	5.76	2.19	4.57	7.52	**7.93**	7.69
	50	8.26	10.64	1.87	8.69	**15.22**	13.39	**15.22**
	100	15.31	34.25	2.19	28.52	**45.93**	31.96	45.73
$beta(2,5)$	5	5.62	5.64	5.83	5.57	5.60	5.97	**6.18**
	10	7.60	7.39	7.88	6.89	8.16	8.48	**8.84**
	15	9.54	10.23	9.025	9.24	**12.65**	11.39	12.53
	20	11.66	14.36	10.20	12.77	**19.17**	14.98	17.12
	25	13.65	19.59	11.07	17.22	**26.05**	18.43	21.90
	30	16.09	25.95	12.23	22.99	**33.92**	22.45	27.29
	50	25.77	55.38	15.94	50.88	**64.74**	39.50	50.03
	100	50.65	95.84	28.71	94.16	**97.69**	76.06	89.84
$beta(5,1.5)$	5	6.76	6.73	6.97	6.62	6.79	7.48	**7.61**
	10	10.80	11.23	11.76	10.39	13.63	13.40	**14.63**
	15	14.93	18.92	15.14	17.24	**25.15**	20.31	23.17
	20	19.45	29.66	18.28	27.04	**39.34**	27.98	33.64
	25	23.68	42.53	21.07	38.62	**53.59**	35.68	43.20
	30	28.32	55.34	23.98	51.31	**66.10**	43.28	52.86
	50	46.55	90.03	35.88	87.72	**94.05**	70.64	83.26
	100	80.22	99.97	76.80	99.96	**99.99**	97.28	99.68
$beta(0.5,0.5)$	5	9.61	10.15	12.24	7.05	7.82	12.63	**13.82**
	10	15.38	17.24	19.85	12.96	18.52	25.74	**29.79**
	15	23.09	31.67	24.36	25.55	38.72	43.81	**52.11**
	20	32.09	50.31	30.17	43.94	62.90	61.72	**72.78**
	25	41.35	69.48	36.32	63.25	81.66	75.94	**86.31**
	30	50.31	84.56	43.03	80.18	92.60	85.78	**94.12**
	50	80.16	99.81	69.58	99.68	99.95	92.36	**99.95**
	100	99.45	**100**	99.12	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	18.37	18.69	19.69	16.56	19.16	**23.31**	21.60
	10	40.65	56.61	43.83	50.34	**65.97**	55.43	60.65
	15	59.81	85.55	56.67	83.38	**91.86**	79.28	84.59
	20	75.24	97.06	75.05	96.34	**98.65**	91.57	95.05
	25	86.18	99.54	86.87	99.35	**99.82**	96.95	98.71
	30	93.13	99.95	94.48	99.92	**99.98**	98.97	99.69
	50	99.83	**100**	99.96	**100**	**100**	99.99	99.99
	100	**100**	**100**	**100**	**100**	**100**	1**100**	**100**
$beta(1,2)$	5	6.49	6.62	7.14	6.09	6.24	7.28	**7.52**
	10	9.69	9.86	10.04	8.59	12.13	12.11	**13.16**
	15	13.53	17.47	10.99	15.04	**24.54**	18.62	20.89
	20	17.88	29.43	12.16	25.98	**40.83**	26.23	30.56
	25	21.93	44.26	13.19	39.65	**57.05**	34.33	40.75
	30	26.66	59.37	14.67	54.48	**71.13**	42.67	51.54
	50	45.59	93.94	25.43	92.08	**96.82**	72.01	84.09
	100	81.93	**99.99**	81.67	**99.99**	**99.99**	98.23	99.86

The bold number is the highest power among the seven tests for each sample size.

Table 2.11: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group III for several sample size n at $\alpha = 0.05$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$Laplace(0,1)$	5	8.94	8.67	7.96	9.27	**9.50**	8.95	8.66
	10	14.15	15.02	15.07	**16.56**	15.44	15.86	15.33
	15	18.27	20.51	**22.04**	21.90	17.48	21.85	21.00
	20	22.28	25.41	**27.39**	26.17	18.00	27.28	26.29
	25	26.23	27.54	31.32	29.67	17.90	**32.52**	31.10
	30	29.60	31.30	36.09	32.55	17.31	**37.41**	35.65
	50	43.33	39.27	48.24	42.48	15.25	**54.71**	52.12
	100	70.39	57.48	70.62	61.98	13.00	**82.89**	79.81
$t(1)$	5	29.54	29.11	27.41	30.08	**30.88**	30.40	29.50
	10	57.83	59.33	58.61	60.92	59.23	**61.45**	59.15
	15	74.30	76.14	75.77	77.22	72.74	**78.36**	76.43
	20	84.47	86.09	85.64	86.50	80.93	**88.07**	86.77
	25	90.66	91.26	91.24	91.91	86.28	**93.45**	92.39
	30	94.50	94.85	95.30	95.12	90.03	**96.46**	95.90
	50	99.40	99.28	99.29	99.39	97.17	**99.73**	99.62
	100	**100**	**100**	**100**	99.99	99.90	**100**	**100**
$t(3)$	5	9.55	9.27	8.55	9.87	**10.10**	9.81	9.43
	10	16.10	17.29	17.99	18.67	18.34	18.70	**18.80**
	15	21.45	24.64	**27.55**	25.85	23.09	26.21	26.62
	20	26.16	30.99	**35.63**	31.57	25.88	32.63	33.93
	25	30.31	34.48	**41.54**	36.13	27.86	38.44	40.24
	30	34.14	39.47	**48.62**	40.52	29.22	43.70	45.87
	50	48.34	50.85	**64.01**	53.39	32.79	60.73	64.00
	100	73.14	71.35	85.84	74.38	39.72	85.03	**87.82**
$t(4)$	5	7.77	7.55	7.30	8.01	**8.19**	7.94	7.64
	10	11.75	12.68	13.79	**13.80**	13.65	13.75	13.72
	15	14.70	17.29	**20.17**	18.85	16.44	18.49	19.43
	20	17.38	23.70	**26.06**	22.03	17.87	22.58	24.28
	25	19.97	23.30	**30.30**	24.73	18.53	26.11	28.37
	30	21.93	26.97	**35.55**	27.79	19.16	29.80	32.67
	50	30.47	33.58	**47.79**	35.95	19.56	41.81	46.70
	100	48.97	47.90	71.08	51.75	19.59	65.06	**71.38**
$t(6)$	5	6.40	6.21	5.78	**6.53**	5.86	6.47	6.48
	10	8.33	8.87	9.34	**9.68**	9.63	9.50	9.55
	15	9.64	11.43	**13.92**	11.99	10.83	11.84	12.77
	20	10.79	13.90	**17.64**	13.91	11.43	13.95	15.35
	25	11.88	14.36	**19.81**	15.31	11.53	15.59	17.48
	30	12.51	16.10	**22.73**	16.50	11.27	17.35	20.22
	50	15.70	18.33	**31.14**	20.05	10.32	23.05	28.40
	100	23.31	23.14	**48.86**	26.30	7.66	35.96	45.19

The bold number is the highest power among the seven tests for each sample size.

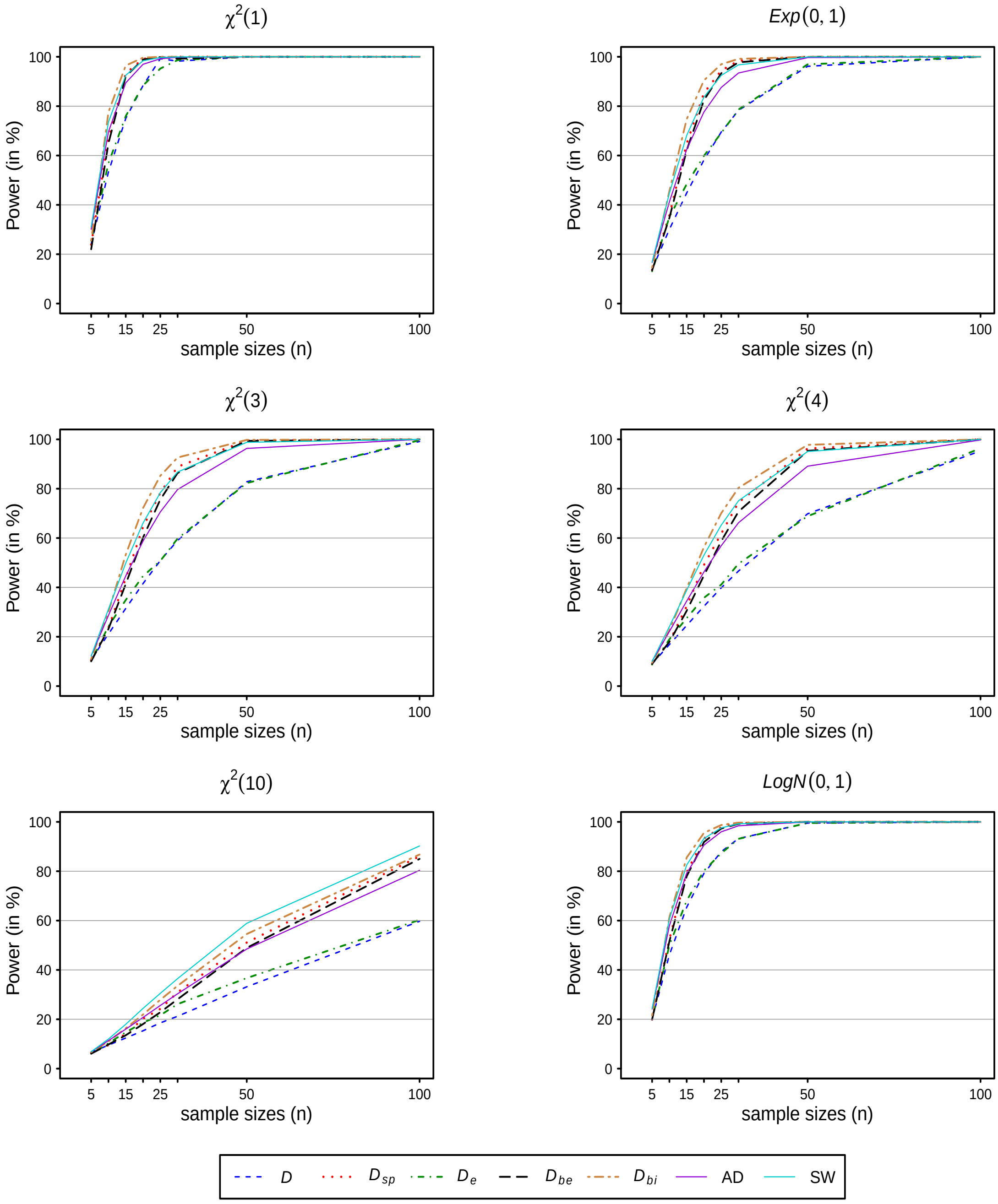

Figure 2.9: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group I for several sample size n at $\alpha = 0.05$

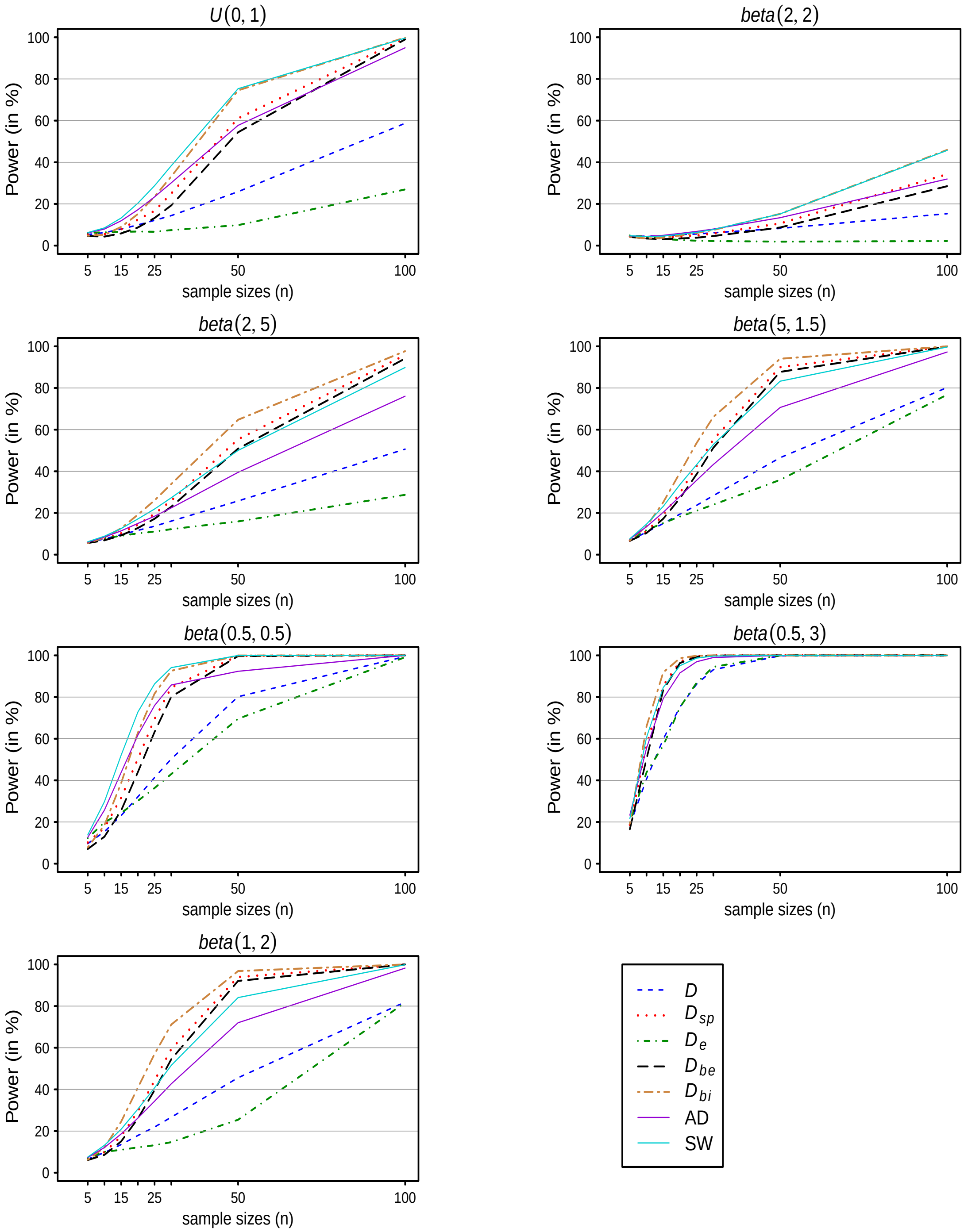

Figure 2.10: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group II for several sample size n at $\alpha = 0.05$

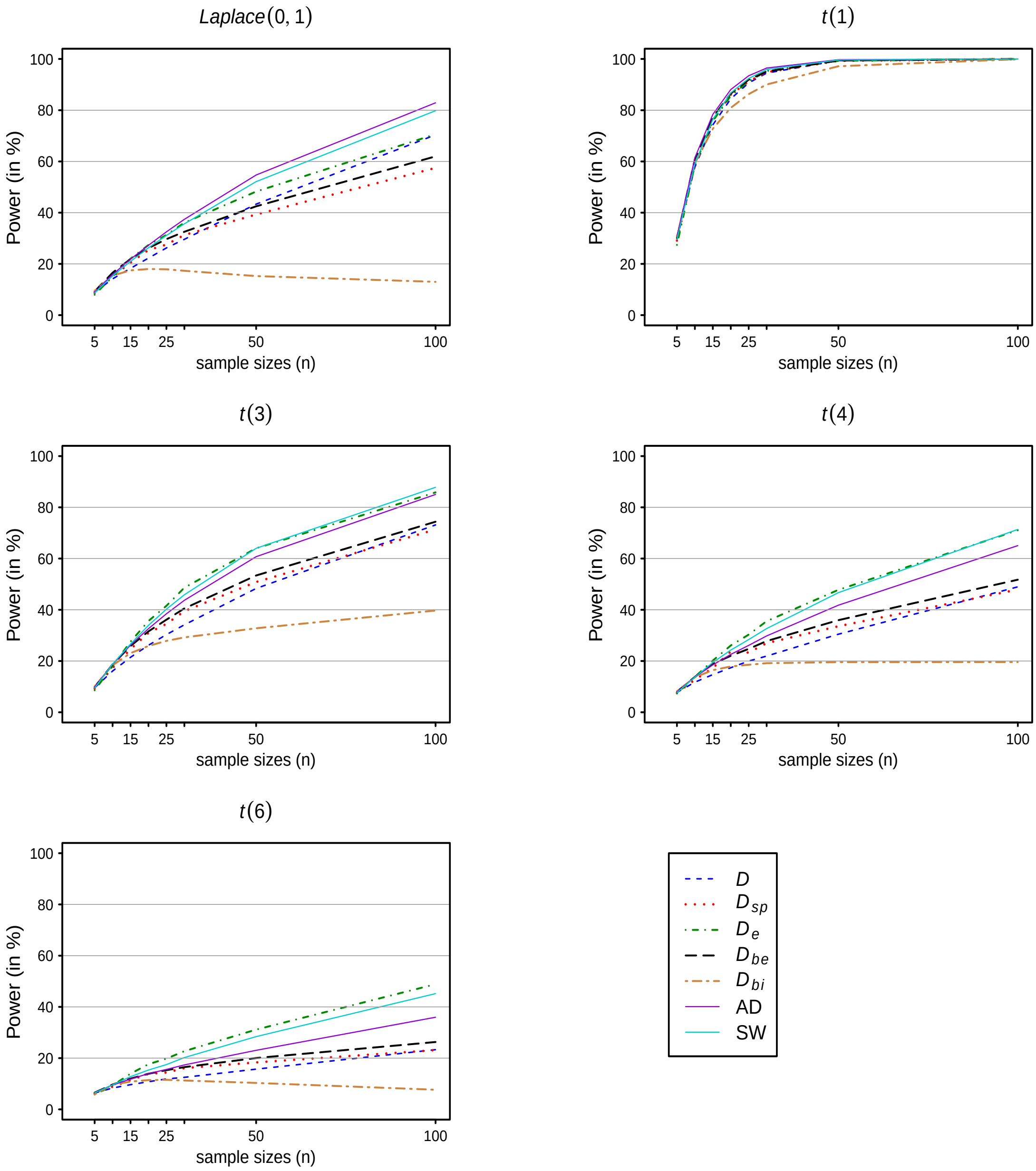

Figure 2.11: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group III for several sample size n at $\alpha = 0.05$

Table 2.12: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group I for several sample size n at $\alpha = 0.1$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$\chi^2(1)$	5	35.24	35.90	36.48	34.81	38.31	42.31	**43.55**
	10	65.71	77.7	69.11	77.39	**84.69**	79.09	82.92
	15	84.27	96.12	86.45	95.99	**98.05**	93.99	96.26
	20	93.80	99.58	95.64	99.53	**99.83**	98.54	99.35
	25	97.85	99.97	98.92	99.96	**99.99**	99.68	99.92
	30	99.33	**100**	99.79	**100**	**100**	99.94	99.98
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	29.15	29.34	29.36	29.38	30.84	33.61	**34.18**
	10	57.41	64.04	60.83	63.50	70.29	67.71	**70.42**
	15	75.46	86.06	77.72	85.58	**90.22**	85.67	88.57
	20	86.52	95.61	88.39	95.17	**97.33**	94.20	96.17
	25	92.83	98.76	94.21	98.65	**99.36**	97.78	98.82
	30	96.41	99.69	97.41	99.66	**99.85**	99.17	99.65
	50	99.81	**100**	99.94	**100**	**100**	99.99	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	5	22.26	22.4	22.63	21.95	23.37	25.78	**26.27**
	10	41.88	49.11	45.63	43.4	56.92	53.10	**57.17**
	15	58.16	74.99	61.41	74.16	**82.49**	73.08	78.57
	20	70.82	90.32	74.25	89.55	**94.33**	85.63	90.34
	25	80.44	96.89	83.95	96.50	**98.45**	92.79	96.17
	30	87.19	99.17	90.93	99.01	**99.65**	96.55	98.54
	50	98.38	**100**	99.59	**100**	**100**	99.88	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	17.92	18.00	18.05	17.76	18.40	20.02	**20.32**
	10	31.70	35.57	35.02	34.96	41.45	39.79	**42.63**
	15	43.92	55.97	47.62	54.98	**64.36**	56.64	61.91
	20	54.63	73.62	58.73	72.15	**81.11**	70.04	76.72
	25	63.84	85.75	67.76	84.56	**90.95**	80.03	86.70
	30	71.60	93.04	75.81	92.24	**96.03**	87.22	92.62
	50	90.51	99.80	94.47	99.72	**99.92**	98.29	99.56
	100	99.73	**100**	99.99	**100**	**100**	99.99	**100**
$\chi^2(4)$	5	15.73	15.77	15.80	15.78	16.19	17.25	**17.39**
	10	26.28	28.70	29.14	28.25	32.90	32.30	**34.69**
	15	36.02	44.18	39.81	43.28	**51.30**	46.22	50.92
	20	44.81	59.49	49.07	57.87	**67.36**	58.3	64.89
	25	52.67	72.21	57.02	70.85	**79.45**	68.24	75.91
	30	59.98	82.23	64.49	80.98	**87.86**	76.54	84.03
	50	80.79	98.09	85.05	97.76	**99.04**	93.95	97.67
	100	98.04	**100**	99.51	**100**	**100**	99.92	99.99
$\chi^2(10)$	5	12.15	12.17	12.13	12.42	12.31	12.63	**12.65**
	10	16.53	17.07	17.96	16.91	18.45	18.77	**19.71**
	15	20.79	22.82	23.35	22.26	25.15	24.86	**26.97**
	20	24.60	28.81	28.44	27.91	32.29	30.57	**34.57**
	25	28.34	34.91	32.84	33.73	39.06	35.99	**41.78**
	30	32.08	41.18	37.22	39.92	45.53	41.59	**48.47**
	50	46.14	63.30	51.74	61.66	67.49	60.42	**70.08**
	100	72.33	92.15	76.39	91.49	93.41	88.10	**94.61**

The bold number is the highest power among the seven tests for each sample size.

Table 2.13: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group II for several sample size n at $\alpha = 0.1$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$U(0,1)$	5	11.29	11.37	12.37	9.71	10.31	12.28	**12.92**
	10	12.65	12.48	14.20	9.49	10.83	15.80	**17.71**
	15	15.75	16.28	15.04	12.63	17.00	22.01	**25.82**
	20	18.84	22.04	16.59	17.65	26.46	29.33	**35.91**
	25	22.35	29.60	18.23	24.89	38.59	37.08	**47.06**
	30	26.12	38.82	20.53	34.04	50.98	45.07	**57.73**
	50	41.58	76.85	30.20	72.49	**88.37**	72.85	88.29
	100	75.54	99.88	64.30	99.77	**99.98**	98.09	99.94
$beta(2,2)$	5	9.53	9.56	**10.02**	8.96	9.03	9.47	9.96
	10	9.31	8.69	9.23	7.30	7.55	9.53	**9.74**
	15	9.85	8.78	7.98	7.22	8.21	10.37	**10.65**
	20	10.73	9.64	7.63	7.68	10.05	11.95	**12.53**
	25	11.41	10.70	7.20	8.69	12.39	13.45	**14.47**
	30	12.35	12.21	7.23	10.07	14.98	15.45	**16.85**
	50	16.00	20.12	7.37	17.38	28.20	23.61	**29.42**
	100	27.04	51.16	9.96	46.12	**66.38**	47.54	65.00
$beta(2,5)$	5	11.27	11.28	11.52	10.89	11.12	11.77	**11.98**
	10	13.89	13.96	14.49	13.17	14.80	15.52	**16.42**
	15	16.87	18.21	16.75	17.04	21.20	19.87	**22.03**
	20	19.98	24.15	19.02	22.22	**29.63**	24.60	28.22
	25	22.85	30.78	20.86	28.56	**38.34**	29.28	34.68
	30	26.09	38.41	23.07	36.15	**47.46**	34.29	41.48
	50	38.40	68.10	31.54	65.19	**77.67**	53.57	65.93
	100	65.05	98.14	59.38	97.55	**99.28**	86.01	95.72
$beta(5,1.5)$	5	13.10	13.14	13.37	12.57	12.99	14.06	**14.21**
	10	18.56	19.57	19.94	18.85	22.39	22.30	**24.29**
	15	24.55	29.85	25.27	28.56	**36.73**	31.39	35.72
	20	30.38	42.59	30.31	40.46	**52.41**	40.49	47.91
	25	35.80	55.92	35.03	53.41	**66.44**	49.26	58.12
	30	41.28	67.99	39.89	65.74	**77.65**	57.34	67.93
	50	60.76	94.52	60.41	93.68	**97.34**	81.55	91.71
	100	89.37	**100**	95.91	99.99	**100**	98.94	99.93
$beta(0.5,0.5)$	5	18.62	18.93	21.03	15.38	17.34	22.55	**24.18**
	10	26.17	30.08	32.62	23.52	29.66	39.86	**46.21**
	15	36.99	48.16	41.24	41.31	54.44	59.60	**69.41**
	20	48.11	67.44	50.78	61.82	77.91	75.59	**85.62**
	25	58.13	82.97	59.32	79.63	91.48	86.35	**94.23**
	30	67.02	92.95	67.70	91.15	97.34	93.00	**97.99**
	50	90.48	99.95	91.16	99.94	99.99	99.74	**99.99**
	100	99.89	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	29.06	29.54	30.34	28.04	30.99	34.79	**36.24**
	10	53.86	66.30	57.04	65.37	**75.50**	68.42	72.96
	15	72.62	91.23	74.74	90.76	**95.18**	87.63	91.54
	20	85.34	98.52	88.40	98.32	**99.37**	95.65	97.79
	25	92.91	99.80	95.77	99.78	**99.93**	98.64	99.54
	30	96.96	99.98	98.79	99.98	**99.99**	99.61	99.93
	50	99.95	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(1,2)$	5	12.81	12.90	13.56	12.38	11.82	14.06	**14.41**
	10	17.34	18.27	18.30	20.62	16.37	21.21	**23.44**
	15	22.82	28.59	20.94	**36.23**	26.40	30.03	34.63
	20	28.48	42.89	24.17	**54.18**	40.15	39.37	46.34
	25	34.02	57.97	27.35	**69.78**	54.98	48.57	57.64
	30	39.85	71.65	31.72	**81.78**	69.05	57.47	68.43
	50	60.42	96.93	56.62	**98.76**	96.31	83.26	92.85
	100	90.89	**100**	98.02	**100**	**100**	99.42	99.97

The bold number is the highest power among the seven tests for each sample size.

Table 2.14: Powers (in %) of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group III for several sample size n at $\alpha = 0.1$

Alternatives	n	D	D_{sp}	D_e	D_{be}	D_{bi}	AD	SW
$Laplace(0,1)$	5	14.96	**17.90**	14.01	16.25	15.92	15.21	14.63
	10	22.21	23.62	23.26	**25.32**	24.13	23.90	22.42
	15	27.64	29.55	31.16	**31.62**	27.32	31.07	28.93
	20	32.49	34.11	**37.91**	36.10	28.76	37.12	34.73
	25	36.77	37.82	**43.10**	39.81	29.42	42.57	40.03
	30	41.22	41.35	**48.04**	43.45	29.72	47.95	45.01
	50	55.96	52.06	62.34	53.97	29.97	**64.97**	61.63
	100	80.87	71.63	82.24	73.47	32.32	**88.77**	86.12
$t(1)$	5	36.93	36.92	35.37	**38.91**	38.78	37.86	36.87
	10	65.15	66.56	65.80	**68.17**	66.61	67.68	65.22
	15	79.89	81.11	81.27	82.38	79.18	**82.86**	80.97
	20	88.50	89.19	89.77	89.96	86.57	**91.01**	89.67
	25	93.47	93.74	**97.31**	94.29	91.03	95.27	94.33
	30	96.30	96.34	96.82	96.69	93.96	**97.56**	97.01
	50	99.64	99.58	99.69	99.64	98.72	**99.83**	99.75
	100	**100**	**100**	**100**	**100**	99.98	**100**	**100**
$t(3)$	5	15.58	15.55	14.74	**16.70**	16.51	15.95	15.47
	10	23.74	25.31	25.98	**26.78**	26.28	26.09	25.67
	15	29.71	32.55	**35.63**	34.45	31.81	34.12	33.79
	20	35.07	38.54	**44.01**	40.42	35.61	40.93	41.24
	25	39.60	43.28	**50.81**	45.20	38.13	46.86	47.73
	30	44.02	47.58	**56.62**	49.56	40.29	52.11	53.24
	50	58.34	60.81	**73.01**	62.28	46.53	68.56	70.82
	100	81.15	80.42	90.93	81.63	57.15	89.59	**91.08**
$t(4)$	5	13.57	13.54	12.87	**14.42**	14.20	13.07	13.59
	10	18.65	20.03	20.65	**21.36**	21.02	20.65	20.08
	15	22.67	25.02	**28.02**	26.48	24.61	26.04	26.39
	20	25.66	28.80	**34.14**	30.50	26.77	30.54	31.43
	25	28.48	32.08	**39.37**	33.76	28.17	34.62	35.89
	30	31.20	34.99	**44.34**	36.88	29.14	38.42	40.33
	50	40.83	44.01	**58.84**	45.74	31.23	51.13	54.44
	100	60.23	60.34	**79.27**	62.14	34.35	73.35	77.55
$t(6)$	5	11.94	11.92	11.52	**12.41**	12.34	12.03	11.84
	10	14.65	15.49	15.88	**16.33**	16.20	15.80	15.56
	15	16.43	18.19	**20.40**	19.32	18.09	18.79	19.11
	20	17.84	20.19	**24.40**	21.35	19.00	21.05	21.91
	25	19.04	21.91	**27.76**	23.07	19.40	23.18	24.40
	30	20.36	23.27	**31.14**	24.78	19.44	25.19	27.41
	50	24.77	27.65	**41.54**	29.04	18.97	32.19	36.33
	100	34.43	35.08	**58.64**	36.95	16.16	46.6	53.66

The bold number is the highest power among the seven tests for each sample size.

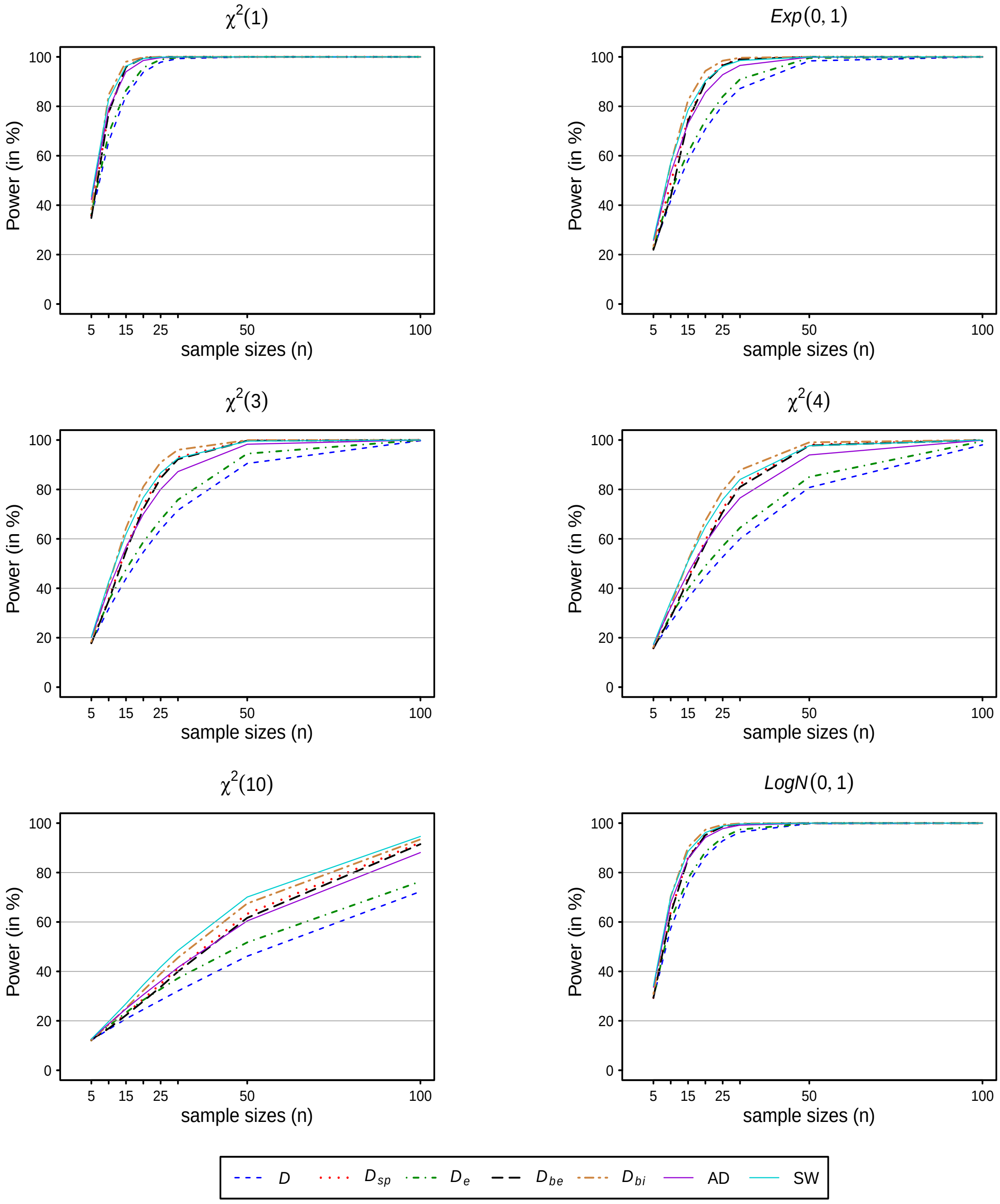

Figure 2.12: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group I for several sample size n at $\alpha = 0.1$

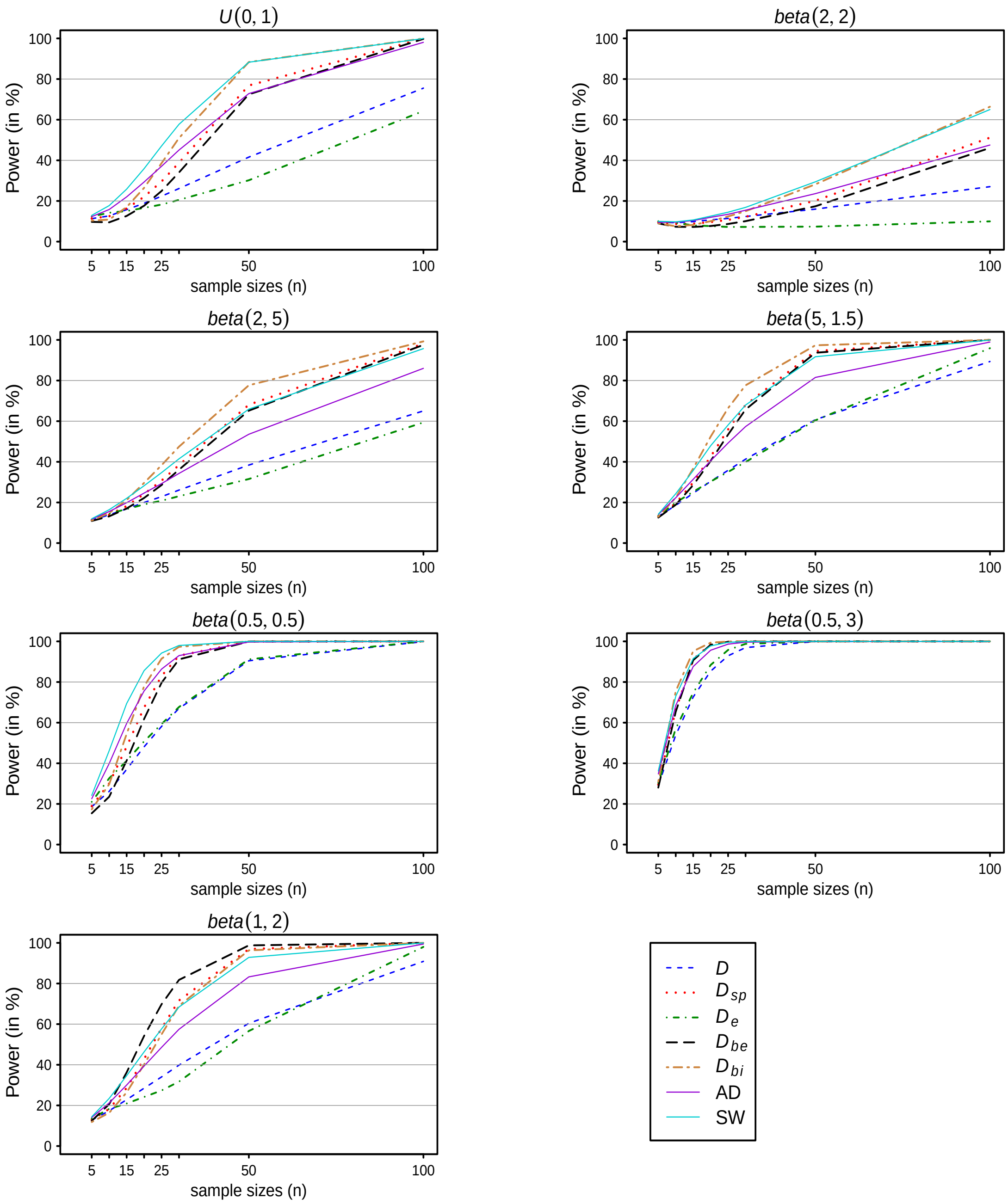

Figure 2.13: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group II for several sample size n at $\alpha = 0.1$

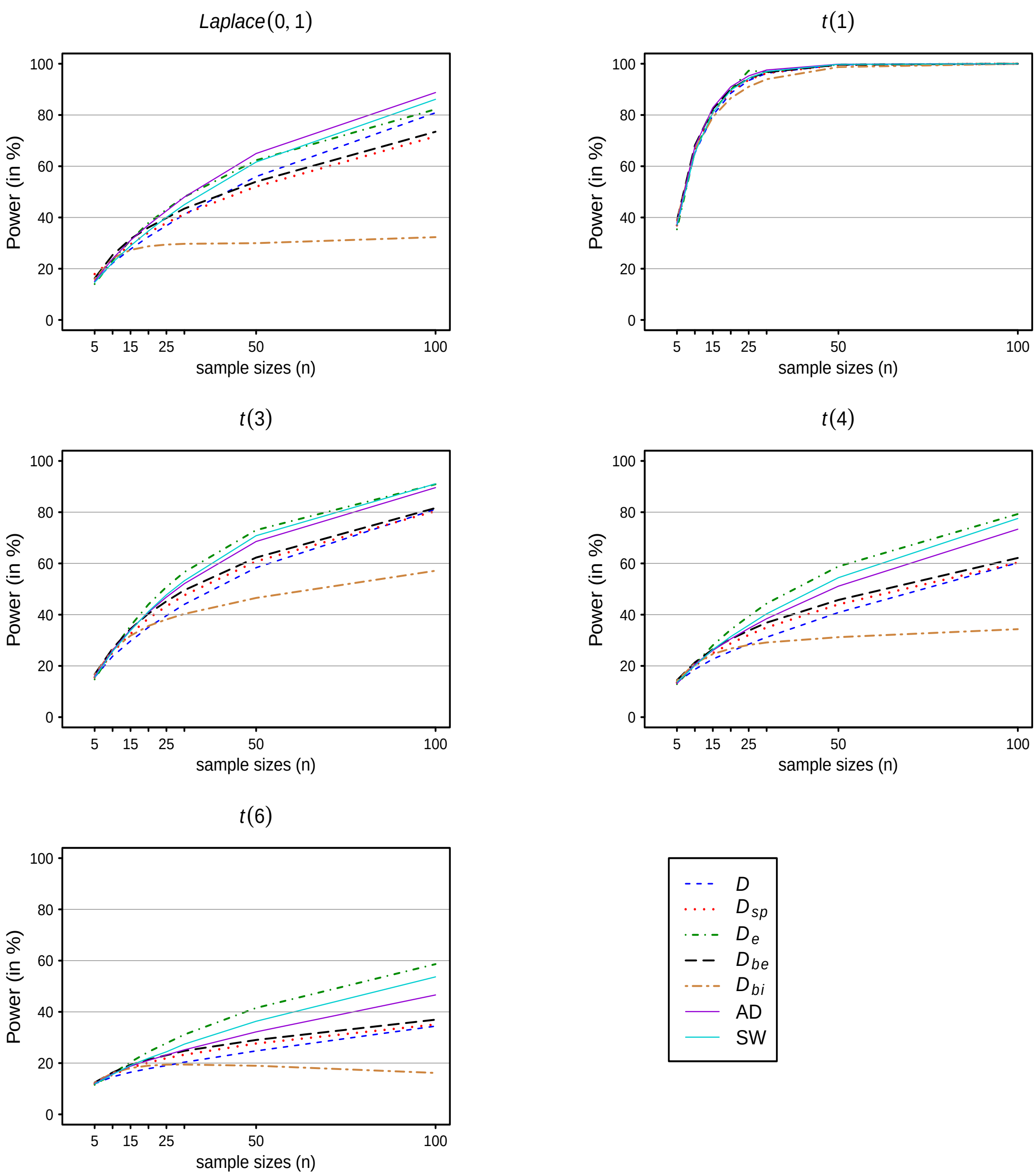

Figure 2.14: Power comparison of the five graphical tests and the two non-graphical tests in detecting non–normality against the alternative distributions from Group III for several sample size n at $\alpha = 0.1$

2.5 Conclusions

Comparison of power against the asymmetrical distributions with their support in $(0, \infty)$ (Group I)

The results in Tables 2.6, 2.9 and 2.12 summarise the simulated powers for selected asymmetrical distributions at the three significance levels. The plots are given in Figures 2.6, 2.9 and 2.12. These tables reveal that if we consider the graphical tests (D, D_{sp}, D_e, D_{be}, and D_{bi}), the performance of the D_{bi} test appears to be better than that of the other tests for all situations. It means that the D_{bi} test offers more power than the existing tests which are the D and D_{sp} tests. When the alternative distribution is $\chi^2(10)$, all tests notably exhibit low power in comparison with other distributions in Group I. For $\chi^2(10)$, the D_{bi} test performs less effectively than the SW test, but better than the AD test. Apart from $\chi^2(10)$, although the SW test is one of the effective non–graphical tests, the performance of the SW test for all circumstances is worse than the power of the D_{bi} test.

However, there does not seem to be so much difference in power between the D test and the other tests when $n \leq 10$. Also, the power of the D_{be} test shows a mediocre performance; that is, it possesses less power than the D_{bi} and SW tests, but more than the D and D_e tests.

Overall, The D test is the least powerful among all the tests. The D_{bi} test is the most powerful among the five graphical tests, and can be more power than the D_{sp} test by over 0.1 (when $n = 10$ and the distribution is $\chi^2(1)$, for example). While the power of D_{be} is similar to that of D_{sp}, D_e is clearly less powerful than D_{sp}. It is interesting to observe that the D_{bi} test is also more powerful than the non-graphical AD and SW tests in most cases.

Comparison of power against the distributions with their supports in $(0, 1)$ (Group II)

The results in Tables 2.7, 2.10 and 2.13, and the plots in Figures 2.7, 2.10 and 2.13. Overall, the D_{bi} test is the most powerful among the five graphical tests, and can be more power than the D_{sp} test by over 0.12 (when $n = 25$ and the distribution is $beta(1, 2)$, for example). The D_e test has the least power against all tests in most situations. The three graphical tests D, D_e and D_{be} are less powerful than D_{sp}. The D_{bi} test is also more powerful than the non-graphical AD and SW tests on most occasions.

Comparison of power against the symmetric non-normal distributions (Group III)

The results in Tables 2.8, 2.11 and 2.14 illustrate the simulated power for symmetrical non-normal distributions at the three significance levels. The plots are given in Figures 2.8, 2.11 and 2.14. These tables reveal that when the sample size increases from 5 to 15; there does not seem to be so much difference in power among the tests. Apart from $t(1)$, the power results of the D_{bi} test slightly increase when sample sizes increase. Also, the powers of the D_{bi} test decrease for larger sample sizes $n \geq 50$; in particular, $t(6)$. For $t(1)$, the AD test is superior to the other tests and the D_{bi} test is inferior to the others. Also, the D_e test has the best performance against $t(3)$, $t(4)$ and $t(6)$, whereas the D_{bi} test has the least power across almost all sample sizes and alternatives. When $t(6)$ is the considered alternative, all tests show the worst performance for all sample sizes in comparison with other alternative distributions. Note that the power of D_{bi} may not increase with sample size n. This phenomenon has been observed with other normality tests (see, for example, Noughabi and Arghami, 2011, which compares D, AD, SW and four other non-graphical tests) and so is not completely surprising in this case. Also, all the distributions in this third group are symmetrical with support $(-\infty, \infty)$ and so are intrinsically harder to detect by a normality test, relative to the first two groups. The two non-graphical tests have a clear advantage over the graphical tests for this group of non-normal distributions based on the power comparison.

The overall conclusions from this power study are as follows. Although not completely dominated, the D test is less powerful than the other four graphical tests in most scenarios and so is not recommended. The D_e test is an improvement over D overall; but it is also less powerful than the other three graphical tests in most scenarios and so it is also not recommended. The power of the D_{be} test is comparable to, but slightly less than, that of the D_{sp} test in the majority of cases and so is not recommended either. If the non-normal distributions in the first and second groups are what one wants to guard against, then D_{bi} is overall more powerful than D_{sp} and the two non-graphical tests AD and SW and so is recommended. On the other hand, if the non-normal distributions in the third group are what one wants to guard against, then D_{sp} is the graphical test one should use. Note however, in this case both D_{sp} and D_{bi} can be substantially less powerful than the two non-graphical tests AD and SW.

CHAPTER 3 Normality tests for the residuals from a linear regression model

3.1 Linear regression model

In this section, the case that $\mathbf{e} = (e_1, ..., e_n)'$ is the vector of residuals from a normal error linear regression model is considered. Normal probability plots are also routinely used to check whether the random errors are independently and normally distributed based on the residuals $\mathbf{e}$. However, the statistical textbooks and statistical software often do not treat the residuals $\mathbf{e}$ and the i.i.d. observations $\mathbf{e}$ differently when producing and interpreting a normal probability plot. The major concern in this part of the thesis is whether, in fact, these two situations should be treated differently. First of all, the theoretical background of a linear regression model is introduced so that it enables us to extend the tests of normality for a simple random sample to the vector of residuals from a linear regression model. It is more convenient to deal with multiple regression models if they are expressed in matrix notation. This allows a very compact display of the model, data, and results. In matrix notation, the model is given by

$$\mathbf{y} = \mathbf{X}\,\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{3.1.1}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1p} \\ 1 & x_{21} & \dots & x_{2p} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \dots & x_{np} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \tag{3.1.2}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}, \mathbf{X} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1p} \\ 1 & x_{21} & \dots & x_{2p} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \dots & x_{np} \end{bmatrix}_{n\times(p+1)}, \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}_{(p+1)\times 1}, \text{ and}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}.$$

In general, $\mathbf{y}$ is an $n \times 1$ vector of the observations, $\mathbf{X}$ is an $n \times (p+1)$ matrix of the levels of the regressor variables or the design matrix, $\boldsymbol{\beta}$ is a $(p+1) \times 1$ vector of the regression coefficients, and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of random errors.
For writing (3.1.2) for each of the n observations, we have

$$\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \ldots + \beta_p x_{1p} + \varepsilon_1 \\
y_2 &= \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \ldots + \beta_p x_{2p} + \varepsilon_2 \\
&\vdots \\
y_n &= \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + \ldots + \beta_p x_{np} + \varepsilon_n.
\end{aligned}$$

The statistical assumptions of the linear model concern the behaviour of the errors; the standard assumptions include:

- **Linearity**: the average error is zero, $\mathrm{E}(\varepsilon_i) = 0$; equivalently, $\mathrm{E}(y_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip}$.

- **Constant error variance**: The variance of the errors is the same for all observations, $\mathrm{Var}(\varepsilon_i) = \sigma^2$; equivalently, $\mathrm{Var}(y_i) = \sigma^2$.

- **Normality**: The errors are normally distributed, and so $\varepsilon_i \sim N(0, \sigma^2)$; equivalently, $y_i \sim N(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip}, \sigma^2)$.

- **Independence**: The errors are independently sampled; that is, ε_i and ε_j are independent for $i \neq j$; equivalent, y_i and y_j are independent.

We wish to find the vector of least-squares estimators, $\hat{\boldsymbol{\beta}}$, that minimises

$$Q(\boldsymbol{\beta}) = \sum_{i=1}^{n} \varepsilon_i^2 = \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}).$$

Note that $Q(\boldsymbol{\beta})$ may be expressed as

$$\begin{aligned}
Q(\boldsymbol{\beta}) &= \mathbf{y}'\mathbf{y} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{y} - \mathbf{y}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} \\
&= \mathbf{y}'\mathbf{y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{y} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}.
\end{aligned}$$

The least-squares estimators must satisfy

$$\left.\frac{\partial Q}{\partial \boldsymbol{\beta}}\right| = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = 0$$

which can be simplified to

$$\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}.$$

The above equations are the **least-squares normal equations.** Thus, the least squares estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

provided that the inverse matrix $(\mathbf{X}'\mathbf{X})^{-1}$ exists. Consequently, the vector of fitted values $\hat{y}_i$ is

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{H}\mathbf{y}.$$

The matrix $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is usually called the **hat matrix**. It maps the vector of observed values into a vector of fitted values. The difference between the observed value y_i and the corresponding fitted value $\hat{y}_i$ is the **residual** $e_i = y_i - \hat{y}_i$. The n residuals may be conveniently written in matrix notation as

$$\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}. \tag{3.1.3}$$

There are several other ways to express the vector of residuals $\mathbf{e}$ that will be useful, including

$$\begin{aligned}
\mathbf{e} &= \mathbf{y} - \hat{\mathbf{y}} \\
&= \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\,\mathbf{y} \\
&= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \\
&= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) \\
&= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\boldsymbol{\varepsilon} \\
&= (\mathbf{I} - \mathbf{H})\boldsymbol{\varepsilon}. \qquad (3.1.4)
\end{aligned}$$

Therefore, if the errors in a linear regression model are i.i.d. $N(0, \sigma^2)$ random variables, then the residuals $\mathbf{e} = (e_1, ..., e_n)'$ have the multivariate normal distribution $N(\mathbf{0}, \sigma^2(\mathbf{I} - \mathbf{H}))$ since $\text{Var}(\mathbf{e}) = \text{Var}[(\mathbf{I} - \mathbf{H})\boldsymbol{\varepsilon}] = (\mathbf{I} - \mathbf{H})\text{Var}(\boldsymbol{\varepsilon})(\mathbf{I} - \mathbf{H})' = \sigma^2(\mathbf{I} - \mathbf{H})$ where $\text{Var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$ and $\mathbf{I} - \mathbf{H}$ is symmetric and idempotent.

So, the covariance matrix of the residual vector is $\sigma^2(\mathbf{I} - \mathbf{H})$, which is not a diagonal matrix. The consequence is that the elements of the residual vector are not independently distributed. The unknown error variance σ^2 is assumed to be estimated by the mean residual sum of squares

$$S_e^2 = \frac{\mathbf{e}'\mathbf{e}}{n-p-1} \tag{3.1.5}$$

and it is clear that S_e^2 and $\mathbf{e}$ are not independent.

We extend the idea of the graphical tests (D, D_{sp}, D_e, and D_{bi}) for a simple random sample to the error vector in a linear regression model in order to conduct statistical tests for normality. Now we describe each graphical test which provides a set of simultaneous intervals for the order statistics of the residual vector in the normal probability plot.

3.2 Type I error rates of the graphical tests for a random sample when applied to the residuals

The main purpose of this section is to investigate whether the graphical tests for normality for a simple random sample presented in Chapter 2 can be applied to the vector of residuals from a linear regression model. If the tests from Chapter 2 can be applied to residuals, they should have Type I error rates close to the specified significance level.

As introduced at the beginning of this chapter, the residual vector $\mathbf{e}$ from a linear regression model is a linear transformation of the error vector $\boldsymbol{\varepsilon}$. It can be generated by

$$\mathbf{e} = (\mathbf{I} - \mathbf{H})\boldsymbol{\varepsilon}. \tag{3.2.1}$$

Since $\mathbf{e} = (\mathbf{I} - \mathbf{H})\boldsymbol{\varepsilon} = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\boldsymbol{\varepsilon}$ regardless of the value taken by $\boldsymbol{\beta}$, there is no need to specify the true or estimated values of $\boldsymbol{\beta}$ for the simulations in the context of testing for normality. However, the distribution of the residual vector $\mathbf{e}$ depends on the design matrix $\mathbf{X}$. In that case, it is very important to be concerned about the construction of $\mathbf{X}$.

This study focuses on a simple linear regression with only one regressor, $\mathbf{x}$, which has n observations. Therefore, concerning the design matrix $\mathbf{X}$, we set two columns so that the first column contains a vector of ones, and the second column is generated by the different approaches to assign values for $\mathbf{x}$ which can be referred to as a "**Design**". Although various options of choosing $\mathbf{x}$ need to be considered, the scope of each value for $\mathbf{x}$ to $[-1, 1]$ will be limited. Generally, the elements of $\mathbf{x}$ should be set to be taken from any $[a, b]$ where in general $a < b$. The proof set out below claims that each value of $\mathbf{x}$ can be taken from $[-1,1]$ instead of $[a,b]$.

Suppose that $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \dots & x_n \end{bmatrix}'_{1\times n}$ and $x_i \in [a,b]$ for $i = 1,\dots,n$. Consequently,

$$\begin{aligned} a \le x_i \le b \quad &\Leftrightarrow \quad \frac{-(b-a)}{2} \le x_i - \frac{b+a}{2} \le \frac{b-a}{2} \\ &\Leftrightarrow \quad -1 \le \frac{x_i - \frac{b+a}{2}}{\frac{b-a}{2}} \le 1 \\ &\Leftrightarrow \quad -1 \le \frac{x_i - m_1}{m_2} \le 1 \quad \text{where } m_1 = \frac{b+a}{2}, m_2 = \frac{b-a}{2}, \text{and } w_i = \frac{x_i - m_1}{m_2} \\ &\Leftrightarrow \quad -1 \le w_i \le 1. \end{aligned}$$

Then, the observation y_i based on a simple linear regression can be expressed as

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + \varepsilon_i \\ &= \beta_0 + \beta_1(m_1 + m_2 w_i) + \varepsilon_i \\ &= (\beta_0 + \beta_1 m_1) + (\beta_1 m_2) w_i + \varepsilon_i \\ &= \alpha_0 + \alpha_1 w_i + \varepsilon_i \quad \text{where } \alpha_0 = \beta_0 + \beta_1 m_1 \quad \text{and} \quad \alpha_1 = \beta_1 m_2. \quad (3.2.2) \end{aligned}$$

This means that y_i can be rewritten in terms of $w_i \in [-1,1]$. As a result of (3.2.2), each $x_i \in [a,b]$ can be drawn from $[-1,1]$ equivalently. Additionally, we need to show that $\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ where $\mathbf{Z} = \begin{bmatrix} \mathbf{1} & c\,\mathbf{x} \end{bmatrix}_{n\times 2}$, $c \neq 0$ is an arbitrary constant and $\mathbf{1} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}'_{1\times n}$. Note that $\sum_{i=1}^{n}$ is represented by $\sum$ in short. Consider the following terms

$$\mathbf{Z}'\mathbf{Z} = \begin{bmatrix} \mathbf{1}' \\ c\,\mathbf{x}' \end{bmatrix} \begin{bmatrix} \mathbf{1} & c\,\mathbf{x} \end{bmatrix} = \begin{bmatrix} n & c\sum x_i \\ c\sum x_i & c^2 \sum x_i^2 \end{bmatrix} \qquad \text{and}$$

$$(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{1}{nc^2 \sum x_i^2 - c^2\big(\sum x_i\big)^2} \begin{bmatrix} c^2 \sum x_i^2 & -c\sum x_i \\ -c\sum x_i & n \end{bmatrix}.$$

To gain the desired result, we further calculate

$$\begin{aligned} (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' &= \frac{1}{nc^2 \sum x_i^2 - c^2\big(\sum x_i\big)^2} \begin{bmatrix} c^2 \sum x_i^2 & -c\sum x_i \\ -c\sum x_i & n \end{bmatrix} \begin{bmatrix} \mathbf{1}' \\ c\,\mathbf{x}' \end{bmatrix} \\ &= \frac{1}{nc^2 \sum x_i^2 - c^2\big(\sum x_i\big)^2} \begin{bmatrix} (c^2 \sum x_i^2)\mathbf{1}' - (c^2 \sum x_i)\mathbf{x}' \\ (-c\sum x_i)\mathbf{1}' + (nc)\mathbf{x}' \end{bmatrix}. \end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' &= \frac{1}{nc^2\sum x_i^2 - c^2\big(\sum x_i\big)^2}\begin{bmatrix}\mathbf{1} & c\,\mathbf{x}\end{bmatrix}\begin{bmatrix}\Big(c^2\sum x_i^2\Big)\mathbf{1}' - \Big(c^2\sum x_i\Big)\mathbf{x}' \\ \Big(-c\sum x_i\Big)\mathbf{1}' + (nc)\mathbf{x}'\end{bmatrix} \\
&= \frac{1}{nc^2\sum x_i^2 - c^2\Big(\sum x_i\Big)^2}\Big[\Big(c^2\sum x_i^2\Big)\mathbf{1}\mathbf{1}' - \Big(c^2\sum x_i\Big)\mathbf{1}\mathbf{x}' \\
&\quad - \Big(c^2\sum x_i\Big)\mathbf{x}\mathbf{1}' + (nc^2)\mathbf{x}\mathbf{x}'\Big] \\
&= \frac{1}{n\sum x_i^2 - \Big(\sum x_i\Big)^2}\Big[\Big(\sum x_i^2\Big)\mathbf{1}\mathbf{1}' - \Big(\sum x_i\Big)\mathbf{1}\mathbf{x}' - \Big(\sum x_i\Big)\mathbf{x}\mathbf{1}' + n\mathbf{x}\mathbf{x}'\Big] \\
&= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'.
\end{aligned}$$

In this study, six different designs are explored; four of these are symmetrical cases and the other two are asymmetrical cases. Each design is described as follows:

1. Symmetrical designs

   (a) Design 1

   The first $\frac{n}{2}$ observations of $\mathbf{x}$ are set equal to -1.

   The remaining $\frac{n}{2}$ observations of $\mathbf{x}$ are set equal to 1.

   (b) Design 2

   The first $\frac{n}{3}$ observations of $\mathbf{x}$ are set equal to -1.

   The second $\frac{n}{3}$ observations of $\mathbf{x}$ are set equal to 0.

   The remaining $\frac{n}{3}$ observations of $\mathbf{x}$ are set equal to 1.

   (c) Design 3

   The first $\frac{n}{4}$ observations of $\mathbf{x}$ are set equal to -1.

   The second $\frac{n}{4}$ observations of $\mathbf{x}$ are set equal to $-\frac{1}{3}$.

   The third $\frac{n}{4}$ observations of $\mathbf{x}$ are set equal to $\frac{1}{3}$.

   The remaining $\frac{n}{4}$ observations of $\mathbf{x}$ are set equal to 1.

   (d) Design 4

   The first observation of $\mathbf{x}$ is set equal to -1.

   The second observation of $\mathbf{x}$ is set equal to $-1 + \frac{2}{n-1}$.

   $\vdots$

   The k^{th} observation of $\mathbf{x}$ is set equal to $-1 + \frac{2(k-1)}{n-1}$ where $k = 1, \ldots, n$.

   $\vdots$

   The n^{th} observation of $\mathbf{x}$ is set equal to 1.

The elements of the vector from Design 4 were generated by Matlab using command *linspace*$(-1, 1, n)$. So this design is an equally-spaced sample.

2. Asymmetrical designs

   (a) Design 5

   The first observation is set equal to -1.

   The second two observations are set equal to $-1 + \frac{2}{n-1}$.

   The remaining observations are set equal to 1.

   (b) Design 6

   The first observation is set equal to -1.

   The remaining observations are set equal to 1.

The study involves a simple linear regression whose design matrix is defined by $\mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{x} \end{bmatrix}_{n\times 2}$ so that the column vector, $\mathbf{x}$, is one of the six designs given above and contains n observations.

Definition 3.2.1. Let R be the number of randomly generated independent samples of sizes n where all R samples follow $N(\mathbf{0}, \mathbf{I} - \mathbf{H})$. The **empirical Type I error rate** $\hat{\alpha}_{n,R}$ of a given test for normality for a given sample size n is given by

$$\hat{\alpha}_{n,R} = \frac{r}{R},$$

where $r \leq R$ is the number of the R tests that reject the null hypothesis of a normally distributed sample at the significance level α.

To evaluate the empirical Type I error rates, the Monte Carlo simulation process is repeated R times and the graphical tests from Chapter 2 are applied, except for the D_{be} test, as follows:

1. Simulate one statistic D by generating $\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$ from multivariate normal distribution $N(\mathbf{0}, \mathbf{I} - \mathbf{H})$ where $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ as defined earlier.

2. Calculate $\overline{Y}, \hat{\sigma}_Y$ and $(Y_{[1]}, ..., Y_{[n]})$, and compute D using formula (2.1.1).

3. Simulate R of independent copies of $D : D_1, ..., D_R$.

4. Compare the statistic D from step 2 with the corresponding critical value c_D at a significance level α.

5. Calculate the proportion of times that the null hypothesis is (falsely) rejected as an empirical Type I error rate.

The other tests can be computed in a similar way to the above. We considered three significance levels $\alpha = 0.01, 0.05$ and 0.1 as usual. Different sample sizes are still a major factor which needs to be recognised. Based on Designs 1,3,4,5 and 6, we considered $n = 4, 8, 12(12)120$. For Design 2, n are set to be 6,9,12(12)120. To compute the Type I error rates, 100,000 independent samples were generated using Matlab for each combination of n, $\mathbf{X}$ and α and the proportion of rejections of the normal hypothesis were recorded.

The empirical Type I error rates of the graphical tests from Chapter 2 against the normal distribution when applied to the residual vectors from a linear regression are given in Tables 3.1 and 3.2.

3.2.1 Results on the empirical Type I error rates

The results on the empirical Type I error rates of the graphical tests (D, D_{sp}, D_e and D_{bi}) for each design show that:

1. Design 1 (see Table 3.1(a))

The empirical Type I error rates for the four tests when $n = 4$ are zero for all nominal significance levels, an exception being the case of the D_e test at $\alpha = 0.1$. Its Type I error rate is 0.0308, so the D_e test is considered conservative. For $n = 8$ and 12, the Type I error rates for all tests are less than their nominal significance levels α. Thus, the four graphical tests seem to be conservative.

2. Design 2 (see Table 3.1(b))

The empirical Type I error rates for four graphical tests when $n = 6$ are all zero for all nominal significance levels. For $n = 9$ at $\alpha = 0.01$ the Type I error rates of the D, D_{sp} and D_e tests are 0.01, 0.0101 and 0.0105, respectively, but the rate of the D_{bi} test is 0.0079, which seems to be conservative. Also, the Type I error rates for all tests ($n = 9$) at $\alpha = 0.05$ and 0.01 are less than the corresponding nominal significance levels, so all tests are considered conservative.

3. Design 3 (see Table 3.1(c))

The empirical Type I error rates for four tests when $n = 4$ are equal to zero, except for the D_e and D_{bi} tests at $\alpha = 0.1$. Their rates are 0.0711 and 0.115, respectively. For $n = 8$ and $n = 12$, the Type I error rates for all tests are less than their corresponding nominal significance levels, so the four graphical tests are conservative.

4. Design 4 (see Table 3.1(d))

For $n = 4$, the Type I error rates for the four tests are all zero at $\alpha = 0.01$ and 0.05. The Type I error rates for the D, D_{sp}, D_e and D_{bi} tests at $\alpha = 0.1$ with $n = 4$ are 0.0547, 0.0547, 0.0711 and 0.0812 respectively, so these tests seem to be conservative.

5. Design 5 (see Table 3.2(e))

Considering $\alpha = 0.01$ with $n = 4$, the empirical Type I error rates of four tests are zero. However, at $\alpha = 0.05$ and 0.1 with $n = 4$. the rates of these tests seem to be liberal. For $n = 8$, the rates are less than the corresponding nominal significance levels; therefore, the four tests are conservative.

6. Design 6 (see Table 3.2(f))

For $n = 4$, the empirical Type I error rates of four graphical tests are zero for all nominal significance levels. For $n = 12$, the empirical Type I error rates for all tests are much greater than their nominal significance levels. In this case, these tests seem to be liberal.

In the case of $n \geq 24$, the empirical Type I error rates of some situations are very close to the corresponding nominal significance levels. Notice that the D, D_{sp}, D_e and D_{bi} tests show a very unreliable performance for $n = 4, 6, 8$ and 12.

From the investigation above, we draw the following conclusions:

1. Normality tests designed for a simple random sample can give misleading Type I error rates when they are applied to the residuals from a linear regression model.

2. Since tests of normality constructed for residuals can easily be used, they are developed in the next section.

Table 3.1: Empirical Type I error rates of **Design 1–4** based on the four graphical tests at significance levels 0.01, 0.05 and 0.1

(a) **Design 1**

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
n	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}
4	0	0	0	0	0	0	0	0	0	0	0.0308	0
8	0.0079	0.0067	0.0078	0.0096	0.0421	0.0409	0.0388	0.0304	0.0902	0.0871	0.0840	0.0773
12	0.0096	0.0092	0.0073	0.0153	0.0475	0.0467	0.0449	0.0457	0.0978	0.0960	0.0925	0.0938
24	0.0104	0.0103	0.0085	0.0149	0.0500	0.0488	0.0472	0.0498	0.1012	0.1003	0.0995	0.0986
36	0.0087	0.0099	0.0084	0.0137	0.0473	0.0512	0.0488	0.0498	0.0974	0.1043	0.0997	0.0975
48	0.0098	0.0110	0.0089	0.0124	0.0493	0.0509	0.0504	0.0499	0.1008	0.1004	0.1003	0.0998
60	0.0101	0.0095	0.0091	0.0113	0.0502	0.0494	0.0482	0.0488	0.1022	0.0976	0.0975	0.0990
72	0.0099	0.0104	0.0091	0.0124	0.0494	0.0503	0.0495	0.0497	0.0978	0.1005	0.0990	0.0990
84	0.0104	0.0099	0.0085	0.0127	0.0493	0.0492	0.0472	0.0495	0.0989	0.0967	0.0982	0.1008
96	0.0104	0.0102	0.0091	0.0116	0.0510	0.0502	0.0501	0.0485	0.1015	0.1001	0.0996	0.0974
108	0.0100	0.0104	0.0090	0.0117	0.0507	0.0485	0.0475	0.0491	0.1000	0.0994	0.0975	0.0993
120	0.0097	0.0089	0.0084	0.0107	0.0494	0.0493	0.0466	0.0470	0.0989	0.0983	0.0979	0.0974

(b) **Design 2**

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
n	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}
6	0	0	0	0	0	0	0	0	0	0	0	0
9	0.0100	0.0101	0.0105	0.0079	0.0473	0.0469	0.0469	0.0429	0.0973	0.0965	0.0925	0.0897
12	0.0098	0.0093	0.0085	0.0084	0.0487	0.0466	0.0459	0.0465	0.0990	0.0965	0.0943	0.0946
24	0.0104	0.0102	0.0086	0.0097	0.0511	0.0490	0.0474	0.0491	0.1027	0.1013	0.0999	0.0978
36	0.0102	0.0095	0.0088	0.0103	0.0499	0.0494	0.0495	0.0498	0.1001	0.1011	0.1001	0.0971
48	0.0102	0.0101	0.0096	0.0096	0.0524	0.0502	0.0507	0.0500	0.1030	0.1014	0.1017	0.0997
60	0.0099	0.0090	0.0092	0.0100	0.0496	0.0477	0.0486	0.0496	0.1007	0.1069	0.0995	0.0993
72	0.0106	0.0102	0.0091	0.0100	0.0487	0.0487	0.0483	0.0491	0.0991	0.0968	0.1005	0.0953
84	0.0107	0.0095	0.0093	0.0091	0.0502	0.0490	0.0477	0.0493	0.0989	0.0980	0.0994	0.0987
96	0.0105	0.0092	0.0092	0.0101	0.0467	0.0493	0.0501	0.0501	0.0969	0.0989	0.1022	0.0998
108	0.0100	0.0098	0.0094	0.0093	0.0501	0.0502	0.0472	0.0487	0.1011	0.1009	0.0967	0.0979
120	0.0105	0.0089	0.0090	0.0085	0.0519	0.0487	0.0468	0.0474	0.1029	0.0995	0.0982	0.0985

(c) **Design 3**

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
n	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}
4	0	0	0	0	0	0	0	0	0	0	0.0711	0.115
8	0.0087	0.0087	0.0087	0.0081	0.0455	0.0454	0.0420	0.0420	0.0958	0.0947	0.0959	0.0869
12	0.0091	0.0088	0.0089	0.0095	0.0472	0.0468	0.0473	0.0473	0.0989	0.0942	0.0945	0.0956
24	0.0100	0.0103	0.0097	0.0097	0.0504	0.0492	0.0494	0.0494	0.1007	0.0970	0.0975	0.0984
36	0.0090	0.0096	0.0088	0.0095	0.0483	0.0504	0.0491	0.0491	0.0979	0.0992	0.0981	0.0977
48	0.0094	0.0093	0.0096	0.0100	0.0502	0.0480	0.0502	0.0502	0.0991	0.1006	0.0971	0.0989
60	0.0010	0.0088	0.0090	0.0102	0.0502	0.0490	0.0487	0.0487	0.1007	0.0980	0.0980	0.0993
72	0.0103	0.0107	0.0092	0.0102	0.0496	0.0502	0.0488	0.0488	0.1011	0.0996	0.0989	0.0971
84	0.0113	0.0104	0.0082	0.0095	0.0483	0.0512	0.0498	0.0498	0.1011	0.0993	0.1000	0.1001
96	0.0094	0.0093	0.0092	0.0100	0.0513	0.0479	0.0478	0.0478	0.0978	0.0978	0.1001	0.0977
108	0.0097	0.0100	0.0089	0.0092	0.0513	0.0470	0.0491	0.0491	0.1000	0.0972	0.0988	0.1001
120	0.0105	0.0096	0.0104	0.0094	0.0547	0.0479	0.0484	0.0484	0.1017	0.1003	0.0967	0.0987

(d) **Design 4**

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
n	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}
4	0	0	0	0	0	0	0	0	0.0547	0.0547	0.0711	0.0812
8	0.0092	0.0091	0.0077	0.0077	0.0472	0.0464	0.0462	0.0442	0.0954	0.0942	0.0943	0.0906
12	0.0104	0.0097	0.0094	0.0094	0.0473	0.0477	0.0482	0.0471	0.0996	0.0976	0.0965	0.0943
24	0.0104	0.0102	0.0097	0.0097	0.0516	0.0499	0.0484	0.0483	0.1028	0.1009	0.1002	0.0978
36	0.0088	0.0097	0.0099	0.0099	0.0491	0.0501	0.0497	0.0503	0.1000	0.1010	0.0991	0.0977
48	0.0094	0.0108	0.0100	0.0100	0.0490	0.0490	0.0485	0.0512	0.0992	0.1000	0.0979	0.1017
60	0.0100	0.0089	0.0108	0.0108	0.0498	0.0482	0.0498	0.0501	0.0994	0.0961	0.1001	0.1025
72	0.0099	0.0110	0.0103	0.0103	0.0503	0.0503	0.0482	0.0498	0.1002	0.1007	0.1007	0.0945
84	0.0106	0.0099	0.0101	0.0101	0.0507	0.0508	0.0491	0.0505	0.1007	0.1001	0.1003	0.1042
96	0.0093	0.0095	0.0091	0.0091	0.0482	0.0478	0.0472	0.0511	0.0974	0.0963	0.0989	0.1023
108	0.0098	0.0097	0.0094	0.0094	0.0503	0.0488	0.0479	0.0486	0.1024	0.1000	0.0980	0.1002
120	0.0102	0.0096	0.0099	0.0099	0.0099	0.0494	0.0492	0.0489	0.0998	0.0990	0.0995	0.1081

Table 3.2: Empirical Type I error rates of **Design 5–6** based on the four graphical tests at significance levels 0.01, 0.05 and 0.1

(e) **Design 5**

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
n	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}
4	0	0	0	0	0.0978	0.0978	0.0978	0.1160	0.2351	0.2351	0.2137	0.1842
8	0.0077	0.007	0.0077	0.0044	0.0452	0.0425	0.0430	0.0435	0.0932	0.0900	0.0903	0.0941
12	0.0095	0.0104	0.0096	0.0119	0.0485	0.0496	0.0509	0.0527	0.0964	0.0988	0.0992	0.1003
24	0.0097	0.011	0.0104	0.0113	0.0503	0.0495	0.0523	0.0545	0.1014	0.1015	0.1061	0.1059
36	0.0099	0.0088	0.0106	0.0115	0.0501	0.0498	0.0535	0.0535	0.1001	0.1005	0.1036	0.1030
48	0.0098	0.0099	0.0100	0.0107	0.0507	0.0487	0.0534	0.0526	0.1008	0.1002	0.1045	0.1022
60	0.0096	0.0085	0.0111	0.0123	0.0512	0.0496	0.0514	0.0524	0.1015	0.0978	0.1036	0.1035
72	0.0106	0.0100	0.0104	0.0111	0.0492	0.0496	0.0529	0.0513	0.0989	0.0985	0.1052	0.1005
84	0.0105	0.0104	0.0101	0.0101	0.0497	0.0490	0.0501	0.0521	0.1001	0.0982	0.1029	0.1040
96	0.0092	0.0098	0.0102	0.0098	0.0466	0.0479	0.0485	0.0503	0.0961	0.0969	0.0994	0.1016
108	0.0101	0.0104	0.0086	0.0093	0.0512	0.0505	0.0488	0.0505	0.1025	0.1012	0.0993	0.1015
120	0.0095	0.0097	0.0105	0.0101	0.0503	0.0489	0.0495	0.0501	0.1017	0.1005	0.0997	0.1010

(f) **Design 6**

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.1$			
n	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}	D	D_{sp}	D_e	D_{bi}
4	0	0	0	0	0	0	0	0	0	0	0	0
8	0.0095	0.0190	0.0127	0.0159	0.0381	0.0503	0.0418	0.0722	0.0630	0.0812	0.0804	0.1330
12	0.0145	0.0145	0.0153	0.0165	0.0683	0.0486	0.0558	0.0659	0.1128	0.1235	0.1196	0.1225
24	0.0096	0.0114	0.0121	0.0134	0.0490	0.0547	0.0568	0.0593	0.1096	0.1046	0.1151	0.1154
36	0.0127	0.0102	0.0114	0.0119	0.0487	0.0519	0.0585	0.0552	0.1035	0.1019	0.1092	0.1076
48	0.0100	0.0092	0.0108	0.0115	0.0480	0.049	0.0556	0.0555	0.1013	0.0990	0.1081	0.1062
60	0.0097	0.0085	0.0104	0.0122	0.0496	0.0482	0.0542	0.054	0.0981	0.0974	0.1065	0.1054
72	0.0105	0.0096	0.0095	0.0111	0.0510	0.0476	0.0547	0.0535	0.1033	0.0973	0.1072	0.1010
84	0.0104	0.0085	0.0101	0.0103	0.0496	0.0482	0.0523	0.0535	0.1015	0.0961	0.1041	0.1074
96	0.0104	0.0092	0.0107	0.0100	0.0485	0.0477	0.0503	0.052	0.0966	0.0956	0.1027	0.1032
108	0.0098	0.0094	0.0097	0.0100	0.0503	0.0471	0.0525	0.0512	0.1016	0.0958	0.1047	0.1050
120	0.0099	0.0090	0.0102	0.0091	0.0508	0.0461	0.0499	0.0495	0.1012	0.0955	0.1039	0.1014

3.3 Graphical tests for residuals

In this section, we emphasise procedures applicable in the linear regression framework and develop tests for normality, namely the D, D_{sp}, D_e and D_{bi} tests, for the residuals. The reason why D_{be} is not included here is because D_{be} has no outstanding performance of power as shown in Chapter 2.

Consider the residual vector $\mathbf{e} = (\mathbf{I} - \mathbf{H})\boldsymbol{\varepsilon}$, where the error vector $\boldsymbol{\varepsilon}$ has the mean zero. Our objective is to test normality of the error vector with the hypothesis

$$H_0 : \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I}). \tag{3.3.1}$$

Then, the corresponding residual vector, $\mathbf{e}$, is exploited in the construction of the following graphical tests.

Let $e_{[1]} \leq ... \leq e_{[n]}$ be the ordered residuals. Also let $\mathbf{W} = (W_1, ..., W_n)'$ be a random vector with distribution $N(\mathbf{0}, \mathbf{I} - \mathbf{H})$ and $W_{[1]} \leq ... \leq W_{[n]}$ be the ordered values of the W_k's. In this section, an interval is provided for each $e_{[k]} (1 \leq k \leq n)$ so that all the $e_{[k]}{}'s$ $(k = 1, ..., n)$ will fall into the corresponding intervals simultaneously with probability $1 - \alpha$. These intervals augment the usual normal probability plot for residuals in judging whether the points fall close to a straight line.

3.3.1 The D test

Based on the D test for a simple random sample, we will obtain the statistic D for errors from a linear regression model based on the corresponding residuals as

$$D = \max_{1 \leq k \leq n} \left| \Phi\Big(\frac{e_{[k]}}{S_e}\Big) - \frac{(k - 0.5)}{n} \right|. \tag{3.3.2}$$

Let c_D be a critical constant so that $P\{D < c_D\} = 1 - \alpha$ when the error vector follows a normal distribution. This probability statement can be rewritten as

$$\begin{aligned} 1 - \alpha &= P\{D < c_D\} \\ &= P\Big\{ \max_{1 \leq k \leq n} \Big| \Phi\big(e_{[k]}/S_e\big) - (k - 0.5)/n \Big| \leq c_D \Big\} \\ &= P\Big\{ -c_D \leq \Phi\big(e_{[k]})/S_e\big) - (k - 0.5)/n \leq c_D \text{ for } k = 1, ..., n \Big\} \\ &= P\Big\{ e_{[k]} \in S_e \Phi^{-1}\big((k - 0.5)/n \pm c_D\big) \text{ for } k = 1, ..., n \Big\}. \end{aligned} \tag{3.3.3}$$

Note that the expression (3.3.3) has nothing to do with the unknown parameter σ^2 and $\boldsymbol{\beta}$ since $\dfrac{\mathbf{e}}{\sigma} = (\dfrac{e_1}{\sigma}, ..., \dfrac{e_n}{\sigma}) \sim N(\mathbf{0}, \mathbf{I} - \mathbf{H})$. Let $W_k = \dfrac{e_k}{\sigma}$ and $W_{[k]} = \dfrac{e_{[k]}}{\sigma}$ for $k = 1, ..., n$. Then $e_{[k]} = \sigma W_{[k]}$ for $k = 1, ..., n$ and

$$\begin{aligned} \frac{e_{[k]}}{S_e} &= \frac{\sigma W_{[k]}}{\sigma S_W} \\ &= \frac{W_{[k]}}{S_W} \end{aligned}$$

where $S_W = \sqrt{\dfrac{\mathbf{W}'\mathbf{W}}{n - p - 1}}$. Therefore, $\dfrac{e_{[k]}}{S_e}$ has nothing to do with σ^2 and we can generate residuals $\mathbf{e} = (e_1, ..., e_n)'$ from $N(\mathbf{0}, \mathbf{I} - \mathbf{H})$ instead of $N(\mathbf{0}, \sigma^2(\mathbf{I} - \mathbf{H}))$.

The required critical constant c_D can be evaluated as the $(1 - \alpha)$-quantile of D by using the following simulations:

1. Simulate one D by generating $\boldsymbol{e} = (e_1, \ldots, e_n)'$ from multivariate normal distribution $N(\mathbf{0}, \mathbf{I} - \mathbf{H})$ where $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ as defined earlier.

2. Calculate $S_e = \sqrt{\dfrac{\mathbf{e}'\mathbf{e}}{n-p-1}}$ and $e_{[1]} \leq ... \leq e_{[n]}$, and compute D using formula (3.3.2).

3. Simulate a large number R of copies of $D : D_1, \ldots, D_R$.

4. Use the $(1-\alpha)$ sample quantile of $D_1, \ldots, D_R$ as an approximation for the critical constant c_D.

Consequently, the simultaneous intervals for $e_{[k]}$ from (3.3.3) are

$$S_e\Phi^{-1}\big((k-0.5)/n \pm c_D\big) \text{ for } k = 1, \ldots, n.$$

3.3.2 The D_{sp} test

Based on the D_{sp} test of Michael (1983), the D_{sp} statistic for errors from a linear regression model is defined by

$$D_{sp} = \max_{1\leq k\leq n} \left| \frac{2}{\pi} \arcsin\sqrt{\Phi\Big(\frac{e_{[k]}}{S_e}\Big)} - \frac{2}{\pi}\arcsin\sqrt{\frac{(k-0.5)}{n}}\ \right|. \qquad (3.3.4)$$

Let c_{sp} be a critical constant so that $P\{D_{sp} < c_{sp}\} = 1-\alpha$ when H_0 is true. The critical constant c_{sp} can be computed by simulation in a similar way to c_D. This probability statement can be rewritten as

$$\begin{aligned} 1-\alpha &= P\Big\{D_{sp} < c_{sp}\Big\} \\ &= P\Big\{\max_{1\leq k\leq n} \Big|\frac{2}{\pi}\arcsin\sqrt{\Phi\Big(\frac{e_{[k]}}{S_e}\Big)} - \frac{2}{\pi}\arcsin\sqrt{\frac{k-0.5}{n}}\ \Big| < c_{sp}\Big\} \\ &= P\Big\{-c_{sp} < \frac{2}{\pi}\arcsin\sqrt{\Phi\Big(\frac{e_{[k]}}{S_e}\Big)} - \frac{2}{\pi}\arcsin\sqrt{\frac{k-0.5}{n}}\ < c_{sp} \text{ for } k=1,...,n\Big\} \\ &= P\Big\{e_{[k]} \in S_e\Phi^{-1}\Big(\sin^2\big(\frac{\pi}{2}c_{sp} \pm \arcsin\sqrt{\frac{k-0.5}{n}}\big)\Big) \text{ for } k=1,\ldots,n\Big\} \quad (3.3.5) \end{aligned}$$

which provides the simultaneous intervals for $e_{[k]}$ for $k = 1, ..., n$. Consequently, the simultaneous interval for $e_{[k]}$ is

$$S_e\Phi^{-1}\Big(\sin^2\big(\frac{\pi}{2}c_{sp} \pm \arcsin\sqrt{\frac{k-0.5}{n}}\big)\Big) \text{ for } k=1,\ldots,n. \qquad (3.3.6)$$

3.3.3 The D_e test

For this test, we still need $\mu_k = \mathrm{E}(W_{[k]})$ and $\sigma_k^2 = \mathrm{Var}(W_{[k]})$ for $k = 1, ..., n$ as the D_e test for a simple random sample. However, no simple formulae for μ_k or σ_k^2, similar to (2.1.11) or (2.1.12), are available since $W_1, ..., W_n$ are not independent in general. Nevertheless, μ_k and σ_k^2 can be computed straightforwardly by simulation in the following way.

1. Generate a large number, R, independent copies of the random vector $\mathbf{W}$.

2. Sort the n components of each copy in ascending order, and denote the R vectors of ordered components as $\mathbf{W}^{[1]}, ..., \mathbf{W}^{[R]}$.

3. Form a R-vector by using the k^{th} components of $\mathbf{W}^{[1]}, ..., \mathbf{W}^{[R]}$.

4. Calculate the mean and variance of this R-vector, and use them as approximations to μ_k and σ_k^2, respectively, for $1 \leq k \leq n$.

If the residuals $\mathbf{e}$ have the distribution $N(\mathbf{0}, \sigma^2(\mathbf{I} - \mathbf{H}))$, then it is clear that $(e_{[1]}, ..., e_{[n]})$ have the same joint distribution as $(\sigma W_{[1]}, ..., \sigma W_{[n]})$. In particular, we have

$$\mathrm{E}(e_{[k]}) = \sigma\mu_k \text{ and } \mathrm{Var}(e_{[k]}) = \sigma^2 \mathrm{Var}(W_{[k]}) = \sigma^2\sigma_k^2 \tag{3.3.7}$$

for $k = 1, ..., n$. A reasonable interval for bounding $e_{[k]}$ should therefore centre around $\sigma\mu_k$ with width proportional to $\sigma\sigma_k$. Since σ is an unknown constant, it is replaced with the available estimation S_e. Specifically, we construct

$$e_{[k]} \in [S_e\mu_k - c_e S_e\sigma_k, S_e\mu_k + c_e S_e\sigma_k] \text{ for } k = 1, ..., n \tag{3.3.8}$$

where c_e is a critical constant chosen so that all the $e_{[k]}$'s will be contained in the corresponding intervals simultaneously with probability $1 - \alpha$.
Clearly, we have

$$\begin{aligned} & P\{e_{[k]} \in [S_e\mu_k - c_e S_e\sigma_k, S_e\mu_k + c_e S_e\sigma_k] \text{ for } k = 1, ..., n\} \\ = \; & P\Big\{ \max_{1 \leq k \leq n} \frac{|e_{[k]}/\sigma - (S_e/\sigma)\mu_k|}{(S_e/\sigma)\sigma_k} \leq c_e \Big\} & (3.3.9) \\ = \; & P\Big\{ \max_{1 \leq k \leq n} \frac{|W_{[k]} - S_W\mu_k|}{S_W\sigma_k} \leq c_e \Big\}. & (3.3.10) \end{aligned}$$

It is clear from (3.3.10) that the simultaneous coverage probability depends on the sample size n, the critical constant c_e and the covariance matrix $\mathbf{I} - \mathbf{H}$ but has nothing to do with the unknown parameter σ^2. The dependence of this critical constant on $\mathbf{I} - \mathbf{H}$ is characteristically different from the critical constant c_e in

Section 2 when $\mathbf{Y}$ is a simple random sample. The required critical constant c_e can be computed as the $(1-\alpha)$-quantile of the random pivot

$$D_W = \max_{1\le k\le n} \frac{|W_{[k]} - S_W\mu_k|}{S_W\sigma_k}. \tag{3.3.11}$$

by using simulation.

It is clear from expression (3.3.9) that the simultaneous intervals in (3.3.8) are based on the statistic

$$D_e = \max_{1\le k\le n} \frac{|e_{[k]} - S_e\mu_k|}{S_e\sigma_k}. \tag{3.3.12}$$

The individual statistic $(e_{[k]} - S_e\mu_k)/(S_e\sigma_k)$ for generating the interval for $e_{[k]}$ has a variance approximately equal to one since the variance of $(e_{[k]} - S_e\mu_k)/(S_e\sigma_k)$ is equal to one from (3.3.7) and $S_e \approx \sigma$ as $n \to \infty$. Additionally, we have both R and Matlab programs to produce the usual normal probability plot with the added simultaneous intervals given in (3.3.8); the inputs are the vector of residuals $\mathbf{e} = (e_1, ..., e_n)$, the covariance matrix $\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and error rate α. Accordingly, the simultaneous interval for $e_{[k]}$ is

$$\mu_k S_e \pm c_e S_e\sigma_k \text{ for } k = 1, \dots, n.$$

3.3.4 The D_{bi} test

The statistic D_{bi} based on the vector of errors for a linear regression model can be defined as

$$D_{bi} = \max_{1\le k\le n} \frac{\left|\Phi\left(e_{[k]}/S_e\right) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}}. \tag{3.3.13}$$

Let c_{bi} be a critical constant so that $P\Big\{D_{bi} < c_{bi}\Big\} = 1-\alpha$; c_{bi} can be determined by simulation as above. This probability statement can be rewritten as

$$\begin{aligned} 1-\alpha &= P\Big\{D_{bi} < c_{bi}\Big\} \\ &= P\Big\{\max_{1\le k\le n} \frac{\left|\Phi\left(e_{[k]}/S_e\right) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi}\Big\} \\ &= P\Big\{-c_{new} < \frac{\Phi\left(e_{[k]}/S_e\right) - (k-0.5)/n}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi} \text{ for } k = 1, \dots, n\Big\} \\ &= P\Big\{e_{[k]} \in S_e\Phi^{-1}\Big((k-0.5)/n \pm c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\Big) \text{ for } k = 1, ..., n\Big\}. \end{aligned}$$

Consequently, the simultaneous interval for $e_{[k]}$ is

$$S_e\Phi^{-1}\Big(\frac{k-0.5}{n} \pm c_{bi}\sqrt{\frac{(k-0.5)(n-k+0.5)}{n^3}}\Big) \text{ for } k = 1, \dots, n. \tag{3.3.14}$$

3.4 Non-graphical tests for residuals

In comparison to graphical tests, we selected one of the non-graphical tests which is the so-called T_n test. This test statistic is based on the distance between the hypothetical characteristic function and the empirical characteristic function of residuals. The T_n test was proposed by Hušková and Meintanis (2007). Later on, Sabolová (2010) pointed out that the T_n test seemed to work better for the larger sample size n than an extension of the Shapiro-Wilk test did. This is the reason why the Shapiro-Wilk test is not selected as a non-graphical test for comparison with the graphical tests for residuals from a linear regression model in this study.

3.4.1 The T_n test

A test for H_0 utilises the scaled residuals, $\hat{e}_k = \dfrac{e_k}{S_e}$ for $k = 1, \ldots, n$ from a regression fit. Consequently, a natural way of testing H_0 is to use a test based on some distance measure of the characteristic function of the scaled residuals under H_0; that is, $\varphi_0(t) = \exp(-\frac{1}{2}t^2\sigma^2)$, from its empirical characteristic function $\varphi_n(t)$.

The empirical characteristic function of scaled residuals is defined as

$$\varphi_n(t) = \frac{1}{n}\sum_{k=1}^{n} \exp(it\hat{e}_k). \tag{3.4.1}$$

Hušková and Meintanis (2007) proposed a test of the composite hypothesis based on the integral

$$T_{n,\omega} = n\int_{-\infty}^{\infty} |\varphi_n(t) - \varphi_0(t)|^2 \omega(t)dt, \tag{3.4.2}$$

where $\omega(.)$ is a non-negative weight function satisfying $\int t^2\omega(t)dt < \infty$ which leads to a convenient representation of the test statistic. For testing a hypothesis of normality, Epps and Pulley (1983) recommended the weight function $\omega(t) = e^{-at^2}$, as it leads to the closed form of the statistic

$$T_{n,a} = \frac{1}{n}\sqrt{\frac{\pi}{a}}\Big(\sum_{j,k=1}^{n} \exp\{-(\hat{e}_j - \hat{e}_k)^2/4a\}\Big) + n\sqrt{\frac{\pi}{1+a}} - \sqrt{\frac{8\pi}{1+2a}}\Big(\sum_{j=1}^{n} \exp\{-\hat{e}_j^2/(2+4a)\}\Big). \tag{3.4.3}$$

Sabolová (2010) suggested that for the sake of simplicity only $a = 1$ was used in (3.4.3). Hence, the test symbol $T_{n,a}$ was represented by T_n throughout this study and

$$T_n = \frac{1}{n}\sqrt{\pi}\Big(\sum_{j,k=1}^{n} \exp\{-(\hat{e}_j - \hat{e}_k)^2/4\}\Big) + n\sqrt{\frac{\pi}{2}} - \sqrt{\frac{8\pi}{3}}\Big(\sum_{j=1}^{n} \exp\{-\hat{e}_j^2/6\}\Big). \tag{3.4.4}$$

Besides, Sabolová (2010) also compared the $T_{n,a}$ test with the extension of the Shapiro-Wilk test for the regression-scale model. The results showed that $T_{n,a}$ was more powerful over all alternative distributions: $t(1)$, $t(5)$, $t(10)$, $logistic(0,1)$, $logistic(0,5)$, $Laplace(0,1)$, and $Laplace(0,5)$.

3.5 Simultaneous intervals

Let $e_{[1]} \leq ... \leq e_{[n]}$ be the ordered residuals and $z_1 < ... < z_n$ be a set of n reference values where $\Phi(z_k) = p_k$ and $p_k = (k-0.5)/n$ for $k = 1, ..., n$. If a point $(z_k, e_{[k]})$ for some $1 \leq k \leq n$ in the normal probability plot is not included in the corresponding vertical interval, then one can claim that H_0 is not supported by the observed data. Now, we construct the simultaneous intervals corresponding to the residuals from a linear regression model for the graphical tests (D, D_{sp}, D_e and D_{bi}). The simultaneous intervals for residuals from a linear regression model shown in Table 3.3 still have nothing to do with the alternative distributions because they depend only on $k = 1, \ldots, n$ and the critical constants of each test.

Table 3.3: Simultaneous intervals for testing normality of residuals from a linear regression model

Graphical tests	Simultaneous intervals for $e_{[k]}$ $k = 1, \ldots, n$
D	$S_e\Phi^{-1}\big((k-0.5)/n \pm c_D\big)$
D_{sp}	$S_e\Phi^{-1}\big(\sin^2[\arcsin\sqrt{(k-0.5)/n} \pm (\pi/2)c_{sp}]\big)$
D_e	$S_e(\mu_k \pm c_e\sigma_k)$
D_{bi}	$S_e\Phi^{-1}\Big((k-0.5)/n \pm c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\Big)$

3.5.1 Illustrative example

Figure 3.1 shows the simultaneous intervals of the D, D_{sp}, D_e and D_{bi} tests based on residuals from a simple linear regression model which were generated from one real dataset.

Example 1

The data are taken from Montgomery et al.,(2001) studying on the rocket propellant manufactured by bonding an igniter propellant and a sustainer propellant together inside a metal housing. The shear strength of the bond between the two types of propellant is an important quality characteristic. It is suspected that shear strength is related to the age in weeks of the batch of sustainer propellant.

Twenty observations on shear strength and the age of the corresponding batch of propellant have been collected, and are shown in Table 3.5. The ordered values of residuals in ascending order from Table 3.5 are −215.98, −213.60, −88.95,−75.32, −67.27, −45.14, −14.59, 8.73, 9.50, 20.37, 37.10, 37.57, 40.06, 48.56, 65.09, 71.17, 80.82, 94.44, 100.68 and 106.75. To estimate the model parameters, we use the least squares method. Therefore, the least squares fit is $\hat{y} = 2627.82 - 37.15x$.

To draw the conclusion that the random errors are i.i.d. normally distributed, by using Figure 3.1, all the values of residuals must lie within the corresponding intervals. Figure 3.1(a) illustrates the simultaneous intervals of residuals based on the D test. It seems that all observations fall inside the simultaneous intervals. However, there is one point which is difficult to assess by eye. Therefore, we need to check numerically whether the 8^{th} ordered residual lies within the 8^{th} vertical interval. The 8^{th} vertical interval from Table 3.6 is $[-77.99, 10.01]$. So, 8.73, the 8^{th} ordered value of residuals of this dataset, is in this interval. Hence the inference is that the random errors are i.i.d. normally distributed based on the D test.
Table 3.4 summarises the conclusion when five tests are applied to assess whether the residuals from Table 3.5 follow a normal distribution.

Table 3.4: Test results for Example 1

Test	Inference
D	Does not reject H_0
D_{sp}	Rejects H_0
D_e	Rejects H_0
D_{bi}	Rejects H_0
T_n	Rejects H_0

Table 3.5: The rocket propellant data

Observation k	Shear Strength(psi) y_k	Age of Propellant(weeks) x_k	Fitted Value $\hat{y}_k$	Residual e_k
1	2158.7	15.5	2051.94	106.75
2	1678.15	23.75	1745.42	-67.27
3	2316	8	2330.59	-14.59
4	2061.3	17	1996.21	65.09
5	2207.5	5.5	2423.48	-215.98
6	1708.3	19	1921.9	-213.6
7	1784.7	24	1736.14	48.56
8	2575	2.5	2534.94	40.06
9	2357.9	7.5	2349.17	8.73
10	2256.7	11	2219.13	37.57
11	2165.2	13	2144.83	20.37
12	2399.55	3.75	2488.5	-88.95
13	1799.8	25	1698.98	80.82
14	2336.75	9.75	2265.58	71.17
15	1765.3	22	1810.44	-45.14
16	2053.5	18	1959.06	94.44
17	2414.4	6	2404.9	9.5
18	2200.5	12.5	2163.4	37.1
19	2654.2	2	2553.52	100.68
20	1753.7	21.5	1829.02	-75.32

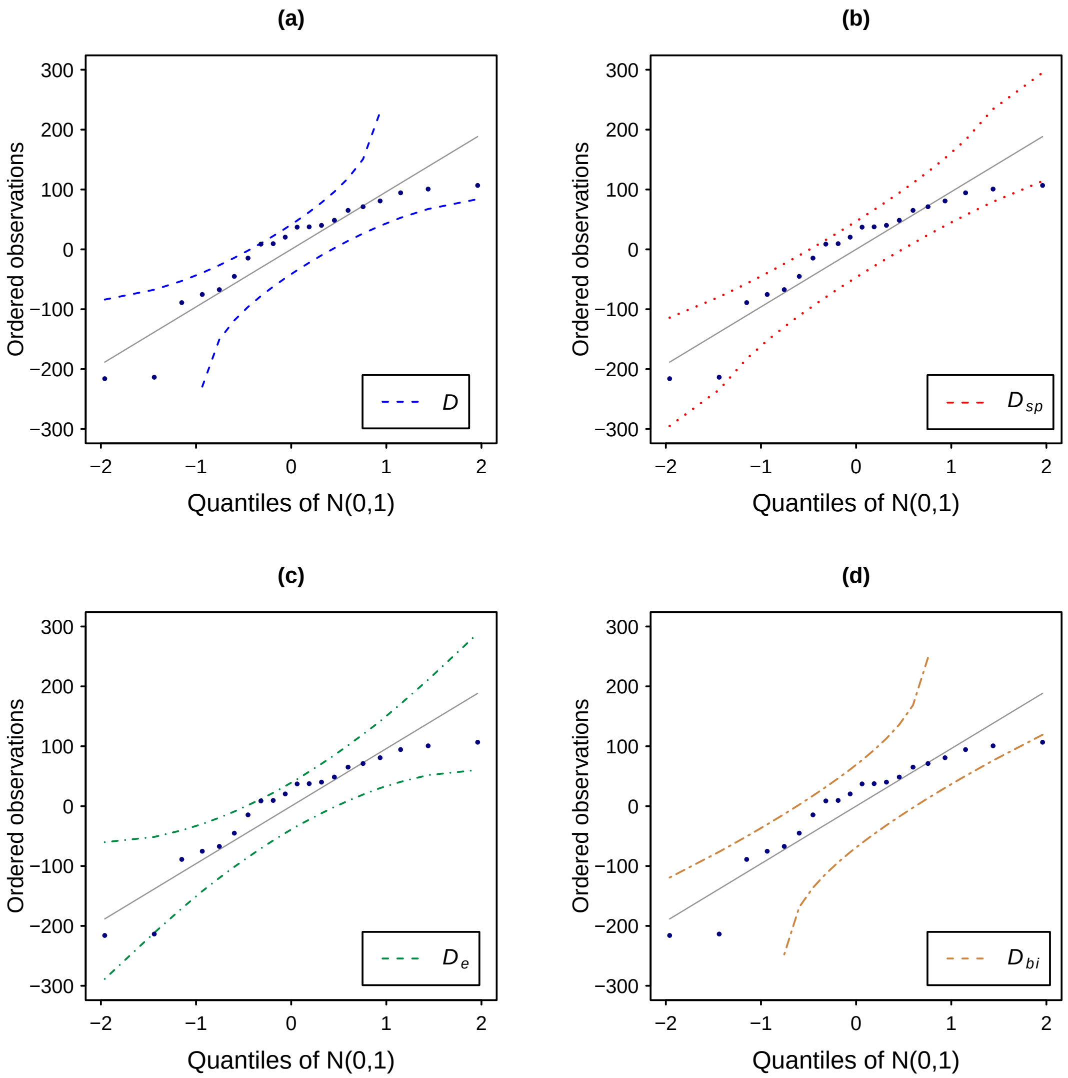

Figure 3.1: Simultaneous intervals for testing normality of the residuals based on the rocket propellant data for four graphical tests (a) D test, (b) D_{sp} test, (c) D_e test and (d) D_{bi} test for $n = 20$ at $\alpha = 0.05$

From Figure 3.1(c), there are two points, the 2^{nd} and 8^{th} ordered values of residuals, which are so close to their corresponding lower bound and upper bound, respectively that we cannot assess by eye whether they lie within their corresponding intervals or not. Again, the numerical values are used. Based on Table 3.3, the simultaneous intervals for the 2^{nd} and 8^{th} ordered values of residuals are $[-211.55, -51.46]$ and $[-70.64, 11.76]$, respectively. Comparing the 2^{nd} ordered value of residuals, -213.6, with its simultaneous interval, we can observe that this value is not in the interval. Although the 8^{th} ordered value, 8.73, falls inside its interval, the inference is that the random errors are not i.i.d. normally distributed based on the D_e test. When Figures 3.1(b) and 3.1(d) are considered, it is obvious that there is one point which lies outside its corresponding interval. Thus, we can draw the conclusion that the errors do not follow a normal distribution based on the D_{sp} and D_{bi} tests.

For the non-graphical tests, the test statistic T_n is 0.3657. Also, the critical value at $\alpha = 0.05$ when $n = 20$ of the T_n statistic is 0.2450. Then, we can conclude that the null hypothesis is rejected at $\alpha = 0.05$ under T_n because the test statistic is larger than its critical value.

Table 3.6: Data residuals with sample size 20 and the corresponding probability intervals for testing normality based on the four graphical tests at $\alpha = 0.05$

			D		D_{sp}		D_e		D_{bi}	
Observations k	Ordered Residuals $e_{[k]}$	Quantiles $N(0,1)$	LB	UB	LB	UB	LB	UB	LB	UB
1	-215.98	-1.96	NaN	-83.84	-295.06	-114.01	-288.60	-60.21	NaN	-119.28
2	-213.6	-1.4395	NaN	-67.42	-234.40	-79.44	-211.55	-51.46	NaN	-76.24
3	-88.95	-1.1503	NaN	-52.76	-182.79	-57.23	-170.86	-40.39	NaN	-50.60
4	-75.32	-0.9346	-229.27	-39.25	-152.26	-39.50	-142.02	-29.84	NaN	-30.56
5	-67.27	-0.7554	-150.62	-26.47	-129.52	-24.11	-119.91	-19.27	-247.75	-13.28
6	-45.14	-0.5978	-118.63	-14.15	-110.81	-10.09	-101.41	-9.04	-168.55	2.47
7	-14.59	-0.4538	-96.15	-2.05	-94.53	3.06	-85.36	1.52	-136.20	17.38
8	8.73	-0.3186	-77.99	10.01	-79.81	15.72	-70.64	11.76	-112.96	31.92
9	9.5	-0.1891	-62.26	22.23	-66.12	28.14	-57.31	22.27	-93.80	46.42
10	20.37	-0.0627	-48.05	34.83	-53.12	40.54	-45.02	33.41	-76.87	61.27
11	37.1	0.0627	-34.82	48.05	-40.54	53.12	-33.26	44.97	-61.27	76.87
12	37.57	0.1891	-22.23	62.27	-28.14	66.13	-22.55	57.61	-46.42	93.80
13	40.06	0.3186	-10.01	77.99	-15.72	79.81	-11.88	70.73	-31.91	112.96
14	48.56	0.4538	2.05	96.15	-3.06	94.53	-1.3995	85.23	-17.39	136.20
15	65.09	0.5978	14.15	118.63	10.098	110.81	8.86	101.52	-2.47	168.55
16	71.17	0.7554	26.47	150.62	24.10	129.52	19.35	119.98	13.28	247.74
17	80.82	0.9346	39.23	229.27	39.50	152.26	30.17	141.64	30.57	NaN
18	94.44	1.1503	52.77	NaN	57.23	182.79	40.57	170.67	50.60	NaN
19	100.68	1.4395	67.42	NaN	79.44	234.40	51.87	211.10	76.24	NaN
20	106.75	1.96	83.85	NaN	114.01	295.06	60.42	288.35	119.28	NaN

NaN stands for Not a Number representing an undefined value.

3.6 Power comparison

In Chapter 2, we introduced the empirical power for simple random samples. Next, we generalise the empirical power for simple random samples to the vector of errors from a linear regression model.

If we give a vector of errors $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ from which we already know that it is not normally distributed, but follows an alternative distribution, then it is simple to describe the principle of an empirical power analysis.

Here, we present the results of the power studies grouped by the classification for the alternative distributions as described above. For each group of all designs, we comment on interesting common facts and compare the results. Noticeably, the D_{be} test was eliminated in this part because its power was rather lower than the D_{bi} tests based on a simple random sample.

In computing the power of a test, the errors are from a given non–normal alternative distributions as mentioned in Chapter 2, which is subtracted by its theoretical mean. In each case, the residual vector, $\mathbf{e}$, is standardised to have a mean of zero. To compute the power, 50,000 simulations are used. Similarly, the $\varepsilon_1^*, \dots, \varepsilon_n^*$ are drawn from the given distribution. Then the error vector $\boldsymbol{\varepsilon} = (\varepsilon_1^* - \mu, \dots, \varepsilon_n^* - \mu)$ where μ is the mean of the distribution. Now, we compute the residual vector $\mathbf{e} = (\mathbf{I} - \mathbf{H})\boldsymbol{\varepsilon}$ and apply the test to $\mathbf{e}$.

One issue that we should take into consideration is that the distribution of $\mathbf{e}$ hinges on the distribution of the error vector $\boldsymbol{\varepsilon}$, the number of regressors p, the elements of the regression matrix $\mathbf{X}$, and the sample size n. Therefore, the performance of any test for normality involves all of these factors. We estimate the empirical power for the investigated tests for several sample sizes, designs and significance levels. The results are tabulated and discussed.

Table 3.7: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group I with Design 1** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$\chi^2(1)$	4	0.73	0.73	0.73	0.73	**3.97**	3.20	3.20	3.20	3.20	**13.60**	10.60	5.93	9.05	5.93	**23.10**
	8	10.60	12.37	7.29	15.51	**18.81**	27.24	32.17	20.01	33.05	**36.78**	40.05	43.91	32.02	44.35	**46.21**
	12	25.75	35.10	20.94	**39.85**	34.90	48.32	56.35	45.71	**59.53**	55.52	59.86	67.66	59.06	**68.35**	67.96
	24	68.64	88.51	70.97	**90.40**	81.52	86.27	95.31	87.82	**96.20**	93.34	92.34	97.36	92.96	**97.84**	96.52
	36	92.14	99.17	92.90	**99.26**	97.59	98.00	99.76	98.02	**99.81**	99.46	99.12	99.88	99.16	**99.93**	99.84
	48	97.89	**99.97**	97.87	**99.97**	99.77	99.69	99.99	99.77	99.99	**100**	99.95	**100**	99.91	99.99	**100**
	60	99.85	**100**	99.73	**100**	99.97	99.98	**100**	99.98	**100**	99.99	**100**	**100**	99.98	**100**	**100**
	72	99.99	**100**	99.94	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	4	0.95	0.95	0.95	0.95	**1.84**	4.36	4.36	4.36	4.36	**7.50**	9.32	7.80	8.85	7.80	**14.85**
	8	4.42	5.24	3.56	6.38	**8.84**	14.97	17.82	12.09	18.59	**23.08**	24.81	27.67	20.94	28.46	**32.46**
	12	12.24	15.68	9.55	18.85	**18.94**	28.19	33.95	26.83	36.12	**36.83**	39.25	45.11	38.53	47.95	**49.23**
	24	34.31	60.05	39.29	**65.64**	55.47	59.03	78.47	63.52	**82.20**	77.31	71.33	85.40	74.73	**88.20**	85.52
	36	59.78	88.82	65.56	**90.53**	83.02	80.65	95.68	83.45	**96.73**	93.47	88.82	97.60	90.79	**98.34**	96.61
	48	76.23	97.71	79.58	**98.12**	94.46	92.21	99.41	93.93	**99.65**	98.77	96.03	99.74	97.16	**99.79**	99.47
	60	89.46	99.70	90.22	**99.75**	98.82	97.34	**99.96**	97.88	**99.96**	99.82	98.96	99.97	99.27	**100**	99.93
	72	95.24	99.97	95.30	**99.98**	99.76	99.13	99.99	99.30	**100**	99.96	99.70	**100**	99.75	**100**	99.97
	84	98.45	**99.99**	98.31	**99.99**	99.89	99.74	99.99	99.82	**100**	99.99	99.94	**100**	99.93	**100**	**100**
	96	99.29	**100**	99.06	**100**	**100**	99.97	**100**	99.94	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	108	99.79	**100**	99.63	**100**	**100**	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.86	**100**	99.90	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	4	1.01	1.01	1.01	1.01	**1.51**	4.38	4.38	4.38	4.38	**6.42**	9.33	8.47	9.00	8.47	**12.74**
	8	3.09	3.46	2.88	4.07	**5.55**	11.95	13.38	9.51	13.44	**16.26**	20.07	21.55	17.49	22.16	**25.06**
	12	7.13	9.23	4.88	10.77	**12.64**	19.14	23.20	18.26	25.40	**27.04**	29.33	33.40	28.18	36.00	**38.51**
	24	20.53	38.20	23.70	**42.76**	39.09	42.03	59.85	47.06	**65.40**	62.62	54.88	71.22	56.68	**75.01**	73.39
	36	38.13	71.07	44.82	**73.74**	65.66	62.44	86.20	67.68	**88.87**	83.93	74.90	91.27	78.61	**93.62**	90.66
	48	52.25	88.48	58.43	**89.97**	82.78	76.67	95.90	80.98	**96.96**	94.30	85.83	97.91	88.86	**98.60**	97.23
	60	68.23	97.03	72.95	**97.55**	93.74	87.10	99.20	90.29	**99.39**	98.18	93.43	99.61	95.20	**99.79**	99.22
	72	78.46	99.17	80.77	**99.30**	97.01	92.96	99.89	95.19	**99.94**	99.35	96.86	99.96	97.98	**99.98**	99.70
	84	87.25	99.78	89.18	**99.81**	98.91	96.40	99.99	97.53	**100**	99.80	98.47	**100**	99.07	**100**	99.98
	96	91.57	99.94	92.26	**99.96**	99.70	98.09	**100**	98.79	99.99	99.95	99.20	**100**	99.56	**100**	99.99
	108	94.83	**100**	95.41	99.99	99.93	99.18	**100**	99.50	**100**	99.99	99.74	**100**	99.88	**100**	99.99
	120	97.08	**100**	97.67	**100**	99.99	99.61	**100**	99.76	**100**	**100**	99.93	**100**	99.97	**100**	**100**
$\chi^2(4)$	4	1.00	1.00	1.00	1.00	**1.18**	4.60	4.60	4.60	4.60	**6.12**	9.36	8.78	8.93	8.78	**12.20**
	8	2.33	2.47	2.20	2.88	**4.20**	9.61	10.81	7.80	10.89	**13.67**	16.53	17.65	14.32	18.99	**21.86**
	12	5.61	6.62	4.16	7.86	**9.43**	15.95	18.22	14.78	20.44	**22.29**	24.81	28.01	23.83	30.50	**32.07**
	24	14.70	25.64	16.99	**29.72**	28.66	32.39	46.53	36.64	51.61	**52.08**	44.68	58.14	48.76	**63.72**	63.20
	36	27.00	53.06	33.47	**56.38**	53.28	49.90	74.14	55.60	**77.53**	73.95	63.56	82.32	68.36	**85.78**	83.44
	48	37.74	74.29	44.41	**76.83**	71.22	63.48	89.12	69.91	**91.21**	87.61	75.54	93.46	80.59	**95.19**	93.15
	60	51.97	89.44	57.88	**90.34**	85.56	75.71	96.31	80.69	**97.15**	94.80	85.52	98.16	89.22	**98.68**	97.19
	72	61.21	95.58	66.37	**95.64**	92.21	83.20	98.92	88.15	**99.27**	97.92	90.94	99.57	94.09	**99.76**	99.15
	84	72.23	98.17	76.71	**98.30**	95.34	89.56	99.56	92.39	**99.73**	99.12	94.53	99.82	96.52	**99.89**	99.70
	96	78.29	99.31	81.14	**99.33**	98.36	93.25	99.91	95.15	**99.89**	99.65	96.67	99.96	98.06	**99.96**	99.91
	108	84.17	**99.69**	86.32	**99.69**	99.09	95.78	99.94	97.10	**99.95**	99.91	98.18	99.97	98.85	**99.99**	99.95
	120	89.09	**99.95**	91.50	99.91	99.63	97.52	**99.98**	98.51	99.97	99.96	99.14	99.98	99.57	99.99	**100**
$\chi^2(10)$	4	1.07	1.07	1.07	1.07	**1.18**	5.28	5.28	5.28	5.28	**5.40**	10.13	9.74	9.90	9.74	**10.88**
	8	1.48	1.50	1.44	1.60	**1.90**	6.66	6.89	6.23	7.05	**8.07**	12.62	13.07	12.04	13.36	**14.16**
	12	2.39	2.64	2.08	2.87	**3.50**	9.15	9.58	8.25	10.23	**12.61**	16.08	17.07	14.84	17.43	**20.14**
	24	5.16	7.22	6.31	8.57	**10.36**	15.33	19.61	17.01	22.63	**26.01**	24.54	30.30	26.80	32.86	**36.89**
	36	9.07	15.37	11.77	15.78	**20.18**	23.59	33.16	27.08	35.08	**39.94**	34.43	44.29	38.65	47.80	**51.88**
	48	12.32	23.38	16.29	23.82	**30.61**	30.23	45.02	35.65	47.48	**52.87**	43.07	57.12	48.64	60.48	**65.11**
	60	17.28	32.94	22.41	33.81	**43.53**	37.65	57.30	44.35	59.07	**64.95**	50.79	69.17	57.49	71.09	**74.81**
	72	22.00	43.06	26.84	42.11	**50.25**	43.89	67.63	51.96	68.45	**73.23**	57.39	77.40	64.49	79.29	**82.22**
	84	27.69	51.53	33.84	50.56	**58.08**	50.95	75.05	57.80	75.25	**80.26**	64.11	84.17	70.28	85.22	**87.73**
	96	31.28	59.71	36.67	57.72	**68.02**	56.25	81.19	63.83	81.05	**85.76**	68.04	89.02	75.56	89.45	**91.61**
	108	35.14	66.27	41.13	64.31	**73.97**	60.88	86.87	68.38	86.59	**89.91**	73.00	92.50	79.88	93.52	**94.23**
	120	39.52	73.59	47.39	71.04	**79.51**	65.36	90.35	72.86	90.20	**92.78**	77.77	94.93	82.83	95.60	**96.11**
$LogN(0,1)$	4	0.78	0.78	0.78	0.78	**1.86**	3.55	3.55	3.55	3.55	**9.45**	8.70	6.80	7.91	6.80	**17.69**
	8	7.56	9.31	4.37	10.71	**18.03**	23.01	26.38	14.16	27.13	**34.00**	33.90	37.48	25.03	38.60	**43.20**
	12	23.84	29.66	16.94	32.47	**34.71**	42.35	48.57	39.19	50.64	**53.12**	53.05	58.61	51.38	60.62	**63.62**
	24	61.57	79.30	65.46	**81.66**	79.64	79.91	89.66	82.24	**91.26**	91.23	87.10	93.42	88.67	94.25	**94.66**
	36	85.57	96.76	88.17	**97.02**	96.19	94.72	98.83	95.61	**99.13**	98.66	97.14	99.40	97.77	**99.56**	99.35
	48	95.16	99.62	95.77	**99.68**	99.49	98.65	99.91	99.03	**99.94**	99.88	99.40	99.95	99.64	**99.99**	99.96
	60	98.72	**99.97**	98.92	99.96	99.90	99.73	99.98	99.83	**100**	99.98	99.87	**100**	99.93	**100**	99.99
	72	99.60	**99.99**	99.67	**99.99**	99.98	99.97	**100**	99.96	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	84	99.89	**100**	99.93	**100**	**100**	99.97	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the five tests for each sample size.

Table 3.8: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group I with Design 2** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$\chi^2(1)$	6	6.18	5.51	4.63	**6.49**	6.41	14.00	13.41	12.07	**14.07**	13.23	19.35	19.91	18.66	**21.16**	18.18
	9	12.75	15.18	10.87	**18.52**	12.99	29.35	34.08	25.72	**37.26**	36.88	41.71	47.12	36.73	49.20	**49.28**
	12	23.68	33.22	19.85	**37.94**	33.22	46.55	55.53	44.18	**58.71**	55.44	57.97	66.11	56.97	**68.10**	66.64
	24	67.23	85.33	70.15	**86.93**	81.79	85.82	93.77	87.09	**94.62**	93.55	91.38	96.25	92.48	**96.81**	96.45
	36	90.75	98.59	92.84	**98.67**	97.56	97.71	99.50	97.99	**99.57**	99.46	98.93	99.77	99.21	**99.82**	99.74
	48	97.92	99.84	98.01	**99.86**	99.71	99.70	99.98	99.74	99.98	**100**	99.96	99.99	99.94	**100**	**100**
	60	99.70	**100**	99.62	**100**	**100**	99.96	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	72	99.97	**100**	99.93	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	99.99	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	6	3.37	3.35	2.48	3.62	**3.98**	9.51	9.55	8.67	9.92	**9.93**	15.38	15.60	14.69	**16.03**	15.47
	9	6.06	7.13	5.04	**9.40**	7.14	17.33	19.71	15.23	22.55	**23.02**	26.61	30.08	24.53	33.05	**33.96**
	12	11.22	15.30	9.13	**18.54**	17.32	27.00	33.43	25.61	**37.11**	36.63	38.81	44.92	38.21	47.89	**48.38**
	24	35.06	60.48	39.83	**64.40**	56.55	58.05	78.43	64.27	**81.98**	77.43	71.58	85.66	74.63	**87.92**	85.48
	36	58.27	88.37	65.24	**89.59**	82.48	80.84	95.30	83.81	**96.23**	93.94	88.43	97.31	90.94	**98.05**	96.73
	48	77.33	97.37	79.81	**97.60**	94.51	92.20	99.18	94.21	**99.35**	98.86	96.26	99.65	97.34	**99.66**	99.54
	60	88.84	99.62	90.54	**99.65**	98.82	97.06	**99.92**	97.95	**99.92**	99.77	99.11	99.97	99.38	**99.98**	99.95
	72	95.11	**99.93**	95.41	**99.93**	99.74	99.04	**99.98**	99.34	**99.98**	99.94	99.72	99.98	99.81	**100**	99.98
	84	98.18	99.98	97.99	99.99	99.90	99.70	**100**	99.85	**100**	99.99	99.91	**100**	99.97	**100**	**100**
	96	99.33	**100**	99.32	**100**	99.99	99.90	**100**	99.92	**100**	**100**	99.97	**100**	**100**	**100**	**100**
	108	99.76	**100**	99.65	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.95	**100**	99.88	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	6	2.53	2.52	1.68	2.56	**2.65**	7.82	7.92	7.51	8.18	**8.27**	13.71	**13.76**	13.27	14.06	13.57
	9	4.33	4.72	3.56	**5.93**	5.19	12.87	14.64	11.79	**17.12**	16.73	21.48	24.42	19.70	**26.56**	26.51
	12	6.44	9.38	5.34	11.01	**11.99**	19.40	23.33	19.01	25.88	**26.62**	29.76	33.61	28.99	36.24	**37.55**
	24	20.79	38.98	24.54	**43.66**	39.79	42.23	60.51	47.73	**65.47**	62.71	56.22	71.27	60.62	**75.63**	73.36
	36	36.64	70.98	45.44	**73.39**	66.05	62.48	85.53	68.98	**87.89**	84.63	74.85	90.89	79.49	**92.83**	90.98
	48	53.26	88.86	58.74	**89.93**	82.91	77.66	96.07	81.21	**96.80**	94.66	86.99	97.74	89.49	**98.31**	97.39
	60	66.94	96.65	73.27	**96.83**	93.57	86.91	99.14	90.60	**99.26**	98.18	93.52	99.53	95.41	**99.68**	99.33
	72	77.87	98.86	81.19	**98.89**	97.23	93.15	99.74	95.08	**99.85**	99.50	96.80	99.86	97.74	**99.96**	99.77
	84	86.23	99.73	88.28	**99.78**	99.05	96.35	**99.97**	97.40	**99.97**	99.81	98.52	99.98	99.01	99.98	99.98
	96	91.18	99.91	93.72	**99.94**	99.72	98.30	**99.99**	98.75	**99.99**	99.97	99.30	99.99	99.58	**100**	**100**
	108	94.94	99.98	95.52	**99.98**	99.88	99.21	**100**	99.45	**100**	99.99	99.73	**100**	99.81	**100**	99.99
	120	97.14	**100**	97.77	**100**	99.98	99.56	**100**	99.75	**100**	**100**	99.91	**100**	99.96	**100**	**100**
$\chi^2(4)$	6	2.03	2.05	1.53	2.06	**2.10**	7.16	7.15	6.75	7.41	**7.70**	12.57	12.84	12.20	**13.08**	12.73
	9	3.39	3.56	2.75	**4.22**	3.77	10.86	11.35	10.05	13.22	**14.21**	18.61	20.01	17.48	**20.94**	22.46
	12	4.99	6.32	3.94	7.52	**8.76**	16.07	18.67	14.65	20.50	**21.75**	25.25	28.05	23.79	29.73	**32.20**
	24	14.93	25.36	17.04	**29.30**	28.79	33.83	46.55	37.37	**51.84**	51.56	45.52	58.64	49.59	**63.51**	63.35
	36	26.66	53.93	33.92	**56.68**	52.19	50.78	74.53	56.71	**78.07**	75.16	63.23	83.29	69.10	**86.05**	83.71
	48	38.69	76.20	44.89	**78.02**	70.68	64.55	89.75	70.29	**91.61**	88.01	76.42	93.97	80.95	**95.32**	93.36
	60	51.44	89.06	57.60	**89.27**	84.44	75.87	96.37	80.76	**97.00**	94.86	85.10	98.13	88.78	**98.58**	97.52
	72	61.49	**95.49**	68.19	95.31	90.95	83.33	98.68	88.06	**99.14**	97.78	90.63	99.36	93.79	**99.68**	99.26
	84	70.21	**98.17**	75.64	98.16	95.90	89.09	99.58	92.34	**99.73**	99.24	94.50	99.81	96.72	**99.90**	99.79
	96	78.69	**99.25**	83.59	99.21	98.39	93.08	**99.87**	95.26	99.86	99.68	96.98	99.93	97.93	**99.96**	99.86
	108	83.89	**99.66**	86.47	99.63	99.24	95.70	**99.97**	97.11	**99.97**	99.89	98.16	99.98	98.94	**100**	99.95
	120	88.48	**99.93**	91.50	**99.93**	99.66	97.70	**100**	98.39	99.99	99.97	99.05	**100**	99.64	**100**	**100**
$\chi^2(10)$	6	1.44	1.31	1.19	1.28	**1.35**	5.63	5.78	5.96	5.94	**6.03**	11.29	11.51	10.95	**11.57**	11.24
	9	1.94	2.07	1.67	**2.39**	2.31	7.65	7.62	7.07	8.26	**9.06**	13.43	13.92	13.09	14.52	**15.77**
	12	2.52	2.56	2.04	3.05	**3.40**	9.66	10.03	9.19	10.14	**11.96**	16.54	17.51	15.67	17.07	**19.21**
	24	4.92	7.12	6.40	7.78	**10.81**	15.55	20.16	17.47	21.51	**25.92**	24.89	29.39	27.18	32.35	**36.20**
	36	8.65	15.40	12.50	15.63	**19.96**	23.70	33.14	27.47	35.01	**40.71**	34.62	44.66	39.00	47.52	**52.63**
	48	12.77	24.11	16.75	25.03	**29.79**	30.30	45.63	36.18	48.10	**53.59**	43.47	57.87	48.71	60.30	**65.49**
	60	16.64	33.71	23.07	34.18	**41.82**	37.01	57.87	44.46	58.62	**64.21**	50.54	68.78	57.59	71.44	**75.45**
	72	21.42	43.37	27.30	41.94	**49.77**	44.31	66.80	51.44	68.57	**72.57**	57.75	77.01	63.84	79.71	**82.48**
	84	26.01	49.40	32.83	50.88	**60.32**	50.36	73.98	58.10	75.75	**80.03**	63.21	83.30	69.99	85.79	**88.03**
	96	31.02	60.63	39.56	57.31	**69.17**	56.60	81.72	63.88	81.23	**86.19**	68.52	88.65	75.52	89.55	**91.67**
	108	34.95	67.40	40.88	65.37	**74.73**	61.30	86.80	68.45	86.55	**89.89**	74.12	92.50	80.07	93.26	**94.13**
	120	39.64	72.05	47.12	70.75	**79.88**	65.79	89.94	72.53	90.17	**92.55**	77.25	94.87	82.92	95.79	**96.08**
$LogN(0,1)$	6	5.32	5.29	3.74	5.56	**5.90**	12.08	12.27	10.11	12.59	**12.87**	17.46	17.57	16.28	**18.92**	17.80
	9	10.97	13.08	8.44	**14.95**	13.10	24.95	28.29	20.03	31.55	**34.70**	36.43	40.88	29.86	42.22	**46.48**
	12	20.83	27.80	15.11	31.21	**33.71**	41.51	48.32	37.84	**90.56**	53.13	52.89	58.66	50.47	60.56	**63.53**
	24	60.97	78.45	65.56	79.92	**80.05**	79.89	89.26	82.59	**98.77**	91.11	87.12	93.40	88.51	94.00	**94.69**
	36	85.27	96.10	88.70	**96.33**	95.89	94.63	98.62	95.61	**99.87**	98.90	96.97	99.30	97.68	99.37	**99.43**
	48	95.11	99.52	95.77	**99.54**	99.40	98.79	99.78	99.01	**99.98**	99.85	99.46	99.90	99.57	**99.96**	99.95
	60	98.56	**99.94**	98.86	**99.94**	99.91	99.67	99.97	99.81	**100**	99.99	99.89	99.99	99.93	**100**	99.99
	72	99.58	**99.98**	99.63	**99.98**	99.99	99.94	**100**	99.97	**100**	**100**	99.97	**100**	99.99	**100**	**100**
	84	99.92	**100**	99.92	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the five tests for each sample size.

Table 3.9: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group I with Design 3** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$\chi^2(1)$	4	1.29	1.29	1.25	0.88	**2.05**	5.78	5.78	5.80	4.99	**8.57**	12.56	11.51	12.62	10.47	**15.13**
	8	11.06	12.32	8.03	**15.33**	12.81	23.2	25.68	20.99	**28.47**	25.38	32.61	35.82	30.95	38.66	**39.24**
	12	23.56	32.23	20.50	**38.56**	33.11	44.48	54.06	43.84	**57.26**	54.98	56.74	64.41	57.03	**67.11**	65.95
	24	67.42	85.79	71.58	**87.07**	82.22	85.05	93.24	87.18	93.87	**99.44**	91.07	95.83	92.51	**96.13**	96.38
	36	90.85	98.07	92.06	**98.39**	97.51	97.42	99.36	97.88	**99.50**	99.37	98.7	99.74	99.03	99.72	**99.99**
	48	97.86	99.80	97.89	**99.82**	99.69	99.62	99.95	99.64	**99.96**	99.96	99.91	99.97	99.86	99.98	**100**
	60	99.75	**100**	99.68	**100**	99.99	99.96	**100**	99.96	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	72	99.95	**100**	99.93	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.99	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	4	1.10	1.10	1.08	0.83	**1.32**	5.84	5.84	5.84	5.25	**5.93**	11.47	10.91	11.53	10.76	**11.77**
	8	5.32	5.82	3.66	**7.30**	7.29	14.77	16.45	12.69	**18.26**	16.90	22.91	24.74	21.28	26.76	**27.01**
	12	11.62	15.59	9.63	**20.29**	17.86	27.35	33.78	26.28	36.98	**37.05**	38.54	44.62	38.22	48.16	**48.62**
	24	34.05	61.22	40.38	**65.81**	55.98	58.13	78.31	63.62	**81.56**	77.17	70.38	85.21	74.43	**87.5**	85.25
	36	58.46	88.45	64.17	**90.29**	83.16	81.25	95.23	84.29	**96.13**	94.04	88.23	97.28	90.45	**97.9**	96.78
	48	76.35	97.09	80.19	**97.32**	94.79	92.24	98.95	93.70	**99.28**	98.76	96.09	99.53	96.89	**99.62**	99.47
	60	89.08	99.45	90.08	**99.50**	98.65	97.32	99.91	97.97	**99.92**	99.80	98.94	**99.97**	99.28	99.96	99.93
	72	95.26	**99.94**	95.78	99.92	99.80	99.11	**99.98**	99.34	99.97	99.95	99.68	**99.99**	99.74	**99.99**	99.98
	84	98.58	**99.96**	98.25	99.95	99.90	99.74	99.99	99.79	**100**	99.98	99.87	**100**	99.94	**100**	99.99
	96	99.30	**100**	99.12	99.99	99.98	99.92	**100**	99.89	**100**	**100**	99.96	**100**	99.98	**100**	**100**
	108	99.77	**100**	99.70	**100**	**100**	99.97	**100**	99.98	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.90	**100**	99.83	**100**	**100**	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	4	**1.14**	**1.14**	1.13	**1.14**	1.12	5.4	5.4	5.38	**5.67**	5.2	10.86	**10.95**	10.87	10.85	10.74
	8	3.84	4.16	2.56	4.82	**5.19**	11.47	12.14	10.34	**13.39**	13.21	18.76	19.7	17.66	21.28	**22.28**
	12	7.14	9.1	5.47	12.26	**12.58**	19.12	24.39	18.41	26.83	**27.6**	29.53	34.44	24.49	37.35	**38.1**
	24	19.94	40.28	25.26	**45.08**	39.54	41.87	61.27	47.74	**66.52**	62.95	55.1	71.93	60.01	**75.98**	73.42
	36	37.65	70.8	44.87	**75.52**	65.68	61.83	86.13	68.32	**88.78**	84.74	73.78	91.31	78.58	**93.46**	90.9
	48	52.73	89.11	59.51	**90.31**	83.92	77.55	95.97	81.79	**96.84**	94.82	86.11	97.8	89.57	**98.22**	97.33
	60	68.19	96.57	72.89	**96.87**	93.39	87.65	98.96	90.33	**99.26**	98.3	93.35	99.57	95.21	**99.68**	99.33
	72	78.75	98.93	82.03	**99.01**	97.31	92.98	99.8	95.13	**99.82**	99.43	96.61	99.94	97.86	**99.96**	99.75
	84	87.33	**99.77**	88.69	99.72	99.01	96.61	**99.99**	97.55	99.97	99.86	98.46	**99.99**	99.03	99.98	99.97
	96	91.55	99.89	92.87	**99.91**	99.69	98.15	99.98	98.8	**99.97**	99.96	99.19	99.99	99.57	99.99	**100**
	108	95.19	99.95	95.67	**99.96**	99.93	99.02	100	99.34	**100**	99.99	99.71	**100**	99.8	**100**	99.99
	120	97.2	**100**	97.35	**100**	99.98	99.74	100	99.78	**100**	**100**	99.9	**100**	99.96	**100**	**100**
$\chi^2(4)$	4	0.99	0.99	1	0.88	**1.25**	5.15	5.15	5.16	5.03	**5.66**	10.51	10.29	10.5	10.07	**10.91**
	8	3.14	3.32	2.17	3.63	**4.16**	9.94	10.69	8.92	**11.73**	11.45	16.54	17.81	15.53	18.52	**19.53**
	12	5.51	6.75	4.06	8.51	**8.93**	15.83	18.42	15.62	20.74	**22.38**	24.73	28.03	24.64	30.18	**32.05**
	24	14.43	27.95	17.78	**31.96**	29.67	31.95	48.19	37.22	**53.59**	52.3	44.42	60.23	49.5	**64.89**	64.14
	36	26.07	53.65	32.77	**58.76**	53.52	50.57	74.37	56.83	**78.49**	75.1	62.8	83.1	68.6	**86.5**	83.84
	48	37.52	75.98	44.4	**78.04**	70.9	64.21	89.49	69.1	**91.9**	88.23	75.9	93.91	80.54	**95.54**	93.13
	60	51.2	88.31	56.92	**89.61**	83.96	75.36	96.26	80.42	**97.22**	94.97	85	98.13	88.72	**98.64**	97.54
	72	61.53	95.65	68.19	**95.67**	92.71	82.93	98.79	87.8	**99.18**	97.94	90.66	99.58	93.98	**99.67**	99.23
	84	72.45	97.95	75.96	**98**	95.73	89.58	99.51	92.61	**99.68**	99.24	94.75	99.84	96.56	**99.89**	99.73
	96	78.64	**99.33**	81.9	99.31	98.16	93.08	**99.88**	95.13	**99.88**	99.65	96.7	**99.96**	97.95	**99.96**	99.86
	108	84.67	**99.74**	87.27	**99.74**	99.28	95.73	99.94	97.22	**99.95**	99.93	98.31	99.97	99.02	99.98	**99.99**
	120	89.47	**99.92**	90.81	99.9	99.66	97.71	**99.99**	98.4	**99.99**	99.97	99.12	**100**	99.57	99.99	**100**
$\chi^2(10)$	4	1.2	1.2	**1.21**	1.1	1.02	**5.47**	**5.47**	**5.47**	5.27	5.32	10.48	**10.54**	10.48	10.45	10.48
	8	1.47	1.45	1.26	1.78	**1.94**	6.63	6.91	6.3	**7.03**	6.85	12.49	12.8	12.14	13.06	**13.28**
	12	2.43	2.72	1.92	3.39	**4.17**	9.23	10.42	8.59	10.9	**12.28**	16.31	17.52	15.58	18.71	**20.11**
	24	4.65	7.96	6.26	9.05	**11.03**	15.5	20.66	17.71	22.78	**26.15**	24.35	30.53	27.38	33.01	**36.86**
	36	8.57	14.66	11.72	17	**20.02**	23.16	31.84	27.95	35.06	**40.88**	33.6	44.22	38.55	47.18	**52.72**
	48	12.05	23.64	16.34	25.46	**29.81**	30.58	44.57	35.09	47.93	**54.07**	42.68	57.14	48.54	60.92	**65.46**
	60	17.29	32.9	22.78	34.17	**41.19**	37.63	56.97	44.24	59.61	**65.02**	50.5	68.68	57.24	71.79	**75.17**
	72	21.82	44.06	27.76	43.26	**52.09**	43.6	66.74	52.17	68.64	**73.21**	56.11	77.41	64.34	80.08	**82.25**
	84	27.29	52.11	33.06	51.46	**59.32**	50.5	75.09	57.72	75.51	**80.77**	63.44	83.94	70.27	85.61	**87.7**
	96	31.47	60.02	38	57.96	**67.19**	55.41	81.39	63.12	81.14	**86.22**	68.12	88.56	74.86	89.39	**91.42**
	108	35.37	67.15	42.8	64.93	**75.08**	60.5	86.66	67.91	86.81	**89.96**	73.21	92.73	78.93	93.1	**94.39**
	120	40.56	73.12	45.79	70.5	**80.29**	66.31	90.62	72.54	90.13	**92.64**	76.85	94.99	83.37	95.59	**96.24**
$LogN(0,1)$	4	1.18	1.18	1.18	0.88	**1.37**	5.73	5.73	5.72	5.45	**6.9**	11.64	11.37	11.67	11.03	**13.36**
	8	10.03	11.02	5.9	12.96	**13.1**	21.26	23.49	17.86	**25.31**	24.21	30.59	33.1	26.67	34.59	**37.51**
	12	21.57	28.2	16.51	33.21	**34.14**	41.23	48.93	38.6	51.5	**53.96**	52.33	59.14	51.28	61.7	**63.96**
	24	60.82	79.1	65.62	**80.99**	80.16	79.32	89.65	82.67	90.67	**91.35**	86.72	93.45	88.68	93.84	**94.73**
	36	84.97	96.06	87.82	**96.29**	96.08	94.29	98.63	95.79	98.65	**98.79**	96.92	99.24	97.62	**99.36**	99.35
	48	94.95	99.35	95.84	99.42	**99.49**	98.71	99.8	99.02	99.85	**99.89**	99.38	99.92	99.6	99.9	**99.93**
	60	98.71	**99.92**	98.98	**99.92**	99.9	99.71	99.96	99.8	**99.99**	99.98	99.88	**99.99**	99.93	**99.99**	**99.99**
	72	99.59	99.97	99.68	99.97	**99.98**	99.93	99.99	99.98	**100**	**100**	99.96	**100**	99.99	**100**	**100**
	84	99.93	**100**	99.91	**100**	**100**	99.99	**100**	99.99	100	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.98	**100**	99.98	**100**	**100**	100	**100**	**100**	**100**	100	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the five tests for each sample size.

Table 3.10: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group I with Design 4** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$\chi^2(1)$	4	1.29	1.29	1.25	0.88	**7.09**	5.78	5.78	5.8	4.99	**8.57**	12.56	11.51	12.62	10.47	**15.13**
	8	9.49	10.7	7.91	13.51	**16.56**	24.03	26.37	19.89	**28.86**	27.93	33.28	37.19	29.97	**39.52**	39
	12	22.6	31.4	19.59	**38.61**	31.87	43	52.66	42.71	**57.72**	54.7	55.81	63.23	56.01	**66.42**	65.58
	24	67.69	84.79	69.44	**85.95**	83.01	85.2	93.05	86.58	**93.86**	93.33	91.14	95.87	92.03	**96.27**	96.59
	36	91.47	97.95	92.03	**97.96**	97.67	97.54	99.31	98.08	**99.35**	99.51	98.91	99.65	99.06	**99.76**	99.75
	48	98.13	99.72	98.28	**99.74**	**99.74**	99.63	99.94	99.65	99.96	**99.97**	99.8	99.97	99.87	99.96	**99.99**
	60	99.59	99.93	99.69	99.97	**100**	99.93	99.99	99.97	**100**	**100**	99.98	**100**	**100**	**100**	**100**
	72	99.94	100	99.93	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	99.99	100	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	4	1.1	1.1	1.08	0.83	**3.88**	5.84	5.84	5.84	5.25	**5.93**	11.74	10.91	11.53	10.76	**11.77**
	8	4.69	5.05	3.61	6.78	**9.07**	14.98	16.36	12.48	17.71	**18.21**	23.16	25.58	20.96	27.13	**27.59**
	12	10.81	14.69	8.95	**20.07**	17.19	25.9	32.34	25.44	**37.49**	36.37	37.25	43.4	37.73	47.16	**48.34**
	24	35.24	61.16	38.96	**65.08**	57.33	59.32	78.27	63.23	**81.86**	77.42	71.31	85.44	74.6	**87.57**	85.43
	36	60.28	88.33	63.87	**89.8**	82.76	80.84	95.15	83.82	**96.10**	94.04	88.48	97.27	90.55	**97.86**	96.66
	48	78.27	97	81.61	**97.21**	95.24	92.19	98.97	94.27	**99.13**	98.79	96.17	99.4	96.99	**99.58**	99.47
	60	89.29	**99.44**	90.29	99.40	98.55	97.41	**99.88**	98	99.86	99.79	98.94	99.92	99.28	99.9	**99.96**
	72	95.4	**99.92**	95.31	99.85	99.74	99.17	**99.98**	99.32	99.97	99.95	99.76	99.98	99.77	**99.99**	**99.99**
	84	98.23	**99.98**	98.12	99.95	99.9	99.73	99.99	99.8	**100**	99.99	99.92	**100**	99.93	**100**	**100**
	96	99.32	**99.99**	99.09	**99.99**	99.98	99.88	**100**	99.9	99.99	**100**	99.96	**100**	99.98	**100**	**100**
	108	99.81	**100**	99.64	**100**	**100**	99.98	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.95	**100**	99.88	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	4	1.14	1.14	1.13	1.14	**2.68**	5.40	5.40	5.38	**5.67**	5.2	10.86	**10.95**	10.87	10.85	10.74
	8	3.24	3.50	2.49	4.04	**6.09**	11.38	11.85	10.14	12.79	**13.61**	18.63	19.74	17.41	21.54	**22.43**
	12	6.52	8.81	5.08	11.88	**11.97**	18.76	22.88	17.88	**27.16**	27.04	28.86	33.48	29.21	37.14	**37.66**
	24	21.1	40.24	24.12	**43.86**	40.77	43.02	61.30	46.59	**66.22**	63.19	55.71	71.89	59.63	**76.03**	73.77
	36	38.69	71.90	44.28	**74.56**	66.18	62.94	86.59	68.94	**88.64**	85.03	74.84	91.28	79.05	**93.17**	90.95
	48	54.84	88.73	61.74	**90.09**	84.79	77.14	95.99	82.42	**97.02**	94.68	86.10	97.94	89.51	**98.37**	97.32
	60	67.79	96.58	73.19	**96.64**	92.97	87.11	99.03	90.77	**99.06**	98.25	93.37	99.58	95.18	**99.67**	99.25
	72	78.80	**98.83**	81.02	98.74	97.12	92.86	99.75	94.90	**99.87**	99.42	96.76	**99.95**	97.76	99.93	99.79
	84	86.09	**99.68**	88.64	99.67	99.09	96.47	99.97	97.34	**99.97**	99.80	98.55	**99.98**	99.09	**99.98**	99.97
	96	91.24	**99.90**	92.40	99.89	99.69	98.23	99.97	98.71	**99.98**	99.95	99.28	99.99	99.55	99.98	**100**
	108	94.92	**99.97**	95.54	99.96	99.88	99.29	**99.98**	99.54	**99.98**	99.99	99.80	99.98	99.88	**100**	**100**
	120	97.20	**100**	97.45	**100**	99.98	99.60	**100**	99.78	**100**	**100**	99.91	100	99.96	**100**	**100**
$\chi^2(4)$	4	0.99	0.99	1.00	0.88	**2.21**	5.15	5.15	5.16	5.03	**5.66**	10.51	10.29	10.50	10.07	**10.91**
	8	2.72	2.84	2.23	3.14	**4.83**	9.63	9.93	8.52	10.80	**11.69**	16.34	17.63	15.26	18.33	**19.51**
	12	5.05	6.20	3.76	8.67	**8.82**	15.27	17.87	14.77	21.05	**21.70**	24.16	27.37	23.94	30.22	**31.60**
	24	14.99	27.45	17.16	30.48	**31.12**	33.03	48.22	36.69	**52.94**	52.30	45.60	59.89	49.14	64.26	**63.82**
	36	27.13	53.73	32.39	**57.74**	52.65	50.77	74.29	56.53	**78.15**	75.19	63.80	83.00	68.54	**86.63**	83.90
	48	39.72	75.13	46.68	**77.52**	71.86	63.83	89.26	70.34	**91.76**	88.05	75.62	93.92	80.65	**95.31**	93.18
	60	51.26	89.13	57.42	**89.36**	83.53	75.73	96.31	80.68	**97.05**	94.98	85.16	97.98	88.79	**98.69**	97.45
	72	61.84	95.45	66.69	**95.50**	92.18	82.68	98.82	87.39	**99.10**	97.93	90.84	99.47	93.31	**99.65**	99.27
	84	70.64	**98.14**	75.78	98.12	96.16	89.18	99.54	92.18	**99.60**	99.25	94.700	99.77	96.38	**99.86**	99.76
	96	77.92	**99.35**	81.15	99.27	98.23	93.20	99.86	95.17	**99.88**	99.64	96.81	99.92	97.94	**99.94**	99.87
	108	84.31	**99.64**	86.42	99.58	99.16	96.03	99.95	97.16	**99.96**	99.91	98.21	**100**	99.02	99.99	99.95
	120	89.09	**99.90**	91.01	99.87	99.64	97.47	**99.99**	98.44	99.98	99.97	98.95	**100**	99.56	**100**	99.99
$\chi^2(10)$	4	1.20	1.20	1.21	1.10	**1.40**	5.47	5.47	5.47	5.27	**5.32**	10.48	10.54	10.48	10.45	**10.48**
	8	1.75	1.85	1.47	1.85	**2.79**	7.02	6.83	6.45	7.31	**7.77**	12.59	13.11	12.23	13.21	**14.37**
	12	2.19	2.73	1.90	3.30	**4.03**	8.66	9.79	8.51	10.81	**12.17**	16	16.44	15.81	17.90	**19.70**
	24	5.17	7.59	5.98	8.42	**11.8**	16.07	20.47	17.67	22.53	**26.33**	25.34	30.59	27.61	33.08	**36.87**
	36	9.23	15.66	11.85	16.82	**20.18**	23.06	32.7	27.15	36.01	**40.96**	34.24	45.07	38.53	48.13	**52.37**
	48	12.86	22.58	17.51	24.35	**31.39**	30.22	44.24	36.3	48.35	**53.56**	42.60	57.10	48.76	59.74	**65.74**
	60	17.37	33.32	23.33	33.74	**40.71**	38.07	57.23	45.14	58.9	**65.05**	50.78	69.53	57.92	72.07	**74.91**
	72	21.95	43.81	26.36	43.18	**50.81**	43.5	66.94	51.5	69.15	**73.11**	56.87	77.35	63.26	79.60	**82.71**
	84	26.31	51.99	32.42	51.68	**61.4**	50.19	74.7	57.29	75.61	**80.30**	63.29	83.88	70.07	85.55	**87.68**
	96	30.58	59.46	37.17	58.20	**67.94**	56.00	81.28	63.10	81.86	**86.20**	68.55	88.69	75.18	89.57	**91.16**
	108	35.15	66.71	41.07	64.51	**73.98**	61.71	86.67	68.80	86.60	**89.86**	73.73	92.65	79.92	93.14	**94.13**
	120	40.36	72.72	47.01	70.57	**79.58**	65.45	90.74	73.29	89.87	**92.72**	77.35	95.43	83.2	95.65	**96.2**
$LogN(0,1)$	4	1.18	1.18	1.18	0.88	**6.20**	5.73	5.73	5.72	5.45	**6.90**	11.64	11.37	11.67	11.03	**13.36**
	8	8.82	10.04	6.20	11.78	**16.07**	21.62	23.37	17.34	24.84	**25.96**	30.43	33.24	26.02	35.03	**36.92**
	12	21.19	27.12	16.09	**33.04**	32.98	39.33	47.53	37.3	51.84	**53.46**	51.5	58.08	50.48	61.01	**63.28**
	24	61.79	79.02	64.89	79.96	**80.84**	79.95	89.43	82.07	90.59	**91.38**	86.99	93.47	88.53	93.72	**94.75**
	36	85.75	96.09	87.76	**96.25**	96.03	94.76	98.68	95.50	98.71	**98.94**	97.18	99.18	97.58	99.30	**99.42**
	48	95.34	99.30	96.23	**99.25**	99.46	98.68	99.77	99.07	99.83	**99.90**	99.45	99.90	99.67	99.91	**99.93**
	60	98.60	99.85	98.99	99.87	**99.91**	99.75	99.97	99.84	99.97	**99.98**	99.90	99.97	99.94	99.98	**100**
	72	99.62	99.97	99.61	99.97	**99.98**	99.95	**100**	99.98	99.99	**100**	99.98	**100**	99.99	**100**	**100**
	84	99.93	**100**	99.90	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.98	**100**	99.98	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the five tests for each sample size.

Table 3.11: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group I with Design 5** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$\chi^2(1)$	4	0.93	1.91	**2.91**	1.83	1.88	5.12	8.62	9.20	7.96	**8.34**	12.05	15.27	14.93	**15.93**	15.55
	8	9.71	12.51	6.62	**18.46**	17.53	26.60	32.30	20.04	**35.84**	33.97	37.95	43.00	31.75	**43.62**	43.06
	12	25.58	38.83	20.84	**48.03**	36.04	46.86	63.58	45.13	**67.86**	56.90	59.51	73.49	58.43	**74.89**	66.82
	24	73.19	**95.16**	71.67	93.54	83.47	89.89	**99.58**	88.87	98.39	93.95	94.36	99.33	93.93	**99.39**	97.10
	36	94.8	**99.86**	91.80	99.51	97.55	99.07	**99.99**	98.37	99.95	99.60	99.62	**100**	99.44	99.99	99.79
	48	99.46	**100**	98.80	99.98	99.81	99.98	**100**	99.90	**100**	99.96	**100**	**100**	99.99	**100**	**100**
	60	99.99	**100**	99.80	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	72	99.99	**100**	99.97	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	4	0.85	1.04	1.45	**1.22**	1.17	5.14	**6.3**	6.28	5.84	6.13	11.07	**12.28**	11.88	12.12	12.10
	8	4.12	4.92	3.08	7.80	**8.15**	15.04	18.13	12.37	20.19	**21.59**	24.14	27.74	21.33	**28.19**	30.21
	12	11.52	16.07	9.00	**22.42**	18.88	26.26	36.4	26.08	**42.77**	37.13	38.03	48.7	38.98	**52.67**	48.82
	24	35.53	71.08	39.98	**75.36**	58.02	60.32	86.25	64.46	**88.65**	78.24	72.84	91.26	75.67	**92.63**	85.89
	36	59.55	**95.35**	62.45	94.92	83.72	82.57	98.46	83.54	**98.47**	94.01	89.71	**99.19**	90.95	**99.19**	97.34
	48	79.14	**99.55**	81.62	98.97	95.54	93.72	**99.83**	94.22	99.80	98.91	97.19	99.92	97.49	**99.94**	99.68
	60	91.36	**99.96**	90.53	99.80	98.92	98.17	**99.99**	97.91	99.98	99.84	99.29	100	99.42	**100**	99.96
	72	96.98	**100**	95.50	99.99	99.84	99.51	**100**	99.4	**100**	99.96	99.8	**100**	99.82	**100**	99.99
	84	99.02	**100**	98.08	**100**	99.91	99.9	**100**	99.8	**100**	**100**	99.98	**100**	99.97	**100**	**100**
	96	99.68	**100**	99.22	**100**	99.98	99.99	**100**	99.95	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	99.95	**100**	99.76	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.97	**100**	99.9	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	4	1.04	1.03	0.99	1.09	**1.10**	5.41	**5.48**	5.33	5.42	**5.48**	10.79	11.21	10.76	**11.94**	11.51
	8	2.89	3.34	2.42	4.59	**5.49**	11.34	12.9	9.34	14.6	**16.02**	18.7	21.13	17.47	22.21	**23.78**
	12	7.03	9.11	5.36	12.47	**12.81**	19.55	24.63	18.63	**29.23**	27.53	29.15	35.40	30.38	**39.66**	38.73
	24	20.66	44.53	24.67	**51.29**	40.46	42.93	67.17	47.13	**73.17**	62.79	56.42	76.92	60.34	**80.78**	74.22
	36	36.84	79.85	42.78	**82.74**	67.18	63.29	91.62	68.36	**93.65**	84.51	74.94	95.18	79.22	**96.00**	91.59
	48	53.09	94.87	61.22	**94.45**	85.56	77.70	**98.44**	82.23	98.44	94.72	87.00	99.14	90.07	**99.32**	97.58
	60	68.65	**98.81**	73.00	98.10	94.07	87.77	**99.74**	90.48	99.65	98.42	93.56	**99.92**	95.58	99.91	99.29
	72	79.57	**99.70**	81.49	99.33	97.53	93.51	**99.96**	94.95	99.89	99.45	97.07	**99.99**	98.05	99.98	99.78
	84	86.49	**99.94**	88.43	99.82	99.08	96.62	**100**	97.42	99.99	99.85	98.74	**100**	99.10	**100**	99.99
	96	92.60	**100**	92.83	99.97	99.72	98.51	**100**	98.92	100	99.98	99.54	**100**	99.76	**100**	**100**
	108	95.55	**100**	95.94	99.98	99.94	99.28	**100**	99.45	100	100	99.81	**100**	99.87	**100**	**100**
	120	97.72	**100**	97.18	100	99.98	99.69	**100**	99.76	100	100	99.94	**100**	99.95	**100**	**100**
$\chi^2(4)$	4	**1.20**	1.05	1.13	1.02	1.06	5.45	5.67	5.37	5.60	**5.71**	10.53	**11.36**	10.68	11.32	10.71
	8	2.05	2.44	1.91	3.32	**4.26**	9.41	10.72	7.92	11.62	**13.13**	16.66	18.30	14.97	18.64	**20.60**
	12	5.33	6.21	4.26	8.54	**9.38**	15.68	18.84	14.59	22.15	**22.29**	24.71	28.46	24.47	31.81	**31.93**
	24	14.54	29.08	17.9	**34.81**	29.85	33.55	51.67	37.97	**58.20**	52.61	46.00	63.39	50.38	**69.01**	64.34
	36	25.79	60.40	31.16	**64.83**	53.73	50.91	79.81	55.64	**83.77**	74.87	63.26	86.84	68.25	**89.63**	84.27
	48	37.92	82.76	46.89	**84.37**	73.34	63.47	93.09	70.54	**94.92**	88.44	76.11	96.29	81.05	**97.39**	93.69
	60	51.34	93.41	57.79	**93.59**	85.56	75.42	98.17	80.97	**98.45**	95.45	84.53	99.2	89.14	**99.25**	97.68
	72	61.80	**97.68**	68.02	97.18	92.98	83.12	99.57	87.98	**99.58**	98.27	90.69	**99.85**	94.11	99.81	99.34
	84	70.43	**99.13**	75.30	98.80	95.87	89.14	99.90	92.10	**99.98**	99.18	94.75	**99.98**	96.33	99.98	99.74
	96	78.85	**99.76**	82.33	99.61	98.13	93.08	**99.98**	95.27	**99.98**	99.67	96.91	**100**	98.26	**100**	99.92
	108	84.52	**99.92**	88.50	99.79	99.27	95.81	**100**	97.35	**99.99**	99.93	98.63	**100**	99.06	**100**	99.99
	120	89.68	**99.99**	90.89	99.93	99.75	97.56	**100**	98.24	**100**	99.96	99.12	**100**	99.42	**100**	99.99
$\chi^2(10)$	4	1.00	0.97	0.98	1.01	**1.02**	5.14	5.46	4.99	**5.70**	5.66	10.07	**11.02**	10.17	10.63	10.37
	8	1.30	1.48	1.21	1.73	**2.04**	6.89	7.19	6.45	7.36	**8.09**	12.80	13.32	12.09	13.50	**14.51**
	12	2.65	2.28	2.01	2.68	**4.14**	8.83	10.07	8.69	10.74	**12.24**	15.86	17.36	16	18.77	**19.81**
	24	5.16	7.59	6.54	8.95	**10.93**	15.51	20.54	18.26	23.13	**25.96**	24.99	30.6	27.74	34.28	**37.30**
	36	8.65	15.51	10.93	17.46	**20.69**	23.16	33.28	26.97	36.26	**40.17**	33.96	44.43	38.89	48.75	**53.25**
	48	12.47	26.63	17.54	27.48	**32.22**	30.43	46.52	36.52	49.61	**53.90**	43.41	58.89	49.22	61.67	**65.77**
	60	17.15	35.30	22.74	36.45	**43.23**	36.99	58.69	44.41	61.09	**65.24**	49.97	70.28	58.02	73.11	**75.53**
	72	21.62	44.77	28.32	44.77	**52.82**	43.50	68.75	51.69	70.57	**74.04**	56.86	79.18	65.02	81.63	**82.68**
	84	25.08	52.97	32.58	52.6	**59.67**	49.71	76.60	57.64	77.56	**80.65**	63.63	85.01	70.09	86.73	**87.79**
	96	31.65	60.55	37.83	61.65	**67.45**	55.61	83.14	63.18	83.32	**85.69**	68.16	89.90	75.10	91.07	**91.51**
	108	35.36	69.02	42.66	67.46	**74.25**	60.66	88.09	68.26	88.25	**89.80**	73.13	93.39	79.84	94.30	**94.51**
	120	40.77	76.32	45.82	73.74	**80.71**	65.33	91.66	72.54	92.30	**93.07**	77.33	95.78	82.87	**96.39**	96.21
$LogN(0,1)$	4	1.11	1.28	**1.77**	1.44	1.33	5.97	6.973	7.21	**7.41**	7.03	12.44	13.44	13.58	**14.02**	13.93
	8	7.80	10.44	4.23	12.76	**17.12**	22.84	27.664	15.43	27.95	**32.16**	33.26	37.84	26.92	38.31	**40.78**
	12	23.56	30.85	16.27	**35.94**	34.89	42.08	52.84	39.44	**54.55**	53.41	54.02	62.58	52.52	**64.79**	63.35
	24	63.58	**85.02**	65.92	84.19	80.33	81.23	**93.59**	82.87	93.33	91.05	88.42	96.1	89.12	**96.37**	95.19
	36	86.55	**98.11**	87.01	97.1	95.9	95.32	**99.51**	95.72	99.21	98.72	97.56	**99.75**	97.9	99.64	99.51
	48	95.88	**99.84**	96.35	99.51	99.52	98.97	**99.97**	99.04	99.94	99.87	99.56	99.97	99.6	**99.98**	99.94
	60	99.02	**100**	99.06	99.99	99.91	99.83	**100**	99.85	**100**	99.98	99.92	**100**	99.95	**100**	99.99
	72	99.79	**100**	99.69	**100**	99.99	99.99	**100**	99.95	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	84	99.97	**100**	99.92	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.99	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the five tests for each sample size.

Table 3.12: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group I with Design 6** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$\chi^2(1)$	4	3.23	3.23	**5.12**	3.23	3.23	12.90	12.90	**13.98**	12.90	12.90	22.42	22.42	**23.18**	22.42	22.42
	8	12.87	14.72	9.28	**24.97**	20.95	25.80	37.88	28.67	**48.74**	35.95	38.35	54.82	48.11	**60.76**	48.23
	12	26.61	51.69	20.01	**66.92**	39.95	48.61	75.47	55.62	**83.17**	62.51	61.89	81.86	70.91	**89.45**	72.78
	24	75.14	99.00	77.03	**99.52**	85.91	91.29	99.73	92.59	**99.88**	95.88	95.30	99.85	96.62	**99.97**	98.06
	36	96.14	100	94.61	**100**	98.60	99.52	**100**	99.18	**100**	99.77	99.76	**100**	99.77	**100**	99.93
	48	99.75	100	99.12	**100**	99.89	99.99	**100**	99.88	**100**	**100**	**100**	**100**	99.98	**100**	**100**
	60	100	**100**	99.87	**100**	**100**	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	72	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Exp(0,1)$	4	1.38	1.38	**2.10**	1.38	1.38	6.99	6.99	**7.46**	6.99	6.99	13.33	13.33	**13.81**	13.33	13.33
	8	6.46	6.58	3.55	9.41	**10.51**	15.23	19.29	15.02	**24.99**	23.03	23.55	31.54	28.64	**36.01**	33.06
	12	11.98	19.36	7.81	**31.07**	21.73	27.57	43.06	29.23	**52.59**	41.58	39.73	53.38	45.61	**63.48**	53.10
	24	36.71	79.07	40.64	**85.86**	59.82	60.30	91.41	69.11	**94.91**	80.73	73.03	94.85	80.10	**97.13**	88.22
	36	60.57	97.80	66.06	**98.62**	85.90	83.42	99.49	86.86	**99.79**	95.36	90.43	99.76	92.66	**99.88**	97.65
	48	81.37	99.86	83.93	**99.93**	95.94	94.35	99.98	95.59	**99.99**	99.15	97.33	99.98	98.01	**100**	99.63
	60	91.79	100	91.24	**100**	99.01	98.39	**100**	98.38	**100**	99.84	99.47	**100**	99.46	**100**	99.97
	72	96.91	100	96.33	**100**	99.84	99.61	**100**	99.47	**100**	99.96	99.83	**100**	99.87	**100**	99.99
	84	99.13	100	98.20	**100**	99.95	99.90	**100**	99.82	**100**	**100**	99.98	**100**	99.97	**100**	**100**
	96	99.45	100	99.35	**100**	99.99	99.99	**100**	99.96	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	99.77	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	99.89	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	4	1.06	1.06	**1.50**	1.06	1.06	5.81	5.81	**6.19**	5.81	5.81	12.12	12.12	**12.48**	12.12	12.12
	8	4.30	4.38	2.64	5.36	**6.88**	11.30	13.25	10.38	16.56	**17.13**	17.96	23.00	20.92	25.57	**25.68**
	12	8.15	10.89	4.87	**16.81**	14.43	20.02	27.59	19.76	**34.66**	30.75	30.51	38.13	33.14	**45.95**	41.91
	24	22.61	50.33	23.76	**60.62**	42.00	41.90	71.31	50.28	**79.89**	65.41	56.15	80.36	64.18	**86.91**	75.09
	36	37.28	84.38	44.01	**87.61**	70.29	63.49	94.23	71.50	**96.56**	86.83	75.73	96.81	82.07	**98.47**	92.27
	48	55.57	96.79	63.27	**97.68**	85.86	78.39	99.21	84.37	**99.58**	95.42	87.14	99.61	91.17	**99.84**	98.04
	60	68.70	99.48	73.65	**99.63**	94.55	87.94	99.94	90.96	**99.98**	98.44	94.22	99.98	96.17	**99.99**	99.29
	72	79.56	99.94	83.55	**99.97**	97.96	93.76	100	95.91	**100**	99.51	97.22	100	98.36	**100**	99.82
	84	87.77	**100**	88.96	**100**	99.24	97.02	**100**	97.87	**100**	99.91	98.82	100	99.27	**100**	99.99
	96	92.72	**100**	93.38	**100**	99.72	98.46	**100**	98.99	**100**	99.99	99.57	100	99.82	**100**	**100**
	108	95.95	**100**	96.02	99.99	99.95	99.31	**100**	99.53	**100**	**100**	99.81	100	99.91	**100**	**100**
	120	99.68	**100**	97.79	**100**	99.98	99.78	**100**	99.90	**100**	**100**	99.96	100	99.98	**100**	**100**
$\chi^2(4)$	4	0.78	0.78	**1.18**	0.78	0.78	5.35	5.35	**5.60**	5.35	5.35	11.52	11.52	**11.85**	11.52	11.52
	8	3.53	3.56	2.06	3.84	**5.07**	9.80	10.71	8.83	12.91	**14.31**	16.12	19.50	17.62	21.29	**21.76**
	12	6.08	7.58	3.30	**11.19**	10.81	16.10	20.37	15.56	**25.31**	24.63	25.77	30.01	26.86	**35.68**	34.87
	24	15.69	32.21	16.10	**41.18**	31.08	32.36	54.57	39.12	**63.49**	54.51	45.35	65.47	53.29	**73.86**	65.71
	36	26.29	64.12	32.03	**68.84**	56.66	50.32	82.49	58.76	**87.65**	77.37	63.64	89.08	71.52	**93.08**	85.16
	48	39.71	85.16	47.80	**87.63**	73.67	64.82	94.59	72.14	**96.77**	89.08	76.38	97.01	82.89	**98.61**	93.9
	60	51.47	94.80	57.66	**96.26**	85.93	75.43	98.87	81.75	**99.40**	95.55	85.27	99.50	89.80	**99.77**	97.88
	72	62.08	98.44	70.24	**99.02**	93.15	83.61	99.73	89.34	**99.92**	98.17	91.68	99.92	94.81	**99.97**	99.40
	84	70.62	99.47	75.57	**99.60**	96.52	89.76	99.93	92.92	**99.98**	99.26	94.91	99.98	96.91	**100**	99.75
	96	78.57	99.95	82.85	**99.95**	98.20	93.20	100	95.68	**100**	99.74	96.94	**100**	98.36	**100**	99.96
	108	86.67	99.99	87.74	**100**	99.41	95.61	100	97.29	**100**	99.94	98.27	**100**	99.09	**100**	99.99
	120	90.05	**100**	91.55	**100**	99.73	97.56	100	98.58	**100**	99.96	99.11	**100**	99.66	**100**	**100**
$\chi^2(10)$	4	0.73	0.73	**0.97**	0.73	0.73	4.97	4.97	**5.01**	4.97	4.97	10.12	10.12	**10.22**	10.12	10.12
	8	2.20	2.21	1.39	2.20	**2.56**	6.92	7.07	6.15	7.75	**8.30**	11.67	13.06	12.84	13.98	**14.54**
	12	2.83	3.11	1.61	3.81	**4.62**	9.29	10.69	8.76	12.23	**13.62**	16.91	17.93	16.57	19.93	**20.97**
	24	5.98	8.09	5.56	10.35	**11.18**	15.67	21.11	17.75	25.07	**27.36**	24.77	31.40	28.57	35.07	**38.24**
	36	9.01	16.43	11.04	17.58	**22.38**	23.39	35.73	28.04	39.31	**42.11**	34.66	47.41	40.31	51.12	**53.78**
	48	13.56	26.34	17.60	27.49	**31.90**	30.70	46.74	36.85	51.22	**54.79**	42.92	59.33	49.67	63.67	**66.48**
	60	17.35	35.45	22.31	37.17	**43.44**	37.31	59.87	44.86	62.81	**65.04**	51.17	70.96	58.46	75.10	**76.44**
	72	22.11	45.94	28.25	48.59	**52.18**	44.05	69.76	52.98	72.71	**74.01**	57.84	79.68	65.65	82.96	**83.05**
	84	26.48	54.41	32.03	54.67	**61.88**	51.23	77.18	58.17	78.92	**80.84**	64.5	85.44	71.19	87.93	**87.95**
	96	31.95	62.39	37.78	63.16	**67.91**	55.23	83.91	64.06	84.62	**85.58**	68.11	90.29	75.59	91.99	**91.66**
	108	36.56	69.30	41.04	69.80	**75.77**	60.71	88.55	68.04	89.55	**90.16**	73.08	93.68	79.80	95.03	**94.72**
	120	41.14	76.17	46.68	75.22	**80.41**	65.78	91.80	73.50	92.44	**92.66**	77.63	95.96	83.85	96.78	**96.31**
$LogN(0,1)$	4	1.42	1.42	**2.32**	1.42	1.42	7.84	7.84	**8.55**	7.84	7.84	15.91	15.91	**16.67**	15.91	15.91
	8	13.15	13.78	4.02	18.88	**20.24**	24.51	30.36	18.65	**36.63**	33.86	33.72	43.11	37.34	**47.67**	44.73
	12	26.12	37.30	9.54	**49.34**	39.64	45.28	59.84	42.41	**67.10**	58.76	56.13	67.93	59.04	**74.86**	68.05
	24	65.44	90.56	66.85	**93.89**	83.04	82.19	96.20	86.28	**97.91**	93.15	88.93	97.93	92.36	**98.85**	96.04
	36	87.18	99.42	88.92	**99.58**	97.08	95.63	99.84	96.95	**99.89**	99.20	97.74	99.91	98.56	**99.96**	99.62
	48	96.61	**99.97**	97.12	**99.97**	99.71	99.18	**99.98**	99.40	**99.98**	99.93	99.68	99.98	99.79	**100**	99.97
	60	98.13	**100**	99.14	**100**	99.96	99.81	**100**	99.86	**100**	99.99	99.94	**100**	99.96	**100**	99.99
	72	99.19	**100**	99.77	**100**	99.99	99.94	**100**	99.95	**100**	**100**	99.98	**100**	99.99	**100**	**100**
	84	**100**	**100**	99.94	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the five tests for each sample size.

Table 3.13: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group II with Design 1** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$U(0,1)$	4	**1.11**	**1.11**	**1.11**	**1.11**	1.08	**5.63**	**5.63**	**5.63**	**5.63**	5.54	**10.62**	10.61	10.52	10.61	10.61
	8	1.08	1.09	**1.71**	1.41	0.96	5.18	5.27	**7.72**	5.99	4.83	10.48	10.89	**13.94**	11.35	9.09
	12	1.24	1.06	**2.88**	1.23	0.49	6.22	5.08	**10.69**	5.95	2.70	11.87	10.99	**18.02**	11.91	5.98
	24	1.76	1.86	**4.40**	3.33	0.12	8.79	10.37	**16.29**	15.45	2.31	16.66	20.32	27.18	**27.66**	5.53
	36	3.57	6.18	7.90	**8.98**	0.22	14.39	23.89	23.71	**33.29**	3.15	25.43	38.49	37.10	**50.98**	10.00
	48	4.73	14.34	9.54	**18.86**	0.44	19.94	42.17	31.47	**53.66**	6.14	33.52	58.84	47.75	**70.92**	20.04
	60	8.41	30.33	14.28	**36.60**	1.12	27.39	62.57	39.87	**71.79**	14.38	43.53	76.63	57.38	**84.80**	36.94
	72	11.29	47.17	16.85	**52.34**	1.80	34.85	78.23	48.36	**85.24**	26.14	52.69	87.66	65.11	**92.99**	55.46
	84	16.70	63.84	24.53	**69.59**	3.83	44.34	88.53	56.48	**92.76**	41.92	61.60	94.39	73.02	**97.35**	72.69
	96	19.92	75.81	26.98	**79.81**	9.12	51.98	94.31	63.89	**96.57**	58.35	68.10	97.50	79.07	**99.04**	83.42
	108	25.96	85.59	33.72	**89.01**	16.08	58.45	97.51	70.45	**98.50**	73.42	74.67	98.87	84.68	**99.62**	91.27
	120	30.04	92.60	40.97	**94.55**	26.43	64.98	98.91	76.25	**99.49**	83.83	81.04	99.69	88.48	**99.86**	95.28
$beta(2,2)$	4	**1.22**	**1.22**	**1.22**	**1.22**	0.99	**5.52**	**5.52**	**5.52**	**5.52**	4.89	**10.57**	10.54	10.54	10.54	9.83
	8	0.98	0.96	**1.26**	0.96	0.90	4.78	4.73	**5.84**	4.86	4.42	9.83	9.80	**11.29**	9.96	8.77
	12	0.80	0.53	**1.27**	0.78	0.39	4.29	3.92	**5.91**	4.14	2.85	8.81	8.28	**11.75**	8.57	6.35
	24	0.91	0.62	**1.42**	1.09	0.12	4.68	4.31	**7.14**	6.37	1.72	9.48	9.38	**13.71**	12.20	4.37
	36	1.11	1.16	**1.85**	1.55	0.21	5.91	6.36	8.57	**8.71**	1.69	11.77	13.04	16.18	**17.22**	4.44
	48	1.19	1.69	1.78	**2.39**	0.10	6.38	9.31	9.27	**13.50**	1.73	13.24	17.39	17.68	**24.84**	5.85
	60	1.70	3.11	2.30	**4.18**	0.23	8.09	13.10	11.02	**18.35**	2.68	15.88	22.94	20.70	**32.60**	8.03
	72	1.79	4.29	2.42	**5.17**	0.21	9.36	18.10	12.35	**24.89**	3.70	18.44	29.82	23.04	**40.82**	11.33
	84	2.60	6.30	2.99	**8.07**	0.30	12.29	23.40	15.10	**31.87**	4.95	22.58	38.35	26.95	**50.53**	16.04
	96	2.88	7.86	3.10	**10.07**	0.55	13.27	29.30	16.81	**38.11**	7.00	23.40	45.11	28.94	**58.85**	21.01
	108	3.50	10.74	3.82	**13.64**	0.62	14.95	37.19	18.42	**46.68**	10.20	25.98	52.82	30.89	**67.79**	27.77
	120	3.71	15.13	4.12	**19.00**	0.83	16.77	42.48	20.09	**54.58**	14.22	29.25	59.97	34.45	**74.60**	33.20
$beta(2,5)$	4	**0.93**	**0.93**	**0.93**	**0.93**	**0.93**	**5.16**	**5.16**	**5.16**	**5.16**	4.99	10.33	10.07	10.10	10.07	**10.34**
	8	1.06	1.14	**1.36**	1.20	1.28	5.81	5.99	6.01	**6.16**	6.02	11.43	11.59	11.82	11.77	**12.02**
	12	1.93	1.80	**2.09**	1.94	1.75	7.71	7.46	8.24	**8.35**	7.85	13.54	14.03	14.60	**14.40**	14.37
	24	3.38	4.70	**3.66**	6.52	3.87	11.30	15.70	13.26	**20.35**	14.61	19.75	25.08	22.15	**30.89**	25.38
	36	5.49	11.91	5.17	**14.27**	8.52	17.11	30.00	18.48	**36.39**	24.95	27.66	42.09	30.59	**50.16**	37.43
	48	6.87	21.26	6.62	**24.78**	13.34	21.59	45.07	22.71	**53.16**	35.91	33.95	58.22	36.05	**66.42**	50.91
	60	11.01	35.53	9.82	**39.13**	23.05	27.70	60.66	29.39	**67.40**	48.20	41.15	72.85	44.16	**79.09**	62.17
	72	14.14	48.92	12.15	**51.21**	29.72	34.24	73.76	34.47	**79.34**	58.21	48.93	83.23	50.03	**87.78**	72.18
	84	19.10	61.31	15.31	**64.50**	36.51	41.83	83.69	40.49	**87.41**	68.05	55.35	90.56	56.32	**93.79**	80.57
	96	21.10	72.23	17.03	**73.59**	47.78	45.89	89.88	45.61	**92.40**	75.79	60.11	94.97	62.07	**96.98**	86.36
	108	25.60	80.46	20.26	**82.64**	55.05	52.34	94.65	51.14	**96.12**	82.37	66.44	97.38	67.36	**98.67**	91.10
	120	29.36	87.47	24.23	**88.92**	62.76	56.95	96.97	55.62	**98.00**	86.87	71.06	98.67	71.34	**99.30**	91.38
$beta(5,1.5)$	4	**1.11**	**1.11**	**1.11**	**1.11**	**1.11**	4.94	4.94	4.94	4.94	**5.40**	9.48	9.22	9.34	9.22	**11.30**
	8	1.79	1.81	1.92	**2.35**	1.79	7.42	7.78	7.37	8.04	**8.76**	13.32	13.88	13.69	14.50	**15.24**
	12	2.89	2.87	2.97	**3.86**	3.26	10.05	10.85	10.77	**13.13**	11.91	17.03	19.15	18.27	**21.00**	19.88
	24	6.22	12.27	7.10	**16.39**	10.01	18.83	30.12	21.34	**37.44**	28.51	29.51	42.93	33.44	**49.85**	42.00
	36	12.10	32.59	12.34	**36.59**	22.00	30.28	56.66	32.94	**63.20**	46.45	43.72	67.95	47.18	**74.68**	60.81
	48	17.51	53.79	17.80	**57.78**	35.94	39.89	76.35	42.27	**81.93**	64.56	54.80	84.76	58.12	**89.08**	77.37
	60	26.03	74.24	24.41	**77.31**	54.68	51.74	89.69	53.15	**91.99**	78.80	65.92	93.85	68.53	**95.85**	87.92
	72	34.21	87.11	32.62	**88.26**	65.60	61.20	96.14	62.90	**97.38**	87.41	74.48	98.06	77.38	**98.92**	93.67
	84	42.88	94.24	38.80	**95.13**	75.45	69.83	98.66	70.58	**99.06**	92.70	81.24	99.37	82.99	**99.65**	96.99
	96	48.78	97.50	43.86	**97.89**	85.89	76.22	99.60	77.15	**99.72**	96.51	86.42	99.79	88.21	**99.96**	98.73
	108	55.89	98.81	50.52	**99.05**	89.63	81.67	99.83	82.44	**99.87**	98.13	90.19	99.89	91.50	**99.96**	99.30
	120	62.94	99.63	57.86	**99.68**	93.91	86.30	99.97	86.82	**99.97**	99.12	93.10	99.99	93.97	**100**	99.73
$beta(0.5,0.5)$	4	1.26	1.26	1.26	1.26	**1.97**	5.52	5.52	5.52	5.52	**7.73**	11.65	10.51	11.46	10.51	**14.58**
	8	2.01	2.18	**4.02**	3.40	1.64	8.22	9.26	**12.96**	9.66	6.73	15.45	16.29	**22.81**	15.99	11.56
	12	3.19	2.75	**7.93**	4.32	0.77	11.97	12.16	**21.75**	14.06	4.01	20.65	21.84	**32.14**	22.56	8.56
	24	9.29	14.20	**21.90**	18.85	0.58	27.74	37.27	**45.77**	45.20	5.25	42.39	53.19	**60.33**	59.84	12.86
	36	22.03	43.18	39.77	**49.61**	1.77	49.81	72.30	68.94	**77.34**	13.84	66.68	82.59	81.02	**86.35**	37.35
	48	34.17	70.56	54.97	**74.95**	3.73	67.73	90.13	83.23	**92.75**	37.84	82.12	94.96	91.57	**96.35**	72.83
	60	54.59	89.00	72.55	**90.89**	16.15	83.22	97.31	92.05	**98.06**	74.44	92.05	98.89	96.97	**99.19**	92.05
	72	68.92	96.12	82.51	**96.51**	33.99	90.87	99.19	96.24	**99.48**	90.90	96.41	99.67	98.65	**99.77**	97.67
	84	82.02	98.89	90.76	**99.20**	58.98	96.42	99.85	98.88	**99.88**	97.76	99.06	99.94	99.73	**99.97**	99.65
	96	88.17	99.57	94.28	**99.67**	84.90	98.47	99.96	99.58	**99.97**	99.43	99.65	99.99	99.87	**99.99**	99.93
	108	93.60	**99.98**	97.34	99.97	93.96	99.37	**100**	99.82	**100**	99.93	99.86	**100**	99.98	**100**	**100**
	120	96.90	99.97	98.93	**99.99**	98.13	99.72	**100**	99.95	**100**	99.96	99.96	**100**	99.98	**100**	**100**
$beta(0.5,3)$	4	0.76	0.76	0.76	0.76	**3.16**	3.74	3.74	3.74	3.74	**11.47**	10.61	7.18	9.32	7.18	**20.06**
	8	7.11	8.33	6.24	10.95	**11.23**	20.39	24.19	17.37	25.46	**26.90**	31.28	35.13	27.56	35.51	**36.22**
	12	16.69	22.79	15.31	**27.62**	20.41	35.88	43.54	35.56	**47.29**	40.38	48.08	55.42	48.06	**58.01**	53.95
	24	47.82	77.62	50.88	**81.58**	61.28	73.27	89.61	75.60	**92.04**	83.48	83.44	93.86	84.99	**95.09**	90.78
	36	78.03	97.12	77.41	**97.35**	88.85	92.29	99.11	92.61	**99.18**	96.93	96.03	99.50	96.46	**99.58**	99.86
	48	91.99	99.69	90.54	**99.74**	97.75	98.23	99.94	98.07	**99.97**	99.63	99.40	99.99	99.32	99.99	**100**
	60	97.64	99.99	96.85	**99.99**	99.75	99.65	**100**	99.60	**100**	99.96	99.85	**100**	99.85	**100**	**100**
	72	99.38	**100**	98.97	**100**	99.98	99.94	**100**	99.88	**100**	**100**	99.98	**100**	99.98	**100**	**100**
	84	99.91	**100**	99.76	**100**	**100**	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.98	**100**	99.98	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(1,2)$	4	1.01	1.01	1.01	1.01	**1.34**	4.94	4.94	4.94	4.94	**6.01**	10.17	9.61	9.78	9.61	**11.43**
	8	1.70	1.77	**2.15**	2.12	1.77	6.66	7.45	**8.03**	7.93	7.17	13.11	13.68	**14.44**	14.09	12.92
	12	2.34	2.34	3.04	**3.16**	1.73	8.64	9.27	10.68	**10.90**	8.25	15.06	16.44	18.31	**18.59**	14.36
	24	5.08	10.25	7.14	**14.8**	4.06	16.46	27.94	21.04	**35.62**	16.53	26.94	40.29	32.34	**48.83**	28.76
	36	10.42	31.73	11.78	**38.08**	10.92	27.25	56.21	31.12	**65.28**	31.67	40.72	68.44	45.82	**76.34**	48.19
	48	15.72	57.10	17.36	**62.37**	19.50	38.48	79.21	41.98	**85.03**	48.98	53.97	86.86	58.42	**91.56**	66.33
	60	24.61	77.74	25.38	**81.20**	34.98	49.19	91.82	53.02	**94.37**	65.39	64.33	95.45	67.84	**97.49**	79.28
	72	31.39	89.86	30.92	**91.30**	46.59	59.45	97.28	61.10	**98.31**	78.07	73.84	98.63	76.26	**99.36**	88.68
	84	41.94	95.81	38.94	**96.71**	58.00	70.30	99.03	70.75	**99.57**	87.40	81.72	99.71	83.75	**99.84**	94.72
	96	48.82	98.43	45.50	**98.69**	72.42	76.11	99.69	77.30	**99.85**	93.01	86.36	99.90	87.92	**99.96**	97.31
	108	56.32	99.38	52.59	**99.53**	80.65	82.82	99.96	83.16	**99.97**	96.29	91.00	100	92.29	**99.99**	98.95
	120	62.87	99.88	59.43	**99.89**	87.17	86.62	99.98	87.18	**99.99**	98.44	93.91	99.99	94.34	**100**	99.45

The bold number is the highest power among the five tests for each sample size.

Table 3.14: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group II with Design 2** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$U(0,1)$	6	1.00	0.96	**1.39**	1.02	0.97	4.80	4.69	**5.54**	4.62	4.44	9.34	9.42	**10.57**	9.72	8.67
	9	0.86	0.73	**1.86**	0.88	0.44	4.98	4.34	**8.63**	4.82	3.04	10.17	9.49	**15.06**	10.35	6.82
	12	0.90	0.60	**2.37**	1.10	0.29	5.65	4.78	**9.93**	5.77	2.26	11.32	10.56	**17.35**	11.06	5.78
	24	1.77	1.93	**4.46**	2.90	0.12	8.76	9.96	**16.43**	14.63	2.16	17.13	19.94	27.01	**27.02**	5.34
	36	2.96	6.42	7.63	**8.56**	0.24	14.66	23.92	24.41	**32.03**	3.56	25.58	39.06	37.63	**49.58**	9.93
	48	5.19	15.55	10.38	**20.18**	0.34	20.89	42.54	31.66	**55.02**	6.68	34.86	59.77	47.91	**72.35**	20.41
	60	7.75	30.65	14.30	**37.20**	0.90	27.80	63.21	40.67	**72.32**	13.32	43.56	76.66	56.56	**84.98**	38.43
	72	11.11	47.45	17.96	**52.58**	2.33	35.00	78.50	47.49	**85.61**	26.19	52.81	87.81	64.70	**93.33**	56.34
	84	15.26	59.94	23.69	**69.50**	4.82	43.01	87.50	57.39	**93.07**	41.19	60.98	94.10	73.12	**97.28**	74.19
	96	20.00	76.57	31.42	**79.49**	10.77	51.92	93.94	63.85	**96.53**	59.08	68.63	97.57	79.38	**98.91**	84.21
	108	25.14	86.02	33.27	**89.55**	17.58	58.92	97.17	69.94	**98.29**	73.37	75.57	98.90	84.81	**99.51**	91.00
	120	29.75	91.37	41.81	**94.22**	27.24	64.79	98.67	75.93	**99.46**	82.81	79.64	99.68	88.57	**99.82**	95.73
$beta(2,2)$	6	0.85	0.74	**1.05**	0.88	0.82	4.60	4.29	**5.21**	4.35	3.89	8.94	8.96	**9.96**	9.12	8.50
	9	0.76	0.65	**1.12**	0.71	0.55	4.31	3.81	**5.42**	4.09	3.14	8.54	8.28	**11.00**	8.99	6.70
	12	0.73	0.59	**1.34**	0.70	0.30	4.32	3.90	**6.34**	4.41	2.35	9.10	8.42	**11.68**	8.45	5.73
	24	0.08	0.72	**1.79**	1.04	0.17	4.76	4.47	**7.81**	5.86	1.59	10.51	9.54	**13.77**	11.94	4.25
	36	1.05	0.96	**2.06**	1.28	0.19	5.70	6.28	8.11	**8.80**	1.48	11.46	12.91	15.40	**17.52**	4.37
	48	1.19	1.78	1.94	**2.59**	0.13	6.84	9.23	9.66	**13.77**	1.98	14.26	17.89	18.72	**24.97**	5.89
	60	1.62	2.75	2.51	**3.79**	0.23	8.15	12.82	10.71	**17.64**	2.59	15.62	22.78	20.49	**31.88**	8.25
	72	1.81	3.95	2.15	**4.90**	0.25	9.92	17.29	12.58	**24.30**	4.11	18.03	28.80	23.42	**41.35**	12.03
	84	2.30	4.97	3.29	**7.90**	0.38	11.22	21.75	14.95	**32.07**	5.00	21.60	36.32	25.75	**50.84**	16.75
	96	2.57	7.99	3.04	**9.40**	0.62	12.94	29.16	15.88	**38.27**	7.44	24.49	45.03	28.40	**58.31**	21.84
	108	3.42	11.26	3.71	**14.85**	0.78	15.41	36.95	18.01	**47.06**	10.45	27.17	53.01	31.78	**67.20**	27.52
	120	4.08	13.38	3.60	**18.38**	1.00	16.76	42.72	19.51	**54.72**	13.41	28.79	59.64	33.75	**74.16**	34.01
$beta(2,5)$	6	1.17	1.30	1.23	1.34	**1.39**	5.61	5.53	5.63	**5.66**	5.39	10.39	10.24	**11.15**	10.52	10.44
	9	1.59	1.44	**1.61**	**1.61**	1.44	6.61	6.47	**6.94**	6.82	6.58	12.25	12.54	**13.27**	12.86	12.42
	12	1.49	1.48	1.71	**2.04**	1.67	7.62	7.61	**8.08**	8.05	7.44	13.52	13.67	**14.97**	14.30	14.06
	24	2.99	4.16	3.80	**5.60**	3.93	11.74	15.00	13.75	**19.83**	14.55	20.46	25.24	22.28	**31.33**	24.97
	36	5.52	12.54	5.77	**14.85**	8.35	17.55	30.09	18.96	**35.34**	26.18	27.62	42.17	29.89	**49.51**	38.99
	48	7.83	22.39	7.15	**26.35**	13.04	22.30	46.24	23.30	**54.42**	36.58	34.58	59.84	37.20	**67.67**	51.35
	60	11.49	34.92	10.36	**39.02**	21.57	28.95	61.34	29.71	**67.49**	48.00	41.38	72.87	44.08	**79.63**	62.53
	72	13.69	49.17	11.91	**51.20**	32.28	33.59	74.56	35.63	**79.72**	59.33	49.02	83.36	50.47	**88.14**	72.82
	84	17.71	58.57	15.74	**63.80**	39.51	41.01	82.4	40.74	**87.53**	68.31	54.62	89.96	55.29	**93.95**	81.24
	96	21.13	73.02	16.30	**73.43**	49.56	46.65	90.54	44.00	**92.88**	76.44	60.84	95.03	61.53	**96.72**	86.73
	108	24.54	80.98	20.64	**83.15**	56.19	51.60	94.75	50.74	**96.22**	82.69	67.03	97.58	67.70	**98.52**	90.89
	120	29.10	86.78	22.83	**89.05**	63.34	57.11	96.53	55.64	**97.90**	86.74	69.99	98.65	70.64	**99.30**	93.40
$beta(5,1.5)$	6	1.32	1.42	1.24	**1.48**	1.41	5.65	5.64	5.76	5.83	**5.92**	10.89	11.15	11.00	**11.36**	11.32
	9	1.94	2.27	2.15	**2.61**	2.42	7.90	8.43	8.19	**9.48**	9.28	14.86	15.20	15.18	**16.57**	16.26
	12	2.84	2.96	2.58	**3.74**	3.31	10.48	11.21	10.85	**12.71**	11.63	17.53	19.24	18.90	**21.43**	19.79
	24	6.91	12.76	7.62	**15.97**	10.54	19.97	30.86	22.39	**37.96**	28.49	31.05	43.92	33.80	**50.38**	41.52
	36	11.78	32.62	12.73	**37.43**	21.75	30.76	56.76	33.57	**63.53**	47.74	43.84	69.13	46.99	**75.34**	62.34
	48	17.80	55.41	18.26	**60.03**	35.59	40.43	77.30	43.78	**82.29**	65.2	55.86	85.42	59.04	**89.87**	77.85
	60	25.70	73.83	26.18	**77.11**	51.37	51.06	89.38	53.10	**91.85**	77.47	64.97	93.78	68.41	**95.91**	87.45
	72	31.97	86.41	30.57	**87.41**	68.34	59.98	95.89	62.78	**96.76**	87.47	74.41	97.94	76.55	**98.49**	93.74
	84	41.15	92.49	39.14	**94.41**	77.3	68.96	98.34	69.99	**98.83**	92.86	80.69	99.24	81.69	**99.60**	97.11
	96	48.98	97.06	42.69	**97.16**	86.79	75.99	99.41	75.74	**99.63**	96.65	86.83	99.76	87.53	**99.90**	98.68
	108	55.65	98.67	50.82	**98.88**	90.35	81.77	99.80	81.78	**99.87**	98.21	90.86	99.90	91.43	**99.93**	99.32
	120	62.63	99.41	56.25	**99.52**	94.04	86.18	99.93	86.40	**99.97**	99.06	92.64	99.98	94.01	**100**	99.73
$beta(0.5,0.5)$	6	1.38	1.32	**2.08**	1.55	1.40	6.73	6.02	**8.46**	5.79	4.52	11.04	**11.46**	14.01	11.45	8.25
	9	1.85	1.62	**4.39**	2.38	0.86	8.32	7.96	**14.27**	9.15	4.25	15.11	15.42	23.88	**16.49**	8.19
	12	2.73	2.46	**7.20**	3.63	0.6	11.50	10.93	**20.53**	13	3.31	19.38	20.17	31.14	**21.53**	7.59
	24	8.64	13.92	**21.68**	18.03	0.58	28.44	37.46	**46.40**	45.43	5.31	43.04	53.64	60.05	**61.20**	12.95
	36	20.14	41.94	39.75	**47.02**	1.29	49.90	71.48	67.76	**76.50**	13.72	65.92	82.90	79.90	**85.92**	37.39
	48	35.76	70.31	55.23	**74.75**	3.47	68.80	89.07	83.32	**91.96**	40.17	82.54	94.58	91.88	**95.74**	73.76
	60	52.72	86.56	73.02	**88.61**	13.20	82.33	96.39	91.63	**97.19**	73.27	91.29	98.39	96.51	**98.84**	92.11
	72	66.85	94.76	81.10	**95.32**	39.98	91.41	98.90	96.85	**99.16**	91.45	96.79	99.55	98.85	**99.69**	97.97
	84	79.80	98.11	91.25	**98.53**	65.97	96.04	99.67	98.79	**99.84**	98.01	98.99	99.91	99.66	**99.96**	99.69
	96	88.25	99.32	94.29	**99.39**	87.12	98.33	99.93	99.49	**99.96**	99.59	99.56	99.98	99.88	**100**	99.98
	108	93.17	99.87	97.30	**99.88**	94.87	99.37	**99.99**	99.82	**99.99**	99.90	99.83	**100**	99.94	**100**	99.97
	120	96.23	99.91	98.65	**99.96**	98.19	99.75	**100**	99.87	**100**	99.98	99.90	**100**	99.97	**100**	**100**
$beta(0.5,3)$	6	4.40	3.76	3.98	**4.31**	4.14	11.51	11.07	10.39	**11.59**	10.71	17.25	17.59	16.93	**18.67**	15.86
	9	8.54	9.98	7.67	**13.23**	8.23	21.95	25.69	20.67	**28.77**	26.06	33.01	36.89	31.39	**39.92**	38.02
	12	15.09	22.30	14.65	**27.11**	19.27	35.36	43.6	34.65	**47.93**	39.96	47.36	55.59	47.86	**58.17**	53.08
	24	48.83	74.49	51.26	**77.77**	62.03	72.85	87.38	75.61	**89.40**	83.69	82.76	92.15	84.37	**93.50**	90.32
	36	76.83	95.64	77.45	**96.03**	89.49	92.27	98.52	92.67	**98.72**	97.33	95.72	99.16	96.38	**99.42**	98.86
	48	91.49	99.37	90.83	**99.49**	97.88	98.12	99.89	98.22	**99.91**	99.58	99.32	99.94	99.26	**99.96**	99.86
	60	97.51	**99.98**	97.01	**99.98**	99.76	99.69	**100**	99.58	**100**	99.98	99.92	**100**	99.89	**100**	99.99
	72	99.34	**100**	98.82	**100**	99.97	99.92	**100**	99.96	**100**	**100**	99.99	**100**	99.98	**100**	**100**
	84	99.84	**100**	99.77	**100**	**100**	**100**	**100**	99.98	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	96	99.99	**100**	99.94	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(1,2)$	6	1.27	1.27	**1.52**	1.44	1.32	5.47	5.53	**6.13**	5.65	5.19	10.74	10.79	**11.59**	11.00	9.92
	9	1.58	1.56	**2.29**	2.04	1.51	6.94	7.16	**8.38**	8.30	6.75	13.03	13.67	14.87	**15.14**	12.38
	12	1.87	2.09	2.94	**3.09**	1.83	8.92	9.27	11.05	**11.25**	7.48	15.70	16.59	18.57	**18.61**	14.18
	24	5.49	10.02	7.25	**14.19**	4.11	17.42	28.03	21.68	**36.41**	16.77	27.80	41.55	33.02	**49.95**	28.20
	36	9.98	32.67	12.03	**38.15**	10.20	27.84	56.49	32.42	**63.95**	31.31	41.48	68.36	45.91	**76.06**	47.20
	48	16.55	59.32	17.95	**64.54**	19.34	39.11	80.29	43.36	**85.47**	49.93	55.00	87.52	58.86	**91.47**	66.99
	60	23.99	77.53	26.11	**80.71**	32.57	50.40	91.02	53.06	**93.34**	66.06	64.71	94.62	68.02	**96.69**	80.72
	72	30.95	89.52	29.95	**90.38**	49.17	59.49	96.93	62.64	**97.79**	78.42	73.89	98.32	76.33	**99.06**	88.58
	84	40.09	94.92	40.47	**96.28**	60.77	69.47	98.80	71.06	**99.29**	87.72	81.71	99.38	83.43	**99.70**	95.19
	96	48.59	98.15	43.73	**98.27**	74.44	77.29	99.56	76.23	**99.71**	93.12	87.02	99.83	87.58	**99.87**	97.49
	108	54.81	99.34	52.99	**99.49**	81.41	81.98	99.92	82.70	**99.95**	96.32	91.49	99.98	92.09	**99.98**	98.96
	120	63.06	99.67	56.45	**99.80**	87.51	87.22	99.97	86.47	**100**	98.21	93.7	**100**	94.24	**100**	99.49

The bold number is the highest power among the five tests for each sample size.

Table 3.15: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group II with Design 3** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$U(0,1)$	4	**1.04**	**1.04**	1.02	0.97	1.02	4.54	4.54	4.54	**5.41**	5.08	9.53	9.77	9.52	**10.42**	9.69
	8	0.86	0.74	**1.44**	0.83	0.61	4.6	4.13	**6.94**	4.59	3.29	9.45	8.94	**13.34**	9.64	7.39
	12	0.86	0.61	**2.58**	0.96	0.37	5.63	5.04	**10.13**	5.48	2.32	11.31	10.36	**17.94**	11.57	5.42
	24	1.76	2.21	**4.84**	3.33	0.08	8.54	10.16	**16.67**	14.8	2.16	16.5	20.16	**27.45**	26.7	5.36
	36	2.88	6.45	7.23	**10.00**	0.23	14.07	23.79	24.7	**33.61**	3.68	24.34	38.77	37.38	**51.83**	10.38
	48	5.02	15.66	9.91	**20.97**	0.42	20.43	42.27	30.7	**55.42**	6.9	33.94	59.31	47.4	**72.33**	19.79
	60	8.02	29.81	14.37	**36.62**	1.05	27.89	62.23	40.23	**73.79**	14.11	44.12	76.92	57.03	**85.98**	37.2
	72	11.47	48.31	19.07	**53.66**	1.64	34.33	77.88	48.98	**85.94**	26.06	51.29	87.88	65.71	**93.28**	54.89
	84	16.53	64.73	23.7	**70.51**	5.69	43.85	88.69	56.56	**93.16**	43.76	61.04	94.77	72.93	**97.51**	72.67
	96	20.85	76.32	29.4	**80.15**	7.89	50.79	94.31	64.1	**96.74**	59.65	67.93	97.6	79.06	**99.00**	83.08
	108	27.05	86.05	35.97	**88.79**	18.82	58.8	97.16	70.64	**98.39**	73.62	74.97	98.94	83.95	**99.44**	91.51
	120	31.91	92.26	39.53	**94.17**	28.98	65.14	98.88	75.6	**99.3**	82.93	80.01	99.59	88.63	**99.86**	95.75
$beta(2,2)$	4	0.95	0.95	0.94	**1.09**	0.94	4.8	4.8	4.79	4.8	**4.87**	9.47	9.58	9.47	**9.92**	9.8
	8	1.00	0.75	**1.19**	0.83	0.69	4.34	4.25	**5.50**	4.51	3.47	8.95	8.79	**10.93**	8.86	7.52
	12	0.7	0.55	**1.41**	0.86	0.39	4.41	3.94	**6.28**	4.09	2.7	8.91	8.23	**12.20**	8.69	6.1
	24	0.7	0.66	**1.68**	0.98	0.11	4.32	3.72	**6.68**	5.38	1.32	9.45	8.89	**13.45**	11.09	3.72
	36	1.06	1.00	**1.9**	1.81	0.21	5.8	5.9	8.77	**8.92**	1.71	11.41	12.24	15.85	**17.23**	4.65
	48	1.32	1.75	1.97	**2.69**	0.14	7.18	9	9.98	**13.78**	2.04	14.06	17.24	18.68	**24.81**	5.82
	60	1.55	2.7	2.21	**3.86**	0.26	8	12.18	10.68	**18.77**	2.58	15.39	22.84	19.92	**32.94**	8.86
	72	1.74	4.31	2.49	**5.38**	0.29	9.17	16.91	12.39	**24.67**	3.68	17.48	29.27	22.86	**41.84**	11.64
	84	2.70	6.11	3.39	**8.09**	0.33	12.3	23.4	15.35	**32.21**	5.32	21.81	38.08	26.49	**51.46**	15.14
	96	2.73	8.2	3.54	**10.07**	0.46	12.83	29.21	16.43	**39.34**	7.3	23.28	44.99	28.22	**58.08**	21.51
	108	3.50	10.64	3.9	**13.36**	0.96	14.93	35.81	18.67	**47.32**	10.24	26.15	54.12	31.72	**66.53**	27.93
	120	3.99	14.40	4.45	**17.70**	1.08	16.53	43.87	20.06	**54.27**	13.3	28.7	60.51	34.54	**74.91**	33.52
$beta(2,5)$	4	**1.09**	**1.09**	1.04	0.96	0.85	4.72	4.72	4.72	**5.33**	4.84	9.63	9.69	9.62	**10.62**	9.75
	8	1.4	1.29	**1.4**	1.26	1.29	5.85	5.87	6.29	**6.13**	5.91	11.45	11.62	**12.09**	11.75	11.52
	12	1.82	1.78	1.75	**2.39**	2.08	7.04	7.57	7.92	**8.72**	7.96	12.85	13.8	14.45	**15.04**	14.1
	24	3.2	5.04	3.89	**6.88**	3.86	11.39	15.98	13.05	**19.95**	14.95	19.63	25.61	22.32	**30.15**	24.73
	36	5.34	11.43	5.75	**15.27**	8.48	17.17	29.56	19.09	**36.76**	25.66	26.82	42.67	29.28	**49.86**	38.36
	48	7.15	22.06	7.04	**26.51**	13.81	22.43	45.16	23.87	**54.06**	36.39	34.36	58.7	36.84	**67.16**	50.32
	60	10.7	34.63	9.34	**39.31**	21.21	29.18	60.53	28.89	**68.53**	47.89	41.88	72.82	43.14	**80.35**	62.95
	72	13.85	49.73	11.99	**52.53**	31.69	34.53	73.48	35.03	**79.81**	58.32	48.3	83.78	50.68	**89.01**	72.47
	84	19.14	62.06	16.74	**64.96**	37.97	41.53	83.62	41.46	**87.64**	68.9	55.14	90.47	56.7	**94.09**	80.15
	96	21.76	72.35	18.99	**74.32**	47.05	44.91	90.59	45.33	**92.82**	76.42	60.34	94.92	61.38	**96.57**	86.75
	108	25.83	80.67	21.18	**82.39**	56.36	51.84	94.04	51.93	**95.93**	82.19	66.21	97.37	66.97	**98.36**	90.43
	120	30.55	82.27	25.2	**88.41**	63.86	56.93	96.9	55.65	**97.78**	86.65	69.99	98.58	71.05	**99.2**	93.29
$beta(5,1.5)$	4	1.01	1.01	0.98	1.08	**1.28**	5.21	5.21	5.18	**5.76**	5.03	10.38	10.67	10.37	**10.91**	10.03
	8	1.87	1.81	1.41	**2.11**	2.05	7.24	7.28	7.10	**8.14**	7.55	13.00	13.64	13.28	**14.72**	14.18
	12	2.46	2.75	2.35	**4.05**	3.59	9.49	11.31	9.73	**13.03**	12.64	16.74	19.3	18.12	**21.65**	20.94
	24	5.97	13.05	6.96	**16.77**	10.27	18.52	30.34	20.53	**37.10**	28.23	29.39	42.86	32.75	**49.31**	41.20
	36	11.40	32.15	12.78	**38.90**	22.2	30.67	56.56	33.83	**64.16**	47.88	42.51	68.57	46.75	**75.54**	61.63
	48	17.62	55.44	17.83	**60.99**	36.19	41.03	77.31	43.23	**82.69**	65.13	54.80	85.23	58.55	**89.91**	76.97
	60	25.16	73.08	23.54	**77.04**	50.40	51.73	89.23	53.09	**92.36**	77.53	65.03	94.02	68.96	**96.01**	87.96
	72	33.97	86.96	31.43	**88.30**	67.52	61.18	95.61	63.2	**96.94**	87.43	74.30	97.76	76.94	**98.59**	93.57
	84	43.28	93.79	41.12	**94.44**	76.31	69.77	98.44	71.49	**98.97**	93.13	81.41	99.36	83.05	**99.62**	96.82
	96	49.88	97.19	47.34	**97.40**	85.63	75.86	99.45	77.05	**99.59**	96.58	86.53	99.78	87.80	**99.84**	98.70
	108	56.65	98.53	51.49	**98.66**	91.21	81.73	99.84	82.40	**99.92**	98.13	90.58	**99.97**	91.42	**99.97**	99.39
	120	63.79	99.44	60.63	**99.44**	94.11	86.08	99.97	87.08	**99.99**	99.03	92.66	**99.99**	94.06	**99.99**	99.73
$beta(0.5,0.5)$	4	0.86	0.86	0.84	1.06	**1.28**	4.66	4.66	4.66	5.12	**5.77**	9.77	9.58	9.81	10.37	**10.93**
	8	1.60	1.28	**3.12**	1.54	0.84	6.51	6.04	**11.27**	7.06	3.57	12.59	12.32	**19.46**	13.37	7.97
	12	2.91	2.63	**7.25**	4.35	0.77	11.52	11.73	**20.52**	14.11	4.05	20.11	21.10	**31.82**	23.22	8.09
	24	8.61	14.89	**22.35**	20.00	0.65	26.95	38.55	45.56	**46.63**	5.09	41.79	54.63	60.01	**61.49**	12.51
	36	20.24	42.59	39.70	**50.89**	1.71	49.63	71.11	68.70	**77.38**	15.13	65.16	82.60	79.96	**86.18**	38.71
	48	34.60	69.94	54.66	**74.73**	3.89	67.98	88.87	82.73	**92.23**	40.76	81.61	94.30	91.29	**96.08**	73.54
	60	54.01	86.48	70.94	**88.88**	12.43	82.25	95.91	92.02	**97.13**	73.47	91.73	98.13	96.67	**98.64**	92.97
	72	67.56	94.32	81.52	**94.96**	38.66	90.48	98.57	96.23	**98.96**	90.92	95.96	99.34	98.43	**99.54**	97.91
	84	81.89	98.03	91.40	**98.33**	62.08	96.54	99.62	98.79	**99.70**	98.08	98.93	99.87	99.67	**99.88**	99.65
	96	88.31	99.20	94.83	**99.33**	84.11	98.37	99.89	99.55	**99.91**	99.54	99.58	99.97	99.94	**99.99**	99.95
	108	93.51	99.73	97.24	**99.76**	95.28	99.34	99.96	99.81	**99.96**	99.89	99.87	**100**	99.97	**100**	99.98
	120	97.07	**99.91**	98.98	**99.91**	98.45	99.73	99.98	99.91	**99.99**	99.95	99.90	99.99	99.99	**100**	**100**
$beta(0.5,3)$	4	0.92	0.92	0.95	0.94	**1.67**	5.29	5.29	5.29	5.14	**7.4**	11.42	10.95	11.44	10.32	**13.94**
	8	7.45	8.04	5.40	**10.31**	8.50	18.59	20.48	17.14	**22.91**	18.92	27.85	30.53	26.76	**32.49**	30.99
	12	15.23	21.13	14.03	**27.89**	19.09	33.82	43.14	33.62	**46.97**	40.81	45.84	54.47	46.94	**58.18**	53.47
	24	46.29	74.57	51.86	**77.81**	61.70	71.19	86.84	73.60	**88.99**	83.54	81.58	91.45	83.75	**92.76**	90.84
	36	76.05	94.70	77.54	**95.70**	88.81	91.84	97.90	92.51	**98.16**	97.09	95.53	98.88	96.22	**99.07**	98.68
	48	91.08	99.27	90.93	**99.31**	98.00	97.97	99.77	98.20	**99.81**	99.72	99.11	99.89	99.16	**99.90**	99.92
	60	97.58	99.91	96.54	**99.92**	99.78	99.69	**100**	99.68	**100**	99.98	99.87	**100**	99.89	**100**	99.99
	72	99.35	99.99	99.09	**99.99**	99.98	99.93	**100**	99.90	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	99.83	**100**	99.79	**100**	**100**	99.99	**100**	99.98	**100**	**100**	99.99	**100**	99.99	**100**	**100**
	96	99.96	**100**	99.96	**100**	**100**	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	99.99	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(1,2)$	4	1.03	1.03	**1.04**	0.96	1.03	5.02	5.02	4.98	**5.09**	5.06	**10.38**	10.03	10.37	10.12	10.12
	8	1.46	1.26	**1.74**	1.54	1.23	6.48	6.35	**7.24**	7.00	5.81	11.67	12.28	**14.01**	12.86	11.63
	12	2.12	1.82	2.88	**3.11**	1.80	8.56	9.37	10.66	**11.32**	7.84	15.10	16.69	**19.17**	19.71	14.56
	24	4.74	10.86	7.29	**15.53**	3.70	16.53	28.47	20.57	**36.25**	15.82	27.07	40.97	32.14	**48.98**	27.72
	36	10.10	32.76	12.50	**41.22**	10.82	27.82	58.14	32.55	**65.87**	33.03	40.67	68.90	46.19	**76.98**	48.52
	48	16.16	57.67	17.55	**63.43**	19.48	39.33	78.40	42.93	**84.57**	49.76	53.47	86.05	58.12	**90.94**	66.31
	60	24.00	77.10	23.45	**80.83**	32.13	50.43	91.08	52.79	**93.91**	66.06	64.86	94.89	67.32	**96.72**	81.17
	72	31.12	89.66	30.40	**90.93**	48.44	59.7	96.38	61.47	**97.94**	77.78	73.2	98.30	76.30	**99.00**	88.62
	84	41.75	95.60	41.25	**96.16**	58.89	69.88	98.73	71.31	**99.12**	88.04	81.51	99.29	83.45	**99.63**	94.60
	96	48.74	98.07	47.57	**98.35**	71.77	75.44	99.51	77.13	**99.68**	93.26	86.23	99.77	87.19	**99.86**	97.44
	108	56.42	99.24	52.29	**99.36**	81.58	82.35	99.89	83.29	**99.98**	96.1	90.63	100	91.83	**99.99**	98.87
	120	63.64	99.73	59.72	**99.76**	87.80	86.69	**99.98**	86.70	**99.98**	98.22	93.57	99.99	94.36	**100**	99.44

The bold number is the highest power among the five tests for each sample size.

Table 3.16: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group II with Design 4** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$U(0,1)$	4	**1.04**	**1.04**	1.02	0.97	0.93	4.54	4.54	4.54	**5.41**	5.08	9.53	9.77	9.52	**10.42**	9.69
	8	0.55	0.58	**1.54**	0.72	0.42	4.35	3.89	**6.66**	4.08	2.93	9.39	8.91	**12.78**	9.22	7.19
	12	0.90	0.47	**2.49**	0.83	0.33	5.26	4.29	**9.84**	5.51	2.30	11.08	9.8	**17.20**	11.04	5.43
	24	1.93	1.91	**4.40**	2.88	0.09	8.74	10.10	**16.05**	14.47	2.21	17.26	19.79	26.43	**26.58**	5.35
	36	3.33	6.24	7.28	**8.75**	0.22	14.27	24.26	24.7	**33.70**	3.61	25.66	39.71	37.98	**51.73**	9.98
	48	5.65	14.94	11.16	**18.48**	0.51	20.64	42.39	32.52	**55.89**	6.5	34.14	59.64	48.13	**71.20**	20.22
	60	8.29	31.18	14.05	**36.76**	0.81	27.97	62.92	40.23	**72.49**	14.08	43.18	76.61	57.42	**85.38**	37.87
	72	11.25	47.96	17.31	**52.89**	1.85	34.63	78.20	47.99	**86.42**	25.64	51.74	88.37	64.37	**93.61**	56.93
	84	14.77	64.21	24.17	**71.14**	5.48	42.74	88.91	57.27	**93.02**	42.08	60.87	94.77	73.88	**97.31**	72.43
	96	19.94	76.15	27.83	**80.29**	9.69	50.95	94.39	62.97	**97.01**	58.74	67.94	97.68	78.51	**98.94**	82.68
	108	26.44	86.05	33.88	**88.93**	16.31	59.49	97.24	70.29	**98.29**	73.26	75.69	98.95	84.92	**99.49**	90.97
	120	30.80	92.09	40.27	**94.06**	26.5	64.05	98.89	76.11	**99.36**	83.18	79.63	99.59	88.01	**99.85**	95.70
$beta(2,2)$	4	0.95	0.95	0.94	**1.09**	0.81	4.80	4.80	4.79	4.80	**4.87**	9.47	9.58	9.47	**9.92**	9.80
	8	0.82	0.73	**1.34**	0.58	0.58	4.65	4.21	**5.81**	3.91	3.00	8.75	8.61	**11.12**	8.36	6.87
	12	0.66	0.42	**1.27**	0.84	0.35	4.14	3.56	**6.41**	4.12	2.65	8.90	7.64	**12.81**	8.29	5.64
	24	0.86	0.62	**1.61**	0.9	0.12	4.60	4.01	**7.23**	5.29	1.39	10.05	8.49	**13.67**	11.02	3.59
	36	1.15	0.97	**1.65**	1.52	0.18	5.78	6.07	8.36	**8.84**	1.58	11.63	12.41	16.06	**17.55**	4.23
	48	1.38	1.85	1.85	**2.68**	0.16	7.23	8.99	10.32	**13.76**	1.88	13.86	17.38	18.11	**24.09**	5.77
	60	1.80	2.87	2.37	**3.69**	0.21	8.21	11.86	11.04	**17.63**	2.56	15.88	23.62	20.47	**32.97**	8.28
	72	1.76	4.21	2.82	**5.26**	0.19	9.21	17.11	12.68	**25.30**	3.69	18.23	29.42	23.55	**40.80**	12.20
	84	2.29	5.94	3.26	**7.86**	0.38	11.80	22.81	15.05	**31.43**	5.11	22.01	37.45	26.33	**50.63**	15.42
	96	2.47	7.69	3.44	**10.47**	0.57	13.22	28.49	16.96	**39.78**	7.03	23.92	44.71	29.12	**58.25**	20.45
	108	3.34	11.23	3.63	**14.03**	0.68	15.59	36.60	18.60	**46.62**	10.16	26.69	52.90	31.91	**66.93**	27.42
	120	3.62	14.26	4.11	**18.35**	0.97	16.03	43.28	19.90	**53.72**	13.53	28.45	60.86	33.93	**74.10**	33.81
$beta(2,5)$	4	1.09	1.09	1.04	0.96	**1.31**	4.72	4.72	4.72	**5.33**	4.84	9.63	9.69	9.62	**10.62**	9.75
	8	1.38	1.38	1.39	1.51	**1.52**	5.86	5.64	**6.45**	6.01	5.75	11.29	11.60	**11.97**	11.61	11.43
	12	1.53	1.43	1.82	**2.12**	2.04	6.75	6.93	7.53	**8.24**	7.74	12.01	12.68	**14.35**	14.23	13.64
	24	3.42	5.08	3.75	**6.28**	4.23	11.64	15.51	13.44	**19.87**	15.11	20.18	25.14	23.14	**29.85**	24.91
	36	6.00	12.36	5.62	**15.15**	8.31	17.53	30.28	19.85	**36.33**	26.69	27.77	42.55	31.00	**50.28**	39.01
	48	8.02	21.57	7.08	**26.10**	14.62	22.03	45.11	24.18	**54.08**	35.65	34.18	58.69	36.25	**66.96**	50.24
	60	11.08	35.76	10.18	**38.97**	20.55	28.21	60.35	29.48	**68.04**	47.80	41.18	72.72	43.47	**79.64**	62.42
	72	14.06	49.93	12.58	**52.63**	29.94	34.29	73.39	35.57	**79.84**	58.14	48.68	83.22	50.29	**88.35**	73.04
	84	17.65	61.80	15.81	**65.04**	40.56	40.44	83.33	39.92	**87.19**	68.15	55.09	90.38	56.05	**93.82**	80.49
	96	21.09	72.37	18.39	**74.75**	48.08	46.00	90.27	45.12	**93.21**	76.3	60.36	94.84	61.09	**96.93**	86.18
	108	26.09	81.21	20.64	**82.92**	55.22	53.29	94.52	51.21	**96.03**	82.55	67.38	97.42	67.44	**98.59**	90.74
	120	29.28	86.81	23.36	**88.87**	62.79	55.8	96.87	54.66	**97.71**	86.82	70.78	98.52	70.66	**99.01**	93.12
$beta(5,1.5)$	4	1.01	1.01	0.98	1.08	**1.56**	5.21	5.21	5.18	**5.76**	5.03	10.38	10.67	10.37	**10.91**	10.03
	8	1.54	1.75	1.79	1.89	**2.26**	7.39	7.21	7.57	**7.74**	7.27	12.81	13.66	13.77	**14.31**	14.29
	12	2.38	2.59	2.41	**4.10**	3.48	9.2	10.73	10.44	**12.85**	12.13	16.84	18.11	18.49	**20.41**	20.10
	24	6.24	12.63	6.75	**15.74**	10.91	19.01	30.21	20.81	**37.22**	28.68	30.2	42.94	32.75	**49.20**	41.39
	36	12.54	33.08	12.29	**38.38**	22.26	30.62	57.05	33.77	**63.93**	48.66	43.87	68.63	47.24	**75.96**	62.30
	48	19.17	55.13	17.53	**60.42**	37.95	41.33	76.95	44.42	**82.93**	64.28	59.94	85.18	57.99	**89.41**	76.79
	60	26.37	75.22	25.14	**76.74**	49.82	52.03	90.07	53.58	**91.85**	77.51	65.89	94.31	68.37	**96.14**	88.21
	72	33.94	87.21	34.14	**88.37**	66.02	61.46	95.68	64.62	**97.10**	87.08	74.55	97.81	77.66	**98.51**	93.91
	84	40.91	93.84	39.99	**94.48**	77.98	69.03	98.34	70.53	**98.74**	92.85	80.91	99.24	82.94	**99.55**	96.94
	96	49.53	97.07	46.81	**97.34**	85.99	76.46	99.4	77.31	**99.59**	96.37	86.61	99.75	88.25	**99.87**	98.57
	108	56.02	98.5	51.21	**98.53**	90.03	82.01	99.78	81.88	**99.83**	98.17	90.58	99.91	91.57	**99.94**	99.34
	120	62.29	99.33	56.29	**99.44**	93.75	85.75	99.87	86.21	**99.98**	99.06	92.91	**99.96**	93.80	**99.96**	99.63
$beta(0.5,0.5)$	4	0.86	0.86	0.84	1.06	**1.18**	4.66	4.66	4.66	5.12	**5.77**	9.77	9.58	9.81	**10.37**	10.93
	8	**1.44**	1.19	3.05	1.41	0.76	6.8	6.15	11.05	**6.49**	3.65	12.96	12.67	19.56	**12.93**	8.00
	12	2.58	2.33	6.94	**4.04**	0.69	10.64	10.64	**20.57**	14.32	3.75	18.73	20.04	31.70	**22.34**	7.8
	24	9.42	14.63	21.64	**18.63**	0.72	27.49	38.23	45.88	**46.35**	5.15	42.72	53.77	60.74	**61.28**	12.37
	36	22.56	42.76	38.84	**49.17**	1.49	49.94	71.51	68.36	**77.73**	14.72	66.02	82.75	80.29	**86.62**	37.36
	48	37.1	69.09	53.82	**73.74**	4.47	67.33	89.19	82.33	**92.12**	39.08	81.26	94.47	90.45	**95.76**	73.01
	60	53.28	86.9	72.44	**88.25**	11.59	83.33	96.17	92.11	**97.06**	73.7	92.27	98.24	96.57	**98.80**	92.58
	72	67.99	94.43	83.11	**95.08**	34.69	90.41	98.53	96.32	**98.96**	90.79	96.08	99.41	98.52	**99.46**	98.07
	84	80.21	97.92	90.52	**98.15**	68.41	96.19	99.60	98.73	**99.70**	97.88	98.86	99.87	99.66	**99.89**	99.67
	96	88.08	99.04	94.9	**99.17**	85.52	98.42	99.85	99.55	**99.93**	99.62	99.58	99.95	99.90	**99.98**	99.93
	108	93.42	99.75	97.28	**99.81**	94.46	99.41	99.98	99.82	**100**	99.88	99.80	**100**	99.95	**100**	99.98
	120	96.79	99.88	98.26	**99.90**	98.08	99.71	**100**	99.93	99.99	99.95	99.99	**100**	99.98	**100**	**100**
$beta(0.5,3)$	4	0.92	0.92	0.95	0.94	**4.12**	5.29	5.29	5.29	5.14	**7.4**	11.42	10.95	11.44	10.32	**13.94**
	8	6.29	6.85	5.64	9.21	**9.75**	18.12	19.68	16.85	**22.04**	19.91	26.65	30.03	26.55	**32.43**	30.61
	12	14.18	19.68	13.84	**27.53**	18.25	31.93	41.06	32.96	**46.38**	39.95	44.60	52.47	46.05	**57.13**	52.55
	24	47.69	74.48	50.91	**76.76**	62.97	71.29	86.68	74.63	**88.66**	83.77	82.29	91.29	83.76	**92.50**	91.06
	36	77.18	94.71	76.37	**95.49**	89.91	91.75	98.07	92.79	**98.36**	97.54	95.80	98.98	96.40	**99.11**	98.98
	48	91.87	98.94	90.85	**99.04**	98.14	97.90	99.70	98.07	**99.76**	99.69	99.13	99.87	99.11	**99.91**	99.89
	60	97.65	99.85	96.92	**99.91**	99.74	99.49	99.97	99.49	**99.98**	99.99	99.82	100	99.78	**100**	99.99
	72	99.29	99.99	99.11	**99.98**	99.99	99.91	**100**	99.92	**100**	**100**	99.97	**100**	99.98	**100**	**100**
	84	99.87	**100**	99.82	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	96	99.95	**100**	99.96	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	120	99.99	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	v100
$beta(1,2)$	4	1.03	1.03	1.04	0.96	**1.12**	5.02	5.02	**4.98**	5.09	5.06	10.38	10.03	**10.37**	10.12	10.12
	8	1.10	1.08	1.69	1.23	**1.37**	6.1	5.65	**7.23**	6.60	5.22	11.61	11.82	**13.99**	12.43	10.54
	12	1.94	1.76	**2.89**	2.86	1.57	8.44	8.60	**10.79**	10.58	7.53	15.04	16.04	**19.00**	18.83	13.88
	24	4.81	10.72	6.80	**14.54**	4.09	16.48	28.14	21.06	**36.44**	15.93	27.38	40.92	33.02	**48.87**	27.98
	36	10.65	33.06	12.03	**39.05**	10.52	27.92	57.12	32.84	**65.17**	32.30	40.97	69.01	46.01	**77.06**	47.25
	48	17.36	56.87	17.45	**63.07**	20.68	38.99	78.75	43.70	**83.69**	49.23	53.39	86.28	57.71	**90.62**	65.89
	60	24.47	77.84	25.4	**80.53**	31.16	49.46	91.61	52.90	**94.01**	65.98	63.89	95.14	67.59	**96.81**	79.52
	72	30.95	89.33	31.62	**90.56**	46.33	59.19	96.43	61.99	**97.62**	77.33	73.59	98.22	75.92	**98.90**	88.98
	84	40.18	95.43	39.95	**96.10**	62.16	69.09	98.71	70.61	**99.09**	87.55	81.92	99.29	83.37	**99.61**	94.76
	96	47.75	97.87	46.99	**98.10**	72.98	75.95	99.48	77.04	**99.66**	92.94	86.74	99.78	87.86	**99.86**	97.28
	108	56.13	99.21	52.48	**99.20**	80.49	82.95	99.81	82.77	**99.93**	96.2	91.56	99.97	92.27	**99.98**	98.92
	120	63.38	99.67	58.77	**99.69**	87.01	87.01	99.97	86.83	**100**	98.19	94.05	**99.98**	94.62	**99.98**	99.35

The bold number is the highest power among the five tests for each sample size.

Table 3.17: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group II with Design 5** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$U(0,1)$	4	1.07	0.98	**1.14**	1.01	0.97	4.94	5.02	**5.40**	5.09	5.14	9.86	**10.57**	10.47	10.47	10.17
	8	1.05	0.77	**1.68**	1.00	0.62	4.90	4.13	**7.88**	4.56	3.08	9.91	8.67	**14.51**	9.83	7.56
	12	1.10	0.57	**2.70**	1.07	0.34	5.22	4.40	**9.91**	5.64	1.87	10.59	9.54	**18.16**	12.46	5.41
	24	1.61	1.92	**4.66**	3.67	0.12	8.52	10.70	16.95	**17.56**	1.77	16.75	21.46	28.21	**30.81**	5.32
	36	3.13	8.29	7.24	**12.69**	0.34	13.90	28.92	24.36	**41.27**	3.10	24.49	44.42	38.91	**59.76**	10.68
	48	4.70	22.63	11.14	**29.12**	0.62	19.79	53.64	33.09	**67.78**	6.28	34.44	70.27	49.51	**82.45**	21.14
	60	7.80	40.79	14.46	**48.48**	1.04	26.84	74.00	40.41	**84.80**	14.56	42.49	86.61	57.89	**93.62**	38.58
	72	10.89	61.03	18.63	**67.62**	2.27	34.21	88.69	48.24	**93.94**	28.03	51.13	95.00	66.22	**98.27**	57.66
	84	14.37	77.38	23.67	**82.97**	4.30	42.33	95.65	55.92	**98.09**	43.55	60.25	98.47	73.17	**99.64**	72.46
	96	20.72	88.51	29.74	**93.35**	8.79	50.60	98.74	64.54	**99.56**	60.32	67.40	99.68	79.49	**99.91**	84.04
	108	25.30	95.76	36.74	**97.04**	16.65	58.23	99.67	71.20	**99.91**	73.19	74.88	99.94	85.08	**99.99**	91.91
	120	31.64	98.47	39.05	**98.93**	30.19	64.14	99.94	75.82	**99.98**	85.28	79.89	99.99	88.41	**100**	95.90
$beta(2,2)$	4	1.02	0.95	0.64	0.95	**0.99**	4.76	4.80	**4.86**	4.78	5.02	9.90	9.58	9.74	**10.01**	9.50
	8	0.85	0.73	**1.13**	0.80	0.64	4.04	4.21	**5.53**	3.74	3.34	8.43	8.61	**10.73**	8.24	7.01
	12	0.75	0.42	**1.39**	0.48	0.29	4.23	3.56	**6.16**	3.50	2.08	8.73	7.64	**11.89**	7.53	5.20
	24	0.73	0.62	**1.35**	1.01	0.08	4.34	4.01	**7.30**	5.93	1.32	9.28	8.49	**13.84**	12.14	3.87
	36	0.91	0.97	1.65	**1.70**	0.19	5.74	6.07	**8.48**	8.33	1.39	11.49	12.41	16.28	**16.97**	4.33
	48	1.27	1.85	2.06	**2.79**	0.11	6.95	8.99	9.79	**13.45**	1.83	13.67	17.38	18.43	**25.24**	5.69
	60	1.63	2.87	2.17	**3.90**	0.21	7.97	13.41	10.8	**18.81**	2.26	15.45	23.62	20.56	**34.12**	8.09
	72	1.78	4.21	2.64	**5.70**	0.28	9.22	17.11	12.43	**25.57**	3.45	17.57	29.42	23.20	**43.08**	11.49
	84	2.13	5.94	3.05	**7.69**	0.32	11.03	22.81	15.35	**32.90**	4.41	21.17	37.45	25.82	**52.64**	14.69
	96	2.73	7.69	3.66	**12.48**	0.44	12.13	28.49	16.2	**40.89**	7.20	22.45	44.71	29.08	**60.80**	20.95
	108	3.32	11.23	3.83	**14.29**	0.94	14.62	36.60	18.56	**49.94**	9.99	26.41	52.90	31.98	**69.63**	26.46
	120	4.04	14.26	4.22	**19.32**	0.87	16.05	43.28	20.08	**59.72**	12.54	28.26	60.86	34.00	**78.70**	33.54
$beta(2,5)$	4	0.83	**1.09**	1.06	0.81	0.95	4.58	4.72	4.78	4.90	**5.20**	9.55	9.69	10.08	**10.67**	10.35
	8	1.04	1.38	1.06	**1.49**	1.46	5.75	5.64	6.08	**6.28**	6.15	11.01	11.60	**11.76**	11.48	11.24
	12	1.44	1.43	1.72	**1.96**	1.91	6.27	6.93	7.33	**8.07**	7.44	12.62	12.68	14.51	**14.68**	13.72
	24	2.92	5.08	3.20	**6.51**	3.90	11.24	15.51	12.97	**21.4**	15.10	19.51	25.14	22.13	**32.88**	25.08
	36	5.10	12.36	5.35	**16.77**	8.93	17.82	30.28	18.96	**39.75**	26.67	27.72	42.55	29.67	**53.91**	39.18
	48	7.18	21.57	7.16	**30.09**	13.92	21.65	45.11	23.69	**58.24**	36.54	33.42	58.69	37.28	**72.11**	51.29
	60	10.63	35.76	9.14	**43.83**	22.23	28.15	60.35	29.57	**74.09**	47.45	40.89	72.72	43.42	**85.14**	62.37
	72	14.32	49.93	13.14	**58.11**	30.63	34.26	73.39	34.82	**85.01**	58.91	48.22	83.22	49.70	**92.58**	73.33
	84	16.95	61.80	14.71	**70.56**	36.88	39.47	83.33	39.62	**91.74**	67.79	54.53	90.38	54.90	**96.23**	80.12
	96	21.96	72.37	18.03	**82.01**	47.38	45.98	90.27	45.66	**95.40**	75.91	59.95	94.84	61.63	**98.26**	85.93
	108	25.44	81.21	20.91	**87.83**	56.48	51.55	94.52	51.57	**97.94**	81.94	66.20	97.42	67.03	**99.28**	90.44
	120	29.06	86.81	22.93	**93.27**	62.62	54.78	96.87	54.43	**99.15**	86.38	69.91	98.52	70.91	**99.74**	93.33
$beta(5,1.5)$	4	0.76	1.01	**1.13**	1.01	1.12	5.17	5.21	5.08	5.04	**5.77**	**10.75**	10.67	10.58	10.19	11.39
	8	1.60	1.75	1.61	2.12	**2.22**	6.81	7.21	6.78	8.22	**8.39**	12.77	13.66	13.36	**14.53**	14.73
	12	2.45	2.59	2.57	3.64	**3.74**	9.47	10.73	10.08	**13.30**	12.07	16.94	18.11	18.84	**21.90**	20.67
	24	6.39	12.63	6.92	**19.46**	10.73	18.91	30.21	21.88	**43.33**	29.51	29.73	42.94	33.84	**56.70**	42.56
	36	11.16	33.08	13.00	**45.12**	22.60	30.12	57.05	34.30	**71.50**	49.25	42.88	68.63	47.98	**82.10**	63.35
	48	17.54	55.13	16.91	**70.00**	37.03	40.12	76.95	42.21	**88.49**	65.09	54.88	85.18	58.49	**94.20**	77.79
	60	25.51	75.22	24.02	**85.65**	53.05	50.84	90.07	52.87	**96.13**	77.91	64.17	94.31	68.42	**98.27**	87.87
	72	33.85	87.21	32.64	**93.44**	67.63	60.91	95.68	62.91	**98.77**	87.65	74.22	97.81	76.23	**99.51**	93.86
	84	40.66	93.84	39.77	**97.27**	75.52	68.40	98.34	70.55	**99.49**	92.94	80.50	99.24	82.69	**99.86**	97.09
	96	48.45	97.07	44.56	**98.96**	84.93	76.09	99.40	77.06	**99.86**	96.46	86.16	99.75	87.78	**99.97**	98.65
	108	55.87	98.50	50.75	**99.42**	91.11	81.49	99.78	82.22	**99.97**	98.39	90.67	99.91	91.25	**99.99**	99.35
	120	62.78	99.33	56.04	**99.76**	93.97	85.74	99.87	86.10	**99.99**	98.94	93.33	99.96	94.34	**100**	99.65
$beta(0.5,0.5)$	4	0.78	0.86	**2.40**	1.17	1.22	4.41	4.66	7.74	5.60	**5.68**	9.45	9.58	**12.89**	10.74	11.03
	8	1.88	1.19	**3.35**	2.59	1.60	7.58	6.15	**13.23**	8.23	5.64	14.25	12.67	21.69	**14.77**	9.31
	12	2.87	2.33	**7.75**	4.76	0.74	10.77	10.64	**22.41**	16.43	4.29	20.06	20.04	**33.92**	27.22	8.84
	24	9.21	14.63	24.31	**30.94**	0.88	28.11	38.23	50.49	**64.58**	6.13	43.40	53.77	64.13	**79.95**	14.56
	36	19.68	42.76	41.71	**74.52**	1.66	50.36	71.51	71.77	**94.30**	15.3	66.83	82.75	83.43	**97.97**	41.30
	48	34.03	69.09	57.71	**94.84**	4.29	68.14	89.19	85.62	**99.43**	42.44	82.37	94.47	93.48	**99.83**	78.15
	60	54.02	86.90	72.46	**99.05**	15.81	82.59	96.17	93.19	**99.95**	76.25	91.76	98.24	97.53	**99.98**	95.34
	72	68.67	94.43	84.77	**99.92**	40.89	91.41	98.53	97.49	**100**	93.78	96.89	99.41	99.22	**100**	99.05
	84	80.33	97.92	91.69	**99.99**	63.14	96.28	99.60	99.10	**100**	98.76	99.03	99.87	99.79	**100**	99.89
	96	89.29	99.04	96.07	**100**	88.38	98.55	99.85	99.74	**100**	99.77	99.70	99.95	99.95	**100**	99.99
	108	94.18	99.75	97.94	**100**	96.80	99.52	99.98	99.93	**100**	99.99	99.91	**100**	99.988	**100**	99.99
	120	97.36	99.88	98.85	**100**	99.03	99.82	**100**	99.99	**100**	**100**	99.99	**100**	99.998	**100**	**100**
$beta(0.5,3)$	4	0.93	0.92	**2.59**	1.61	1.81	4.96	5.29	**8.41**	7.01	7.62	11.22	10.95	**14.11**	13.26	13.26
	8	5.87	6.85	5.08	**12.40**	10.30	19.27	19.68	16.69	**26.88**	14.10	29.46	30.03	27.03	**37.33**	33.44
	12	15.31	19.68	13.49	**34.74**	19.50	34.17	41.06	34.45	**56.40**	41.64	47.37	52.47	47.78	**67.13**	54.48
	24	49.36	74.48	50.24	**89.42**	63.52	75.73	86.68	76.17	**96.26**	85.00	85.87	91.29	85.71	**98.15**	92.01
	36	81.44	94.71	77.60	**98.36**	90.81	95.61	98.07	93.70	**99.69**	98.16	98.08	98.98	97.11	**99.92**	99.24
	48	95.45	98.94	91.52	**99.93**	98.47	99.38	99.70	98.59	**99.99**	99.81	99.83	99.87	99.60	**100**	99.98
	60	99.44	99.85	97.04	**100**	99.76	99.93	99.97	99.73	**100**	99.98	99.99	100	99.98	**100**	**100**
	72	99.92	99.99	99.26	**100**	99.99	99.99	**100**	99.95	**100**	**100**	100	100	99.99	**100**	**100**
	84	**100**	**100**	99.88	**100**	**100**	**100**	**100**	**100**	**100**	100	100	100	**100**	**100**	100
	96	**100**	**100**	99.99	**100**	**100**	**100**	**100**	**100**	**100**	100	100	100	**100**	**100**	100
	108	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	100	100	**100**	**100**	100
	120	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	100	100	**100**	**100**	100
$beta(1,2)$	4	0.78	1.03	**1.12**	1.03	0.84	4.82	5.02	5.27	5.15	**5.42**	9.73	10.03	**10.46**	10.43	10.34
	8	1.30	1.08	1.60	**2.05**	1.61	6.62	5.65	**7.76**	7.08	6.45	12.28	11.82	**14.01**	13.40	11.49
	12	2.09	1.76	2.73	**3.16**	1.76	8.32	8.60	11.01	**12.11**	7.59	14.96	16.04	19.39	**20.37**	14.12
	24	4.63	10.72	6.67	**19.44**	3.88	15.93	28.14	20.81	**44.15**	15.91	26.22	40.92	33.096	**58.22**	27.97
	36	9.22	33.06	11.85	**51.20**	10.26	27.54	57.12	31.41	**76.96**	32.71	39.86	69.01	46.131	**85.96**	48.46
	48	15.68	56.87	17.64	**76.83**	19.96	38.30	78.75	43.34	**92.25**	49.69	53.02	86.28	58.608	**96.18**	67.15
	60	23.77	77.84	24.09	**90.72**	34.62	49.79	91.61	53.10	**97.84**	65.92	64.39	95.14	67.732	**99.24**	81.01
	72	31.16	89.33	31.17	**96.46**	48.19	58.78	96.43	61.91	**99.52**	78.04	72.95	98.22	76.227	**99.90**	89.70
	84	39.60	95.43	38.95	**98.52**	58.89	68.10	98.71	69.36	**99.83**	87.69	81.64	99.29	82.554	**99.99**	94.82
	96	48.16	97.87	46.29	**99.63**	72.38	76.08	99.48	77.43	**99.98**	93.01	86.89	99.78	88.355	**100**	97.20
	108	55.80	99.21	53.11	**99.83**	81.47	82.24	99.81	83.33	**100**	96.50	91.01	99.97	92.423	**100**	98.90
	120	64.85	99.67	58.76	**99.94**	87.30	87.24	99.97	87.11	**100**	98.07	94.4	99.98	94.909	**100**	99.36

The bold number is the highest power among the five tests for each sample size.

Table 3.18: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group II with Design 6** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$U(0,1)$	4	**1.20**	**1.20**	1.13	1.02	**1.20**	**5.88**	**5.88**	5.68	**5.88**	**5.88**	11.49	11.49	**11.51**	11.49	11.49
	8	0.52	0.52	**2.49**	0.62	0.57	2.44	2.70	**8.88**	3.72	2.73	5.36	6.72	**15.59**	8.38	6.06
	12	0.22	0.41	**3.34**	1.04	0.24	2.52	3.40	**12.64**	5.29	1.84	6.79	7.41	**21.29**	11.31	4.66
	24	0.10	1.85	**6.11**	4.39	0.10	5.23	10.41	**19.46**	19.18	1.85	11.63	20.34	31.60	**32.25**	4.69
	36	0.28	8.36	8.44	**12.32**	0.30	10.88	29.46	26.66	**43.44**	3.06	20.06	46.03	41.07	**62.34**	8.67
	48	0.54	22.51	11.67	**28.70**	0.51	16.94	53.98	33.77	**70.02**	5.81	29.57	70.77	50.70	**84.38**	17.68
	60	0.73	40.64	14.86	**50.25**	0.88	22.73	76.6	41.91	**87.31**	11.26	39.91	87.93	60.19	**94.99**	34.04
	72	1.81	63.39	18.91	**74.44**	1.66	31.72	89.97	50.20	**95.56**	23.79	48.49	95.72	67.53	**98.66**	53.64
	84	3.88	79.47	23.97	**85.49**	4.64	41.23	96.31	58.48	**98.70**	40.20	58.75	98.74	75.55	**99.72**	69.88
	96	7.67	90.47	29.30	**94.45**	7.67	47.43	99.04	65.23	**99.75**	55.76	65.59	99.78	81.01	**99.96**	82.47
	108	17.70	96.13	35.04	**97.93**	17.40	55.77	99.75	71.67	**99.98**	72.39	73.05	99.97	85.95	**99.99**	90.59
	120	24.88	98.65	41.18	**99.34**	25.59	62.67	99.97	78.00	**100**	81.63	78.71	**100**	90.01	**100**	95.46
$beta(2,2)$	4	0.90	0.90	**1.01**	0.90	0.90	4.84	4.84	**4.85**	4.84	4.84	10.02	10.02	**10.15**	10.02	10.02
	8	0.49	0.49	**1.47**	0.51	0.50	2.92	2.97	**6.63**	3.41	2.66	6.19	6.79	**12.09**	7.14	6.13
	12	0.35	0.37	**1.45**	0.59	0.22	2.34	2.56	**7.02**	3.28	2.05	6.96	6.38	**13.21**	7.33	4.59
	24	0.65	0.57	**2.22**	1.13	0.11	3.14	3.44	**8.41**	5.98	1.44	7.60	7.56	**15.62**	11.77	3.83
	36	0.67	0.74	**2.48**	1.18	0.14	4.17	5.38	**10.05**	8.73	1.22	8.76	11.62	**17.38**	17.18	3.69
	48	0.97	1.80	**2.46**	2.45	0.11	5.57	8.11	11.10	**13.70**	1.55	11.23	16.23	19.91	**24.85**	5.04
	60	0.90	2.60	2.30	**3.72**	0.14	5.90	12.42	11.32	**19.31**	2.08	12.91	22.21	21.35	**34.28**	6.86
	72	1.33	4.24	3.05	**7.04**	0.29	7.53	17.77	13.66	**27.12**	3.08	15.39	30.15	24.16	**44.52**	10.03
	84	1.62	6.00	3.26	**8.22**	0.29	9.64	23.37	15.89	**34.47**	4.12	19.24	37.90	27.24	**53.32**	12.78
	96	2.10	8.56	3.33	**12.56**	0.43	10.13	30.87	16.73	**41.71**	5.71	20.02	45.89	29.13	**61.51**	18.81
	108	2.80	10.68	3.93	**15.67**	0.88	12.66	37.77	18.57	**52.22**	8.83	23.27	55.19	33.08	**70.86**	23.67
	120	2.83	14.81	4.30	**20.37**	0.71	14.42	45.45	20.83	**58.89**	11.38	25.70	62.71	34.98	**78.11**	30.24
$beta(2,5)$	4	1.18	1.18	**1.24**	1.18	1.18	5.12	5.12	**5.30**	5.12	5.12	10.41	10.41	**10.58**	10.41	10.41
	8	1.36	1.37	1.28	1.37	**1.44**	4.99	5.20	**6.33**	5.77	5.93	9.02	10.27	**12.99**	11.08	11.46
	12	1.45	1.45	1.60	**1.81**	1.64	6.92	6.73	7.60	**8.70**	7.85	11.84	13.13	14.64	**15.00**	13.98
	24	3.04	4.70	4.25	**7.80**	3.82	10.22	16.33	14.90	**22.88**	15.60	18.51	26.15	24.90	**35.15**	25.69
	36	4.80	13.21	6.83	**16.61**	9.19	15.87	33.34	20.95	**41.78**	26.74	26.58	45.92	32.50	**55.54**	39.63
	48	7.16	25.86	8.45	**30.21**	15.46	20.66	49.84	25.55	**60.41**	36.73	32.15	63.05	38.68	**73.76**	51.31
	60	8.95	38.91	9.94	**45.29**	21.26	26.56	67.59	31.62	**76.30**	47.60	39.98	78.62	46.00	**86.44**	62.58
	72	12.20	55.55	14.71	**63.27**	31.72	32.91	80.64	37.29	**87.34**	59.10	47.49	88.76	51.60	**93.94**	72.98
	84	16.25	68.22	16.89	**73.37**	38.80	39.02	88.84	42.66	**93.30**	68.67	53.60	94.16	57.08	**97.10**	80.32
	96	20.10	79.28	19.06	**83.63**	47.69	43.03	94.07	47.49	**96.36**	75.14	58.18	96.95	63.21	**98.59**	85.86
	108	25.02	86.71	22.72	**90.14**	57.45	49.52	97.35	52.33	**98.69**	81.75	64.45	98.90	68.31	**99.66**	90.46
	120	27.46	92.60	26.02	**94.74**	63.58	54.95	98.72	57.34	**99.41**	86.46	69.25	99.50	72.66	**99.87**	93.07
$beta(5,1.5)$	4	**0.95**	**0.95**	0.62	**0.95**	**0.95**	**5.21**	**5.21**	4.95	**5.21**	**5.21**	**10.86**	**10.86**	10.54	**10.86**	**10.86**
	8	1.69	1.73	1.73	1.80	**2.03**	6.34	6.76	8.06	**8.42**	8.40	11.11	13.98	**15.80**	15.51	15.18
	12	2.59	2.79	2.91	**4.72**	3.51	5.66	11.58	11.69	**15.81**	13.42	16.55	19.49	20.73	**25.18**	21.74
	24	6.07	15.14	7.93	**23.38**	10.62	17.01	36.39	24.44	**48.02**	30.14	28.08	49.45	37.24	**61.14**	43.46
	36	10.45	40.99	14.89	**48.19**	23.94	28.57	67.33	36.74	**76.53**	49.99	41.67	78.04	50.85	**85.55**	63.45
	48	17.13	68.27	20.59	**73.14**	39.39	40.45	86.14	46.60	**91.45**	65.77	53.89	91.87	60.99	**95.89**	78.04
	60	24.05	85.50	25.32	**89.14**	51.93	49.27	96.15	55.58	**97.79**	78.10	64.05	97.89	70.21	**99.18**	88.17
	72	31.87	94.87	38.30	**96.85**	68.84	60.17	99.02	67.56	**99.54**	87.99	74.04	99.59	80.04	**99.93**	94.23
	84	39.59	98.46	42.45	**98.96**	77.37	68.30	99.80	72.61	**99.92**	93.60	80.49	99.94	84.40	**99.97**	97.29
	96	47.09	99.38	46.88	**99.66**	85.17	74.63	99.96	78.72	**100**	96.39	85.65	**100**	88.68	**100**	98.80
	108	56.27	99.93	54.18	**99.96**	91.72	80.54	**100**	83.37	**100**	98.36	90.09	**100**	92.42	**100**	99.43
	120	61.08	99.98	60.34	**100**	94.32	85.42	**100**	87.98	**100**	99	92.68	**100**	94.96	**100**	99.68
$beta(0.5,0.5)$	4	2.35	2.35	**2.38**	2.35	2.35	**10.14**	**10.14**	9.84	**10.14**	**10.14**	**18.03**	**18.03**	18.02	**18.03**	**18.03**
	8	0.94	0.99	**6.10**	2.28	1.56	3.49	5.34	**18.29**	9.58	4.68	8.62	13.54	**29.47**	17.65	8.77
	12	1.45	2.55	**10.63**	7.22	0.75	5.45	13.73	**29.34**	22.42	4.18	12.72	21.34	**43.54**	34.36	8.44
	24	6.00	26.41	27.93	**40.45**	0.69	23.17	56.61	55.26	**74.80**	5.41	36.06	73.89	69.70	**87.14**	12.93
	36	17.78	71.07	46.27	**79.82**	1.66	47.91	93.94	76.16	**97.74**	13.75	64.00	97.78	86.99	**99.31**	35.42
	48	35.05	95.69	61.72	**97.49**	4.55	68.21	99.57	88.01	**99.86**	38.30	81.81	99.86	94.34	**99.96**	74.80
	60	51.41	99.63	73.45	**99.81**	11.57	82.10	99.97	94.52	**99.99**	72.81	92.35	99.99	98.10	**99.99**	94.36
	72	67.07	99.99	87.13	**99.99**	37.89	91.32	**100**	97.90	**100**	92.84	96.77	**100**	99.31	**100**	98.97
	84	80.53	**100**	92.23	**100**	64.06	96.76	**100**	99.32	**100**	98.76	99.16	**100**	99.88	**100**	99.89
	96	89.41	**100**	96.44	**100**	86.10	98.74	**100**	99.84	**100**	99.67	99.7	**100**	99.98	**100**	99.99
	108	94.62	**100**	98.43	**100**	96.59	99.47	**100**	99.98	**100**	99.97	99.92	**100**	100	**100**	99.99
	120	96.87	**100**	99.34	**100**	98.81	99.77	**100**	99.97	**100**	99.99	99.98	**100**	99.99	**100**	**100**
$beta(0.5,3)$	4	2.77	2.77	**4.45**	2.77	2.77	11.38	11.38	**12.17**	11.38	11.38	19.30	19.30	**19.90**	19.30	19.30
	8	7.21	8.18	8.24	**15.99**	12.29	17.36	26.51	25.01	**36.51**	26.18	28.09	42.85	40.48	**49.16**	37.83
	12	14.43	33.57	16.73	**51.93**	21.88	36.68	63.47	44.05	**73.14**	47.56	48.70	71.58	59.68	**81.27**	60.16
	24	50.90	96.24	59.09	**97.98**	66.29	77.54	98.85	82.99	**99.52**	88.13	87.00	99.41	90.95	**99.83**	93.97
	36	84.41	99.97	83.91	**99.98**	92.76	96.43	99.99	96.10	**100**	98.54	98.35	**100**	98.22	**100**	99.47
	48	97.17	100	94.61	**100**	98.97	99.62	**100**	99.16	**100**	99.90	99.92	**100**	99.77	**100**	99.97
	60	99.57	100	97.72	**100**	99.89	99.98	**100**	99.81	**100**	**100**	**100**	**100**	99.95	**100**	**100**
	72	99.96	100	99.47	**100**	99.99	99.99	**100**	99.98	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	84	100	100	99.92	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	**100**	**100**	**100**	**100**
	96	100	100	99.99	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	**100**	**100**	**100**	**100**
	108	100	100	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**	100	**100**	**100**	**100**	**100**
	120	100	100	100	**100**	**100**	**100**	**100**	**100**	v100	**100**	**100**	**100**	**100**	**100**	**100**
$beta(1,2)$	4	1.07	1.07	**1.36**	1.07	1.07	5.54	5.54	**5.60**	5.54	5.54	11.28	11.28	**11.59**	11.28	11.28
	8	1.32	1.32	**2.34**	1.62	1.66	4.48	5.13	**9.67**	6.98	6.12	8.85	11.16	16.81	**13.38**	10.85
	12	1.41	1.65	3.65	**3.74**	1.60	7.51	8.99	13.38	**13.81**	8.05	12.59	16.22	22.91	**22.67**	14.63
	24	4.06	15.18	9.51	**24.10**	4.43	13.20	36.04	25.62	**50.17**	17.97	22.98	49.69	38.02	**63.69**	29.75
	36	7.56	45.79	15.16	**53.28**	11.09	23.89	71.00	35.96	**80.76**	32.66	36.80	81.34	50.20	**88.96**	47.78
	48	14.37	75.97	21.84	**80.24**	21.35	35.95	91.20	47.30	**95.12**	49.59	50.39	95.16	61.23	**97.72**	66.91
	60	20.20	91.41	26.98	**93.99**	32.43	46.22	98.12	57.01	**99.03**	65.63	62.42	99.00	72.01	**99.66**	80.86
	72	27.27	97.96	35.69	**98.85**	49.05	57.22	99.69	65.25	**99.89**	78.17	72.86	99.89	78.60	**99.97**	89.72
	84	37.77	99.50	42.24	**99.67**	60.56	67.90	99.94	73.14	**99.97**	88.09	81.17	99.97	84.66	**99.99**	94.81
	96	45.63	99.93	48.75	**99.97**	71.83	74.02	**100**	78.75	**100**	92.61	86.31	**100**	89.57	**100**	97.24
	108	55.42	99.98	56.23	**100**	82.31	81.02	**100**	84.16	**100**	96.38	90.73	**100**	93.10	**100**	98.87
	120	60.75	**100**	61.63	**100**	87.74	86.31	**100**	88.28	**100**	98.16	93.71	**100**	95.55	**100**	99.54

The bold number is the highest power among the five tests for each sample size.

Table 3.19: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group III with Design 1** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$Laplace(0,1)$	4	1.07	1.07	1.07	1.07	**1.27**	4.55	4.55	4.55	4.55	**6.10**	9.48	9.06	9.35	9.06	**11.80**
	8	2.08	2.25	1.18	2.29	**3.50**	8.38	9.47	5.63	8.71	**11.24**	15.71	16.53	11.22	16.11	**18.14**
	12	4.35	5.15	1.85	4.00	**6.29**	12.91	14.07	8.40	12.74	**17.26**	21.05	22.07	14.84	20.07	**25.82**
	24	8.48	9.30	7.93	4.38	**13.80**	21.58	22.28	18.92	14.49	**29.50**	31.69	32.08	27.90	24.51	**40.14**
	36	15.07	13.39	15.39	3.49	**22.44**	31.56	30.73	29.83	14.82	**39.64**	43.69	41.20	39.96	26.90	**51.31**
	48	18.66	15.06	20.57	2.67	**28.88**	38.54	35.15	38.02	13.43	**49.18**	51.44	47.54	49.22	26.03	**60.05**
	60	25.86	18.64	26.92	2.51	**37.90**	47.30	40.37	46.48	12.70	**57.26**	60.41	53.33	58.31	26.04	**67.60**
	72	31.08	20.77	32.33	1.62	**42.68**	54.33	43.68	54.56	11.36	**64.31**	67.08	58.81	66.47	26.15	**74.04**
	84	38.24	22.59	38.82	1.44	**47.82**	62.19	48.44	61.04	10.87	**70.35**	73.28	63.00	72.44	27.39	**80.08**
	96	42.05	23.77	41.58	1.00	**55.21**	66.84	51.45	66.23	10.26	**75.28**	77.43	67.00	76.89	27.42	**83.75**
	108	47.86	27.07	46.88	0.87	**60.39**	72.81	58.21	72.15	10.43	**80.80**	82.10	71.28	82.06	30.10	**88.42**
	120	52.78	30.41	53.89	0.98	**65.45**	76.53	60.34	75.94	10.94	**83.67**	85.94	74.54	84.67	31.46	**90.11**
$t(1)$	4	0.66	0.66	0.66	0.66	**3.70**	3.14	3.14	3.14	3.14	**13.42**	10.75	6.45	9.21	6.45	**22.66**
	8	16.70	19.69	2.11	19.70	**29.51**	32.23	36.53	16.44	36.55	**42.38**	42.26	46.29	27.15	45.77	**49.96**
	12	39.62	44.43	25.61	40.02	**47.48**	54.77	57.91	45.87	54.51	**61.18**	62.85	65.52	54.77	62.71	**68.90**
	24	77.06	79.09	73.97	67.55	**82.47**	86.26	87.52	84.20	81.01	**90.29**	90.35	90.95	88.54	87.13	**93.16**
	36	92.25	91.91	91.53	80.52	**95.14**	96.02	96.01	95.94	91.45	**97.70**	97.53	97.50	97.35	95.22	**98.57**
	48	97.26	97.07	96.69	86.89	**98.41**	98.96	98.82	98.72	95.78	**99.39**	99.48	99.31	99.41	98.07	**99.68**
	60	99.23	98.91	98.85	92.52	**99.67**	99.76	99.64	99.72	98.00	**99.89**	99.88	99.81	99.82	99.16	**99.96**
	72	99.81	99.59	99.68	95.18	**99.93**	99.99	99.97	99.96	98.98	**98.98**	99.99	99.99	99.98	99.68	**100**
	84	99.94	99.84	99.88	97.57	**99.93**	99.97	99.96	99.95	99.57	**99.98**	99.97	99.97	99.97	99.87	**100**
	96	99.99	99.92	99.97	98.30	**100**	**100**	**100**	**100**	99.81	**100**	**100**	**100**	**100**	99.94	**100**
	108	**100**	99.96	99.98	99.16	**100**	**100**	**100**	**100**	99.87	**100**	**100**	**100**	**100**	99.95	**100**
	120	**100**	**100**	**100**	99.49	**100**	**100**	**100**	**100**	99.95	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	4	**1.01**	**1.01**	**1.01**	**1.01**	0.96	4.60	4.60	4.60	4.60	**5.41**	8.81	8.70	8.72	8.70	**11.95**
	8	2.10	2.60	1.11	2.42	**5.50**	8.85	10.39	5.33	9.80	**13.85**	16.28	17.76	11.22	17.33	**20.76**
	12	6.06	7.62	2.40	6.48	**10.88**	15.37	17.30	11.13	16.14	**22.37**	23.62	25.26	18.69	24.09	**30.39**
	24	14.62	17.28	17.36	11.74	**24.59**	27.12	30.24	27.83	24.06	**38.91**	36.19	39.19	36.34	33.89	**47.83**
	36	22.34	24.55	29.44	13.22	**36.35**	38.05	40.48	42.16	27.53	**51.49**	48.10	50.19	49.86	38.97	**60.54**
	48	27.59	29.24	38.88	14.35	**46.11**	44.88	46.79	52.38	29.96	**61.45**	55.42	56.84	60.46	41.81	**70.51**
	60	35.59	35.17	47.52	14.90	**55.95**	53.06	53.85	61.01	31.14	**70.02**	63.11	63.69	68.81	45.02	**76.86**
	72	40.97	38.89	53.96	14.32	**61.65**	59.04	59.14	68.34	32.55	**75.24**	69.51	68.78	75.62	46.99	**81.60**
	84	48.60	43.61	62.91	15.52	**67.97**	66.52	64.16	75.15	34.64	**81.81**	75.15	73.86	81.50	50.84	**87.36**
	96	53.82	48.35	68.12	15.60	**75.39**	72.20	69.72	80.46	36.78	**86.26**	79.86	79.13	86.19	54.28	**90.68**
	108	57.38	51.06	71.72	16.68	**78.85**	75.61	72.75	83.41	38.28	**89.09**	82.89	81.08	88.39	56.91	**92.93**
	120	62.64	54.98	77.01	17.58	**83.12**	79.44	75.61	87.76	40.94	**91.86**	86.02	84.37	91.68	59.58	**94.71**
$t(4)$	4	0.90	0.90	0.90	0.90	**1.19**	4.56	4.56	4.56	4.56	**5.49**	9.35	8.91	9.23	8.91	**11.27**
	8	1.60	1.88	1.13	1.70	**3.36**	7.40	8.36	5.22	7.81	**10.51**	14.19	15.11	10.52	15.15	**17.38**
	12	3.91	5.01	1.65	4.18	**7.10**	11.67	13.08	8.12	12.43	**16.91**	18.80	20.39	14.77	19.50	**24.91**
	24	7.59	9.88	10.06	6.69	**15.43**	17.40	20.28	19.18	15.94	**28.08**	25.97	28.90	26.26	24.71	**37.22**
	36	11.10	13.01	18.31	6.54	**22.57**	23.30	25.62	28.55	17.24	**36.46**	32.59	35.32	36.76	26.98	**46.46**
	48	13.85	15.57	25.08	6.76	**29.25**	28.36	31.04	37.45	18.22	**45.32**	39.52	41.51	45.73	28.60	**55.37**
	60	18.10	18.20	30.54	7.01	**36.08**	33.76	35.05	44.42	17.92	**52.61**	44.83	45.69	52.81	28.72	**61.40**
	72	21.20	20.69	36.68	6.50	**40.76**	38.29	39.15	51.26	18.64	**58.11**	50.20	50.24	60.08	30.23	**67.26**
	84	25.67	22.83	43.37	6.05	**45.76**	44.15	42.83	57.93	17.57	**64.16**	55.32	55.01	66.29	31.31	**73.03**
	96	27.88	24.30	46.61	5.78	**51.94**	48.49	45.44	62.46	17.95	**68.60**	58.85	58.90	70.48	32.40	**76.46**
	108	31.01	26.20	50.76	5.99	**55.56**	51.24	48.47	66.03	18.69	**72.86**	63.05	60.33	74.04	33.05	**80.37**
	120	34.13	28.27	56.25	5.59	**60.52**	55.96	51.55	70.41	19.44	**76.38**	67.12	64.85	77.63	34.20	**82.85**
$t(6)$	4	0.91	0.91	0.91	0.91	**1.19**	5.23	5.23	5.23	5.23	**5.57**	10.39	9.98	10.10	9.98	**10.90**
	8	1.28	1.43	0.97	**1.26**	2.14	6.38	6.89	5.19	6.75	**7.83**	12.47	12.97	10.25	12.67	**14.50**
	12	2.08	2.61	1.04	2.29	**3.53**	8.06	8.88	5.99	8.03	**10.82**	14.67	15.30	11.91	14.42	**17.72**
	24	3.46	5.02	4.76	3.49	**7.75**	11.34	13.03	11.89	10.32	**17.73**	18.54	20.39	18.66	17.71	**26.42**
	36	5.13	6.11	8.72	3.22	**11.09**	13.19	15.19	16.92	10.03	**22.74**	21.46	23.85	24.07	17.41	**31.50**
	48	5.58	6.37	12.43	2.69	**14.55**	15.49	17.46	22.25	9.84	**27.33**	23.77	26.61	30.26	17.76	**37.39**
	60	6.47	6.91	15.23	2.67	**17.24**	17.47	18.26	26.08	8.41	**31.42**	26.63	27.80	34.39	16.08	**41.18**
	72	7.57	7.38	18.35	2.23	**19.61**	18.97	20.31	30.56	8.24	**34.86**	29.14	30.21	38.80	16.12	**44.78**
	84	9.18	7.85	22.36	2.11	**20.99**	22.77	21.87	34.69	8.20	**38.62**	32.44	32.75	43.85	16.50	**49.26**
	96	9.49	7.75	24.59	1.72	**24.74**	23.95	22.21	38.79	7.18	**42.27**	33.72	34.12	47.85	15.44	**53.25**
	108	10.28	7.89	**27.21**	1.58	26.41	25.74	23.78	42.12	6.53	**45.01**	36.93	34.71	50.89	14.98	**56.08**
	120	11.77	8.78	**31.90**	1.43	30.21	27.54	25.09	46.29	6.67	**49.18**	39.59	37.42	55.23	15.02	**59.42**

The bold number is the highest power among the five tests for each sample size.

Table 3.20: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group III with Design 2** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$Laplace(0,1)$	6	2.08	2.03	1.25	1.97	**2.26**	7.16	7.21	5.80	6.98	**7.46**	12.67	12.76	11.10	12.52	**13.37**
	9	2.92	3.33	1.62	3.25	**3.54**	9.66	11.00	6.60	10.26	**12.50**	17.23	17.57	12.69	17.42	**20.16**
	12	4.05	5.27	1.87	4.24	**6.22**	13.62	15.02	8.40	12.52	**16.78**	21.61	22.97	15.07	20.33	**25.61**
	24	8.31	9.16	7.99	4.23	**14.21**	22.63	22.50	19.69	13.79	**29.49**	33.02	32.81	28.72	24.08	**39.87**
	36	14.32	13.70	16.15	3.59	**22.07**	31.59	30.04	30.31	14.17	**40.18**	43.83	42.23	40.49	26.50	**51.38**
	48	18.86	15.41	21.03	2.56	**28.73**	39.54	35.25	38.41	13.46	**49.59**	52.65	48.33	50.36	26.76	**60.47**
	60	25.64	18.58	27.01	2.38	**36.04**	47.40	39.83	46.37	12.04	**57.26**	59.73	52.95	58.03	26.33	**68.31**
	72	30.69	19.72	32.78	1.45	**44.52**	54.77	44.54	54.77	11.62	**64.80**	67.29	58.41	66.32	26.45	**74.24**
	84	36.85	20.29	38.25	1.53	**49.80**	61.06	49.99	61.66	11.31	**70.58**	73.09	63.74	73.06	28.13	**80.65**
	96	41.22	24.61	45.15	1.04	**56.74**	66.30	52.31	66.74	9.94	**75.80**	78.09	66.75	77.51	27.55	**84.03**
	108	47.79	27.76	46.86	1.02	**61.03**	72.65	58.29	71.76	11.15	**80.61**	82.87	71.96	82.05	29.60	**88.21**
	120	53.38	28.20	53.54	0.95	**64.99**	76.28	61.19	76.09	10.66	**83.45**	85.08	75.38	84.87	30.06	**90.20**
$t(1)$	6	8.68	8.34	6.33	8.40	**8.94**	14.91	15.11	12.42	15.13	**15.89**	20.81	20.81	17.45	20.50	**20.93**
	9	18.59	21.75	10.97	18.49	**23.23**	35.07	37.78	19.36	37.54	**44.71**	45.17	48.64	30.12	47.93	**53.97**
	12	35.64	42.06	18.37	38.33	**46.74**	53.11	57.64	43.99	53.57	**60.54**	62.51	64.98	53.89	62.07	**68.53**
	24	76.96	78.40	73.96	66.62	**82.81**	86.40	87.32	84.40	80.72	**90.18**	90.41	91.12	88.55	86.86	**93.04**
	36	91.58	91.81	91.10	79.58	**94.46**	96.05	95.95	95.44	90.53	**97.34**	97.50	97.36	96.97	94.46	**98.36**
	48	97.01	97.19	96.82	87.43	**98.49**	98.97	98.79	98.77	95.82	**99.40**	99.48	99.31	99.34	98.07	**99.62**
	60	99.15	98.75	98.78	92.29	**99.60**	99.68	99.66	99.74	97.81	**99.89**	99.84	99.83	99.85	99.11	**99.97**
	72	99.79	99.66	99.69	94.95	**99.93**	99.93	99.87	99.99	99.06	**99.90**	99.96	99.91	99.99	99.64	**100**
	84	99.92	99.85	99.90	97.49	**99.94**	99.98	99.96	99.95	99.55	**99.97**	99.99	**99.98**	99.97	99.86	**99.98**
	96	99.98	99.94	99.98	98.14	**100**	**100**	**100**	**100**	99.76	**100**	**100**	**100**	**100**	99.95	**100**
	108	99.99	99.96	99.97	99.10	**100**	**100**	**100**	**100**	99.89	**100**	**100**	**100**	**100**	99.95	**100**
	120	**100**	**100**	**100**	99.39	**100**	**100**	**100**	**100**	99.97	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	6	2.36	2.33	1.39	2.20	**2.44**	7.58	7.49	6.21	**7.64**	**7.64**	12.64	12.81	11.31	12.70	**12.81**
	9	3.81	4.50	2.09	4.42	**4.97**	10.98	12.10	7.35	11.72	**15.43**	18.38	19.46	13.15	19.24	**23.26**
	12	5.68	7.79	2.92	6.63	**10.65**	15.44	17.46	10.87	12.14	**21.48**	23.68	25.32	18.22	24.13	**29.74**
	24	14.09	17.32	17.63	11.24	**25.02**	27.47	30.02	28.74	15.95	**39.06**	36.82	39.18	36.71	33.90	**48.06**
	36	21.23	24.94	31.07	13.15	**37.11**	36.85	39.88	43.13	23.94	**53.07**	47.65	50.42	51.13	39.12	**61.63**
	48	28.39	30.21	39.37	14.85	**46.02**	45.25	47.64	52.88	27.77	**62.13**	56.6	57.41	61.22	42.58	**70.77**
	60	34.32	34.24	47.37	13.94	**54.41**	52.97	53.45	61.17	30.32	**69.68**	63.25	63.34	69.02	44.53	**76.89**
	72	41.43	38.31	54.04	14.22	**63.15**	59.95	58.87	68.47	30.5	**75.86**	68.92	68.83	75.71	47.57	**82.03**
	84	47.27	41.81	62.66	15.34	**69.41**	65.50	62.65	75.59	32.55	**82.24**	74.89	73.30	81.65	51.89	**87.73**
	96	53.57	48.64	69.6	15.28	**75.98**	72.17	69.10	80.96	35.54	**86.44**	80.45	78.68	86.35	54.44	**90.64**
	108	57.31	51.67	71.67	17.12	**79.30**	75.70	72.66	82.96	37.11	**89.15**	83.54	81.24	88.42	55.85	**92.69**
	120	62.47	53.57	77.38	17.35	**83.27**	79.62	75.82	87.52	38.87	**91.72**	85.76	84.24	91.58	58.71	**94.93**
$t(4)$	6	1.67	1.67	1.15	1.65	**1.74**	6.11	6.22	5.55	6.48	**6.58**	11.73	11.74	10.69	11.59	**12.16**
	9	2.64	2.76	1.44	2.77	**3.42**	8.71	9.13	6.01	9.31	**11.77**	15.33	15.94	11.39	15.94	**18.65**
	12	3.35	4.77	1.93	4.33	**6.94**	10.81	12.98	8.03	11.82	**16.28**	18.60	20.22	14.21	19.15	**24.26**
	24	7.58	9.54	10.08	5.98	**15.74**	18.12	20.28	19.32	15.54	**28.20**	27.05	28.64	26.79	24.58	**36.65**
	36	11.14	13.21	18.18	6.37	**21.69**	24.04	25.36	28.69	16.41	**36.43**	32.99	35.55	36.68	26.19	**46.40**
	48	14.46	16.71	25.05	7.41	**29.14**	29.17	31.29	37.76	18.79	**45.57**	40.31	41.78	46.46	29.34	**55.53**
	60	18.02	17.80	31.53	6.52	**35.29**	34.81	35.19	44.94	17.75	**52.93**	45.24	46.16	53.63	29.06	**61.86**
	72	20.52	19.52	37.14	6.48	**42.06**	37.85	38.22	51.37	18.38	**58.73**	49.19	49.10	59.86	30.6	**67.63**
	84	24.00	20.77	42.97	6.10	**47.37**	42.90	41.48	58.45	18.52	**64.47**	54.92	53.41	66.58	32.00	**73.42**
	96	28.12	24.41	48.68	5.37	**52.94**	47.79	45.65	62.8	18.34	**69.01**	59.13	58.33	70.67	32.32	**76.61**
	108	30.09	26.84	50.70	6.29	**56.25**	51.21	48.62	65.9	19.06	**72.82**	63.29	60.93	74.14	32.79	**80.15**
	120	33.39	27.17	56.10	5.56	**60.92**	54.28	51.28	70.19	19.58	**76.15**	66.75	64.63	77.58	33.70	**83.18**
$t(6)$	6	1.38	1.35	1.03	1.47	**1.51**	5.80	5.73	5.21	5.65	**6.14**	10.87	11.21	9.91	11.06	**11.53**
	9	1.83	2.13	1.11	2.38	**2.60**	7.22	7.66	6.00	7.74	**9.03**	13.94	14.22	11.71	13.74	**15.66**
	12	2.19	2.85	1.22	2.45	**3.44**	8.31	9.05	6.12	8.47	**10.23**	14.43	15.01	12.06	14.38	**17.28**
	24	3.36	4.85	4.82	3.33	**7.58**	11.12	13.12	12.36	10.12	**18.08**	18.93	20.67	18.84	17.62	**26.18**
	36	4.42	6.14	9.20	3.20	**11.09**	13.42	15.59	17.20	9.62	**23.17**	21.70	24.23	24.57	17.69	**32.26**
	48	5.75	6.62	12.35	2.75	**14.36**	16.62	17.82	22.62	10.53	**28.20**	24.85	26.99	30.74	18.55	**37.66**
	60	6.57	7.12	16.36	2.55	**17.27**	17.37	18.26	27.01	9.04	**32.14**	26.48	27.70	35.35	16.90	**41.64**
	72	7.15	7.19	18.49	2.08	**20.60**	18.26	19.86	30.56	8.54	**35.13**	29.62	29.12	39.09	16.75	**45.20**
	84	8.49	7.21	22.12	2.15	**22.08**	21.32	20.05	35.20	8.46	**38.83**	31.66	31.15	43.74	16.57	**49.99**
	96	9.83	7.85	26.07	1.59	**25.50**	23.93	22.03	39.19	7.23	**43.10**	34.62	33.89	48.18	15.65	**53.51**
	108	9.76	8.43	27.35	1.66	**27.14**	24.98	23.88	41.57	6.87	**44.73**	37.42	35.52	50.79	14.93	**55.56**
	120	10.88	8.06	31.66	1.41	**30.50**	26.40	24.89	45.88	6.55	**48.78**	38.15	37.30	55.09	14.99	**59.42**

The bold number is the highest power among the five tests for each sample size.

Table 3.21: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group III with Design 3** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$Laplace(0,1)$	4	1.08	1.08	1.07	0.91	**1.23**	5.55	5.55	5.54	5.14	**5.74**	11.33	10.94	**11.35**	10.33	11.07
	8	2.76	3.06	1.32	2.97	**3.62**	8.85	9.45	6.25	9.43	**10.23**	15.06	15.72	12.08	15.62	**17.24**
	12	4.37	5.30	1.83	4.64	**6.58**	13.50	15.08	8.85	12.88	**17.05**	21.60	22.93	15.62	21.09	**26.08**
	24	8.48	9.89	8.49	4.44	**14.32**	21.41	22.45	19.41	14.06	**29.27**	31.94	32.66	28.38	23.52	**39.83**
	36	14.34	13.48	15.28	4.19	**22.88**	32.16	30.34	30.79	14.69	**40.58**	43.00	42.03	40.33	26.60	**51.94**
	48	19.31	15.84	21.47	2.88	**28.58**	39.55	34.59	38.49	13.48	**50.21**	52.21	47.55	50.21	26.69	**60.95**
	60	25.45	18.22	26.90	2.48	**37.13**	47.09	39.65	46.71	13.25	**57.27**	60.12	52.13	58.45	27.38	**68.30**
	72	31.35	21.16	32.49	1.70	**42.59**	54.16	44.28	54.83	11.54	**64.22**	66.65	58.68	66.11	27.60	**74.35**
	84	38.10	22.59	38.77	1.38	**51.32**	61.85	50.10	61.39	11.52	**70.98**	73.55	63.51	72.73	28.46	**79.93**
	96	42.09	24.22	42.94	1.06	**54.13**	66.07	52.77	66.43	10.43	**75.63**	76.97	66.90	77.43	26.96	**83.80**
	108	49.22	27.41	49.50	0.81	**61.78**	72.37	58.05	72.76	10.83	**80.68**	82.33	72.48	82.13	29.25	**88.33**
	120	53.76	29.53	51.96	0.78	**66.43**	76.42	60.33	75.64	10.49	**83.27**	85.29	74.82	84.70	30.95	**90.14**
$t(1)$	4	1.25	1.25	1.24	0.79	**2.58**	6.58	6.58	6.54	4.57	**9.10**	14.62	12.94	14.60	9.90	**15.53**
	8	16.01	16.86	9.37	16.57	**18.01**	25.34	26.68	18.97	**25.78**	24.48	33.38	35.90	25.94	34.41	**45.31**
	12	36.14	41.19	20.77	38.36	**46.82**	53.28	56.76	44.05	52.98	**61.68**	62.15	64.73	54.76	62.61	**68.64**
	24	74.69	77.20	73.11	65.38	**81.56**	85.25	86.30	83.19	79.10	**89.21**	89.56	90.09	87.76	85.73	**92.47**
	36	92.24	92.35	91.52	81.08	**95.24**	96.52	96.36	96.22	91.48	**97.87**	97.76	97.73	97.33	95.14	**98.64**
	48	97.05	96.73	96.62	87.57	**98.23**	98.73	98.48	98.56	95.44	**99.24**	99.11	99.13	99.24	97.62	**99.56**
	60	99.05	98.64	98.84	92.32	**99.59**	99.76	99.69	99.74	98.00	**99.92**	99.86	99.82	99.89	99.10	**99.99**
	72	99.79	99.61	99.65	95.15	**99.91**	99.96	99.94	99.96	99.08	**99.99**	99.98	**99.99**	**99.99**	99.71	**99.99**
	84	99.95	99.87	99.89	97.53	**99.94**	99.98	99.97	99.97	99.58	**99.98**	99.98	99.98	99.98	99.88	**100**
	96	99.99	99.97	99.98	98.15	**100**	**100**	**100**	**100**	99.77	**100**	**100**	**100**	**100**	99.97	**100**
	108	**100**	99.94	99.97	98.82	99.99	**100**	99.99	**100**	99.85	**100**	**100**	**100**	**100**	99.94	**100**
	120	**100**	99.99	**100**	99.39	**100**	**100**	**100**	v 100	99.96	**100**	**100**	**100**	**100**	99.99	**100**
$t(3)$	4	1.13	1.13	1.14	0.86	**1.24**	5.40	5.40	5.39	4.72	**5.61**	10.83	10.74	10.79	10.23	**11.18**
	8	3.37	3.74	1.68	3.76	**4.82**	9.91	10.59	6.87	10.51	**11.67**	16.02	16.73	12.63	17.01	**19.59**
	12	6.38	7.71	3.36	7.16	**10.61**	15.37	17.79	11.35	16.34	**21.89**	23.74	25.59	18.85	24.29	**30.24**
	24	13.58	17.26	17.58	11.79	**24.98**	25.99	29.87	28.40	23.49	**38.69**	35.72	39.46	36.15	33.08	**47.74**
	36	21.89	23.94	29.52	13.68	**36.34**	37.44	39.42	42.75	27.13	**52.34**	47.16	49.49	50.37	38.65	**60.84**
	48	28.65	30.82	39.63	14.74	**46.88**	45.98	47.26	53.07	30.84	**62.88**	56.70	57.65	61.32	43.36	**70.62**
	60	34.27	34.02	47.01	14.11	**54.33**	52.84	52.21	61.35	31.57	**69.66**	63.04	63.24	69.19	45.47	**76.77**
	72	40.62	39.00	54.16	14.75	**62.58**	59.55	58.69	68.25	32.81	**75.33**	69.29	69.05	75.89	48.26	**81.66**
	84	48.39	44.15	62.79	15.43	**68.36**	66.38	64.25	75.57	35.54	**82.18**	75.29	74.37	81.63	52.27	**87.43**
	96	54.10	48.42	68.70	15.43	**75.14**	71.55	69.67	80.95	37.31	**86.43**	79.60	78.95	86.04	54.16	**90.48**
	108	58.26	50.67	73.52	15.88	**79.52**	75.81	71.88	84.51	38.13	**89.69**	83.58	81.98	89.05	55.95	**93.16**
	120	63.16	54.67	76.55	17.26	**83.39**	79.50	76.22	87.33	39.93	**91.69**	85.58	84.75	91.53	59.32	**94.95**
$t(4)$	4	1.08	1.08	1.08	**1.09**	1.03	**5.74**	**5.74**	5.71	5.27	5.60	11.09	11.03	11.11	10.86	**11.36**
	8	2.56	2.78	1.27	2.78	**3.65**	8.06	8.81	5.90	8.78	**9.55**	13.99	14.48	11.55	14.97	**16.17**
	12	3.55	4.60	1.76	4.25	**6.33**	11.27	13.11	8.26	12.02	**16.20**	18.71	20.01	14.67	19.24	**24.19**
	24	7.23	10.17	10.63	6.60	**15.26**	18.53	20.99	19.82	15.85	**28.52**	27.38	30.16	27.60	24.25	**37.02**
	36	10.93	13.53	18.41	7.36	**22.76**	23.90	26.05	28.98	17.39	**37.48**	32.31	35.83	36.40	27.22	**46.79**
	48	14.24	16.09	25.12	7.14	**28.86**	29.39	30.61	37.36	18.21	**45.80**	39.57	41.07	46.19	29.00	**54.62**
	60	17.15	17.49	31.22	6.33	**34.83**	34.23	34.36	44.72	18.20	**52.46**	44.82	46.33	53.57	29.98	**61.78**
	72	21.34	21.27	37.40	6.81	**41.91**	38.58	38.58	51.41	18.79	**58.04**	49.58	50.43	59.41	31.52	**67.21**
	84	25.67	22.61	43.59	6.12	**46.34**	44.18	42.64	58.21	18.65	**64.75**	55.27	55.24	66.63	32.20	**72.79**
	96	28.21	24.37	47.32	5.74	**51.59**	47.48	45.56	62.20	18.31	**69.26**	58.78	58.23	70.02	32.05	**76.42**
	108	30.87	25.87	51.86	6.03	**55.80**	50.87	48.00	66.86	18.05	**72.26**	62.49	60.82	74.41	32.51	**80.25**
	120	34.86	28.26	55.65	5.57	**61.20**	55.90	52.35	70.10	19.03	**75.90**	66.63	65.21	77.81	34.58	**83.06**
$t(6)$	4	**1.13**	**1.13**	1.11	0.92	0.90	**5.57**	**5.57**	**5.57**	4.80	5.05	10.77	10.51	**10.79**	9.88	10.53
	8	1.62	1.79	1.02	1.86	**2.29**	6.90	7.20	5.26	7.40	**7.90**	12.33	12.86	10.63	12.83	**13.93**
	12	2.57	2.98	1.34	2.80	**3.99**	8.53	9.32	6.27	9.06	**11.42**	14.93	16.02	12.50	15.42	**18.78**
	24	3.30	5.00	5.21	3.48	**7.69**	10.87	13.01	12.33	10.44	**18.01**	18.33	20.94	19.35	17.20	**26.31**
	36	4.48	5.65	8.54	3.11	**11.05**	13.82	14.84	17.70	9.98	**23.27**	21.15	23.49	25.54	17.44	**31.91**
	48	5.73	7.07	12.77	3.15	**14.34**	16.03	17.42	22.80	10.59	**28.95**	24.87	26.74	31.18	18.93	**38.02**
	60	6.79	6.88	16.27	2.60	**16.83**	17.51	18.07	26.90	9.49	**32.17**	26.79	27.97	35.36	17.65	**41.69**
	72	7.56	7.51	18.56	2.34	**20.39**	19.02	19.29	30.94	8.82	**35.04**	28.56	29.78	38.94	16.97	**44.89**
	84	9.18	7.98	22.37	2.17	**21.51**	21.90	21.38	34.91	8.42	**38.81**	32.13	32.21	44.08	16.76	**49.43**
	96	9.60	7.68	25.24	1.79	**24.35**	23.10	22.12	38.99	7.46	**43.00**	34.43	33.69	47.94	15.34	**52.91**
	108	10.05	7.99	28.43	1.57	**27.25**	25.05	22.55	41.90	6.62	**45.53**	36.69	35.10	50.86	14.16	**56.32**
	120	12.10	8.51	30.90	1.46	**30.81**	27.46	25.24	45.57	6.32	**48.45**	38.55	37.45	54.94	15.19	**59.52**

The bold number is the highest power among the five tests for each sample size.

Table 3.22: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group III with Design 4** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$Laplace(0,1)$	4	1.08	1.08	1.07	0.91	**2.30**	5.55	5.55	5.54	5.14	**5.74**	11.33	10.94	**11.35**	10.33	11.07
	8	2.45	3.06	1.23	2.85	**4.41**	8.93	9.68	6.26	9.15	**10.76**	15.44	16.10	11.66	15.87	**17.97**
	12	4.09	5.37	1.80	4.49	**6.59**	12.98	14.88	8.64	13.28	**16.72**	21.38	22.83	15.81	20.68	**25.56**
	24	8.67	9.45	7.93	4.13	**14.64**	22.15	22.51	19.08	13.77	**29.42**	33.16	32.34	27.95	24.07	**40.28**
	36	15.27	13.47	15.44	3.77	**22.64**	31.71	30.87	30.92	14.49	**40.74**	43.48	42.16	40.85	26.87	**52.36**
	48	20.13	15.34	22.34	2.91	**30.16**	39.85	35.48	39.26	13.40	**49.54**	52.01	48.02	50.41	26.58	**60.57**
	60	25.61	18.22	27.26	2.32	**35.24**	47.64	39.86	47.27	12.31	**57.19**	60.00	53.25	59.39	26.40	**67.82**
	72	31.55	19.64	32.06	1.67	**43.11**	54.55	44.36	54.51	11.88	**64.27**	67.28	58.51	65.70	26.85	**74.06**
	84	36.81	22.41	37.54	1.44	**50.83**	61.05	49.20	60.84	11.17	**70.34**	73.12	63.33	72.45	27.76	**79.76**
	96	41.72	23.44	42.62	1.15	**55.47**	67.26	52.60	66.34	10.67	**75.65**	77.52	66.71	77.15	27.43	**83.78**
	108	48.27	27.55	47.21	0.87	**60.55**	73.43	58.12	72.13	11.12	**80.44**	89.92	72.03	82.18	29.14	**88.07**
	120	53.23	29.75	52.59	0.83	**65.70**	75.95	61.27	76.27	10.17	**83.33**	85.25	74.93	84.69	29.91	**90.03**
$t(1)$	4	1.25	1.25	1.24	0.79	**8.85**	6.58	6.58	6.54	4.57	**9.10**	14.62	12.94	14.60	9.90	**15.53**
	8	15.45	16.65	8.32	16.30	**26.37**	28.59	30.12	19.07	27.79	**34.16**	37.24	39.20	26.70	38.20	**43.14**
	12	34.40	38.57	22.31	34.86	**45.30**	50.51	54.80	41.04	52.55	**60.83**	60.68	63.54	52.96	61.25	**68.41**
	24	75.37	77.01	72.46	64.29	**81.69**	85.59	86.30	83.24	79.01	**89.45**	89.89	90.21	87.70	85.55	**92.59**
	36	91.99	91.73	90.75	79.52	**94.56**	95.87	95.79	95.43	90.85	**97.45**	97.42	97.35	97.20	94.63	**98.38**
	48	97.26	96.60	96.79	86.50	**98.34**	98.66	98.48	98.65	95.44	**99.23**	99.11	99.10	99.26	97.55	**99.56**
	60	99.16	98.98	98.81	92.36	**99.59**	99.80	99.72	99.73	97.80	**99.93**	99.90	99.86	99.89	99.23	**100**
	72	99.78	99.64	99.67	95.01	**99.90**	99.95	99.92	99.96	99.02	**99.99**	99.98	99.97	**99.99**	99.64	**99.99**
	84	99.92	99.88	99.91	97.62	**99.94**	99.98	99.95	**100**	99.58	**99.97**	99.98	99.98	99.98	99.88	**100**
	96	99.98	99.96	99.97	98.24	**100**	**100**	**100**	**100**	99.74	**100**	**100**	**100**	**100**	99.95	**100**
	108	**100**	99.95	99.97	99.08	**100**	**100**	**100**	**100**	99.87	**100**	**100**	**100**	**100**	99.96	**100**
	120	**100**	**100**	**100**	99.32	**100**	**100**	**100**	**100**	99.97	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	4	1.13	1.13	1.14	0.86	**2.37**	5.40	5.40	5.39	4.72	**5.61**	10.83	10.74	10.79	10.23	**11.18**
	8	3.17	3.56	1.72	3.79	**5.87**	10.25	10.59	6.86	10.46	**12.80**	16.27	17.31	12.51	17.24	**19.83**
	12	6.00	7.32	3.02	6.86	**10.37**	15.02	16.94	11.05	16.36	**21.73**	23.36	24.96	18.75	23.73	**29.88**
	24	14.37	17.19	16.91	11.05	**25.32**	27.19	30.20	28.15	23.28	**39.03**	36.98	39.47	36.12	33.18	**47.62**
	36	23.17	24.97	30.34	13.62	**37.34**	37.83	40.68	43.21	28.07	**53.51**	48.70	50.66	51.29	39.73	**61.52**
	48	29.76	30.45	40.39	14.34	**47.82**	46.73	47.04	53.48	30.98	**62.22**	56.76	57.63	61.34	42.46	**70.75**
	60	35.40	35.12	47.29	13.94	**53.82**	53.73	53.48	61.57	31.00	**69.61**	62.95	63.72	69.73	45.19	**76.93**
	72	40.95	38.93	53.48	14.53	**62.00**	59.33	58.41	68.17	32.96	**75.18**	69.28	68.58	75.12	47.07	**82.15**
	84	47.07	43.93	61.91	15.27	**69.88**	65.71	64.16	74.71	34.73	**82.04**	75.09	73.78	81.21	51.72	**87.54**
	96	53.73	48.11	68.45	15.84	**75.51**	72.06	69.00	80.85	37.72	**86.25**	80.10	78.56	86.27	54.20	**90.42**
	108	57.44	51.00	71.87	16.46	**79.06**	75.97	72.51	83.39	38.22	**89.09**	83.37	80.93	88.55	55.71	**92.77**
	120	61.98	53.74	76.94	17.21	**83.03**	79.05	75.69	87.77	39.57	**91.78**	86.33	84.15	91.30	57.27	**94.55**
$t(4)$	4	1.08	1.08	1.08	1.09	**1.83**	**5.74**	**5.74**	5.71	5.27	5.60	11.09	11.03	11.11	10.86	**11.36**
	8	2.15	2.50	1.36	2.54	**3.94**	8.54	8.60	6.08	8.40	**10.19**	14.58	15.23	11.57	15.01	**17.19**
	12	3.34	4.17	1.62	3.85	**6.18**	11.04	12.48	7.86	11.99	**15.84**	18.29	19.70	14.65	19.19	**23.24**
	24	7.98	10.11	10.24	6.05	**15.81**	18.58	20.94	19.53	15.91	**28.55**	27.91	29.62	27.11	24.30	**37.43**
	36	11.63	13.04	17.93	6.54	**22.03**	23.11	25.45	28.79	16.63	**37.01**	32.94	35.54	36.67	26.72	**46.10**
	48	15.20	15.48	25.36	6.51	**29.58**	29.13	30.15	37.67	18.17	**45.31**	39.68	41.31	46.27	27.72	**54.74**
	60	18.27	18.28	31.64	6.87	**34.47**	33.91	35.07	45.33	17.82	**52.80**	44.91	46.17	54.01	29.25	**61.52**
	72	21.37	20.82	36.63	6.72	**40.96**	38.29	38.17	50.96	18.92	**57.79**	49.57	49.71	59.17	30.54	**67.84**
	84	24.19	22.59	42.73	6.27	**48.03**	43.57	42.45	57.35	17.96	**64.44**	54.74	54.58	66.25	31.52	**72.88**
	96	27.92	24.31	46.99	5.69	**52.17**	47.89	45.34	62.31	18.41	**68.97**	58.68	57.48	69.78	32.13	**76.24**
	108	31.17	26.70	50.89	5.90	**55.67**	52.04	48.81	66.40	18.90	**72.85**	63.29	61.04	74.32	32.57	**80.04**
	120	34.36	27.39	55.90	5.48	**60.49**	55.55	51.53	70.57	18.71	**76.13**	66.90	64.45	77.89	32.84	**83.32**
$t(6)$	4	1.13	1.13	1.11	0.92	**1.14**	**5.57**	**5.57**	**5.57**	4.80	5.05	**10.77**	10.51	10.79	9.88	10.53
	8	1.33	1.43	1.10	1.46	**2.58**	6.63	6.66	5.28	6.64	**7.61**	11.89	12.63	10.34	12.78	**13.97**
	12	2.37	2.89	1.15	2.44	**3.88**	8.48	9.15	6.31	8.83	**11.26**	14.68	15.75	12.20	15.47	**18.25**
	24	3.46	4.79	4.84	3.22	**7.97**	11.43	13.27	12.13	10.27	**18.11**	19.25	20.80	18.90	17.68	**26.52**
	36	5.15	5.99	8.98	3.08	**11.55**	13.89	15.41	17.40	9.88	**23.46**	22.12	23.87	24.80	17.83	**32.05**
	48	6.07	6.76	13.04	2.92	**14.77**	15.96	17.27	22.82	10.57	**28.50**	24.74	26.25	30.80	18.73	**38.01**
	60	6.55	6.98	16.60	2.52	**16.69**	17.66	18.06	27.31	9.40	**32.25**	26.92	27.46	35.75	16.99	**41.32**
	72	7.40	7.47	18.04	2.19	**19.63**	19.03	19.12	30.41	8.92	**34.79**	28.83	29.61	38.24	16.72	**45.68**
	84	8.27	7.90	21.81	2.20	**22.68**	21.54	21.10	34.08	8.06	**38.40**	31.89	31.89	43.34	16.59	**49.35**
	96	9.25	7.57	25.04	1.76	**24.74**	23.44	21.58	38.71	7.44	**42.65**	34.47	33.19	47.76	15.27	**52.57**
	108	10.52	8.12	27.28	1.48	**26.62**	26.07	23.46	41.95	6.71	**44.55**	37.42	35.28	51.15	14.66	**55.61**
	120	11.14	8.18	31.33	1.30	**30.20**	26.49	23.87	46.28	6.28	**48.57**	38.39	35.84	55.00	14.91	**59.14**

The bold number is the highest power among the five tests for each sample size.

Table 3.23: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group III with Design 5** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$Laplace(0,1)$	4	1.04	1.04	0.99	**1.19**	1.13	**5.63**	**5.63**	5.03	5.44	5.48	11.39	**11.48**	10.99	10.86	11.12
	8	2.32	2.70	1.07	2.65	**3.42**	9.21	10.02	6.42	9.39	**11.62**	16.24	16.93	12.47	16.25	**18.74**
	12	4.57	5.90	1.72	4.62	**6.86**	13.69	15.93	8.61	13.67	**17.48**	22.17	23.54	15.62	22.19	**26.67**
	24	10.12	10.23	8.93	4.93	**15.01**	24.47	25.06	20.70	15.14	**31.03**	35.33	34.53	29.42	25.52	**42.23**
	36	15.49	15.23	16.10	4.16	**23.54**	34.23	32.15	31.53	15.13	**41.95**	46.01	43.87	41.94	27.08	**53.42**
	48	20.94	17.14	22.41	3.45	**30.94**	41.83	37.26	40.37	14.34	**51.08**	56.64	49.09	51.21	27.49	**61.73**
	60	28.05	20.11	27.36	2.63	**38.00**	49.75	42.14	47.99	13.26	**57.99**	61.94	55.16	59.83	28.60	**69.28**
	72	33.43	21.08	33.68	1.70	**45.23**	56.18	46.12	55.50	12.72	**65.45**	68.35	59.98	67.05	29.31	**75.66**
	84	37.98	25.20	38.01	1.74	**48.77**	62.72	50.84	61.03	11.81	**70.55**	74.54	64.56	72.30	29.09	**80.49**
	96	44.57	25.28	43.72	1.32	**56.28**	68.13	54.20	66.68	11.38	**76.54**	79.20	68.33	77.72	29.03	**84.34**
	108	50.35	29.89	49.71	0.97	**62.38**	74.02	59.39	72.46	12.30	**81.59**	83.90	72.99	82.26	31.55	**88.44**
	120	55.45	29.39	51.88	0.73	**66.61**	78.15	60.81	75.64	11.12	**84.04**	86.76	75.20	85.07	31.21	**90.86**
$t(1)$	4	1.13	2.44	**2.62**	2.40	2.42	5.98	9.02	8.42	9.10	**9.27**	14.01	**15.43**	14.93	14.56	15.38
	8	16.20	22.60	4.16	23.35	**28.24**	34.65	40.10	17.71	38.66	**41.83**	44.68	48.39	33.16	45.62	**48.43**
	12	44.12	45.79	28.96	41.14	**46.58**	58.59	60.41	48.24	56.26	**60.37**	66.84	67.83	58.26	65.14	**66.71**
	24	79.23	80.81	74.92	69.05	**83.27**	88.29	88.42	84.76	81.95	**90.44**	91.93	91.95	88.88	86.96	**92.84**
	36	93.12	93.42	91.38	82.68	**95.37**	96.68	96.87	96.02	92.28	**97.75**	97.88	98.11	97.33	95.13	**98.53**
	48	97.63	97.52	97.15	88.99	**98.44**	99.08	98.78	98.76	95.98	**99.30**	99.48	99.28	99.40	98.04	**99.58**
	60	99.42	99.06	99.03	93.13	**99.61**	99.82	99.74	99.73	98.23	**99.94**	99.9	99.86	99.90	99.26	**99.97**
	72	99.80	99.67	99.74	95.55	**99.92**	99.95	99.95	99.94	99.06	**99.99**	99.96	**99.99**	**99.99**	99.73	**99.99**
	84	99.95	99.89	99.90	97.68	**99.94**	99.97	99.96	**99.97**	99.67	**99.97**	99.99	99.98	**100**	99.90	99.99
	96	99.98	99.96	99.96	98.77	**99.99**	**99.99**	**99.99**	**99.99**	99.81	**99.99**	**100**	99.99	**100**	99.96	**100**
	108	**100**	99.96	99.97	98.94	**100**	**100**	99.99	**100**	99.9	**100**	**100**	**100**	**100**	99.97	**100**
	120	**100**	99.99	**100**	99.48	**100**	**100**	**100**	**100**	99.97	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	4	0.98	1.03	0.97	**1.19**	1.13	5.54	**5.87**	5.05	5.76	5.76	10.47	**11.35**	10.17	10.93	10.41
	8	2.54	3.29	1.08	3.59	**5.46**	10.94	11.63	6.14	11.39	**14.20**	17.17	18.83	12.60	17.58	**21.00**
	12	7.12	8.21	3.05	7.14	**11.64**	16.66	19.21	11.41	17.53	**22.28**	24.83	26.88	19.02	25.78	**29.99**
	24	16.01	19.00	17.54	12.86	**25.79**	29.32	32.06	28.58	25.03	**39.93**	38.51	41.21	36.68	34.46	**48.38**
	36	23.53	25.68	30.67	14.79	**37.46**	39.53	41.35	43.18	28.65	**52.28**	49.86	50.81	51.68	40.25	**62.70**
	48	30.44	33.58	40.32	15.73	**48.59**	47.94	49.74	53.32	32.21	**63.78**	58.51	59.65	61.27	43.56	**71.51**
	60	35.92	36.29	46.98	15.08	**56.00**	54.97	54.23	61.11	32.80	**70.85**	63.56	65.07	68.71	46.04	**77.27**
	72	41.72	39.92	54.36	15.40	**63.45**	62.51	60.18	68.16	34.08	**76.31**	69.52	70.35	75.84	49.64	**82.57**
	84	48.24	44.79	62.30	15.99	**69.69**	68.04	64.87	75.17	36.27	**82.36**	75.81	74.88	80.99	52.62	**87.34**
	96	55.82	48.77	67.95	17.62	**75.43**	72.53	71.09	79.87	38.51	**86.82**	80.74	79.72	85.50	55.04	**90.73**
	108	59.33	52.8	73.41	16.62	**79.34**	76.92	73.54	83.99	39.89	**89.81**	84.01	82.29	88.65	58.21	**93.50**
	120	63.43	57.11	75.02	18.05	**83.79**	80.56	77.18	86.40	43.05	**92.24**	86.7	85.48	90.63	58.33	**94.56**
$t(4)$	4	0.93	1.06	0.87	1.16	**1.21**	5.33	**5.35**	4.65	5.21	5.28	10.68	10.49	10.28	10.67	**10.81**
	8	1.82	2.36	1.21	2.70	**3.69**	8.59	9.31	5.71	9.44	**11.33**	14.26	16.12	11.30	15.44	**17.82**
	12	4.17	5.30	1.60	4.65	**7.25**	12.23	13.92	7.98	12.74	**16.69**	18.89	21.02	14.89	19.91	**23.47**
	24	8.83	10.64	10.83	6.90	**16.56**	19.61	22.19	19.92	17.18	**28.89**	28.66	30.69	27.18	25.13	**38.07**
	36	11.44	14.46	18.10	7.76	**23.85**	25.63	27.61	29.14	18.09	**37.04**	34.8	36.78	37.64	28.35	**47.51**
	48	15.12	18.38	25.13	7.57	**30.99**	30.73	32.67	37.36	18.57	**46.19**	40.84	42.77	45.88	29.02	**55.70**
	60	18.50	19.31	31.87	6.84	**37.10**	35.62	36.72	44.88	18.66	**53.79**	45.44	48.07	53.85	30.57	**62.70**
	72	22.24	21.58	37.53	6.95	**42.59**	40.81	40.24	51.40	19.59	**59.33**	50.28	51.78	59.86	32.65	**67.83**
	84	24.95	23.03	42.92	6.05	**47.49**	45.44	43.73	57.30	18.86	**64.86**	55.49	55.92	65.92	32.38	**73.24**
	96	28.90	24.70	47.38	6.88	**51.81**	48.27	46.33	61.62	19.00	**68.88**	58.85	58.45	69.57	33.00	**76.35**
	108	31.52	26.84	51.83	5.97	**55.54**	52.99	49.03	66.39	18.86	**72.48**	63.08	61.33	74.10	33.94	**80.47**
	120	35.49	30.16	54.70	5.94	**62.29**	56.57	53.74	70.60	20.62	**77.29**	67.77	65.99	77.87	33.75	**83.41**
$t(6)$	4	1.02	0.98	0.92	**1.04**	0.98	5.27	5.20	5.13	5.22	**5.37**	10.67	10.35	10.13	**11.11**	10.79
	8	1.21	1.62	0.98	1.66	**2.02**	6.89	7.37	5.64	7.40	**8.17**	12.61	13.32	11.12	12.71	**14.08**
	12	2.79	2.82	1.34	2.45	**3.82**	8.52	9.34	6.28	8.99	**11.19**	16.05	15.88	13.17	15.99	**18.27**
	24	3.89	5.24	5.24	3.41	**8.23**	11.91	13.40	12.19	10.63	**18.37**	19.03	21.39	18.78	18.67	**26.81**
	36	5.18	6.36	8.87	3.28	**11.64**	14.39	16.06	17.98	10.54	**22.98**	22.35	24.32	24.87	18.43	**33.57**
	48	5.75	8.01	13.25	3.40	**15.72**	16.44	19.38	23.21	10.84	**29.42**	25.50	28.03	30.93	18.43	**38.80**
	60	7.32	7.65	16.18	2.58	**18.14**	18.27	19.04	26.69	9.53	**32.73**	26.79	29.14	35.14	17.62	**41.84**
	72	7.82	8.07	18.92	2.21	**21.00**	20.98	20.90	30.63	9.34	**36.04**	30.41	31.73	39.23	17.87	**45.46**
	84	8.62	8.15	21.93	2.07	**22.08**	22.80	21.98	34.37	8.49	**39.24**	32.30	32.95	43.01	16.85	**49.84**
	96	10.26	8.44	24.56	2.05	**25.43**	24.06	23.78	37.97	7.84	**43.00**	34.78	35.10	47.72	16.62	**52.75**
	108	10.59	8.40	27.93	1.63	**26.79**	26.20	23.67	41.40	6.85	**45.44**	36.76	35.95	50.35	15.30	**56.05**
	120	12.18	9.77	29.43	1.49	**31.94**	28.04	27.04	44.09	7.16	**50.28**	39.08	39.05	53.55	14.32	**59.29**

The bold number is the highest power among the five tests for each sample size.

Table 3.24: Empirical percentage of powers for the five tests when errors generated from the alternatives in **Group III with Design 6** for $\alpha = 0.01, 0.05$ and 0.1

		0.01					0.05					0.1				
Alternatives	n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n	D	D_{sp}	D_e	D_{bi}	T_n
$Laplace(0,1)$	4	1.06	1.06	**1.08**	1.06	1.06	**5.24**	**5.24**	5.16	**5.24**	**5.24**	10.48	10.48	**10.63**	10.48	10.48
	8	3.92	3.70	0.75	3.75	**4.13**	12.76	**13.19**	4.56	12.93	13.09	20.36	**21.65**	11.73	20.95	20.86
	12	6.97	6.56	0.84	6.53	**7.29**	15.70	18.45	6.39	16.76	**19.02**	27.53	28.01	13.42	25.31	**28.13**
	24	13.00	12.86	6.26	5.88	**15.72**	29.42	28.45	17.72	18.12	**33.12**	40.44	39.42	28.16	29.66	**44.03**
	36	19.11	16.75	13.69	4.32	**25.32**	39.29	35.62	29.17	17.46	**44.01**	51.64	48.16	40.35	30.85	**55.11**
	48	26.08	20.12	21.03	3.41	**33.10**	47.66	41.02	39.42	15.92	**53.04**	59.74	53.65	51.51	30.90	**64.10**
	60	31.00	22.59	25.87	2.81	**38.61**	54.18	45.88	46.70	15.58	**60.04**	67.47	59.39	59.11	32.08	**70.84**
	72	36.44	25.01	32.14	2.37	**46.67**	60.81	50.84	55.05	15.13	**67.15**	73.07	63.71	66.90	32.75	**77.10**
	84	43.32	27.03	36.46	1.65	**51.72**	67.90	53.63	61.40	14.26	**72.94**	78.36	68.32	73.09	32.24	**82.02**
	96	48.76	28.60	42.63	1.61	**57.26**	71.35	57.54	66.82	12.86	**77.41**	82.01	71.66	78.31	32.15	**85.56**
	108	55.02	33.22	48.01	1.17	**64.62**	76.97	61.69	72.41	14.40	**82.56**	85.79	75.57	82.63	34.41	**89.26**
	120	57.45	35.04	52.56	1.09	**68.35**	80.65	65.43	76.21	13.10	**85.02**	88.35	78.63	85.54	35.60	**91.55**
$t(1)$	4	**2.49**	**2.49**	2.34	**2.49**	**2.49**	**10.29**	**10.29**	10.19	**10.29**	**10.29**	**18.03**	**18.03**	18.01	**18.03**	**18.03**
	8	30.61	29.95	1.86	30.89	**31.76**	43.53	45.15	17.11	**45.51**	44.26	52.26	**54.17**	38.64	54.07	53.03
	12	50.06	52.47	10.62	49.51	**51.11**	62.85	64.95	44.88	62.99	**64.63**	72.65	**72.99**	57.02	69.84	71.86
	24	84.63	83.47	72.11	73.64	**84.63**	90.55	90.40	84.63	85.00	**91.85**	93.42	93.50	89.73	89.99	**94.46**
	36	95.29	93.47	90.37	82.84	**95.42**	97.32	97.04	95.70	92.90	**97.89**	98.36	98.27	97.21	95.99	**98.68**
	48	98.62	97.72	97.00	89.67	**98.55**	99.24	98.99	98.76	96.51	**99.40**	99.57	99.38	99.35	98.41	**99.67**
	60	99.63	99.12	98.99	93.80	**99.64**	99.86	99.77	99.73	98.53	**99.94**	99.94	99.91	99.87	99.46	**99.96**
	72	99.96	99.76	99.71	96.67	**99.96**	99.99	99.95	99.94	99.28	**100**	100	99.99	99.98	99.83	**100**
	84	99.96	99.91	99.89	97.98	**99.96**	99.99	99.98	99.96	99.76	99.98	100	99.98	**100**	99.92	**100**
	96	**99.99**	99.96	99.96	98.90	**99.99**	99.99	99.99	**100**	99.88	**100**	**100**	99.99	**100**	99.97	**100**
	108	**100**	99.98	99.98	99.10	**100**	**100**	99.99	**100**	99.94	**100**	**100**	**100**	**100**	99.98	**100**
	120	**100**	**100**	**100**	99.56	**100**	**100**	**100**	**100**	99.97	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	4	**0.99**	**0.99**	0.96	**0.99**	**0.99**	**5.32**	**5.32**	5.23	**5.32**	**5.32**	11.04	11.04	**11.09**	11.04	11.04
	8	5.93	5.61	0.65	5.49	**6.51**	14.70	15.00	4.58	14.86	**15.44**	21.18	22.88	12.84	22.37	**23.28**
	12	12.22	10.51	0.93	9.53	**12.95**	18.61	21.59	8.54	20.50	**23.97**	28.87	30.40	17.25	28.78	**32.34**
	24	25.98	20.70	15.16	14.23	**25.98**	32.03	34.05	26.70	27.43	**41.76**	41.54	43.36	35.69	37.30	**50.47**
	36	39.18	27.78	28.39	15.15	**39.66**	43.55	43.88	41.77	30.85	**55.43**	52.95	54.81	50.54	42.62	**64.04**
	48	49.63	34.73	39.60	15.77	**49.00**	52.04	51.53	52.99	33.41	**65.09**	62.05	61.18	61.27	45.76	**72.57**
	60	55.84	36.87	46.11	15.16	**56.51**	55.78	56.57	60.24	34.45	**70.90**	66.79	66.51	68.60	48.69	**78.48**
	72	64.37	41.45	53.44	17.18	**64.03**	63.10	62.15	67.85	36.27	**76.70**	72.71	71.63	75.49	52.10	**83.52**
	84	70.64	46.09	61.46	16.41	**71.45**	69.71	66.46	75.13	38.00	**83.17**	78.10	76.16	81.46	54.12	**88.05**
	96	**76.28**	50.54	67.68	17.93	**76.28**	74.35	72.26	79.88	39.72	**87.15**	82.40	80.50	85.81	57.18	**91.36**
	108	**80.94**	53.15	72.61	17.68	80.86	78.28	74.37	83.87	42.33	**90.53**	85.45	83.15	88.98	59.35	**93.87**
	120	83.85	57.02	76.33	18.71	**83.98**	81.51	78.17	87.49	42.91	**92.31**	87.30	85.92	91.76	61.98	**95.33**
$t(4)$	4	**0.92**	**0.92**	0.85	**0.92**	**0.92**	**4.85**	**4.85**	4.83	**4.85**	**4.85**	10.46	10.46	**10.56**	10.46	10.46
	8	3.86	3.88	0.76	3.54	**4.29**	11.26	11.35	4.42	11.26	**12.00**	17.95	19.01	11.44	18.06	**19.20**
	12	7.24	6.06	0.74	5.44	**7.95**	12.73	15.10	6.30	14.73	**17.32**	22.36	22.93	13.28	21.88	**25.10**
	24	**16.66**	11.72	8.42	7.92	**16.66**	22.26	24.23	18.31	18.55	**30.01**	31.00	33.47	26.60	28.26	**39.01**
	36	23.99	14.98	16.40	7.37	**24.50**	27.98	29.06	28.41	19.38	**39.27**	37.15	39.60	36.48	29.80	**48.42**
	48	**31.75**	18.88	24.71	7.32	31.13	33.68	33.65	37.38	19.72	**47.39**	44.18	44.30	46.13	30.75	**56.93**
	60	36.57	19.61	30.63	6.91	**37.34**	37.27	38.40	44.41	19.57	**54.18**	49.43	49.26	53.27	32.31	**63.77**
	72	**43.33**	22.19	36.61	7.86	42.91	42.08	41.71	51.15	20.80	**59.95**	53.76	53.61	59.73	33.78	**69.04**
	84	48.56	24.12	41.83	6.17	**49.68**	47.58	44.63	57.27	20.07	**65.92**	58.86	56.92	66.27	34.00	**74.29**
	96	52.92	26.30	46.88	6.91	**52.92**	50.13	47.64	61.69	19.67	**69.37**	61.14	59.46	69.75	34.21	**77.58**
	108	**57.52**	27.27	50.52	6.49	57.41	53.86	49.64	65.84	20.16	**73.63**	65.57	63.04	74.38	35.35	**81.14**
	120	62.10	29.72	55.18	6.36	**62.30**	58.42	54.57	70.66	20.30	**77.02**	68.93	66.57	77.48	36.29	**84.07**
$t(6)$	4	**0.86**	**0.86**	**0.86**	**0.86**	**0.86**	**4.86**	**4.86**	4.78	**4.86**	**4.86**	9.46	9.46	**9.53**	9.46	9.46
	8	2.33	2.41	0.91	2.07	**2.59**	8.19	8.31	4.65	8.09	**8.40**	14.27	15.09	10.44	14.69	**15.63**
	12	4.28	3.72	0.90	3.32	**4.69**	8.88	10.33	5.48	9.95	**11.99**	16.95	17.60	11.03	16.97	**18.66**
	24	8.42	5.80	3.96	3.92	**8.42**	12.85	14.72	10.95	11.93	**19.48**	20.90	22.76	17.99	19.81	**27.92**
	36	12.30	6.87	8.02	3.23	**12.59**	15.83	16.96	16.94	10.90	**24.59**	24.41	26.60	24.04	19.14	**33.75**
	48	16.40	8.42	12.94	3.15	**15.92**	18.46	19.83	22.91	11.21	**30.20**	27.67	28.76	30.96	19.69	**39.63**
	60	17.73	7.72	15.57	2.56	**18.25**	19.26	20.23	25.87	10.40	**33.14**	30.01	29.89	34.54	18.92	**43.24**
	72	**21.23**	8.33	18.04	2.71	21.00	21.51	21.99	30.21	9.69	**36.47**	32.39	32.46	38.67	18.88	**46.95**
	84	22.74	8.61	21.06	2.08	**23.73**	24.47	22.74	34.39	9.03	**40.16**	35.34	33.66	43.18	17.36	**50.91**
	96	**26.18**	9.11	24.35	2.05	**26.18**	25.66	24.79	37.98	8.03	**43.38**	36.86	36.03	47.44	17.29	**54.02**
	108	**28.73**	8.43	26.95	1.73	**28.66**	27.41	24.56	40.88	7.54	**46.88**	39.47	37.05	50.22	15.88	**57.14**
	120	31.44	9.24	30.77	1.56	**31.62**	29.39	27.47	46.28	6.90	**49.88**	41.29	39.29	55.14	16.59	**60.76**

The bold number is the highest power among the five tests for each sample size.

3.6.1 Results on the empirical power comparisons

Comparison of power against the asymmetric distributions with support $(0, \infty)$ **(Group I)**

The results for the alternative distributions from Group I are given in Tables 3.7–3.12. The first thing we notice from these tables is that in most situations, the performance of the D test is similar to that of the D_e test. The power of the D_{sp} test is similar to that of the D_{bi} and T_n tests. Also, the D_{sp}, D_{bi} and T_n tests have higher power than the D and D_e tests. For $\chi^2(10)$, the T_n test is the most powerful test whereas the D test is the least powerful one. However, the power of all tests for $\chi^2(10)$ seems to be lower in comparison to the power of other distributions in Group I. It is also notable that the D test is least powerful in almost all sample sizes, except for n = 8 and 12. For those two sample sizes, the D_e test shows the worst performance. In Group I, we cannot assess which test performs best across all combinations of n, α and alternatives because the power of the D_{sp}, D_{bi} and T_n tests are competitive and no test performs consistently higher than the others. The fact which is observed from Group I is that the power of all tests improves with the increase of the sample sizes. It is very interesting that there is a slight difference in power between the D_{bi} and T_n tests which are the graphical test and the non–graphical test, respectively.

Comparison of power against the distributions with their supports in $(0, 1)$ **(Group II)**

The results for the alternative distributions from Group II are given in Tables 3.13–3.18. First, it is obvious that the power study results for $beta(2, 2)$ are the most interesting because almost all tests gave low performance regardless of sample sizes considered. The power comparison in Group II results in a unified view which is that the D_{bi} test is the most powerful test, followed by the D_{sp} test in all circumstances. Similar to Group I, the power of the D test is similar to that of the D_e test for almost all considered distributions, except for $U(0, 1)$ and $beta(0.5, 0.5)$. It is remarkable that there is such a massive difference between the D and D_e tests in all sample sizes for $U(0, 1)$ and $beta(0.5, 0.5)$. The T_n test presents low power for $U(0, 1)$ over all situations. It is worth investigating the power of the D, D_{sp}, D_{bi} and T_n tests because we may observe the interesting results, in which these tests present decreasing powers while the sample sizes increase for $U(0,1)$, $beta(0.5,0.5)$ and $beta(2,2)$.

Comparison of power against the symmetric non-normal distributions with support $(\infty,-\infty)$ (Group III)

The results for the alternative distributions from Group III are given in Tables 3.19−3.24. It is also undeniable that the D_e test was least powerful for sample sizes $n \leq 12$ but the D_{bi} test has the worst performance for $n > 12$ in all designs. But the D_e test is a good choice when $n > 12$ because it gives higher power among all graphical tests in most situations. The results of the power comparison of the D, D_{sp} and D_e tests are similar when sample sizes $n \geq 24$. The highest power is achieved by the T_n test across all situations. However, the power of all tests for $t(6)$ seems to be lower in comparison to the power of other distributions in this group, in particular for the t distribution with higher degrees of freedom. For a Laplace distribution, the power of the D_{bi} test is extremely poor; even for larger sample sizes, the power still did not increase. Furthermore, the power of the D_{bi} test decreases as the sample size increases for $t(4)$ and $t(6)$ over all designs.

CHAPTER 4 Tests for the Weibull distribution

4.1 Background

The two most widely used distributions in reliability are the Weibull and exponential (see, Kimber (1985)). The use of the Weibull distribution in reliability–type studies began with the paper of Weibull published in 1939. Today, Weibull distribution is the most popular distribution in reliability and life studies. It has been used in numerous applications relating to reliability, failure time and aging studies. The literature on the statistical methods involving the Weibull distribution is considered.

For more than half a century, the Weibull distribution has attracted the attention of statisticians working on theory and methods as well as those involved in various fields of applied statistics. According to Rinne (2008), hundreds or perhaps even thousands of papers have been written on this distribution and the research is ongoing. Together with the normal, exponential, χ^2, t, and F distributions the Weibull distribution is, without any doubt, the most popular model in modern statistics. It is of utmost interest to theory–oriented statisticians because of its great number of special features and to practitioners because of its ability to fit to data from various fields, ranging from life data to weather data or observations made in economics and business administration, in hydrology, in biology or in the engineering sciences.

Kimber (1985) used the D_{sp} statistic developed by Michael (1983), the Kolmogorov-Smirnov statistic, and the Shapiro–Wilk statistic to test for the exponential, Gumbel, and Weibull distributions. In addition, he used the Downton's estimates, which are approximately best linear unbiased (BLUE), for unknown location and scale parameters based on the Gumbel distribution.

Littell et al. (1979) compared the powers of five tests for the two–parameter Weibull distribution; these are the Kolmogorov-Smirnov statistic, the Cramér-von Mises statistic, the Anderson-Darling statistic, Smith and Bain's statistic, and Fertig's statistic. The parameters of the first three tests listed above were replaced by maximum likelihood estimators (MLE). The authors advised that the Anderson-Darling and Cramér-von Mises statistics were reasonably good for all alternative distributions.

Stirling (1982) developed a concentration band following the idea of Quesenberry and Hales (1980) that $F(Y_{[k]})$ is a beta variate with parameters k and $n-k+1$ to compute a $(1-\alpha)$ confidence interval (A_i^α, B_i^α) for each $F(Y_{[k]})$. Similar to the $1-\alpha$ confidence interval for $Y_{[k]}, k=1,\ldots,n$, $(F^{-1}(A_i^\alpha), F^{-1}(B_i^\alpha))$ can be displayed on Q–Q plots. Due to their lack of advanced computing power, the author failed to illustrate that their concentration band was a $1-\alpha$ simultaneous confidence interval for the whole probability plot.

Coles (1989) investigated the D_{sp}^* statistic and the D_{sp} statistic for testing the extreme value distribution of minima with unknown parameters. The parameters were estimated by using Blom's procedure (1958), where he showed that due to improved estimation, the D_{sp}^* statistic has higher power than Kimber's (1985) proposed test statistic.

The study of Liao and Shimokawa (1999) introduced a new statistic, L_n, for testing the goodness-of-fit of type-I-extreme-value and two-parameter Weibull distributions with estimated parameters. The author used the maximum likelihood estimators (MLE) and graphical plotting techniques (GPT) to estimate the population parameters from a complete sample. The power comparisons reveal that the L_n statistic with GPT is the most powerful test in most situations.

Castro-Kuriss (2011) used the D_{sp} statistic to test the two–parameter exponential distribution with unknown parameters based on censored data, and gives an example by assessing whether all points from a sample fall inside the acceptance bands by using probability-probability (P–P) plots and stabilised probability (S–P) plots. Pirouzi-Fard and Holmquist (2013) studied testing the extreme value distribution based on six statistics. Three of the six statistics are the Anderson–Darling statistic, the statistic proposed by Liao and Shimokawa (1999) and the D_{sp} statistic by Michael (1983). The authors proposed the approximate expressions of the means and the variances of the order statistics of the standard extreme value distribution (see Pirouzi-Fard and Holmquist, 2007, 2008) and used their results to compute the best linear unbiased estimates (BLUE) of the unknown location and scale parameters.

Moreover, Estudilo-Martínez et al. (2013a, 2013b) reviewed the key features of the three main probability plots; Q–Q plots, P–P plots and S–P plots. Moreover, R script was constructed for the graph S–P plot to make the production of the plots easily.

Consequently, the simultaneous probability intervals of the order statistics in a sample can be easily obtained by replacing the normal distribution function with the distribution function of interest. Chapter 2 highlights the construction of simultaneous probability intervals based on the normal distribution. In this chapter, that study is extended to other distributions in a location-scale family. Although the Weibull distribution is not a member of a location-scale family, the log-transformation is used to change it into the smallest extreme value distribution which is one of such a family. This chapter also considers graphical tests for testing the Weibull distribution. We first review the literature on existing tests for the Weibull distribution and in particular, for the exponential distribution, which is a special feature of the Weibull. Then, both the graphical and non-graphical tests for the three–parameter Weibull distribution are investigated. Eventually, the performance of graphical tests for the two–parameter exponential distribution is examined in Chapter 5.

The use of the location–scale family of distributions is popular in applied statistics, thanks to the theoretical and computational simplifications it provides. Probability plots are therefore often used to assess the validity of such a distribution when its parameters are usually unspecified.

In probability theory, a location-scale family is a family of univariate probability distributions which offers useful interpretations for modeling as well as convenient mathematical properties. We start with its definition from Casella and Berger (2002).

Definition 4.1.1. Let $f(y)$ be any pdf. Then for any $-\infty < \mu < \infty$ and $\sigma > 0$, the family of pdfs $\frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$, indexed by the parameter (μ, σ), is called ***the location–scale family*** with standard pdf $f(y)$; μ is called *the location parameter* and σ is called *the scale parameter*.

Note that μ is not necessarily the mean and σ is not necessarily the standard deviation of Y which has pdf $\frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$, an exception being the normal distribution. The distributions of Y obtained when μ and σ taken on all values of their domains generate a location–scale family. The random variable $\dfrac{Y-\mu}{\sigma}$ has pdf $f(\cdot)$ which is free of any parameter. The model under consideration has unspecified parameters;

therefore, the parameter estimation must be considered before calculation.

Let X be a random variable with probability density function

$$f(x|a,b,c) = \frac{c}{b}\Big(\frac{x-a}{b}\Big)^{c-1}\exp\Big\{-\Big[\frac{x-a}{b}\Big]^c\Big\}, \quad x>a,\ b>0,\ c>0 \tag{4.1.1}$$

with the location parameter a, the scale parameter b, and the shape parameter c. The distribution of X is known as the Weibull distribution, $Wbl(a,b,c)$.

The cumulative distribution function of $Wbl(a,b,c)$ is given by

$$F(x|a,b,c) = 1-\exp\Big\{-\Big[\frac{x-a}{b}\Big]^c\Big\}, \quad x>a,\ b>0,\ c>0. \tag{4.1.2}$$

One recognises that the exponential distribution is a special case (with $c=1$) of the Weibull distribution.

Here, we set the transformation of $X \sim Wbl(a,b,c)$:

$$Y = \ln(X-a); \quad X \geq a.$$

As known from (4.1.2),

$$F_X(t|a,b,c) = P(X\leq t) = 1-\exp\Big\{-\Big[\frac{t-a}{b}\Big]^c\Big\},$$

and we obtain

$$\begin{aligned} F_Y(t) &= P(Y\leq t) \\ &= P\Big[\ln(X-a)\leq t\Big] \\ &= P\Big[X-a\leq e^t\Big] \\ &= P\Big[X\leq a+e^t\Big] \\ &= 1-\exp\left[-(e^t/b)^c\right] \\ &= 1-\exp[-\exp\{(t-\ln b)c\}] \\ &= 1-\exp\Big[-\exp\Big\{\Big(\frac{t-\ln b}{1/c}\Big)\Big\}\Big] \end{aligned}$$

with the location parameter

$$\mu = \ln b \tag{4.1.3}$$

and the scale parameter

$$\sigma = \frac{1}{c}. \tag{4.1.4}$$

In order to derive the additional properties of a log-Weibull distribution, we introduce the variable

$$Z = \frac{Y - \mu}{\sigma}$$

with

$$G_Z(z) = 1 - \exp(-\exp(z))$$

$$g_Z(z) = G_Z'(z) = \exp(z - \exp(z)).$$

Thus,

$$\begin{aligned} f(y) &= F'(y) \\ &= \frac{d}{dy} G\Big(\frac{y-\mu}{\sigma}\Big) \\ &= \frac{1}{\sigma} g\Big(\frac{y-\mu}{\sigma}\Big) \\ &= \frac{1}{\sigma} \exp\Big(\frac{y-\mu}{\sigma} - \exp\Big(\frac{y-\mu}{\sigma}\Big)\Big). \end{aligned}$$

Likewise, the log-Weibull distribution is the so-called **smallest extreme value (SEV) distribution**, which is defined by $Y = \ln(X - a)$ provided a is known. Its pdf is given by

$$f(y|\mu, \sigma) = \frac{1}{\sigma} \exp\Big(\frac{y-\mu}{\sigma} - \exp\Big(\frac{y-\mu}{\sigma}\Big)\Big), \quad -\infty < y < \infty, \tag{4.1.5}$$

where the location parameter $-\infty < \mu < \infty$ and the scale parameter $\sigma > 0$. Also, the cumulative distribution function is given by

$$F(y|\mu, \sigma) = 1 - \exp\Big(- \exp\Big(\frac{y-\mu}{\sigma}\Big)\Big), \quad -\infty < y < \infty. \tag{4.1.6}$$

In short, $Y \sim SEV(\mu, \sigma)$. The null hypothesis in this section is

$$H_0 : X_1, \ldots, X_n \text{ comes from } Wbl(a, b, c) \text{ where } a \text{ is known.}$$

The transformed set $\{X_1 - a, \ldots, X_n - a\}$ is equivalently generated from $Wbl(0, b, c)$ under H_0. For testing H_0, the smallest extreme value distribution, $SEV(\mu, \sigma)$ can be used instead of the Weibull distribution, $Wbl(0, b, c)$. A test for H_0 is made by testing whether $Y_k = \ln(X_k - a)$ for $k = 1, \ldots, n$ has the smallest extreme value distribution with the unknown parameters μ and σ.

Moreover, the **p^{th} quantile** of the smallest extreme value distribution of a random variable $\frac{Y-\mu}{\sigma}$ is given by $F^{-1}(p) = \ln(-\ln(1 - p))$ where $0 < p < 1$. Then, we obtain

$$\begin{aligned} Y &= \mu + \sigma F^{-1}(p) \\ &= \mu + \sigma \ln(-\ln(1-p)). \end{aligned} \tag{4.1.7}$$

A probability plot of $Y_1, \ldots, Y_n$ based on the smallest extreme value distribution consists of plotting $Y_{[k]}$ versus $\ln(-\ln(1-p_k))$ where $p_k = \frac{k-0.5}{n}$, $k = 1, \ldots, n$. Note that the natural log of the data is used on the horizontal axis rather than the data value directly for Weibull plotting. Probability plots in their most common form are used with the location–scale parameter models (Lawless, 1982).

To be emphasised, a Q–Q plot can be constructed as a plot of the ordered sample observations against the expected values of the standardised null distribution (scale parameter of one and location parameter of zero). If a given sample follows the null distribution, the plot will be close to a straight line.

Thus, the model is

$$Y_{[k]} = \mu + \sigma F^{-1}(p_k) \tag{4.1.8}$$

where $F^{-1}(p_k)$ is the quantile corresponding to the null distribution with $\mu = 0$ and $\sigma = 1$ and $Y_{[k]}{}'s, k = 1, \ldots, n$ are the ordered sample observations. A practitioner then has to decide whether the plotted points fall close to a straight line or not because the probability of a Type I error is not specified, for a detailed description, see Balakrishnan and Basu (1995).

Hazen (1930) suggested $p_k = \frac{k-0.5}{n}$ as an alternative. Taking plotting position $p_k = \frac{k}{n}$, this is unused since it is not possible to plot the largest observation on Q–Q plots. Note that $p_k = \frac{k-0.5}{n}$ is the mean of the interval $\frac{k-1}{n}$ to $\frac{k}{n}$ and is still very widely used. Barnett (1975) suggested that for most purposes whatever p_k is applied does not cause any problem. However, when the aim is to estimate μ and σ from the probability plot, the choice of p_k can make a difference.

If the parameter a of the original Weibull distribution is known, then the log–transformation brings about a distribution of the location–scale family which is easy to deal with. Therefore, the parameters b and c of $Wbl(a, b, c)$ are estimated using a log–transformation of the observed Weibull data.

4.2 Parameter estimation

The exact distribution of a test statistic for a composite hypothesis testing is very difficult to find, but Monte Carlo studies can be used as a good approximation for the null distribution. According to D'Agostino and Stephens (1986), as long as an appropriate method is applied to estimate the location and scale parameters, the distribution of the goodness-of-fit statistic does not depend on the true parameters.

Since both the location and scale parameters of $SEV(\mu,\sigma)$ are unknown, we have to estimate them. Three estimators are proposed: the maximum likelihood estimators (MLE), the best linear unbiased estimators (BLUE) and the best linear invariant estimators (BLIE). They become the choices for attaining the highest powers of the tests in the next part of the thesis. Moreover, all estimators must have the invariant property according to the following definition (see, Lawless (1982) p.147).

Definition 4.2.1. Under the location–scale transformations, an estimator $\hat{\mu}$ of the location parameter is called invariant if and only if

$$\hat{\mu}_{p+qZ} = p + q\hat{\mu}_Z \quad \text{for } -\infty < p < \infty,\ q > 0.$$

An estimator $\hat{\sigma}$ of the scale parameter is called invariant if and only if

$$\hat{\sigma}_{p+qZ} = q\hat{\sigma}_Z \quad \text{for } -\infty < p < \infty,\ q > 0.$$

4.2.1 Maximum likelihood estimators

Let $\mathbf{X} = (X_1,\ldots,X_n)$ with $X_k \overset{i.i.d.}{\sim} Wbl(a,b,c)$ for $k = 1,\ldots,n$ and $Y = \ln(X-a)$ with a specified parameter a. Then $Y \overset{i.i.d.}{\sim} SEV(\mu,\sigma)$. The corresponding likelihood function is

$$\begin{aligned} L(\mu,\sigma|Y_1,\ldots,Y_n) &= \prod_{k=1}^{n} f(Y_k|\mu,\sigma) \\ &= \frac{1}{\sigma^n}\exp\Big(\sum_{k=1}^{n}\frac{Y_k-\mu}{\sigma} - \sum_{k=1}^{n}\exp\Big(\frac{Y_k-\mu}{\sigma}\Big)\Big). \end{aligned} \tag{4.2.1}$$

Then, the log-likelihood function of the log-transformed Weibull distribution becomes

$$\begin{aligned} \mathcal{L}(\mu,\sigma|Y_1,\ldots,Y_n) &= \ln L(\mu,\sigma|Y_1,\ldots,Y_n) \\ &= -n\ln\sigma + \sum_{k=1}^{n}\Big(\frac{Y_k-\mu}{\sigma}\Big) - \sum_{k=1}^{n}\exp\Big(\frac{Y_k-\mu}{\sigma}\Big) \end{aligned}$$

with the following system of normal equations:

$$\frac{\partial\mathcal{L}(\mu,\sigma)}{\partial\mu} = -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{k=1}^{n}\exp\Big(\frac{Y_k-\mu}{\sigma}\Big) = 0 \tag{4.2.2}$$

$$\frac{\partial\mathcal{L}(\mu,\sigma)}{\partial\sigma} = -\frac{n}{\sigma} - \sum_{k=1}^{n}\Big(\frac{Y_k-\mu}{\sigma^2}\Big) + \sum_{k=1}^{n}\Big(\frac{Y_k-\mu}{\sigma^2}\Big)\exp\Big(\frac{Y_k-\mu}{\sigma}\Big) = 0. \tag{4.2.3}$$

With regards to the two parameters of the log-transformed Weibull distribution, we obtain from (4.2.2) that

$$\begin{aligned}
n &= \sum_{k=1}^{n} \exp\Big(\frac{Y_k-\mu}{\sigma}\Big) \\
n &= \exp\Big(-\frac{\mu}{\sigma}\Big)\sum_{k=1}^{n} \exp\Big(\frac{Y_k}{b}\Big) \\
n\exp\Big(\frac{\mu}{\sigma}\Big) &= \sum_{k=1}^{n} \exp\Big(\frac{Y_k}{\sigma}\Big) \\
\exp(\mu) &= \Big(\frac{1}{n}\sum_{k=1}^{n} \exp\Big(\frac{Y_k}{\sigma}\Big)\Big)^{\sigma} \\
\tilde{\mu} &= \tilde{\sigma}\ln\Big(\frac{1}{n}\sum_{k=1}^{n} \exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)\Big). \qquad (4.2.4)
\end{aligned}$$

Simplify (4.2.3) and substitute (4.2.4) into (4.2.3), and we have

$$\begin{aligned}
-n-\sum_{k=1}^{n}\frac{Y_k}{\sigma}+\frac{n\mu}{\sigma}+\frac{1}{\sigma}\exp\Big(-\frac{\mu}{\sigma}\Big)\Big[\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\sigma}\Big)-\mu\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\sigma}\Big)\Big] &= 0 \\
-\sum_{k=1}^{n}\frac{Y_k}{\sigma}+\frac{n\mu}{\sigma}+\frac{1}{\sigma}\exp\Big(-\frac{\mu}{\sigma}\Big)\Big[\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\sigma}\Big)-\mu\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\sigma}\Big)\Big] &= n \\
-\bar{Y}+\mu+\frac{1}{n}\exp\Big(-\frac{\mu}{\sigma}\Big)\Big[\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\sigma}\Big)-\mu\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\sigma}\Big)\Big] &= \sigma \\
-\bar{Y}+\mu+\frac{1}{\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\sigma}\Big)}\Big[\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\sigma}\Big)-\mu\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\sigma}\Big)\Big] &= \sigma \\
-\bar{Y}+\mu+\Big[\frac{\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\sigma}\Big)}{\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\sigma}\Big)}-\mu\Big] &= \sigma \\
-\bar{Y}+\frac{\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)}{\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)} &= \tilde{\sigma}.
\end{aligned}$$

Therefore, the maximum likelihood estimators (MLE) of μ and σ are

$$\tilde{\mu}_Y = \tilde{\sigma}_Y\ln\Big(\frac{1}{n}\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\tilde{\sigma}_Y}\Big)\Big), \qquad (4.2.5)$$

$$\tilde{\sigma}_Y = -\bar{Y}+\frac{\sum_{k=1}^{n} Y_k\exp\Big(\frac{Y_k}{\tilde{\sigma}_Y}\Big)}{\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\tilde{\sigma}_Y}\Big)}. \qquad (4.2.6)$$

Now, we want to show the invariant property of $\tilde{\mu}_Y$ and $\tilde{\sigma}_Y$ where $\tilde{\sigma}_Z$ is MLE of σ from $SEV(0,1)$ and $-\infty < p < \infty$, $q > 0$. Let $Y_k = p + qZ_k$. Consequently,

$$\begin{aligned}
-(p+q\bar{Z}) + \frac{\sum_{k=1}^{n}(p+qZ_k)\exp\left(\frac{p+qZ_k}{\tilde{\sigma}_{p+qZ}}\right)}{\sum_{k=1}^{n}\exp\left(\frac{p+qZ_k}{\tilde{\sigma}_{p+qZ}}\right)} &= \tilde{\sigma}_{p+qZ} \\
-q\bar{Z} + \frac{q\sum_{k=1}^{n} Z_k\exp\left(\frac{p+qZ_k}{\tilde{\sigma}_{p+qZ}}\right)}{\sum_{k=1}^{n}\exp\left(\frac{p+qZ_k}{\tilde{\sigma}_{p+qZ}}\right)} &= \tilde{\sigma}_{p+qZ} \\
-\bar{Z} + \frac{\sum_{k=1}^{n} Z_k\exp\left(\frac{qZ_k}{\tilde{\sigma}_{p+qZ_k}}\right)}{\sum_{k=1}^{n}\exp\left(\frac{qZ_k}{\tilde{\sigma}_{p+qZ}}\right)} &= \frac{\tilde{\sigma}_{p+qZ}}{q} \\
-\bar{Z} + \frac{\sum_{k=1}^{n} Z_k\exp\left(\frac{Z_k}{(\tilde{\sigma}_{p+qZ})/q}\right)}{\sum_{k=1}^{n}\exp\left(\frac{Z_k}{(\tilde{\sigma}_{p+qZ})/q}\right)} &= \frac{\tilde{\sigma}_{p+qZ}}{q}
\end{aligned}$$

Therefore, $\tilde{\sigma}_Z = \frac{\tilde{\sigma}_{p+qZ}}{q}$. Also, we need to show $\tilde{\mu}_{p+qZ} = p + q\tilde{\mu}_Z$:

$$\begin{aligned}
\tilde{\mu}_{p+qZ} &= \tilde{\sigma}_{p+qZ}\ln\Big(\frac{1}{n}\sum_{k=1}^{n}\exp\big(\frac{p+qZ_k}{\tilde{\sigma}_{p+qZ}}\big)\Big) \\
&= q\tilde{\sigma}_Z\ln\Big(\frac{1}{n}\sum_{k=1}^{n}\exp\Big(\frac{p+qZ_k}{q\tilde{\sigma}_Z}\Big)\Big) \\
&= q\tilde{\sigma}_Z\ln\Big(\frac{1}{n}\exp(\frac{p}{q\tilde{\sigma}_Z})\cdot\sum_{k=1}^{n}\exp(\frac{Z_k}{\tilde{\sigma}_Z})\Big) \\
&= q\tilde{\sigma}_Z\Big[\ln\big(\frac{1}{n}\exp(\frac{p}{q\tilde{\sigma}_Z})\big) + \ln\sum_{k=1}^{n}\exp(\frac{Z_k}{\tilde{\sigma}_Z})\Big] \\
&= p + q\tilde{\sigma}_Z\Big[\ln\frac{1}{n} + \ln\sum_{k=1}^{n}\exp(\frac{Z_k}{\tilde{\sigma}_Z})\Big] \\
&= p + q\tilde{\sigma}_Z\Big[\ln\big(\frac{1}{n}\sum_{k=1}^{n}\exp(\frac{Z_k}{\tilde{\sigma}_Z})\big)\Big] \\
&= p + q\tilde{\mu}_Z.
\end{aligned}$$

Since $\frac{Y_k-\mu}{\sigma} \sim SEV(0,1)$, let $Z_k = \frac{Y_k-\mu}{\sigma}$ and $Z_{[k]} = \frac{Y_{[k]}-\mu}{\sigma}$ for $k = 1, ..., n$. Then, $Y_{[k]} = \mu + \sigma Z_{[k]}$ and

$$\begin{aligned}
\frac{Y_{[k]} - \tilde{\mu}_Y}{\tilde{\sigma}_Y} &= \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma\tilde{\mu}_Z)}{\sigma\tilde{\sigma}_Z} \\
&= \frac{Z_{[k]} - \tilde{\mu}_Z}{\tilde{\sigma}_Z}.
\end{aligned}$$

Therefore, $\frac{Y_{[k]}-\tilde{\mu}_Y}{\tilde{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1, \ldots, Y_n$ from $SEV(0,1)$ instead of $SEV(\mu, \sigma)$.

4.2.2 Best linear unbiased estimators

Pirouzi–Fard and Holmquist (2013) considered the statistic D_{sp} such that the estimators of μ and σ in $SEV(\mu, \sigma)$ are obtained by the generalised least squares (GLS) regression of the order statistics from $SEV(0,1)$.

Let $Z_{[1]} \leq \ldots \leq Z_{[n]}$ be the ordered values from $SEV(0,1)$ with

$$\mu_k = \mathrm{E}(Z_{[k]});\ k = 1, \ldots, n \tag{4.2.7}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]});\ k = 1, \ldots, n. \tag{4.2.8}$$

Many statisticians have made much effort to compute means and variances of the order statistics from $SEV(0,1)$. For example, Lieblein and Zelen (1956) calculated them for $n = 1(1)6$. Lieblein and Salzer (1957) showed a table of expected values of order statistics for $n = 1(1)10, 15(5)25$. Moreover, White (1969) tabulated means and variances of order statistics for $n = 1(1)50, 55(5)100$. Later on, Balakrishnan and Chan (1992) presented tables of means, variances and covariances of the order statistics up to $n = 30$. To obtain μ_k and σ_k^2, Pirouzi–Fard and Holmquist (2007) proposed

$$\mu_k \approx \left\{ \begin{array}{ll} -\ln(n) - \gamma, & \text{for } k = 1; \\ \ln(-\ln(1 - [\frac{k-0.4866}{n+0.1840}])), & \text{for } k = 2, \ldots, n \end{array} \right\} \tag{4.2.9}$$

where $\gamma \approx 0.577215665$ which is Euler's constant. Also approximate expressions of the variances and covariances of the order statistics from $SEV(0,1)$ are given by Pirouzi–Fard and Holmquist (2008) as

$$\sigma_{rk}^2 \approx \left\{ \begin{array}{ll} \frac{\pi^2}{6}, & \text{for } r = k = 1; \\ \frac{(k-0.469)([n+0.831-k][n+0.073])^{-1}}{\ln(\frac{n+0.831-k}{n+0.356})\ln(\frac{n+0.779-k}{n+0.356})}, & \text{for } 1 \leq r \leq k \leq n \end{array} \right\} \tag{4.2.10}$$

where $\sigma_{rk}^2 = \sigma_{kr}^2$ is the covariance of the r^{th} and k^{th} order statistics of $SEV(0,1)$ and if $r = k$, we use σ_k^2 as the variance of the k^{th} order statistic. Table 4.1 compares the expected values and variances of the order statistics from $SEV(0,1)$ when $n = 10$ proposed by Balakrishnan and Chan (1992), and Pirouzi–Fard and Holmquist (2007, 2008).

Table 4.1: Expected values and variances of order statistics from $SEV(0,1)$ when $n = 10$ by Balakrishnan and Chan (1992) and Pirouzi–Fard and Holmquist (2007, 2008)

	Balakrishnan and Chan (1992)		Pirouzi–Fard and Holmquist (2007, 2008)	
k	$\text{E}(Z_{[k]})$	$\text{Var}(Z_{[k]})$	$\text{E}(Z_{[k]})$	$\text{Var}(Z_{[k]})$
1	-2.8798	1.64449	-2.8798	1.6449
2	-1.8262	0.64586	-1.8271	0.6540
3	-1.2672	0.39702	-1.2608	0.4012
4	-0.8681	0.28739	-0.8601	0.2910
5	-0.5436	0.22686	-0.5352	0.2302
6	-0.2575	0.18958	-0.2491	0.1927
7	0.0120	0.16581	0.0203	0.1688
8	0.2837	0.15192	0.2916	0.1548
9	0.5846	0.14879	0.5920	0.1515
10	0.9899	0.17143	1.0008	0.1744

Subsequently, we apply the generalised least squares method to obtain the BLUE. Let

$$\begin{aligned}\boldsymbol{\mu} &= n \times 1\,\text{vector of } \mu_k = [\mu_1 \dots \mu_n]' \\ V &= n \times n\,\text{matrix of } \sigma^2_{rk} \\ \mathbf{Y} &= n \times 1\,\text{vector of ordered values} = [Y_{[1]} \dots Y_{[n]}]' \text{ from } SEV(\mu, \sigma).\end{aligned}$$

Then,

$$\begin{aligned}Y_{[k]} &= \mu + \sigma Z_{[k]} \\ \text{E}(Y_{[k]}) &= \mu + \sigma E(Z_{[k]}) = \mu + \sigma \mu_k \\ \text{Var}(Y_{[k]}) &= \sigma^2 \sigma^2_k.\end{aligned}$$

Define $Y_{[k]} = \text{E}(Y_{[k]}) + \varepsilon$. We then have a regression model

$$Y_{[k]} = \mu + \sigma\mu_k + \varepsilon_k \quad ; k = 1, \dots, n. \tag{4.2.11}$$

So, the n points $(Y_{[k]}, \mu_k)$ should be on a straight line. The parameters (μ, σ) in (4.2.11) can be estimated by a suitable method with

$$\text{Var}(Y_{[k]}) = \sigma^2 \sigma^2_k \tag{4.2.12}$$

$$\text{Cov}(Y_{[r]}, Y_{[k]}) = \sigma^2 \text{Cov}(Z_{[r]}, Z_{[k]}) = \sigma^2 \sigma^2_{rk}. \tag{4.2.13}$$

Since $Y_{[k]}$'s are heteroscedastic and autocorrelated, we have to apply the generalised least squares (GLS) method to ensure the validity of the Gauss-Markov theorem.

Theorem 4.2.2. *Gauss-Markov theorem : If* $E(\boldsymbol{Y}) = \boldsymbol{X}\beta$ *and* $Cov(\boldsymbol{Y}) = \sigma^2\boldsymbol{I}$, *the least squares estimators* $\dot{\beta}_j; j = 1,\ldots,p$ *have the minimum variance among all unbiased estimators. The resulting minimum variance estimator of* $\dot{\beta}$ *is* $\dot{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$ *is BLUE of* $\boldsymbol{\beta}$.

Lloyd (1952) was the first to apply the GLS method for estimating the parameters of a location–scale distribution. We write (4.2.11) in the matrix form

$$\begin{aligned} \mathbf{Y} &= \begin{bmatrix} \mathbf{1} & \boldsymbol{\mu} \end{bmatrix} \begin{bmatrix} \mu \\ \sigma \end{bmatrix} + \boldsymbol{\varepsilon} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ \mathrm{Cov}(\boldsymbol{\varepsilon}) &= \mathrm{Cov}(\mathbf{Y}) = \sigma^2 V. \end{aligned} \tag{4.2.14}$$

For the model in (4.2.14), we obtain the following results:

1. The best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$

$$\dot{\boldsymbol{\beta}} = (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y} \tag{4.2.15}$$

2. The covariance matrix for $\dot{\boldsymbol{\beta}}$ is

$$\mathrm{Cov}(\dot{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}'V^{-1}\mathbf{X})^{-1}. \tag{4.2.16}$$

To prove (4.2.15) and (4.2.16), the following theorem (see Rencher (2000), p.23) is useful.

Theorem 4.2.3. *A symmetric matrix* V *is a positive definite if and only if there exists* P *such that* $PP' = V$.

To obtain (4.2.15), let $V = PP'$ and $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. Then, $P^{-1}\mathbf{Y} = P^{-1}\mathbf{X}\boldsymbol{\beta} + P^{-1}\boldsymbol{\varepsilon}$.

$$\begin{aligned} \text{Thus,} \quad \mathrm{E}(P^{-1}\boldsymbol{\varepsilon}) &= P^{-1}\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0} \\ \mathrm{Cov}(P^{-1}\boldsymbol{\varepsilon}) &= P^{-1}\mathrm{Cov}(\boldsymbol{\varepsilon})(P^{-1})' \\ &= P^{-1}(\sigma^2 V)(P^{-1})' \\ &= \sigma^2 P^{-1}(PP')(P^{-1})' \\ &= \sigma^2\mathbf{I}. \end{aligned}$$

From the Gauss-Markov theorem, the least squares estimators $\dot{\boldsymbol{\beta}} = \begin{bmatrix} \dot{\mu} & \dot{\sigma} \end{bmatrix}'$ have the minimum variance among all linear unbiased estimators. We will have the best linear unbiased estimators (BLUE) of $\boldsymbol{\beta}$ as

$$\begin{aligned}
\dot{\boldsymbol{\beta}} &= \left[(P^{-1}\mathbf{X})'(P^{-1}\mathbf{X})\right]^{-1}(P^{-1}\mathbf{X})'P^{-1}\mathbf{Y} \\
&= [\mathbf{X}'(P')^{-1}P^{-1}\mathbf{X}]^{-1}\mathbf{X}'(P')^{-1}P^{-1}\mathbf{Y} \\
&= \left[\mathbf{X}'(PP')^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'(PP')^{-1}\mathbf{Y} \\
&= \left[\mathbf{X}'V^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'V^{-1}\mathbf{Y}.
\end{aligned}$$

Now, $\dot{\beta}$ can be simplified as

$$\begin{aligned}
\dot{\boldsymbol{\beta}} &= (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y} \\
&= \left(\begin{bmatrix}\mathbf{1}' \\ \boldsymbol{\mu}'\end{bmatrix} V^{-1}\begin{bmatrix}\mathbf{1} & \boldsymbol{\mu}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{1}' \\ \boldsymbol{\mu}'\end{bmatrix} V^{-1}\mathbf{Y} \\
&= \begin{bmatrix}\mathbf{1}'V^{-1}\mathbf{1} & \mathbf{1}'V^{-1}\boldsymbol{\mu} \\ \boldsymbol{\mu}'V^{-1}\mathbf{1} & \boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1}'V^{-1} \\ \boldsymbol{\mu}'V^{-1}\end{bmatrix}\mathbf{Y} \\
&= \frac{1}{\triangle}\begin{bmatrix}\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu} & -\mathbf{1}'V^{-1}\boldsymbol{\mu} \\ -\boldsymbol{\mu}'V^{-1}\mathbf{1} & \mathbf{1}'V^{-1}\mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{1}'V^{-1} \\ \boldsymbol{\mu}'V^{-1}\end{bmatrix}\mathbf{Y} \\
&= \frac{1}{\triangle}\begin{bmatrix}(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu})\mathbf{1}'V^{-1} & + & (-\mathbf{1}'V^{-1}\boldsymbol{\mu})\boldsymbol{\mu}'V^{-1} \\ (-\boldsymbol{\mu}'V^{-1}\mathbf{1})\mathbf{1}'V^{-1} & + & (\mathbf{1}'V^{-1}\mathbf{1})\boldsymbol{\mu}'V^{-1}\end{bmatrix}\mathbf{Y} \\
&= \frac{1}{\triangle}\begin{bmatrix}\{(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu})\mathbf{1}' & - & (\mathbf{1}'V^{-1}\boldsymbol{\mu})\boldsymbol{\mu}'\}V^{-1} \\ \{(-\boldsymbol{\mu}'V^{-1}\mathbf{1})\mathbf{1}' & + & (\mathbf{1}'V^{-1}\mathbf{1})\boldsymbol{\mu}'\}V^{-1}\end{bmatrix}\mathbf{Y} \\
&= \frac{1}{\triangle}\begin{bmatrix}\{(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu})\mathbf{1}' & - & (\boldsymbol{\mu}'V^{-1}\mathbf{1})\boldsymbol{\mu}'\}V^{-1} \\ \{(\mathbf{1}'V^{-1}\mathbf{1})\boldsymbol{\mu}' & - & (\mathbf{1}'V^{-1}\boldsymbol{\mu})\mathbf{1}'\}V^{-1}\end{bmatrix}\mathbf{Y} \\
&= \frac{1}{\triangle}\begin{bmatrix}\boldsymbol{\mu}'V^{-1}\{\boldsymbol{\mu}\mathbf{1}' & - & \mathbf{1}\boldsymbol{\mu}'\}V^{-1} \\ \mathbf{1}'V^{-1}\{\mathbf{1}\boldsymbol{\mu}' & - & \boldsymbol{\mu}\mathbf{1}'\}V^{-1}\end{bmatrix}\mathbf{Y}
\end{aligned}$$

where $\triangle = (\mathbf{1}'V^{-1}\mathbf{1})(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}) - (\mathbf{1}'V^{-1}\boldsymbol{\mu})^2$. Accordingly,

$$\dot{\mu} = \frac{1}{\triangle}(\boldsymbol{\mu}'V^{-1}\{\boldsymbol{\mu}\mathbf{1}' - \mathbf{1}\boldsymbol{\mu}'\}V^{-1})\mathbf{Y} \tag{4.2.17}$$

$$\dot{\sigma} = \frac{1}{\triangle}(\mathbf{1}'V^{-1}\{\mathbf{1}\boldsymbol{\mu}' - \boldsymbol{\mu}\mathbf{1}'\}V^{-1})\mathbf{Y}. \tag{4.2.18}$$

Also, (4.2.16) can be shown as

$$\begin{aligned}
\mathrm{Cov}(\dot{\boldsymbol{\beta}}) &= \mathrm{Cov}((\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y}) \\
&= (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}(\sigma^2 V)((\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1})'
\end{aligned}$$

$$\begin{aligned}
&= \sigma^2(\mathbf{X}'V^{-1}\mathbf{X})^{-1} \\
&= \frac{\sigma^2}{\triangle}\begin{bmatrix} \boldsymbol{\mu}'V^{-1}\boldsymbol{\mu} & -(\mathbf{1}'V^{-1}\boldsymbol{\mu}) \\ -(\mathbf{1}'V^{-1}\boldsymbol{\mu}) & \mathbf{1}'V^{-1}\mathbf{1} \end{bmatrix} \qquad (4.2.19)\\
&= \sigma^2\begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}.
\end{aligned}$$

Now, we need to show the invariant property of the parameters $\dot{\mu}$ and $\dot{\sigma}$ which is obtained from the generalised least squares method. Firstly, suppose that $\boldsymbol{p} = [p,\ldots,p]' = p[1,\ldots,1]' = p\mathbf{1}, -\infty < p < \infty$ and $q > 0$

$$\begin{aligned}
\dot{\mu}_{p+q\mathbf{Y}} &= \frac{1}{\triangle}\Big[\boldsymbol{\mu}'V^{-1}\{(\boldsymbol{\mu}\mathbf{1}' - \mathbf{1}\boldsymbol{\mu}'\}V^{-1}\Big][p\mathbf{1} + q\mathbf{Y}] \\
&= \frac{p}{\triangle}\Big[\boldsymbol{\mu}'V^{-1}\{\boldsymbol{\mu}\mathbf{1}' - \mathbf{1}\boldsymbol{\mu}'\}V^{-1}\mathbf{1}\Big] + q\dot{\mu}_{\mathbf{Y}} \\
&= \frac{p}{\triangle}\Big[\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}\mathbf{1}'V^{-1}\mathbf{1} - (\boldsymbol{\mu}'V^{-1}\mathbf{1})^2\Big] + q\dot{\mu}_{\mathbf{Y}} \\
&= \frac{p}{\triangle}\Big[(\mathbf{1}'V^{-1}\mathbf{1})(\boldsymbol{\mu}'V^{-1}\boldsymbol{\mu}) - (\boldsymbol{\mu}'V^{-1}\mathbf{1})^2\Big] + q\dot{\mu}_{\mathbf{Y}} \\
&= \frac{p}{\triangle}\cdot\triangle + q\dot{\mu}_{\mathbf{Y}} \\
&= p + q\dot{\mu}_{\mathbf{Y}}.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\dot{\sigma}_{p+q\mathbf{Y}} &= \frac{1}{\triangle}\Big[\mathbf{1}'V^{-1}\{\mathbf{1}\boldsymbol{\mu}' - \boldsymbol{\mu}\mathbf{1}'\}V^{-1}\Big][p\mathbf{1} + q\mathbf{Y}] \\
&= \frac{p}{\triangle}\Big[(\mathbf{1}'V^{-1}\mathbf{1})(\boldsymbol{\mu}'V^{-1}\mathbf{1}) - (\mathbf{1}'V^{-1}\boldsymbol{\mu})(\mathbf{1}'V^{-1}\mathbf{1})\Big] + q\dot{\sigma}_{\mathbf{Y}} \\
&= \frac{p}{\triangle}\Big[(\boldsymbol{\mu}'V^{-1}\mathbf{1})'(\mathbf{1}'V^{-1}\mathbf{1})' - (\mathbf{1}'V^{-1}\boldsymbol{\mu})(\mathbf{1}'V^{-1}\mathbf{1})\Big] + q\dot{\sigma}_{\mathbf{Y}} \\
&= \frac{p}{\triangle}\Big[(\mathbf{1}'V^{-1}\boldsymbol{\mu})(\mathbf{1}'V^{-1}\mathbf{1}) - (\mathbf{1}'V^{-1}\boldsymbol{\mu})(\mathbf{1}'V^{-1}\mathbf{1})\Big] + q\dot{\sigma}_{\mathbf{Y}} \\
&= q\dot{\sigma}_{\mathbf{Y}}.
\end{aligned}$$

Since $\frac{Y_k-\mu}{\sigma} \sim SEV(0,1)$, let $Z_k = \frac{Y_k-\mu}{\sigma}$ and $Z_{[k]} = \frac{Y_{[k]}-\mu}{\sigma}$ for $k = 1, ..., n$. Then $Y_{[k]} = \mu + \sigma Z_{[k]}$ and

$$\begin{aligned}
\frac{Y_{[k]} - \dot{\mu}_Y}{\dot{\sigma}_Y} &= \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma\dot{\mu}_Z)}{\sigma\dot{\sigma}_Z} \\
&= \frac{Z_{[k]} - \dot{\mu}_Z}{\dot{\sigma}_Z}.
\end{aligned}$$

Therefore, $\frac{Y_{[k]}-\dot{\mu}_Y}{\dot{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1,\ldots,Y_n$ from $SEV(0,1)$ instead of $SEV(\mu,\sigma)$.

4.2.3 Best linear invariant estimators

Although the best linear unbiased estimators (BLUE) have some very nice properties, they often have larger mean square errors than some other linear estimators where mean square error is the sum of variance and bias^2. Mann (1967) developed the best linear invariant estimators (BLIE) as an alternative to the BLUE.

The best linear invariant (in the sense of minimum mean squared error and invariance) estimators (BLIE) $\ddot{\mu}$ and $\ddot{\sigma}$ according to Mann (1969) are respectively

$$\ddot{\mu} = \dot{\mu} - \dot{\sigma}\Big(\frac{E_{12}}{1+E_{22}}\Big) \quad (4.2.20)$$

$$\ddot{\sigma} = \frac{\dot{\sigma}}{1+E_{22}} \quad (4.2.21)$$

where $\dot{\mu}$ and $\dot{\sigma}$ are the BLUE of μ and σ as in (4.2.17) and (4.2.18), respectively. The values of E_{11}, E_{12} and E_{22} can be obtained from (4.2.19) as

$$\begin{aligned} E_{11} &= \frac{\boldsymbol{\mu}' V^{-1} \boldsymbol{\mu}}{\triangle}, \\ E_{22} &= \frac{\mathbf{1}' V^{-1} \mathbf{1}}{\triangle}, \\ E_{12} &= \frac{-(\mathbf{1}' V^{-1} \boldsymbol{\mu})}{\triangle}. \end{aligned}$$

As seen in (4.2.20) and (4.2.21), the best linear invariant estimators (BLIE) which have minimum mean square errors among all linear estimators result from the best linear unbiased estimators (BLUE) by a simple linear transformation. Then, the invariant property for the parameters $\ddot{\mu}$ and $\ddot{\sigma}$ are shown as:

$$\begin{aligned} \ddot{\mu}_Y = \ddot{\mu}_{p+qZ} &= \dot{\mu}_{p+qZ} - \dot{\sigma}_{p+qZ}(\frac{E_{12}}{1+E_{22}}) \\ &= (p + q\dot{\mu}_Z) - q\dot{\sigma}_Z(\frac{E_{12}}{1+E_{22}}) \\ &= p + q(\dot{\mu}_Z - \dot{\sigma}_Z(\frac{E_{12}}{1+E_{22}})) \\ &= p + q\ddot{\mu}_Z \end{aligned}$$

$$\text{and } \ddot{\sigma}_Y = \ddot{\sigma}_{p+qZ} = \frac{\dot{\sigma}_{p+qZ}}{1+E_{22}} = \frac{q\dot{\sigma}_Z}{1+E_{22}} = q\ddot{\sigma}_Z.$$

Since $\frac{Y_k-\mu}{\sigma} \sim SEV(0,1)$, let $Z_k = \frac{Y_k-\mu}{\sigma}$ and $Z_{[k]} = \frac{Y_{[k]}-\mu}{\sigma}$ for $k = 1, ..., n$, then, $Y_{[k]} = \mu + \sigma Z_{[k]}$ and

$$\frac{Y_{[k]} - \ddot{\mu}_Y}{\ddot{\sigma}_Y} = \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma\ddot{\mu}_Z)}{\sigma\ddot{\sigma}_Z}$$

$$= \frac{Z_{[k]} - \ddot{\mu}_Z}{\ddot{\sigma}_Z}.$$

Therefore, $\frac{Y_{[k]} - \ddot{\mu}_Y}{\ddot{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1, \ldots, Y_n$ from $SEV(0,1)$ instead of $SEV(\mu, \sigma)$.

4.3 Graphical tests for the Weibull distribution

As shown earlier, if we consider the three–parameter Weibull distribution, provided the location parameter is known, and it is not a member of the location–scale family, the log-transformation is required. Consequently, the smallest extreme value distribution is one starting point for the construction of simultaneous probability intervals of the order statistics from the Weibull distribution. The cumulative distribution function (cdf) $F(\cdot)$ follows the smallest extreme value distribution as in (4.1.6). Now, the main objective is to check whether a sample is taken from the Weibull distribution.

Suppose that $X_1, \ldots, X_n$ are n i.i.d. random variables from $Wbl(a, b, c)$ with a specified location parameter a. The corresponding order statistics are $X_{[1]} \leq \ldots \leq X_{[n]}$. Then, $Y_k = \ln(X_k - a)$ for $k = 1, \ldots, n$ are random variables from $SEV(\mu, \sigma)$. Therefore, we denote that $\hat{\mu}$ and $\hat{\sigma}$ are the estimators of μ and σ. In the previous section, the three estimators are introduced, which are MLE $(\tilde{\mu}, \tilde{\sigma})$, BLUE $(\dot{\mu}, \dot{\sigma})$ and BLIE $(\ddot{\mu}, \ddot{\sigma})$. The following $(\hat{\mu}, \hat{\sigma})$ can therefore be substituted by $(\tilde{\mu}, \tilde{\sigma})$, $(\dot{\mu}, \dot{\sigma})$ or $(\ddot{\mu}, \ddot{\sigma})$.

4.3.1 The Kolmogorov-Smirnov test (The D test)

The Kolmogorov-Smirnov statistic is

$$D = \max_{1 \leq k \leq n} \Big| F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k - 0.5)/n \Big|. \tag{4.3.1}$$

Let c_D be a critical constant so that $P\{D < c_D\} = 1 - \alpha$ when the sample follows the Weibull distribution. This probability statement can be rewritten as

$$\begin{aligned}
1 - \alpha &= P\{D < c_D\} \\
&= P\Big\{ \max_{1 \leq k \leq n} \Big| F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k - 0.5)/n \Big| \leq c_D \Big\} \\
&= P\Big\{ - c_D \leq F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k - 0.5)/n \leq c_D \text{ for } k = 1, ..., n \Big\} \\
&= P\Big\{ \frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y} \in F^{-1}\big((k - 0.5)/n \pm c_D\big) \text{ for } k = 1, ..., n \Big\}
\end{aligned}$$

$$= \quad P\Big\{Y_{[k]} \in \hat{\mu}_Y + \hat{\sigma}_Y \ln[-\ln\big(1-(k-0.5)/n \mp c_D\big)] \text{ for } k=1,...,n\Big\}.$$

Note that the expression (4.3.1) has nothing to do with the unknown parameters μ and σ since $\frac{Y_1-\mu}{\sigma},...,\frac{Y_n-\mu}{\sigma} \overset{i.i.d.}{\sim} SEV(0,1)$. Let $Z_k = \frac{Y_k-\mu}{\sigma}$ and $Z_{[k]} = \frac{Y_{[k]}-\mu}{\sigma}$ for $k=1,...,n$. Then $Y_{[k]} = \mu + \sigma Z_{[k]}$ for $k=1,...,n$. Let us consider

$$\begin{aligned} \frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y} &= \frac{(\mu+\sigma Z_{[k]})-\hat{\mu}_{\mu+\sigma Z}}{\hat{\sigma}_{\mu+\sigma Z}} \\ &= \frac{(\mu+\sigma Z_{[k]})-(\mu+\sigma\hat{\mu}_Z)}{\sigma\hat{\sigma}_Z} \\ &= \frac{Z_{[k]}-\hat{\mu}_Z}{\hat{\sigma}_Z}. \end{aligned}$$

The required critical constant c_D can be evaluated straightforwardly as the $(1-\alpha)$-quantile of D by using the following simulations:

1. Simulate one D by generating i.i.d. $X_1,...,X_n$ from $Wbl(0,1,1)$ or $Exp(0,1)$, and transform by using $Y_k = \ln X_k$ for $k=1,\ldots,n$ which results in $Y_1,...,Y_n$ from $SEV(0,1)$.

2. Calculate $\hat{\mu}_Y, \hat{\sigma}_Y$ and $(Y_{[1]},...,Y_{[n]})$, and compute D using formula (4.3.1).

3. Simulate a large number R of independent copies of $D: D_1,...,D_R$.

4. Use the $(1-\alpha)$ sample quantile of $D_1,...,D_R$ as an approximation of the critical constant c_D.

From (4.3.1), $Y_{[k]}$ should fall in the corresponding interval

$$\hat{\mu}_Y + \hat{\sigma}_Y \ln[-\ln\big(1-(k-0.5)/n \mp c_D\big)] \text{ for } k=1,...,n \tag{4.3.2}$$

with a simultaneous probability $1-\alpha$ if the sample follows a Weibull distribution.

4.3.2 The D_{sp} test

Based on the D_{sp} test of Michael (1983), the D_{sp} test for testing Weibull distribution is defined by

$$D_{sp} = \max_{1\le k\le n}\left|\frac{2}{\pi}\arcsin\sqrt{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)} - \frac{2}{\pi}\arcsin\sqrt{(k-0.5)/n}\right|. \tag{4.3.3}$$

Let c_{sp} be a critical constant so that $P\{D_{sp} < c_{sp}\} = 1-\alpha$ when the sample is taken from the Weibull population. The critical constant c_{sp} can be computed by simulation in a similar way to c_D. This probability statement can be rewritten as

$$\begin{aligned} 1-\alpha &= P\Big\{D_{sp} < c_{sp}\Big\} \\ &= P\Big\{\max_{1\le k\le n}\Big|\frac{2}{\pi}\arcsin\sqrt{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)}-\frac{2}{\pi}\arcsin\sqrt{(k-0.5)/n}\Big| < c_{sp}\Big\} \\ &= P\Big\{-c_{sp} < \frac{2}{\pi}\arcsin\sqrt{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)}-m < c_{sp} \text{ for } k=1,\ldots,n\Big\} \\ &= P\Big\{\frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y} \in F^{-1}\big(\sin^2\big(\frac{\pi}{2}(m\pm c_{sp})\big)\big) \text{ for } k=1,\ldots,n\Big\} \\ &= P\Big\{Y_{[k]} \in \hat{\mu}_Y+\hat{\sigma}_Y\ln[-\ln\big(1-\sin^2\big(\frac{\pi}{2}(m\pm c_{sp})\big)\big)] \text{ for } k=1,\ldots,n\Big\} \quad (4.3.4) \end{aligned}$$

where $m = \frac{2}{\pi}\arcsin\sqrt{(k-0.5)/n}$. The expression (4.3.4) provides the simultaneous intervals for $Y_{[k]}$, $k=1,\ldots,n$. Consequently, the simultaneous interval for $Y_{[k]}$ is

$$\hat{\mu}_Y+\hat{\sigma}_Y\ln[-\ln\big(1-\sin^2\big(\arcsin\sqrt{(k-0.5)/n}\pm\frac{\pi}{2}c_{sp}\big)\big)] \text{ for } k=1,\ldots,n.$$

4.3.3 The D_e test

From the previous section, the two–parameter Weibull distribution with scale parameter b and shape parameter c is considered, the logarithmic transformation converts this scale–shape parameter distribution into a more tractable location–scale parameter distribution. Thus, we clarify the order statistics $Z_{[k]}$ for $k=1,\ldots,n$ of the variate $Z=\dfrac{Y-\mu}{\sigma}$ having

$$\begin{aligned} F_Z(z) &= 1-\exp\{-\exp(z)\}, \quad -\infty\le z\le\infty & (4.3.5) \\ f_Z(z) &= \exp\{z-\exp(z)\}, \quad -\infty\le z\le\infty. & (4.3.6) \end{aligned}$$

Let $Z_1,\ldots,Z_n$ be a simple random sample drawn from the standard smallest extreme value distribution $SEV(0,1)$ and $Z_{[1]}\le\ldots\le Z_{[n]}$ be the ordered values. The expected value and variance of $Z_{[k]}$ for $k=1,\ldots,n$ are given by

$$\begin{aligned} \mu_k &= \mathrm{E}(Z_{[k]}) & (4.3.7) \\ \sigma_k^2 &= \mathrm{Var}(Z_{[k]}) = \mathrm{E}(Z_{[k]}^2)-\mu_k^2 & (4.3.8) \end{aligned}$$

where $f_k(z)$ is the probability density function of $Z_{[k]}$ and is defined by

$$\begin{aligned} f_k(z) &= \frac{n!}{(k-1)!(n-k)!}\,(F_Z(z))^{k-1}\big(1-F_Z(z)\big)^{n-k}f_Z(z)\ ,\ -\infty\le z\le\infty \\ &= \frac{n!}{(k-1)!(n-k)!}(1-\exp\{-e^z\})^{k-1}(\exp\{-e^z\})^{n-k}\exp\{z-e^z\} \end{aligned}$$

$$\begin{aligned} &= \frac{n!}{(k-1)!(n-k)!}(1-\exp\{-e^z\})^{k-1}\exp\{-(n-k)e^z\}\exp\{z-e^z\} \\ &= \frac{n!}{(k-1)!(n-k)!}(1-\exp\{-e^z\})^{k-1}\exp\{z-e^z(n-k+1)\}. \end{aligned} \quad (4.3.9)$$

It is obvious that $(Y_{[1]},...,Y_{[n]})$ have the same joint distribution as $(\mu+\sigma Z_{[1]},...,\mu+\sigma Z_{[n]})$. In particular, we have

$$\mathrm{E}(Y_{[k]}) = \mu+\sigma\mu_k \quad (4.3.10)$$

$$\mathrm{Var}(Y_{[k]}) = \sigma^2\sigma_k^2. \quad (4.3.11)$$

Specially, we construct the statistic

$$D_e = \max_{1\le k\le n}\left|\frac{Y_{[k]}-(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)}{\hat{\sigma}_Y\sigma_k}\right|. \quad (4.3.12)$$

Let c_e be a critical constant chosen so that all the $Y_{[k]}$'s will be contained in the corresponding intervals simultaneously with probability $1-\alpha$.

The probability statement can be rewritten as

$$\begin{aligned} 1-\alpha &= P\Big\{D_e < c_e\Big\} \\ &= P\Big\{\max_{1\le k\le n}\frac{|Y_{[k]}-(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)|}{\hat{\sigma}_Y\sigma_k}\le c_e\Big\} \\ &= P\Big\{Y_{[k]}\in(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)\pm\hat{\sigma}_Y\sigma_k c_e \text{ for } k=1,\dots,n\Big\}. \end{aligned} \quad (4.3.13)$$

Consider the term $\frac{Y_{[k]}-(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)}{\hat{\sigma}_Y\sigma_k}$ in (4.3.12). Since $\frac{Y_1-\mu}{\sigma},...,\frac{Y_n-\mu}{\sigma}\overset{i.i.d.}{\sim}SEV(0,1)$. Let $Z_k=\frac{Y_k-\mu}{\sigma}$ and $Z_{[k]}=\frac{Y_{[k]}-\mu}{\sigma}$ for $k=1,...,n$. Then $Y_{[k]}=\mu+\sigma Z_{[k]}$ for $k=1,...,n$ and

$$\begin{aligned} \frac{Y_{[k]}-(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)}{\hat{\sigma}_Y\sigma_k} &= \frac{\big(\mu+\sigma Z_{[k]}\big)-(\hat{\mu}_{\mu+\sigma Z}+\hat{\sigma}_{\mu+\sigma Z}\mu_k)}{\hat{\sigma}_{\mu+\sigma Z}\sigma_k} \\ &= \frac{\big(\mu+\sigma Z_{[k]}\big)-(\mu+\sigma\hat{\mu}_Z+\sigma\hat{\sigma}_Z\mu_k)}{\sigma\hat{\sigma}_Z\sigma_k} \\ &= \frac{Z_{[k]}-(\hat{\mu}_Z+\hat{\sigma}_Z\mu_k)}{\hat{\sigma}_Z\sigma_k}. \end{aligned} \quad (4.3.14)$$

It is clear from (4.3.14) that the simultaneous coverage probability in (4.3.13) only depends on the sample size and critical constant but has nothing to do with the unknown parameters μ and σ. The required critical constants c_e can be computed as similar to the computation for c_D.

4.3.4 The D_{be} test

According to Theorem 2.1.1, $U_k = F\Big(\dfrac{Y_k - \mu}{\sigma}\Big) \sim U(0,1)$. Consequently, $U_{[k]} = F\Big(\dfrac{Y_{[k]} - \mu}{\sigma}\Big) \sim beta(k, n-k+1)$ if $F(\cdot)$ is a cdf of the smallest extreme value distribution and we apply the D_{be} statistic for testing normality to the Weibull distribution by the following steps:

- **Step 1**. Construct p^* level highest-density probability interval for $U_{[k]}$. That is, $[L(p^*,k,n), U(p^*,k,n)]$ is the shortest probability interval for $U_{[k]}$ among all the p^* level probability intervals for $U_{[k]}$.

- **Step 2**. Construct simultaneous probability intervals for $Y_{[1]} \leq ... \leq Y_{[n]}$ based on

$$L(p^*,k,n) \leq F\Big(\frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y}\Big) \leq U(p^*,k,n) \ \text{ for } k = 1,...,n, \tag{4.3.15}$$

where p^* is chosen so that

$$K(p^*) \equiv P\Big\{F^{-1}(L) \leq \frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y} \leq F^{-1}(U) \ \text{ for } k = 1,...,n\Big\} \ = \ 1-\alpha$$
$$P\Big\{\ln[-\ln(1-L)] \leq \frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y} \leq \ln[-\ln(1-U)] \ \text{ for } k = 1,...,n\Big\} \ = \ 1-\alpha.$$

- **Step 3**. Such a p^* can be found by simulation :
(i) for each p^*, find $K(p^*)$;
(ii) search over $p^* \in (1-\frac{\alpha}{n}, 1-\alpha)$ so that $K(p^*) = 1-\alpha$.
Therefore, the simultaneous probability intervals for $Y_{[1]} \leq ... \leq Y_{[n]}$ can be expressed as

$$\hat{\mu}_Y + \hat{\sigma}_Y \ln[-\ln(1 - L(p^*,k,n))] \leq Y_{[k]} \leq \hat{\mu}_Y + \hat{\sigma}_Y \ln[-\ln(1 - U(p^*,k,n))] \ \text{ for } k = 1,\ldots,n.$$

4.3.5 The D_{bi} test

The D_{bi} statistic for testing the Weibull distribution can be defined as

$$D_{bi} \ = \ \max_{1\leq k\leq n} \frac{\big|F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k-0.5)/n\big|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}}.$$

Let c_{bi} be a critical constant so that $P\Big\{D_{bi} < c_{bi}\Big\} = 1-\alpha$; c_{bi} can be determined by simulation as c_D from the D test. The probability statement for $Y_{[1]} \leq ... \leq Y_{[n]}$

is given by

$$
\begin{aligned}
1-\alpha &= P\Big\{D_{bi} < c_{bi}\Big\} \\
&= P\Big\{\max_{1\le k\le n} \frac{\big|F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)-(k-0.5)/n\big|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi}\Big\} \\
&= P\Big\{-c_{bi} < \frac{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)-(k-0.5)/n}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi} \text{ for } k=1,\ldots,n\Big\} \\
&= P\Big\{F^{-1}\Big(\frac{k-0.5}{n}-dc_{bi}\Big) < \frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y} < F^{-1}\Big(\frac{k-0.5}{n}+dc_{bi}\Big) \text{ for } k=1,\ldots,n\Big\} \\
&= P\Big\{Y_{[k]} \in \hat{\mu}_Y + \hat{\sigma}_Y \ln[-\ln\big(1-\frac{k-0.5}{n}\mp dc_{bi}\big)] \text{ for } k=1,\ldots,n\Big\}
\end{aligned}
$$

where $d = \sqrt{(k-0.5)(n-k+0.5)/n^3}$. Therefore, the simultaneous probability intervals of $Y_{[k]}$ for $k = 1,\ldots,n$ are

$$\hat{\mu}_Y + \hat{\sigma}_Y \ln[-\ln\big(1-\frac{k-0.5}{n}\mp c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\big)]. \tag{4.3.16}$$

After declaring all graphical tests for the Weibull distribution, the next stage is to compute power comparisons between the graphical and non–graphical tests.

4.4 Non–graphical tests for the Weibull distribution

Littell et al. (1979) advised that without prior knowledge of the alternative distributions, the Anderson-Darling and Cramér-von Mises statistics are the most powerful when the other statistics considered are those of Kolmogorov-Smirnov, Smith and Bain, and Mann, Scheuer and Fertig. Their powers are good for testing the two–parameter Weibull distribution over the alternatives *logistic*$(0,1)$, *Laplace*(0.1), $N(0,1)$, *Cauchy*$(0,1)$, $\chi^2(1)$, and $\chi^2(4)$. Sürücü (2008) reported the simulation results for testing normality, exponentiality and the Weibull distribution. For testing the exponential distribution, and the Anderson-Darling statistic is the most powerful and they have better power than the Shapiro-Wilk statistic. For testing the extreme value distribution, the statistic based on generalised sample spacings and the Anderson-Darling statistic provide the most powerful test. Therefore, the Anderson-Darling and Cramér-von Mises statistics are selected to compare powers with the graphical tests for testing both an exponential and a Weibull distributions.

4.4.1 The Anderson–Darling test

The Anderson-Darling test statistic (AD) is

$$\mathrm{AD} = -\sum_{k=1}^{n}\Big[\frac{(2k-1)\{\ln(F_k)+\ln(1-F_{n+1-k})\}}{n}\Big] - n, \qquad (4.4.1)$$

where $F_k = 1 - \exp\Big(-\exp\Big(\frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y}\Big)\Big)$. The critical constant c_{AD} which satisfies $P\{\mathrm{AD} < c_{\mathrm{AD}}\} = 1-\alpha$ can be determined by simulation as the critical constant c_D for the D test.

4.4.2 The Cramér-von Mises test

The Cramér-von Mises statistic (CvM) is

$$\mathrm{CvM} = \sum_{k=1}^{n}\Big[F_k - \frac{2k-1}{2n},\Big]^2 + \frac{1}{12n} \qquad (4.4.2)$$

where $F_k = 1 - \exp\Big(-\exp\Big(\frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y}\Big)\Big)$. The critical constant c_{CvM} which satisfies $P\{\mathrm{CvM} < c_{\mathrm{CvM}}\} = 1-\alpha$ can be determined by simulation as the critical constant c_D for the D test.

4.5 Power comparison

To evaluate the power of the tests for a Weibull distribution, the Monte Carlo simulation is used. The simulation study is undertaken in two parts. In the first part of the simulation study, the critical values under the two–parameter Weibull distribution, $Wbl(0, b, c)$, are calculated. In the second part, the powers are computed based on simulating data from several alternative distributions. The many alternatives considered in this power study are selected to be representatives of the various types of distributions. They are used in many extensive reviews of power studies for goodness-of-fit tests. These alternative distributions are categorised into three groups.

The first group (Group I) of seven alternative distributions is asymmetrical on the support $(0, \infty)$ and includes:

- $\chi^2(1)$;
- $\chi^2(3)$;
- $\chi^2(4)$;
- $\chi^2(6)$;
- $\chi^2(10)$;
- $LogN(0, 1)$;
- $HN(0, 1)$: If Y is $N(0, 1)$, $|Y|$ has a half-normal distribution, denoted by $HN(0, 1)$; see, Stephens (1978).

The second group (Group II) of seven distributions is on the interval $(0, 1)$ and comprises:

- $U(0, 1)$;
- $beta(2, 2)$;
- $beta(2, 5)$;
- $beta(5, 1.5)$;
- $beta(0.5, 0.5)$;
- $beta(0.5, 3)$;
- $beta(1, 2)$.

The third group (Group III) of eight distributions lies symmetrical on the support $(-\infty, \infty)$ and comprises:

- $Laplace(0, 1)$;
- $logistic(0, 1)$;
- $Cauchy(0, 1)$;
- $N(0, 1)$
- $t(1)$;
- $t(3)$;
- $t(4)$;
- $t(6)$.

Some of the referred alternative distributions have been used by Littell et al. (1979), Tiku and Singh (1981), and Castro-Kusiss (2011) as well as Pirouzi-Fard and Holmquist et al., (2013). The critical values and the powers of our seven tests are computed for all possible combinations of $\alpha = 0.01, 0.05$ and 0.1, three estimators (MLE, BLUE, and BLIE), and sample size $n = 10, 25, 40, 100, 150, 200, 250, 300, 350, 400$ and 500 for the alternative distributions from Group I and Group II. For the alternative distributions from Group III, the considered sample sizes are 5(5)30, 40, 50, 100, 150, and 200 because the speed of convergence in power against alternatives from Group III is very fast and reports 100% power at moderate sample sizes (see Tables 4.8, 4.11 and 4.14).

The critical constant (see Tables 4.2–4.4) and power of each test for each scenario are computed based on 300,000 and 100,000 simulations, respectively. As the observations made from this power comparison study are similar for $\alpha = 0.01, 0.05$ and 0.1 in Tables 4.6–4.14, only the results of powers for $\alpha = 0.05$ are analysed. Tables of power comparisons for $\alpha = 0.01$ and 0.1 are presented at the end of this chapter in Tables 4.20–4.37.

Table 4.2: Critical values for testing Weibull distribution of the D, D_{sp}, D_e, D_{bi}, AD and CvM tests using MLE at $\alpha = 0.01, 0.05$ and 0.1 for several sample sizes n

(a) $\alpha = 0.01$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2998	0.1957	1.7737	1.4657	0.9018	0.1544
10	0.2505	0.1712	2.4990	2.1217	0.9895	0.1672
15	0.2180	0.1529	2.7654	2.5626	0.9920	0.1686
20	0.1939	0.1411	2.8701	2.8450	1.0055	0.1702
25	0.1778	0.1314	2.9857	3.1409	1.0190	0.1736
30	0.1657	0.1251	3.0529	3.3082	1.0148	0.1710
40	0.1454	0.1141	3.1714	3.6682	1.0308	0.1743
50	0.1327	0.1054	3.2541	3.8933	1.0316	0.1728
100	0.0967	0.0832	3.4808	4.6151	1.0266	0.1728
150	0.0806	0.0720	3.5595	5.0740	1.0330	0.1743
200	0.0703	0.0639	3.6339	5.2773	1.0435	0.1743
250	0.0626	0.0583	3.6712	5.3372	1.0420	0.1741
300	0.0577	0.0545	3.7290	5.559	1.0326	0.1747
350	0.0538	0.0517	3.7570	5.7259	1.0375	0.1732
400	0.0503	0.0484	3.7756	5.5995	1.0464	0.1763
500	0.0453	0.0439	3.7771	5.8770	1.0523	0.1767

(b) $\alpha = 0.05$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2523	0.0533	2.7403	3.7151	0.7533	0.0459
10	0.211	0.0488	2.8239	3.8076	0.7589	0.0431
15	0.1839	0.0454	2.8389	3.9604	0.7524	0.0388
20	0.1644	0.0425	2.8638	3.9677	0.7558	0.1666
25	0.1506	0.0402	2.8905	4.0081	0.7614	0.1454
30	0.1402	0.0367	2.9565	4.0879	0.7594	0.1313
40	0.1244	1.416	1.1868	0.6931	0.1154	0.12
50	0.1132	1.7926	1.6141	0.726	0.1198	0.1117
100	0.0829	1.9704	1.9036	0.7341	0.1213	0.1061
150	0.0689	2.0681	2.1083	0.739	0.1215	0.0962
200	0.0598	2.1394	2.2965	0.7415	0.1221	0.1232
250	0.054	2.2184	2.4337	0.7503	0.1236	0.1238
300	0.0496	2.3043	2.6481	0.7566	0.1237	0.124
350	0.0893	2.3849	2.8059	0.7524	0.1245	0.1237
400	0.0694	2.5878	3.2887	0.7566	0.1236	0.1245
500	0.0598	2.6652	3.5474	0.7571	0.1238	0.1245

(c) $\alpha = 0.1$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2275	0.1509	1.2514	1.0627	0.5968	0.0978
10	0.1908	0.1334	1.5382	1.4133	0.6164	0.1001
15	0.1667	0.1199	1.6913	1.6451	0.6212	0.1008
20	0.1496	0.1099	1.7878	1.8116	0.6233	0.101
25	0.1372	0.1027	1.8571	1.9436	0.6259	0.101
30	0.1279	0.0972	1.9177	2.0533	0.6283	0.1013
40	0.1135	0.0881	1.9958	2.2192	0.6297	0.1021
50	0.1032	0.0816	2.0651	2.3576	0.6304	0.1017
100	0.0757	0.0633	2.2343	2.7484	0.6353	0.1023
150	0.0631	0.0545	2.3339	2.9512	0.6338	0.1019
200	0.055	0.0485	2.3915	3.0884	0.6296	0.1008
250	0.0496	0.0445	2.4509	3.172	0.636	0.1023
300	0.0454	0.0413	2.4729	3.2313	0.634	0.1017
350	0.0422	0.0387	2.5058	3.286	0.6344	0.1017
400	0.0396	0.0367	2.5304	3.3129	0.635	0.102
500	0.0356	0.0334	2.5744	3.4007	0.6368	0.102

Table 4.3: Critical constants for testing Weibull distribution of the D, D_{sp}, D_e, D_{bi}, AD and CvM tests using BLUE at $\alpha = 0.01, 0.05$ and 0.1 for several sample sizes n

(a) $\alpha = 0.01$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2960	0.1926	1.3862	1.4772	0.8419	0.1523
10	0.2499	0.1732	2.2015	2.2441	0.9796	0.1668
15	0.2175	0.1557	2.5196	2.7193	0.9908	0.1686
20	0.1941	0.1438	2.6978	3.0109	1.0029	0.1705
25	0.1780	0.1342	2.854	3.3101	1.0151	0.1737
30	0.1654	0.1273	2.9395	3.4984	1.0178	0.1724
40	0.1456	0.1155	3.0473	3.7771	1.0252	0.1750
50	0.1328	0.1068	3.1965	4.0231	1.0374	0.1737
100	0.0970	0.084	3.4273	4.7180	1.0257	0.1731
150	0.0804	0.0726	3.5146	5.1239	1.0292	0.1739
200	0.0703	0.0644	3.6287	5.3158	1.0435	0.1741
250	0.0627	0.0586	3.6557	5.3686	1.0396	0.1747
300	0.0578	0.0547	3.7166	5.6009	1.0343	0.1750
350	0.0539	0.0519	3.7502	5.7573	1.0339	0.1731
400	0.0503	0.0485	3.7460	5.6270	1.0432	0.177
500	0.0454	0.0440	3.7604	5.8640	1.0532	0.1766

(b) $\alpha = 0.05$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2409	0.1585	1.1456	1.1911	0.6190	0.1090
10	0.2086	0.1460	1.6022	1.7336	0.6948	0.1177
15	0.1825	0.1323	1.8184	2.0321	0.7196	0.1207
20	0.1644	0.1217	1.9443	2.2422	0.7250	0.1211
25	0.1505	0.1133	2.0388	2.4084	0.7351	0.1221
30	0.1401	0.1075	2.1254	2.5372	0.7432	0.1234
40	0.1243	0.0975	2.2243	2.7385	0.7529	0.1240
50	0.1133	0.0904	2.3163	2.8881	0.7499	0.1246
100	0.0829	0.0700	2.5448	3.3544	0.7572	0.1238
150	0.0689	0.0603	2.6247	3.6071	0.7567	0.1239
200	0.0597	0.0537	2.7086	3.7660	0.7551	0.1232
250	0.0539	0.0491	2.7936	3.8528	0.7608	0.1243
300	0.0497	0.0456	2.8097	3.9950	0.7524	0.1240
350	0.0459	0.0427	2.8401	3.9978	0.7552	0.1235
400	0.0431	0.0404	2.8657	4.0456	0.7621	0.1245
500	0.0388	0.0368	2.9361	4.1211	0.7599	0.1247

(c) $\alpha = 0.1$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2152	0.1424	1.0233	1.0550	0.5195	0.0911
10	0.1876	0.1324	1.3952	1.5080	0.5820	0.0972
15	0.1654	0.1209	1.5771	1.7626	0.6032	0.0994
20	0.1489	0.1109	1.6886	1.9325	0.6103	0.1003
25	0.1368	0.1039	1.7648	2.0624	0.6147	0.1007
30	0.1275	0.0982	1.8356	2.1646	0.6197	0.1011
40	0.1134	0.0889	1.9184	2.3213	0.6233	0.1019
50	0.1031	0.0824	1.9959	2.4566	0.6264	0.1018
100	0.0756	0.0638	2.1862	2.8202	0.6349	0.1024
150	0.0630	0.0549	2.2883	3.0111	0.6329	0.102
200	0.0550	0.0488	2.3499	3.1406	0.6278	0.1011
250	0.0496	0.0447	2.4187	3.2195	0.6367	0.1025
300	0.0454	0.0415	2.4426	3.2803	0.634	0.1018
350	0.0422	0.0389	2.4774	3.3306	0.6342	0.1019
400	0.0397	0.0368	2.5064	3.3550	0.6359	0.1022
500	0.0356	0.0336	2.5519	3.4395	0.6376	0.1022

Table 4.4: Critical constants for testing Weibull distribution of the D, D_{sp}, D_e, D_{bi}, AD and CvM tests using BLIE at $\alpha = 0.01, 0.05$ and 0.1 for several sample sizes n

(a) $\alpha = 0.01$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.3176	0.2073	1.8773	1.5497	0.9619	0.1646
10	0.2565	0.1738	2.5415	2.0540	1.0346	0.1726
15	0.2217	0.1537	2.7669	2.4681	1.0224	0.1715
20	0.1961	0.1407	2.8980	2.7676	1.0223	0.1741
25	0.1796	0.1311	3.0113	3.0746	1.0342	0.1764
30	0.1671	0.1243	3.0783	3.2812	1.0308	0.1748
40	0.1459	0.1133	3.1626	3.5933	1.0417	0.1750
50	0.1334	0.1049	3.2847	3.8561	1.0446	0.1749
100	0.0971	0.0830	3.4786	4.6126	1.0335	0.1734
150	0.0809	0.0720	3.5584	5.0312	1.0351	0.1744
200	0.0703	0.0638	3.6638	5.2429	1.0450	0.1748
250	0.0629	0.0582	3.6841	5.3089	1.0386	0.1743
300	0.0579	0.0545	3.7374	5.5449	1.0348	0.1751
350	0.0538	0.0516	3.7680	5.7051	1.0376	0.1740
400	0.0504	0.0483	3.7661	5.5811	1.0456	0.1769
500	0.0454	0.0438	3.7794	5.8298	1.0519	0.1763

(b) $\alpha = 0.05$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2663	0.1772	1.5367	1.2494	0.7282	0.1235
10	0.2163	0.1483	1.8549	1.6128	0.7438	0.1248
15	0.1868	0.1327	2.0140	1.8855	0.7472	0.1244
20	0.1662	0.1207	2.1081	2.0813	0.7503	0.1239
25	0.1523	0.1123	2.1752	2.2581	0.7519	0.1248
30	0.1413	0.1063	2.2487	2.3898	0.7588	0.1255
40	0.1252	0.0961	2.3249	2.6084	0.7628	0.1251
50	0.1139	0.0892	2.4027	2.7689	0.7596	0.1259
100	0.0832	0.0693	2.5958	3.2669	0.7602	0.1246
150	0.0691	0.0598	2.6669	3.5376	0.7587	0.1244
200	0.0598	0.0533	2.7390	3.7080	0.7566	0.1235
250	0.0540	0.0487	2.8223	3.8035	0.7610	0.1243
300	0.0496	0.0453	2.8321	3.9492	0.7534	0.1241
350	0.0459	0.0425	2.8602	3.9563	0.7576	0.1239
400	0.0431	0.0402	2.8838	4.0082	0.7626	0.1244
500	0.0388	0.0367	2.9550	4.0892	0.7598	0.1248

(c) $\alpha = 0.1$

n	D	D_{sp}	D_e	D_{bi}	AD	CvM
5	0.2403	0.1617	1.3695	1.1175	0.6181	0.1041
10	0.1961	0.1359	1.6049	1.4243	0.6258	0.1030
15	0.1696	0.1210	1.7421	1.6439	0.6297	0.1031
20	0.1514	0.1108	1.8246	1.8107	0.6297	0.1032
25	0.1388	0.1032	1.8811	1.9376	0.6306	0.1028
30	0.1292	0.0974	1.9432	2.0473	0.6332	0.1028
40	0.1144	0.0882	2.0076	2.208	0.6334	0.1035
50	0.1038	0.0818	2.0731	2.3539	0.6336	0.1029
100	0.0759	0.0633	2.2331	2.7475	0.6382	0.1029
150	0.0632	0.0545	2.3266	2.9505	0.6352	0.1023
200	0.0551	0.0486	2.3813	3.0893	0.6294	0.1012
250	0.0496	0.0445	2.4417	3.1742	0.6381	0.1024
300	0.0454	0.0413	2.4656	3.2395	0.6346	0.1020
350	0.0422	0.0388	2.4962	3.2938	0.6352	0.1021
400	0.0396	0.0367	2.5234	3.3211	0.6360	0.1023
500	0.0356	0.0335	2.5663	3.4114	0.6375	0.1023

4.5.1 The best choice of parameter estimation

To examine the power of each test with three estimators (MLE, BLUE and BLIE), plots should be constructed by plotting the power of a test with three different estimators in the same plot. Then, we investigate which estimator produces the highest power under the considered tests. Figures 4.1–4.3 reveal plots of resulting power based on the D test by comparing powers among three estimators at $\alpha = 0.05$ against the alternative distributions from Group I, Group II and Group III, respectively. Also, Figures 4.4–4.21 show the power comparisons of the specific test using the three estimators at $\alpha = 0.05$ from Group I, II and III.

Table 4.5: Combination of goodness-of-fit test statistics and the estimators

	Estimators		
Test statistics	MLE	BLUE	BLIE
D	$D(1)$	$D(2)$	$D(3)$
D_{sp}	$D_{sp}(1)$	$D_{sp}(2)$	$D_{sp}(3)$
D_e	$D_e(1)$	$D_e(2)$	$D_e(3)$
D_{be}	$D_{be}(1)$	$D_{be}(2)$	$D_{be}(3)$
D_{bi}	$D_{bi}(1)$	$D_{bi}(2)$	$D_{bi}(3)$
AD	AD(1)	AD(2)	AD(3)
CvM	CvM(1)	CvM(2)	CvM(3)

From the analysis of the results in Figures 4.1– 4.21, the remarkable results emerging from these figures are that no substantial noteworthy differences in powers are found over the alternative distributions from Group I, Group II and Group III for the D, D_{sp}, D_{be}, AD and CvM statistics when three different estimators are used.

Based on Figures 4.7–4.9, the powers of $D_e(1)$ are slightly less than the powers of $D_e(2)$ and $D_e(3)$ for all alternative distributions. For the D_{bi} test, there is no significant difference in powers among three different estimators under the alternatives from Group I. However, the powers of all tests using MLE against Group II and III are slightly greater than those using BLUE and BLIE. Interestingly, powers of a test with BLUE are often as good as powers of a test with BLIE.

From the evidence above, we may conclude that all three estimators are likely to be used as replacements of the parameters because they cause little difference in powers for each test. To test the Weibull distribution, the BLUE is chosen as a representative of the estimators in order to compare powers of the seven tests in the next section.

Table 4.6: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$\chi^2(1)$	10	5.8	5.64	**8.78**	5.12	4.25	6.63	5.79
	25	7.73	6.28	**12.63**	5.27	4.13	9.02	7.56
	40	8.32	7.23	**15.06**	6.45	4.83	10.77	9.23
	100	14.82	13.37	**21.21**	13.50	10.06	20.58	17.35
	150	17.99	18.81	**27.80**	18.85	14.41	27.8	22.55
	200	24.32	24.75	31.33	26.45	17.18	**36.44**	29.9
	250	28.02	31.73	34.94	33.44	22.11	**43.49**	35.85
	300	33.46	37.42	39.22	39.42	24.91	**52.05**	43.07
	350	39.88	43.94	44.86	46.98	29.56	**59.45**	50.15
	400	42.42	48.36	47.22	51.96	31.16	**65.06**	55.99
	500	52.5	59.16	53.33	61.43	39.58	**74.81**	65.86
$\chi^2(3)$	10	5.20	5.40	4.02	5.22	**6.34**	5.11	5.33
	25	5.37	6.60	3.21	6.62	**7.66**	5.31	5.55
	40	5.61	7.59	3.01	7.74	**8.25**	6.10	6.12
	100	6.46	9.96	3.86	**10.36**	10.19	7.82	7.30
	150	7.70	11.47	5.19	**11.59**	10.95	9.78	9.02
	200	9.06	13.21	5.83	**13.44**	12.36	11.81	10.65
	250	9.42	14.98	5.68	**15.34**	13.38	12.48	11.32
	300	10.66	16.6	6.35	**17.11**	13.32	14.68	12.65
	350	11.92	18.46	7.78	**18.54**	14.06	17.18	14.83
	400	12.44	19.4	8.16	**20.40**	14.92	18.17	15.66
	500	14.89	22.75	8.98	**23.16**	16.17	22	18.55
$\chi^2(4)$	10	5.53	5.90	3.07	5.74	**7.08**	5.28	5.87
	25	5.97	8.31	2.50	8.23	**9.92**	6.58	6.68
	40	6.84	10.38	3.00	10.63	**12.33**	7.56	7.41
	100	9.62	16.94	5.13	**17.17**	16.29	8.77	11.76
	150	12.39	**21.23**	7.91	21.18	19.39	13.2	15.83
	200	15.89	26.12	9.65	**26.51**	21.91	18.22	19.81
	250	18.28	30.93	10.84	**31.64**	25.72	23.51	23.55
	300	20.82	35.15	13.30	**35.72**	26.10	33.53	27.87
	350	24.4	37.94	15.70	**40.42**	28.74	38.04	31.88
	400	26.3	42.62	17.94	**45.82**	31.86	41.76	34.69
	500	32.19	49.46	20.60	51.73	36.09	**52.09**	43.2
$\chi^2(6)$	10	5.63	5.87	2.62	6.31	**7.90**	5.54	6.07
	25	7.26	10.9	2.44	11.04	**13.44**	8.42	8.37
	40	8.85	14.35	3.43	15.19	**16.20**	10.97	10.7
	100	15.14	28.71	8.89	**29.05**	28.3	22.83	20.13
	150	21.63	38.38	13.58	**38.90**	34.52	33.73	28.37
	200	28.02	48.00	17.96	**48.86**	40.83	44.25	37.53
	250	33.89	56.82	21.12	**57.84**	46.56	54.49	46.07
	300	39.25	63.20	26.92	**64.66**	50.68	63.64	53.5
	350	45.98	69.66	33.2	**71.28**	55.4	70.64	60.43
	400	50.08	75.48	36.54	**76.38**	60.38	76.02	65.84
	500	60.07	83.29	44.99	84.93	68.07	**85.34**	77.03
$\chi^2(10)$	10	6.37	6.84	2.30	7.1367	**9.64**	6.13	6.74
	25	8.86	14.58	2.66	14.51	**17.91**	10.81	10.62
	40	12.63	20.33	4.69	21.38	**23.45**	15.70	14.93
	100	23.68	44.82	14.25	**45.24**	43.13	37.71	32.14
	150	33.61	59.47	23.23	**59.88**	53.75	53.43	45.99
	200	44.79	71.35	29.96	**72.39**	63.52	68.63	59.49
	250	53.58	80.83	37.19	**81.81**	72.16	79.37	70.77
	300	61.51	86.93	46.04	**87.85**	76.42	86.52	78.32
	350	69.50	91.64	55.80	**91.74**	82.1	91.73	85.08
	400	75.14	94.56	63.18	**94.94**	86.74	94.35	89.24
	500	84.89	97.87	74.42	**98.23**	92.45	98.13	95.17
$LogN(0,1)$	10	9.68	11.30	1.93	11.84	**16.19**	10.07	11.02
	25	19.42	33.52	7.29	34.04	**40.56**	28.33	26.13
	40	29.53	52.31	15.90	52.14	**57.70**	44.19	39.42
	100	65.56	**91.97**	46.49	91.84	90.79	87.19	80.82
	150	83.76	98.51	69.26	**98.57**	97.85	97.22	94.43
	200	93.73	99.83	85.28	**99.85**	99.45	99.6	98.7
	250	97.78	**100**	93.79	99.98	99.92	99.94	99.75
	300	99.14	99.98	98.12	**99.98**	99.96	99.98	99.94
	350	99.72	**100**	99.54	**100**	**100**	99.99	99.97
	400	99.82	**100**	99.88	**100**	**100**	**100**	**100**
	500	99.99	**100**	99.99	**100**	**100**	**100**	**100**
$HN(0,1)$	10	5.96	5.70	**9.04**	5.05	4.41	6.72	5.87
	25	7.15	6.30	**12.47**	5.18	3.54	8.50	7.32
	40	8.51	6.99	**14.61**	6.07	4.76	10.39	9.31
	100	14.12	13.04	**21.00**	12.23	10.23	19.22	16.14
	150	18.72	18.76	27.10	19.36	14.36	**28.42**	23.42
	200	24.74	25.53	31.44	26.72	17.45	**36.91**	30.52
	250	29.75	31.31	34.25	32.45	21.64	**44.46**	37.12
	300	33.90	37.47	39.41	39.74	24.71	**53.03**	44.13
	350	39.80	43.56	43.92	45.89	29.22	**59.37**	50.72
	400	43.66	48.96	47.70	51.36	33.06	**64.93**	55.62
	500	53.25	59.42	53.88	61.16	39.44	**75.85**	66.57

The bold number is the highest power among the seven tests for each sample size.

Table 4.7: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$U(0,1)$	10	13.48	13.37	**19.85**	10.43	7.99	19.50	15.91
	25	30.27	36.57	36.47	31.42	44.01	**47.30**	37.97
	40	45.41	72.07	48.22	68.43	**81.58**	69.24	57.81
	100	87.53	**100**	92.93	99.97	**100**	99.06	96.27
	150	97.67	100	99.93	**100**	**100**	**100**	99.78
	200	99.69	**100**	**100**	**100**	**100**	**100**	**100**
	250	99.96	**100**	**100**	**100**	**100**	**100**	**100**
	300	99.99	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	6.6	6.27	**9.68**	5.03	3.80	8.03	6.96
	25	9.4	8.04	**14.08**	6.55	5.86	13.28	10.76
	40	12.24	12.07	16.64	10.28	13.22	**18.10**	14.92
	100	26.56	50.56	26.07	43.89	**60.04**	46.71	36.7
	150	39.27	79.60	34.49	74.92	**86.96**	66.49	53.33
	200	53.39	94.11	43.21	92.42	**97.15**	81.80	68.51
	250	64.78	98.89	54.17	97.91	**99.62**	90.81	80.10
	300	72.1	99.82	69.82	99.70	**99.96**	96.12	87.82
	350	80.76	99.98	82.74	**100**	**100**	98.18	92.9
	400	86.44	**100**	91.30	**100**	**100**	99.36	99.06
	500	93.14	**100**	97.66	**100**	**100**	99.94	99
$beta(2,5)$	10	4.82	4.63	4.73	4.10	4.50	4.87	**4.95**
	25	5.18	4.93	**5.39**	3.87	4.35	5.29	5.33
	40	4.62	4.17	**4.87**	4.32	4.30	4.75	4.87
	100	5.45	5.09	3.92	4.46	**6.23**	5.97	5.87
	150	5.64	5.58	3.95	5.52	**7.14**	6.20	6.10
	200	6.19	7.45	4.10	6.60	**8.91**	7.71	7.05
	250	6.7	8.59	3.20	6.98	**10.87**	8.78	8.14
	300	6.68	8.99	3.42	8.51	**12.35**	9.28	8.23
	350	7.61	10.80	3.40	10.50	**14.26**	10.63	9.39
	400	7.74	13.66	3.38	11.26	**18.44**	11.3	10.16
	500	9.5	16.25	2.78	13.40	**21.62**	13.74	12.3
$beta(5,1.5)$	10	7.95	7.53	**11.90**	5.91	4.19	10.36	8.42
	25	14.41	13.35	20.80	10.58	12.28	**21.76**	17.29
	40	19.63	25.46	25.57	22.45	**32.53**	32.22	25.09
	100	48.54	89.71	45.88	85.83	**94.05**	76.59	63.29
	150	68.00	99.36	67.67	98.81	**99.81**	93.05	83.7
	200	82.8	99.99	87.84	99.97	**100**	98.57	94.05
	250	91.58	**100**	96.55	**100**	**100**	99.71	97.98
	300	96.05	**100**	99.67	**100**	**100**	99.96	99.37
	350	98.36	**100**	99.96	**100**	**100**	99.99	99.91
	400	99.28	**100**	100	**100**	**100**	**100**	99.96
	500	99.88	**100**	100	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	10	31.50	31.29	35.31	25.84	28.12	**45.25**	38.23
	25	69.25	88.81	66.10	86.72	**94.39**	88.36	80.23
	40	88.76	99.73	88.72	99.66	**99.88**	98.72	96.13
	100	99.98	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	7.49	7.02	**11.57**	5.85	4.45	9.12	7.72
	25	11.32	9.96	**18.34**	7.79	6.64	15.41	12.43
	40	15.08	14.94	**23.65**	12.30	13.61	22.35	18.12
	100	32.97	48.67	39.53	44.15	51.17	**53.11**	43.43
	150	46.86	71.87	50.84	70.99	**75.65**	71.54	60.66
	200	60.33	88.58	60.77	87.20	**90.58**	85.66	75.08
	250	71.65	96.07	68.92	94.82	**97.02**	93.22	85.54
	300	79.91	98.76	78.68	98.34	**99.07**	97.12	91.89
	350	86.97	99.62	86.63	99.51	**100**	98.88	95.92
	400	91.12	99.88	91.80	99.92	**100**	99.58	97.52
	500	96.36	**100**	99.88	100	**100**	99.96	99.40
$beta(1,2)$	10	6.90	6.59	**10.31**	5.12	3.85	8.23	7.12
	25	10.84	9.81	**17.24**	7.80	7.38	15.79	12.54
	40	15.44	16.18	21.32	13.29	17.34	**22.63**	18.5
	100	33.16	60.71	34.41	55.27	**69.18**	56.14	44.85
	150	47.68	87.58	47.38	84.56	**92.71**	76.76	63.68
	200	63.85	97.68	60.32	96.41	**98.83**	89.83	78.41
	250	73.67	99.71	70.95	99.54	**99.85**	95.69	88.08
	300	83.42	99.93	84.09	99.90	**99.97**	98.55	93.81
	350	88.76	99.98	92.88	99.99	**100**	99.49	96.79
	400	93.06	**100**	97.18	**100**	**100**	99.94	99.5
	500	97.92	**100**	99.80	**100**	**100**	**100**	99.74

The bold number is the highest power among the seven tests for each sample size.

Table 4.8: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$Laplace(0,1)$	5	8.89	8.62	5.16	10.06	**10.36**	7.55	9.23
	10	19.11	22.66	7.88	24.46	**28.02**	21.82	22.82
	15	28.93	33.66	12.10	35.97	**36.78**	34.73	35.36
	20	38.68	44.28	22.39	**47.19**	44.92	45.96	46.02
	25	47.90	53.48	32.41	56.38	50.37	**56.42**	55.98
	30	56.62	60.71	41.16	62.60	56.27	**65.23**	65.06
	40	68.72	71.12	53.96	73.66	63.28	**77.91**	77.43
	50	77.99	78.40	65.15	80.12	68.89	**85.55**	84.91
	100	97.25	96.15	92.88	96.73	87.11	**98.78**	98.70
	150	99.74	99.48	98.86	99.50	94.56	**99.94**	99.92
	200	99.98	99.92	99.82	97.70	97.70	**99.98**	99.98
$logistic(0,1)$	5	6.68	6.48	3.18	7.23	**7.60**	5.36	6.73
	10	12.25	15.11	3.65	16.32	**21.06**	14.16	15.14
	15	17.41	22.95	4.73	24.85	**29.36**	22.36	21.85
	20	23.08	32.35	10.52	34.18	**37.72**	30.85	29.53
	25	29.07	40.85	16.96	42.67	**43.48**	38.85	37.08
	30	35.22	47.80	22.86	48.53	**49.59**	46.76	44.22
	40	42.94	57.84	32.16	**59.70**	57.66	58.81	55.50
	50	51.87	66.50	41.99	67.65	64.38	**68.04**	63.77
	100	82.56	90.87	74.16	91.78	85.24	**93.56**	91.13
	150	94.12	97.58	90.36	98.02	93.6	**98.87**	98.18
	200	98.18	99.24	96.58	99.58	96.96	**99.82**	99.64
$Cauchy(0,1)$	5	27.65	27.06	23.20	29.19	**29.66**	27.6	28.42
	10	56.61	58.71	47.12	60.44	**60.72**	59.70	59.89
	15	73.75	75.19	64.81	76.56	74.72	**77.91**	77.50
	20	83.98	85.13	78.30	86.60	82.53	**87.83**	87.72
	25	90.62	91.17	87.01	92.48	87.40	**93.40**	93.33
	30	95.11	95.10	92.26	95.48	91.60	**96.84**	96.66
	40	98.22	98.02	96.66	98.30	95.5	**99.25**	99.22
	50	99.36	99.26	99.01	99.39	97.75	**99.66**	99.67
	100	99.99	**100**	**100**	**100**	99.92	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	5.97	5.81	2.66	6.24	**6.64**	4.77	6.03
	10	9.68	11.30	1.93	11.97	**16.19**	10.07	11.02
	15	12.20	17.87	2.02	18.84	**24.91**	15.74	15.78
	20	16.32	25.74	4.38	25.70	**33.80**	22.06	20.99
	25	19.42	33.52	7.29	34.10	**40.56**	28.33	26.13
	30	22.46	40.20	9.73	40.52	**46.84**	33.60	29.86
	40	29.16	52.12	15.70	51.48	**58.36**	44.19	39.42
	50	35.56	63.10	20.95	62.64	**67.43**	54.80	48.10
	100	65.56	**91.97**	46.49	91.96	90.79	87.19	80.82
	150	83.86	98.66	69.66	**98.70**	97.82	97.22	94.43
	200	93.46	99.86	85.26	**99.88**	99.22	99.60	98.70
$t(1)$	5	28.11	27.60	23.79	29.31	**30.32**	28.17	28.95
	10	55.70	57.83	47.30	**60.40**	60.07	59.37	59.92
	15	73.62	75.02	65.14	76.76	74.29	**77.95**	77.79
	20	84.22	85.12	78.82	86.59	82.30	**87.67**	87.46
	25	90.71	91.18	87.02	92.15	87.52	**93.42**	93.33
	30	94.50	94.54	91.84	95.38	91.02	**96.36**	96.30
	40	98.06	97.86	96.72	98.36	95.34	**99.21**	99.2
	50	99.56	99.46	99.15	99.50	98.06	**99.78**	99.77
	100	**100**	99.99	**100**	99.99	99.97	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	9.58	9.16	5.60	10.57	**10.94**	8.84	10.07
	10	20.24	23.42	10.84	24.89	**28.33**	23.06	23.63
	15	29.63	35.3	16.52	37.18	**39.46**	36.39	35.98
	20	39.28	46.54	28.40	**48.34**	47.60	47.19	46.19
	25	45.97	52.91	37.43	**55.56**	51.98	55.25	53.69
	30	54.41	61.15	45.78	63.04	58.63	**63.96**	62.4
	40	65.02	70.84	59.02	74.38	65.78	**76.23**	75
	50	75.26	79.43	69.58	80.65	72.09	**84.28**	82.79
	100	95.65	95.86	93.68	96.54	89.39	**98.26**	97.93
	150	99.32	99.4	98.94	99.64	95.54	**99.84**	99.78
	200	99.90	99.96	99.70	99.96	98.10	**99.99**	**99.99**
$t(4)$	5	8.05	7.84	4.56	8.97	**9.10**	7.18	8.46
	10	16.03	19.24	6.83	20.59	**24.40**	18.01	18.93
	15	24.49	30.55	11.04	31.95	**35.77**	30.56	30.26
	20	31.06	39.65	19.28	41.77	**42.39**	39.38	38.34
	25	37.69	47.84	28.67	**49.71**	47.96	47.96	46.40
	30	45.81	55.54	37.19	56.73	54.37	**56.74**	54.75
	40	55.80	65.38	49.40	66.22	61.90	**68.81**	66.56
	50	66.42	74.06	60.05	75.13	68.64	**78.57**	76.23
	100	91.04	93.41	88.13	94.22	85.65	**96.44**	95.56
	150	98.18	98.62	97.30	98.76	93.94	**99.63**	99.49
	200	99.50	99.56	99.26	99.72	97.44	**99.98**	99.95
$t(6)$	5	6.90	6.61	3.63	7.73	**7.93**	5.97	7.62
	10	13.00	16.09	4.39	17.18	**21.43**	14.59	15.51
	15	19.91	26.08	6.62	27.28	**31.85**	25.15	24.98
	20	25.41	34.74	13.04	35.89	**38.95**	33.18	32.05
	25	30.59	42.20	20.08	43.95	**44.50**	40.90	38.85
	30	37.16	49.17	26.76	**50.63**	50.35	48.70	46.23
	40	46.66	61.16	38.08	**61.60**	59.70	61.19	58.14
	50	56.03	68.64	47.47	69.63	65.29	**71.00**	67.52
	100	84.99	91.05	78.86	91.79	84.67	**94.26**	92.42
	150	95.58	97.72	93.36	98.04	93.34	**98.99**	98.46
	200	98.77	99.28	97.84	99.60	96.78	**99.86**	99.79

The bold number is the highest power among the seven tests for each sample size.

Table 4.9: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$\chi^2(1)$	10	5.88	5.27	**8.26**	5.35	3.72	5.96	6.01
	25	7.92	5.28	**12.20**	4.69	3.62	8.90	7.60
	40	8.59	6.08	**15.10**	5.50	4.14	10.54	9.45
	100	15.12	11.76	**21.55**	11.83	8.97	20.02	17.65
	150	18.26	17.09	28.31	18.26	13.23	**30.18**	22.92
	200	24.79	23.03	32.11	24.50	15.90	**36.20**	30.45
	250	28.68	29.80	36.00	31.73	20.39	**44.26**	36.02
	300	33.80	35.66	40.64	39.01	23.22	**52.24**	43.40
	350	40.00	42.05	44.83	44.21	27.60	**59.38**	50.78
	400	44.94	48.46	49.11	49.36	31.80	**64.56**	56.43
	500	53.12	57.82	54.76	59.67	37.64	**75.64**	66.21
$\chi^2(3)$	10	4.95	5.56	4.26	5.30	**6.35**	5.22	5.36
	25	5.19	6.95	3.49	6.86	**7.90**	4.90	5.45
	40	5.69	8.05	3.41	7.80	**8.61**	5.22	5.92
	100	6.42	10.25	4.15	**10.42**	10.25	7.72	7.23
	150	7.60	11.63	5.42	**12.32**	10.91	10.08	9.00
	200	8.90	13.49	6.17	**14.26**	12.48	11.82	10.62
	250	9.53	15.12	6.08	**15.46**	13.48	12.28	11.15
	300	10.46	16.67	6.61	**17.40**	13.29	15.54	12.76
	350	11.89	18.54	7.58	**18.94**	14.85	17.26	14.84
	400	12.59	19.61	8.46	**20.36**	15.34	18.60	15.64
	500	14.85	23.04	9.59	**23.24**	16.16	22.08	18.56
$\chi^2(4)$	10	5.38	6	3.61	5.94	**7.15**	5.10	5.78
	25	5.86	8.51	3.28	7.82	**9.94**	5.76	6.65
	40	6.52	10.8	3.69	10.56	**12.73**	7.30	7.25
	100	9.46	16.89	5.73	**16.46**	16.36	12.88	11.57
	150	11.94	**21.11**	8.48	21.04	19.32	18.50	15.73
	200	15.81	26.15	10.32	**26.42**	21.74	22.88	19.73
	250	18.3	31.00	11.59	**32.24**	25.62	26.9	23.20
	300	20.29	35.16	14.34	**36.96**	26.09	34.22	27.74
	350	23.49	39.30	15.93	**40.20**	29.91	38.24	31.85
	400	25.85	42.15	18.41	**45.44**	30.54	42.80	34.45
	500	31.83	49.39	21.69	**52.28**	35.57	51.98	42.92
$\chi^2(6)$	10	5.27	6.31	3.25	7.16	**8.16**	5.64	6.11
	25	6.89	11.60	3.55	10.16	**13.83**	8.02	8.22
	40	8.68	14.66	4.67	14.64	**16.47**	10.06	10.47
	100	14.87	**28.75**	9.88	27.98	27.93	22.66	19.81
	150	20.91	38.06	15.08	**39.04**	33.85	34.52	28
	200	27.7	47.71	19.75	**48.26**	40.11	43.62	37.24
	250	33.44	56.15	23.16	**56.50**	45.65	52.74	45.45
	300	38.48	62.91	28.89	**65.20**	49.38	64.00	53.22
	350	44.89	69.20	33.12	**71.16**	55.36	69.82	60.17
	400	49.49	74.85	39.38	75.80	59.42	**76.5**	65.55
	500	59.44	82.92	46.73	84.48	66.89	**85.54**	76.69
$\chi^2(10)$	10	6.05	7.42	3.25	7.62	**9.90**	6.58	6.94
	25	8.45	15.36	4.53	15.18	**18.09**	10.70	10.44
	40	12.10	20.96	6.79	20.46	**23.40**	15.80	14.34
	100	22.95	**44.68**	16.41	43.98	42.34	37.64	31.43
	150	32.45	58.86	25.65	**60.14**	52.66	54.08	45.41
	200	43.97	70.82	32.42	**72.00**	62.19	68.92	58.92
	250	52.61	80.28	40.01	**81.68**	70.69	78.58	69.77
	300	60.13	86.40	49.10	**87.44**	75.20	86.82	77.76
	350	69.08	91.21	57.34	91.68	81.42	**91.76**	84.74
	400	74.35	94.05	65.87	**94.66**	85.55	94.38	88.95
	500	84.01	97.62	76.72	98.10	91.39	**98.30**	95.04
$LogN(0,1)$	10	9.34	12.40	3.97	12.43	**16.88**	10.36	11.25
	25	18.62	34.49	11.94	33.89	**40.13**	28.40	25.46
	40	28.12	52.20	20.64	51.58	**56.83**	43.92	38.57
	100	63.39	**91.66**	51.29	91.59	90.10	86.94	79.81
	150	81.86	98.38	73.18	**98.48**	97.35	97.18	94.17
	200	93.03	99.80	87.30	**99.80**	99.33	99.56	98.54
	250	97.35	**100**	94.65	99.98	99.89	99.90	99.74
	300	98.93	**99.98**	98.58	**99.98**	99.95	99.98	99.94
	350	99.71	99.99	99.60	99.99	**100**	99.98	99.97
	400	99.88	**100**	99.88	**100**	**100**	**100**	**100**
	500	99.99	**100**	99.99	**100**	**100**	**100**	**100**
$HN(0,1)$	10	5.93	5.43	**8.56**	5.25	4.00	6.62	5.99
	25	7.15	5.28	**12.18**	4.51	3.05	8.64	7.45
	40	8.75	5.91	**14.58**	5.38	4.06	10.40	9.28
	100	14.39	11.21	**21.20**	10.78	9.02	19.24	16.33
	150	18.87	17.01	27.84	17.83	13.15	**28.12**	23.71
	200	25.35	23.58	32.23	25.03	16.05	**36.26**	31.01
	250	30.30	29.43	35.25	30.53	20.49	**43.74**	37.25
	300	34.27	35.62	40.75	38.35	23.16	**52.94**	44.61
	350	40.56	41.92	45.11	44.39	27.38	**59.48**	51.30
	400	44.67	47.59	48.83	49.75	30.69	**65.56**	56.10
	500	53.89	58.04	55.33	59.65	37.40	**75.70**	66.90

The bold number is the highest power among the seven tests for each sample size.

Table 4.10: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$U(0,1)$	10	14.21	12.00	**20.92**	9.97	5.44	19.68	15.94
	25	32.05	31.54	42.77	25.61	36.30	**48.62**	39.43
	40	49.17	66.52	57.26	60.33	**77.42**	72.12	60.46
	100	91.16	99.99	92.79	99.93	**100**	99.32	97.21
	150	98.66	100	99.86	**100**	**100**	**100**	99.89
	200	99.88	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	6.59	5.50	**9.59**	4.80	3.14	7.52	6.88
	25	9.56	6.47	**14.47**	5.53	4.15	13.38	10.62
	40	12.84	9.65	18.01	8.36	10.64	**18.16**	15.15
	100	28.22	46.57	29.62	39.38	**56.22**	49.62	38.56
	150	41.99	76.58	39.77	70.98	**84.94**	70.94	56.61
	200	56.74	92.79	47.98	90.94	**96.39**	83.74	71.93
	250	68.90	98.58	57.59	97.30	**99.50**	92.04	82.95
	300	75.78	99.76	70.69	99.55	**99.92**	97.06	90.43
	350	84.35	99.98	81.58	99.89	**99.99**	98.90	94.79
	400	89.27	**100**	89.49	99.99	**100**	99.46	97.29
	500	94.82	**100**	97.48	**100**	**100**	99.96	99.25
$beta(2,5)$	10	4.62	4.41	4.78	4.28	4.39	**4.86**	4.85
	25	**5.20**	4.62	5.16	3.64	4.27	4.34	5.10
	40	**4.57**	3.98	4.56	3.96	4.30	4.54	4.65
	100	5.37	4.85	3.94	4.23	**6.14**	5.50	5.64
	150	5.67	5.27	4.07	5.30	6.67	**7.26**	6.04
	200	6.24	6.88	4.16	6.22	**8.42**	7.28	7.01
	250	6.71	8.16	3.29	6.39	**10.59**	8.06	7.99
	300	6.57	8.53	3.49	8.11	**11.78**	9.66	8.18
	350	7.58	10.32	3.54	9.15	**14.76**	10.70	9.57
	400	8.96	12.58	3.90	10.45	**16.46**	11.28	10.85
	500	9.14	15.62	3.29	12.60	**20.97**	14.32	11.99
$beta(5,1.5)$	10	8.01	6.43	**11.80**	5.54	3.14	9.92	8.34
	25	15.01	11.07	**22.14**	8.52	8.50	21.30	17.66
	40	20.65	21.21	29.10	17.81	27.54	**35.22**	26.04
	100	52.08	87.37	52.56	82.54	**92.98**	79.70	66.78
	150	72.15	99.11	70.78	98.45	**99.75**	94.98	86.69
	200	86.56	99.98	86.87	99.96	**100**	99.22	95.78
	250	93.99	**100**	95.62	**100**	**100**	99.78	98.62
	300	97.42	**100**	98.47	**100**	**100**	**100**	99.68
	350	99.16	**100**	99.93	**100**	**100**	**100**	99.93
	400	99.59	**100**	**100**	**100**	**100**	**100**	99.98
	500	99.96	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	10	33.39	30.47	42.47	25.68	19.10	**46.44**	38.96
	25	73.77	85.45	79.02	80.91	**92.48**	90.24	82.42
	40	92.10	99.60	93.11	99.37	**99.84**	99.30	97.09
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	7.64	6.09	**10.92**	5.59	3.40	8.78	7.67
	25	11.56	8.25	**18.65**	6.62	4.76	17.96	12.59
	40	15.57	12.17	**24.85**	10.12	10.80	21.82	18.44
	100	34.31	44.73	43.08	40.37	46.97	**53.34**	44.78
	150	48.78	68.59	55.84	67.44	72.02	**74.86**	62.26
	200	62.84	86.65	65.45	85.04	**88.69**	86.86	76.82
	250	74.08	95.30	73.12	93.69	**96.18**	93.42	87.03
	300	81.81	98.42	81.59	97.86	**98.85**	97.84	93.06
	350	88.79	99.57	88.39	99.41	**99.71**	98.88	96.52
	400	92.53	99.81	92.62	99.82	**99.94**	99.74	98.00
	500	97.01	100	97.31	99.97	**100**	99.98	99.68
$beta(1,2)$	10	6.89	5.53	**10.12**	4.95	3.19	9.56	6.88
	25	11.17	8.10	**17.98**	6.36	5.07	15.70	12.68
	40	15.92	12.97	**23.00**	10.83	13.73	22.46	18.94
	100	35.17	56.25	39.37	50.40	**65.34**	57.34	46.89
	150	50.68	84.91	53.03	81.43	**91.37**	79.70	66.85
	200	67.18	97.10	64.43	95.51	**98.58**	90.42	81.10
	250	77.27	99.60	72.79	99.35	**99.83**	96.64	89.98
	300	86.15	99.93	84.35	99.88	**99.96**	98.92	95.14
	350	90.96	**100**	92.33	99.99	**100**	99.66	97.74
	400	94.92	**100**	97.08	**100**	**100**	99.96	99.07
	500	98.28	**100**	99.71	**100**	**100**	99.98	99.89

The bold number is the highest power among the seven tests for each sample size.

Table 4.11: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group III for sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$Laplace(0,1)$	5	9.73	9.71	8.24	10.60	**11.44**	8.42	9.82
	10	19.56	24.02	12.10	24.88	**27.00**	23.03	24.12
	15	28.93	34.3	21.59	35.23	34.12	35.57	**35.92**
	20	37.33	43.57	32.12	46.18	41.05	**47.03**	46.43
	25	46.92	52.22	40.26	54.51	46.24	**57.06**	56.08
	30	55.42	58.61	48.28	60.53	51.40	**65.73**	65.04
	40	67.43	68.97	61.76	70.24	57.72	**78.08**	76.93
	50	76.62	75.98	71.28	77.85	62.10	**85.56**	84.58
	100	96.73	95.10	94.00	95.95	79.88	**98.80**	98.60
	150	99.68	98.97	98.98	99.27	89.72	**99.93**	99.91
	200	99.91	99.84	99.84	99.91	95.01	**99.98**	99.98
$logistic(0,1)$	5	6.92	6.96	6.11	7.49	**8.39**	6.00	6.92
	10	12.14	16.66	6.23	16.95	**21.01**	15.21	15.6
	15	17.08	24.35	11.14	24.59	**28.35**	22.95	22.02
	20	21.99	32.40	17.64	33.89	**36.11**	31.65	29.29
	25	27.67	40.51	23.56	**41.59**	41.50	39.05	36.56
	30	33.29	46.99	29.12	**47.51**	46.99	47.12	43.69
	40	41.67	57.45	39.37	57.19	55.02	**58.72**	54.4
	50	49.23	64.32	47.57	65.94	60.65	**67.92**	62.79
	100	80.52	89.51	77.33	90.48	81.39	**93.46**	90.59
	150	93.54	96.73	92.68	97.40	91.26	**98.8**	98.01
	200	98.02	99.20	97.08	99.43	95.98	**99.8**	99.58
$Cauchy(0,1)$	5	29.08	29.11	26.82	30.47	**31.18**	28.53	30.07
	10	57.8	59.99	51.95	60.90	58.71	61.20	**61.36**
	15	74.67	75.52	71.1	76.32	72.27	78.68	**78.78**
	20	84.26	84.86	82.86	86.39	80.08	**88.66**	88.40
	25	90.94	90.79	89.62	92.15	85.79	**93.77**	93.73
	30	95.14	94.82	93.96	95.26	90.04	**97.01**	96.83
	40	98.28	98.12	98.01	98.04	94.80	**99.30**	99.24
	50	99.33	99.22	99.15	99.33	97.02	99.67	**99.68**
	100	99.99	**100**	**100**	**100**	99.89	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	6.12	6.14	5.35	6.35	**7.10**	5.19	6.33
	10	9.34	12.40	3.97	12.31	**16.88**	10.66	11.25
	15	11.99	19.50	6.00	19.18	**25.58**	15.97	15.54
	20	15.08	26.60	9.04	26.15	**33.13**	22.62	20.79
	25	18.62	34.49	11.94	34.17	**40.13**	28.44	25.46
	30	21.20	40.80	14.26	40.64	**46.48**	33.81	29.35
	40	28.12	52.20	20.64	51.58	**56.83**	44.20	38.57
	50	34.00	63.23	25.66	62.38	**66.57**	54.84	47.11
	100	63.39	91.66	51.29	**91.65**	90.10	87.06	79.81
	150	81.86	98.38	73.18	**98.48**	97.35	97.11	94.17
	200	93.03	99.80	87.30	**99.80**	99.33	99.59	98.54
$t(1)$	5	29.51	29.54	27.90	30.42	**31.14**	29.09	30.53
	10	57.46	59.16	52.82	60.89	58.12	60.83	**61.63**
	15	74.7	75.48	71.96	76.67	72.05	78.63	**78.87**
	20	84.57	84.78	82.93	86.41	79.99	**88.29**	88.27
	25	90.87	90.96	89.64	91.88	85.82	**93.61**	93.57
	30	94.48	94.24	94.23	**98.23**	89.47	96.59	96.57
	40	98.48	98.15	98.16	98.04	95.13	99.24	**99.25**
	50	99.55	99.39	99.17	99.46	97.38	**99.82**	99.81
	100	99.99	99.99	99.99	99.99	99.95	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	10.22	10.22	9.11	11.07	**11.78**	9.47	10.68
	10	20.44	24.62	14.82	25.56	**26.67**	24.35	24.79
	15	29.91	35.99	25.70	36.78	**37.79**	37.32	36.57
	20	38.28	46.30	36.23	47.67	44.65	**48.31**	46.71
	25	44.87	52.07	44.71	54.12	48.93	**55.70**	53.75
	30	53.21	59.70	51.88	61.79	55.00	**64.49**	62.31
	40	65.06	70.69	65.83	71.58	61.89	**76.47**	74.48
	50	73.21	77.30	73.89	78.96	67.17	**84.37**	82.35
	100	94.82	94.91	95.01	95.87	84.33	**98.24**	97.79
	150	99.17	99.05	99.08	99.25	92.34	**99.81**	99.78
	200	99.90	99.82	99.79	99.91	96.78	**99.99**	**99.99**
$t(4)$	5	8.42	8.34	7.19	9.38	**10.01**	7.86	9.03
	10	16.23	20.52	10.06	21.19	**24.02**	19.26	19.71
	15	24.30	31.57	18.27	31.69	**34.57**	31.35	30.48
	20	29.66	39.30	**27.64**	**41.26**	39.82	40.30	38.41
	25	36.57	46.94	34.70	48.43	44.96	**48.65**	46.28
	30	43.92	54.04	42.72	55.34	50.89	**57.18**	54.20
	40	54.16	63.65	55.10	65.51	57.76	**68.83**	65.64
	50	63.98	71.89	64.72	73.25	63.75	**78.45**	75.41
	100	89.52	92.18	90.06	93.15	80.28	**96.31**	95.22
	150	97.67	98.05	97.77	98.56	89.32	**99.62**	99.50
	200	99.71	99.67	99.44	99.69	95.12	**99.98**	99.94
$t(6)$	5	7.36	7.44	6.39	8.06	**8.94**	6.60	8.00
	10	13.12	17.30	7.71	17.84	**21.37**	15.56	16.08
	15	19.79	27.33	13.72	27.05	**31.09**	25.84	25.03
	20	24.37	34.85	20.20	35.55	**37.02**	34.02	32.06
	25	28.99	41.57	25.69	**42.88**	42.01	41.23	38.45
	30	35.57	48.13	32.68	**49.40**	47.53	48.97	45.51
	40	45.09	59.16	44.56	60.42	55.11	**61.44**	57.07
	50	52.67	66.73	53.43	67.79	60.94	**70.89**	66.24
	100	82.40	89.59	81.38	90.57	80.02	**94.11**	91.83
	150	94.14	96.94	94.13	97.61	89.50	**98.94**	98.26
	200	98.59	99.17	98.32	99.32	94.79	**99.84**	99.74

The bold number is the highest power among the seven tests for each sample size.

Table 4.12: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group I for sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$\chi^2(1)$	10	5.04	5.68	**8.28**	4.77	4.04	6.12	5.08
	25	6.22	6.10	**12.80**	5.01	3.85	8.71	6.75
	40	6.58	6.34	**14.92**	6.03	4.43	10.51	8.78
	100	13.44	14.02	**22.66**	12.87	9.14	20.47	16.61
	150	18.12	18.54	**28.44**	19.21	13.32	27.75	21.80
	200	24.00	24.46	32.32	25.66	15.94	**36.24**	29.57
	250	29.16	31.20	36.22	32.77	20.40	**43.50**	35.35
	300	33.25	37.90	42.84	40.05	23.25	**52.24**	42.75
	350	38.97	43.34	46.38	45.23	27.63	**59.49**	49.78
	400	44.13	47.26	48.44	50.33	31.85	**65.26**	55.73
	500	52.46	58.20	54.44	60.33	37.62	**74.83**	65.63
$\chi^2(3)$	10	4.98	5.48	3.48	5.70	**6.03**	5.31	5.45
	25	5.74	6.52	3.24	6.52	**7.68**	5.42	5.66
	40	5.76	6.70	3.00	7.13	**8.43**	6.08	6.41
	100	6.94	9.84	3.32	10.10	**10.18**	7.72	7.39
	150	7.78	11.22	5.68	**11.99**	10.9	9.63	9.21
	200	8.38	13.10	5.86	**13.68**	12.47	11.74	10.87
	250	9.60	14.74	6.08	**15.09**	13.48	12.58	11.54
	300	11.12	16.34	7.32	**16.91**	13.30	14.74	13.19
	350	12.28	18.62	8.16	**19.25**	14.86	17.22	15.05
	400	13.17	19.08	8.48	**20.17**	15.34	18.27	16.02
	500	15.49	22.30	8.96	**22.92**	16.16	22.08	18.93
$\chi^2(4)$	10	5.96	6.26	3.18	6.09	**6.98**	5.49	6.26
	25	7.16	8.1	3.42	8.68	**9.67**	6.59	7.08
	40	7.86	10.02	3.58	9.98	**12.37**	7.56	7.77
	100	9.86	**17.28**	5.42	16.48	16.28	13.15	12.00
	150	12.64	21.04	8.90	**21.18**	19.28	18.09	16.00
	200	15.48	25.02	10.80	**26.71**	21.71	23.29	20.30
	250	18.82	29.40	11.46	**30.58**	25.57	27.91	23.90
	300	21.44	33.84	15.08	**35.50**	26.07	33.57	28.29
	350	24.40	37.68	16.52	**40.21**	29.93	38.22	32.41
	400	27.02	42.14	18.72	**43.48**	30.56	41.70	35.21
	500	32.85	49.46	20.40	50.29	35.61	**52.11**	43.54
$\chi^2(6)$	10	6.5	6.64	**2.60**	6.76	**8.01**	5.83	6.53
	25	8.28	10.04	3.60	10.85	**13.44**	8.46	8.76
	40	9.32	13.70	4.40	14.01	**16.17**	10.99	11.11
	100	15.32	27.52	9.38	**27.99**	27.75	22.74	20.39
	150	21.64	38.48	15.20	**38.84**	33.85	33.71	28.79
	200	27.26	46.42	19.00	**48.45**	40.08	44.08	38.07
	250	34.00	54.48	23.20	**56.47**	45.62	54.44	46.52
	300	39.82	63.58	30.5	**65.02**	49.41	63.57	54.09
	350	45.96	69.08	34.78	**70.93**	55.42	70.66	60.85
	400	50.70	74.80	37.68	75.30	59.44	**75.93**	66.51
	500	60.56	82.32	46.60	83.40	66.92	**85.25**	77.28
$\chi^2(10)$	10	6.94	6.96	2.48	7.86	**9.40**	6.73	7.36
	25	10.16	13.26	3.62	13.63	**17.57**	10.89	11.11
	40	12.76	20.3	6.02	20.07	**23.04**	15.64	15.29
	100	22.68	43.24	14.42	**43.37**	42.24	37.38	32.46
	150	33.92	**59.18**	25.02	59.11	52.62	53.48	46.27
	200	45.47	70.18	32.18	**72.19**	62.21	68.31	59.83
	250	53.75	80.16	39.94	**80.84**	70.67	79.34	70.73
	300	61.47	86.32	50.72	**87.15**	75.21	86.66	78.48
	350	69.93	91.16	57.82	91.57	81.46	**91.69**	85.24
	400	75.18	94.36	64.28	**94.73**	85.57	94.39	89.34
	500	84.65	97.40	75.86	97.66	91.41	**98.14**	95.21
$LogN(0,1)$	10	12.18	11.82	2.32	12.93	**16.07**	10.60	12.02
	25	20.95	31.68	11.16	32.25	**39.08**	28.46	26.90
	40	30.20	50.20	19.70	50.18	**56.32**	43.84	40.34
	100	64.55	91.30	49.18	**91.31**	90.09	86.74	79.98
	150	82.23	98.46	72.44	**98.46**	97.35	97.14	94.52
	200	93.51	**99.82**	86.26	99.79	99.33	99.48	98.36
	250	97.35	**100**	94.38	99.98	99.89	99.92	99.74
	300	99.02	**99.98**	98.70	99.98	99.95	99.98	99.90
	350	99.74	**100**	99.60	99.99	**100**	99.98	99.94
	400	99.88	**100**	99.86	**100**	**100**	**100**	100
	500	99.99	**100**	99.98	**100**	**100**	**100**	100
$HN(0,1)$	10	5.14	5.24	**8.98**	4.41	3.95	6.12	4.78
	25	6.12	6.02	**12.96**	4.79	3.26	8.36	6.60
	40	7.70	6.66	**13.92**	5.80	4.34	10.04	8.56
	100	13.42	12.42	**20.80**	12.24	9.19	18.9	15.58
	150	17.79	18.06	**28.16**	18.88	13.22	27.86	22.58
	200	24.27	23.40	31.46	26.06	16.08	**35.44**	28.78
	250	29.17	29.48	35.68	31.53	20.51	**44.00**	35.96
	300	33.52	35.84	41.56	39.21	23.18	**52.70**	43.62
	350	39.38	43.36	45.32	45.09	27.41	**59.70**	50.56
	400	43.90	47.98	48.78	50.84	30.70	**65.60**	55.94
	500	53.14	57.50	54.54	60.45	37.41	**76.18**	66.96

The bold number is the highest power among the seven tests for each sample size.

Table 4.13: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$U(0,1)$	10	10.71	12.18	**21.74**	9.00	6.40	18.50	13.50
	25	29.31	33.27	41.46	27.32	39.05	**49.08**	38.02
	40	46.61	67.25	57.94	61.43	**78.62**	73.52	60.42
	100	90.42	99.99	90.28	99.93	**100**	99.26	96.70
	150	98.59	**100**	99.88	**100**	**100**	**100**	99.82
	200	99.86	**100**	99.98	**100**	**100**	**100**	100
	250	**100**	**100**	**100**	**100**	**100**	**100**	100
	300	**100**	**100**	**100**	**100**	**100**	**100**	100
	350	**100**	**100**	**100**	**100**	**100**	**100**	100
	400	**100**	**100**	**100**	**100**	**100**	**100**	100
	500	**100**	**100**	**100**	**100**	**100**	**100**	100
$beta(2,2)$	10	5.50	5.68	**10.03**	4.49	3.30	7.60	5.95
	25	8.47	7.80	**14.74**	6.23	4.80	13.56	9.99
	40	11.65	11.38	18.16	9.52	11.38	**18.96**	14.75
	100	26.95	45.76	29.89	40.89	**56.67**	49.54	38.02
	150	41.00	76.54	39.65	71.12	**85.00**	69.93	56.21
	200	56.17	93.10	47.96	90.97	**96.41**	84.54	71.64
	250	68.24	98.26	56.83	97.35	**99.50**	92.92	82.98
	300	75.76	99.68	69.64	99.56	**99.92**	97.27	90.51
	350	84.20	99.94	80.45	99.89	**99.99**	98.70	94.69
	400	89.19	**100**	88.24	99.99	**100**	99.57	97.34
	500	94.75	**100**	97.19	**100**	**100**	99.96	99.21
$beta(2,5)$	10	4.91	4.76	4.91	4.33	4.27	**4.93**	4.81
	25	5.20	4.36	**5.23**	3.93	4.28	5.22	5.10
	40	4.71	4.38	4.66	4.16	4.25	**4.85**	5.00
	100	5.32	4.78	3.98	4.51	6.16	**6.20**	5.83
	150	5.71	6.04	4.06	5.34	6.71	6.61	6.33
	200	6.36	6.80	4.19	6.26	**8.42**	8.10	7.16
	250	6.93	7.02	3.32	6.41	**10.59**	9.20	8.40
	300	6.90	9.38	3.51	8.16	**11.79**	9.74	8.43
	350	7.71	10.12	3.58	9.12	**14.77**	11.12	9.85
	400	9.13	12.86	3.89	10.56	**16.49**	12.64	11.21
	500	9.17	15.02	3.26	12.64	**20.96**	14.26	12.33
$beta(5,1.5)$	10	6.24	6.96	**12.50**	5.19	3.33	10.09	7.40
	25	13.13	12.24	**22.61**	9.83	9.72	22.2	16.40
	40	19.03	23.56	29.14	19.62	28.82	**33.93**	25.37
	100	50.62	87.34	52.62	83.08	**93.11**	79.90	66.46
	150	71.27	99.22	69.35	98.42	**99.77**	94.74	86.47
	200	86.31	99.96	85.47	99.98	**100**	99.20	95.75
	250	93.85	**100**	94.79	**100**	**100**	99.81	98.63
	300	97.43	**100**	99.32	**100**	**100**	99.97	99.68
	350	99.15	**100**	99.91	**100**	**100**	**100**	99.93
	400	99.59	**100**	99.97	**100**	**100**	**100**	99.97
	500	99.96	**100**	**100**	**100**	**100**	**100**	100
$beta(0.5,0.5)$	10	28.16	30.76	42.39	24.17	22.46	**45.96**	35.41
	25	71.15	85.94	78.40	81.43	**93.26**	90.25	81.40
	40	91.49	99.68	92.43	99.34	**99.86**	99.06	96.82
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
	250	**100**	**100**	**100**	**100**	**100**	**100**	100
	300	**100**	**100**	**100**	**100**	**100**	**100**	100
	350	**100**	**100**	**100**	**100**	**100**	**100**	100
	400	**100**	**100**	**100**	**100**	**100**	**100**	100
	500	**100**	**100**	**100**	**100**	**100**	**100**	100
$beta(0.5,3)$	10	5.96	6.80	**11.56**	5.29	3.94	8.66	6.53
	25	9.92	9.82	**18.86**	7.33	5.36	15.11	11.50
	40	14.01	14.08	**25.04**	11.48	11.45	22.56	17.42
	100	32.50	45.48	43.33	42.25	47.28	**54.49**	43.66
	150	47.25	71.66	55.89	68.20	72.16	**73.25**	61.51
	200	61.82	87.86	65.53	85.27	**88.73**	87.10	76.27
	250	73.19	94.98	72.93	93.78	**96.19**	94.17	86.74
	300	81.58	98.30	81.47	97.89	**98.86**	97.63	93.01
	350	88.46	99.56	88.16	99.41	**99.70**	99.14	96.48
	400	92.25	99.81	92.39	99.83	**99.94**	99.59	97.94
	500	96.92	**100**	97.18	99.97	**100**	99.91	99.67
$beta(1,2)$	10	5.67	5.58	**10.64**	4.59	3.32	8.06	5.98
	25	9.60	8.86	**18.14**	7.30	5.69	16.03	11.80
	40	14.44	14.66	23.16	12.09	14.62	**23.50**	18.23
	100	33.91	56.38	39.59	51.58	**65.73**	58.91	45.99
	150	49.39	86.74	52.86	81.62	**91.43**	79.38	66.29
	200	66.43	96.86	63.92	95.49	**98.59**	91.50	80.92
	250	76.64	99.58	72.06	99.34	**99.83**	96.73	89.86
	300	85.94	99.92	83.45	99.87	**99.96**	99.08	95.14
	350	90.73	99.98	91.63	99.99	**100**	99.67	97.68
	400	94.81	**100**	96.65	**100**	**100**	99.95	99.07
	500	98.25	**100**	99.58	**100**	**100**	99.99	99.88

The bold number is the highest power among the seven tests for each sample size.

Table 4.14: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group III for sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$Laplace(0,1)$	5	11.37	10.50	5.81	12.33	**12.63**	8.81	11.26
	10	21.69	23.17	9.58	25.46	**28.25**	22.74	24.21
	15	30.35	32.97	19.97	35.75	**36.08**	35.22	35.97
	20	39.61	42.92	29.35	**47.47**	43.34	45.88	46.35
	25	47.85	51.67	38.35	54.68	48.17	**56.26**	55.77
	30	56.52	58.60	46.97	60.45	53.31	**64.90**	64.84
	40	68.06	69.29	60.59	70.74	59.34	**77.52**	76.79
	50	77.12	76.52	70.04	78.48	63.38	**85.03**	84.22
	100	96.74	95.38	94.41	95.90	80.64	**98.74**	98.58
	150	99.69	99.02	98.93	99.27	90.34	**99.92**	99.91
	200	99.91	99.88	99.84	99.92	95.41	**99.98**	99.98
$logistic(0,1)$	5	8.62	8.16	3.84	9.38	**9.45**	6.46	8.22
	10	14.64	15.42	4.68	16.73	**20.35**	15.14	16.37
	15	18.73	22.26	10.01	23.34	**28.13**	23.03	22.58
	20	24.26	30.66	16.26	33.69	**36.44**	31.08	30.18
	25	29.04	38.62	22.10	40.07	**41.64**	38.73	37.17
	30	35.23	45.33	28.35	45.00	**47.11**	46.56	44.18
	40	43.26	56.39	38.55	56.17	55.07	**58.25**	55.12
	50	50.26	63.72	46.96	65.11	60.63	**67.41**	62.90
	100	80.65	89.24	77.40	90.07	81.51	**93.29**	90.76
	150	93.73	96.70	92.29	97.36	91.38	**98.76**	98.05
	200	98.05	99.20	96.98	99.43	95.98	**99.81**	99.60
$Cauchy(0,1)$	5	28.71	27.64	23.88	**30.30**	30.22	28.11	29.73
	10	56.61	58.14	49.78	59.96	**60.88**	59.66	59.72
	15	73.38	74.71	69.66	76.32	74.57	**77.62**	77.22
	20	83.48	84.52	81.46	86.75	81.95	**87.54**	87.32
	25	89.96	90.41	88.83	91.49	87.04	**93.24**	93.04
	30	94.70	94.71	93.29	95.01	91.03	**96.69**	96.49
	40	98.22	98.13	97.87	97.99	95.37	**99.21**	99.10
	50	99.27	99.18	99.18	99.34	97.32	**99.63**	99.63
	100	99.99	**100**	**100**	99.99	99.89	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	7.99	7.73	3.58	8.37	**8.55**	5.67	7.86
	10	12.18	11.82	2.66	12.93	**16.07**	10.98	12.35
	15	14.34	16.92	5.00	17.26	**24.09**	16.42	17.16
	20	17.90	24.29	8.11	25.71	**32.60**	22.36	22.42
	25	20.95	31.94	11.16	32.25	**39.08**	28.40	26.72
	30	23.72	38.04	13.53	38.14	**45.98**	33.64	30.69
	40	30.20	50.59	19.91	50.18	**56.32**	44.29	40.29
	50	35.71	61.55	24.93	60.66	**66.25**	54.77	48.31
	100	64.55	91.43	50.54	**91.31**	90.09	87.14	80.73
	150	82.23	98.32	71.55	**98.46**	97.35	97.13	94.42
	200	93.51	**99.80**	86.28	**99.79**	99.33	99.61	98.62
$t(1)$	5	29.31	28.42	24.78	29.56	**30.62**	28.77	30.05
	10	56.14	57.48	49.37	59.86	**60.06**	59.22	59.47
	15	73.01	74.35	70.19	75.71	74.06	**77.61**	77.36
	20	83.71	84.37	81.82	86.56	81.88	**87.28**	87.15
	25	90.25	90.73	88.81	92.01	87.27	**93.18**	93.14
	30	94.08	94.21	93.02	95.02	90.63	**96.21**	96.05
	40	98.34	98.13	97.93	98.11	95.63	**99.16**	99.13
	50	99.52	99.41	99.31	99.47	97.69	**99.78**	99.76
	100	99.99	99.99	**100**	**100**	99.95	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	11.72	11.06	6.36	12.32	**12.53**	9.79	11.98
	10	22.08	23.25	12.49	25.28	**27.93**	23.76	24.56
	15	30.59	34.28	24.26	36.01	**38.31**	36.83	36.56
	20	39.22	45.04	34.81	**47.28**	46.08	47.03	46.2
	25	45.04	51.15	42.77	54.16	50.04	**54.75**	53.14
	30	53.52	58.77	51.00	61.04	56.29	**63.60**	61.72
	40	64.91	70.50	65.24	71.19	62.88	**75.65**	74.23
	50	73.25	77.11	73.07	79.13	67.96	**83.71**	81.99
	100	94.73	95.03	94.59	95.73	84.74	**98.15**	97.71
	150	99.16	99.09	99.05	99.24	92.57	**99.81**	99.78
	200	99.88	99.8	99.79	99.91	96.84	**99.99**	**99.99**
$t(4)$	5	10.1	9.45	5.3	**11.08**	10.95	8.24	10.11
	10	17.8	19.58	8.41	20.9	**24.48**	18.95	20.07
	15	25.55	29.47	18.51	29.92	**34.55**	31.19	31.07
	20	31.27	37.87	25.56	**41.38**	40.71	39.51	38.46
	25	37.42	45.20	33.98	47.07	45.53	**47.75**	45.88
	30	44.97	52.94	42.69	54.45	51.77	**56.19**	54.06
	40	54.81	63.57	54.32	65.06	58.60	**68.33**	65.46
	50	64.09	71.60	64.32	72.75	64.17	**77.82**	75.04
	100	89.62	92.09	89.75	93.09	80.66	**96.18**	95.20
	150	97.69	98.09	97.67	98.55	89.51	**99.61**	99.50
	200	99.70	99.66	99.43	99.70	95.27	**99.98**	99.94
$t(6)$	5	9.05	8.44	4.23	**10.08**	9.96	7.20	9.34
	10	15.14	16.14	5.81	17.28	**20.79**	15.64	16.94
	15	21.25	24.98	12.46	25.71	**30.83**	25.82	25.92
	20	26.43	33.08	18.87	35.59	**37.54**	33.39	32.89
	25	30.46	39.78	25.17	41.65	**42.28**	40.67	38.85
	30	36.97	46.39	31.76	47.19	47.76	**48.20**	45.85
	40	46.16	58.44	43.86	59.64	55.57	**60.87**	57.43
	50	53.52	66.17	52.15	68.13	61.32	**70.27**	66.47
	100	82.72	89.50	81.51	90.39	80.35	**93.90**	91.95
	150	94.14	96.84	93.95	97.59	89.58	**98.91**	98.28
	200	98.66	99.13	98.22	99.28	94.93	**99.84**	**99.84**

The bold number is the highest power among the seven tests for each sample size.

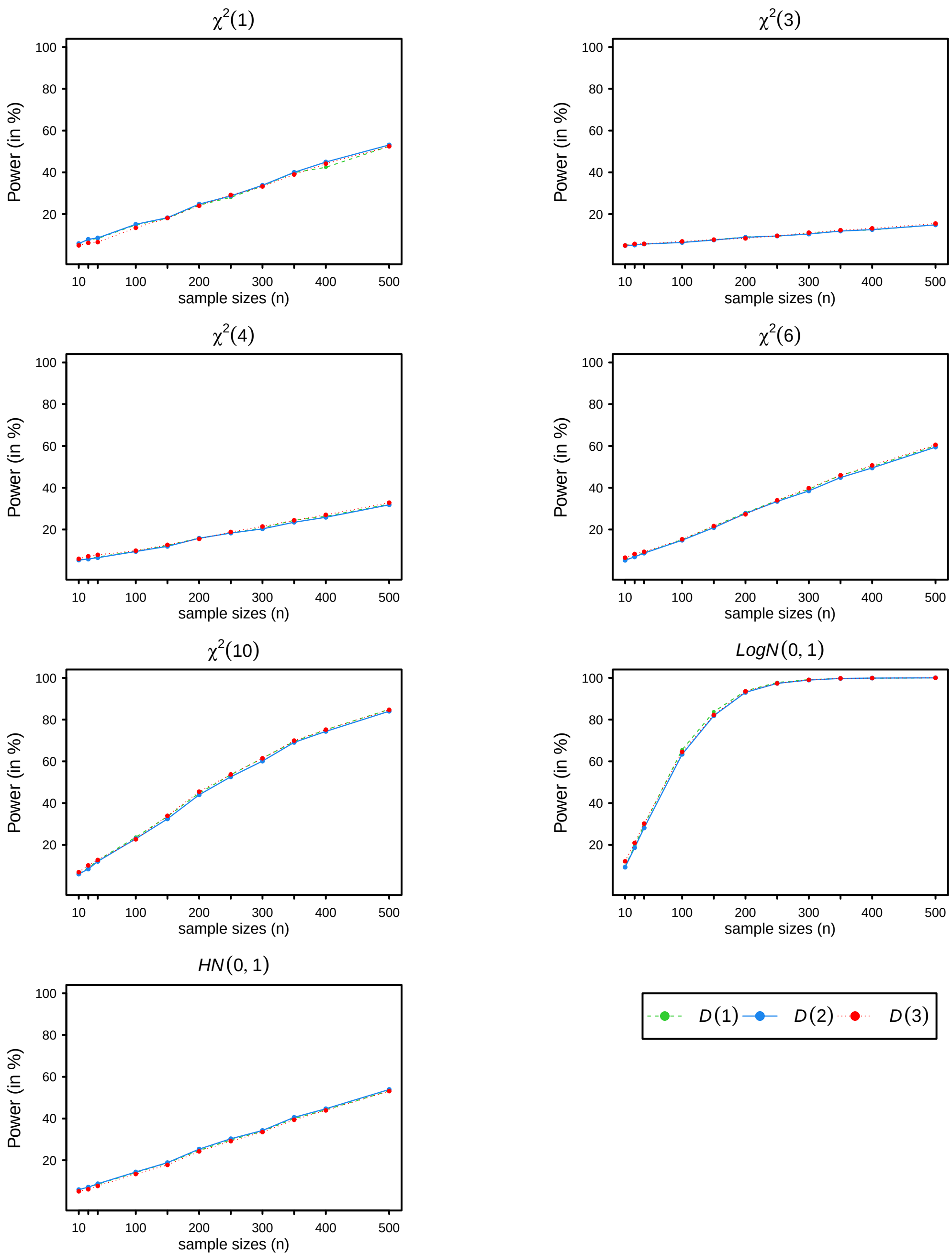

Figure 4.1: Power comparison of the D statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

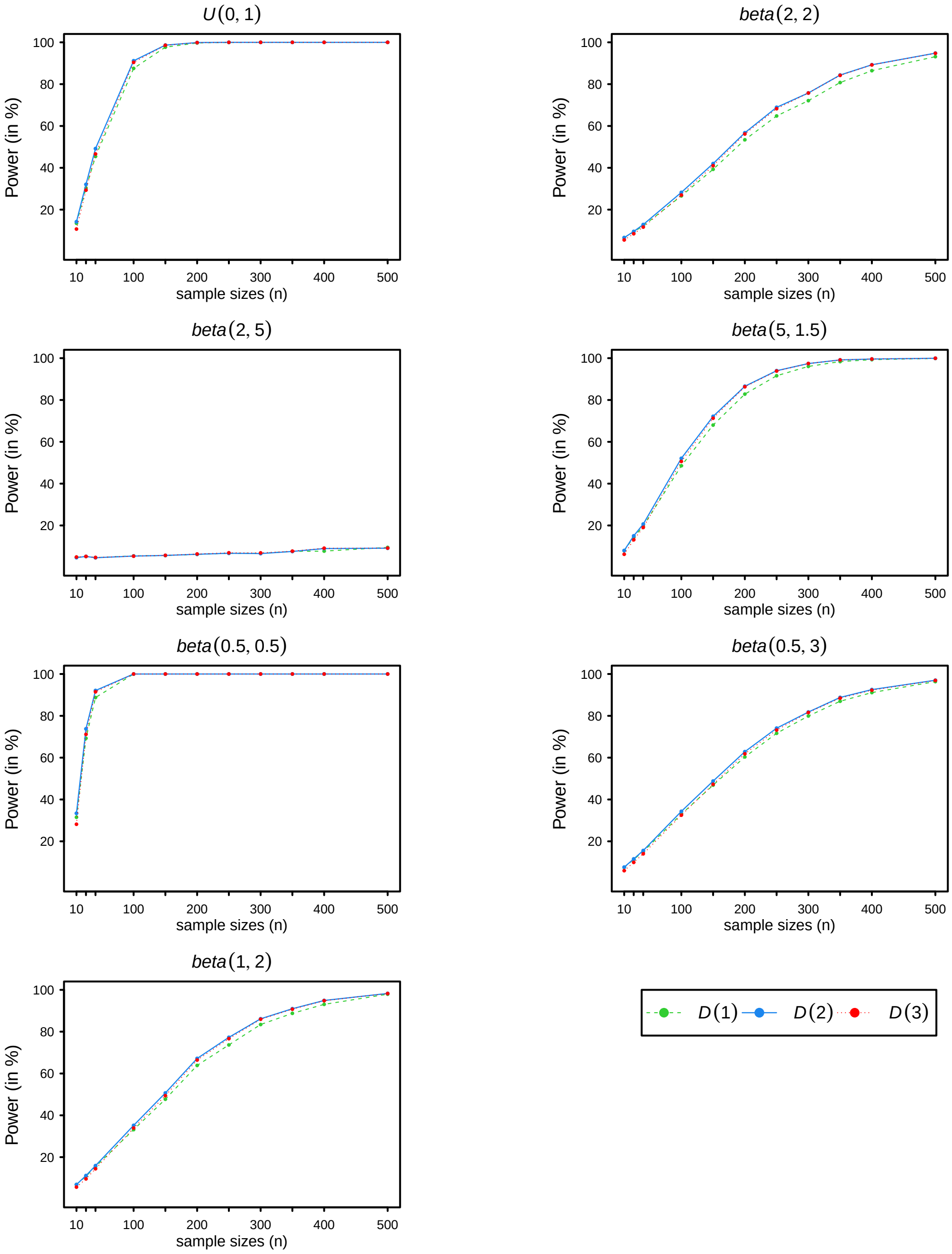

Figure 4.2: Power comparison of the D statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several samples sizes n

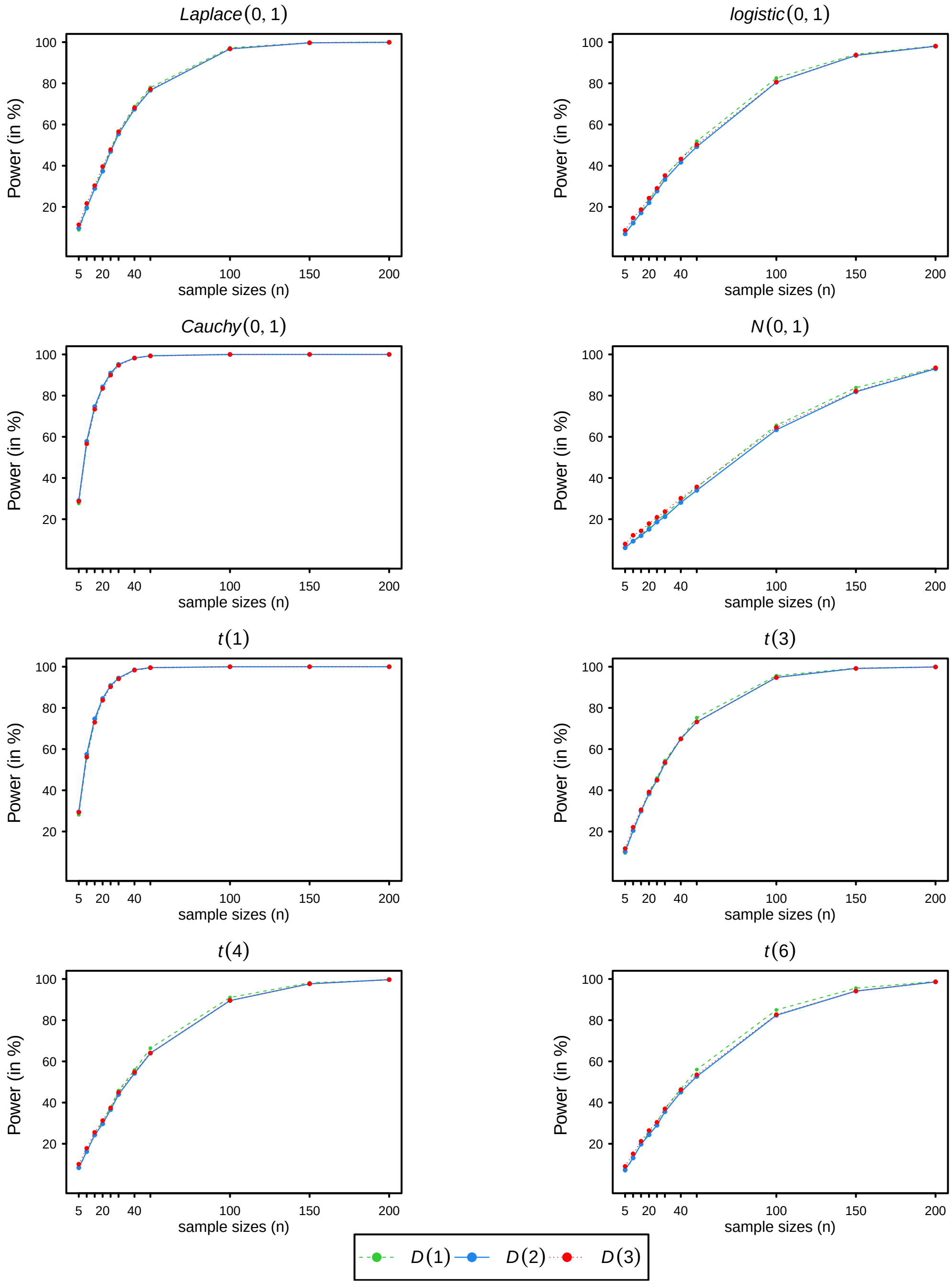

Figure 4.3: Power comparison of the D statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

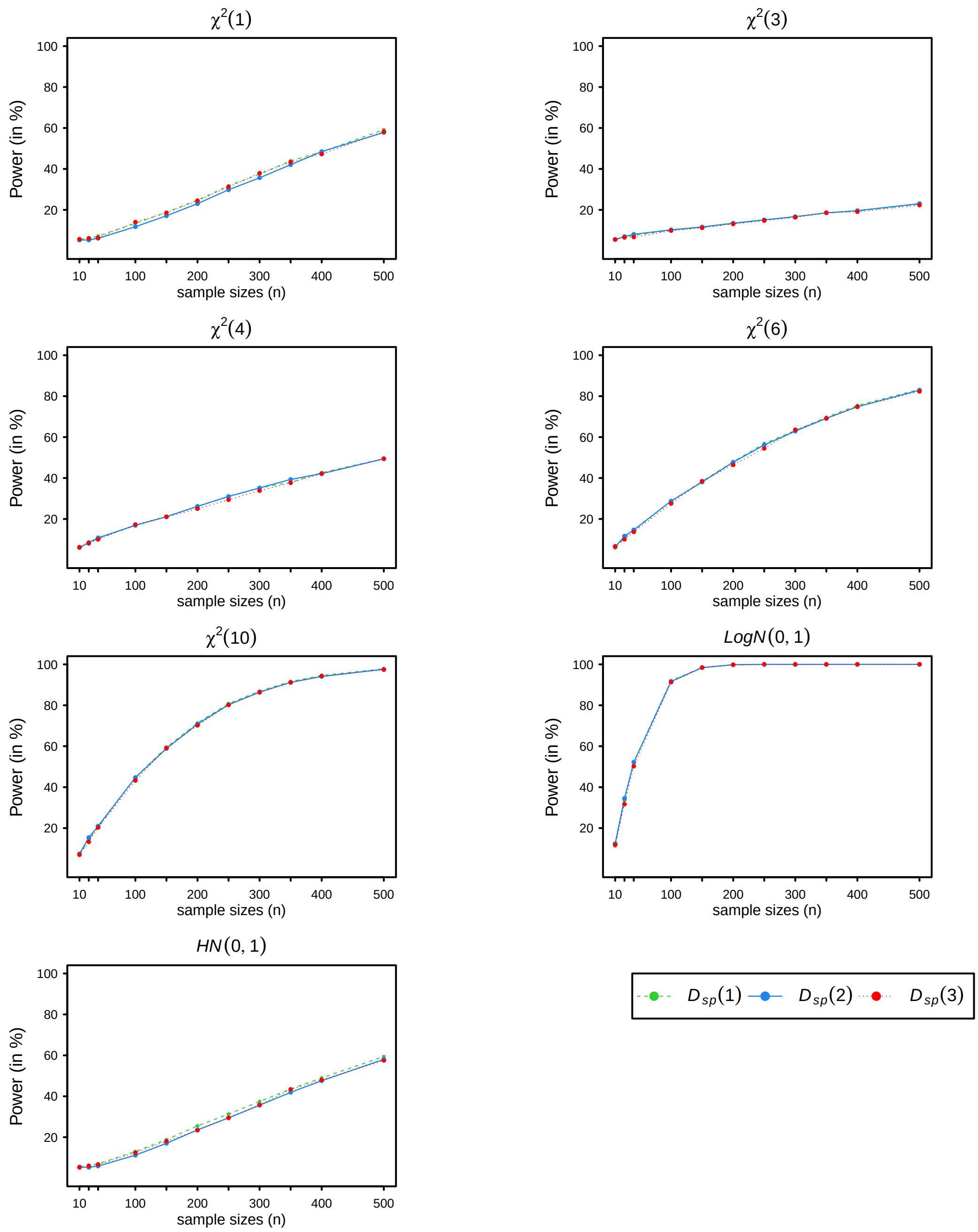

Figure 4.4: Power comparison of the D_{sp} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

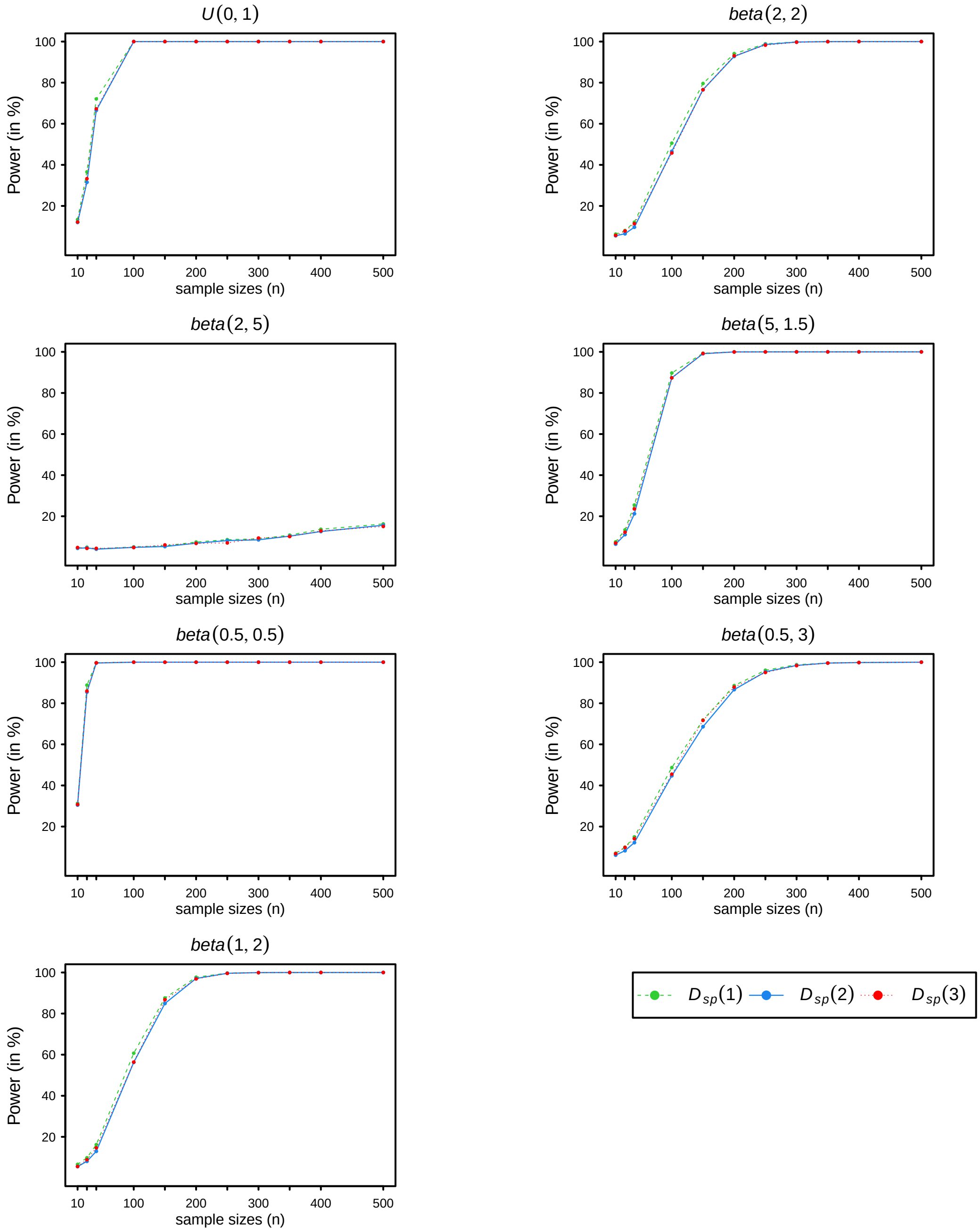

Figure 4.5: Power comparison of the D_{sp} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

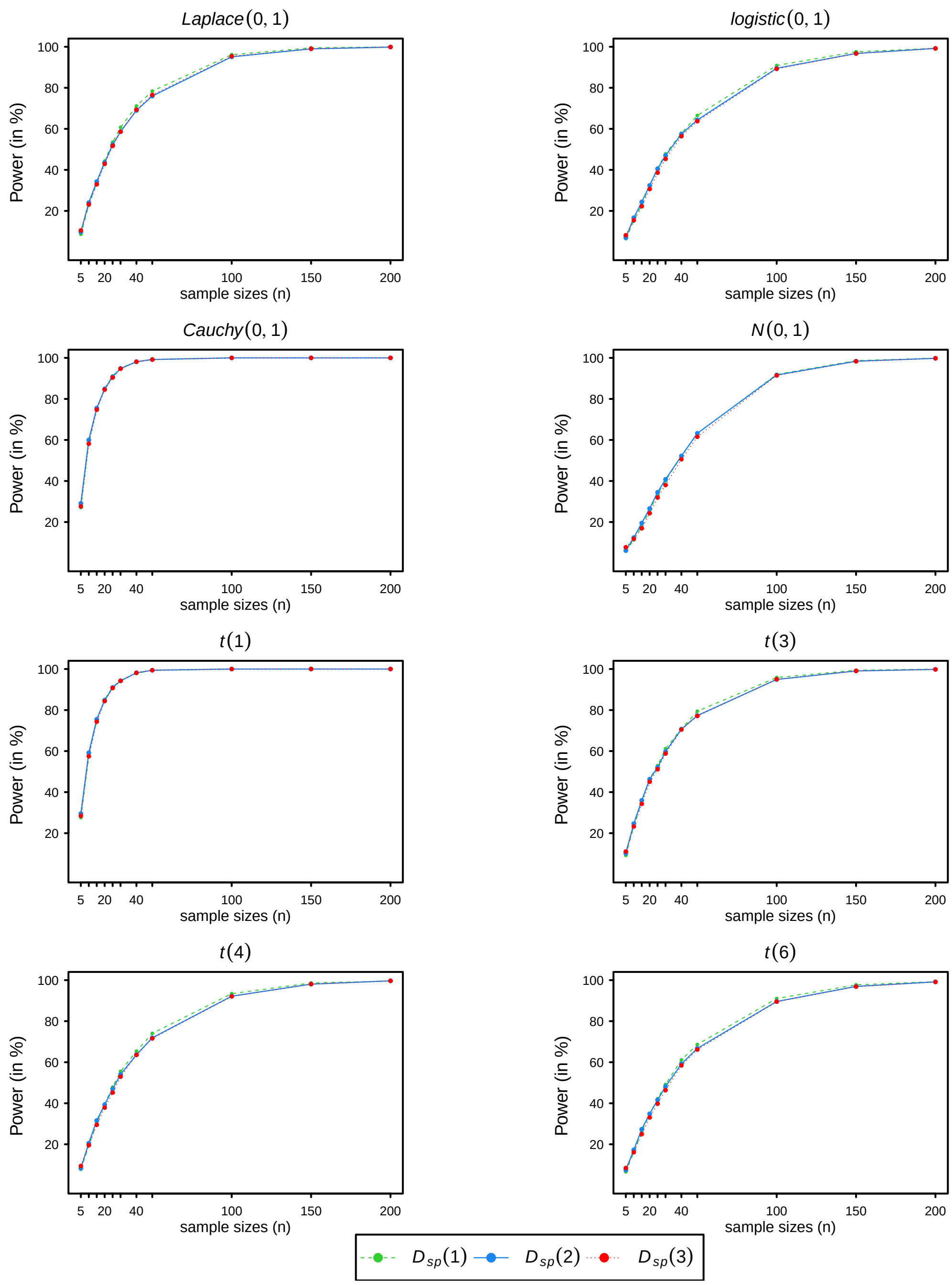

Figure 4.6: Power comparison of the D_{sp} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

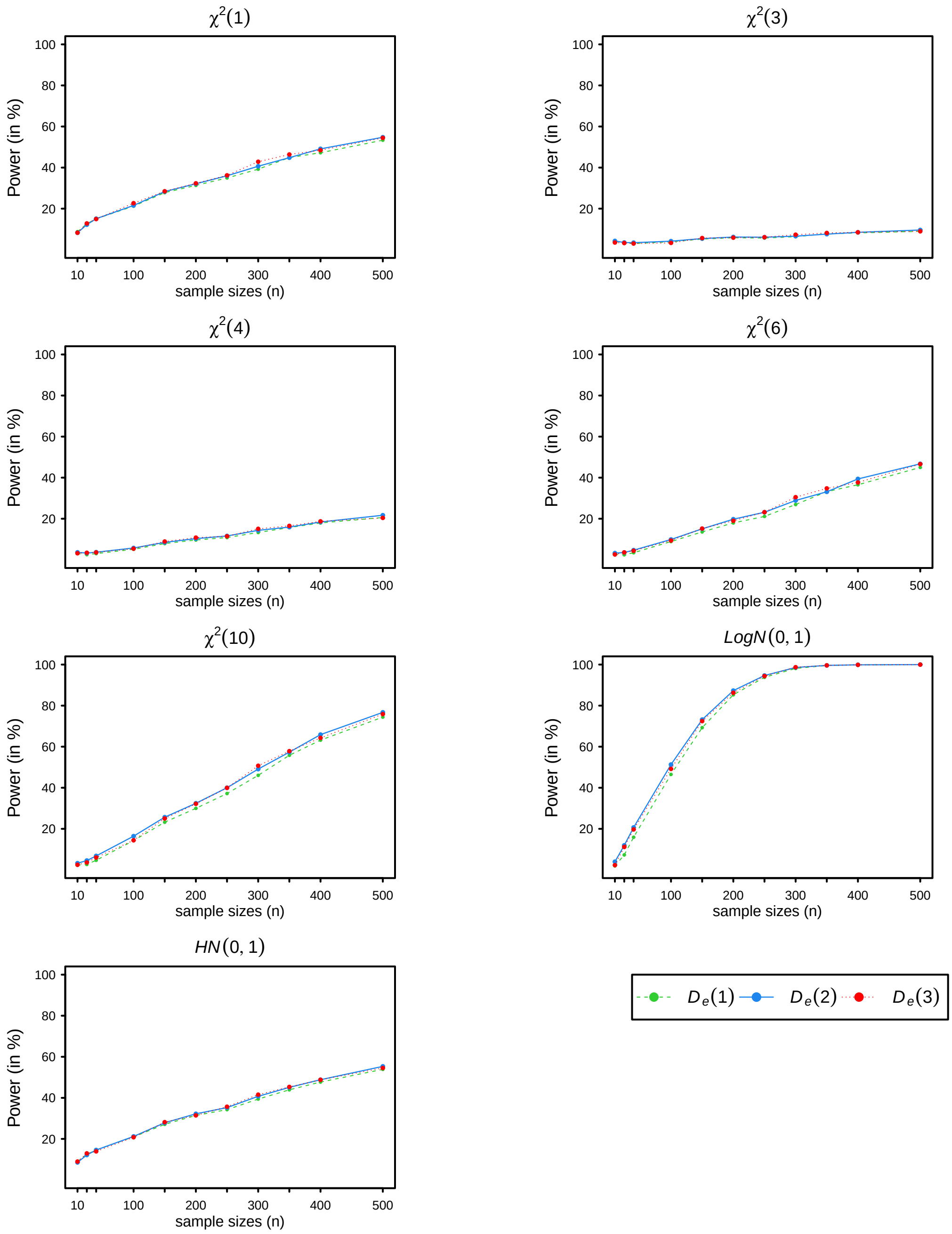

Figure 4.7: Power comparison of the D_e statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

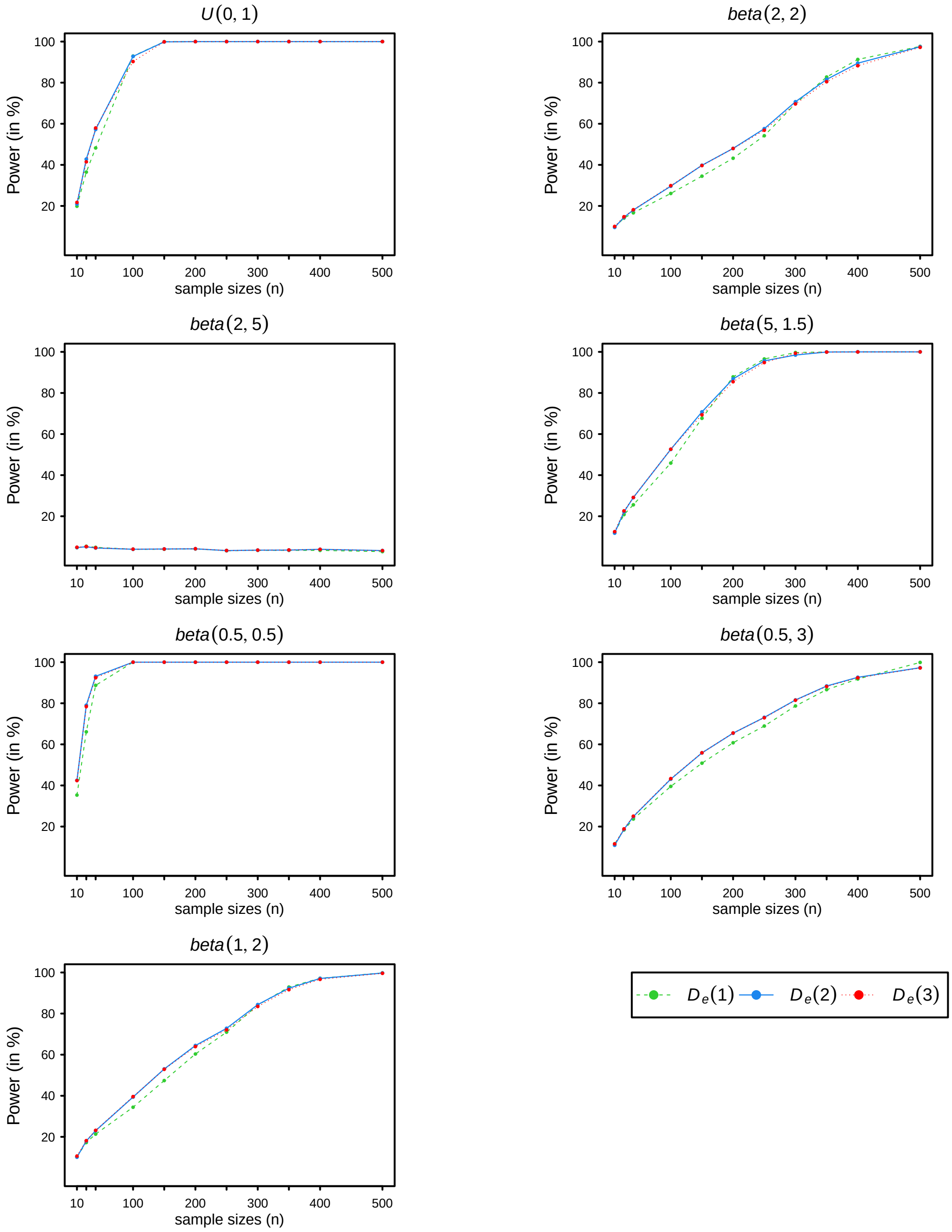

Figure 4.8: Power comparison of the D_e statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

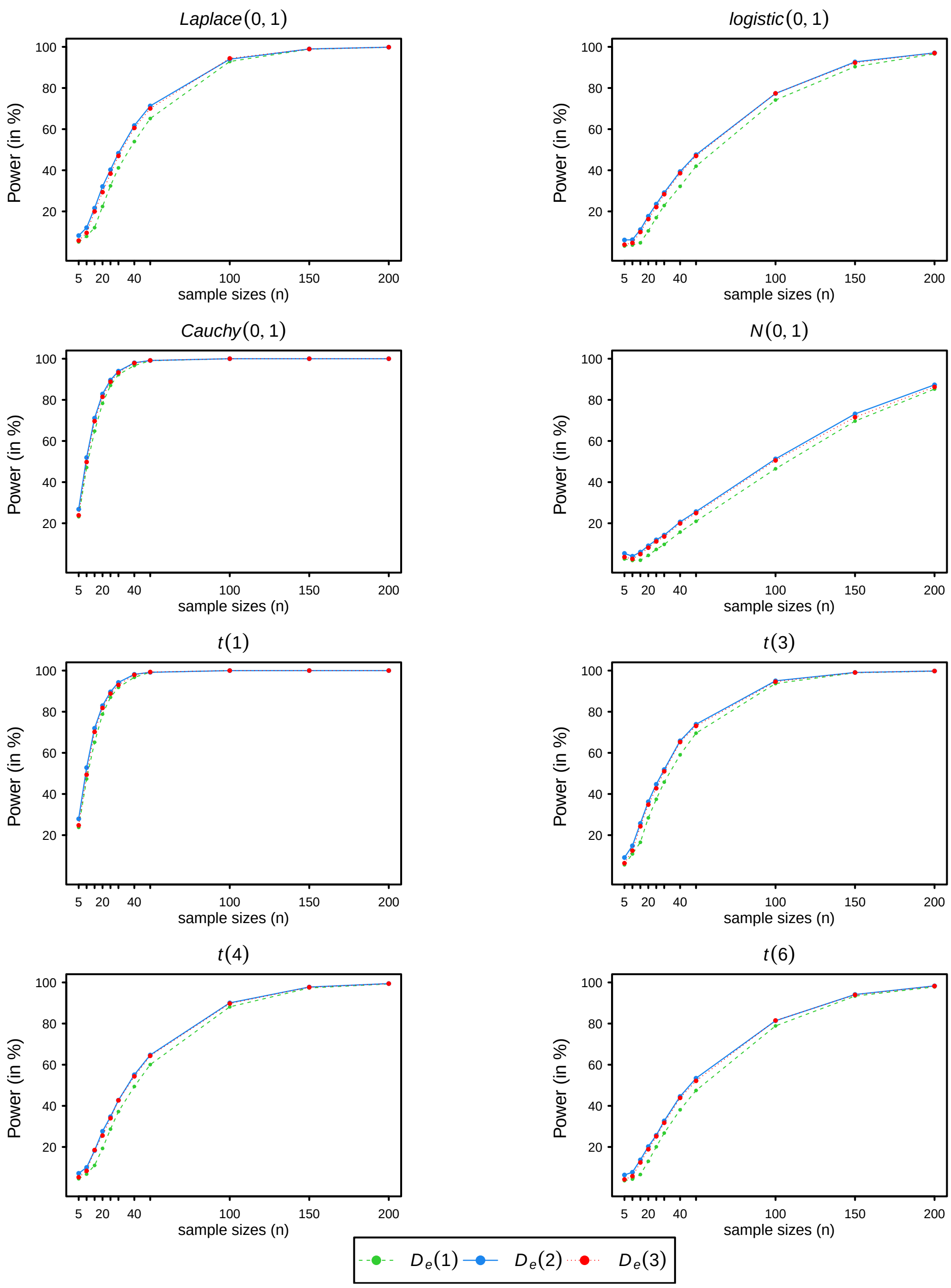

Figure 4.9: Power comparison of the D_e statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

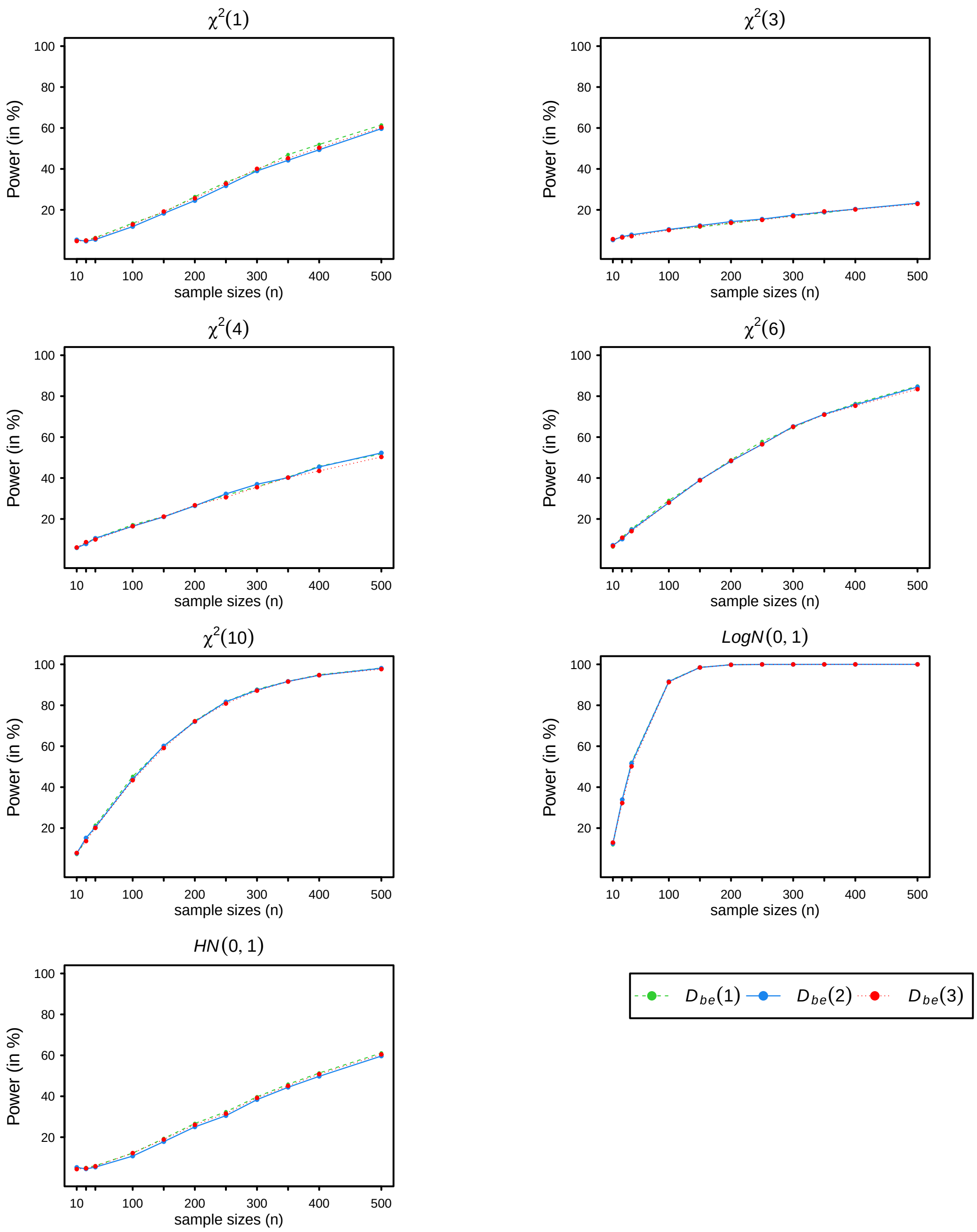

Figure 4.10: Power comparison of the D_{be} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

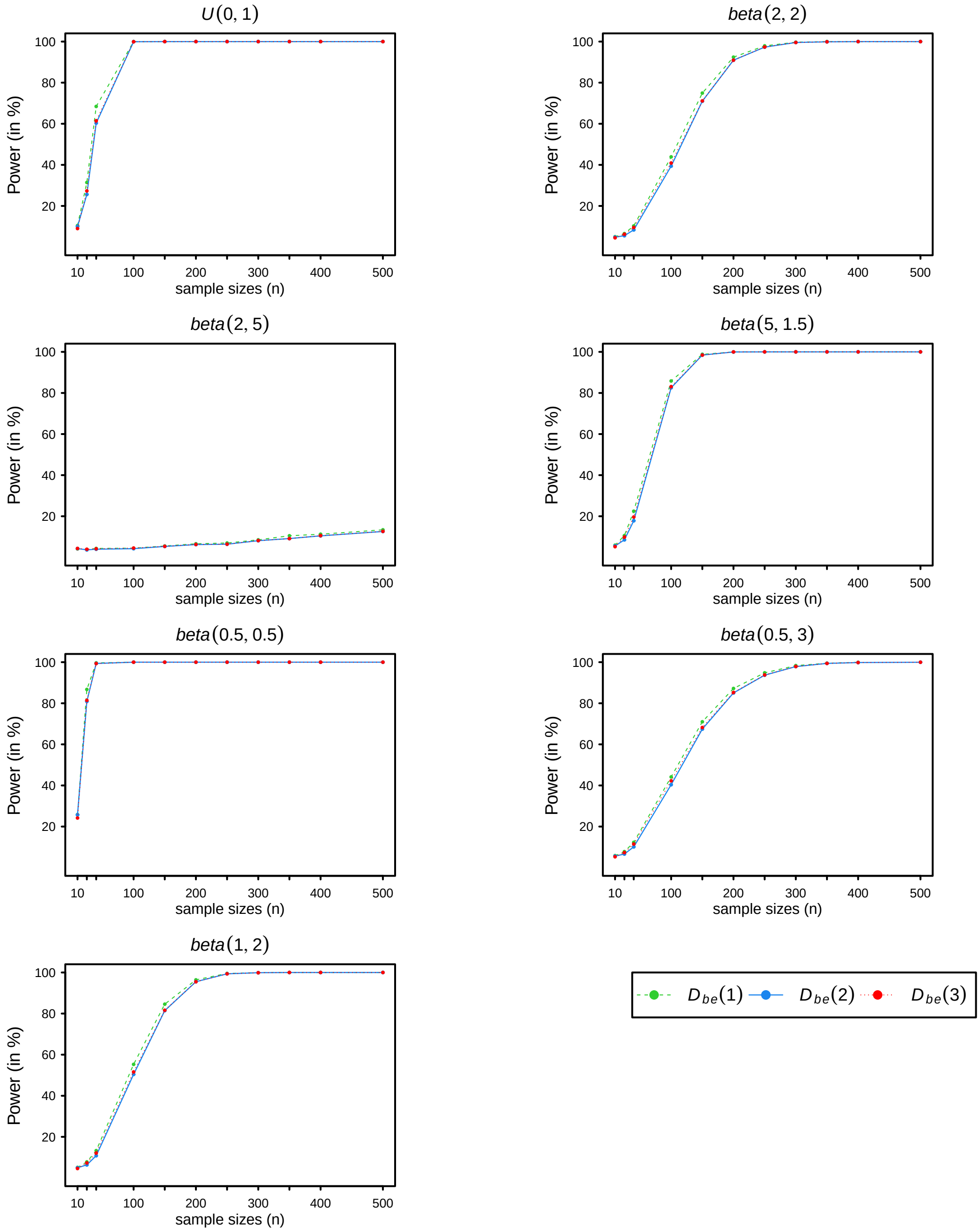

Figure 4.11: Power comparison of the D_{be} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

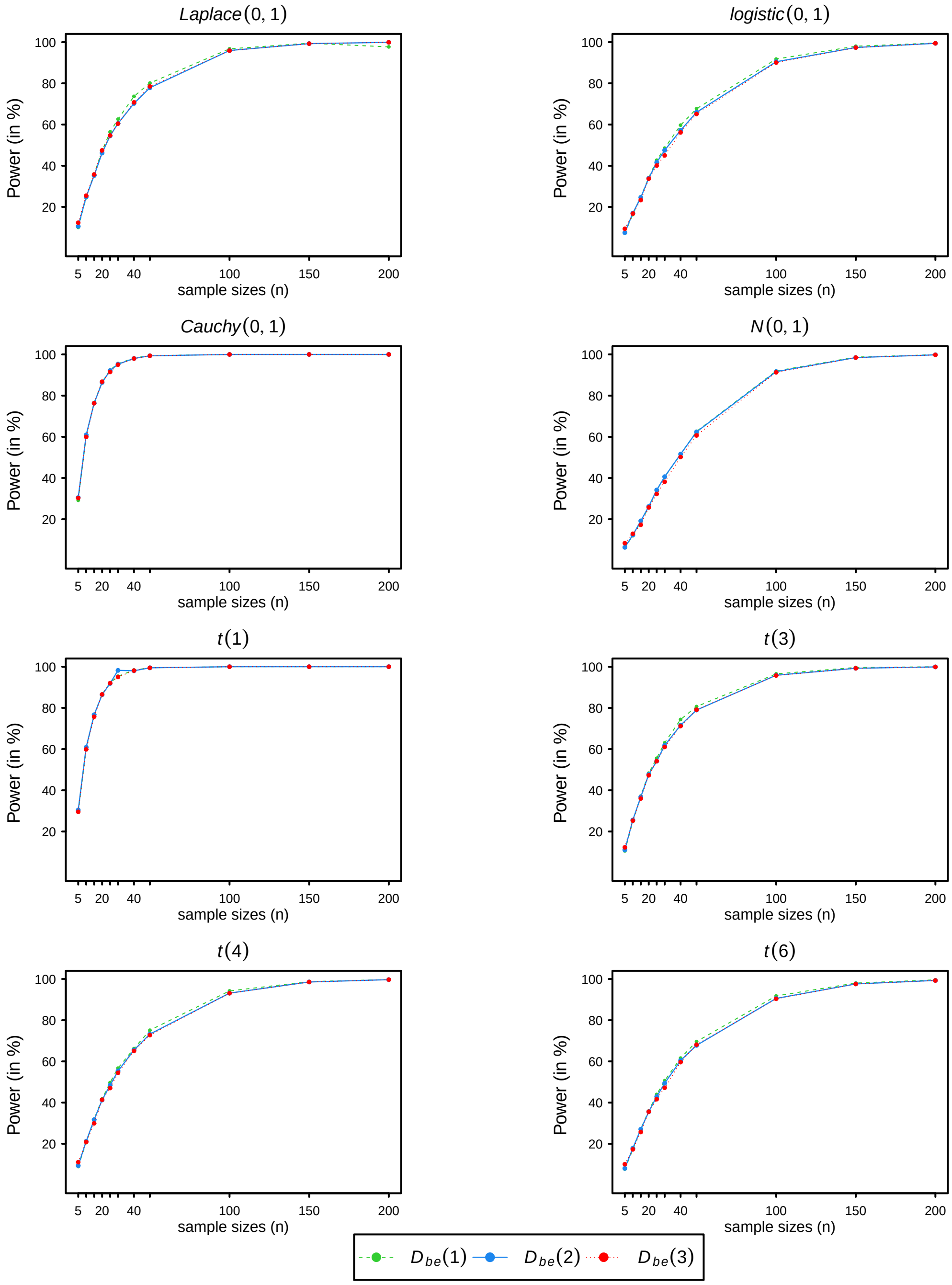

Figure 4.12: Power comparison of the D_{be} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

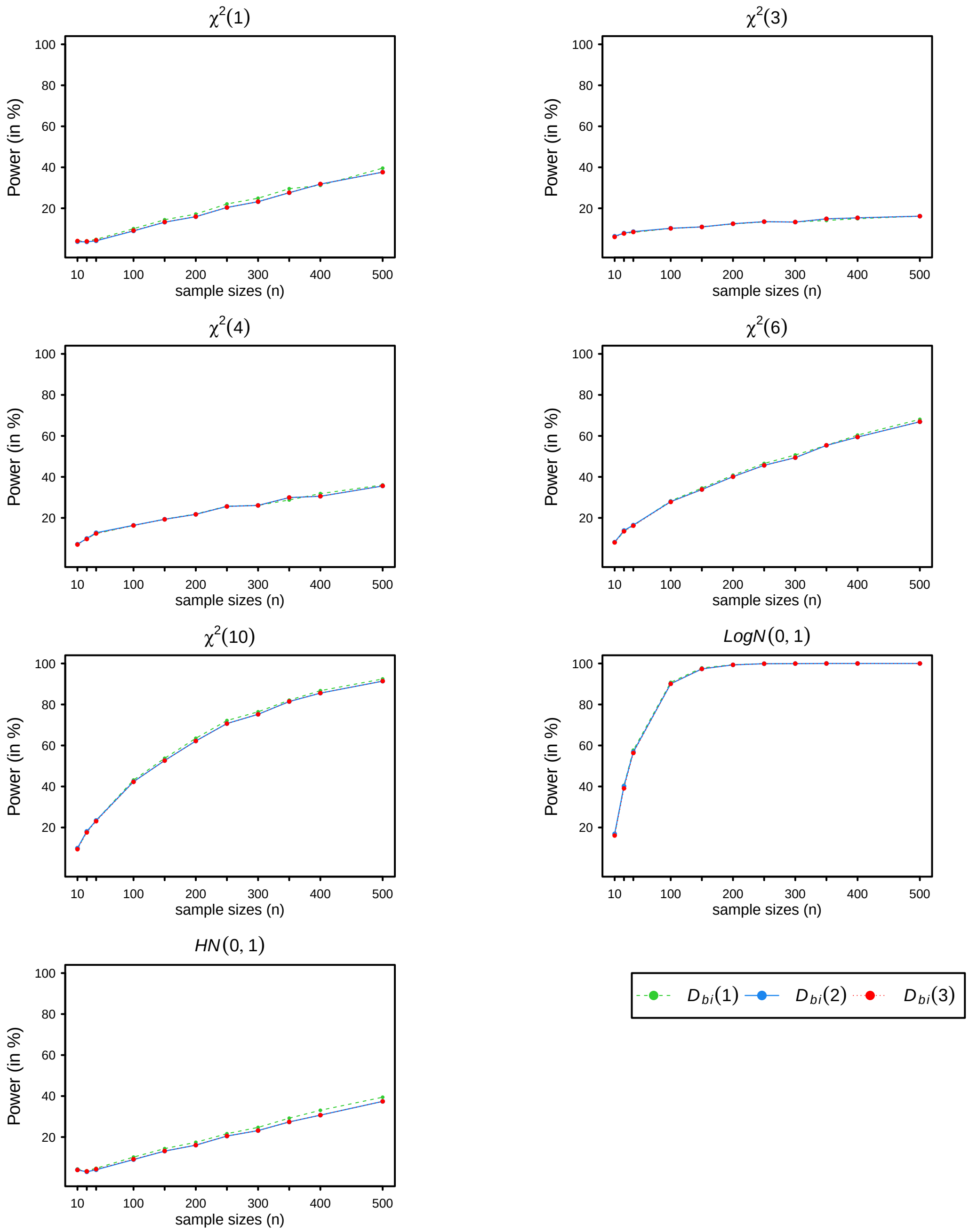

Figure 4.13: Power comparison of the D_{bi} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

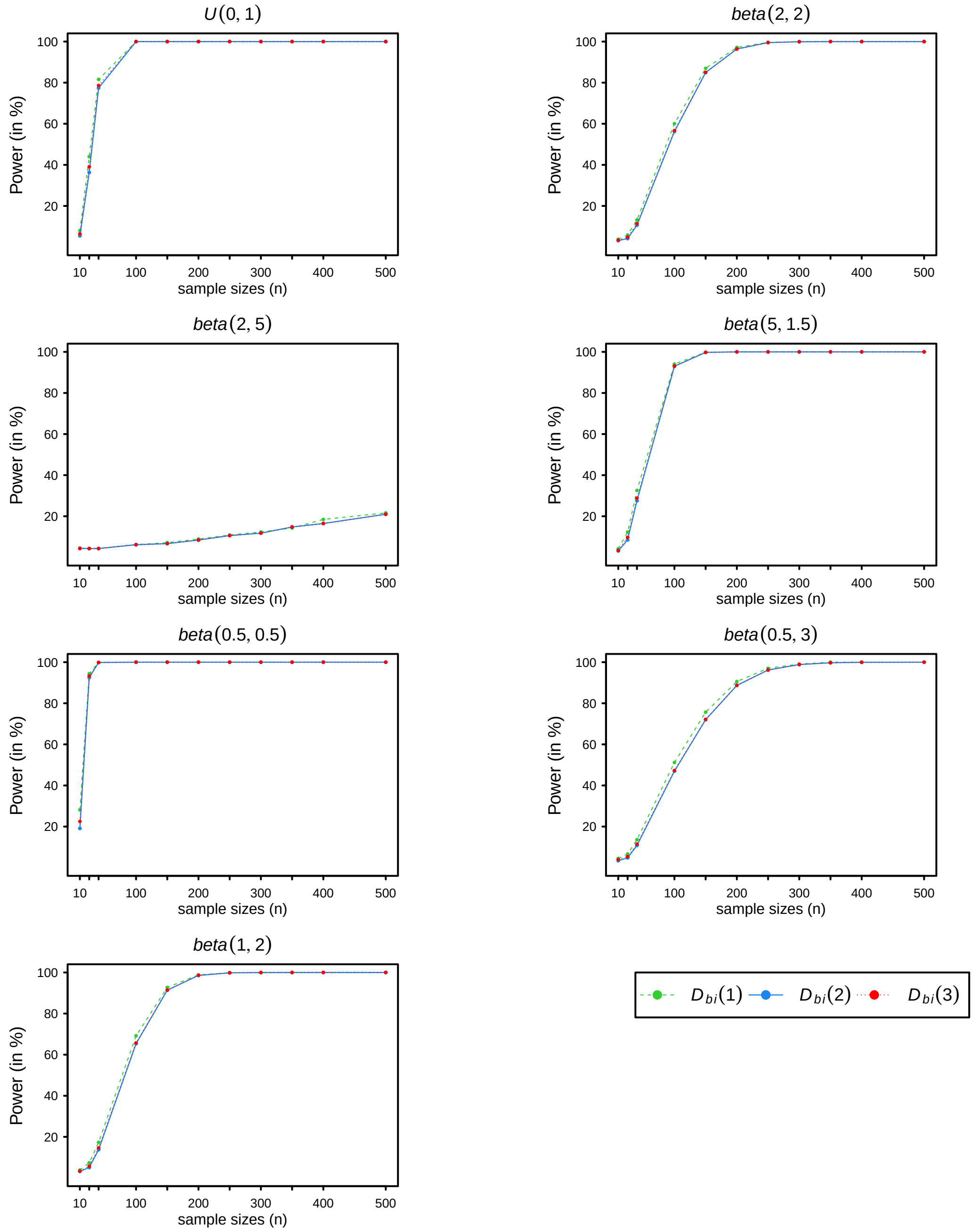

Figure 4.14: Power comparison of the D_{bi} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

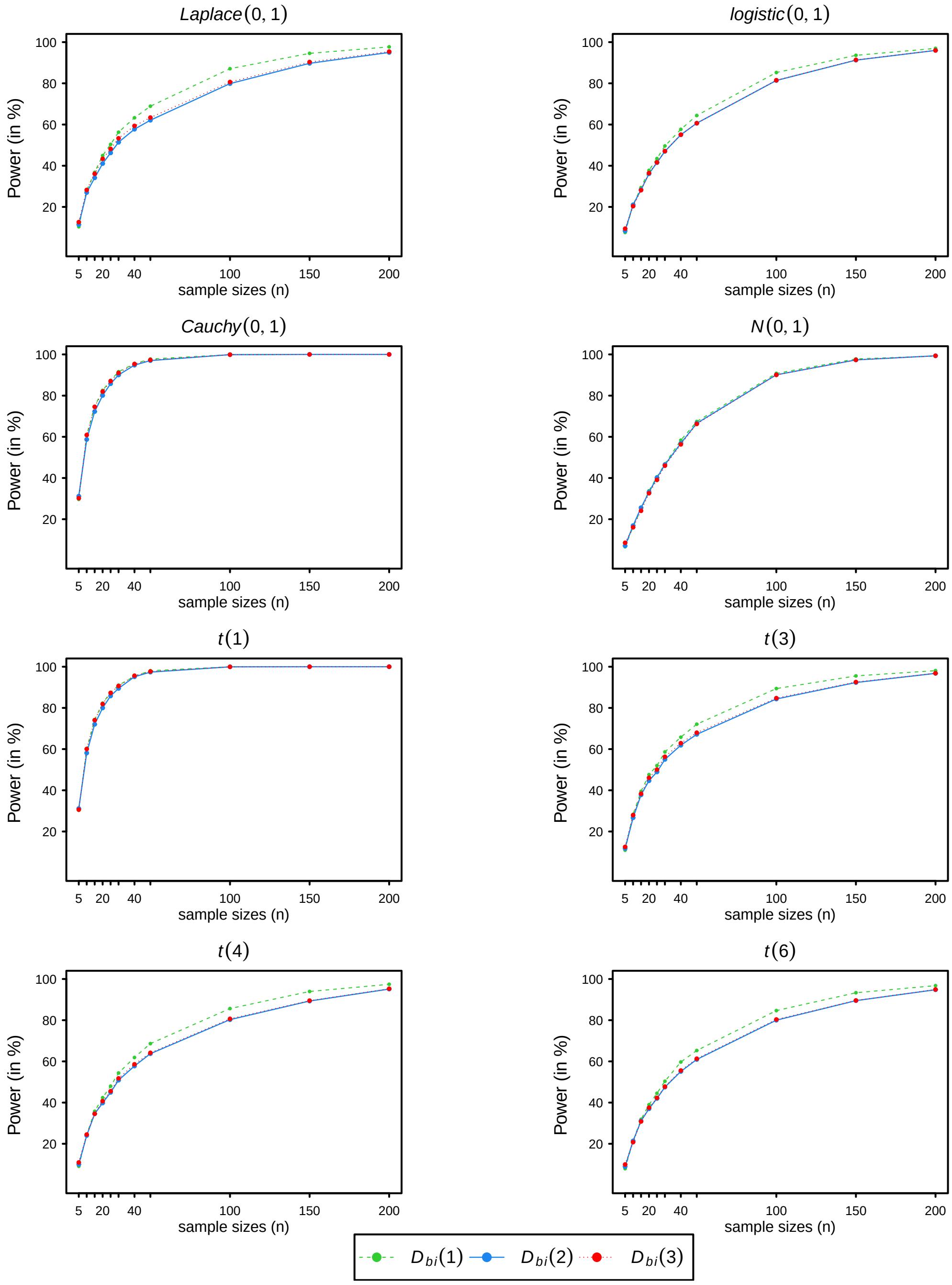

Figure 4.15: Power comparison of the D_{bi} statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

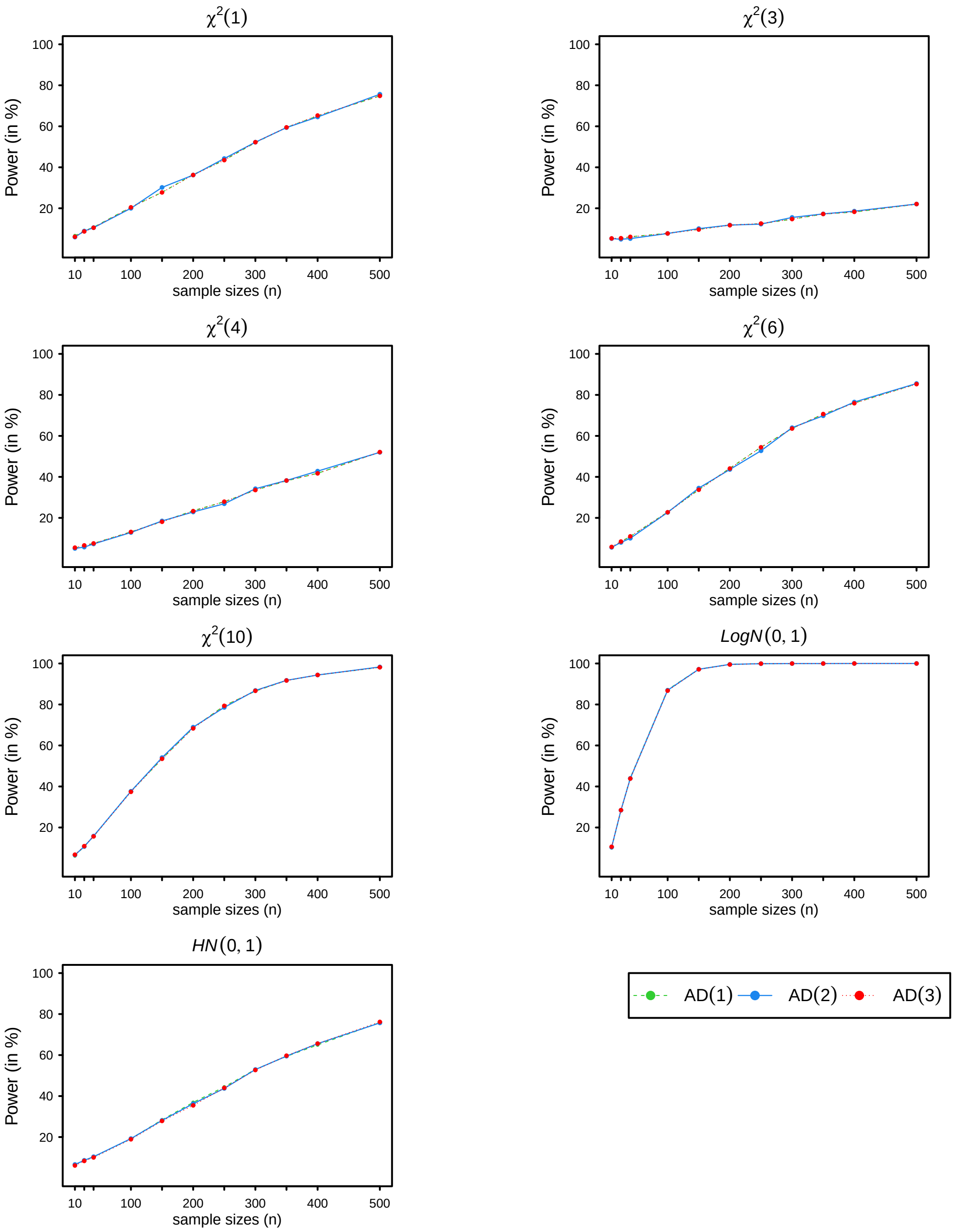

Figure 4.16: Power comparison of the AD statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

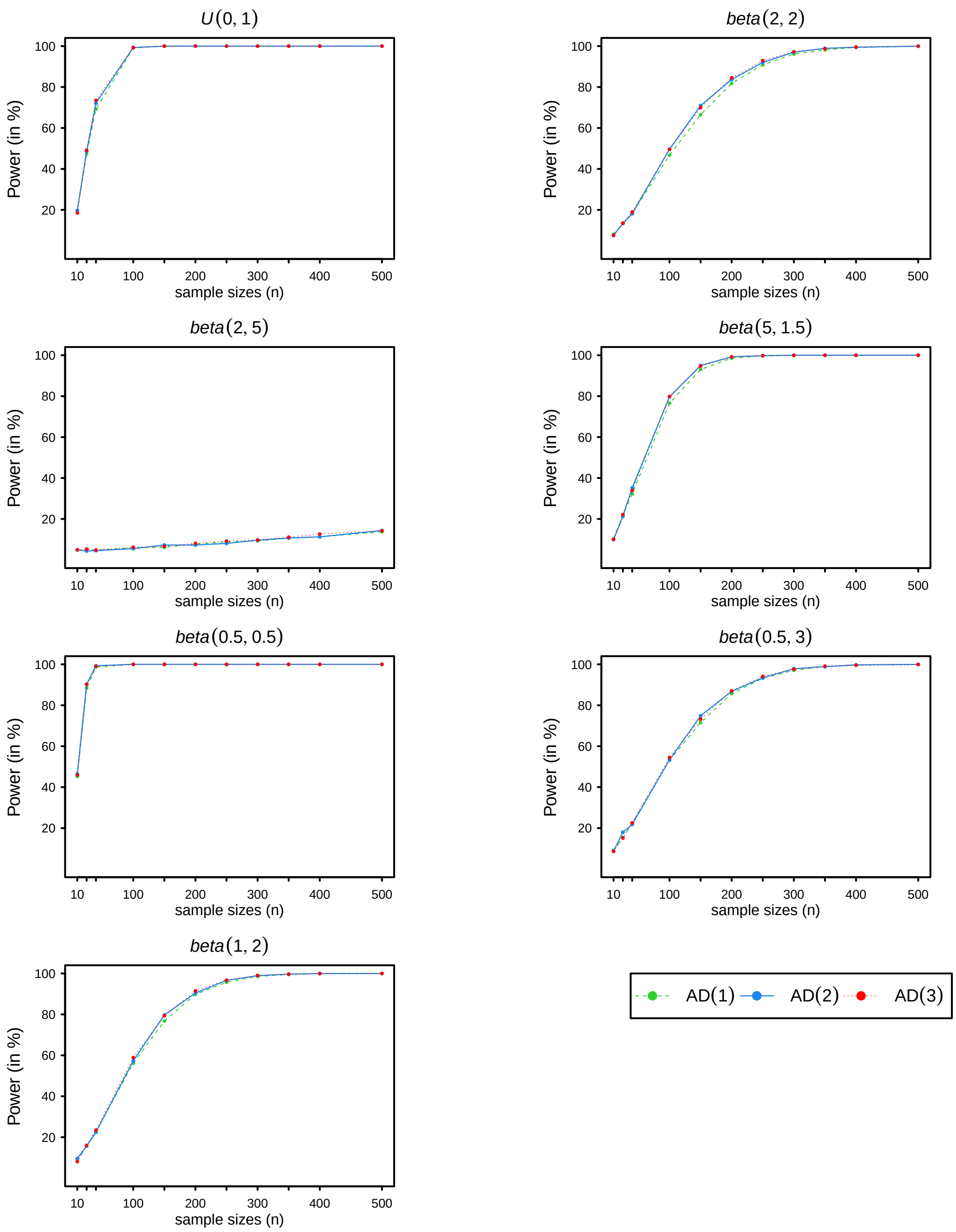

Figure 4.17: Power comparison of the AD statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

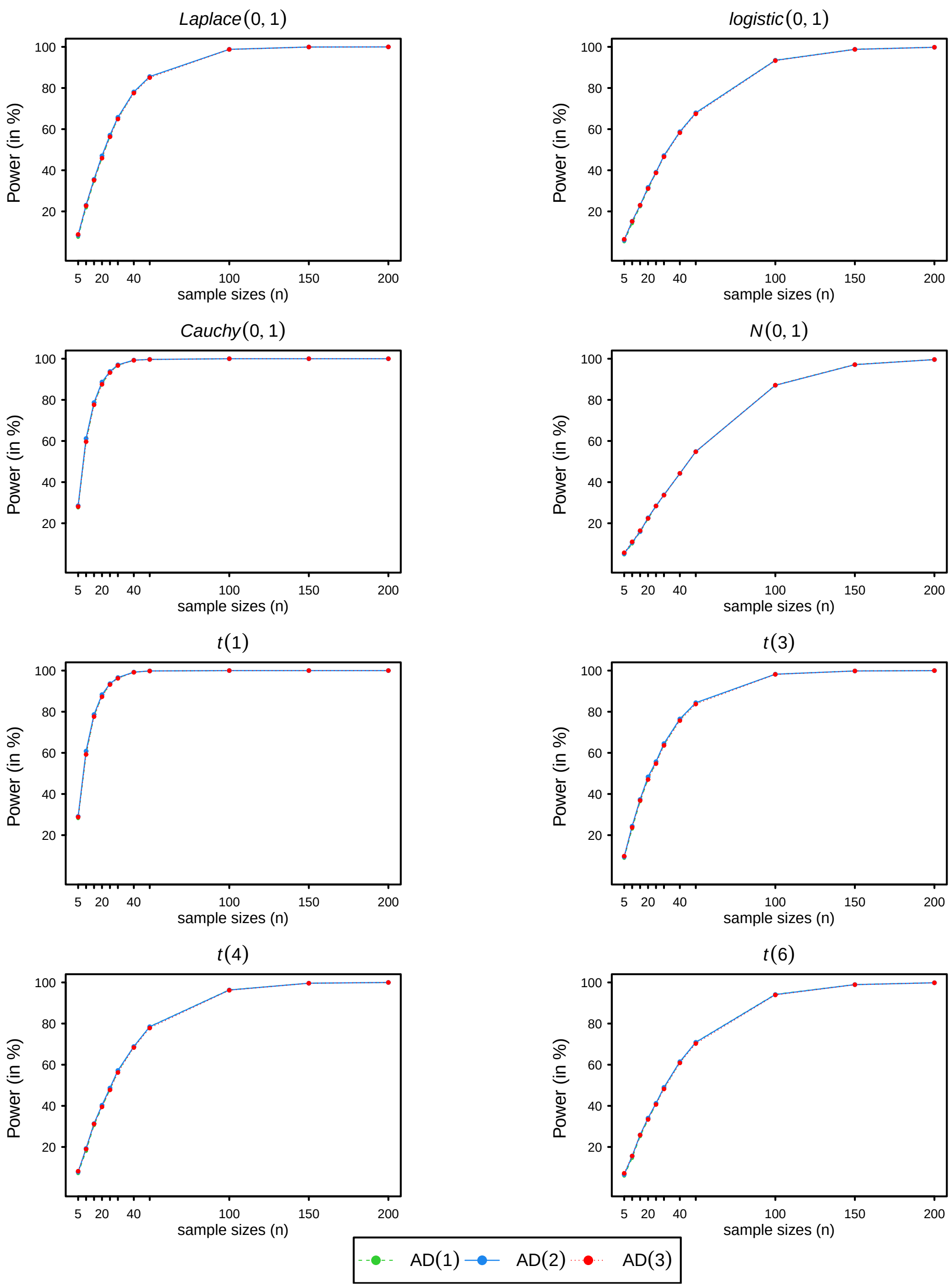

Figure 4.18: Power comparison of the AD statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

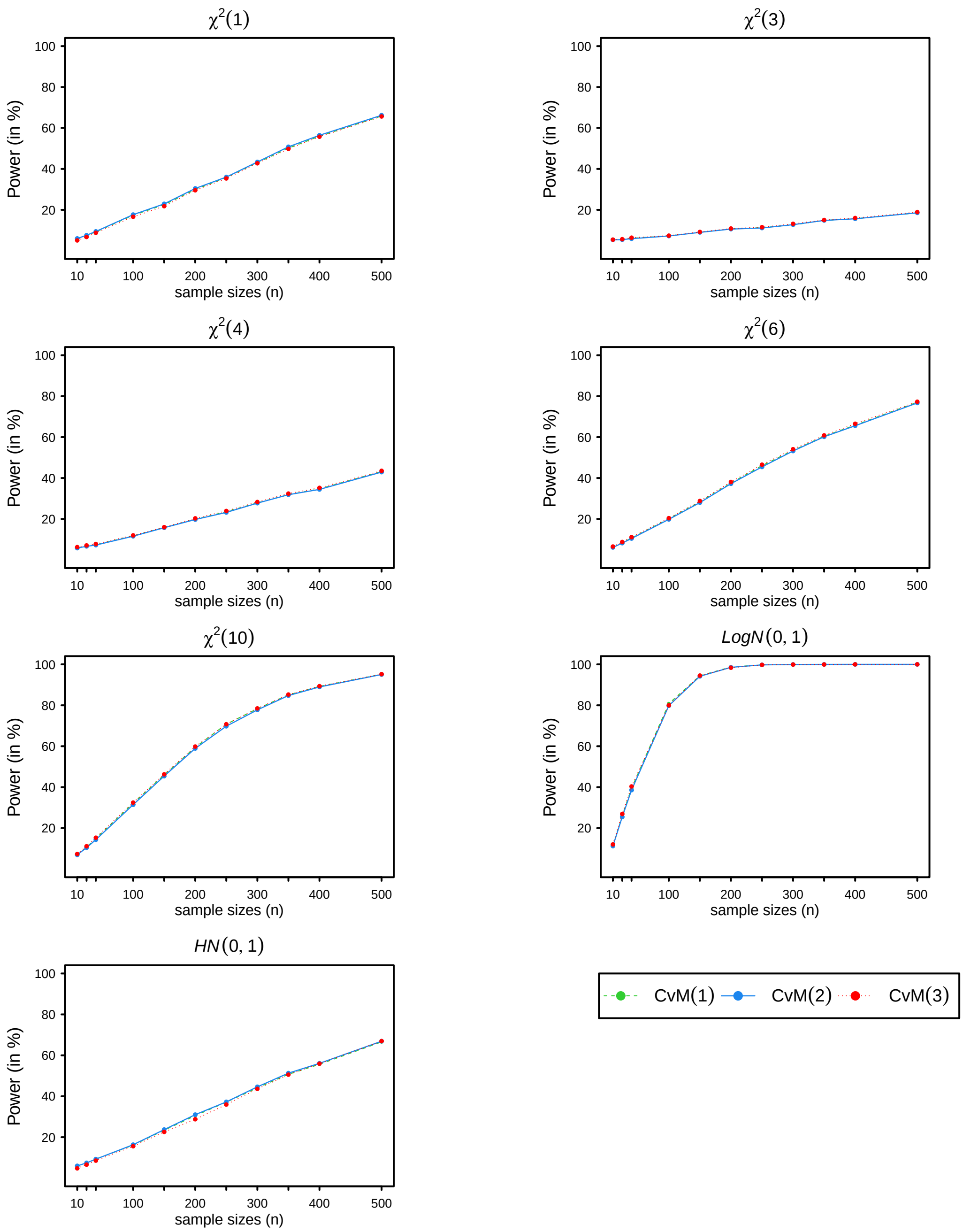

Figure 4.19: Power comparison of the CvM statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

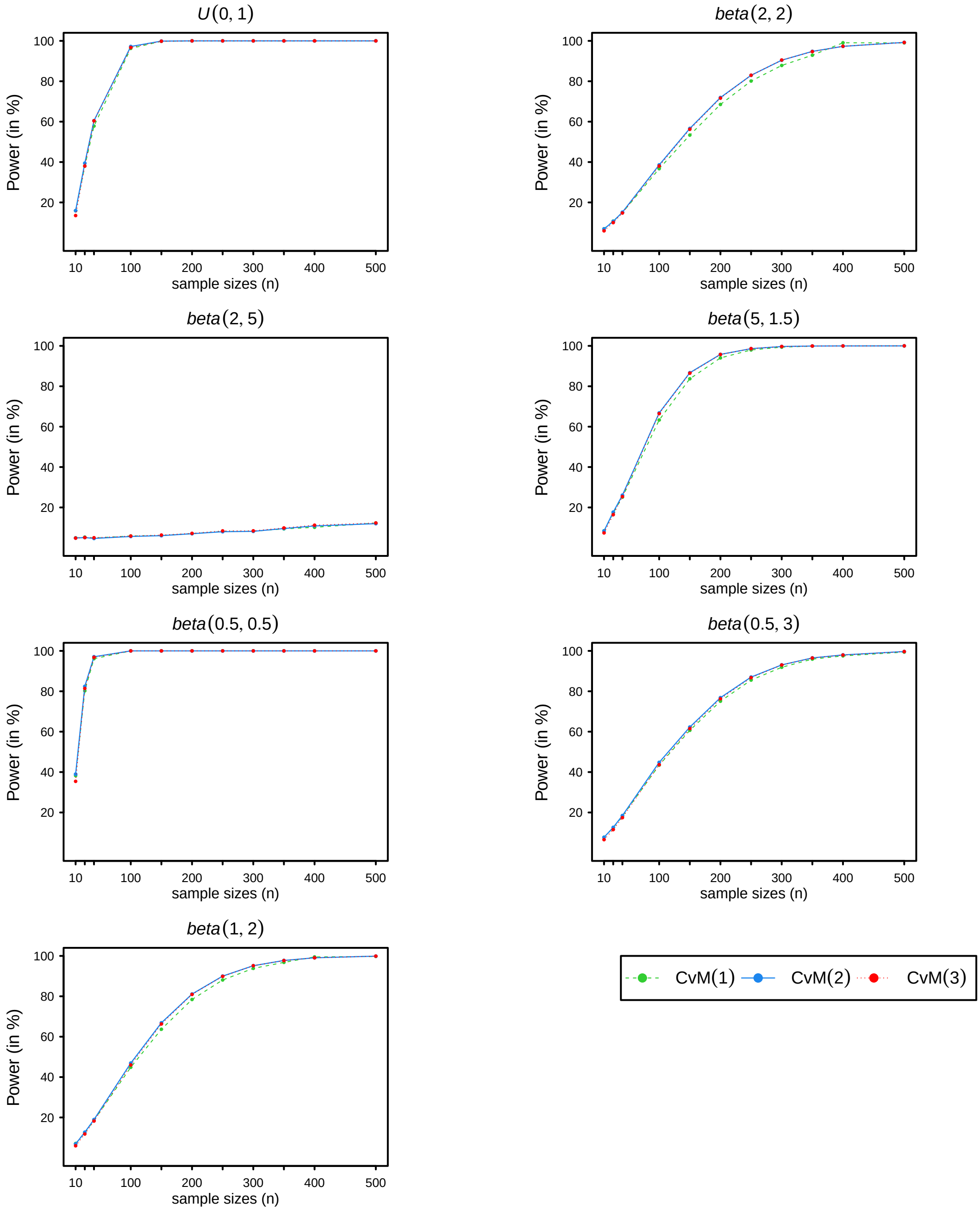

Figure 4.20: Power comparison of the CvM statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

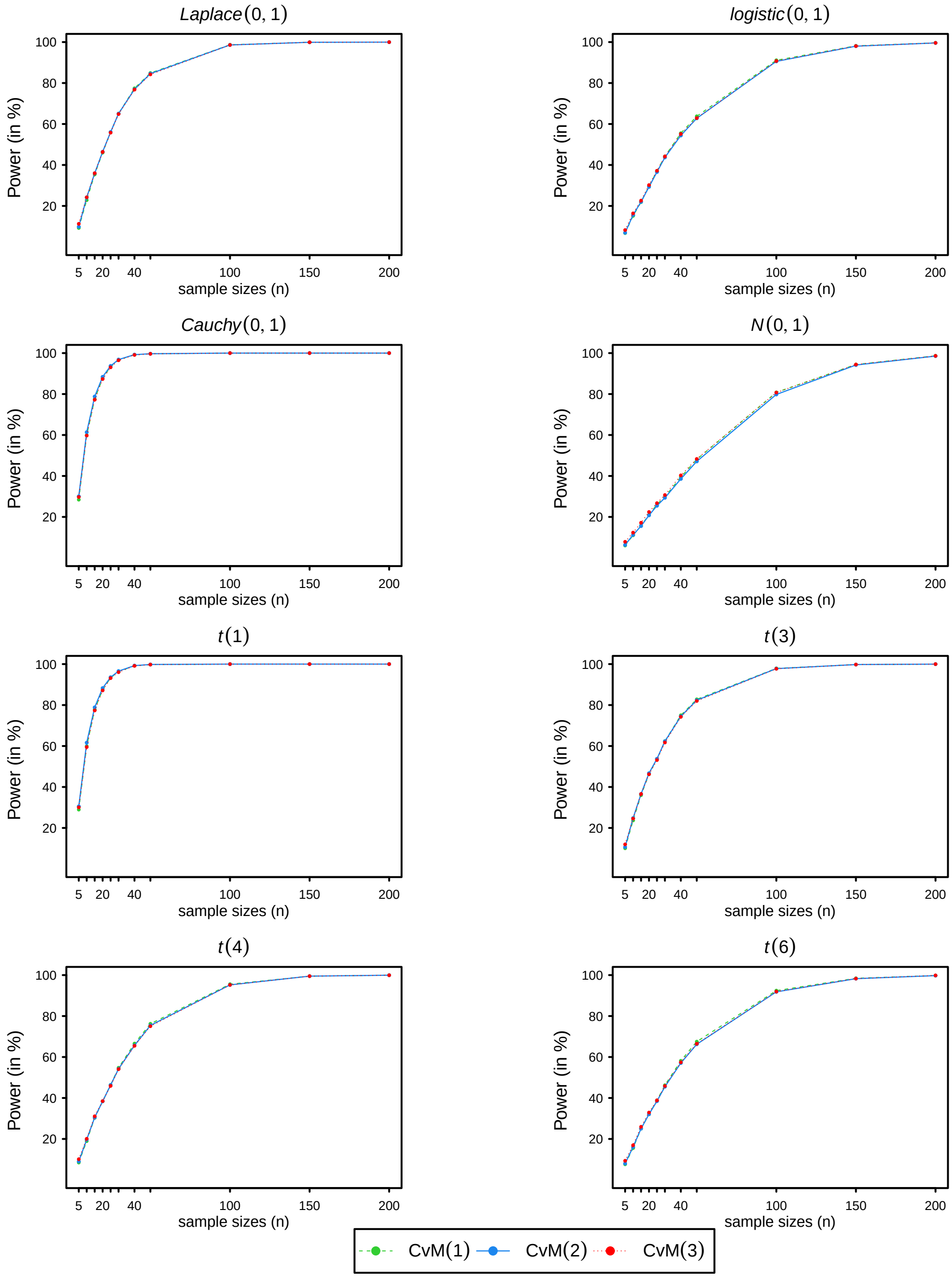

Figure 4.21: Power comparison of the CvM statistic by using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

4.5.2 Results on power

After the BLUE is chosen as the estimators of the location and scale parameters from $SEV(0,1)$, the powers of the seven tests are compared.

From the power results given in Table 4.9 and Figure 4.22 for **the first group** of alternative distributions, the following observations can be made. It emerges that the D_{be} test provides superior power against the alternative distribution in Group I, except for $\chi^2(1)$ and $HN(0,1)$. In addition, the D_{be} test has greater powers than the non-graphical AD and CvM tests. Moreover, the D_e test is the best choice against $\chi^2(1)$ and $HN(0,1)$ when $n < 150$. When $n \geq 150$, the AD test is generally the best against $\chi^2(1)$ and $HN(0,1)$, followed by the CvM test. Also, the D_{bi} test has the worst powers against $\chi^2(1)$ and $HN(0,1)$. It can also be observed from this simulation study that the powers of the D_{be} test are similar to those of the D_{sp} test against all alternative distributions from Group I. Overall, the D_e test has the worst powers among all tests on most occasions, and is even worse than the D test.

From the power results given in Table 4.10 and Figure 4.23 for **the second group** of alternative distributions, the D_{bi} test has often the best powers whereas the D and D_e tests generally have least powers. However, when $n \leqslant 50$, D_e seems to have greater powers than all the other tests. Additionally, the powers of D_{sp} and D_{be} are close to each other, but the powers of D_{sp} are slightly higher than those of the D_{be} test. All tests have almost no power in detecting the departure of the Weibull distribution under $beta(2,5)$. Also, the D_{bi} test is more powerful than the non-graphical AD and CvM tests on most occasions.

From the power results given in Table 4.11 and Figure 4.24 for **the third group** of alternative distributions, when considering all the tests for alternative distributions for Group III, the AD and CvM tests are more powerful than the other tests. Nonetheless, for $N(0,1)$, the D_{bi} test is more powerful than the AD and CvM tests. It can be seen that the D, D_e and D_{bi} tests show low power over the distributions in Group III and the powers of the D_{bi} test are less than those of the D and D_e tests for the larger sample sizes. In terms of graphical tests, the D_{sp} and D_{be} tests appear to be the best choices when the whole range of sample sizes is considered. Obviously, the two non-graphical AD and CvM tests are the most powerful tests over the distributions in this group.

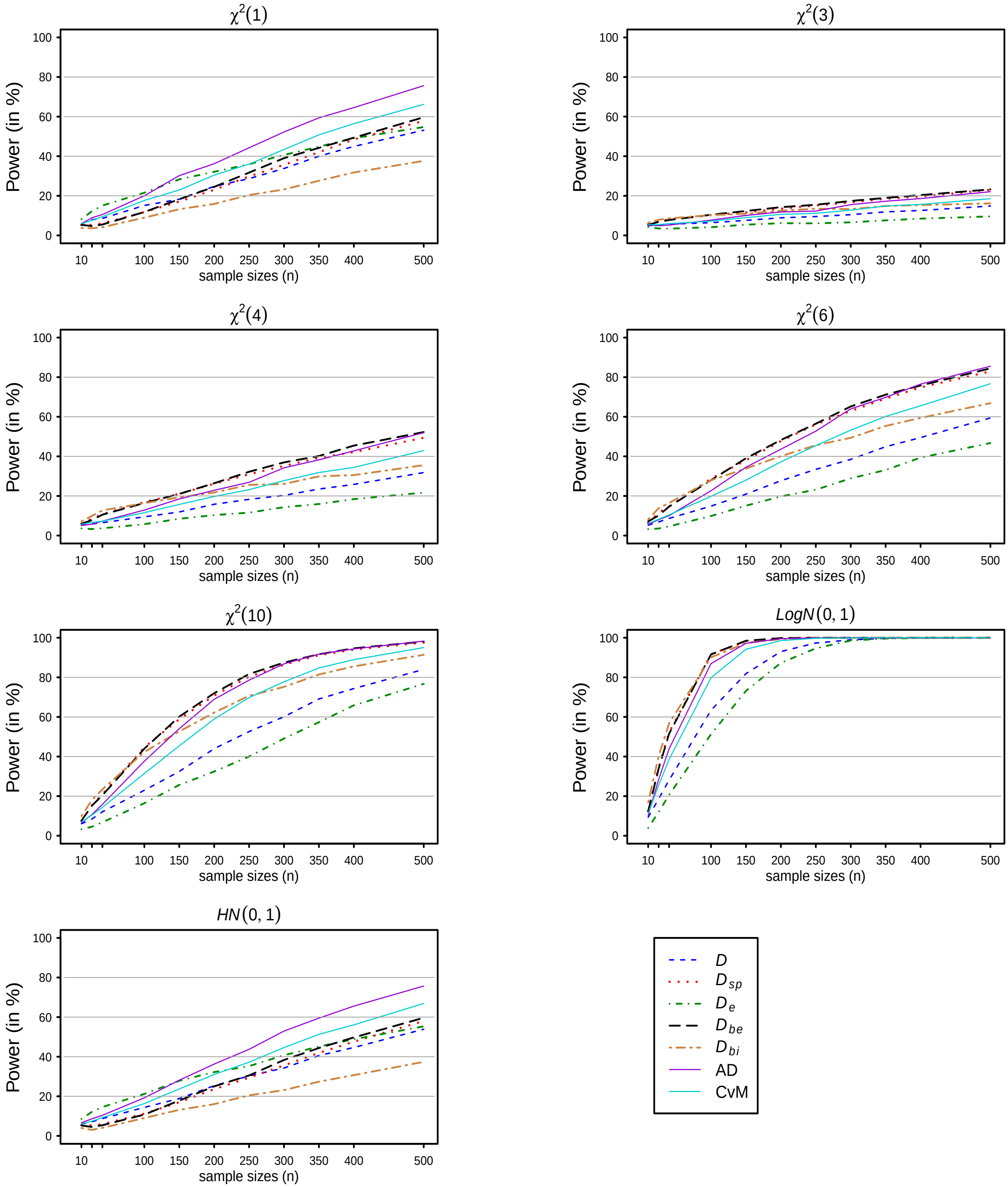

Figure 4.22: Power comparison of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

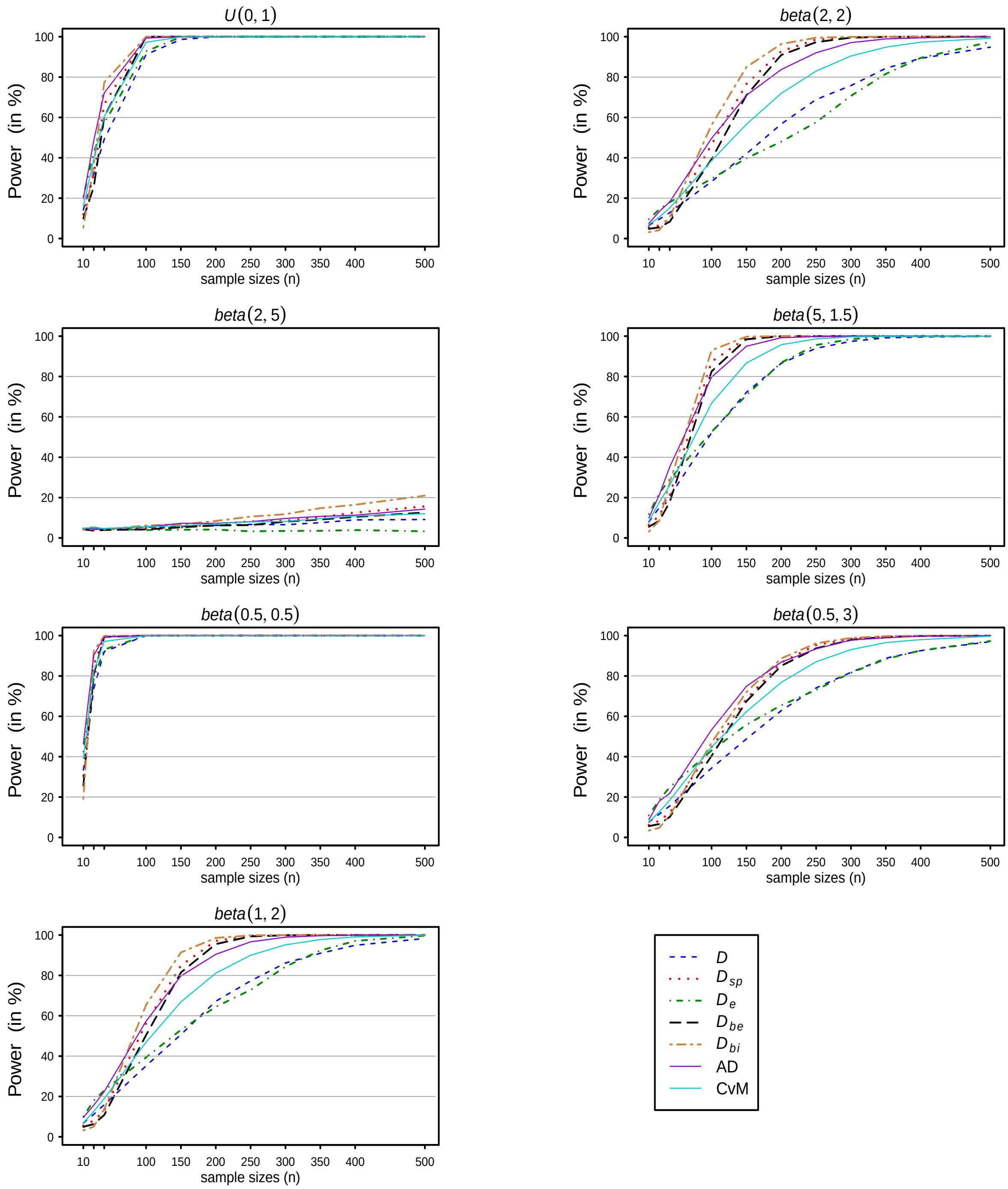

Figure 4.23: Power comparison of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

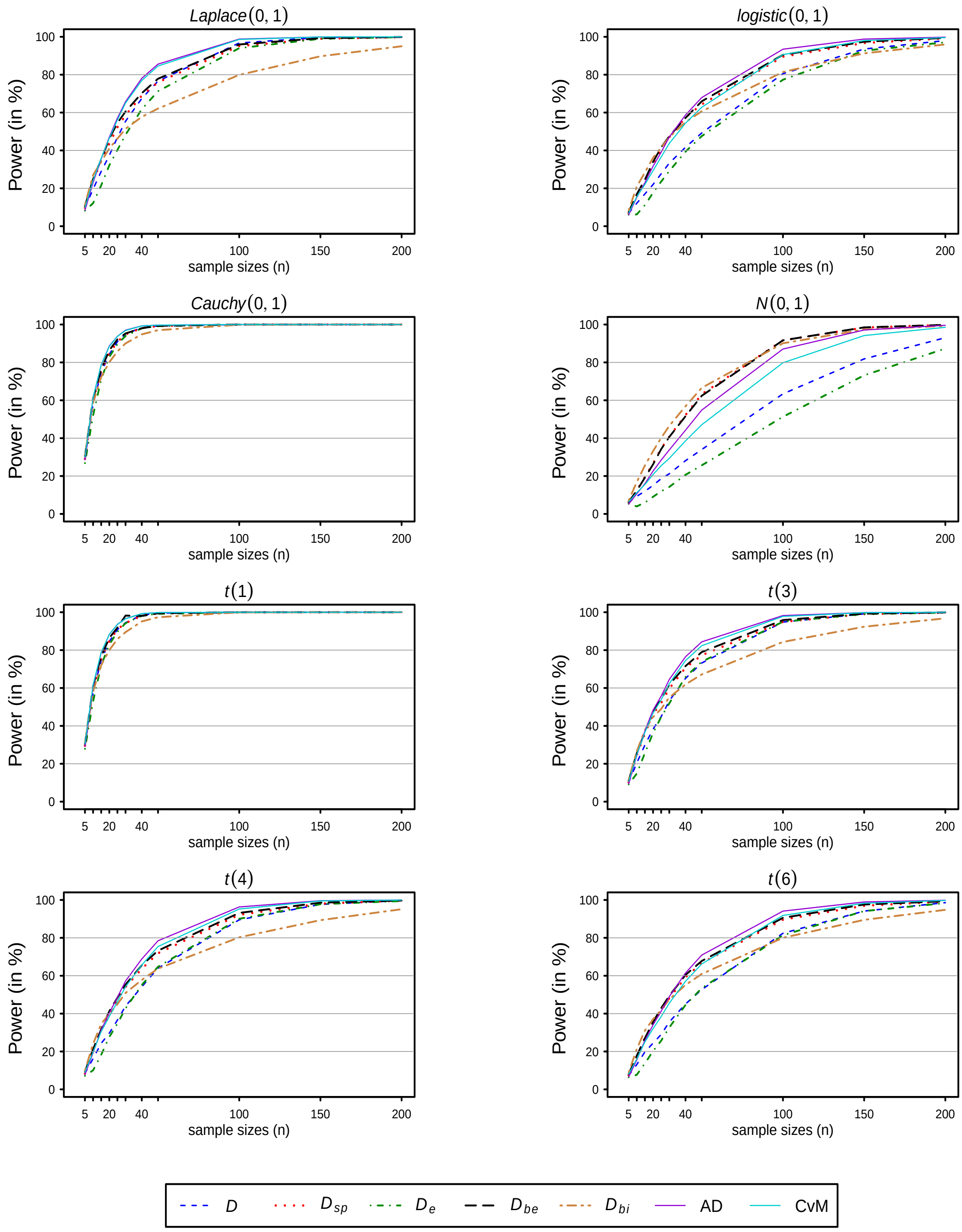

Figure 4.24: Power comparison of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

4.6 Simultaneous intervals

The null hypothesis is H_0 : a random sample $X_1, \ldots, X_n$ is from the Weibull distribution $Wbl(a, b, c)$ where a is known. The a is equal to zero, without loss of generality, so we assume that

H_0 : a random sample $X_1, \ldots, X_n$ comes from the Weibull distribution $Wbl(0, b, c)$.

If a is not equal to zero then we consider $X_1 - a, \ldots, X_n - a$ instead of the original $X_1, \ldots, X_n$. Define $Y_k = \ln(X_k)$, $k = 1, \ldots, n$. Then each Y_k has the smallest extreme value distribution, $SEV(\mu, \sigma)$, with $\mu = \ln b$ and $\sigma = \frac{1}{c}$, as discussed above. Hence a test of H_0 is equivalent to a test of $H_0 : Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} SEV(\mu, \sigma)$. Any of the five graphical tests can then be used to test this H_0 in the following way.

1. Choose a significance level α and choose the graphical test.

2. Calculate $Y_k = \ln X_k$, $k = 1, \ldots, n$ and compute the BLUE of the unknown parameters μ and σ of $SEV(\mu, \sigma)$, say $\dot{\mu}$ and $\dot{\sigma}$. Also, calculate the critical constant, which depends on α, sample size n and the number of simulations.

3. Sort the $Y_k's$ in ascending order $Y_{[1]} \leq \ldots \leq Y_{[n]}$ and plot the points $(\ln(-\ln(1-p_k)), Y_{[k]})$ for $k = 1, \ldots, n$ where $p_k = \frac{k-0.5}{n}$. This produces the Q-Q plot.

4. For each k, plot the vertical interval based on the graphical test considered.

5. Join the upper limits of the k vertical intervals and join the lower limits of the k vertical intervals to obtain a band.

6. Reject the null hypothesis H_0 at level α if at least one point $(\ln(-\ln(1 - p_k)), Y_{[k]})$ falls outside the corresponding vertical interval.

Table 4.15 gives the k simultaneous vertical intervals corresponding to each of the five graphical tests D, D_{sp}, D_e, D_{be} and D_{bi}.

These simultaneous intervals have nothing to do with the alternative distributions because they depend only on $k = 1, \ldots, n$ and their critical constants of each test.

Table 4.15: Simultaneous intervals for testing Weibull distribution based on the five graphical tests by using BLUE as the estimators

Graphical tests	Simultaneous intervals of $Y_{[k]}$ for $k = 1, \ldots, n$
D	$\dot{\mu} + \dot{\sigma}\ln[-\ln\left(1 - (k-0.5)/n \mp c_D\right)]$
D_{sp}	$\dot{\mu} + \dot{\sigma}\ln[-\ln\left(1 - \sin^2\left(\arcsin\sqrt{(k-0.5)/n} \pm c_{sp}\right)\right)]$
D_e	$(\dot{\mu} + \dot{\sigma}\mu_k) \pm \dot{\sigma}\sigma_k c_e$
D_{be}	$(\dot{\mu} + \dot{\sigma}\ln[-\ln(1-L)], \dot{\mu} + \dot{\sigma}\ln[-\ln(1-U)])$
D_{bi}	$\dot{\mu} + \dot{\sigma}\ln[-\ln\left(1 - \frac{k-0.5}{n} \mp c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\right)]$

Example 1

An illustrative example is presented to apply the simultaneous probability intervals associated with the five graphical tests D, D_{sp}, D_e, D_{be} and D_{bi} for testing the Weibull distribution at significance level $\alpha = 0.05$. Figure 4.25 shows the Q-Q plots constructed from the observations simulated from $LogN(0,1)$ with $n = 40$ and their corresponding probability intervals for the five graphical tests of the ordered observations. Therefore, the blue triangles represent the pairs $(\ln(-\ln(1-p_k)), Y_{[k]})$ and the straight line results from plotting $\dot{\mu} + \dot{\sigma}\ln(-\ln(1-p_k))$ against $\ln(-\ln(1-p_k))$. In each plot, the simultaneous probability intervals are constructed corresponding to the considered graphical test.

From Figure 4.25, the first ordered observation based on D_{bi} lies outside the corresponding vertical interval. Furthermore, there exists one point from the plot of the D_{sp} and D_{be} tests which seems to fall outside its corresponding vertical interval, but it is rather subjective to judge that both points fall outside their corresponding intervals. Therefore, the numerical comparison must be used to assess whether points lie within their corresponding intervals whenever points cannot be distinguished by eye. Table 4.17 illustrates 40 ordered observations generated from $LogN(0,1)$ and log of these observations, as well as the corresponding probability intervals. This table will be helpful when any points cannot be assessed by eye. Based on the D_{sp} test, the probability interval of the 1^{st} ordered observation is $[-5.3681, -1.8762]$. Comparing the 1^{st} ordered value, $(Y_{[1]} = -1.8088)$, with its interval, we can observe that the corresponding interval does not cover the 1^{st} ordered value. Therefore, these data do not follow the Weibull distribution based on the D_{sp} test. Similarly, the probability interval of the 1^{st} ordered value from the D_{be} test is $[-\infty, -1.8374]$ but the point goes beyond the corresponding interval. Therefore, this sample is not taken from the Weibull distribution based on the D_{be} test.

All observations fall inside the bands based on the D and D_e tests. Therefore, we can claim that this sample is taken from a Weibull distribution when the D and D_e tests are considered.

For the non-graphical tests, the test statistics of AD and CvM tests computed by (4.4.1) and (4.4.2) are 0.5111 and 0.0783, respectively. Also, the corresponding critical values of AD and CvM statistics are 0.7529 and 0.1240, respectively. Then, comparing the test statistics with their critical values, it can be concluded that the hypothesis is not rejected at $\alpha = 0.05$ under AD and CvM statistics since the test statistic of each test is less than its corresponding critical value. Therefore, the findings support the conclusion that such a dataset follows a Weibull distribution by the AD and CvM tests.

It can be observed that the null hypothesis is rejected according to the D_{sp}, D_{be} and D_{bi} tests, but not according to the D, D_e, AD and CvM tests. Interestingly, even though the AD and CvM tests are non–graphical tests, they cannot detect the non-Weibull data in this case. Table 4.16 summarises the results when seven tests are applied to assess whether the dataset follows the Weibull distribution.

Table 4.16: Test results for Example 1

Test	Inference
D	Does not reject H_0
D_{sp}	Rejects H_0
D_e	Does not reject H_0
D_{be}	Rejects H_0
D_{bi}	Rejects H_0
AD	Does not reject H_0
CvM	Does not reject H_0

Table 4.17: Data generated from Lognormal(0,1) with sample size 40 and the corresponding probability intervals for testing the Weibull distribution based on the five graphical tests using BLUE (D, D_{sp}, D_e, D_{be} and D_{bi}) at $\alpha = 0.05$

			D		D_{sp}		D_e		D_{be}		D_{bi}	
$X_{[k]}$	$Y_{[k]} = \ln(X_{[k]})$	Quantiles	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
0.1638	-1.8088	-4.3757	NaN	-1.1979	-5.3681	-1.8762	-6.0585	-0.7279	$-\infty$	-1.8374	NaN	-1.9975
0.176	-1.7375	-3.2644	NaN	-1.0278	-5.3452	-1.3601	-4.1167	-0.7897	-5.6592	-1.3885	NaN	-1.3313
0.2208	-1.5104	-2.7405	NaN	-0.8798	-3.7183	-1.0659	-3.2675	-0.6667	-3.8551	-1.0964	NaN	-0.9936
0.2697	-1.3105	-2.3907	NaN	-0.7485	-2.9856	-0.8517	-2.7454	-0.5382	-3.0618	-0.8785	NaN	-0.7583
0.2872	-1.2477	-2.1257	NaN	-0.6298	-2.5166	-0.6805	-2.3707	-0.4194	-2.5652	-0.7036	NaN	-0.5744
0.2976	-1.212	-1.9111	-3.4463	-0.5212	-2.1728	-0.5362	-2.0792	-0.3111	-2.2062	-0.5563	NaN	-0.4216
0.3782	-0.9723	-1.7297	-2.4403	-0.4208	-1.9015	-0.4105	-1.8404	-0.2118	-1.9255	-0.4281	-4.9098	-0.2897
0.3851	-0.9542	-1.572	-1.9575	-0.327	-1.6772	-0.2984	-1.638	-0.1201	-1.6947	-0.314	-3.1267	-0.1727
0.4464	-0.8066	-1.4317	-1.6335	-0.2388	-1.4856	-0.1967	-1.462	-0.0346	-1.4985	-0.2105	-2.5129	-0.0669
0.4934	-0.7064	-1.3051	-1.3876	-0.1553	-1.318	-0.1031	-1.306	0.0457	-1.3275	-0.1154	-2.1222	0.0302
0.4946	-0.704	-1.1891	-1.188	-0.0756	-1.1687	-0.016	-1.1656	0.1215	-1.1755	-0.027	-1.8311	0.1205
0.5341	-0.6272	-1.0818	-1.0193	0.0007	-1.0338	0.0658	-1.0377	0.1937	-1.0384	0.0559	-1.5969	0.2052
0.5413	-0.6138	-0.9816	-0.8724	0.0742	-0.9103	0.1431	-0.9199	0.2627	-0.9132	0.1342	-1.3997	0.2855
0.6063	-0.5004	-0.8874	-0.7418	0.1453	-0.7961	0.2168	-0.8105	0.3291	-0.7976	0.2088	-1.2283	0.3621
0.631	-0.4605	-0.7981	-0.6237	0.2144	-0.6898	0.2874	-0.708	0.3932	-0.69	0.2803	-1.076	0.4357
0.6395	-0.4471	-0.7129	-0.5156	0.2818	-0.5898	0.3554	-0.6115	0.4554	-0.589	0.3491	-0.9383	0.5069
0.8083	-0.2128	-0.6313	-0.4156	0.3479	-0.4954	0.4213	-0.5199	0.516	-0.4937	0.4158	-0.8123	0.5761
0.829	-0.1875	-0.5528	-0.3222	0.413	-0.4056	0.4855	-0.4327	0.5753	-0.403	0.4806	-0.6955	0.6438
0.9798	-0.0204	-0.4767	-0.2342	0.4772	-0.3197	0.5481	-0.3491	0.6335	-0.3164	0.5439	-0.5862	0.7104
1.0765	0.0737	-0.4028	-0.1509	0.541	-0.2372	0.6096	-0.2687	0.6908	-0.2332	0.6061	-0.4832	0.7761
1.2162	0.1957	-0.3306	-0.0714	0.6045	-0.1575	0.6703	-0.191	0.7475	-0.1529	0.6673	-0.3854	0.8415
1.2174	0.1967	-0.2599	0.0047	0.6681	-0.0802	0.7303	-0.1156	0.8038	-0.0751	0.7279	-0.2919	0.9069
1.5189	0.418	-0.1903	0.0781	0.732	-0.0049	0.7899	-0.0421	0.8599	0.0008	0.7882	-0.202	0.9726
1.539	0.4312	-0.1216	0.1491	0.7967	0.0687	0.8494	0.0299	0.916	0.0749	0.8483	-0.1149	1.0391
1.7137	0.5387	-0.0534	0.2181	0.8624	0.141	0.9091	0.1005	0.9723	0.1478	0.9086	-0.0303	1.1069
1.7962	0.5857	0.0146	0.2854	0.9298	0.2124	0.9692	0.1701	1.0292	0.2196	0.9694	0.0525	1.1766
2.1652	0.7725	0.0827	0.3515	0.9994	0.2831	1.0301	0.2391	1.0868	0.2908	1.0309	0.1338	1.2491
2.4304	0.8881	0.1511	0.4165	1.0719	0.3535	1.0921	0.3076	1.1454	0.3617	1.0937	0.2143	1.3254
2.445	0.894	0.2204	0.4807	1.1484	0.424	1.1557	0.3761	1.2055	0.4327	1.158	0.2944	1.407
2.6073	0.9583	0.2908	0.5444	1.2304	0.4948	1.2213	0.4448	1.2675	0.5041	1.2245	0.3745	1.4964
2.772	1.0196	0.363	0.6079	1.3202	0.5666	1.2897	0.5141	1.3319	0.5765	1.2939	0.4552	1.5978
2.8333	1.0415	0.4375	0.6715	1.4215	0.6398	1.3617	0.5846	1.3995	0.6503	1.3671	0.5373	1.7198
2.9133	1.0693	0.5152	0.7355	1.5417	0.715	1.4385	0.6566	1.4711	0.7262	1.4453	0.6214	1.8854
3.1765	1.1558	0.5972	0.8002	1.6982	0.7932	1.5219	0.7309	1.5481	0.8051	1.5304	0.7088	2.2494
3.6735	1.3011	0.6852	0.866	1.9617	0.8755	1.6145	0.8084	1.6324	0.8882	1.6253	0.8007	NaN
4.2328	1.4429	0.7815	0.9335	NaN	0.9638	1.7208	0.8903	1.727	0.9773	1.735	0.8996	NaN
4.3731	1.4755	0.8904	1.0032	NaN	1.0607	1.8498	0.9785	1.8371	1.0752	1.8692	1.0088	NaN
4.4028	1.4822	1.0198	1.0759	NaN	1.1716	2.0222	1.0761	1.9727	1.1866	2.0512	1.1354	NaN
4.5422	1.5134	1.1889	1.1527	NaN	1.3082	2.3205	1.1888	2.1581	1.3206	2.3686	1.2956	NaN
7.4225	2.0045	1.4775	1.235	NaN	1.5128	2.324	1.3309	2.4876	1.4987	∞	1.5556	NaN

NaN stands for Not a Number representing an undefined value.

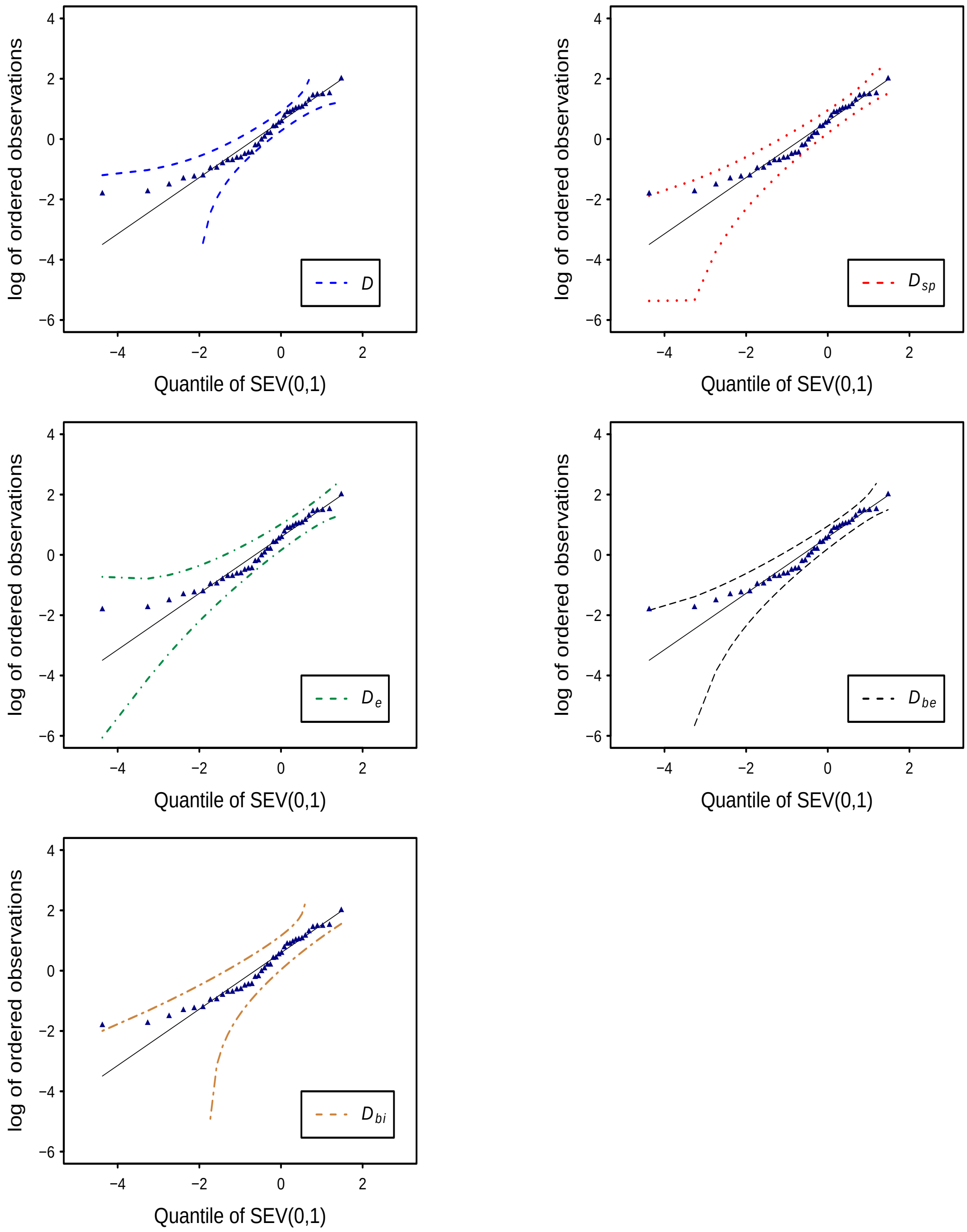

Figure 4.25: Simultaneous probability intervals of the data generated from $LogN(0, 1)$ with $n = 40$ for testing the Weibull distribution based on the five graphical tests (D, D_{sp}, D_e, D_{be} and D_{bi}) by using BLUE at $\alpha = 0.05$

Example 2

The following data are failure times of the air conditioning system of an airplane (see Linhart and Zucchini, 1986): 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, and 95. Table 4.19 shows the data in ascending order, and the upper and lower bounds of each observation from the five graphical tests using BLUE at $\alpha = 0.05$. Figure 4.26 illustrates the Q-Q plots which are plotted using the data from Table 4.19. Undoubtedly, all points fall inside their corresponding probability intervals simultaneously based on the D, D_{sp}, D_e, D_{be} and D_{bi} tests; therefore, this dataset follows a Weibull distribution under the five graphical tests. For the non-graphical tests, the test statistics AD and CvM are 0.5352 and 0.0945, respectively. The critical values at $\alpha = 0.05$ with sample size 30 of the AD and CvM tests are 0.7432 and 0.1234, respectively. Thus, the conclusion is that we cannot reject the null hypothesis at $\alpha = 0.05$ under AD and CvM tests because the test statistics are less than their critical values. Therefore, these data follow a Weibull distribution under all tests.

Table 4.18: Test results for Example 2

Test	Inference
D	Does not reject H_0
D_{sp}	Does not reject H_0
D_e	Does not reject H_0
D_{be}	Does not reject H_0
D_{bi}	Does not reject H_0
AD	Does not reject H_0
CvM	Does not reject H_0

Table 4.19: Failure times of the air conditioning system of an airplane and their corresponding probability intervals for testing the Weibull distribution based on the five graphical tests using BLUE (D, D_{sp}, D_e, D_{be} and D_{bi}) at $\alpha = 0.05$

			D		D_{sp}		D_e		D_{be}		D_{bi}	
$X_{[k]}$	$Y_{[k]} = \ln(X_{[k]})$	Quantiles	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
1	0	-4.086	NaN	1.8909	-3.7374	1.1303	-4.0225	2.5100	$-\infty$	1.1727	NaN	0.9694
3	1.0986	-2.9702	NaN	2.1453	-2.8694	1.8131	-1.5856	2.4960	-3.4286	1.7717	NaN	1.8419
5	1.6094	-2.4417	NaN	2.3631	-0.9895	2.2032	-0.5114	2.6796	-1.2074	2.1627	NaN	2.2879
7	1.9459	-2.0870	NaN	2.5549	-0.0962	2.4882	0.1529	2.8613	-0.2162	2.4545	NaN	2.6012
11	2.3979	-1.8170	-1.5092	2.7272	0.4869	2.7175	0.6321	3.0270	0.4104	2.6896	NaN	2.848
11	2.3979	-1.5969	0.2743	2.8846	0.9198	2.9118	1.0071	3.1777	0.8672	2.8887	-2.5756	3.0548
11	2.3979	-1.4098	0.9798	3.0303	1.2648	3.0823	1.316	3.3161	1.2272	3.063	-0.3546	3.2350
12	2.4849	-1.2459	1.4343	3.1667	1.5527	3.2355	1.5793	3.4444	1.5252	3.2194	0.4374	3.3965
14	2.6391	-1.0992	1.774	3.2957	1.8007	3.3757	1.8098	3.5647	1.7806	3.3624	0.9467	3.5441
14	2.6391	-0.9656	2.0477	3.4188	2.0195	3.5059	2.0155	3.6784	2.005	3.4949	1.3292	3.6813
14	2.6391	-0.8422	2.2788	3.5371	2.2163	3.6283	2.202	3.7868	2.2061	3.6194	1.6392	3.8106
16	2.7726	-0.7269	2.4801	3.6517	2.3958	3.7446	2.3733	3.8910	2.3891	3.7375	1.9023	3.9339
16	2.7726	-0.618	2.6596	3.7635	2.5617	3.856	2.5325	3.9919	2.5580	3.8507	2.1328	4.0528
20	2.9957	-0.5144	2.8225	3.8735	2.7168	3.9637	2.6818	4.0901	2.7155	3.9600	2.3395	4.1686
21	3.0445	-0.415	2.9726	3.9823	2.8630	4.0686	2.823	4.1864	2.8639	4.0664	2.5281	4.2825
23	3.1355	-0.3188	3.1126	4.0910	3.0022	4.1715	2.9577	4.2814	3.005	4.1708	2.7029	4.3957
42	3.7377	-0.2250	3.2444	4.2003	3.1357	4.2732	3.0871	4.3757	3.1403	4.2741	2.8670	4.5094
47	3.8501	-0.1330	3.3697	4.3114	3.2648	4.3746	3.2123	4.4700	3.271	4.3769	3.0228	4.6250
52	3.9512	-0.0420	3.4898	4.4255	3.3905	4.4763	3.3342	4.5649	3.3982	4.4803	3.1722	4.7443
62	4.1271	0.0486	3.6058	4.5443	3.5140	4.5794	3.4538	4.6612	3.5231	4.5850	3.3170	4.8694
71	4.2627	0.1397	3.7186	4.6702	3.6361	4.6848	3.572	4.7597	3.6467	4.6923	3.4587	5.0039
71	4.2627	0.2320	3.8292	4.8067	3.7581	4.7939	3.6897	4.8615	3.7701	4.8035	3.5989	5.1536
87	4.4659	0.3266	3.9384	4.9598	3.881	4.9083	3.8079	4.9678	3.8946	4.9204	3.7392	5.3301
90	4.4998	0.4249	4.0470	5.1413	4.0064	5.0303	3.9278	5.0806	4.0216	5.0455	3.8814	5.5641
95	4.5539	0.5285	4.1560	5.3819	4.1361	5.1636	4.0509	5.2024	4.1529	5.1828	4.0280	6.0531
120	4.7875	0.6403	4.2662	5.8423	4.2727	5.3139	4.179	5.3373	4.2914	5.3388	4.1822	NaN
120	4.7875	0.7647	4.3789	NaN	4.4205	5.4924	4.3152	5.4922	4.4411	5.5262	4.3494	NaN
225	5.4161	0.9102	4.4956	NaN	4.5867	5.7248	4.4641	5.6802	4.6089	5.7754	4.5391	NaN
246	5.5053	1.0972	4.6183	NaN	4.7879	6.1053	4.6346	5.9335	4.8078	6.1988	4.7740	NaN
261	5.5645	1.4096	4.7499	NaN	5.0838	6.2475	4.8499	6.3751	5.0672	∞	5.1452	NaN

NaN stands for Not a Number representing an undefined value.

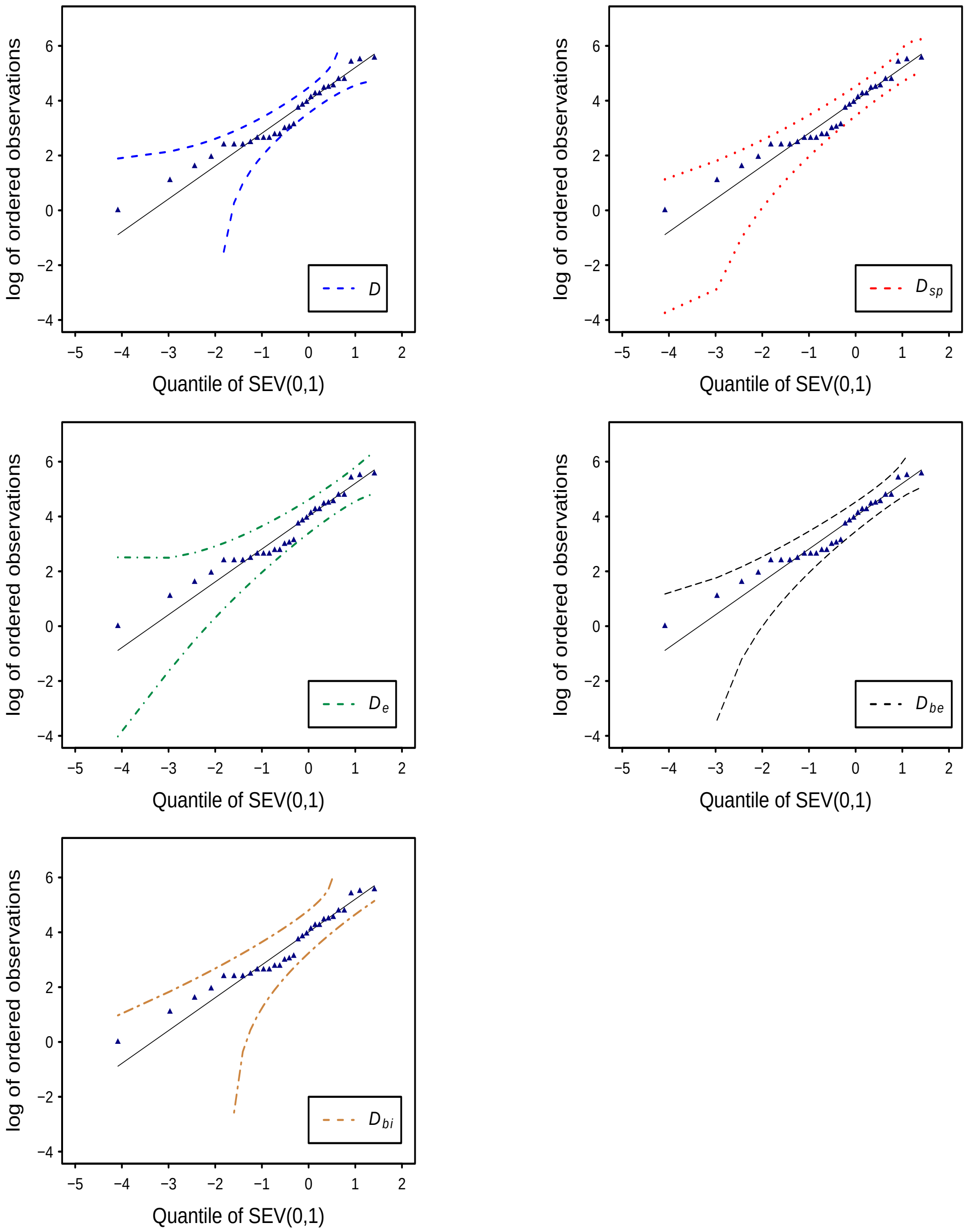

Figure 4.26: Simultaneous probability intervals of failure times of the air conditioning system of an airplane for testing the Weibull distribution based on the five graphical tests (D, D_{sp}, D_e, D_{be} and D_{bi}) using BLUE at $\alpha = 0.05$

Table 4.20: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.01$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$\chi^2(1)$	10	1.38	1.10	**2.15**	1.37	0.58	1.49	1.31
	25	1.64	1.11	**3.6**	1.06	0.62	2.29	1.80
	40	2.29	1.18	**4.81**	1.3	0.68	3.10	2.24
	100	4.58	2.77	6.98	3.83	2.06	**8.10**	5.93
	150	6.29	4.30	9.61	7.47	3.41	**12.3**	8.99
	200	8.81	6.99	10.73	9.27	4.98	**16.81**	12.78
	250	11.84	9.46	12.81	13.52	7.46	**22.75**	17.29
	300	15.07	12.40	13.58	15.98	8.69	**29.92**	22.3
	350	18.90	14.14	17.26	18.79	9.84	**35.81**	27.72
	400	22.86	18.32	18.94	23.03	11.86	**41.38**	33.44
	500	27.60	27.07	22.57	32.25	15.44	**52.78**	42.44
$\chi^2(3)$	10	1.05	1.11	0.81	**1.24**	1.21	0.97	1.11
	25	1.17	1.80	0.39	1.54	**1.88**	1.10	1.11
	40	1.29	**2.3**	0.37	1.77	2.28	1.51	1.46
	100	1.59	3.04	0.38	**3.42**	2.82	2.14	1.97
	150	1.67	3.50	0.61	**4.09**	3.08	2.91	2.44
	200	2.14	4.19	0.80	**4.37**	3.23	3.52	3.07
	250	2.58	**5.08**	0.99	5.07	4.24	3.94	3.47
	300	3.19	**5.51**	0.96	4.62	3.97	5.14	4.01
	350	3.3	4.78	1.26	5.03	3.50	**6.10**	4.96
	400	3.82	**6.56**	1.74	6.50	4.24	6.28	5.42
	500	4.34	8.02	1.89	8.12	4.73	**8.58**	6.82
$\chi^2(4)$	10	1.08	1.32	0.42	1.34	**1.69**	1.10	1.23
	25	1.21	2.26	0.31	2.52	**2.52**	1.34	1.46
	40	1.70	3.35	0.19	2.76	**3.52**	2.10	1.91
	100	2.50	**5.95**	0.52	5.65	5.66	2.31	2.16
	150	3.69	7.76	1.25	**8.73**	6.39	4.26	3.62
	200	4.76	**10.41**	1.62	10.02	7.94	6.85	5.51
	250	6.18	**13.47**	2.07	13.23	9.91	9.15	7.54
	300	7.49	14.46	2.58	14.23	10.05	**15.66**	11.78
	350	9.18	15.12	3.52	16.85	10.54	**18.17**	14.10
	400	10.72	19.44	3.70	19.36	11.96	**21.40**	17.20
	500	12.84	24.33	4.91	26.55	13.43	**28.74**	21.90
$\chi^2(6)$	10	1.14	1.30	0.26	1.55	**1.81**	1.03	1.35
	25	1.69	3.70	0.15	3.32	**4.37**	2.12	2.12
	40	2.37	5.27	0.10	4.73	**5.60**	3.47	3.10
	100	4.87	**12.64**	1.19	12.45	10.90	9.88	7.87
	150	7.37	18.15	2.48	**20.30**	14.48	16.31	12.52
	200	10.21	24.83	3.55	**25.10**	17.80	23.49	18.01
	250	14.42	30.99	5.28	**32.40**	21.47	30.73	23.88
	300	17.26	37.12	6.5	37.91	25.59	**40.15**	30.23
	350	22.24	41.24	8.32	43.94	26.22	**47.84**	37.91
	400	27.06	49.5	9.84	50.01	30.46	**53.08**	44.2
	500	33.43	60.05	12.88	62.66	35.47	**68.08**	55.86
$\chi^2(10)$	10	1.33	1.7	0.2	1.8	**2.45**	1.3	1.64
	25	2.39	5.12	0.08	4.84	**5.71**	3.25	2.97
	40	3.48	8.30	0.08	7.05	**9.13**	5.65	4.91
	100	9.10	**23.06**	2.51	22.94	19.89	19.24	15.03
	150	13.97	34.31	5.06	**37.12**	27.06	32.3	24.74
	200	20.39	47.09	7.70	**47.43**	34.59	45.44	36.33
	250	28.72	58.82	11.15	**59.7**	43.37	58.69	47.16
	300	35.45	65.83	13.31	67.19	47.59	**70.27**	57.48
	350	43.66	72.24	17.88	75.09	51.58	**78.98**	67.29
	400	51.54	80.32	21.46	81.52	57.94	**85.00**	74.96
	500	62.15	89.14	29.06	90.95	66.48	**93.12**	85.12
$LogN(0,1)$	10	2.27	3.31	0.02	3.60	**5.31**	2.34	2.94
	25	6.68	17.52	0.06	16.26	**19.75**	12.18	10.58
	40	12.53	31.46	1.35	28.14	**33.17**	23.96	20.21
	100	39.76	78.58	14.19	**78.71**	73.59	72.40	62.72
	150	60.92	93.08	27.31	**94.18**	88.04	91.69	84.08
	200	78.41	98.53	39.20	**98.63**	95.55	97.77	94.59
	250	89.91	99.75	53.04	**99.79**	98.49	99.67	98.58
	300	94.89	99.94	64.74	**99.95**	99.43	99.91	99.56
	350	97.76	**99.98**	80.84	99.96	99.80	**99.98**	99.86
	400	99.08	**100**	88.82	**100**	99.88	**100**	99.98
	500	99.88	**100**	97.84	**100**	**100**	**100**	**100**
$HN(0,1)$	10	1.41	1.10	**2.37**	1.29	0.64	1.67	1.34
	25	1.86	1.06	**3.33**	1.28	0.52	2.11	1.69
	40	2.31	1.11	**4.30**	1.10	0.65	3.25	2.41
	100	4.49	2.61	6.86	3.74	2.21	**7.65**	5.76
	150	6.83	4.22	9.01	6.94	3.39	**12.49**	9.52
	200	9.63	6.66	10.99	8.99	4.87	**17.55**	13.65
	250	12.32	9.82	13.27	12.99	6.95	**23.34**	17.78
	300	15.35	12.45	14.59	15.60	8.55	**30.14**	22.75
	350	18.14	14.28	17.54	18.92	8.86	**36.38**	28.26
	400	23.22	18.04	19.22	23.43	11.2	**42.02**	32.19
	500	27.85	26.85	23.39	33.16	15.39	**52.97**	42.26

The bold number is the highest power among the seven tests for each sample size.

Table 4.21: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.01$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$U(0,1)$	10	4.25	3.38	6.06	3.46	1.03	**6.78**	4.89
	25	12.72	12.76	13.94	10.94	14.54	**23.96**	17.18
	40	22.93	42.06	20.00	33.93	**50.38**	44.49	32.70
	100	68.26	99.86	43.37	99.38	**99.92**	95.55	86.85
	150	89.21	**100**	70.52	**100**	**100**	99.82	98.17
	200	97.15	**100**	94.33	**100**	**100**	**100**	99.85
	250	99.65	**100**	99.62	**100**	**100**	**100**	99.98
	300	99.93	**100**	**100**	**100**	**100**	**100**	99.99
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	1.69	1.13	**2.4**	1.31	0.47	1.75	1.48
	25	2.45	1.35	**3.61**	1.52	0.67	3.58	2.71
	40	3.55	1.93	4.37	1.90	2.07	**6.24**	4.31
	100	10.37	20.42	7.42	17.30	**25.51**	23.42	16.11
	150	17.14	48.20	10.12	44.95	**55.34**	40.61	28.19
	200	25.65	76.63	11.37	68.70	**81.05**	57.85	42.93
	250	37.22	92.63	14.63	87.90	**95.09**	73.76	57.39
	300	45.53	97.98	17.16	95.54	**98.82**	84.40	68.21
	350	54.07	99.39	20.03	98.64	**99.66**	91.50	78.18
	400	63.60	99.98	25.52	99.71	**99.98**	99.56	86.52
	500	76.89	**100**	44.42	**100**	**100**	99.98	93.70
$beta(2,5)$	10	1.04	0.93	0.83	**1.08**	0.66	1.03	1.03
	25	**1.12**	1.08	1.10	0.74	0.89	0.97	0.88
	40	**0.99**	0.61	0.97	0.57	0.79	0.9	0.83
	100	1.01	0.98	0.84	0.81	1.07	**1.26**	1.19
	150	1.12	1.06	0.83	1.12	1.24	**1.38**	1.30
	200	1.13	1.46	0.72	1.10	1.58	**1.65**	1.59
	250	1.35	1.82	0.66	1.30	**2.46**	1.94	1.76
	300	1.49	1.95	0.67	1.45	**2.75**	2.41	1.97
	350	1.55	2.59	0.57	1.77	**3.26**	2.64	2.33
	400	2.05	3.26	0.79	2.05	**4.70**	2.84	2.84
	500	2.03	4.81	0.61	3.12	**5.68**	3.74	3.12
$beta(5,1.5)$	10	1.71	1.37	**2.88**	1.56	0.54	2.63	1.96
	25	4.43	2.64	6.04	2.54	1.84	**7.61**	5.49
	40	6.98	6.5	7.67	5.1	8.2	**13.54**	9.19
	100	24.05	65.86	15.07	60.42	**73.02**	53.46	38.71
	150	40.64	94.84	20.7	93.17	**96.69**	79.10	62.08
	200	58.59	99.69	27.59	99.39	**99.85**	92.54	80.66
	250	73.35	99.97	38.13	**99.98**	99.97	98.00	91.15
	300	83.34	**100**	51.74	**100**	**100**	99.48	96.39
	350	90.39	**100**	71.30	**100**	**100**	99.89	98.85
	400	94.70	**100**	87.70	**100**	**100**	**100**	99.64
	500	98.39	**100**	99.35	**100**	**100**	**100**	99.96
$beta(0.5,0.5)$	10	14.31	12.19	13.63	11.26	7.40	**23.67**	18.56
	25	45.24	70.14	31.01	64.83	**77.98**	71.44	60.22
	40	72.03	98.30	47.42	96.93	**99.13**	94.14	86.81
	100	99.74	**100**	98.73	**100**	**100**	**100**	99.99
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	1.98	1.38	**3.31**	1.46	0.56	2.4	1.77
	25	3.31	1.82	**5.51**	2.10	0.91	5.00	3.49
	40	5.31	2.89	7.77	2.93	2.60	**8.22**	6.05
	100	13.85	18.89	14.72	18.65	20.39	**30.32**	21.61
	150	22.67	39.03	19.41	41.76	41.00	**49.35**	36.50
	200	33.67	63.95	25.23	60.55	65.30	**66.30**	51.7
	250	46.96	81.55	32.13	78.58	**82.94**	80.20	66.42
	300	56.00	91.14	36.13	87.9	**91.89**	89.22	77.08
	350	65.25	96.10	43.00	94.54	**96.25**	94.66	86.12
	400	73.17	98.56	49.23	97.77	**98.91**	95.94	91.52
	500	85.46	**99.9**	64.71	99.77	99.87	98.58	96.58
$beta(1,2)$	10	1.62	1.26	**2.61**	1.28	0.60	2.18	1.63
	25	3.11	1.71	**4.71**	2.04	0.96	4.62	3.08
	40	5.25	3.06	6.02	2.78	3.35	**8.46**	5.92
	100	13.57	28.46	11.15	25.15	**33.97**	31.47	21.86
	150	23.30	61.06	15.98	58.78	**66.53**	52.5	37.75
	200	35.81	87.02	19.73	80.86	**89.87**	70.99	55.04
	250	48.40	96.91	24.32	94.87	**98.15**	84.52	69.48
	300	59.43	99.31	28.50	98.48	**99.61**	92.55	80.79
	350	67.57	99.89	35.24	99.67	**99.94**	96.52	90.68
	400	76.91	99.99	44.26	99.95	**100**	99.92	93.76
	500	87.34	**100**	68.26	**100**	**100**	**100**	97.92

The bold number is the highest power among the seven tests for each sample size.

Table 4.22: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.01$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$Laplace(0,1)$	5	2.26	2.22	0.69	2.1867	2.31	1.52	**2.54**
	10	7.91	10.85	0.68	10.86	**13.08**	9.60	11.21
	15	14.35	20.25	0.81	20.72	19.22	**21.9**	20.99
	20	22.07	28.44	1.53	30.20	25.44	**31.52**	30.72
	25	29.79	36.39	2.19	38.26	29.49	**40.46**	39.76
	30	37.02	43.71	6.51	44.78	35.13	**52.08**	50.69
	40	49.98	51.86	15.48	53.30	33.78	**64.92**	63.96
	50	62.19	62.56	27.25	65.25	45.23	74.42	**74.84**
	100	92.46	88.5	67.85	90.9867	65.78	97.06	**97.07**
	150	98.82	97.08	89.30	98.20	77.84	99.78	**99.80**
	200	99.86	99.36	96.06	99.54	86.28	**100**	99.94
$logistic(0,1)$	5	1.34	1.26	0.33	1.3433	1.37	0.86	**1.58**
	10	3.92	5.76	0.18	5.7467	**8.36**	4.66	5.50
	15	6.47	11.99	0.17	11.60	**13.38**	11.56	10.49
	20	9.59	17.98	0.22	18.13	**19.72**	16.64	15.21
	25	12.87	**24.54**	0.32	24.22	24.33	22.06	21.00
	30	16.95	**30.62**	1.96	29.65	29.62	30.32	27.78
	40	24.08	37.90	5.70	36.54	34.84	**40.72**	36.95
	50	31.58	48.20	12.85	**49.03**	41.51	49.02	47.36
	100	65.38	77.86	41.26	80.15	65.28	**86.66**	82.15
	150	84.62	92.32	65.68	94.06	78.86	**96.56**	94.99
	200	93.42	96.90	77.06	97.86	86.56	**99.18**	98.67
$Cauchy(0,1)$	5	16.12	15.98	8.04	15.91	16.42	15.69	**17.37**
	10	43.03	46.09	24.17	46.44	45.19	47.40	**48.26**
	15	62.57	66.12	38.87	66.61	59.01	68.71	**69.01**
	20	74.63	76.75	52.48	78.83	68.7	**80.96**	80.79
	25	83.95	84.77	63.51	86.69	74.79	**88.42**	88.27
	30	90.52	90.24	74.68	91.59	81.49	93.97	**94.04**
	40	96.14	95.42	86.52	95.78	87.32	**98.16**	98.1
	50	98.56	98.02	94.31	98.60	92.48	99.29	**99.31**
	100	99.99	99.94	99.91	99.99	99.48	**100**	**100**
	150	**100**	**100**	**100**	**100**	99.98	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	1.06	1.02	0.29	1.0867	1.08	0.73	**1.19**
	10	2.27	3.31	0.02	3.19	**5.31**	2.34	2.94
	15	3.52	7.46	0.05	6.77	**10.19**	5.66	5.54
	20	5.11	12.12	0.05	10.7967	**15.25**	8.62	8.06
	25	6.68	17.52	0.06	15.5367	**19.75**	12.18	10.58
	30	7.47	21.73	0.14	20.0633	**25.19**	16.24	14.40
	40	12.18	30.92	0.90	27.26	**33.18**	23.96	20.21
	50	15.60	**41.82**	3	40.44	41.49	33.12	27.71
	100	39.76	**78.58**	14.19	78.4967	73.59	72.40	62.72
	150	61.88	94.04	28.60	**94.62**	88.74	91.69	84.08
	200	77.58	98.60	36.24	**98.78**	95.76	97.77	94.59
$t(1)$	5	17.02	16.87	8.19	16.54	17.20	16.16	**18.02**
	10	41.92	44.76	24.15	46.20	43.70	46.02	**46.95**
	15	62.06	65.61	40.06	66.79	59.14	**68.81**	68.59
	20	75.38	76.92	52.63	78.80	68.76	81.11	**81.13**
	25	84.28	85.16	63.78	86.50	74.87	**89.09**	88.75
	30	89.67	89.59	74.23	91.36	80.09	**93.53**	93.37
	40	96.24	95.24	86.56	96.26	86.96	**98.27**	98.21
	50	98.97	98.55	94.59	98.77	92.53	**99.53**	99.51
	100	99.99	99.98	99.93	99.99	99.49	**100**	99.99
	150	**100**	**100**	**100**	**100**	99.98	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$t(3)$	5	2.77	2.63	0.96	2.68	2.83	2.26	**3.24**
	10	9.21	12.54	2.08	12.39	**14.94**	11.98	12.74
	15	16.46	**23.92**	3.65	23.52	23.71	23.37	23.16
	20	24.13	32.80	5.86	32.90	31.15	**34.33**	33.12
	25	30.33	39.57	8.37	40.53	34.25	**41.27**	39.54
	30	37.75	47.39	16.07	48.09	40.64	**51.55**	49.70
	40	44.58	56.70	27.18	58.48	46.26	**65.21**	62.93
	50	60.45	66.61	40.36	68.46	53.26	**74.96**	73.05
	100	90.21	89.99	78.79	92.01	74.31	**96.17**	95.62
	150	98.12	97.48	93.80	98.62	84.94	**99.60**	99.38
	200	99.60	99.38	97.60	99.62	91.58	**99.95**	99.91
$t(4)$	5	1.93	1.87	0.66	1.9933	1.96	1.43	**2.13**
	10	6.16	9.32	0.96	9.09	**11.66**	8.19	9.13
	15	11.60	19.27	1.57	17.88	**20.39**	18.30	17.78
	20	16.44	25.49	2.42	25.83	25.29	**25.38**	24.33
	25	21.97	32.62	3.72	33.11	28.96	**33.25**	31.09
	30	28.60	39.93	9.11	40.31	36.36	**42.89**	40.72
	40	38.54	49.32	19.02	47.60	41.28	**54.60**	51.41
	50	49.66	59.89	29.97	60.86	49.41	**66.88**	64.08
	100	81.25	84.05	65.95	86.95	67.86	**92.92**	90.93
	150	94.90	95.8	86.46	96.48	80.34	**98.96**	98.45
	200	98.54	98.32	93.90	99.10	88.24	**99.86**	99.79
$t(6)$	5	1.60	1.55	0.55	1.47	1.62	1.14	**1.85**
	10	4.66	6.86	0.36	6.43	**9.24**	5.89	6.58
	15	7.75	14.54	0.48	13.78	**16.21**	13.13	12.61
	20	11.58	20.71	0.87	20.12	**22.38**	19.19	18.22
	25	15.10	**26.68**	1.09	26.5267	25.49	25.11	22.87
	30	20.29	32.62	4.37	32.57	30.98	**33.5**	31.25
	40	27.48	42.10	10.20	40.32	37.34	**44.77**	41.26
	50	36.16	51.05	18.46	52.25	43.73	**55.61**	51.58
	100	69.57	79.43	50.14	81.75	65.35	**88.04**	84.44
	150	87.86	93.30	74.88	94.44	78.50	**97.34**	95.84
	200	95.66	97.56	84.20	97.96	86.86	**99.48**	99.19

The bold number is the highest power among the seven tests for each sample size.

Table 4.23: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.01$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$\chi^2(1)$	10	1.5	1.17	**2.22**	1.46	0.60	1.52	1.10
	25	1.81	0.99	**3.64**	0.88	0.31	2.54	2.04
	40	2.55	0.96	**4.99**	1.07	0.36	3.38	2.74
	100	4.82	2.01	7.14	3.31	1.44	**8.12**	5.82
	150	6.73	3.21	10.12	6.57	2.56	**13.60**	10.36
	200	9.25	5.83	10.73	8.28	3.98	**17.72**	13.48
	250	12.44	7.99	13.12	12.00	6.20	**24.00**	18.68
	300	15.61	10.69	14.04	14.96	7.24	**30.66**	23.28
	350	17.89	12.87	16.36	17.61	8.68	**37.18**	29.22
	400	22.26	17.65	19.54	21.93	12.15	**41.94**	33.04
	500	28.49	25.19	23.85	31.54	13.91	**54.08**	44.40
$\chi^2(3)$	10	1.03	1.03	0.75	1.21	**1.38**	0.86	0.94
	25	1.12	1.71	0.40	1.41	**1.83**	0.94	0.80
	40	1.29	**2.37**	0.48	1.67	2.36	1.16	1.10
	100	1.47	3.02	0.54	**3.55**	2.82	2.20	2.06
	150	1.57	3.52	0.90	**4.12**	3.11	2.62	2.26
	200	2.01	4.16	0.97	**4.48**	3.37	3.32	3.00
	250	2.51	**5.08**	1.14	5.02	4.30	3.72	3.02
	300	3.02	**5.56**	1.19	4.89	4.07	5.16	4.38
	350	3.21	5.60	1.24	5.15	4.24	**6.16**	4.86
	400	3.61	6.19	1.49	**6.56**	4.92	6.18	5.16
	500	4.23	8.07	2.04	**8.39**	4.81	8.10	6.38
$\chi^2(4)$	10	1.02	1.31	0.47	1.34	**1.78**	0.92	1.02
	25	1.14	2.16	0.39	2.37	**2.46**	1.06	0.96
	40	1.54	3.39	0.39	2.64	**3.59**	1.92	1.66
	100	2.3	**5.94**	0.97	5.86	5.52	4.22	3.26
	150	3.66	7.66	1.78	**8.69**	6.39	6.40	5.08
	200	4.51	10.22	2.06	**10.27**	8.05	9.00	7.38
	250	5.96	**13.36**	2.53	12.94	9.98	11.56	8.66
	300	7.18	14.28	3.05	14.37	9.82	**14.94**	11.14
	350	8.1	16.05	3.36	16.76	10.95	**18.16**	13.94
	400	9.54	18.62	4.04	19.19	12.15	**21.02**	15.90
	500	12.41	24.13	5.66	26.86	13.38	**27.88**	20.90
$\chi^2(6)$	10	0.97	1.27	0.30	1.55	**1.86**	1.20	1.18
	25	1.51	3.67	0.33	3.10	**3.98**	2.24	1.90
	40	2.07	5.25	0.58	4.28	**5.51**	3.42	2.80
	100	4.38	12.16	2.07	**12.63**	10.45	8.64	6.32
	150	7.11	17.43	3.56	**20.06**	14.35	15.54	11.08
	200	9.77	23.91	4.66	**24.78**	17.50	23.26	17.92
	250	13.85	29.96	6.49	**31.11**	21.08	30.12	22.72
	300	16.52	36.27	7.76	37.68	24.84	**38.70**	28.64
	350	19.94	40.90	9.15	43.02	25.88	**47.80**	37.06
	400	24.53	47.33	11.41	49.23	30.92	**53.20**	41.66
	500	32.24	58.65	14.95	62.51	34.72	**66.74**	54.22
$\chi^2(10)$	10	1.12	1.74	0.21	1.62	**2.45**	1.22	1.34
	25	2.11	5.00	0.31	4.39	**5.43**	2.98	2.40
	40	3.1	8.14	1.02	6.45	**8.67**	5.48	4.42
	100	8.27	22.03	3.88	**22.63**	18.95	17.82	13.40
	150	13.27	32.70	7.18	**36.29**	26.17	32.18	23.60
	200	19.4	44.99	9.94	**46.64**	33.09	44.24	34.26
	250	27.45	56.99	13.73	**57.97**	41.89	57.12	44.8
	300	33.74	64.20	16.25	66.23	45.90	**68.66**	55.96
	350	40.75	71.33	19.80	73.85	50.84	**77.88**	65.52
	400	47.09	79.44	24.18	80.53	59.12	**83.52**	72.60
	500	60.28	88.00	33.07	90.66	64.87	**92.96**	85.04
$LogN(0,1)$	10	1.92	3.21	0.08	3.33	**5.30**	2.30	2.56
	25	5.73	16.48	2.04	14.85	**17.97**	11.34	9.40
	40	10.73	30.26	5.37	26.02	**31.55**	23.76	18.36
	100	35.38	76.54	21.65	**78.01**	70.18	70.26	58.46
	150	57.03	92.25	36.42	**93.71**	86.07	91.48	82.74
	200	74.70	98.23	47.38	**98.45**	94.44	97.52	94.12
	250	87.76	**99.73**	60.82	99.72	97.94	99.64	98.20
	300	93.74	**99.94**	71.26	99.93	99.26	99.88	99.48
	350	96.98	**99.96**	81.62	99.95	99.69	99.94	99.82
	400	98.82	**100**	90.34	**100**	99.91	**100**	99.94
	500	99.83	**100**	98.18	**100**	99.99	**100**	**100**
$HN(0,1)$	10	1.61	1.17	**2.34**	1.41	0.68	1.74	1.46
	25	1.93	0.89	**3.29**	1.15	0.36	2.34	1.86
	40	2.58	0.87	**4.39**	0.96	0.36	3.86	3.08
	100	4.64	2.03	7.08	3.35	1.31	**7.66**	5.82
	150	7.31	3.26	9.37	6.12	2.45	**12.52**	9.56
	200	10.07	5.38	10.96	8.14	3.92	**17.94**	14.04
	250	12.89	8.42	13.57	11.46	5.74	**23.66**	18.10
	300	15.87	10.87	15.02	14.72	7.22	**30.66**	23.54
	350	18.60	12.72	16.70	17.71	7.97	**36.98**	28.68
	400	22.11	16.95	19.62	22.05	11.07	**41.98**	33.20
	500	28.88	25.03	24.59	32.04	13.95	**53.04**	43.38

The bold number is the highest power among the seven tests for each sample size.

Table 4.24: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.01$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$U(0,1)$	10	4.9	3.57	7.29	3.75	0.68	**7.50**	5.82
	25	14.83	8.47	17.15	8.77	4.42	**26.72**	20.02
	40	26.87	29.54	26.96	22.96	34.8	**50.52**	38.28
	100	75.08	99.62	58.72	99.04	**99.83**	97.04	89.82
	150	93.57	**100**	78.44	**100**	**100**	99.92	99.16
	200	98.67	**100**	91.21	**100**	**100**	**100**	99.92
	250	99.83	**100**	98.79	**100**	**100**	**100**	**100**
	300	99.95	**100**	99.91	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	1.68	1.23	**2.4**	1.33	0.50	1.90	1.52
	25	2.72	1.08	3.8	1.4	0.20	**4.94**	3.54
	40	3.92	1.39	5.11	1.67	0.81	**7.58**	5.54
	100	11.46	14.45	8.92	13.03	18.23	**26.00**	17.72
	150	19.77	39.33	12.33	37.32	**47.08**	46.10	32.66
	200	29.22	69.90	14.12	62.10	**75.97**	63.96	48.38
	250	41.89	89.82	19.00	83.20	**93.25**	76.98	60.88
	300	50.31	96.92	22.09	93.44	**98.23**	88.72	74.12
	350	59.36	99.13	26.24	97.94	**99.51**	94.16	83.88
	400	69.32	99.92	31.87	99.49	**99.98**	96.54	88.48
	500	81.82	100	45.29	99.99	**100**	99.4	96.14
$beta(2,5)$	10	1.02	0.84	0.87	**1.09**	0.71	0.98	0.92
	25	**1.08**	0.98	0.99	0.67	0.87	0.88	0.82
	40	**0.98**	0.69	0.98	0.57	0.77	0.88	0.76
	100	0.94	0.87	0.82	0.77	0.89	**1.36**	1.16
	150	1.18	0.94	0.83	1.01	1.07	**1.40**	1.2
	200	1.17	1.21	0.68	0.96	1.36	1.76	**1.78**
	250	1.36	1.49	0.65	1.03	**2.15**	1.9	1.64
	300	1.53	1.70	0.68	1.30	2.48	**2.80**	2.28
	350	1.60	2.21	0.57	1.49	2.95	**3.02**	2.56
	400	2.14	2.84	0.82	1.74	**4.30**	2.72	2.44
	500	2.10	4.24	0.61	2.86	**5.33**	3.94	3.32
$beta(5,1.5)$	10	2.04	1.36	**3.30**	1.71	0.54	2.94	2.2
	25	5.11	2.09	6.84	2.50	0.31	**8.60**	6.12
	40	7.81	3.94	9.41	3.90	3.32	**16.82**	11.54
	100	27.49	56.61	19.61	51.52	**63.92**	58.02	42.4
	150	46.88	92.19	28.41	90.14	**94.98**	83.96	69.08
	200	65.14	99.45	37.10	98.86	**99.71**	94.9	85.64
	250	79.59	99.97	47.28	99.97	**99.97**	98.78	93.64
	300	88.31	**100**	56.34	**100**	**100**	99.8	97.76
	350	93.84	**100**	68.07	**100**	**100**	99.94	99.32
	400	96.99	**100**	82.48	**100**	**100**	**100**	99.72
	500	99.34	**100**	97.98	**100**	**100**	**100**	99.98
$beta(0.5,0.5)$	10	16.83	13.81	19.83	13.27	3.18	**25.72**	20.1
	25	52.73	54.51	45.49	50.06	59.59	**77.28**	66.5
	40	79.99	96.36	68.19	92.83	**97.96**	96.22	91.00
	100	99.92	**100**	97.92	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	2.28	1.52	**3.51**	1.71	0.45	2.36	1.84
	25	3.62	1.53	**5.73**	1.81	0.26	5.40	4.18
	40	5.84	1.91	8.61	2.36	1.02	**8.84**	6.78
	100	14.95	14.03	16.64	15.84	14.78	**31.58**	22.24
	150	24.82	32.03	22.81	36.17	34.49	**52.46**	39.20
	200	36.28	56.83	28.49	54.82	58.43	**69.44**	55.56
	250	49.96	76.36	36.99	73.25	78.26	**81.20**	68.20
	300	59.47	88	42.23	84.89	88.44	**91.78**	80.54
	350	68.55	94.48	49.83	92.54	94.35	**95.62**	87.88
	400	76.62	97.9	56.92	96.75	**98.16**	97.5	92.52
	500	87.79	99.8	71.15	99.63	**99.81**	99.62	97.38
$beta(1,2)$	10	1.79	1.34	**2.86**	1.44	0.58	2.46	1.78
	25	3.47	1.22	5.10	1.81	0.27	**5.64**	4.16
	40	5.74	2.13	7.17	2.24	1.26	**9.26**	6.50
	100	15.15	21.49	13.40	20.26	25.50	**34.26**	23.64
	150	26.44	52.55	19.77	51.18	**59.27**	56.94	42.46
	200	39.65	82.09	24.01	75.30	**86.39**	75.06	59.84
	250	53.17	95.31	30.61	92.03	**97.24**	87.74	73.96
	300	64.02	98.90	35.98	97.71	**99.40**	94.70	85.00
	350	72.33	99.78	43.21	99.46	**99.91**	98.16	91.86
	400	81.72	99.98	51.48	99.90	**100**	99.10	94.68
	500	90.59	**100**	68.56	**100**	**100**	99.94	98.90

The bold number is the highest power among the seven tests for each sample size.

Table 4.25: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.01$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$Laplace(0,1)$	5	1.74	1.76	2.07	1.70	2.17	1.78	**2.37**
	10	7.13	10.37	1.10	10.42	**12.03**	9.97	10.83
	15	13.09	18.08	5.14	18.70	15.83	**20.08**	19.64
	20	19.75	24.83	10.77	27.02	20.79	**30.02**	28.88
	25	26.81	32.05	15.76	34.30	23.49	**39.69**	37.68
	30	33.82	38.33	22.31	40.06	27.19	**49.91**	47.87
	40	46.45	47.25	34.19	47.82	31.60	**63.82**	61.49
	50	57.53	55.8	42.50	59.70	34.45	**73.81**	72.58
	100	90.65	83.17	78.04	88.16	49.23	**97.01**	96.56
	150	98.66	94.29	93.84	96.97	59.80	**99.8**	99.75
	200	99.75	98.4	97.91	99.44	69.82	**99.95**	99.94
$logistic(0,1)$	5	1.07	1.07	1.33	1.0567	1.24	1.00	**1.41**
	10	3.36	5.67	0.30	5.36	**7.82**	4.56	5.19
	15	5.65	11.02	1.81	10.46	**11.86**	10.36	9.6
	20	8.03	16.13	4.57	16.25	**17.08**	15.65	13.69
	25	11.02	22.04	7.17	21.60	20.90	**22.04**	18.71
	30	14.75	27.06	10.75	26.43	24.70	**29.07**	24.95
	40	20.93	35.76	17.51	33.01	31.08	**39.64**	33.32
	50	27.26	43.18	23.16	44.40	35.39	**49.46**	44.12
	100	59.39	72.26	52.90	76.51	56.14	**85.05**	79.99
	150	80.21	88.06	74.11	91.81	69.74	**96.54**	94
	200	91.39	95.25	85.74	96.99	79.62	**99.13**	98.39
$Cauchy(0,1)$	5	15.38	15.35	15.69	15.25	16.37	15.99	**17.31**
	10	42.88	45.94	30.09	46.71	42.99	47.65	**48.69**
	15	62.27	64.68	51	65.69	55.02	68.80	**69.27**
	20	74.1	75.26	66.82	77.61	63.96	**81.08**	80.8
	25	83.23	83.18	76.54	85.80	70.03	**88.52**	88.31
	30	89.95	88.96	84.23	90.66	76.24	93.93	**93.95**
	40	96.23	95.15	93.11	95.29	84.62	**98.17**	98.06
	50	98.43	97.69	96.49	98.33	89.21	99.27	**99.29**
	100	99.99	99.92	99.97	99.98	98.80	**100**	**100**
	150	**100**	**100**	**100**	**100**	99.89	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	0.8	0.81	0.81	0.84	1.03	0.82	**1.12**
	10	1.92	3.21	0.08	2.98	**5.3**	2.34	2.59
	15	2.92	7.30	0.37	6.38	**9.30**	5.26	4.90
	20	4.25	11.64	1.18	10.21	**13.95**	8.02	7.19
	25	5.73	16.48	2.04	14.67	**17.97**	11.35	9.18
	30	6.47	20.30	2.87	18.80	**22.59**	15.27	12.57
	40	10.73	30.26	5.37	26.02	**31.55**	22.97	18.11
	50	13.20	**40.08**	7.57	38.50	38.94	31.30	25.19
	100	35.38	76.54	21.65	**77.07**	70.18	71.40	60.13
	150	57.03	92.25	36.42	**93.71**	86.07	91.44	82.60
	200	74.70	98.23	47.38	**98.45**	94.44	97.58	94.03
$t(1)$	5	16.04	15.94	16	15.6667	17	16.3	**17.83**
	10	42.04	44.79	30.14	46.47	41.43	46.31	**47.62**
	15	62.14	64.55	51.45	65.85	55.29	**68.81**	37.37
	20	74.69	75.41	66.84	77.74	63.94	81.06	**81.16**
	25	83.57	83.73	76.3	85.51	69.9	**89.20**	88.96
	30	89.09	88.27	84.7	90.46	74.62	**93.48**	93.36
	40	96.42	95.41	93.51	95.25	84.85	**98.32**	98.21
	50	98.76	98.08	96.91	98.51	89.4	**99.54**	99.52
	100	**99.99**	**99.99**	99.98	99.99	98.89	**99.99**	**99.99**
	150	**100**	**100**	**100**	**100**	99.9	**100**	**100**
	200	**100**	**100**	**100**	**100**	99.99	**100**	**100**
$t(3)$	5	2.29	2.27	2.24	2.1967	2.74	2.42	**3.05**
	10	8.56	12.35	3.15	12.08	**13.70**	12.11	12.42
	15	15	22.12	9.9	21.79	20.75	**23.01**	22.18
	20	21.87	29.74	17.13	30.28	26.58	**33.78**	31.62
	25	27.19	35.64	23.84	37.14	28.66	**40.42**	37.61
	30	34.36	42.85	31.24	44.24	33.41	**50.13**	47.38
	40	46.23	52.86	45.23	53.48	39.32	**64.01**	60.07
	50	55.68	61.02	54.29	63.73	43.4	**73.55**	70.38
	100	87.10	86.17	85.30	89.64	60.08	**95.90**	94.94
	150	97.14	95.39	95.71	97.60	69.98	**99.58**	99.27
	200	99.42	98.77	98.72	99.36	80.19	**99.95**	99.91
$t(4)$	5	1.47	1.48	1.63	1.59	1.91	1.48	**1.98**
	10	5.66	8.99	1.34	8.78	**10.61**	8.28	8.61
	15	10.54	**17.87**	5.55	16.55	17.66	17.67	16.81
	20	14.64	23.09	11.62	23.45	21.50	**24.63**	22.53
	25	19.14	28.84	15.83	30.04	24.63	**32.15**	28.98
	30	25.09	35.99	22.17	36.54	29.80	**41.15**	37.92
	40	34.61	44.99	33.49	44.95	34.61	**53.15**	48.30
	50	43.90	53.99	42.44	55.78	40.24	**64.92**	60.79
	100	76.63	78.80	75.46	83.78	54.98	**92.10**	89.42
	150	93.03	91.69	90.73	95.53	66.10	**98.87**	98.05
	200	98.05	97.44	96.48	98.49	76.15	**99.84**	99.74
$t(6)$	5	1.27	1.28	1.26	1.2067	1.57	1.24	**1.69**
	10	3.93	6.80	0.56	6.2467	**8.77**	5.80	6.21
	15	6.86	13.51	3.02	12.6167	**14.29**	12.55	11.54
	20	9.69	18.87	6.43	18.22	**19.17**	18.34	16.49
	25	12.93	23.64	9.40	23.86	21.64	**24.05**	20.72
	30	17.31	29.35	14.65	29.1167	25.83	**31.93**	28.21
	40	24.17	38.36	22.86	37.77	32.59	**43.12**	37.97
	50	31.02	45.86	29.51	47.5467	36.65	**53.32**	47.96
	100	63.10	74.43	61.06	78.11	55.54	**87.05**	82.12
	150	84.23	88.28	79.97	92.86	67.41	**97.02**	95.04
	200	93.59	95.61	89.97	97.14	76.99	**99.42**	98.99

The bold number is the highest power among the seven tests for each sample size.

Table 4.26: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.01$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$\chi^2(1)$	10	1.42	1.09	**1.96**	1.12	0.56	1.24	1.10
	25	1.74	1.13	**3.34**	1.03	0.39	2.42	2.04
	40	1.78	1.27	**4.34**	1.38	0.43	3.28	2.74
	100	4.10	2.45	6.62	3.86	1.49	**7.96**	5.82
	150	6.28	3.69	9.00	7.30	2.67	**13.34**	10.36
	200	7.96	6.37	10.04	8.89	4.04	**17.74**	13.48
	250	12.34	8.69	13.02	12.93	6.22	**24.08**	18.68
	300	15.78	11.27	14.66	15.68	7.25	**30.56**	23.28
	350	18.90	13.38	16.22	18.26	8.68	**37.40**	29.22
	400	22.44	18.18	19.4	22.99	12.15	**42.16**	33.04
	500	28.28	26.05	23.02	32.11	13.85	**54.22**	44.40
$\chi^2(3)$	10	0.76	1.2	0.52	1.19	**1.27**	0.96	0.94
	25	0.96	1.72	0.5	1.39	**1.83**	0.94	0.80
	40	1.12	2.22	0.52	1.69	**2.34**	1.24	1.1
	100	1.68	2.94	0.56	**3.44**	2.78	2.22	2.06
	150	1.76	3.40	0.90	**4.06**	3.11	2.6	2.26
	200	2.04	4.16	0.92	**4.39**	3.37	3.4	3.00
	250	2.46	**5.05**	1.08	4.94	4.25	3.84	3.02
	300	2.90	**5.45**	**1.18**	4.85	4.06	5.18	4.38
	350	3.42	5.51	1.56	4.99	4.24	**6.36**	4.86
	400	3.86	6.16	1.62	**6.53**	4.92	6.28	5.16
	500	4.74	7.99	2.16	8.15	4.81	**8.18**	6.38
$\chi^2(4)$	10	1.16	1.27	0.50	1.26	**1.63**	1.00	1.02
	25	1.58	2.06	0.44	2.34	**2.46**	1.12	0.96
	40	1.82	3.22	0.56	2.55	**3.60**	1.82	1.66
	100	2.38	**5.68**	0.94	5.58	5.46	4.06	3.26
	150	3.40	7.51	1.76	**8.59**	6.4	6.32	5.08
	200	4.60	**10.20**	2.12	9.87	8.04	9.02	7.38
	250	6.06	**13.29**	2.56	12.74	9.97	11.62	8.66
	300	7.80	14.15	3.06	14.19	9.83	**15.00**	11.14
	350	9.20	15.89	3.44	16.42	10.96	**18.34**	13.94
	400	10.82	19.60	4.52	19.02	12.15	**21.22**	15.90
	500	13.40	23.88	5.7	26.34	13.31	**28.10**	20.90
$\chi^2(6)$	10	1.28	1.36	0.26	**1.74**	1.71	1.32	1.18
	25	1.8	3.48	0.18	2.88	**4.06**	2.34	1.90
	40	2.32	4.44	0.44	4.05	**5.48**	3.22	2.80
	100	4.76	11.3	1.88	**12.14**	10.38	8.70	6.32
	150	7.26	17.8	3.56	**19.77**	14.35	15.48	11.08
	200	9.94	23.6	4.60	**24.32**	17.49	23.28	17.92
	250	14.34	29.44	6.16	**30.76**	21.03	30.36	22.72
	300	18.36	36.8	7.02	37.22	24.86	**38.94**	28.64
	350	22	40.28	9.16	42.58	25.86	**48.06**	37.06
	400	27.04	46.72	11.38	49.18	30.93	**53.72**	41.66
	500	34.34	58.14	15.24	61.85	34.67	**67.04**	54.22
$\chi^2(10)$	10	1.46	1.50	0.10	1.82	**2.36**	1.22	1.34
	25	2.48	5.02	0.24	4.23	**5.48**	3.02	2.40
	40	3.3	7.82	0.90	6.27	**8.62**	5.36	4.42
	100	8.00	21.62	3.52	**21.91**	18.80	17.80	13.40
	150	13.68	33.96	7.46	**35.95**	26.20	32.08	23.60
	200	19.58	45.30	10.10	**45.59**	33.05	44.30	34.26
	250	28.64	55.76	12.70	**57.67**	41.84	57.50	44.80
	300	34.67	64.96	16.44	65.83	45.91	**68.72**	55.96
	350	42.01	71.18	19.18	73.52	50.84	**78.00**	65.52
	400	47.62	78.48	23.96	80.39	59.14	**83.82**	72.60
	500	60.99	88.06	88.06	90.38	64.83	**93.10**	85.04
$LogN(0,1)$	10	2.90	3.3	0.02	3.56	**4.92**	2.45	2.56
	25	6.74	16.32	2.00	13.62	**18.10**	11.40	9.40
	40	12.55	28.04	5.31	25.24	**31.51**	22.58	18.36
	100	36.97	**75.54**	21.73	36.19	70.02	71.11	58.46
	150	57.19	**92.92**	36.30	77.20	86.13	91.39	82.74
	200	47.25	**98.38**	47.25	93.57	94.46	97.61	94.12
	250	87.97	99.58	60.47	98.31	97.95	**99.65**	98.20
	300	93.87	**99.92**	70.90	99.70	99.27	99.90	99.48
	350	97.13	**99.98**	81.11	99.93	99.69	99.97	99.82
	400	98.82	**100**	89.89	99.95	99.91	**100**	99.94
	500	99.83	**100**	98.01	**100**	99.99	**100**	**100**
$HN(0,1)$	10	1.19	1.20	**2.35**	1.17	0.71	1.54	1.46
	25	1.63	1.54	**3.37**	1.24	0.41	2.04	1.86
	40	2.14	1.02	**4.42**	1.21	0.41	3.78	3.08
	100	4.25	2.18	7.13	3.85	1.33	**7.26**	5.82
	150	6.48	4.16	9.38	6.85	2.49	**12.22**	9.56
	200	9.63	5.40	10.96	8.64	3.95	**17.30**	14.04
	250	11.96	8.54	13.61	12.25	5.75	**24.70**	18.10
	300	15.13	10.14	15.20	15.38	7.24	**30.56**	23.54
	350	18.00	13.10	16.80	18.35	7.99	**38.00**	28.68
	400	21.05	16.5	19.70	23.08	11.07	**42.30**	33.20
	500	28.07	25.82	24.63	32.92	13.9	**53.32**	43.38

The bold number is the highest power among the seven tests for each sample size.

Table 4.27: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.01$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$U(0,1)$	10	3.51	3.49	**7.33**	2.97	1.13	6.96	5.82
	25	12.99	11.00	17.43	10.13	6.84	**27.14**	20.02
	40	25.34	31.71	27.1	25.79	38.61	**49.78**	38.28
	100	74.17	99.64	59.25	99.04	**99.84**	97.09	89.82
	150	93.02	**100**	78.52	**100**	**100**	99.94	99.16
	200	98.64	**100**	90.47	**100**	**100**	**100**	99.92
	250	99.81	**100**	98.47	**100**	**100**	**100**	**100**
	300	99.95	**100**	99.9	**100**	**100**	**100**	**100**
	350	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	1.32	1.2	**2.39**	1.12	0.5	1.78	1.52
	25	2.23	1.36	3.87	1.58	0.31	**4.12**	3.54
	40	3.45	1.76	5.14	2.17	1.00	**7.22**	5.54
	100	10.93	14.6	9.07	13.86	18.69	**26.55**	17.72
	150	18.49	43.26	12.36	38.25	**47.65**	45.59	32.66
	200	28.52	69.76	14.19	62.04	**76.13**	63.56	48.38
	250	40.87	88.88	19.19	83.34	**93.25**	78.85	60.88
	300	49.68	95.88	22.51	93.34	**98.23**	88.47	74.12
	350	59.02	99.04	26.54	97.95	**99.52**	94.22	83.88
	400	68.35	99.9	32.03	99.52	**99.98**	97.22	88.48
	500	81.51	**100**	44.83	99.99	**100**	99.45	96.14
$beta(2,5)$	10	1.01	**1.08**	0.87	1.04	0.69	**1.08**	0.92
	25	**1.1**	0.9	1.04	0.73	0.88	1.05	0.82
	40	0.96	0.66	0.99	0.57	0.79	**1.01**	0.76
	100	0.98	0.84	0.84	0.81	0.89	**1.49**	1.16
	150	1.16	0.84	0.82	1.15	1.10	**1.52**	1.20
	200	1.16	1.32	0.68	0.96	1.36	**1.90**	1.78
	250	1.36	1.14	0.65	1.08	2.15	**2.24**	1.64
	300	1.49	1.64	0.69	1.32	2.46	**2.66**	2.28
	350	1.58	1.98	0.59	1.48	2.97	**2.99**	2.56
	400	2.10	2.48	0.82	1.78	**4.3**	3.7	2.44
	500	2.09	3.88	0.61	2.81	**5.3**	4.06	3.32
$beta(5,1.5)$	10	1.38	1.70	**3.3**	1.27	0.59	2.55	2.20
	25	4.28	3.08	6.95	2.88	0.65	**8.47**	6.12
	40	7.10	4.96	9.44	5.19	4.36	**15.5**	11.54
	100	26.55	57.66	19.87	52.29	**64.5**	58.72	42.40
	150	45.10	93.72	28.63	90.32	**95.13**	84.08	69.08
	200	64.47	99.58	37.33	98.82	**99.71**	94.97	85.64
	250	78.84	**99.98**	47.54	99.97	99.97	98.94	93.64
	300	87.95	**100**	56.62	**100**	**100**	99.74	97.76
	350	93.76	**100**	67.73	**100**	**100**	99.98	99.32
	400	96.84	**100**	81.27	**100**	**100**	99.99	99.72
	500	99.31	**100**	97.56	**100**	**100**	**100**	99.98
$beta(0.5,0.5)$	10	13.64	13.98	19.89	11.61	4.76	**25.37**	20.1
	25	50.13	60.52	45.91	52.5	66.26	**76.81**	66.5
	40	78.99	96.18	68.12	93.65	**98.36**	96.16	91
	100	99.90	**100**	97.74	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	1.48	1.4	**3.52**	1.13	0.57	2.3	1.84
	25	2.88	2.08	**5.83**	2.10	0.37	5.15	4.18
	40	5.10	2.74	5.10	2.91	1.28	**9.06**	6.78
	100	13.96	15.28	16.91	16.98	15.15	**32.44**	22.24
	150	23.23	38.44	22.87	37.87	34.98	**52.11**	39.20
	200	35.34	57.14	28.59	55.29	58.59	**69.15**	55.56
	250	48.52	76.16	37.24	73.87	78.27	**83.05**	68.20
	300	58.68	86.9	42.55	85.06	88.49	**91.18**	80.54
	350	68.03	93.82	50.3	92.69	94.36	**95.9**	87.88
	400	75.62	97.48	57.15	96.87	**98.16**	97.8	92.52
	500	87.45	99.74	71.11	99.62	**99.8**	99.66	97.38
$beta(1,2)$	10	1.19	1.1	**2.86**	1.01	0.62	2.08	1.78
	25	2.79	2.50	**5.18**	2.13	0.34	5.03	4.16
	40	5.23	2.86	7.21	3.03	1.66	**9.48**	6.50
	100	14.30	22.08	13.60	21.33	26.04	**35.11**	23.64
	150	24.91	57.18	19.80	52.33	**59.79**	57.67	42.46
	200	38.91	81.10	24.16	75.40	**86.45**	75.71	59.84
	250	51.86	95.20	30.76	92.17	**97.24**	88.01	73.96
	300	63.21	98.86	36.29	97.69	**99.41**	94.51	85.00
	350	71.98	99.80	43.66	99.49	**99.91**	97.73	91.86
	400	81.16	99.98	51.65	99.90	**100**	99.20	94.68
	500	90.35	**100**	68.01	**100**	**100**	99.94	98.90

The bold number is the highest power among the seven tests for each sample size.

Table 4.28: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.01$ against the alternative distributions from Group III for sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$Laplace(0,1)$	5	3.3	3.29	0.77	3.24	**3.30**	2.14	3.29
	10	8.67	10.26	0.78	11.00	**12.62**	10.02	11.38
	15	14.26	18.56	4.50	19.86	17.56	19.85	**20.14**
	20	21.57	26.10	10.37	28.35	22.39	**29.70**	28.60
	25	28.43	33.23	15.58	34.77	24.76	**38.99**	37.47
	30	34.93	39.38	22.36	40.44	28.53	**49.21**	47.53
	40	48.25	48.29	33.71	49.61	32.38	**62.88**	61.73
	50	58.49	56.96	43.1	60.17	35.20	**73.25**	72.22
	100	90.92	84.10	78.8	88.51	49.53	**96.91**	96.49
	150	98.63	94.74	93.78	97.10	60.3	**99.79**	99.75
	200	99.76	98.54	97.87	99.46	70.24	**99.95**	99.94
$logistic(0,1)$	5	2.01	2.03	0.46	1.93	2.01	1.25	**2.17**
	10	4.39	5.33	0.20	5.70	**7.81**	4.75	5.62
	15	6.59	10.33	1.37	10.99	**12.19**	10.30	10.18
	20	9.17	15.44	4.14	16.8	**17.35**	15.57	13.94
	25	12.16	21.18	7.03	20.88	21.29	**21.76**	18.90
	30	15.49	26.9	10.61	25.28	25.05	**28.58**	25.21
	40	22.53	35.18	17.30	32.92	31.22	**38.86**	34.17
	50	28.23	42.84	23.37	43.21	35.60	**49.15**	44.18
	100	60.46	72.07	52.66	76.99	55.99	**84.70**	80.13
	150	80.01	87.83	74.01	91.87	69.89	**96.46**	94.00
	200	91.72	95.33	85.69	96.95	79.73	**99.13**	98.41
$Cauchy(0,1)$	5	17.24	16.96	10.64	17.01	17.24	16.57	**18.28**
	10	42.89	45.42	27.33	46.3	45.69	46.78	**47.80**
	15	61.26	64.75	49.76	65.86	59.13	67.90	**68.25**
	20	73.19	75.35	65.33	78.65	67.31	**80.26**	79.37
	25	82.83	83.45	75.86	85.65	72.90	**87.78**	87.41
	30	89.26	89.46	83.86	90.92	78.40	**93.38**	93.38
	40	96.02	95.46	92.97	95.39	85.99	**97.99**	97.84
	50	98.29	97.75	96.72	98.37	90.18	**99.22**	99.21
	100	99.99	99.92	99.96	99.99	98.88	**100**	**100**
	150	**100**	**100**	**100**	**100**	99.9	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	1.80	1.75	0.37	1.70	**1.80**	1.05	1.77
	10	2.90	3.19	0.02	3.56	**4.92**	2.45	3.23
	15	3.75	6.14	0.31	5.92	**9.32**	5.38	5.70
	20	5.50	10.57	1.07	10.31	**13.97**	8.16	7.62
	25	6.74	15.54	2	13.62	**18.10**	11.40	10.00
	30	7.54	19.24	2.8	17.22	**22.65**	15.31	13.26
	40	12.55	28.99	5.31	25.24	**31.51**	22.58	19.39
	50	14.85	38.86	7.6	36.19	**38.98**	31.38	26.19
	100	36.97	75.98	21.73	**77.20**	70.02	71.11	61.24
	150	57.19	91.99	36.30	**93.57**	86.13	91.39	83.21
	200	75.53	98.24	47.25	**98.31**	94.46	97.61	94.12
$t(1)$	5	18.14	17.93	10.93	17.54	18.14	16.83	**18.99**
	10	42.09	44.31	27.03	46.32	44.48	45.38	**46.57**
	15	61.04	64.28	51.06	64.8	59.15	**67.60**	67.57
	20	73.85	76.02	65.96	78.22	67.17	**80.32**	79.8
	25	82.73	84.05	76.13	85.54	72.59	**88.40**	87.96
	30	88.25	88.48	83.31	90.71	77.06	**93.07**	92.73
	40	96.24	95.58	93.35	95.36	86.26	**98.09**	98.03
	50	98.63	98.20	96.62	98.21	90.44	**99.50**	99.46
	100	99.99	99.99	99.97	**100**	99.02	99.99	99.99
	150	**100**	**100**	**100**	**100**	99.92	**100**	**100**
	200	**100**	**100**	**100**	**100**	99.99	**100**	**100**
$t(3)$	5	3.64	3.61	1.09	3.57	3.64	2.85	**3.85**
	10	9.91	11.70	2.29	12.71	**14.36**	12.04	12.84
	15	15.95	21.51	9.35	22.02	22.22	**22.83**	22.42
	20	22.82	30.03	17.21	30.48	28.08	**33.24**	31.10
	25	27.52	35.63	23.26	37.32	29.81	**39.69**	36.94
	30	34.68	43.34	31.15	44.33	34.67	**49.43**	46.70
	40	46.75	53.27	44.97	53.84	40.32	**63.02**	59.96
	50	55.93	61.21	53.35	63.98	44.10	**72.92**	69.94
	100	87.09	86.35	85.60	89.49	60.48	**95.68**	94.85
	150	96.97	95.42	95.68	97.65	70.45	**99.55**	99.26
	200	99.42	98.83	98.70	99.35	80.5	**99.95**	99.91
$t(4)$	5	2.94	2.92	0.73	2.71	2.94	1.92	**3.05**
	10	6.82	8.67	1.12	8.77	**11.21**	8.24	9.09
	15	11.45	16.80	5.62	16.03	**18.91**	17.56	17.18
	20	15.52	22.61	10.37	**24.71**	22.61	24.36	22.24
	25	20.17	28.75	16.03	29.54	25.34	**31.55**	28.73
	30	25.53	35.72	22.78	36.31	30.63	**40.48**	37.59
	40	35.38	45.05	33.25	45.24	35.14	**52.01**	48.51
	50	44.33	54.09	43.14	54.67	40.78	**64.34**	60.66
	100	76.85	78.87	74.97	84.42	55.17	**91.88**	89.40
	150	92.76	91.62	90.71	95.55	66.34	**98.77**	98.02
	200	98.07	97.54	96.45	98.50	76.28	**99.83**	99.74
$t(6)$	5	2.29	2.28	0.64	**2.48**	2.29	1.57	2.47
	10	5.26	6.18	0.45	6.40	**8.74**	5.90	6.73
	15	7.79	12.53	2.76	11.86	**15.09**	12.48	12.07
	20	10.84	18.06	6.03	18.62	**19.81**	18.09	16.49
	25	13.92	23.12	9.70	23.14	22.01	**23.60**	20.95
	30	17.95	28.89	14.5	28.42	26.33	**31.58**	28.32
	40	25.61	38.01	22.63	37.49	32.86	**42.32**	38.57
	50	31.88	45.37	29.92	47.37	36.86	**52.88**	48.10
	100	63.87	74.29	60.89	78.64	55.56	**86.67**	82.38
	150	83.74	88.22	79.95	92.84	67.62	**96.89**	95.04
	200	93.81	95.64	89.89	97.17	77.07	**99.41**	98.99

The bold number is the highest power among the seven tests for each sample size.

Table 4.29: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.1$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$\chi^2(1)$	10	10.92	10.6	**15.30**	10.32	9.04	12.09	11.34
	25	13.8	12.35	**19.94**	10.21	9.23	16.03	13.84
	40	15.1	13.84	**23.79**	12.05	10.15	18.57	15.16
	100	23.75	24.28	**33.20**	22.73	17.92	31.16	26.04
	150	27.85	31.51	**40.04**	30.95	24.09	39.57	34.36
	200	35.4	39.39	45.59	39.05	28.37	**49.03**	42.50
	250	40.11	46.46	50.73	47.55	34.2	**55.93**	49.82
	300	46.5	53.57	55.57	54.93	39.54	**63.62**	56.18
	350	53.88	61.68	61.06	61.17	44.62	**70.43**	63.18
	400	55.72	64.42	64.08	64.14	46.48	**76.24**	67.18
	500	66.44	74.18	71.43	75.21	54.89	**83.58**	76.76
$\chi^2(3)$	10	10.58	10.32	8.85	10.83	**11.51**	9.99	9.92
	25	10.47	11.78	7.85	11.87	**13.71**	10.62	11.18
	40	11.18	13.21	7.56	13.23	**14.99**	11.65	10.36
	100	12.48	16.63	9.34	17.1	**17.23**	13.89	13.30
	150	13.86	18.91	10.49	**19.41**	18.77	16.63	14.98
	200	15.66	21.47	12.46	**22.00**	20.35	19.27	17.10
	250	16.75	23.37	12.14	**24.36**	21.62	20.68	17.90
	300	18.53	25.38	13.83	**26.59**	22.91	23.30	20.94
	350	20.36	29.1	16.12	**30.17**	24.16	26.31	23.84
	400	22.04	29.64	17.22	**29.86**	25.82	28.43	25.36
	500	24.44	33.6	19.07	**33.90**	27.00	33.12	28.68
$\chi^2(4)$	10	10.76	10.96	7.91	11.56	**12.60**	10.53	10.98
	25	11.44	13.84	7.31	14.95	**16.29**	12.33	12.48
	40	12.63	17.16	8.17	17.29	**19.66**	14.00	13.74
	100	17.3	25.62	12.15	**25.97**	25.91	21.56	19.26
	150	20.52	31.09	16.16	**32.10**	30.99	27.97	23.98
	200	24.98	37.49	19.06	**38.21**	33.78	34.09	29.2
	250	28.32	43.11	22.04	**43.78**	37.99	40.07	33.84
	300	32.04	47.79	25.55	**48.83**	41.44	45.61	39.44
	350	35.74	52.84	29.34	**54.60**	43.62	50.19	44.12
	400	38.06	56.12	33.94	**58.66**	46.40	54.76	47.20
	500	44.92	63.88	38.59	57.04	51.57	**63.90**	55.98
$\chi^2(6)$	10	10.92	11.07	7.34	12.64	**14.03**	10.76	11.18
	25	13.08	17.40	7.59	18.44	**21.57**	14.82	14.80
	40	15.83	21.69	9.91	23.14	**26.03**	18.85	17.74
	100	25.11	39.63	18.85	**40.60**	40.33	33.92	29.44
	150	32.85	50.71	25.83	**51.95**	48.28	46.26	40.36
	200	40.50	60.79	32.49	**61.29**	56.20	57.04	48.70
	250	47.12	68.96	39.34	**70.06**	61.98	66.68	56.70
	300	54.06	74.99	47.21	**76.64**	67.53	73.63	66.12
	350	60.14	80.72	54.32	**82.19**	72.82	80.21	72.92
	400	64.10	84.12	59.76	**84.56**	76.06	84.55	76.64
	500	73.55	90.51	69.77	91.26	82.53	**91.39**	86
$\chi^2(10)$	10	11.84	12.04	6.67	13.90	**15.64**	11.37	12.46
	25	15.88	21.6	8.68	22.68	**27.42**	18.68	18.34
	40	20.98	29.64	12.66	31.54	**35.12**	25.17	23.72
	100	35.91	56.14	28.06	**56.99**	56.88	49.39	42.44
	150	46.87	70.49	39.84	**71.72**	68.02	65.58	59.12
	200	58.13	81.81	50.35	**82.69**	77.54	79.03	70.52
	250	67.58	88.57	60.44	**89.44**	84.10	86.93	79.76
	300	75.32	92.78	70.01	**93.16**	88.79	92.10	86.3
	350	82.10	**96.08**	78.44	95.95	91.90	95.66	91.56
	400	85.62	97.30	84.60	**97.68**	94.80	97.29	94.66
	500	92.06	**99.30**	91.97	99.17	97.39	99.13	98.02
$LogN(0,1)$	10	16.87	18.6	7.24	21.04	**24.48**	17.04	18.06
	25	30.19	43.63	19.26	45.41	**52.55**	39.52	37.18
	40	42.15	62.91	31.24	64.09	**70.16**	56.58	52.28
	100	77.75	95.65	68.58	95.7	**95.80**	92.62	87.30
	150	91.17	99.32	87.19	**99.40**	99.14	98.77	97.00
	200	97.07	**99.94**	96.26	**99.94**	99.91	99.86	99.22
	250	99.11	**100**	98.96	**100**	**100**	99.99	99.88
	300	99.79	99.98	99.79	99.98	**99.99**	**99.99**	99.96
	350	99.92	**100**	99.98	**100**	**100**	**100**	99.98
	400	99.98	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$HN(0,1)$	10	11.15	10.87	**16.00**	9.97	9.27	12.76	11.18
	25	13.21	11.91	**19.93**	9.95	8.31	15.18	14.34
	40	15.27	13.92	**23.00**	11.95	10.51	18.16	15.66
	100	22.89	23.57	**32.89**	21.74	17.94	29.48	24.76
	150	28.90	31.63	39.26	30.87	23.59	**39.54**	32.98
	200	35.68	40.03	45.70	38.76	28.38	**49.54**	41.32
	250	41.75	46.42	50.53	47.08	34.14	**57.02**	48.30
	300	47.25	54.05	55.95	54.18	39.11	**64.57**	56.76
	350	54.02	60.3	60.18	60.93	44.54	**70.84**	64.54
	400	57.48	65.42	64.86	65.4	47.62	**76.04**	68.86
	500	66.53	75.17	71.93	75.3	55.04	**84.06**	77.44

The bold number is the highest power among the seven tests for each sample size.

Table 4.30: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.1$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$U(0,1)$	10	22.21	21.87	**31.21**	17.54	15.83	29.92	25.36
	25	42.34	51.04	51.05	46.18	59.42	**60.06**	50.47
	40	59.44	82.4	66.09	81.21	**90.34**	81.33	70.69
	100	93.77	**100**	99.16	99.99	**100**	99.71	98.34
	150	99.19	**100**	99.99	**100**	**100**	**100**	99.95
	200	99.95	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	12.45	11.70	**17.05**	10.35	8.50	14.63	13.40
	25	16.97	15.34	**23.15**	12.11	12.81	21.51	18.28
	40	21.18	22.3	27.42	19.67	24.63	**29.56**	23.98
	100	39.60	65.99	42.99	61.59	**75.40**	60.37	48.24
	150	53.55	89.17	55.84	86.91	**94.49**	78.71	67.26
	200	66.78	97.72	71.27	96.94	**99.14**	89.87	79.06
	250	77.42	99.6	83.47	99.41	**99.91**	95.44	87.86
	300	84.09	99.98	93.06	99.95	**100**	98.37	93.9
	350	89.90	99.99	97.5	**100**	**100**	99.4	96.78
	400	93.48	**100**	99.18	**100**	**100**	99.86	98.48
	500	96.92	**100**	99.94	**100**	**100**	99.98	99.62
$beta(2,5)$	10	9.74	9.09	**10.21**	9.03	8.94	9.63	9.78
	25	9.93	9.66	**10.24**	8.11	8.78	10.19	9.78
	40	9.82	8.78	9.44	8.36	9.33	**10.53**	10.10
	100	10.87	10.64	8.35	9.29	**12.60**	11.59	10.58
	150	11.34	11.36	8.03	10.97	**14.82**	12.71	12.38
	200	12.32	13.86	8.26	12.41	**17.46**	14.53	13.08
	250	12.81	15.92	7.44	14.16	**20.11**	16.20	13.52
	300	13.42	17.41	7.81	16.50	**23.12**	16.84	16.90
	350	14.29	20.01	7.81	18.85	**26.55**	18.56	18.44
	400	15.91	22.93	8.06	20.65	**29.85**	20.88	18.62
	500	16.54	27.17	7.73	24.23	**34.86**	23.32	20.80
$beta(5,1.5)$	10	14.89	14.05	**20.32**	11.75	9.78	18.23	15.46
	25	24.09	23.41	31.3	19.19	24.11	**32.83**	27.02
	40	31	40.41	38.64	36.3	**48.68**	46.77	37.52
	100	63.14	94.87	69.87	93.26	**97.81**	86.18	75.34
	150	80.55	99.84	90.01	99.62	**99.95**	96.97	91.44
	200	90.76	**100**	98.48	99.99	**100**	99.64	97.26
	250	96.16	**100**	99.76	**100**	**100**	99.89	98.90
	300	98.46	**100**	**100**	**100**	**100**	99.99	99.86
	350	99.53	**100**	**100**	**100**	**100**	**100**	99.94
	400	99.74	**100**	**100**	**100**	**100**	**100**	99.96
	500	99.97	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	10	42.93	43.65	50.03	37.38	41.44	**57.12**	49.36
	25	79.68	93.77	82.41	93.28	**97.52**	93.61	89.78
	40	94.36	99.84	97.51	99.9	**99.99**	99.41	98.54
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	13.32	12.69	**19.14**	10.97	9.45	16.08	13.9
	25	18.8	18.47	**28.24**	14.32	14.13	24.83	21.28
	40	24.49	25.41	**35.53**	21.74	24.17	33.76	28.62
	100	46.66	64.15	56.28	60.55	**66.70**	66.04	54.78
	150	60.86	83.33	68.9	83.53	**87.04**	81.9	74.04
	200	72.75	94.87	79.87	93.79	**96.19**	92.24	85.7
	250	82.42	98.51	87.2	98.07	**99.15**	96.76	91.4
	300	88.88	99.71	93.63	99.56	**99.85**	98.8	95.64
	350	93.29	99.96	97.02	99.89	**99.96**	99.59	98.06
	400	95.87	99.99	98.65	99.97	**100**	99.84	99.06
	500	98.7	**100**	99.8	**100**	**100**	99.96	99.78
$beta(1,2)$	10	13.05	12.16	**17.78**	9.96	8.95	14.68	13.58
	25	18.8	18.21	**26.54**	14.71	15.16	25.24	21.12
	40	24.86	27.33	33.16	23.61	29.84	**35.39**	28.94
	100	47.09	74.33	53.26	71.2	**82.38**	69.02	56.32
	150	62.31	94.35	69.02	92.85	**97.30**	86.49	76.90
	200	75.94	99.13	83.52	98.73	**99.67**	94.94	87.24
	250	84.51	99.88	92.97	99.85	**99.95**	98.20	94.00
	300	91.23	99.99	97.42	99.99	**100**	99.48	96.82
	350	94.64	**100**	99.47	**100**	**100**	99.83	99.00
	400	97.05	**100**	99.79	**100**	**100**	**100**	99.50
	500	99.16	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 4.31: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.1$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$Laplace(0,1)$	5	15.46	14.96	10.4	17.0433	**17.30**	14.92	16.18
	10	28.36	30.57	16.71	34.01	**37.43**	30.08	31.70
	15	38.92	42.66	28.08	46.40	**47.10**	44.42	43.55
	20	48.34	52.54	39.17	**56.96**	55.15	55.20	54.65
	25	57.89	62.12	49.27	65.33	62.25	**65.50**	64.48
	30	66.13	69.17	58.11	71.36	68.05	**73.52**	72.66
	40	77.5	79	69.92	80.12	74.14	82.80	**83.19**
	50	84.93	84.99	79.1	86.43	79.82	**89.58**	89.22
	100	98.47	98.12	97.38	98.29	94.22	**99.32**	99.24
	150	99.84	99.8	99.68	99.72	98.24	**99.96**	99.95
	200	**100**	**100**	99.98	99.98	99.52	**100**	99.99
$logistic(0,1)$	5	12.23	12.02	7.65	13.3733	**13.43**	10.68	12.65
	10	19.9	22.41	10.13	25.18	**29.10**	21.76	22.74
	15	26.45	31.85	15.8	34.88	**38.70**	30.67	30.81
	20	32.98	41.97	23.71	44.91	**47.59**	39.96	38.62
	25	39.72	49.36	30.8	52.76	**54.81**	48.31	46.52
	30	45.99	56.95	38.21	58.99	**61.03**	56.17	54.25
	40	54.44	66.64	48.54	**69.04**	68.62	67.78	64.87
	50	63.5	75.11	58.03	**76.78**	75.37	75.94	72.84
	100	89.33	94.66	86.84	**95.12**	92.07	95.12	94.29
	150	97.04	98.8	95.82	99.2	97.02	**99.41**	99.1
	200	99.1	99.72	99.28	99.76	99.04	**99.91**	99.85
$Cauchy(0,1)$	5	35.48	34.97	31.29	37.7667	**38.11**	35.83	36.89
	10	63.78	64.83	57.97	67.58	**67.75**	66.31	65.99
	15	78.99	79.88	75.88	81.82	80.54	**82.41**	81.9
	20	88.12	88.87	86.03	90.17	87.9	**90.84**	90.52
	25	93.41	93.56	91.98	94.54	92.11	**95.24**	95.18
	30	96.51	96.9	95.98	96.88	95.23	**97.80**	97.59
	40	**99.78**	98.74	98.38	98.92	97.62	99.51	99.49
	50	99.61	99.53	99.60	99.64	99.04	**99.81**	99.75
	100	**100**	**100**	**100**	**100**	99.98	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	11.55	11.34	7.2	12.12	**12.59**	10.14	12.02
	10	16.87	18.6	7.24	20.74	**24.48**	17.04	18.7
	15	21.2	26.45	9.99	29.06	**34.93**	24.3	24.49
	20	26.26	35.66	14.55	37.40	**44.21**	31.67	30.47
	25	30.19	43.63	19.26	45.32	**52.55**	39.52	36.72
	30	33.57	50.39	22.74	52.39	**59.03**	44.78	41.63
	40	41.72	62.7	31.36	63.64	**69.96**	56.58	51.91
	50	49.18	73.25	37.68	74.10	**78.22**	66.58	60.78
	100	77.75	95.65	68.58	95.71	**95.80**	92.62	87.78
	150	91.02	99.44	87.78	**99.46**	99.2	98.77	97.02
	200	96.86	99.94	96.58	**99.94**	99.9	99.86	99.37
$t(1)$	5	35.81	35.21	31.82	37.55	**37.84**	35.42	36.47
	10	63.34	64.29	57.62	**67.75**	67.4	65.87	65.62
	15	79.26	80.19	76.14	81.96	80.65	**82.52**	82.06
	20	88.29	88.88	86.41	89.98	87.81	**90.69**	90.41
	25	93.54	93.41	92.04	94.33	92.07	**95.03**	94.91
	30	96.34	96.19	95.62	96.85	94.70	**97.49**	97.43
	40	98.8	98.58	98.42	98.90	97.68	**99.46**	99.41
	50	99.8	99.75	99.65	99.74	99.25	**99.91**	99.90
	100	**100**	**100**	**100**	99.9967	99.99	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	16.11	15.81	11.31	**17.68**	14.46	15.08	17.1
	10	28.2	30.35	19.18	34.05	**36.82**	30.77	31.38
	15	38.89	43.49	31.08	46.83	**48.55**	44.43	43.67
	20	48.19	54.33	43.34	57.32	**57.36**	55.52	54.19
	25	55.19	60.55	51.29	**63.73**	62	62.44	61.42
	30	62.9	67.94	59.63	**70.76**	67.45	70.4	69.44
	40	73.36	77.08	71.5	81.04	74.6	**81.95**	80.47
	50	82	85.27	80.29	86.20	81.32	**88.19**	87.03
	100	97.35	97.5	97.16	97.97	94.34	**98.84**	98.71
	150	99.68	99.66	99.6	99.82	98.26	**99.89**	99.87
	200	99.92	99.98	99.94	**100**	99.58	**100**	**100**
$t(4)$	5	14.54	14.13	9.45	15.72	**15.77**	13.61	15.31
	10	23.79	26.26	14.41	29.66	**32.35**	25.72	26.74
	15	33.28	39.12	24.32	41.63	**45.06**	39.05	38.53
	20	40.30	48.12	33.76	51.46	**51.93**	47.75	46.92
	25	48.06	55.71	42.78	58.80	**59.18**	56.17	54.78
	30	55.24	63.63	51.90	**65.64**	65.15	65.03	63.17
	40	65.70	72.96	63.20	74.72	71.28	**75.82**	73.63
	50	74.90	80.60	72.73	82.06	77.98	**84.18**	82.30
	100	94.46	96.09	93.97	96.37	92.26	**97.59**	97.03
	150	99.12	99.18	99.04	99.30	97.74	**99.79**	99.73
	200	99.74	99.78	99.80	99.80	99.20	**99.98**	99.98
$t(6)$	5	12.79	12.43	8.19	14.15	**14.24**	11.81	13.80
	10	20.39	22.65	11.42	26.17	**28.87**	22.23	23.07
	15	28.96	34.87	18.29	37.09	**41.04**	33.74	33.63
	20	34.83	43.50	26.25	46.22	**48.87**	41.97	41.12
	25	40.88	50.61	33.97	53.68	**55.15**	50.07	48.30
	30	47.78	57.93	41.73	60.54	**61.11**	57.94	55.91
	40	57.82	68.82	52.84	**70.66**	69.44	69.71	66.85
	50	66.42	76.72	62.80	77.94	75.87	**78.34**	75.50
	100	90.77	94.55	89.09	95.00	91.51	**96.25**	95.00
	150	92.22	98.90	97.36	98.96	96.90	**99.46**	99.16
	200	99.44	99.70	99.32	99.86	98.80	**99.91**	99.88

The bold number is the highest power among the seven tests for each sample size.

Table 4.32: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.1$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$\chi^2(1)$	10	10.9	9.99	**14.22**	10.10	8.76	11.74	10.94
	25	13.65	10.98	**19.57**	9.48	8.69	15.24	14.16
	40	15.23	12.34	**23.72**	10.69	9.52	17.66	15.96
	100	24.06	22.49	**33.55**	21.27	17.05	29.74	26.77
	150	28.3	29.57	41.13	29.09	22.98	**41.44**	33.90
	200	35.49	37.56	47.01	37.54	27.08	**49.04**	42.43
	250	40.61	45.04	51.84	46.13	33.03	**56.80**	48.33
	300	46.95	52.20	57.00	53.47	37.83	**63.40**	56.07
	350	53.08	58.63	61.70	59.6	42.42	**70.28**	62.70
	400	57.88	64.56	65.52	62.99	46.75	**75.36**	68.30
	500	66.63	73.06	72.60	74.02	53.14	**83.88**	76.85
$\chi^2(3)$	10	10.59	10.78	9.29	11.25	**11.89**	9.68	10.52
	25	10.35	12.41	8.36	12.13	**13.77**	10.62	10.60
	40	11.01	13.84	8.16	13.68	**15.08**	11.32	11.45
	100	12.64	16.99	9.61	**17.50**	17.42	13.56	13.28
	150	14.01	19.26	10.81	**19.70**	18.67	16.90	15.59
	200	15.49	21.96	12.94	**22.43**	20.32	19.64	18.17
	250	16.79	23.81	12.55	**25.10**	20.42	20.70	18.83
	300	18.65	25.77	14.29	**27.15**	21.52	23.64	21.15
	350	19.83	28.78	16.03	**30.35**	22.79	26.16	23.04
	400	20.89	30.18	17.40	**30.46**	24.15	28.36	24.96
	500	24.48	33.82	19.53	**34.05**	25.27	32.94	28.93
$\chi^2(4)$	10	10.9	11.58	8.63	11.86	**12.98**	10.4	11.18
	25	11.42	14.56	8.31	15.59	**16.58**	11.82	12.32
	40	12.53	18.11	8.80	17.78	**19.77**	14.22	13.53
	100	17.33	26.41	12.63	**26.76**	25.88	21.26	20
	150	20.56	31.48	16.95	**32.45**	30.59	28.3	24.64
	200	24.89	37.78	20.12	**38.45**	33.37	34.36	29.75
	250	28.36	43.52	22.95	**44.34**	37.66	38.2	34.71
	300	32.06	47.97	26.85	**49.14**	40.62	45.22	39.75
	350	34.71	52.35	30.26	**54.63**	44.26	50.26	43.32
	400	37.98	55.88	33.53	**57.26**	45.50	56.26	47.32
	500	44.73	63.78	39.92	64.18	50.66	**64.34**	56.36
$\chi^2(6)$	10	11.28	12.36	8.44	12.96	**15.02**	11.04	11.86
	25	12.99	18.47	8.74	18.78	**21.77**	15.34	14.84
	40	15.78	22.8	11.04	23.83	**25.82**	18.46	17.76
	100	25.08	40.19	19.84	**41.09**	39.76	32.82	30.15
	150	32.69	50.99	27.61	**52.18**	47.4	47.00	40.06
	200	40.00	60.91	34.7	**61.33**	55.11	56.76	49.79
	250	46.71	68.97	41.24	**70.23**	60.87	64.88	58.49
	300	53.76	75.07	49.10	**76.67**	66.36	74.02	65.68
	350	59.36	80.24	55.20	**82.09**	71.36	79.94	72.17
	400	63.88	84.58	61.75	84.59	75.93	**85.18**	77.20
	500	73.07	90.31	71.53	91.05	81.47	**91.52**	85.61
$\chi^2(10)$	10	11.94	13.64	8.29	14.49	**16.63**	11.70	12.77
	25	15.94	23.09	10.53	23.42	**27.30**	19.14	18.00
	40	20.80	31.20	14.30	32.26	**34.56**	25.68	23.54
	100	35.61	56.82	29.93	**57.47**	55.91	49.46	43.68
	150	46.37	70.27	42.34	**71.31**	66.89	66.94	58.24
	200	57.15	81.79	53.02	**82.61**	76.41	79.68	71.52
	250	66.93	88.36	63.01	**89.34**	82.93	86.5	79.91
	300	74.48	92.61	72.44	**93.15**	87.84	92.54	86.55
	350	80.40	95.65	79.74	**95.96**	91.59	95.6	91.39
	400	84.71	97.37	85.76	**97.64**	93.96	97.46	93.72
	500	91.57	**99.25**	92.9	99.16	96.94	99.08	97.53
$LogN(0,1)$	10	17.07	21.00	10.73	22.37	**25.97**	17.70	19.29
	25	29.86	45.30	22.61	45.99	**52.08**	40.12	36.96
	40	40.99	64.47	34.38	64.65	**69.46**	57.62	51.87
	100	76.59	95.63	71.86	**95.66**	95.27	92.50	87.53
	150	90.27	99.33	89.41	**99.38**	99.05	98.82	96.91
	200	96.65	99.94	97.08	**99.95**	99.85	99.84	99.34
	250	98.94	**100**	99.28	**100**	99.99	99.98	99.88
	300	99.72	99.98	99.87	99.98	99.99	**100**	99.97
	350	99.92	**100**	99.97	**100**	**100**	**100**	99.99
	400	99.98	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$HN(0,1)$	10	11.24	10.29	**14.77**	9.77	8.74	12.74	11.39
	25	12.97	10.42	**19.56**	8.8	7.71	15.40	13.51
	40	15.15	12.55	**23.18**	11.09	9.89	17.46	16.09
	100	23.07	22.01	**33.22**	20.42	17.02	29.44	25.11
	150	29.38	30.02	**40.61**	28.94	22.58	39	34.04
	200	35.93	38.04	46.93	37.07	27.16	**48.36**	43.06
	250	42.03	45.06	51.56	45.65	32.78	**55.98**	49.37
	300	47.70	52.50	57.19	52.84	37.64	**64.26**	57.13
	350	53.82	58.41	61.61	59.37	41.79	**71.22**	63.47
	400	57.74	64.21	65.71	64.22	46.00	**76.48**	68.21
	500	66.89	74.05	73.13	74.46	53.21	**83.94**	77.38

The bold number is the highest power among the seven tests for each sample size.

Table 4.33: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.1$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$U(0,1)$	10	22.43	19.91	**31.47**	17.54	12.86	29.13	24.42
	25	43.87	47.10	57.29	46.18	55.62	**60.52**	50.91
	40	62.16	80.66	73.28	81.21	**89.23**	82.51	72.33
	100	95.71	**100**	99.28	99.99	**100**	99.82	98.81
	150	99.56	**100**	**100**	**100**	**100**	**100**	99.97
	200	99.97	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	12.41	10.48	**16.20**	9.50	7.41	14	12.62
	25	16.96	12.94	**23.80**	10.35	11.25	21.84	17.81
	40	21.31	19.80	**29.29**	16.5	22.41	28.32	24.48
	100	41.49	63.64	47.66	58.56	**73.42**	62.5	51.44
	150	56.02	87.87	60.62	85.09	**93.69**	81.36	68.86
	200	69.69	97.30	74.01	96.28	**98.99**	91.36	81.86
	250	80.03	99.58	84.15	99.27	**99.91**	96.38	89.98
	300	86.87	99.96	93.21	99.93	**99.99**	98.58	95.21
	350	91.93	99.99	97.39	**100**	**100**	99.56	97.46
	400	94.93	**100**	99.13	**100**	**100**	99.84	98.86
	500	98.06	**100**	99.93	**100**	**100**	99.98	99.77
$beta(2,5)$	10	9.73	9.37	**10.00**	9.04	8.74	9.94	9.47
	25	9.8	9.27	**10.28**	7.82	8.75	9.36	9.69
	40	9.48	8.68	9.4	8.33	9.32	9.64	**9.65**
	100	10.78	10.30	8.5	8.94	**12.41**	10.90	10.75
	150	11.09	10.82	8.37	10.42	**14.30**	14.04	11.69
	200	12.10	13.19	8.64	11.93	**17.15**	14.36	13.29
	250	12.79	15.36	7.69	13.73	**19.73**	14.44	14.62
	300	13.34	16.95	8.13	15.89	**22.52**	16.88	15.01
	350	14.28	19.36	8.22	18.24	**26.18**	18.58	16.93
	400	15.85	22.35	8.35	20.02	**29.13**	19.14	18.82
	500	16.70	26.54	7.95	23.24	**34.18**	23.46	20.64
$beta(5,1.5)$	10	14.58	12.58	**19.69**	10.29	8	16.92	15.24
	25	24.27	20.33	**33.51**	15.88	20.93	31.84	27.11
	40	31.78	36.92	43.20	32.21	45.72	**48.10**	37.81
	100	66.36	94.37	73.84	92.14	**97.44**	87.38	77.18
	150	83.52	99.80	90.99	99.59	**99.94**	97.88	92.84
	200	92.99	**100**	98.57	99.99	**100**	99.72	98.11
	250	97.21	**100**	99.73	**100**	100	99.94	99.55
	300	99.22	**100**	**100**	**100**	100	**100**	99.88
	350	99.80	**100**	**100**	**100**	100	**100**	99.98
	400	99.84	**100**	**100**	**100**	100	**100**	99.99
	500	99.99	**100**	**100**	**100**	100	**100**	**100**
$beta(0.5,0.5)$	10	43.67	41.64	55.28	34.34	36.66	**57.86**	50.05
	25	82.24	92.82	89.47	91.06	**96.95**	94.72	89.01
	40	96.12	99.82	98.26	99.82	**99.99**	99.64	98.72
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	13.32	11.77	**17.67**	10.4	8.26	15.00	13.58
	25	19.02	15.73	28.62	12.22	12.07	**29.36**	20.63
	40	24.71	22.71	**37.13**	18.89	22.08	33.10	27.67
	100	48.05	62.05	59.93	57.86	64.19	**65.22**	56.92
	150	62.36	81.86	72.39	81.31	**85.63**	84.12	73.48
	200	74.43	94.14	82.69	92.86	**95.58**	93.02	85.82
	250	84.19	98.37	89.00	97.83	**98.92**	96.48	92.72
	300	90.21	99.65	94.61	99.46	**99.78**	99.06	96.77
	350	94.24	99.92	97.37	99.87	**99.96**	99.66	98.33
	400	96.48	99.99	98.80	99.97	**100**	99.92	99.31
	500	98.99	**100**	99.79	**100**	**100**	**100**	99.87
$beta(1,2)$	10	12.73	11.1	**16.78**	9.33	7.70	16	12.52
	25	18.59	15.45	**27.67**	12.20	13.16	25.14	20.72
	40	25.21	24.75	**35.82**	20.40	27.22	33.22	28.88
	100	49.18	72.07	58.23	68.69	**80.84**	69.62	59.73
	150	65.04	93.61	72.38	91.42	**96.86**	87.52	77.72
	200	78.30	98.99	85.21	98.39	**99.62**	95.30	89.46
	250	86.86	99.87	93.22	99.80	**99.95**	98.44	94.84
	300	93.01	99.98	97.43	99.99	**100**	99.56	97.78
	350	95.94	**100**	99.39	**100**	**100**	99.92	99.08
	400	97.94	**100**	99.82	**100**	**100**	99.98	99.72
	500	99.55	**100**	**100**	**100**	**100**	**100**	99.99

The bold number is the highest power among the seven tests for each sample size.

Table 4.34: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.1$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$Laplace(0,1)$	5	16.72	17.10	14.37	18.70	**19.31**	16.03	17.31
	10	30.33	33.95	22.24	35.64	**37.13**	32.31	33.98
	15	40.19	43.85	33.86	**46.61**	45.13	44.96	45.67
	20	48.98	53.46	44.81	**56.58**	52.91	55.99	56.17
	25	58.06	62.07	54.12	64.85	58.82	**66.07**	65.58
	30	66.27	68.71	61.50	70.33	64.35	**74.02**	73.59
	40	77.11	77.81	74.11	79.09	71.14	**84.24**	83.67
	50	84.23	83.95	81.87	85.40	75.33	**89.87**	89.36
	100	98.27	97.71	97.58	97.96	91.19	**99.31**	99.31
	150	99.85	99.67	99.78	99.71	96.88	**99.95**	99.95
	200	99.95	99.96	99.96	**99.99**	98.82	99.98	**99.99**
$logistic(0,1)$	5	12.89	13.15	11.53	14.58	**15.54**	12.12	13.60
	10	20.55	25.57	13.98	26.92	**30.48**	23.16	24.31
	15	26.69	33.71	20.57	35.58	**38.57**	32.01	31.85
	20	32.59	43.67	28.44	45.1267	**46.64**	41.01	39.45
	25	38.95	50.55	35.3	52.93	**53.15**	49.47	47.10
	30	45.39	57.63	41.4	58.75	**59.38**	57.60	54.54
	40	53.99	67.84	53.36	68.52	67.28	**68.64**	64.98
	50	62.15	74.58	61.45	75.77	72.77	**76.45**	72.51
	100	88.33	94.31	87.99	94.64	90.05	**95.90**	94.13
	150	96.67	98.72	96.83	98.80	96.39	**99.44**	99.07
	200	99.2	99.68	99.14	99.72	98.51	**99.92**	99.84
$Cauchy(0,1)$	5	37.27	37.87	34.64	39.4567	**39.49**	37.5	38.37
	10	65.53	67.04	61.80	68.45	67.23	67.95	**68.79**
	15	80.27	80.71	78.39	82.03	79.23	83.28	**83.28**
	20	88.88	89.10	88.23	90.17	86.6	**91.47**	91.23
	25	93.81	93.69	93.69	94.45	90.76	**95.65**	95.67
	30	96.84	96.85	96.45	96.88	94.34	97.96	**98.00**
	40	99.12	98.93	99.08	98.86	97.66	99.57	**99.60**
	50	99.62	99.54	99.61	99.62	98.70	99.81	**99.82**
	100	**100**	**100**	**100**	**100**	99.94	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	12.05	12.36	10.65	12.8733	**13.60**	11.05	12.14
	10	17.07	21.00	10.73	21.99	**25.97**	18.12	19.29
	15	20.95	28.89	13.64	30.13	**35.48**	25.25	25.26
	20	25.85	38.08	18.13	38.26	**44.55**	32.67	30.79
	25	29.86	45.30	22.61	46.33	**52.08**	40.45	36.96
	30	32.94	52.35	25.85	53.13	**58.58**	45.68	41.55
	40	40.99	64.47	34.38	64.65	**69.46**	57.46	51.87
	50	47.83	73.87	41.32	74.06	**77.08**	67.09	60.65
	100	76.59	**95.63**	71.86	95.59	95.27	92.64	87.53
	150	90.27	99.33	89.41	**99.38**	99.05	98.79	96.91
	200	96.65	99.94	97.08	**99.95**	99.85	99.86	99.34
$t(1)$	5	37.34	37.81	35.17	39.24	**39.53**	36.94	38.27
	10	65.45	66.61	62.28	**68.71**	66.66	67.75	68.30
	15	80.55	80.93	79.41	82.25	79.18	83.50	**83.61**
	20	88.94	89.28	88.42	89.97	86.38	**91.34**	91.28
	25	93.80	93.50	93.41	94.35	91.08	**95.44**	95.32
	30	96.62	96.20	96.98	96.83	93.94	97.7	**97.75**
	40	99.09	98.83	98.96	98.78	97.55	99.47	**99.47**
	50	99.82	99.74	99.58	99.72	98.97	**99.94**	99.92
	100	**100**	**100**	**100**	99.9967	99.99	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	16.96	17.06	15.27	19.13	**19.39**	16.53	17.7
	10	29.66	33.3	24.1	35.55	**37.22**	32.56	33.43
	15	39.58	45.14	36.35	**47.46**	47.44	45.81	45.49
	20	48.61	55.21	46.71	**57.28**	55.39	57.01	55.69
	25	55.19	60.97	55.42	63.62	59.61	**63.67**	62.22
	30	62.78	67.91	62.8	70.26	64.96	**71.43**	70.11
	40	74.00	78.37	75.39	79.43	73.34	**82.67**	80.83
	50	81.24	84.4	82.58	85.31	77.7	**88.73**	87.12
	100	97.04	97.23	97.62	97.66	92.02	**98.93**	98.65
	150	99.69	99.61	99.67	99.63	97.32	**99.89**	99.87
	200	99.97	99.93	99.95	99.97	99.23	**100**	**100**
$t(4)$	5	15.2	15.44	13.45	17.1567	**17.70**	14.78	15.92
	10	25.02	29.10	18.95	31.33	**33.28**	27.49	28.63
	15	33.96	40.77	28.65	42.32	**44.36**	40.52	40.24
	20	40.80	49.00	38.36	**51.46**	50.71	49.41	48.39
	25	47.75	56.37	46.13	**58.73**	56.78	57.61	55.54
	30	54.96	63.90	54.47	65.27	62.44	**66.44**	63.78
	40	65.37	73.10	66.82	74.40	69.58	**76.65**	73.78
	50	73.73	79.95	75.28	81.06	74.56	**84.73**	82.34
	100	93.81	95.53	94.82	96.02	89.37	**97.60**	97.04
	150	98.89	99.13	99.03	99.22	96	**99.79**	99.71
	200	99.92	99.87	99.82	99.83	98.64	**99.98**	**99.98**
$t(6)$	5	13.54	13.94	11.73	15.54	**16.17**	13.26	14.42
	10	21.19	25.55	15.88	27.57	**30.05**	23.61	24.57
	15	29.35	36.32	23.24	37.82	**40.63**	34.89	35.07
	20	34.89	45.18	30.50	46.45	**47.96**	43.43	42.01
	25	40.63	51.36	37.50	53.71	**53.78**	51.48	48.83
	30	47.21	58.09	45.03	**60.22**	59.03	58.90	56.16
	40	57.06	68.76	57.31	**70.76**	67.26	70.30	67.09
	50	64.83	75.91	66.38	76.98	72.74	**78.82**	75.29
	100	89.71	94.01	90.17	94.48	89.28	**96.25**	94.85
	150	97.15	98.56	97.64	98.80	95.61	**99.47**	99.08
	200	99.44	99.76	99.50	99.72	98.34	**99.92**	99.84

The bold number is the highest power among the seven tests for each sample size.

Table 4.35: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.1$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$\chi^2(1)$	10	9.54	10.10	**15.01**	9.44	8.36	11.68	10.09
	25	11.16	11.42	**19.96**	9.48	8.94	15.46	13.30
	40	12.32	12.68	**24.09**	11.29	10.08	17.97	14.88
	100	21.56	24.34	**33.90**	22.25	17.19	30.38	25.41
	150	27.96	31.24	**41.34**	30.42	23.12	39.08	32.79
	200	34.92	39.86	47.05	38.51	27.09	**48.68**	41.6
	250	41.36	46.54	52.04	46.95	33.11	**55.49**	47.45
	300	45.83	53.46	57.18	54.42	37.93	**63.29**	55.15
	350	52.09	61.02	61.92	60.43	42.50	**70.26**	61.92
	400	56.93	64.12	65.55	63.75	46.80	**76.02**	67.57
	500	65.78	74.64	72.66	74.54	53.11	**83.50**	76.30
$\chi^2(3)$	10	10.82	10.78	9.01	11.15	**11.72**	10.15	10.58
	25	11.32	11.36	8.12	11.69	**13.75**	10.88	11.32
	40	10.94	12.74	7.85	13.36	**15.13**	12.10	11.68
	100	13.36	16.12	9.46	16.93	**17.20**	13.91	13.87
	150	14.24	18.48	10.49	**19.46**	18.59	16.78	16.05
	200	15.76	21.18	12.72	**22.11**	20.29	19.65	18.64
	250	17.20	23.48	12.39	**24.65**	21.51	20.96	19.24
	300	19.31	25.72	14.06	**26.58**	22.80	23.67	21.68
	350	20.47	29.14	15.84	**30.13**	24.17	26.66	23.40
	400	21.80	29.72	17.13	**29.99**	25.28	28.67	25.60
	500	24.98	33.44	19.33	**33.70**	26.76	33.39	29.38
$\chi^2(4)$	10	11.54	11.66	8.60	11.85	**12.84**	10.91	11.84
	25	12.66	13.78	7.80	14.72	**16.43**	12.76	13.23
	40	13.46	16.38	8.44	17.18	**19.76**	14.19	14.25
	100	18.06	25.68	12.36	**25.97**	25.73	21.82	20.61
	150	21.66	31.04	16.62	**31.83**	30.53	28.39	25.51
	200	24.44	37.46	19.66	**37.76**	33.29	34.5	30.67
	250	28.30	42.20	22.58	**43.80**	37.68	40.27	35.83
	300	33.24	46.74	26.23	**48.55**	40.64	46.00	40.48
	350	35.74	52.42	29.77	**54.27**	44.30	50.55	44.03
	400	39.45	55.84	32.83	**56.84**	45.50	55.13	48.05
	500	45.55	63.36	39.26	63.76	50.65	**64.11**	57.16
$\chi^2(6)$	10	11.98	12.04	7.98	13.09	**14.13**	11.42	12.93
	25	14.70	16.5	8.31	17.90	**21.67**	15.37	15.75
	40	16.70	22.14	10.59	22.86	**25.52**	19.22	18.6
	100	25.04	38.96	19.42	**40.13**	39.54	34.33	31.13
	150	33.30	50.14	26.81	**51.37**	47.35	46.54	41.24
	200	39.46	59.78	33.74	**60.90**	55.10	57.89	51.07
	250	47.00	68.5	40.36	**69.73**	60.88	66.78	59.58
	300	54.80	74.58	48.12	**76.17**	66.4	73.91	66.49
	350	60.66	80.46	54.04	**81.76**	71.39	80.53	72.78
	400	65.19	83.90	60.75	84.31	75.97	**84.64**	77.78
	500	73.64	90.50	70.76	90.80	81.48	**91.45**	86.07
$\chi^2(10)$	10	12.94	13.10	7.40	**14.78**	16.02	12.31	14.23
	25	17.82	20.88	9.87	22.09	**27.16**	19.49	19.45
	40	20.98	29.68	13.6	31.01	**34.33**	25.58	24.8
	100	36	54.98	29.21	**56.20**	55.73	49.96	44.79
	150	48.36	70.88	41.11	**71.03**	66.89	66.01	59.39
	200	58.06	81.40	51.80	**82.15**	76.38	79.48	72.6
	250	67.56	87.98	61.85	**89.04**	82.94	87.03	80.76
	300	75.33	92.88	70.99	**92.92**	87.86	92.26	87.11
	350	81.18	**95.88**	78.49	95.85	91.59	95.72	91.73
	400	85.39	97.14	84.85	**97.57**	93.99	97.31	94.02
	500	91.86	99.08	92.35	99.13	96.94	**99.14**	97.65
$LogN(0,1)$	10	19.91	19.86	9.23	22.16	**25.07**	18.06	21.72
	25	32.49	42.38	21.48	44.26	**51.47**	40.8	38.67
	40	43.38	61.82	33.3	62.85	**69.23**	57.68	53.52
	100	77.47	95.18	69.99	72.65	**95.23**	92.54	88.2
	150	90.46	**99.36**	87.94	95.39	99.03	98.84	97.15
	200	96.81	**99.94**	96.53	99.36	99.85	99.84	99.38
	250	99.02	**100**	99.07	99.94	99.99	99.98	99.88
	300	99.75	99.98	99.84	**100**	99.99	**100**	99.97
	350	99.92	**100**	99.97	99.98	**100**	**100**	99.99
	400	99.98	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$HN(0,1)$	10	9.61	10.48	**15.54**	9.40	8.81	12.20	10.20
	25	11.35	11.50	**19.95**	9.18	7.94	15.34	12.23
	40	13.70	13.60	**23.44**	11.48	10.32	17.30	14.76
	100	21.39	22.24	**33.57**	21.39	17.12	28.98	24.16
	150	27.81	32.04	**40.76**	30.26	22.66	38.46	33.00
	200	34.97	37.94	46.98	38.21	27.23	**47.64**	42.12
	250	40.78	45.94	51.76	46.64	32.84	**55.94**	48.54
	300	46.36	52.72	57.22	53.57	37.68	**63.98**	56.20
	350	52.83	59.84	61.70	60.19	41.87	**71.68**	62.49
	400	56.87	64.96	65.88	65.08	46.07	**76.80**	67.49
	500	65.97	73.24	73.19	74.97	53.26	**84.08**	77.04

The bold number is the highest power among the seven tests for each sample size.

Table 4.36: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.1$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$U(0,1)$	10	17.98	19.81	**32.62**	14.70	13.61	28.74	22.03
	25	40.53	48.60	57.27	40.95	57.28	**60.79**	49.06
	40	59.85	80.37	72.67	77.8	**89.59**	82.91	71.02
	100	95.43	**100**	98.86	90.67	**100**	99.85	98.76
	150	99.53	**100**	99.99	99.99	**100**	**100**	99.96
	200	99.96	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	10.04	11.52	**16.92**	9.26	7.94	13.92	11.81
	25	14.89	14.74	**24.37**	11.14	11.55	21.37	17.13
	40	19.85	21.4	29.71	17.79	22.97	**29.93**	23.83
	100	40.23	63.36	47.46	58.66	**73.50**	62.58	51.44
	150	54.82	88	59.75	85.3	**93.70**	81	68.58
	200	68.86	97.48	72.59	96.23	**98.99**	91.68	81.87
	250	79.60	99.54	83.11	99.27	**99.91**	96.28	90.1
	300	86.59	99.94	92.39	99.93	**99.99**	98.81	95.21
	350	91.75	**100**	96.93	**100**	**100**	99.63	97.41
	400	94.84	**100**	98.97	**100**	**100**	99.87	98.85
	500	97.99	**100**	99.92	**100**	**100**	99.99	99.77
$beta(2,5)$	10	9.58	9.98	**10.37**	9.16	9.02	9.83	9.6
	25	9.93	8.70	**10.33**	7.82	8.65	10.17	9.84
	40	9.62	8.96	9.51	8.26	9.29	**10.68**	9.98
	100	10.75	9.20	8.57	9.20	**12.45**	11.67	11.16
	150	11.00	11.26	8.39	10.68	**14.29**	12.91	12.06
	200	12.36	13.26	8.61	12.06	**17.14**	14.91	13.87
	250	12.90	14.24	7.76	13.92	**19.76**	16.34	15.16
	300	13.07	17.68	8.16	15.97	**22.52**	17.34	15.48
	350	14.59	19.58	8.24	18.47	**26.17**	19.08	17.45
	400	16.04	22.90	8.43	19.85	**29.14**	21.52	19.23
	500	16.90	25.88	7.91	23.18	**34.10**	23.92	21.18
$beta(5,1.5)$	10	11.41	12.90	**20.72**	9.70	8.59	17.24	13.44
	25	21.54	21.68	**34**	17.04	22.01	32.89	25.80
	40	29.52	37.26	43.30	33.17	46.48	**47.83**	36.85
	100	65.20	93.76	73.03	91.96	**97.45**	88.04	76.92
	150	82.79	99.78	89.36	99.58	**99.94**	97.75	92.69
	200	92.76	99.98	98.14	99.99	**100**	99.78	98.14
	250	97.14	**100**	99.66	**100**	**100**	99.92	99.54
	300	99.19	**100**	**100**	**100**	**100**	99.99	99.88
	350	99.79	**100**	**100**	**100**	**100**	**100**	99.98
	400	99.83	**100**	**100**	**100**	**100**	**100**	99.99
	500	99.99	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	10	37.58	41.22	55.23	33.76	37.64	**57.81**	47.01
	25	80.07	93.36	88.89	90.84	**97.18**	94.61	88.14
	40	95.54	99.88	97.74	99.83	**99.99**	99.6	98.6
	100	100	**100**	**100**	**100**	**100**	**100**	**100**
	150	100	**100**	**100**	**100**	**100**	**100**	**100**
	200	100	**100**	**100**	**100**	**100**	**100**	**100**
	250	100	**100**	**100**	**100**	**100**	**100**	**100**
	300	100	**100**	**100**	**100**	**100**	**100**	**100**
	350	100	**100**	**100**	**100**	**100**	**100**	**100**
	400	100	**100**	**100**	**100**	**100**	**100**	**100**
	500	100	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	11.12	11.72	**18.96**	9.49	8.54	15.01	12.24
	25	16.46	17.20	**29.39**	13.06	12.92	24.01	18.69
	40	22.4	25.18	**37.29**	20.27	22.78	33.49	26.21
	100	45.98	62.16	60.14	58.63	64.28	**66.65**	55.76
	150	60.74	84.00	72.34	81.92	**85.69**	82.93	72.79
	200	73.82	94.44	82.32	92.86	**95.60**	92.88	85.52
	250	83.63	98.28	88.72	97.88	**98.93**	97.14	92.57
	300	89.89	99.58	94.26	99.49	**99.78**	98.96	96.67
	350	94.00	99.92	97.14	99.88	**99.96**	99.71	98.31
	400	96.35	99.98	98.70	99.97	**100**	99.83	99.31
	500	98.96	**100**	99.77	**100**	**100**	99.98	99.87
$beta(1,2)$	10	10.52	11.26	**17.90**	9.07	8.11	14.14	11.73
	25	16.41	16.66	**28.33**	12.88	13.58	24.82	19.88
	40	23.15	26.24	**35.94**	21.62	28.10	35.62	27.77
	100	47.71	72.46	58.10	68.64	**80.91**	70.73	59.04
	150	63.72	93.66	71.51	91.62	**96.86**	88.19	77.34
	200	77.71	98.78	83.87	98.32	**99.62**	95.85	89.40
	250	86.55	99.90	92.52	99.8	**99.95**	98.63	94.86
	300	92.75	**100**	97.10	99.99	**100**	99.62	97.72
	350	95.80	**100**	99.21	**100**	**100**	99.92	99.09
	400	97.91	**100**	99.82	**100**	**100**	**100**	99.72
	500	99.49	**100**	**100**	**100**	**100**	**100**	99.99

The bold number is the highest power among the seven tests for each sample size.

Table 4.37: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.1$ against the alternative distributions from Group III for sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$Laplace(0,1)$	5	18.63	17.44	11.68	**20.91**	19.94	15.89	18.96
	10	32.15	32.34	19.69	36.03	**38.46**	31.95	34.51
	15	41.51	43.16	32.09	47.05	**47.25**	44.19	45.57
	20	50.13	52.27	42.29	**57.75**	54.48	55.18	55.84
	25	58.48	61.68	52.1	64.39	61.05	65.01	**65.26**
	30	66.37	68.43	60.55	70.36	66.01	**73.35**	73.26
	40	77.55	77.80	73.08	78.86	72.95	**83.60**	83.28
	50	84.48	84.01	80.56	85.46	76.73	**89.60**	89.08
	100	98.25	97.82	97.61	97.71	91.80	99.27	**99.31**
	150	99.86	99.67	99.74	99.71	97.09	99.95	**99.95**
	200	99.95	99.95	99.96	99.98	98.91	99.98	**99.99**
$logistic(0,1)$	5	15.77	14.63	8.82	**17.09**	16.29	12.67	15.73
	10	23.30	23.45	12.35	26.71	**30.21**	23.31	25.73
	15	28.72	31.84	19.44	34.87	**38.75**	32.03	33.13
	20	35.01	41.27	26.41	44.83	**46.71**	41.10	40.56
	25	41.15	49.02	33.24	50.83	**53.52**	49.13	48.24
	30	46.74	56.12	40.70	57.36	**59.62**	57.36	55.49
	40	55.14	66.48	52.35	67.42	67.75	**68.30**	65.37
	50	63.17	73.53	60.41	75.23	72.92	**76.25**	73.07
	100	88.66	94.19	87.89	94.25	90.11	**95.81**	94.26
	150	96.77	98.6	96.64	98.69	96.42	**99.41**	99.09
	200	99.22	99.69	99.01	99.71	98.53	**99.91**	99.86
$Cauchy(0,1)$	5	36.43	34.97	32.45	**38.93**	37.73	36.49	37.57
	10	64.02	64.54	59.84	**67.68**	67.61	66.52	66.85
	15	78.96	79.88	77.59	81.57	80.45	**82.31**	81.76
	20	87.81	88.43	87.19	90.37	87.81	**90.73**	90.31
	25	93.20	93.32	92.56	93.82	91.65	**95.27**	95.15
	30	96.37	96.68	96.26	96.73	95.01	**97.72**	97.57
	40	99.01	98.98	98.92	98.85	97.84	**99.50**	99.49
	50	99.55	99.49	99.59	99.69	98.83	**99.80**	99.76
	100	**100**	**100**	**100**	99.99	99.94	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	14.89	13.99	8.56	**15.65**	15.21	11.70	14.60
	10	19.91	19.69	9.23	22.16	**25.07**	18.80	21.72
	15	23.97	26.9	12.29	28.64	**34.60**	25.85	27.24
	20	28.45	34.61	17.02	36.97	**43.46**	33.07	32.68
	25	32.49	43.04	21.48	44.26	**51.47**	40.74	38.67
	30	35.48	49.48	24.77	51.10	**57.62**	45.77	43.42
	40	43.38	62.05	33.30	62.85	**69.23**	57.55	53.52
	50	50.12	72.03	40.08	72.65	**76.75**	67.29	62.13
	100	77.47	95.39	69.99	**95.39**	95.23	92.71	88.20
	150	90.46	99.24	87.94	**99.36**	99.03	98.81	97.15
	200	96.81	99.94	96.53	**99.94**	99.85	99.86	99.38
$t(1)$	5	36.65	35.38	32.59	**38.37**	37.9	36.25	37.57
	10	63.54	64.04	59.85	67.70	**67.73**	66.18	66.33
	15	79.30	79.85	77.87	81.79	80.62	**82.44**	82.24
	20	87.87	88.45	87.34	90.19	87.62	**90.59**	90.25
	25	93.19	93.20	92.62	94.14	92.02	**94.97**	94.78
	30	96.06	95.96	96.04	96.86	94.53	**97.47**	97.35
	40	98.97	98.83	98.94	98.70	97.86	**99.45**	99.43
	50	99.79	99.73	99.70	99.69	99.13	**99.91**	99.9
	100	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	19.22	18.12	12.22	**20.7**	19.84	16.82	19.36
	10	30.77	31.49	22.38	34.81	**37.22**	32.19	34.09
	15	40.92	43.48	35.02	46.21	**48.71**	45.2	45.61
	20	49.26	53.55	46.07	**56.54**	56.25	56	55.18
	25	55.27	59.39	53.75	**62.75**	60.79	62.75	61.66
	30	62.71	67.17	61.45	69.75	65.99	**70.72**	69.96
	40	73.82	77.68	**74.64**	78.78	74.49	**82.05**	80.51
	50	80.99	84.06	81.75	85.10	78.49	**88.24**	86.76
	100	96.98	97.26	97.38	97.67	92.31	**98.87**	98.62
	150	99.69	99.58	99.66	99.6	97.45	**99.89**	99.87
	200	99.97	99.93	99.94	99.97	99.28	**100**	**100**
$t(4)$	5	17.36	16.16	10.85	**18.74**	18.37	15.05	17.82
	10	26.72	27.05	17.3	30.90	**33.41**	27.36	29.34
	15	35.41	38.74	28.12	40.86	**44.96**	40.07	40.67
	20	42.00	47.10	36.52	**51.60**	51.11	48.85	48.73
	25	48.51	54.74	45.58	56.84	**57.36**	56.61	55.57
	30	55.86	62.49	54.31	64.71	63.16	**65.44**	63.68
	40	65.63	72.33	65.83	73.82	70.17	**76.09**	73.69
	50	73.88	79.23	74.52	80.88	75.02	**84.30**	82.11
	100	93.93	95.48	94.53	96.14	89.64	**97.51**	96.90
	150	98.85	99.08	98.96	99.21	96.14	**99.79**	99.71
	200	99.92	99.87	99.79	99.81	98.68	**99.98**	99.98
$t(6)$	5	16.31	15.12	9.3	**17.93**	17.16	13.65	16.57
	10	23.40	23.98	13.72	26.85	**29.70**	23.69	25.89
	15	31.60	34.99	22.00	36.89	**40.97**	34.87	35.93
	20	36.77	42.60	29.14	46.46	**48.05**	43.12	42.72
	25	41.80	49.61	36.58	51.80	**53.97**	51.06	49.42
	30	48.73	56.60	43.81	59.14	**59.28**	58.54	56.92
	40	58.05	67.48	56.29	69.90	67.74	**69.92**	67.41
	50	65.62	74.95	64.91	77.16	73.01	**78.43**	75.65
	100	89.93	93.94	89.90	94.26	89.36	**96.12**	94.93
	150	97.06	98.52	97.48	98.82	95.64	**99.46**	99.09
	200	99.43	99.74	99.46	99.71	98.36	**99.91**	99.84

The bold number is the highest power among the seven tests for each sample size.

CHAPTER 5 Tests of the exponential distribution

5.1 Background

The probability density function (pdf) of the two-parameter exponential distribution with the location parameter μ and the scale parameter σ is given by

$$f(y|\mu,\sigma) = \frac{1}{\sigma}\exp\Big(-\frac{y-\mu}{\sigma}\Big) \quad y>\mu,\, \sigma>0, \tag{5.1.1}$$

and cumulative distribution function

$$F(y|\mu,\sigma) = 1-\exp\Big(-\frac{y-\mu}{\sigma}\Big) \quad y>\mu,\, \sigma>0. \tag{5.1.2}$$

It is referred to as $Exp(\mu,\sigma)$. Also, the notation $Y \sim Exp(0,1)$ is used for the standard exponential random variable. As mentioned above, the exponential distribution is one of the location-scale distributions as a normal distribution.

If $Y_1,...,Y_n \overset{i.i.d.}{\sim} Exp(\mu,\sigma)$, then $Y_{[1]} \leq Y_{[2]} \leq ... \leq Y_{[n]}$ are their ordered values. Let the $\boldsymbol{p^{th}}$ **quantile** of the distribution of a random variable Y, denoted by y_p, be defined as follows. If Y is continuous, then the unique y_p is defined by

$$P(Y \leq y_p) = p. \tag{5.1.3}$$

Specifically, the $\boldsymbol{p^{th}}$ **quantile** of a random variable $Y \sim Exp(\mu,\sigma)$ is given by

$$F^{-1}(p) = \mu - \sigma\ln(1-p). \tag{5.1.4}$$

In particular, $F^{-1}(p) = -\ln(1-p)$ is the p^{th} quantile of the random variable $\frac{Y-\mu}{\sigma} \sim Exp(0,1)$.

Now we describe each graphical test which provides a set of simultaneous intervals for $Y_{[k]}$ $(k=1,...,n)$ in the Q-Q plot.

5.2 Parameter estimation

The three estimators MLE, BLUE and BLIE (as described in Chapter 4) are again investigated in order to find the best estimator which establish good powers.

5.2.1 Maximum likelihood estimators

If $Y_1, \ldots, Y_n$ are i.i.d. $Exp(\mu, \sigma)$, then the likelihood function is

$$L(\mu, \sigma | Y_1, \ldots, Y_n) = \frac{1}{\sigma^n} \exp\Big[-\frac{1}{\sigma} \sum_{k=1}^{n} (Y_k - \mu) \Big] I(Y_{[1]} \geq \mu) \tag{5.2.1}$$

and the log-likelihood function is

$$\begin{aligned} \mathcal{L}(\mu, \sigma | Y_1, \ldots, Y_n) &= \ln L(\mu, \sigma | Y_1, \ldots, Y_n) \\ &= \Big[-n \ln \sigma - \frac{1}{\sigma} \sum_{k=1}^{n} (Y_k - \mu) \Big] I(Y_{[1]} \geq \mu). \end{aligned}$$

For any fixed $\sigma > 0$, the log-likelihood function is maximised by maximising μ. Therefore, $\hat{\mu} = Y_{[1]}$. Then, the profile log-likelihood which is

$$\mathcal{L}(\sigma | Y_{[1]}) = -n \ln \sigma - \frac{1}{\sigma} \sum_{k=1}^{n} (Y_k - Y_{[1]}) \tag{5.2.2}$$

is maximised by $\tilde{\sigma} = \frac{1}{n} \sum_{k=1}^{n} (Y_k - Y_{[1]})$. Thus, the maximum likelihood estimators

$$(\tilde{\mu}, \tilde{\sigma}) = \Big(Y_{[1]}, \frac{1}{n} \sum_{k=1}^{n} (Y_k - Y_{[1]}) \Big) = \big(Y_{[1]}, \bar{Y} - Y_{[1]} \big). \tag{5.2.3}$$

Now, we want to show the invariant property of $\tilde{\mu}_Y$ and $\tilde{\sigma}_Y$ where $\tilde{\sigma}_Z$ is MLE of σ if $Z \sim Exp(0, 1)$ and $-\infty < p < \infty$, $q > 0$. Let $Y_k = p + qZ_k$. Consequently,

$$\begin{aligned} \tilde{\mu}_{p+qZ} &= p + qZ_{[1]} = p + q\tilde{\mu}_Z \\ \tilde{\sigma}_{p+qZ} &= (p + q\bar{Z}) - (p + qZ_{[1]}) = q(\bar{Z} - Z_{[1]}) = q\tilde{\sigma}_Z. \end{aligned}$$

Then,

$$\begin{aligned} \frac{Y_{[k]} - \tilde{\mu}_Y}{\tilde{\sigma}_Y} &= \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma\tilde{\mu}_Z)}{\sigma\tilde{\sigma}_Z} \\ &= \frac{Z_{[k]} - \tilde{\mu}_Z}{\tilde{\sigma}_Z}. \end{aligned}$$

Therefore, $\frac{Y_{[k]} - \tilde{\mu}_Y}{\tilde{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1, \ldots, Y_n$ from $Exp(0, 1)$ instead of $Exp(\mu, \sigma)$.

5.2.2 Best linear unbiased estimators

Let $Z_{[1]} \leq \ldots \leq Z_{[n]}$ be the ordered sample from $Exp(0,1)$ with

$$\mu_k = \mathrm{E}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{n-i+1} \quad ; k = 1, \ldots, n \tag{5.2.4}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2} \quad ; k = 1, \ldots, n \tag{5.2.5}$$

$$\sigma_{rk}^2 = \mathrm{Cov}(Z_{[r]}, Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2} \quad ; 1 \leq r \leq k \leq n. \tag{5.2.6}$$

where $\sigma_{rk}^2 = \sigma_{kr}^2$ is the covariance of the r^{th} and k^{th} order statistics of $Exp(0,1)$ (see Ahsanullah and Hamedani, 2010). Similarly, we can use the generalised least squares method based on the Weibull distribution from obtaining BLUE $(\dot{\mu}, \dot{\sigma})$ of $Exp(\mu, \sigma)$. Let

$$\begin{aligned}
\boldsymbol{\mu} &= n \times 1 \,\text{vector of}\, \mu_k = [\mu_1, \ldots, \mu_n]' \\
V &= n \times n \,\text{matrix of}\, \sigma_{rk}^2 \\
\mathbf{Y} &= n \times 1 \,\text{vector of ordered values} = [Y_{[1]}, \ldots, Y_{[n]}]' \,\text{from}\, Exp(\mu, \sigma) \\
\mathbf{X} &= [\mathbf{1} \;\; \boldsymbol{\mu}] \text{ where } \mathbf{1} = [1 \ldots 1]'_{1 \times n}.
\end{aligned}$$

Therefore, $\dot{\boldsymbol{\beta}}$ has the simple closed form (see, Sarhan (1954)) as

$$\begin{aligned}
\dot{\boldsymbol{\beta}} &= \begin{bmatrix} \dot{\mu} \\ \dot{\sigma} \end{bmatrix} = (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y} \\
&= \frac{1}{n(n-1)} \begin{bmatrix} n^2-1 & -1 & \cdots & -1 \\ -n(n-1) & n & \cdots & n \end{bmatrix}_{2 \times n} \begin{bmatrix} Y_{[1]} \\ \vdots \\ Y_{[n]} \end{bmatrix}_{n \times 1} \\
&= \frac{1}{n(n-1)} \begin{bmatrix} n^2 Y_{[1]} - \sum_{k=1}^{n} Y_{[k]} \\ -n^2 Y_{[1]} + n \sum_{k=1}^{n} Y_{[k]} \end{bmatrix} \\
&= \begin{bmatrix} \frac{nY_{[1]} - \bar{Y}}{n-1} \\ \frac{n(Y_{[1]} - \bar{Y})}{n-1} \end{bmatrix}.
\end{aligned}$$

Consequently, the best linear unbiased estimators of the location and scale parameters μ and σ are respectively

$$\dot{\mu} = \frac{nY_{[1]} - \bar{Y}}{n-1} \tag{5.2.7}$$

and

$$\dot{\sigma} = \frac{n(Y_{[1]} - \bar{Y})}{n-1}. \tag{5.2.8}$$

Also, (4.2.16) can be shown as

$$\begin{aligned}
\mathrm{Cov}(\dot{\boldsymbol{\beta}}) &= \mathrm{Cov}((\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y}) \\
&= (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}(\sigma^2 V)((\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1})' \\
&= \sigma^2(\mathbf{X}'V^{-1}\mathbf{X})^{-1} \\
&= \frac{\sigma^2}{\triangle}\begin{bmatrix} \boldsymbol{\mu}'V^{-1}\boldsymbol{\mu} & -(\mathbf{1}'V^{-1}\boldsymbol{\mu}) \\ -(\mathbf{1}'V^{-1}\boldsymbol{\mu}) & \mathbf{1}'V^{-1}\mathbf{1} \end{bmatrix} \qquad (5.2.9) \\
&= \frac{\sigma^2}{n(n-1)}\begin{bmatrix} 1 & -1 \\ -1 & n \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}.
\end{aligned}$$

Therefore, on simplification we get

$$E_{11} = \frac{1}{n(n-1)} \tag{5.2.10}$$

$$E_{22} = \frac{1}{n-1} \tag{5.2.11}$$

$$E_{12} = -\frac{1}{n(n-1)}. \tag{5.2.12}$$

Now, we need to show that $\dot{\mu}$ and $\dot{\sigma}$ are invariant:

$$\begin{aligned}
\dot{\mu}_{p+qZ} &= \frac{n(p+qZ_{[1]}) - (p+q\bar{Z})}{n-1} \\
&= \frac{(n-1)p + qnZ_{[1]} - q\bar{Z}}{n-1} \\
&= p + q\frac{nZ_{[1]} - \bar{Z}}{n-1} \\
&= p + q\dot{\mu}_Z
\end{aligned}$$

$$\dot{\sigma}_{p+qZ} = \frac{n[(p+qZ_{[1]}) - (p+q\bar{Z})]}{n-1}$$

$$\begin{aligned} &= q\frac{[n(Z_{[1]} - \bar{Z})]}{n-1} \\ &= q\dot{\sigma}_Z. \end{aligned}$$

Then,

$$\begin{aligned} \frac{Y_{[k]} - \dot{\mu}_Y}{\dot{\sigma}_Y} &= \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma \dot{\mu}_Z)}{\sigma \dot{\sigma}_Z} \\ &= \frac{Z_{[k]} - \dot{\mu}_Z}{\dot{\sigma}_Z}. \end{aligned}$$

Therefore, $\frac{Y_{[k]} - \dot{\mu}_Y}{\dot{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1, \ldots, Y_n$ from $Exp(0, 1)$ instead of $Exp(\mu, \sigma)$.

5.2.3 Best linear invariant estimators

The best linear invariant (in the sense of minimum mean squared error and invariance) estimators (BLIE) $\ddot{\mu}$ and $\ddot{\sigma}$ according to Mann (1969) are

$$\begin{aligned} \ddot{\mu} &= \dot{\mu} - \dot{\sigma}\Big(\frac{E_{12}}{1 + E_{22}}\Big), \\ \ddot{\sigma} &= \frac{\dot{\sigma}}{1 + E_{22}} \end{aligned}$$

where $\dot{\mu}$ and $\dot{\sigma}$ are the BLUE of μ and σ obtained from (5.2.7) and (5.2.8), respectively

Using E_{12} and E_{22} obtained from (5.2.10), (5.2.11) and (5.2.12), BLIE become

$$\ddot{\mu} = (1 + \frac{1}{n})Y_{[1]} - \frac{\bar{Y}}{n} \quad (5.2.13)$$

$$\ddot{\sigma} = \bar{Y} - Y_{[1]}. \quad (5.2.14)$$

Thus, the invariant property for the estimators is shown as:

$$\begin{aligned} \ddot{\mu}_{p+qZ} &= (1 + \frac{1}{n})(p + qZ_{[1]}) - \frac{(p + q\bar{Z})}{n} \\ &= p + q\big((1 + \frac{1}{n})Z_{[1]} - \frac{\bar{Z}}{n}\big) \\ &= p + q\ddot{\mu}_Z \\ \ddot{\sigma}_{p+qZ} &= (p + q\bar{Z}) - (p + qZ_{[1]}) \\ &= q(\bar{Z} - Z_{[1]}) \\ &= q\ddot{\sigma}_Z. \end{aligned}$$

Then, $Y_{[k]} = \mu + \sigma Z_{[k]}$ and

$$\begin{aligned}\frac{Y_{[k]} - \ddot{\mu}_Y}{\ddot{\sigma}_Y} &= \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma \ddot{\mu}_Z)}{\sigma \ddot{\sigma}_Z} \\ &= \frac{Z_{[k]} - \ddot{\mu}_Z}{\ddot{\sigma}_Z}.\end{aligned}$$

Therefore, $\frac{Y_{[k]} - \ddot{\mu}_Y}{\ddot{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1, \ldots, Y_n$ from $Exp(0,1)$ instead of $Exp(\mu, \sigma)$.

5.3 Graphical tests for the exponential distribution

In this section, we continue to investigate the graphical tests for the exponential distribution which are generalised from the graphical tests for normality. Note that $(\hat{\mu}, \hat{\sigma})$ can be replaced by $(\tilde{\mu}, \tilde{\sigma})$, $(\dot{\mu}, \dot{\sigma})$ and $(\ddot{\mu}, \ddot{\sigma})$.

5.3.1 The Kolmogorov-Smirnov test (The D test)

The Kolmogorov-Smirnov statistic is

$$D = \max_{1 \le k \le n} \Big| F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k - 0.5)/n \Big|. \tag{5.3.1}$$

Let c_D be a critical constant so that $P\{D < c_D\} = 1 - \alpha$ when a sample follows an exponential distribution $Exp(\mu, \sigma)$. This probability statement can be rewritten as

$$\begin{aligned}1 - \alpha &= P\{D < c_D\} \\ &= P\Big\{ \max_{1 \le k \le n} \Big| F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k - 0.5)/n \Big| \le c_D \Big\} \\ &= P\Big\{ -c_D \le F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k - 0.5)/n \le c_D \text{ for } k = 1, ..., n \Big\} \\ &= P\Big\{ \frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y} \in F^{-1}\big((k - 0.5)/n \pm c_D\big) \text{ for } k = 1, ..., n \Big\} \\ &= P\Big\{ Y_{[k]} \in \hat{\mu}_Y - \hat{\sigma}_Y \ln\big(1 - (k - 0.5)/n \mp c_D\big) \text{ for } k = 1, ..., n \Big\}. \end{aligned} \tag{5.3.2}$$

Note that the expression (5.3.2) has nothing to do with the unknown parameters μ and σ since $\frac{Y_1 - \mu}{\sigma}, ..., \frac{Y_n - \mu}{\sigma} \overset{i.i.d.}{\sim} Exp(0,1)$. Let $Z_k = \frac{Y_k - \mu}{\sigma}$ and $Z_{[k]} = \frac{Y_{[k]} - \mu}{\sigma}$ for $k = 1, ..., n$. Using the invariant property, then

$$\begin{aligned}\frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y} &= \frac{(\mu + \sigma Z_{[k]}) - (\mu + \sigma \hat{\mu}_Z)}{\sigma \hat{\sigma}_Z} \\ &= \frac{Z_{[k]} - \hat{\mu}_Z}{\hat{\sigma}_Z}.\end{aligned}$$

Therefore, $\frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y}$ has nothing to do with μ and σ and we can generate the random sample $Y_1, ..., Y_n$ from $Exp(0,1)$ instead of $Exp(\mu,\sigma)$ in order to find the critical constant c_D.

The required critical constant c_D can be evaluated straightforwardly as the $(1-\alpha)$-quantile of D by using the following simulations:

1. Simulate one D by generating i.i.d. $Y_1, ..., Y_n$. from $Exp(0,1)$.
2. Calculate $\hat{\mu}_Y, \hat{\sigma}_Y$ and $(Y_{[1]}, ..., Y_{[n]})$, and compute D using formula (5.3.1).
3. Simulate a large number R of independent copies of $D : D_1, ..., D_R$.
4. Use the $(1-\alpha)$ sample quantile of $D_1, ..., D_R$ as an approximation of the critical constant c_D.

From (5.3.2), $Y_{[k]}$ should fall in the corresponding interval

$$\hat{\mu}_Y - \hat{\sigma}_Y \ln\big(1-(k-0.5)/n \mp c_D\big) \text{ for } k = 1, ..., n \tag{5.3.3}$$

with a simultaneous probability $1-\alpha$ if the sample follows an exponential distribution.

5.3.2 The D_{sp} test

Based on the D_{sp} test of Michael (1983), the statistic D_{sp} for testing exponentiality is defined by

$$D_{sp} = \max_{1\le k\le n}\left|\frac{2}{\pi}\arcsin\sqrt{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)} - \frac{2}{\pi}\arcsin\sqrt{(k-0.5)/n}\right|. \tag{5.3.4}$$

Let c_{sp} be a critical constant so that $P\{D_{sp} < c_{sp}\} = 1-\alpha$ when the sample follows an exponential distribution. The critical constant c_{sp} can be computed by simulation in a similar way as is the case with c_D. This probability statement can be rewritten as

$$\begin{aligned}
1-\alpha &= P\Big\{D_{sp} < c_{sp}\Big\} \\
&= P\Big\{\max_{1\le k\le n}\left|\frac{2}{\pi}\arcsin\sqrt{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)} - \frac{2}{\pi}\arcsin\sqrt{(k-0.5)/n}\right| < c_{sp}\Big\} \\
&= P\Big\{-c_{sp} < \frac{2}{\pi}\arcsin\sqrt{F\big((Y_{[k]}-\hat{\mu}_Y)/\hat{\sigma}_Y\big)} - m < c_{sp} \text{ for } k = 1, \ldots, n\Big\} \\
&= P\Big\{\frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y} \in F^{-1}(\sin^2(\frac{\pi}{2}(m \pm c_{sp}))) \text{ for } k = 1, \ldots, n\Big\} \\
&= P\Big\{Y_{[k]} \in \hat{\mu}_Y - \hat{\sigma}_Y \ln\big(1-\sin^2(\frac{\pi}{2}(m \pm c_{sp}))\big) \text{ for } k = 1, \ldots, n\Big\},
\end{aligned} \tag{5.3.5}$$

where $m = \frac{2}{\pi} \arcsin \sqrt{(k-0.5)/n}$. The expression (5.3.5) provides the simultaneous intervals for $Y_{[k]}$, $k = 1, \ldots, n$. Consequently, the simultaneous interval for $Y_{[k]}$ is

$$\hat{\mu}_Y - \hat{\sigma}_Y \ln[1 - \sin^2\big(\arcsin \sqrt{(k-0.5)/n} \pm \tfrac{\pi}{2} c_{sp})] \text{ for } k = 1, \ldots, n.$$

5.3.3 The D_e test

Let $Z_1, \ldots, Z_n$ be a simple random sample drawn from the standard exponential distribution $Exp(0,1)$ and $Z_{[1]} \le \ldots \le Z_{[n]}$ be their ordered values. Then, the expected value and variance of $Z_{[k]}$ for $k = 1, \ldots, n$ are given by

$$\mu_k = \mathrm{E}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{n-i+1} \tag{5.3.6}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2}, \tag{5.3.7}$$

where $f_k(z)$ is the probability density function of $Z_{[k]}$ and is defined as

$$f_k(z) = \frac{n!}{(k-1)!(n-k)!} F(z)^{k-1} f(z) \big(1 - F(z)\big)^{n-k}, \; -\infty \le z \le \infty. \tag{5.3.8}$$

Note that $F(z)$ and $f(z)$ are the cdf and pdf of $Exp(0,1)$, respectively. Since the Y_k's are taken from the distribution $Exp(\mu, \sigma)$, it is clear that $(Y_{[1]}, ..., Y_{[n]})$ have the same joint distribution as $(\mu + \sigma Z_{[1]}, ..., \mu + \sigma Z_{[n]})$. In particular, we have

$$\mathrm{E}(Y_{[k]}) = \mu + \sigma \mu_k = \mu + \sigma \sum_{i=1}^{k} \frac{1}{n-i+1} \quad \text{and} \tag{5.3.9}$$

$$\mathrm{Var}(Y_{[k]}) = \sigma^2 \sigma_k^2 = \sigma^2 \sum_{i=1}^{k} \frac{1}{(n-i+1)^2}. \tag{5.3.10}$$

Specially, we construct the D_e statistic as

$$D_e = \max_{1 \le k \le n} \frac{|Y_{[k]} - (\hat{\mu}_Y + \hat{\sigma}_Y \mu_k)|}{\hat{\sigma}_Y \sigma_k}. \tag{5.3.11}$$

Let c_e be a critical constant chosen so that all the $Y_{[k]}$'s will be contained in the corresponding intervals simultaneously with probability $1-\alpha$. The probability statement can be rewritten as

$$\begin{aligned} 1 - \alpha &= P\Big\{ D_e < c_e \Big\} \\ &= P\Big\{ \max_{1 \le k \le n} \frac{|Y_{[k]} - (\hat{\mu}_Y + \hat{\sigma}_Y \mu_k)|}{\hat{\sigma}_Y \sigma_k} \le c_e \Big\} \\ &= P\Big\{ Y_{[k]} \in (\hat{\mu}_Y + \hat{\sigma}_Y \mu_k) \pm \hat{\sigma}_Y \sigma_k c_e \Big\}. \end{aligned} \tag{5.3.12}$$

Consider the term $\frac{Y_{[k]}-(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)}{\hat{\sigma}_Y\sigma_k}$ in (5.3.12), where $\hat{\mu}_Y$ and $\hat{\sigma}_Y$ are the estimators of the location and scale parameters, respectively. Since $\frac{Y_1-\mu}{\sigma},...,\frac{Y_n-\mu}{\sigma} \overset{i.i.d.}{\sim} Exp(0,1)$. Let $Z_k = \frac{Y_k-\mu}{\sigma}$ and $Z_{[k]} = \frac{Y_{[k]}-\mu}{\sigma}$ for $k = 1,...,n$. Then $Y_{[k]} = \mu + \sigma Z_{[k]}$ for $k = 1,...,n$ and

$$\begin{aligned}\frac{Y_{[k]}-(\hat{\mu}_Y+\hat{\sigma}_Y\mu_k)}{\hat{\sigma}_Y\sigma_k} &= \frac{(\mu+\sigma Z_{[k]})-(\mu+\sigma\hat{\sigma}_Z+\sigma\hat{\sigma}_Z\mu_k)}{\sigma\hat{\sigma}_Z\sigma_k} \\ &= \frac{Z_{[k]}-(\hat{\mu}_Z+\hat{\sigma}_Z\mu_k)}{\hat{\sigma}_Z\sigma_k}. \end{aligned} \tag{5.3.13}$$

It is clear from (5.3.13) that the simultaneous coverage probability in (5.3.12) depends only on the sample size and the critical constant, and has nothing to do with the unknown parameters μ and σ. Furthermore, the critical constant c_e can be estimated in a similar way as was the case with c_D.

5.3.4 The D_{be} test

According to Theorem 2.1.1, $U_k = F\Big(\frac{Y_k-\mu}{\sigma}\Big) \sim U(0,1)$. Consequently, $U_{[k]} = F\Big(\frac{Y_{[k]}-\mu}{\sigma}\Big) \sim beta(k, n-k+1)$ if $F(\cdot)$ is a cdf of exponential distribution. We apply the D_{be} statistic for testing normality to exponentiality by the following steps:

- **Step 1**. Construct p^* level highest-density probability interval for $U_{[k]}$. That is, $[L(p^*,k,n), U(p^*,k,n)]$ is the shortest probability interval for $U_{[k]}$ among all the p^* level probability intervals for $U_{[k]}$.

- **Step 2**. Construct simultaneous probability intervals for $Y_{[1]} \le ... \le Y_{[n]}$ based on

$$L(p^*,k,n) \le F\Big(\frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y}\Big) \le U(p^*,k,n) \text{ for } k = 1,...,n,$$

  where p^* is chosen so that

$$K(p^*) \equiv P\Big\{F^{-1}(L(p^*,k,n)) \le \frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y} \le F^{-1}(U(p^*,k,n)) \text{ for } k = 1,...,n\Big\} = 1-\alpha.$$

$$P\Big\{-\ln(1-L(p^*,k,n)) \le \frac{Y_{[k]}-\hat{\mu}_Y}{\hat{\sigma}_Y} \le -\ln(1-U(p^*,k,n)) \text{ for } k = 1,...,n\Big\} = 1-\alpha.$$

- **Step 3**. Such a p^* can be found by simulation :
  (i) for each p^*, find $K(p^*)$;
  (ii) search over $p^* \in (1-\frac{\alpha}{n}, 1-\alpha)$ so that $K(p^*) = 1-\alpha$.
  Therefore, the simultaneous probability intervals for $Y_{[1]} \le ... \le Y_{[n]}$ can be

expressed as

$$\hat{\mu}_Y - \hat{\sigma}_Y \ln(1-L) \le Y_{[k]} \le \hat{\mu}_Y - \hat{\sigma}_Y \ln(1-U) \text{ for } k = 1, \ldots, n. \quad (5.3.14)$$

5.3.5 The D_{bi} test

Similar to the D_{bi} statistic for testing normality, the statistic D_{bi} for testing exponential distribution can be defined as

$$D_{bi} = \max_{1\le k\le n} \frac{\left|F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}}. \quad (5.3.15)$$

Let c_{bi} be a critical constant so that $P\Big\{D_{bi} < c_{bi}\Big\} = 1-\alpha$; c_{bi} can be determined by simulation as was c_D from the D test. The probability statement for $Y_{[1]} \le ... \le Y_{[n]}$ is given by

$$\begin{aligned}
1-\alpha &= P\Big\{D_{bi} < c_{bi}\Big\} \\
&= P\Big\{\max_{1\le k\le n} \frac{\left|F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k-0.5)/n\right|}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi}\Big\} \\
&= P\Big\{-c_{bi} < \frac{F\big((Y_{[k]} - \hat{\mu}_Y)/\hat{\sigma}_Y\big) - (k-0.5)/n}{\sqrt{(k-0.5)(n-k+0.5)/n^3}} < c_{bi} \text{ for } k = 1, \ldots, n\Big\} \\
&= P\Big\{F^{-1}\Big(\frac{k-0.5}{n} - dc_{bi}\Big) < \frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y} < F^{-1}\Big(\frac{k-0.5}{n} + dc_{bi}\Big)\Big\} \\
&= P\Big\{Y_{[k]} \in \hat{\mu}_Y - \hat{\sigma}_Y \ln\big(1 - \frac{k-0.5}{n} \mp dc_{bi}\big) \text{ for } k = 1, \ldots, n\Big\}
\end{aligned}$$

where $d = \sqrt{(k-0.5)(n-k+0.5)/n^3}$. Therefore, the simultaneous interval of $Y_{[k]}$ is

$$\hat{\mu}_Y - \hat{\sigma}_Y \ln\big(1 - \frac{k-0.5}{n} \mp c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\big) \text{ for } k = 1, \ldots, n. \quad (5.3.16)$$

5.4 Non-graphical tests for the exponential distribution

The selected non-graphical tests for testing an exponential distribution are the Anderson-Darling and the Cramer von Mises tests which are similar to the non-graphical tests used for testing a Weibull distribution.

5.4.1 The Anderson-Darling test

The Anderson-Darling test statistic (AD) is

$$\mathrm{AD} = -\sum_{k=1}^{n} \Big[\frac{(2k-1)\{\ln(F_k) + \ln(1 - F_{n+1-k})\}}{n} \Big] - n, \tag{5.4.1}$$

where $F_k = 1 - \exp\Big(-\frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y}\Big)$. The critical constant c_{AD} which satisfies $P\{\mathrm{AD} < c_{\mathrm{AD}}\} = 1 - \alpha$ can be determined by simulation as was the critical constant c_D from the D test.

5.4.2 The Crarńer-von Mises test

The Crarńer-von Mises statistic (CvM) is

$$\mathrm{CvM} = \sum_{k=1}^{n} \Big[F_k - \frac{2k-1}{2n} \Big]^2 + \frac{1}{12n} \tag{5.4.2}$$

where $F_k = 1 - \exp\Big(-\frac{Y_{[k]} - \hat{\mu}_Y}{\hat{\sigma}_Y}\Big)$. The critical constant c_{CvM} which satisfies $P\{\mathrm{CvM} < c_{\mathrm{CvM}}\} = 1 - \alpha$ can be determined by simulation as was the critical constant c_D from the D test.

5.5 Power comparison

To estimate powers of the tests for the exponential distribution, the Monte Carlo simulation is applied. In the first part of the simulation study, the critical values under the two-parameter exponential distribution are computed. In the second part, the powers are estimated based on simulation data from 23 alternative distributions. They are used in many power studies for goodness-of-fit tests. These alternative distributions are still categorised into three groups.

The first group (Group I) of nine distributions is asymmetrical on the support $(0, \infty)$ and includes:

- $\chi^2(1)$;
- $\chi^2(3)$;
- $\chi^2(4)$;
- $\chi^2(6)$;
- $\chi^2(10)$;

- $LogN(0,1)$;
- $HN(0,1)$;
- $Wbl(0,0.5,0.5)$;
- $Wbl(0,2,2)$.

The second group (Group II) of seven distributions is on the interval $(0,1)$ and consists of:

- $U(0,1)$;
- $beta(2,2)$;
- $beta(2,5)$;
- $beta(5,1.5)$;
- $beta(0.5,0.5)$;
- $beta(0.5,3)$;
- $beta(1,2)$.

The third group (Group III) of eight distributions is symmetrical on the support $(-\infty,\infty)$ and includes:

- $Laplace(0,1)$;
- $logistic(0,1)$;
- $Cauchy(0,1)$;
- $N(0,1)$;
- $t(1)$;
- $t(3)$;
- $t(4)$;
- $t(6)$.

Some of the referred alternative distributions have been used by Kimber (1985) and Sürücü (2008). The critical values and powers of the seven tests are computed for all possible combinations of $\alpha = 0.01$, 0.05 and 0.1, three estimators (MLE, BLUE and BLIE), and sample sizes $n = 5(5)30$, 40, 50, 100, 150 and 200. The largest selected sample size is 200 at which the power against almost all alternative distributions reaches 100%. The goodness-of-fit tests investigated are the combinations of the test statistics and parameter estimators listed in Table 5.1. Before carrying out the power study, the critical values of the test statistics with the exception of the D_{be} statistic using three estimators are computed based on 300,000 simulations drawn from the standard exponential distribution for each sample size under consideration (see Tables 5.2 - 5.4). The powers of each test for each circumstance are calculated based on 100,000 simulations. As the observations made from this power comparison study are similar for $\alpha = 0.01$, 0.05 and 0.1, only the power for $\alpha = 0.05$ is analysed here in Tables 5.5 – 5.13. The corresponding tables for power comparison considering significance levels $\alpha = 0.01$ and 0.1 are available at the end of this chapter in Tables 5.19 – 5.27 and Tables 5.28 – 5.30, respectively.

Table 5.1: Combination of goodness-of-fit test statistics and the estimators

	Estimators		
Test statistics	MLE	BLUE	BLIE
D	$D(1)$	$D(2)$	$D(3)$
D_{sp}	$D_{sp}(1)$	$D_{sp}(2)$	$D_{sp}(3)$
D_e	$D_e(1)$	$D_e(2)$	$D_e(3)$
D_{be}	$D_{be}(1)$	$D_{be}(2)$	$D_{be}(3)$
D_{bi}	$D_{bi}(1)$	$D_{bi}(2)$	$D_{bi}(3)$
AD	AD(1)	AD(2)	AD(3)
CvM	CvM(1)	CvM(2)	CvM(3)

Table 5.2: Critical values for testing exponential distribution of the D, D_{sp}, D_e, D_{bi}, AD and CvM tests by using MLE at $\alpha = 0.01, 0.05$ and 0.1 for several sample sizes

(a) $\alpha = 0.01$

n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
5	0.4409	0.3450	1.8271	2.0460	1.9165	0.3526
10	0.3405	0.2570	2.5282	2.4175	1.9523	0.3506
15	0.2868	0.2186	2.8129	2.7170	1.9508	0.3488
20	0.2550	0.1945	2.9712	2.9352	1.9643	0.3463
25	0.2315	0.1786	3.1068	3.1416	1.9704	0.3450
30	0.2140	0.1652	3.2134	3.2596	1.993	0.346
40	0.1881	0.1436	3.2821	3.5385	1.9648	0.3442
50	0.1686	0.1324	3.3920	3.7668	1.9773	0.3393
100	0.1235	0.0972	3.5935	4.3471	1.9719	0.3448
150	0.1005	0.0807	3.6869	4.5253	1.9182	0.3334
200	0.0879	0.0718	3.7643	4.8550	1.9458	0.3398

(b) $\alpha = 0.05$

n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
5	0.3515	0.2926	1.5291	1.6360	1.4621	0.2336
10	0.2785	0.2215	1.9421	1.9410	1.3944	0.2271
15	0.2364	0.1876	2.1357	2.1360	1.3722	0.2225
20	0.2104	0.1658	2.2708	2.2958	1.3648	0.2237
25	0.1930	0.1515	2.3502	2.3985	1.3613	0.2265
30	0.1775	0.1397	2.4194	2.5002	1.3518	0.2220
40	0.1569	0.1234	2.4900	2.6396	1.3462	0.2247
50	0.1416	0.1127	2.5656	2.7928	1.3410	0.2229
100	0.1030	0.0835	2.7538	3.1105	1.3334	0.2231
150	0.0851	0.0700	2.8333	3.2746	1.3167	0.2210
200	0.0737	0.0615	2.9068	3.3824	1.3307	0.2240

(c) $\alpha = 0.1$

n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
5	0.3012	0.2566	1.3956	1.4427	1.2433	0.1819
10	0.2469	0.2012	1.7167	1.7301	1.1492	0.1795
15	0.2121	0.1713	1.8793	1.8936	1.1224	0.1769
20	0.1891	0.1526	1.9870	2.0251	1.1088	0.1767
25	0.1731	0.1393	2.0494	2.1160	1.0982	0.1775
30	0.1598	0.1285	2.1067	2.1969	1.0935	0.1763
40	0.141	0.1136	2.1900	2.2970	1.0893	0.1761
50	0.1273	0.1032	2.2524	2.3962	1.0867	0.1758
100	0.0928	0.0768	2.4127	2.6307	1.0711	0.1755
150	0.0769	0.0642	2.5098	2.7692	1.0690	0.1748
200	0.0669	0.0565	2.5633	2.8377	1.0696	0.1747

Table 5.3: Critical values for testing exponential distribution of the D, D_{sp}, D_e, D_{bi}, AD and CvM tests using BLUE at $\alpha = 0.01, 0.05$ and 0.1 for several sample sizes

(a) $\alpha = 0.01$

n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
5	0.3449	0.2246	1.6935	1.6832	0.9838	0.1960
10	0.3103	0.2109	2.4551	2.4579	1.3397	0.2612
15	0.2726	0.1905	2.8168	2.9110	1.4733	0.2856
20	0.2449	0.1756	3.0484	3.1963	1.5874	0.2975
25	0.2245	0.1646	3.2326	3.4526	1.6386	0.3035
30	0.2083	0.1532	3.3305	3.5633	1.7018	0.3103
40	0.1841	0.1369	3.3955	3.8211	1.7424	0.3176
50	0.1663	0.1267	3.5310	4.0456	1.7715	0.3203
100	0.1224	0.0954	3.7490	4.5665	1.8727	0.3323
150	0.0998	0.0800	3.8591	4.7378	1.8384	0.3244
200	0.0875	0.0709	3.9295	5.0245	1.8978	0.3321

(b) $\alpha = 0.05$

n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
5	0.2863	0.1879	1.2870	1.3823	0.7360	0.1420
10	0.2559	0.1746	1.8752	1.9501	0.9308	0.1769
15	0.2254	0.1577	2.1398	2.2560	1.0225	0.1914
20	0.2041	0.1455	2.3098	2.4693	1.0771	0.1992
25	0.1872	0.1351	2.4021	2.5957	1.1185	0.2047
30	0.1739	0.1266	2.4790	2.7217	1.1386	0.2064
40	0.1541	0.1143	2.5800	2.8657	1.1755	0.2107
50	0.1396	0.1055	2.6806	3.0425	1.1961	0.2143
100	0.1023	0.0801	2.8511	3.3564	1.2487	0.2170
150	0.0847	0.0677	2.9553	3.4836	1.2629	0.2164
200	0.0736	0.0599	3.0123	3.5975	1.2842	0.2198

(b) $\alpha = 0.1$

n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
5	0.2540	0.1669	1.1345	1.2057	0.6212	0.1162
10	0.2292	0.1561	1.6251	1.7080	0.7654	0.1426
15	0.2022	0.1420	1.8495	1.9617	0.8269	0.1510
20	0.1829	0.1307	1.9898	2.1331	0.8680	0.1576
25	0.1683	0.1217	2.0721	2.2499	0.9005	0.1621
30	0.1558	0.1145	2.1447	2.3472	0.9142	0.1632
40	0.1388	0.1036	2.2400	2.4707	0.9427	0.1658
50	0.1258	0.0958	2.3121	2.5789	0.9671	0.1681
100	0.0922	0.0727	2.4889	2.8282	1.0028	0.1709
150	0.0766	0.0617	2.5959	2.9628	1.0187	0.1719
200	0.0667	0.0545	2.6445	3.0116	1.0316	0.1733

Table 5.4: Critical values for testing exponential distribution of the D, D_{sp}, D_e, D_{bi}, AD and CvM tests using BLIE at $\alpha = 0.01, 0.05$ and 0.1 for several sample sizes

(a) $\alpha = 0.01$

n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
5	0.3975	0.2602	2.3136	1.9394	1.2168	0.2507
10	0.3285	0.2235	2.8737	2.6455	1.4542	0.2967
15	0.284	0.1984	3.1335	3.0082	1.5618	0.3124
20	0.2518	0.1814	3.2929	3.2414	1.6247	0.3169
25	0.2308	0.1679	3.4455	3.4467	1.6814	0.3238
30	0.2127	0.1566	3.5041	3.5692	1.7351	0.3272
40	0.1863	0.1386	3.5429	3.7632	1.7702	0.3307
50	0.1684	0.1278	3.6597	3.9588	1.804	0.3294
100	0.1231	0.0953	3.8198	4.4788	1.8677	0.3404
150	0.1001	0.0801	3.903	4.6847	1.8492	0.3269
200	0.0876	0.0708	3.9692	4.9646	1.8964	0.3375

(a) $\alpha = 0.05$

n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
5	0.3336	0.2197	1.8840	1.6261	0.9427	0.1885
10	0.2753	0.1866	2.2580	2.0770	1.0558	0.2073
15	0.2361	0.1639	2.4292	2.3167	1.1074	0.2136
20	0.2113	0.1496	2.5378	2.4796	1.1427	0.2166
25	0.1920	0.1377	2.5998	2.5907	1.1664	0.2180
30	0.1775	0.1286	2.6459	2.6890	1.1857	0.2189
40	0.1567	0.1153	2.7151	2.8344	1.2088	0.2198
50	0.1416	0.1060	2.7940	3.0026	1.2248	0.2226
100	0.1027	0.0800	2.9223	3.3090	1.2631	0.2203
150	0.0851	0.0677	3.0022	3.4491	1.2718	0.2196
200	0.0739	0.0599	3.0498	3.5663	1.2916	0.2219

(b) $\alpha = 0.1$

n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
5	0.2999	0.1989	1.6946	1.4159	0.8215	0.1569
10	0.2475	0.1678	1.9899	1.7993	0.8783	0.1693
15	0.2127	0.1480	2.1196	1.9982	0.9052	0.1702
20	0.1896	0.1343	2.2119	2.1350	0.9331	0.1742
25	0.1729	0.1238	2.2573	2.2336	0.9500	0.1734
30	0.1596	0.1161	2.3093	2.3245	0.9622	0.1735
40	0.1411	0.1045	2.3700	2.4445	0.9785	0.1742
50	0.1280	0.0959	2.4178	2.5426	0.9913	0.1747
100	0.0928	0.0727	2.5523	2.8017	1.0137	0.1746
150	0.0770	0.0616	2.6381	2.9377	1.0292	0.1741
200	0.0670	0.0545	2.6798	2.9924	1.0344	0.1753

5.5.1 The best choice of parameter estimation

As previously outlined, the three estimators (MLE, BLUE and BLIE) are derived in detail. The objective of this section is to find the best choice of the parameter estimation which gives the highest powers of the tests. Then, the selected estimator will be used in order to construct the simultaneous probability intervals for the ordered data associated with a considered test in the next section. Figures 5.1–5.3 illustrate plots of power comparisons for the D test using three estimators against the alternative distributions from Groups I, II and III, respectively. These figures are produced by using the powers of the D test in Tables 5.5–5.13. Also, plots of power comparisons for the other tests using three estimators are shown in Figures 5.4–5.21. From Figures 5.1–5.21, one can identify the best choice of the estimators that gives the highest power.

- For the D test (see Figures 5.1–5.3), in most situations over the alternative distributions considered, the powers of $D(3)$ are larger than those of $D(1)$ and $D(2)$. However, for some distributions ($\chi^2(1)$, $LogN(0,1)$, $Wbl(0,0.5,0.5)$ and $beta(0.5,3)$), $D(1)$ gives the most power out of all sample sizes.

- For the D_{sp} test (see Figures 5.4–5.6), it can be seen that $D_{sp}(3)$ provides superior power compared to $D_{sp}(1)$ and $D_{sp}(2)$ in most cases under Group I. However, $D_{sp}(1)$ has higher power than $D_{sp}(2)$ and $D_{sp}(3)$ against $\chi^2(1)$, $Wbl(0,0.5,0.5)$ and $beta(0.5,3)$. Also $D_{sp}(2)$ is the best choice against $LogN(0,1)$. Generally, $D_{sp}(3)$ for small sample sizes appears to have larger power than $D_{sp}(1)$ and $D_{sp}(2)$ over the alternatives from Group II and Group III, but the powers of all tests grow closer to one another as n increases.

- For the D_e test (see Figures 5.7–5.9), among the alternative distributions from Group I, $D_e(1)$ seems to be the most powerful for $\chi^2(1)$, $LogN(0,1)$, $Wbl(0,0.5,0.5)$ and $beta(0.5,0.5)$. Evidently, $D_e(2)$ has larger power than $D_e(1)$ and $D_e(3)$; however, the power of $D_e(2)$ is slightly greater than that of $D_e(3)$ by less than 1%.

- For the D_{be} test (see Figures 5.10–5.12), the $D_{be}(3)$ provides superior power when compared with $D_{be}(1)$ and $D_{be}(2)$ in most cases under Group I. However, $D_{be}(1)$ has more power than $D_{be}(2)$ and $D_{be}(3)$ against $\chi^2(1)$, $Wbl(0,0.5,0.5)$ and $beta(0.5,3)$. Also $D_{be}(2)$ is the best choice against $LogN(0,1)$. Generally,

$D_{be}(3)$ for small sample sizes ($n = 5, 10, 15$ and 20) appears to have greater power than $D_{be}(1)$ and $D_{be}(2)$ over the alternatives from Group II and Group III, but it can be observed that the powers of all tests are close to one another when n increases.

- For the D_{bi} test (see Figures 5.13–5.15), the results suggest that $D_{bi}(2)$ and $D_{bi}(3)$ are the most powerful in most situations when all distributions are considered. Nonetheless, $D_{be}(1)$ has greater power than $D_e(2)$ and $D_e(3)$ against $\chi^2(1)$, $Wbl(0, 0.5, 0.5)$, $beta(0.5, 3)$ and LogN(0,1).

- For the AD test (see Figures 5.16–5.18), it can be seen that $D_{bi}(2)$ and $D_{bi}(3)$ are the most powerful in most situations when all distributions are considered. On the other hand, AD(1) has more power than AD(2) and AD(3) against $\chi^2(1)$, $Wbl(0, 0.5, 0.5)$, $beta(0.5, 3)$ and $LogN(0, 1)$.

- For the CvM test (see Figures 5.19–5.21), CvM(1) seems to have the most power for $\chi^2(1)$, $LogN(0, 1)$, $Wbl(0, 0.5, 0.5)$ and $logistic(0, 1)$. For small sample sizes ($n = 5, 10, 15$ and 20), the powers of CvM(1) are substantially less than that of CvM(2) and CvM(3). In general, CvM(3) provides better power than CvM(1) and CvM(2) in most cases when all distributions are considered.

Based on the simulation study and the observations above, the following conclusions can be drawn.

1. MLE results in the highest power when the considered alternative distributions are $\chi^2(1)$, $LogN(0, 1)$, $Wbl(0, 0.5, 0.5)$ and $beta(0.5, 3)$.

2. In many cases, the BLUE and BLIE produce little difference in power.

3. For most alternative distributions, a test using BLIE is the most powerful test.

Overall, BLIE should therefore be used as the best estimator for testing the exponential distribution.

Table 5.5: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$\chi^2(1)$	5	14.24	15.85	11.34	**16.30**	13.64	9.58	15.17
	10	27.36	31.98	18.05	**32.79**	22.71	23.45	30.18
	15	38.42	**45.63**	24.08	42.77	27.9	37.88	42.69
	20	48.52	**57.75**	30.36	55.95	32.35	50.94	53.56
	25	56.31	**65.92**	37.13	65.62	36.96	61.87	62.69
	30	65.01	73.8	43.21	**74.25**	41.78	70.85	71.06
	40	77.23	**85.57**	60.14	85.2	53.4	84.4	82.61
	50	86.34	**92.31**	71.3	91.72	60.05	92.18	90.26
	100	99.15	99.75	97.47	99.7	91.91	**99.81**	99.58
	150	99.92	**100**	99.84	99.99	98.65	99.99	99.95
	200	**100**	**100**	**100**	**100**	99.94	**100**	**100**
$\chi^2(3)$	5	3.19	3.34	4.56	3	3.54	**5.90**	2.84
	10	4.02	2.72	5.83	3.17	6.15	**7.76**	2.99
	15	5.09	3.26	7.57	2.71	10.2	**10.50**	4.65
	20	6.25	3.81	8.61	3.4	**12.28**	12.24	6.17
	25	7.7	5.26	10.94	4.16	**16.20**	14.72	7.65
	30	9.9	7.47	13.26	6.06	**20.08**	18.5	10.72
	40	13.79	11.88	19.16	9.47	**26.46**	25.43	15.83
	50	19.79	17.04	24.92	12.54	30.98	**32.61**	22.3
	100	46.11	46.33	52.9	41.71	54.16	**65.83**	53.91
	150	69.81	70.7	74.65	64.1	72.41	**86.05**	78.29
	200	85.04	85.81	87.47	82.11	83.77	**94.99**	90.63
$\chi^2(4)$	5	2.44	2.93	4.2	2.53	2.85	**6.43**	1.94
	10	4.48	2.47	7.76	2.54	8.08	**11.61**	3.38
	15	8	4.11	11.63	3.01	15.7	**17.25**	7.56
	20	12.06	7.48	16.2	5.93	22.58	**23.27**	12.01
	25	16.5	11.75	22.59	9.58	30.68	**30.95**	17.97
	30	22.8	18.09	28.68	14.57	37.74	**38.80**	25.66
	40	34.7	30.89	42.54	24.33	50.59	**53.60**	39.62
	50	47.47	43.97	54.82	35.35	60.01	**67.66**	54.6
	100	87.63	86.93	89.01	83.87	89.44	**96.04**	92.69
	150	98.19	97.88	98.2	96.87	97.76	**99.77**	99.38
	200	99.75	99.67	99.7	99.63	99.57	**99.97**	99.92
$\chi^2(6)$	5	2.25	2.18	4.93	2.14	2.99	**7.58**	1.57
	10	6.81	3.04	10.96	2.83	12.22	**17.46**	5.68
	15	13.93	7.55	19.67	5.58	25.68	**29.18**	14.73
	20	23.89	15.93	30.3	12.48	38.87	**42.53**	25.79
	25	33.63	25.46	41.12	20.87	50.96	**54.14**	37.36
	30	45	37.47	51.74	31.67	61.08	**65.30**	50.82
	40	64.2	59.32	70.74	51.56	76.54	**82.28**	71.95
	50	78.68	74.68	81.61	67.22	84.88	**90.88**	85.33
	100	99.08	98.75	98.9	98.12	98.93	**99.80**	99.63
	150	99.98	99.98	99.99	99.97	99.98	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	2.28	2	5.25	2.05	3.12	**8.89**	1.33
	10	9.67	3.92	15.43	3.67	17.61	**25.12**	8.61
	15	23.17	12.68	30.47	10.06	37.97	**43.42**	24.99
	20	37.99	27.01	45.03	22.29	53.84	**59.77**	42.5
	25	51.84	42.58	58.62	35.87	67.37	**72.48**	58.2
	30	65.47	57.54	69.64	51.01	77.28	**82.44**	72.99
	40	83.11	78.72	85.75	72.95	89.53	**93.47**	89.06
	50	93.21	90.56	93.93	85.85	95.4	**97.93**	96.23
	100	99.92	99.84	99.89	99.83	99.9	**100**	99.99
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	7.96	6.42	6.97	6.25	7.87	5.75	**8.26**
	10	11.15	8.13	**12.34**	8.17	11.02	8.34	12.3
	15	13.21	8.84	**16.24**	7.94	11.53	10.13	15.22
	20	14.87	9.87	**19.74**	8.72	12.21	11.79	16.75
	25	15.89	10.15	**23.15**	9.3	12.25	12.94	17.95
	30	17.44	11.2	**25.81**	10.26	12.15	15.12	20.39
	40	20.62	13.27	**31.88**	11.4	13.73	18.65	23.05
	50	23.14	13.9	**36.87**	11.52	13.84	21.75	26.12
	100	38.11	26.82	**59.87**	25.12	22.15	41.06	42.9
	150	51.2	44.8	**76.54**	41.14	35.01	62.06	58.78
	200	64.74	62.51	**86.76**	59.74	47.73	78.08	72.09
$HN(0,1)$	5	2.46	3.3	4.1	3.13	2.85	**6.37**	2.08
	10	4.26	2.92	6.79	3.11	6.97	**10.54**	3.3
	15	6.79	3.69	9.34	3.06	12.79	**14.09**	6.37
	20	9.4	5.24	11.8	4.46	18.01	**18.18**	9.4
	25	12.42	7.39	14.6	6.02	**24.00**	21.9	12.88
	30	16.27	9.86	17.47	7.85	**29.50**	27.11	17.86
	40	22.76	16.11	22.75	11.61	**40.71**	37.12	26.67
	50	30.39	22.62	27.72	14.84	**49.00**	46.38	36.81
	100	63.49	59.05	50.24	48.66	78.34	**79.91**	74.47
	150	83.42	82.37	68.83	72.61	91.85	**94.06**	92.08
	200	93.9	93.58	81.13	88.87	96.76	**98.30**	97.57
$Wbl(0,2,2)$	5	2.16	1.75	5.61	1.45	3.04	**9.63**	1.18
	10	10.14	4.07	16.69	3.73	18.35	**26.48**	9.2
	15	24.2	13.05	31.1	10.25	38.23	**44.79**	26.36
	20	38.81	26.12	44.72	21.54	57.2	**61.79**	44.39
	25	52.52	40.86	58.35	34.59	71.4	**75.17**	61.27
	30	67.25	57.52	70.29	48.86	82.21	**85.29**	76.41
	40	83.9	79.04	84.84	71.64	92.52	**94.83**	90.95
	50	93.14	90.61	92.81	84.71	97.01	**98.48**	96.96
	100	99.97	99.95	99.93	99.93	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	28.88	30.07	24.85	30.8	27.83	20.85	**31.27**
	10	58.29	61.29	45.32	60.64	53.35	54.66	**62.56**
	15	76.02	79.36	62.61	77.87	67.53	76.83	**80.63**
	20	87.07	89.48	74.66	88.96	77.4	88.95	**90.81**
	25	92.64	94.98	84.28	94.56	84.51	95	**95.20**
	30	96.5	97.95	90.55	97.43	89.95	**97.99**	97.92
	40	99.06	99.51	97.35	99.48	96.02	**99.61**	99.51
	50	99.83	99.95	99.13	99.9	98.2	**99.97**	99.93
	100	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.6: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$U(0,1)$	5	3.34	2.13	8.57	2.08	4.72	**14.57**	1.71
	10	13.71	5.08	21.51	4.93	23.16	**33.40**	14.24
	15	27.09	11.64	33.08	9.15	**57.40**	51.09	32.61
	20	40.16	28.51	42.75	17.66	**83.24**	66.25	50.47
	25	51.58	53.84	52.61	31.15	**95.15**	77.99	64.41
	30	63.78	77.88	61.28	54.51	**98.92**	86.6	77.54
	40	79.29	97.3	75.66	88.25	**99.94**	95.93	91.35
	50	89.95	99.78	84.6	98.08	**100**	98.91	97.3
	100	99.91	**100**	99.15	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	2.78	1.56	8.56	1.47	4.15	**14.91**	1.24
	10	18.06	6.99	26.94	6.26	29.08	**42.15**	18.1
	15	39.81	21.22	47.28	16.75	63.61	**66.60**	46.68
	20	59.63	43.91	63.53	33.87	**86.44**	83.94	70.59
	25	73.73	67.42	76.51	54.14	**96.46**	92.43	84.61
	30	85.33	85.49	85.14	72.62	**99.07**	96.87	93.39
	40	95.52	98.35	94.9	94.44	**99.97**	99.58	98.99
	50	99.19	99.89	98.82	99.32	**99.99**	99.96	99.87
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	1.99	1.94	5.08	1.88	2.81	**8.72**	1.11
	10	8.3	3.53	13.59	3.01	14.92	**21.40**	7.21
	15	17.68	8.99	23.58	7.5	30.98	**36.82**	19.59
	20	30.31	18.74	36.03	14.98	49.74	**53.45**	34.55
	25	42.91	31.28	48.63	24.56	65.37	**67.45**	50.51
	30	55.16	44.97	58.76	36.81	75.67	**77.30**	65.03
	40	74.04	68.26	75.77	58.18	89.6	**90.93**	83.61
	50	86.12	83.87	85.85	73.83	95.69	**96.36**	92.98
	100	99.7	99.88	99.61	99.71	99.99	**99.99**	99.95
	150	99.99	100	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	6.37	1.19	17.2	1.01	9.58	**28.60**	1.96
	10	44.24	22.48	55.62	21.12	61.17	**73.92**	50.36
	15	77.36	58.82	81.76	50.37	**94.37**	93.43	85.74
	20	91.82	89.12	92.96	79.7	**99.52**	98.64	96.54
	25	97.51	98.36	97.51	94.74	**99.99**	99.79	99.4
	30	99.41	99.85	99.26	99.12	**100**	99.96	99.9
	40	99.98	100	99.99	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	6.43	5.41	13.05	5.26	7.68	**17.45**	4.63
	10	14.49	11.61	22.35	12.2	23.46	**30.64**	17.42
	15	24.4	19.22	28.81	16.55	**55.14**	44.16	31.87
	20	33.67	38.17	35.19	25.5	**78.87**	56.5	44.33
	25	41.41	60.23	41.9	41.61	**91.07**	67.87	54.9
	30	52.12	80.99	48.54	63.36	**96.55**	78.59	68.51
	40	66.92	97.18	61.82	90.15	**99.70**	90.8	81.93
	50	80.18	99.75	72.65	98.06	**99.98**	96.97	92.39
	100	99.32	**100**	97.1	**100**	**100**	**100**	99.94
	150	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	11.51	**13.78**	9.18	12.48	11.03	8.84	12.16
	10	18.12	24.33	8.99	**25.02**	14.23	15.99	19.99
	15	25.12	**35.13**	11.25	33.86	16.58	25.71	28.23
	20	31.75	44.36	12.53	**44.66**	18.47	34.07	35.13
	25	36.89	51.46	15.28	**52.67**	21	42.93	40.91
	30	44.76	60.8	19.81	**61.69**	24.74	52.96	50.21
	40	56.29	**73.86**	30.71	73.67	31.66	67.43	61.26
	50	66.2	**82.35**	41.23	81.14	36.69	78.33	71.4
	100	92.9	98.24	83.93	**98.50**	71.39	97.9	94.63
	150	98.9	99.83	97.44	**99.88**	91.77	99.85	99.36
	200	99.83	99.97	99.69	**100**	98.23	99.99	99.88
$beta(1,2)$	5	2.74	2.98	4.78	2.8	3.16	**7.47**	2.15
	10	4.87	3.16	7.99	3.56	8.13	**12.73**	4.09
	15	8.8	4.81	11.88	3.95	17.38	**18.39**	9.13
	20	12.56	6.62	14.85	5.54	**28.08**	24.43	13.53
	25	16	9.29	17.74	7.68	**41.05**	30.52	18.22
	30	22.27	14.91	21.73	9.98	**53.90**	37.36	26.41
	40	30.53	29.25	28.3	16.31	**74.56**	49.41	37.89
	50	38.91	45.5	32.75	25.2	**87.08**	59.52	49.24
	100	75.93	98	58.81	91.27	**99.95**	92.35	87.8
	150	93.31	**100**	78.84	99.85	**100**	99.22	98.48
	200	98.49	**100**	90.58	**100**	**100**	99.94	99.91

The bold number is the highest power among the seven tests for each sample size.

Table 5.7: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$Laplace(0,1)$	5	5.7	1.39	13.77	1.31	8.49	**22.23**	2.28
	10	40.73	24.68	49.58	22.96	51.66	**60.76**	41.16
	15	72.89	59.64	77.3	53.86	79.8	**84.67**	75.28
	20	88.48	81.08	90.11	78.01	91.12	**94.04**	90.27
	25	94.92	91.27	95.94	90.31	96.24	**97.90**	96.3
	30	98	96.49	98.1	95.95	98.27	**99.23**	98.71
	40	99.81	99.58	99.77	99.38	99.77	**99.95**	99.9
	50	99.98	99.96	99.97	99.93	99.98	**99.99**	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	3.98	1.47	10.89	1.36	6.16	**18.50**	1.61
	10	31.84	16.9	41.19	15.46	43.76	54.96	**67.43**
	15	62.41	47.33	68.33	41.19	73.72	80.07	**85.83**
	20	81.67	70.86	84.01	66.58	87.98	91.67	**93.88**
	25	90.46	85.26	91.61	82.71	94.56	96.72	**97.64**
	30	95.86	93.45	96.21	91.81	97.53	98.63	**99.80**
	40	99.52	98.91	99.45	98.35	99.65	99.86	**99.98**
	50	99.94	99.82	99.93	99.68	99.92	99.98	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	21.57	5.01	29.66	4.88	25.17	**35.30**	15.57
	10	62.45	50.65	68.29	49.45	70	**71.57**	62.44
	15	85.22	79.88	88.49	76.38	89.4	**89.73**	86.38
	20	94.23	92.07	95.77	90.39	96.02	**96.35**	95
	25	97.6	96.71	98.4	96.24	98.48	**98.91**	98.23
	30	98.98	98.66	99.39	98.71	99.29	**99.59**	99.36
	40	99.89	99.87	99.92	99.76	99.9	**99.97**	99.94
	50	99.98	99.96	99.98	99.95	99.97	**99.99**	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	3.38	1.33	9.46	1.24	5.16	**16.55**	1.07
	10	26	11.82	35.9	10.93	38.75	**51.48**	27.58
	15	55.1	37.89	61.61	32.52	70.72	**76.74**	61.41
	20	76.43	64.34	79.74	57.58	87.52	**90.61**	82.9
	25	88.01	80.91	89.52	75.85	94.61	**96.20**	92.66
	30	94.97	91.4	94.94	88.05	98.02	**98.74**	97.48
	40	99.11	98.64	99.15	97.54	99.84	**99.84**	99.69
	50	99.82	99.76	99.69	99.5	99.96	**99.99**	99.94
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	21.98	5.4	30.37	4.98	25.69	**35.07**	15.99
	10	62.93	50.39	68.64	49.67	70.29	**71.54**	62.93
	15	85.29	79.6	88.39	76.39	**89.77**	89.75	86.46
	20	94.07	91.89	95.54	90.41	95.85	**96.12**	95.11
	25	97.43	96.76	98.24	96.16	98.3	**98.60**	97.96
	30	99.16	98.88	99.47	98.58	99.42	**99.69**	99.41
	40	99.91	99.83	99.94	99.8	99.91	**99.96**	99.95
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	5.94	1.52	14.26	1.44	8.67	**21.95**	2.39
	10	38.77	24.25	47.49	23.38	49.36	**58.65**	39.45
	15	68.02	55.85	73.13	51.67	76.51	**80.97**	71.11
	20	85.43	78.61	87.49	74.95	89.23	**92.39**	87.72
	25	93.38	90.31	94.67	87.61	95.5	**97.01**	95.06
	30	97.27	95.72	97.62	94.51	97.8	**98.87**	98.06
	40	99.57	99.32	99.64	98.93	99.62	**99.84**	99.77
	50	99.89	99.83	99.84	99.78	99.86	**99.95**	99.92
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	4.79	1.23	12.32	1.23	7.11	**19.68**	1.91
	10	36.12	21.07	45.4	20.39	47.53	**57.34**	37.11
	15	65.31	51.92	70.54	47.43	75.01	**79.72**	68.79
	20	83.65	75.91	85.7	72.04	88.65	**91.50**	86.81
	25	92.83	88.48	94.02	85.73	95.2	**96.93**	94.74
	30	96.81	94.54	97.14	93.56	97.66	**98.86**	97.94
	40	99.41	99.03	99.48	98.65	99.52	**99.87**	99.73
	50	99.95	99.85	99.94	99.8	99.94	**100**	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	3.92	1.24	11.06	1.26	5.99	**18.88**	1.49
	10	33.15	17.83	42.26	17.11	44.7	**55.41**	34
	15	62.78	47.27	68.42	41.6	73.35	**79.34**	67.04
	20	81.33	71.93	84.07	67.82	88.01	**91.43**	85.39
	25	91.69	86.58	93.05	83.23	94.82	**96.80**	94.19
	30	96.55	94.02	96.81	91.82	97.73	**98.75**	97.81
	40	99.49	99.02	99.46	98.5	99.58	**99.83**	99.69
	50	99.97	99.88	99.96	99.64	99.94	**99.99**	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.8: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$\chi^2(1)$	5	7.91	8.35	4.63	8.19	6.71	**10.00**	8.26
	10	17.34	17.84	10.46	16.29	10.36	**23.07**	19.25
	15	26.92	27.68	14.92	25.05	11.34	**36.48**	30.68
	20	37.12	38.32	19.34	35.13	12.77	**49.26**	41.48
	25	45.64	47.45	23.98	47.36	15.46	**59.83**	51.36
	30	54.96	57.54	29.28	58.18	17.71	**69.49**	61.07
	40	70.09	73.98	41.24	72.41	26.44	**83.52**	75.64
	50	80.96	83.8	52.3	83.58	32.21	**91.50**	85.55
	100	98.66	99.28	92.86	99.31	76.52	**99.77**	99.29
	150	99.82	99.95	99.24	99.99	95.53	**99.98**	99.95
	200	**100**	**100**	**100**	**100**	99.66	**100**	**100**
$\chi^2(3)$	5	5.59	5.42	**6.42**	5.77	5.83	5.33	5.74
	10	7.41	7.76	8.68	8	**9.24**	6.77	7.59
	15	9.18	10.38	10.86	10.49	**12.99**	9.25	10.16
	20	10.27	11.79	11.9	12.23	**14.43**	10.43	11.73
	25	12.49	14.8	14.76	15.56	**18.18**	12.97	14.13
	30	14.45	18.81	18.57	18.84	**21.95**	16.17	17.56
	40	19.65	25.07	24.29	24.7	**28.30**	23.07	23.94
	50	25.72	31.48	29.89	31.14	**31.86**	29.95	29.78
	100	52.33	60.67	57.72	61.24	53.84	**63.67**	61.94
	150	73.88	80.45	77.32	80.28	72.66	**84.75**	82.56
	200	87.58	91.06	88.73	91.36	83.27	**94.44**	93.05
$\chi^2(4)$	5	5.91	5.59	**7.44**	6.69	6.17	5.35	6.12
	10	9.87	10.45	11.53	10.42	**12.26**	9.55	11.31
	15	14.66	16.32	17.06	16.98	**19.62**	14.68	16.36
	20	19.19	21.67	22.13	22.71	**25.72**	20.64	22.42
	25	24.64	29.05	29.14	29.39	**33.80**	27.78	29.36
	30	31.06	36.25	35.65	37.22	**39.32**	35.75	37.27
	40	43.19	49.77	48.58	49.65	**51.97**	50.41	51.1
	50	55.93	61.83	59.84	61.64	60.02	**64.87**	64.27
	100	90.76	92.41	90.97	92.92	88.85	**95.48**	95.06
	150	98.67	98.98	98.34	98.9	97.57	**99.71**	99.62
	200	99.83	99.86	99.7	99.87	99.46	**99.97**	99.94
$\chi^2(6)$	5	7.07	6.81	**8.55**	7.67	7.25	6.56	7.32
	10	14.83	15.36	17.13	15.77	**18.36**	14.81	16.89
	15	23.9	26.42	27.22	27.22	**30.93**	25.58	27.98
	20	34.27	37.38	37.61	36.93	**41.80**	38.41	40.93
	25	44.62	49.03	48.93	48.86	**53.30**	50.52	52.39
	30	54.8	59.9	58.66	59.7	61.97	62.18	**63.69**
	40	72.63	76.28	75.14	74.95	76.73	80.16	**80.70**
	50	83.9	86.01	84.1	85.35	84.36	89.59	**89.86**
	100	99.39	99.37	99.08	99.33	98.75	99.74	**99.75**
	150	99.99	99.99	99.99	99.99	99.95	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	8.42	8.06	**10.32**	8.86	8.86	7.55	8.55
	10	20.65	21.31	23.28	22.2	**25.10**	21.37	24.11
	15	35.74	37.56	38.7	38.33	**42.65**	39.46	42.04
	20	50.19	52.42	52.71	52.02	56.78	56.08	**58.20**
	25	63.13	65.89	65.28	65.36	69.11	69.56	**71.19**
	30	73.78	76.06	74.74	76.25	77.79	80.27	**81.47**
	40	88.01	88.94	87.92	88.44	89.34	92.68	**93.07**
	50	95.23	95.79	94.83	95.3	95.04	**97.60**	97.58
	100	99.95	99.95	99.91	99.98	99.88	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	6.41	6.34	4.97	6.63	6.51	**6.67**	6.32
	10	9.84	9.38	9.92	9.31	8.16	**11.56**	10.69
	15	11.83	11.01	13.34	11.17	8.97	**13.57**	12.75
	20	13.48	12.77	**17.45**	12.75	9.74	15.64	15.17
	25	15.06	13.66	**20.56**	13.85	9.99	17.31	16.73
	30	16.65	15.03	**23.45**	15.36	10.2	19.37	18.88
	40	19.8	17.74	**29.71**	17.82	11.73	22.62	22.59
	50	22.76	19.97	**34.63**	20.49	12.36	25.67	25.32
	100	37.93	37.18	**60.02**	39.64	22.02	45.03	43.9
	150	51.55	56.12	**76.62**	57.98	36.96	64.12	60.73
	200	65.26	71.8	**87.64**	74.84	49.14	79.44	74.33
$HN(0,1)$	5	5.72	5.51	**7.15**	6.1	5.99	5.36	5.99
	10	9.12	9.05	10.43	9.5	**10.68**	8.75	10.04
	15	11.93	12.09	12.99	12.45	**15.69**	11.88	13.42
	20	14.78	15.16	15.42	14.92	**20.20**	15.87	17.43
	25	18.36	18.76	18.18	18.4	**25.02**	19.37	21.67
	30	22.07	22.8	21.08	22.27	**29.73**	24.4	26.78
	40	29.15	29.65	25.74	27.5	**39.73**	34.01	36.35
	50	36.9	37.38	29.6	34.93	**46.25**	43.45	45.37
	100	67.84	70.17	51.62	67.85	75.48	77.99	**79.35**
	150	85.65	88.24	68.05	85.67	90.48	93.15	**93.84**
	200	94.63	95.89	80.03	95.08	95.68	98.05	**98.13**
$Wbl(0,2,2)$	5	9.13	8.61	**11.43**	10.1	9.65	7.97	9.59
	10	21.59	22.29	24.73	22.87	**26.39**	22.96	25.65
	15	36.51	37.51	38.75	37.95	43.36	40.56	**43.44**
	20	50.63	52.51	51.92	52.52	59.58	58.31	**60.06**
	25	63.85	65.68	64.1	65.58	72.51	72.28	**73.98**
	30	75.48	77.71	75.21	76.33	82.06	83.52	**84.28**
	40	88.71	89.8	87.07	88.65	92.14	94.15	**94.36**
	50	95.32	95.95	93.51	95.39	96.35	98.07	**98.19**
	100	99.97	**100**	99.95	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	18.23	19.24	10.1	18.03	15.07	**22.11**	18.72
	10	45.8	46.82	32.72	43.59	33.6	**55.01**	49.86
	15	66.65	67.14	49.41	65.16	45.92	**76.44**	71.7
	20	80.9	81.41	61.54	80.56	55.88	**89.07**	85.08
	25	88.97	89.54	73.4	89.63	65.53	**94.82**	92.16
	30	94.23	95.2	81.71	94.76	73.93	**97.91**	96.2
	40	98.45	98.82	93.07	98.8	87.25	**99.55**	99.16
	50	99.64	99.72	97.78	99.77	93.8	**99.96**	99.82
	100	**100**	**100**	**100**	**100**	99.96	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.9: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$U(0,1)$	5	12.07	11.69	**15.41**	13.04	12.54	12.58	13.74
	10	25.21	23.8	27.91	23.33	**31.89**	30.55	31.82
	15	38.09	38.7	37.72	35.6	**61.99**	48.25	49.02
	20	49.02	60.02	46.2	51.95	**84.12**	63.8	63.76
	25	59.92	80.01	55.04	71.66	**95.11**	76.03	75.42
	30	70.35	91.99	62.93	86.93	**98.78**	85.24	84.43
	40	83.57	99.17	75.52	97.93	**99.93**	95.26	94.49
	50	92.22	99.96	83.27	99.83	**100**	98.76	98.17
	100	99.92	100	98.98	**100**	**100**	**100**	99.99
	150	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	13.51	12.79	**17.04**	13.66	14.13	12.65	14.52
	10	32.98	31.74	35.9	32.54	38.03	38.3	**40.62**
	15	53.16	53.42	54.13	51.81	**67.93**	63.57	64.87
	20	69.38	73.19	68.5	70.02	**87.26**	81.85	82.39
	25	81.26	87.72	79.86	84.83	**96.28**	91.33	91.7
	30	89.82	95.64	87.36	93.51	**99.01**	96.26	96.23
	40	97.16	99.7	95.34	99.09	**99.96**	99.54	99.51
	50	99.49	99.98	98.84	99.98	**99.99**	99.95	99.91
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	8.63	8.17	**10.86**	9.31	9.17	7.29	8.49
	10	18.05	18.45	20.53	18.78	**21.67**	18.46	20.96
	15	28.75	29.2	30.92	31.36	**36.07**	32.95	34.99
	20	41.3	43.8	43.44	43.11	**52.46**	49.61	51.6
	25	54.02	57.02	54.97	55.76	**66.90**	63.67	65.22
	30	64.55	68.24	64.6	67.16	**75.40**	74.45	75.31
	40	80.2	84.02	78.77	82.16	89.11	89.37	**89.66**
	50	89.99	92.39	87.17	91.48	95.08	**95.60**	95.48
	100	99.83	99.93	99.61	99.96	99.97	**99.99**	99.99
	150	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	24.61	23.77	**28.64**	25.49	25.49	25.82	27.94
	10	61.11	59.47	64.07	59.89	70.79	71.3	**72.56**
	15	85.39	87.24	85.29	85.21	**95.49**	92.48	92.68
	20	94.81	97.7	94.14	96.54	**99.53**	98.39	98.38
	25	98.5	99.74	97.91	99.51	**99.99**	99.75	99.72
	30	99.68	99.97	99.38	99.94	**100**	99.96	99.96
	40	99.99	**100**	99.97	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	12.81	12.61	**17.79**	12.68	12.94	14.88	15.29
	10	20.78	18.85	23.68	18.43	**28.80**	27.62	26.73
	15	28.77	33.46	28	27.64	**56.65**	40.79	37.25
	20	36.1	54.73	32.07	43.92	**78.27**	52.79	47.7
	25	44.55	74.29	37.4	64.17	**90.14**	64.42	57.95
	30	53.7	87.74	43.72	80.9	**95.90**	75.6	68.57
	40	68.07	98.07	54.26	96.06	**99.56**	89.11	82.31
	50	81.21	99.79	64.4	99.49	**99.97**	96.34	91.65
	100	99.36	**100**	95.21	**100**	**100**	**100**	99.93
	150	**100**	**100**	99.85	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	6.98	7.44	4.66	6.85	5.86	**8.19**	6.56
	10	10.59	10.81	4.83	9.89	5.49	**14.12**	11.61
	15	15.78	16.58	5.86	15.66	5.37	**22.33**	17.37
	20	21.02	22.99	5.7	22.23	5.29	**30.81**	23.64
	25	26.62	30.2	6.77	29.41	6.39	**38.73**	30.21
	30	33.9	39.61	8.72	38.66	7.72	**49.06**	38.56
	40	46.3	53.7	13.5	53.18	11.28	**63.49**	51.16
	50	57.26	65.61	19.25	65.31	13.3	**75.12**	62.14
	100	89.68	95.72	67.81	96.03	44.7	**97.31**	92.64
	150	98.3	99.42	92.67	99.65	78.16	**99.84**	99.05
	200	99.77	99.95	99.02	99.99	93.85	**99.99**	99.85
$beta(1,2)$	5	6.99	6.73	**8.62**	7.66	7.35	6.35	7.11
	10	10.38	10.17	12.1	10.46	**12.33**	10.93	11.98
	15	14.8	14.36	15.55	14.55	**20.33**	16.19	17.31
	20	18.4	18.43	17.97	18.01	**29.81**	21.52	23.22
	25	22.35	24.24	20.9	23.16	**41.54**	27.27	28.86
	30	28.52	32.71	25.35	27.93	**52.72**	34.36	35.8
	40	37.13	48.47	30.3	38.97	**73.08**	46.14	47.1
	50	45.09	62.78	33.66	54.54	**85.34**	56.36	56.81
	100	78.89	99.16	58.43	97.47	**99.94**	91.3	90.47
	150	94.36	**100**	76.52	99.98	**100**	99.11	98.83
	200	98.8	**100**	88.94	**100**	**100**	99.94	99.91

The bold number is the highest power among the seven tests for each sample size.

Table 5.10: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$Laplace(0,1)$	5	20.9	20.07	**23.74**	21.88	21.61	20.01	21.97
	10	55.55	55.94	58.29	57.09	58.99	58.21	**60.71**
	15	80.83	80.8	81.9	80.54	81.85	82.88	**84.91**
	20	91.95	91.85	92.02	91.69	91.7	93.43	**94.04**
	25	96.66	96.75	96.79	96.85	96.28	97.52	**97.87**
	30	98.66	98.61	98.57	98.9	98.15	99.14	**99.27**
	40	99.9	99.86	99.86	99.81	99.73	99.94	**99.94**
	50	99.98	99.97	99.97	99.98	99.97	**99.99**	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	17.27	16.57	**20.00**	18.18	17.92	16.42	18.36
	10	47.54	47.48	50.46	48.56	51.9	51.79	**54.29**
	15	73.09	72.65	73.42	72.46	76.46	78.13	**79.62**
	20	86.75	86.99	86.96	86.58	88.55	90.46	**91.31**
	25	93.82	93.8	93.39	94.07	94.69	96.22	**96.53**
	30	97.12	97.29	96.89	97.46	94.69	98.43	**98.58**
	40	99.66	99.61	99.52	99.51	97.41	99.85	**99.90**
	50	99.96	99.94	99.9	99.95	99.65	99.98	**99.98**
	100	**100**	**100**	**100**	**100**	**99.91**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	35.49	34.81	35.93	36.66	36.11	35.45	**36.75**
	10	71.47	72.01	72.37	72.43	**73.72**	72.07	73.61
	15	88.99	90.05	90.12	89.97	90.24	90.34	**90.93**
	20	95.88	96.24	96.23	96.06	96.14	96.65	**96.87**
	25	98.4	98.58	98.63	98.55	98.39	99.03	**99.07**
	30	99.29	99.47	99.56	99.58	99.25	99.61	**99.65**
	40	99.92	99.93	99.93	99.91	99.9	99.96	**99.99**
	50	99.98	99.97	99.98	**99.99**	99.95	**99.99**	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	15.64	15.03	**18.65**	16.49	16.37	14.28	16.16
	10	43.02	42.65	46.11	43.33	47.56	47.92	**50.30**
	15	67.41	67.1	68.27	67.03	73.8	74.01	**75.65**
	20	83.69	84.07	83.45	83.41	88.11	89.14	**89.95**
	25	91.97	92.63	91.57	92.16	94.76	95.54	**95.93**
	30	96.59	96.91	96.01	96.61	97.87	98.5	**98.71**
	40	99.44	99.59	99.34	99.5	99.81	99.81	**99.83**
	50	99.87	99.94	99.74	99.91	99.97	99.96	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	35.69	35.19	35.99	36.27	36.4	35.39	**36.61**
	10	71.62	72.16	72.36	73.06	**73.99**	72.14	73.95
	15	89.29	90.1	89.74	89.19	90.37	90.46	**90.85**
	20	95.74	96.08	96.03	96	95.76	96.38	**96.58**
	25	98.13	98.48	98.45	98.53	98.32	98.71	**98.79**
	30	99.44	99.54	99.56	99.55	99.36	**99.70**	99.68
	40	99.94	99.94	99.94	99.93	99.88	**99.96**	99.96
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	20.66	19	**23.40**	21.6	21.48	19.81	21.69
	10	53.44	53.4	56.05	55.29	57.15	55.8	**58.53**
	15	76.47	77.15	77.9	77.52	78.98	79.28	**81.09**
	20	89.6	89.69	89.88	89.19	89.73	91.68	**92.49**
	25	95.43	95.61	95.69	95.82	95.62	96.74	**97.05**
	30	98.12	98.11	98	97.93	97.73	98.72	**98.91**
	40	99.76	99.73	99.69	99.66	99.55	99.83	**99.87**
	50	99.91	99.9	99.86	99.95	99.87	99.94	**99.96**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	18.7	18.03	**21.61**	20.54	19.58	17.45	19.55
	10	51.77	51.6	54.76	51.66	55.21	54.34	**57.10**
	15	74.57	74.8	75.83	75.57	77.49	77.91	**79.33**
	20	88.11	88.14	88.1	87.78	89.12	90.73	**91.53**
	25	95.07	95.25	95.35	95.02	95.31	96.62	**96.98**
	30	97.82	97.78	97.56	98.01	97.49	98.65	**98.88**
	40	99.62	99.6	99.54	99.53	99.48	99.86	**99.87**
	50	99.99	99.97	99.94	99.93	99.93	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	17.72	17.07	**20.79**	18.49	18.61	16.3	18.71
	10	48.97	48.83	51.84	49.4	52.58	51.82	**54.66**
	15	73.02	72.89	73.94	72.55	76.13	76.9	**78.69**
	20	87.16		87.1	86.66	88.51	90.47	**91.27**
	25	94.55	94.47	94.45	94.41	94.82	96.39	**96.74**
	30	97.7	97.77	97.53	97.36	97.56	98.52	**98.76**
	40	99.65	99.59	99.5	99.64	99.53	99.79	**99.82**
	50	99.98	99.96	99.94	99.93	99.92	**99.99**	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.11: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$\chi^2(1)$	5	3.88	3.93	6.43	4.22	3.92	**6.91**	4.38
	10	10.66	10.95	11.78	10.29	5.98	**17.95**	12.18
	15	19.85	21.14	15.45	19.33	8.04	**31.62**	22.77
	20	29.58	31.39	19.63	29.14	10.39	**44.49**	34.35
	25	39.12	42.05	23.29	41.13	13.56	**55.70**	44.23
	30	48.89	52.44	28.02	52.94	16.41	**66.29**	55.15
	40	65.77	70.48	38.14	68.94	25.37	**81.21**	71.65
	50	77.49	81.98	47.76	81.63	32.17	**90.29**	82.33
	100	98.44	99.19	91.11	99.25	77.25	**99.76**	99.16
	150	99.8	99.95	99.04	99.98	95.65	99.97	99.95
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	6.71	6.68	6.03	6.5	6.72	6.55	**7.13**
	10	9.14	9.38	8.09	9.31	**9.99**	8.35	9.85
	15	11.41	12.33	10.13	12	**13.27**	11.11	12.82
	20	12.79	13.49	11.58	14.08	**15.26**	12.9	14.53
	25	14.86	16.54	14.33	17.05	**18.98**	15.67	17.36
	30	17.48	20.55	18.15	20.5	**23.20**	19.07	20.93
	40	22.36	27.02	23.75	26.2	**29.30**	26.17	27.44
	50	28.63	33.3	29.57	33.12	32.83	32.79	**33.31**
	100	55.11	62.31	57.49	62.72	55.78	**66.06**	64.9
	150	75.82	81.42	77.27	80.94	73.85	**85.92**	84.15
	200	88.34	91.63	88.78	91.78	84.15	**94.96**	93.84
$\chi^2(4)$	5	7.53	7.52	6.83	**8.40**	7.55	7.31	8.38
	10	12.49	12.71	10.84	12.52	13.65	12.47	**14.34**
	15	17.87	18.81	15.87	19.37	20.18	18.21	**20.34**
	20	22.79	24.45	21.38	25.45	26.9	24.29	**27.25**
	25	28.83	31.47	28.2	31.62	**34.64**	32.04	34.48
	30	35.45	38.87	35	39.77	41.44	39.55	**42.07**
	40	47.26	52.1	48.1	52	53.31	54.19	**55.67**
	50	59.32	64.21	59.55	63.68	62.12	67.96	**68.35**
	100	91.96	93.2	90.83	93.39	89.79	**96.13**	95.92
	150	98.9	99.05	98.33	98.97	97.78	**99.76**	99.67
	200	99.86	99.85	99.71	99.88	99.48	**99.97**	99.94
$\chi^2(6)$	5	9.01	8.76	7.76	9.82	9.09	8.82	**9.91**
	10	18.46	18.81	15.91	18.81	20.24	18.83	**21.69**
	15	28.86	29.97	25.94	30.47	31.66	30.47	**33.50**
	20	39.04	40.25	36.76	40.41	42.5	43.67	**46.71**
	25	49.3	52.05	47.95	51.09	54.19	55.2	**57.70**
	30	59.48	62.14	58.25	62.56	63.71	66.24	**68.38**
	40	75.66	78.24	74.9	76.43	78	82.77	**83.69**
	50	86.15	86.82	83.99	86.31	85.44	91.07	**91.42**
	100	99.54	99.47	99.07	99.45	98.92	99.8	**99.81**
	150	99.99	99.99	99.99	**100**	99.96	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	11.25	10.8	9.24	11.59	11.31	10.15	11.63
	10	25.03	25.23	21.74	25.71	26.69	26.66	**29.69**
	15	40.73	41.76	37.59	42.23	42.87	45.36	**48.66**
	20	55.33	55.86	51.79	55.47	57.69	61.25	**63.83**
	25	67.44	68.15	64.6	67.48	69.71	73.69	**75.90**
	30	77.25	77.87	74.37	78.26	78.83	83.11	**84.53**
	40	89.81	89.72	87.86	89.06	89.67	93.82	**94.33**
	50	95.93	96.11	94.85	95.74	95.32	98.07	**98.18**
	100	99.97	99.96	99.91	99.98	99.89	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	4.94	4.86	5.46	5.07	4.92	**5.60**	4.98
	10	6.6	6.45	**10.54**	6.48	6.05	8.55	7.11
	15	8.41	8.25	**13.68**	8.24	7.28	10.84	9.22
	20	9.98	9.73	**17.84**	9.63	8.53	12.6	11.09
	25	11.5	10.76	**20.89**	10.67	8.9	13.91	12.43
	30	13.09	12.49	**23.86**	12.69	10.04	16.17	14.93
	40	16.56	15.92	**29.88**	15.67	11.7	19.52	18.62
	50	19.22	18.29	**34.85**	18.23	12.62	22.86	21.12
	100	34.56	36.09	**59.85**	38.55	23.17	41.63	40.03
	150	48.77	55.58	**76.67**	57.09	37.67	62.01	57.11
	200	62.53	71.6	**87.66**	74.74	50.32	77.89	71.49
$HN(0,1)$	5	7.71	7.56	6.47	**8.05**	7.73	7.14	8.01
	10	11.95	11.51	9.81	11.89	11.87	11.56	**13.63**
	15	15.4	14.68	12.49	14.9	15.01	15.28	**17.56**
	20	18.66	17.42	15.07	17.37	19.53	19.5	**21.97**
	25	22.44	20.66	17.79	20.32	23.72	23.84	**26.89**
	30	26.36	24.34	20.92	24.39	28.17	28.53	**31.49**
	40	33.68	31.05	25.74	29.35	37.97	38.52	**42.03**
	50	41.27	38.31	29.78	35.99	44.46	47.83	**50.45**
	100	71.14	70.53	51.8	68.05	74.89	80.67	**82.66**
	150	87.36	87.97	68.69	85.55	90.08	94.22	**94.95**
	200	95.33	95.82	80.58	95.21	95.53	98.39	**98.51**
$Wbl(0,2,2)$	5	12.13	11.97	10.16	13.08	12.18	11.35	**13.10**
	10	26.38	26.68	23.32	27.15	27.95	28.51	**31.66**
	15	42	41.66	37.49	42.47	42.88	46.8	**49.91**
	20	56.21	55.18	51.39	56.12	58.68	63.61	**66.42**
	25	69.1	67.8	63.63	68.26	71.4	76.72	**78.60**
	30	79.46	78.94	75.08	78.38	81.81	86.19	**87.49**
	40	90.65	90.41	86.89	89.52	92.1	95.26	**95.75**
	50	96.17	96.11	93.6	95.57	96.25	98.61	**98.75**
	100	99.99	**100**	99.95	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	8.09	8.08	14.47	8.58	8.09	**15.83**	9.42
	10	34.64	35.48	33.86	33.55	23.84	**48.41**	38.89
	15	58.36	59.64	48.38	57.67	39.07	**72.17**	64.11
	20	75.35	76.39	59.78	75.66	51.32	**86.42**	80
	25	85.75	86.82	70.47	86.84	62.04	**93.62**	89.27
	30	92.51	93.55	79.13	93.13	72.78	**97.25**	94.6
	40	97.97	98.55	91.08	98.43	86.55	**99.45**	98.89
	50	99.52	99.64	97.03	99.67	93.63	**99.93**	99.71
	100	**100**	**100**	**100**	**100**	99.96	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.12: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$U(0,1)$	5	15.67	15.55	14.28	16.24	15.71	16.46	**17.66**
	10	31.33	29.25	27.39	28.75	27.13	37.18	**38.87**
	15	45.01	40.55	37.26	39.61	47.03	55.29	**56.27**
	20	56.22	52.96	46.61	50.21	**75.36**	69.99	70.11
	25	66.49	72.22	55.16	64.58	**91.51**	81.01	80.71
	30	76.06	87.6	63.65	80.84	**97.90**	88.46	88.06
	40	86.99	98.51	76.27	96.24	**99.93**	96.62	96.14
	50	93.95	99.94	83.99	99.64	**100**	99.18	98.96
	100	99.95	**100**	99.03	**100**	**100**	**100**	**100**
	150	**100**	**100**	99.97	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	17.46	17.34	15.33	17.35	17.48	17.36	**19.58**
	10	39.35	37.79	34.76	38.38	37.23	45.71	**48.30**
	15	59.77	56.97	53.34	56.85	59.87	69.63	**71.17**
	20	74.61	72.36	68.37	71.85	82.53	85.9	**86.51**
	25	85.39	85.76	79.74	83.82	93.97	93.73	**94.01**
	30	92.35	94.41	87.41	92.36	**98.52**	97.28	97.43
	40	97.95	99.47	95.49	98.83	**99.92**	99.64	99.57
	50	99.66	99.96	98.94	99.96	**99.99**	99.96	99.96
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	11.54	11.21	9.58	11.68	11.55	10.35	12.07
	10	22.41	22.2	19.41	22.78	23.13	23.4	**26.49**
	15	34.2	33.89	29.72	35.67	34.7	38.87	**41.73**
	20	46.85	46.9	42.85	46.37	50.73	55.33	**58.30**
	25	59.29	59.51	54.71	58.35	65.09	69.04	**70.72**
	30	69.49	69.56	64.45	68.86	74.73	78.33	**79.83**
	40	83.18	84.2	78.88	82.76	88.58	91.46	**92.18**
	50	91.68	92.59	87.17	91.88	94.65	**96.61**	96.51
	100	99.89	99.95	99.62	99.97	99.97	**99.99**	**99.99**
	150	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	29.41	29.25	26.77	30.17	29.43	32.5	**34.46**
	10	67.79	65.79	63.03	66.24	63.23	77.22	**78.49**
	15	88.88	87.03	84.94	86.69	91.03	94.62	**94.72**
	20	96.36	96.32	94.22	95.58	**99.06**	98.85	98.88
	25	99.04	99.53	97.9	99.24	**99.93**	99.83	99.83
	30	99.77	99.94	99.41	99.92	**100**	99.96	99.97
	40	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	15.37	17.09	17.51	15.55	15.02	**19.08**	18.55
	10	25.68	23.86	24.3	23.7	17.77	**33.07**	30.86
	15	34.68	30.67	29.02	29.62	39.82	**46.82**	42.86
	20	42.05	43.44	33.96	37.41	**67.97**	58.65	52.89
	25	51.63	64.34	39.1	53.45	**85.32**	69.81	63.54
	30	60.5	82.04	46.19	72.82	**94.24**	79.64	73.03
	40	73.47	96.83	56.8	93.4	**99.34**	91.34	85.51
	50	84.76	99.67	66.83	99.02	**99.97**	97.12	93.44
	100	99.58	**100**	95.72	**100**	**100**	**100**	99.94
	150	**100**	**100**	99.8	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	4.2	4.29	5.74	4.68	4.22	**6.20**	4.43
	10	6.89	7.12	5.54	6.57	3.41	**11.14**	7.24
	15	11.39	12.59	6.15	11.45	3.69	**19.21**	12.77
	20	16.4	18.64	5.8	18.03	4.06	**27.30**	18.89
	25	21.82	25.79	6.1	25.21	5.23	**35.77**	25.36
	30	29.42	35.96	7.52	34.63	7.15	**46.15**	33.48
	40	41.8	50.45	10.36	49.5	10.79	**61.25**	47.45
	50	53.19	63.21	14.96	62.88	13.39	**73.41**	58.37
	100	88.7	95.48	62	95.76	46.13	**97.09**	91.98
	150	98.05	99.38	90.74	99.59	79.03	**99.79**	98.87
	200	99.75	99.94	98.74	**99.99**	94.15	**99.99**	99.85
$beta(1,2)$	5	9.27	9	7.91	9.37	9.29	8.25	**9.51**
	10	13.61	13.05	11.31	13.28	12.79	13.94	**16.16**
	15	18.9	17.48	15.04	17.5	17.18	19.96	**21.98**
	20	22.78	20.16	18.07	20.65	24.65	26.83	**28.78**
	25	27.42	24.72	20.68	24.74	34.97	33.24	**35.41**
	30	33.8	31.55	25.3	28.85	**46.92**	39.23	41.61
	40	42.05	45	30.49	37.17	**68.24**	51.83	53.67
	50	50.05	59.11	33.95	51.04	**82.03**	61.94	62.77
	100	82.37	98.78	58.94	96.71	**99.93**	92.94	92.8
	150	95.21	**100**	77.37	99.97	**100**	99.26	99.11
	200	99.06	**100**	89.27	**100**	**100**	99.94	99.93

The bold number is the highest power among the seven tests for each sample size.

Table 5.13: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$Laplace(0,1)$	5	25.61	24.7	21.48	26.16	25.67	25.04	**27.65**
	10	60.42	60.3	56.88	61.58	60.62	62.09	**65.41**
	15	83.94	83.34	81.32	82.93	82.41	85.7	**87.23**
	20	93.38	92.83	91.86	92.78	92.59	94.33	**95.17**
	25	97.35	97.23	96.71	97.19	96.87	98.02	**98.31**
	30	98.95	98.74	98.56	99.07	98.51	99.26	**99.39**
	40	99.92	99.89	99.85	99.83	99.75	99.95	**99.96**
	50	99.99	99.97	99.97	99.98	99.98	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$logistic(0,1)$	5	21.7	20.94	18.16	22.39	21.76	21.18	23.72
	10	53	52.19	48.92	53.43	52.87	56.92	**60.18**
	15	77.37	75.92	72.94	75.67	76.04	81.49	**83.42**
	20	89.05	88.33	86.86	88.07	88.63	92.34	**93.20**
	25	94.98	94.46	93.29	94.59	94.89	96.95	**97.39**
	30	97.64	97.47	**99.51**	97.89	97.6	98.76	99
	40	99.75	99.65	**99.92**	99.57	99.62	99.88	99.92
	50	99.97	99.96	**100**	99.95	99.92	99.98	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$Cauchy(0,1)$	5	36.91	36.15	34.7	37.87	36.95	37.42	**38.16**
	10	71.92	72.42	71.82	72.93	73.18	72.74	**73.67**
	15	89.01	90.12	89.82	89.63	90.32	90.3	**90.66**
	20	95.56	96.1	96.2	96	96.2	96.62	**96.64**
	25	98.37	98.53	98.63	98.44	98.43	**98.96**	98.84
	30	99.26	99.48	99.53	99.57	99.33	**99.61**	99.59
	40	99.92	99.93	99.93	99.91	99.9	**99.97**	**99.97**
	50	99.98	99.96	99.98	99.98	99.95	**99.99**	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$N(0,1)$	5	19.96	19.38	16.72	20.56	20	19.36	**21.68**
	10	48.77	48.2	44.51	48.66	47.92	53.85	**56.93**
	15	72.26	70.56	67.53	70.91	70.84	78.64	**80.52**
	20	86.74	85.46	83.21	85.28	86.97	91.58	**92.48**
	25	93.69	93.11	91.48	92.8	94.22	96.61	**97.10**
	30	97.4	96.97	95.94	96.93	97.79	98.89	**98.97**
	40	99.53	99.6	99.36	99.54	99.7	99.87	**99.87**
	50	99.9	99.93	99.74	99.9	99.97	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$t(1)$	5	36.95	36.49	35.15	37.38	36.99	37.63	38.10
	10	71.96	72.62	71.82	73.13	73.33	72.87	**73.95**
	15	88.84	89.8	89.55	88.73	90.01	90.29	**90.42**
	20	95.55	95.99	95.99	95.8	95.96	96.28	**96.40**
	25	98.1	98.47	98.38	98.38	98.5	98.73	**98.75**
	30	99.44	99.57	99.54	99.55	99.46	**99.70**	99.64
	40	99.94	99.94	99.94	99.91	99.89	**99.96**	**99.96**
	50	**100**	**100**	**100**	99.99	**100**	**100**	100
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$t(3)$	5	24.99	24.26	21.54	25.92	25.03	24.09	**26.51**
	10	57.86	57.82	54.6	59.09	58.19	60.54	**63.44**
	15	79.54	79.53	77.17	79.71	79.48	81.94	**83.82**
	20	91.11	90.99	89.6	90.34	90.63	92.85	**93.56**
	25	96.16	96.11	95.66	96.26	95.85	97.23	**97.55**
	30	98.54	98.27	98.01	98.24	97.99	99.02	**99.15**
	40	99.78	99.71	99.69	99.74	99.66	99.84	**99.87**
	50	99.92	99.95	99.86	99.94	99.88	**99.96**	**99.96**
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$t(4)$	5	23.11	22.34	19.63	24.39	23.22	22.4	24.80
	10	56.85	56.46	53.18	56.42	56.33	59.36	**62.35**
	15	77.96	77.74	75.06	78.02	77.67	80.79	**82.63**
	20	89.87	89.29	88.05	89.12	89.5	92.19	**92.92**
	25	96.04	95.82	95.24	95.62	95.75	97.18	**97.59**
	30	98.27	98	97.55	98.3	97.75	98.93	**99.11**
	40	99.67	99.62	99.54	99.61	99.54	99.88	**99.91**
	50	**100**	99.98	99.95	99.94	99.94	**100**	100
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100
$t(6)$	5	22.14	21.71	18.71	22.41	22.19	21.57	**24.02**
	10	54.25	53.75	50.25	54.45	53.79	57.52	**60.85**
	15	76.8	76.15	73.28	75.82	75.88	80.59	**82.37**
	20	89.28	88.48	86.95	88.44	88.94	91.94	**92.83**
	25	95.62	95.19	94.29	95.04	95.05	97.1	**97.37**
	30	98.15	98.08	97.46	97.75	97.92	98.83	**99.06**
	40	99.72	99.64	99.52	99.72	99.6	99.83	**99.87**
	50	99.98	99.98	99.95	99.94	99.94	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	100
	150	**100**	**100**	**100**	**100**	**100**	**100**	100
	200	**100**	**100**	**100**	**100**	**100**	**100**	100

The bold number is the highest power among the seven tests for each sample size.

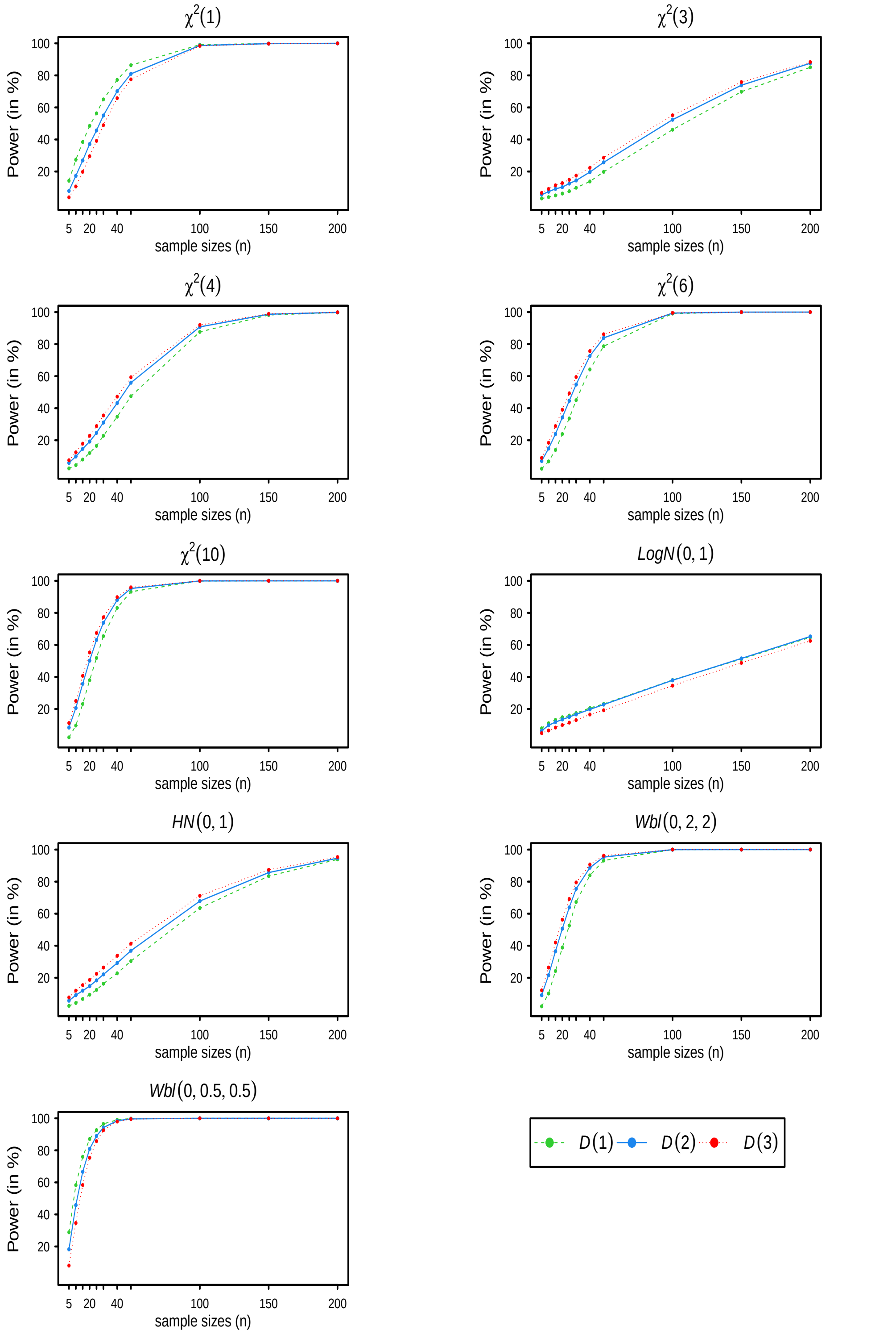

Figure 5.1: Power comparison of the D statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

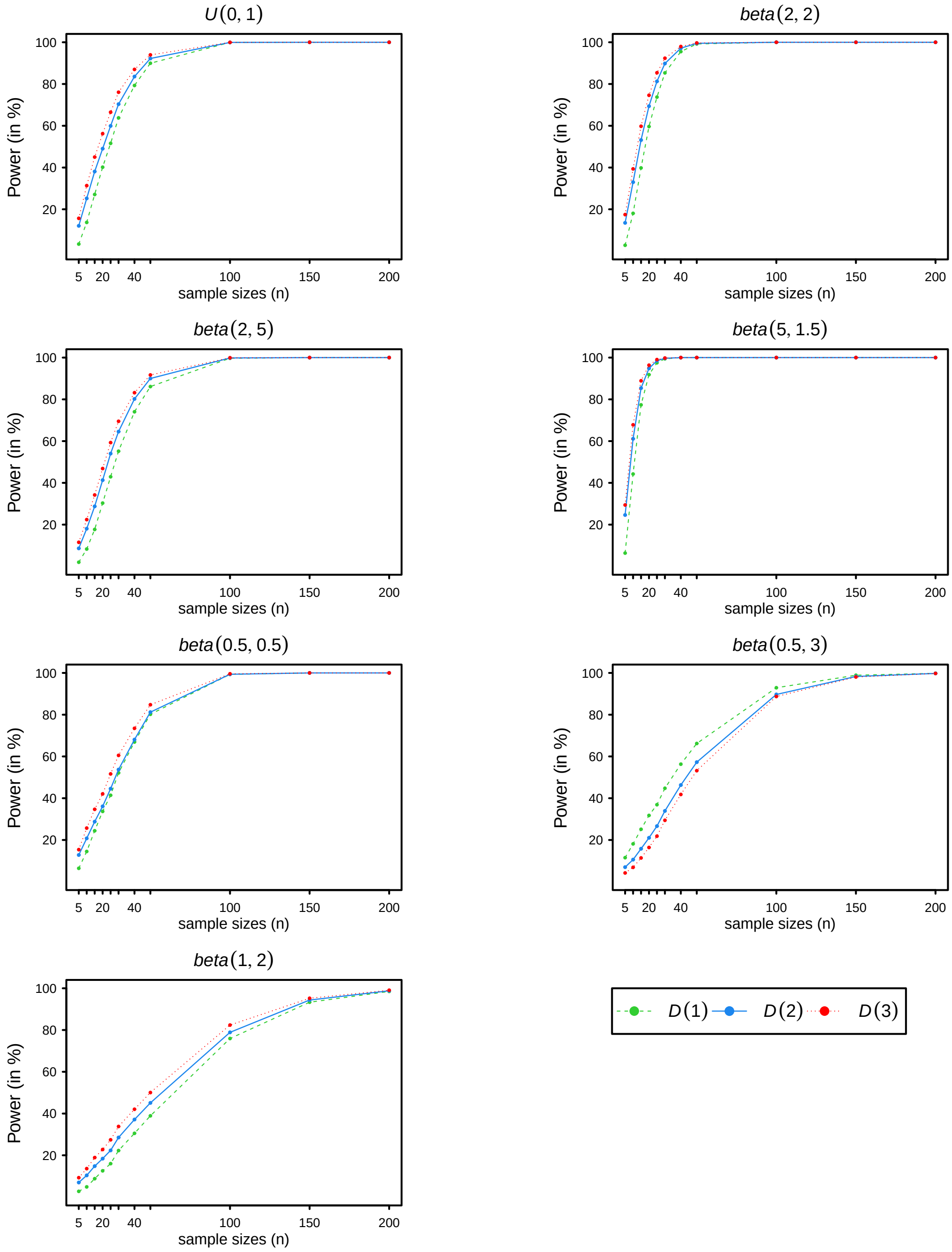

Figure 5.2: Power comparison of the D statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

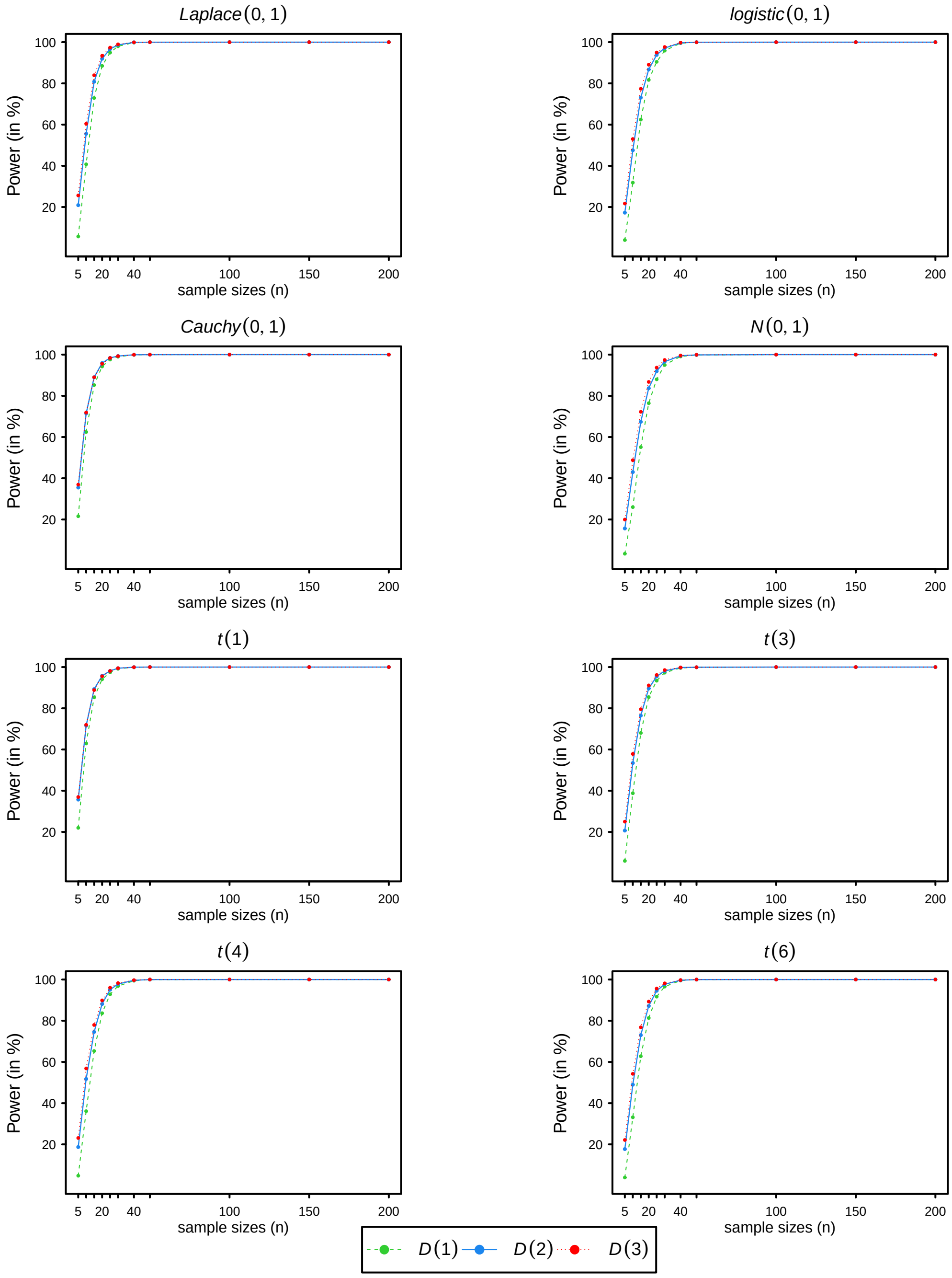

Figure 5.3: Power comparison of the D statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

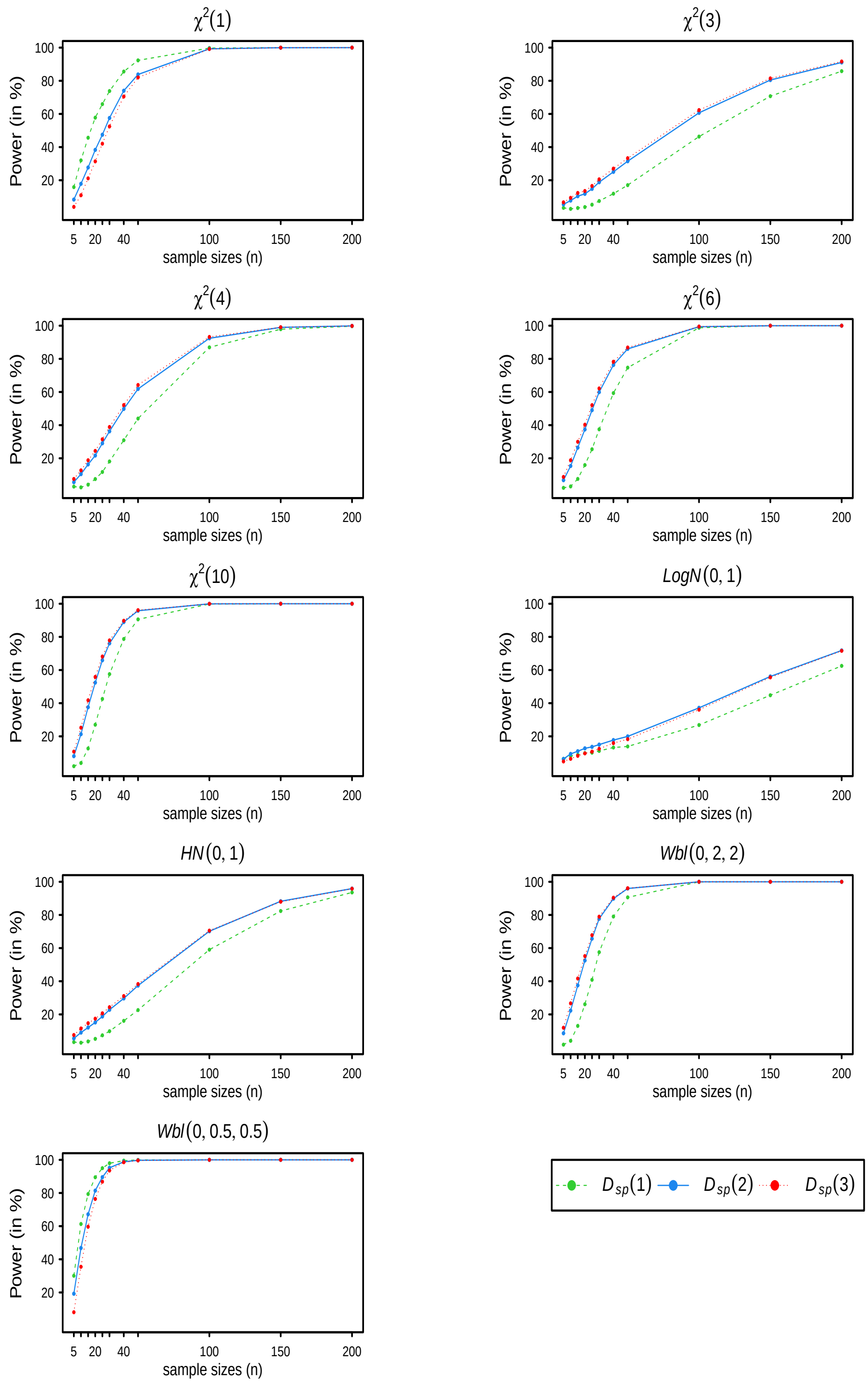

Figure 5.4: Power comparison of the D_{sp} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

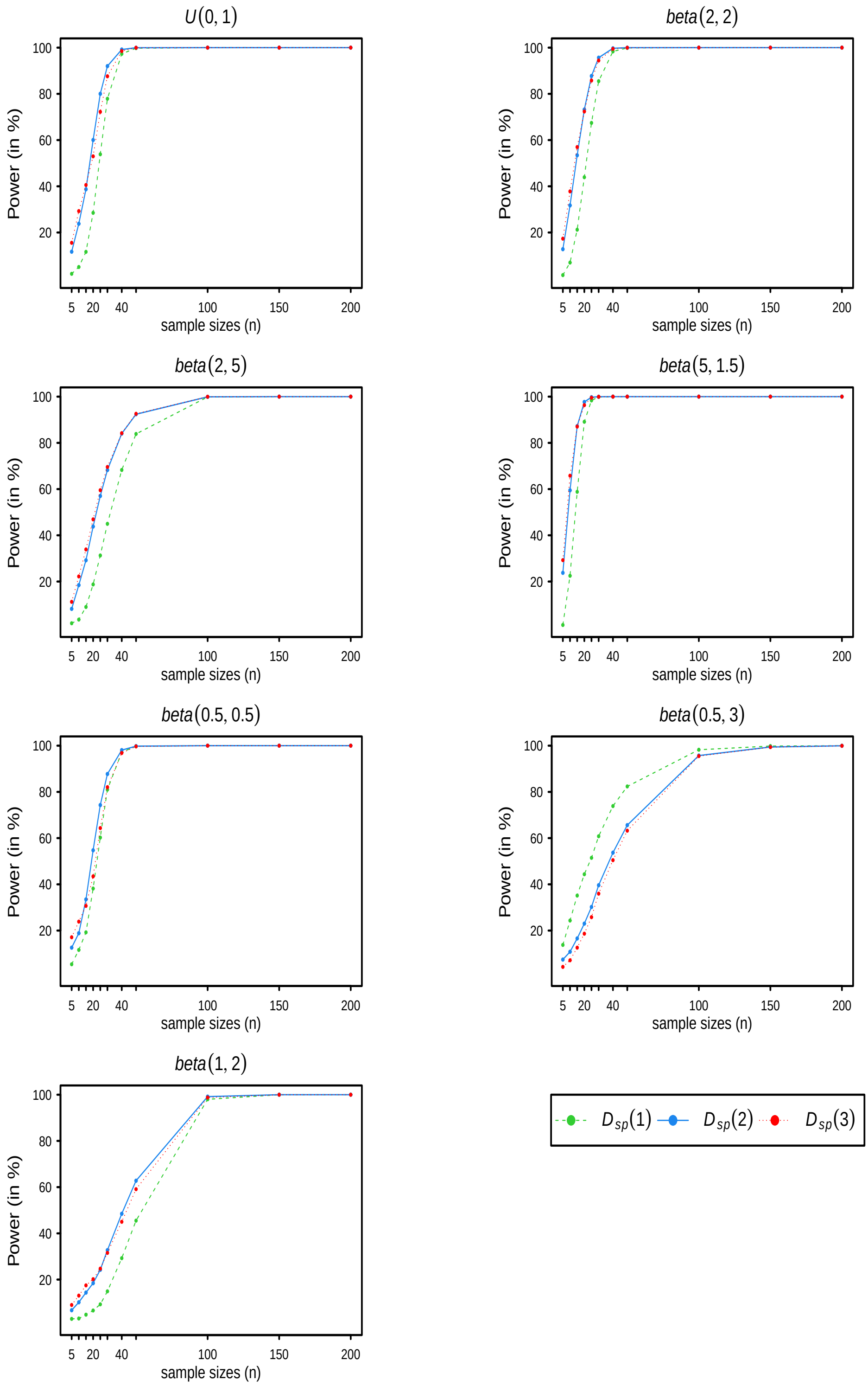

Figure 5.5: Power comparison of the D_{sp} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

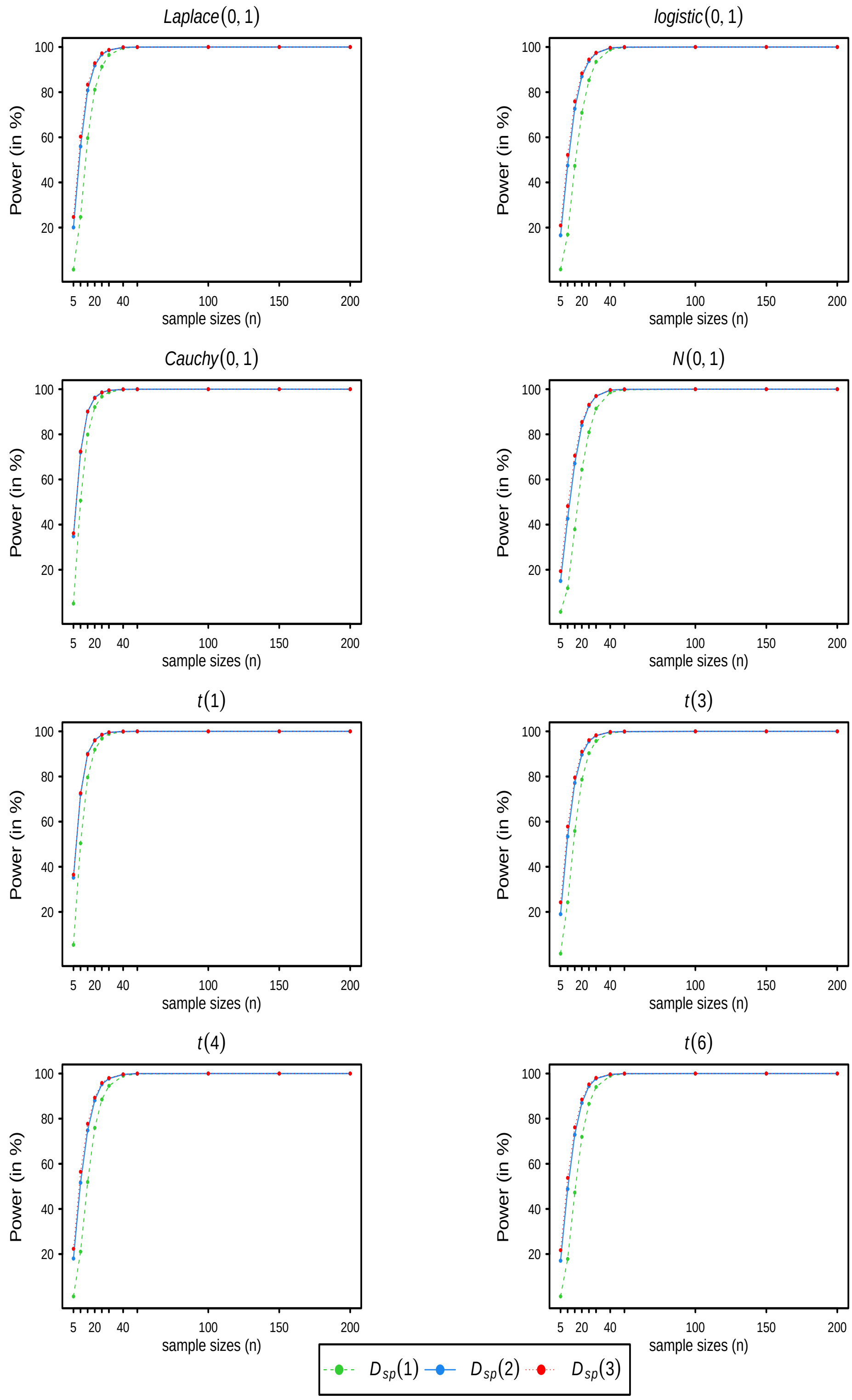

Figure 5.6: Power comparison of the D_{sp} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

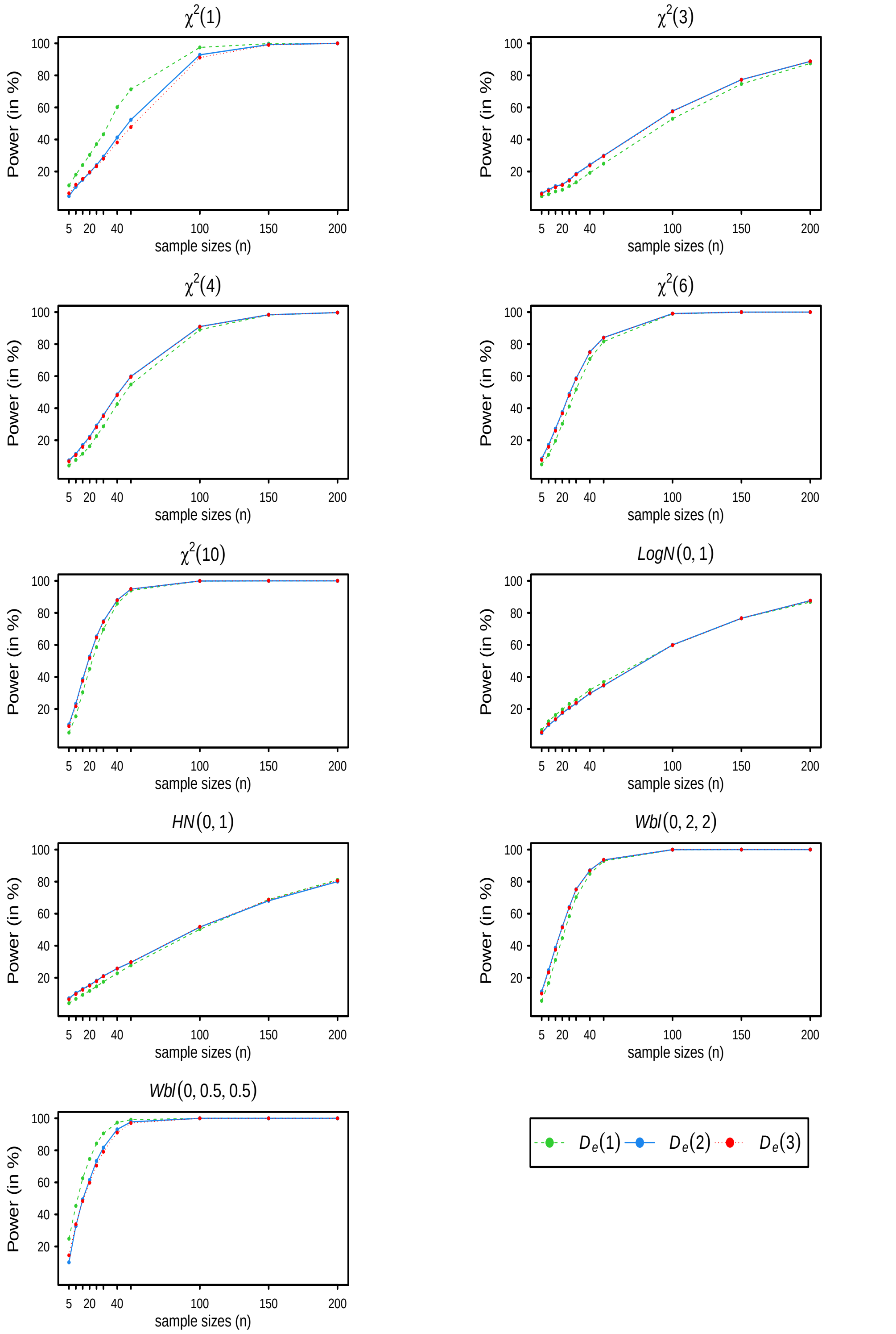

Figure 5.7: Power comparison of the D_e statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

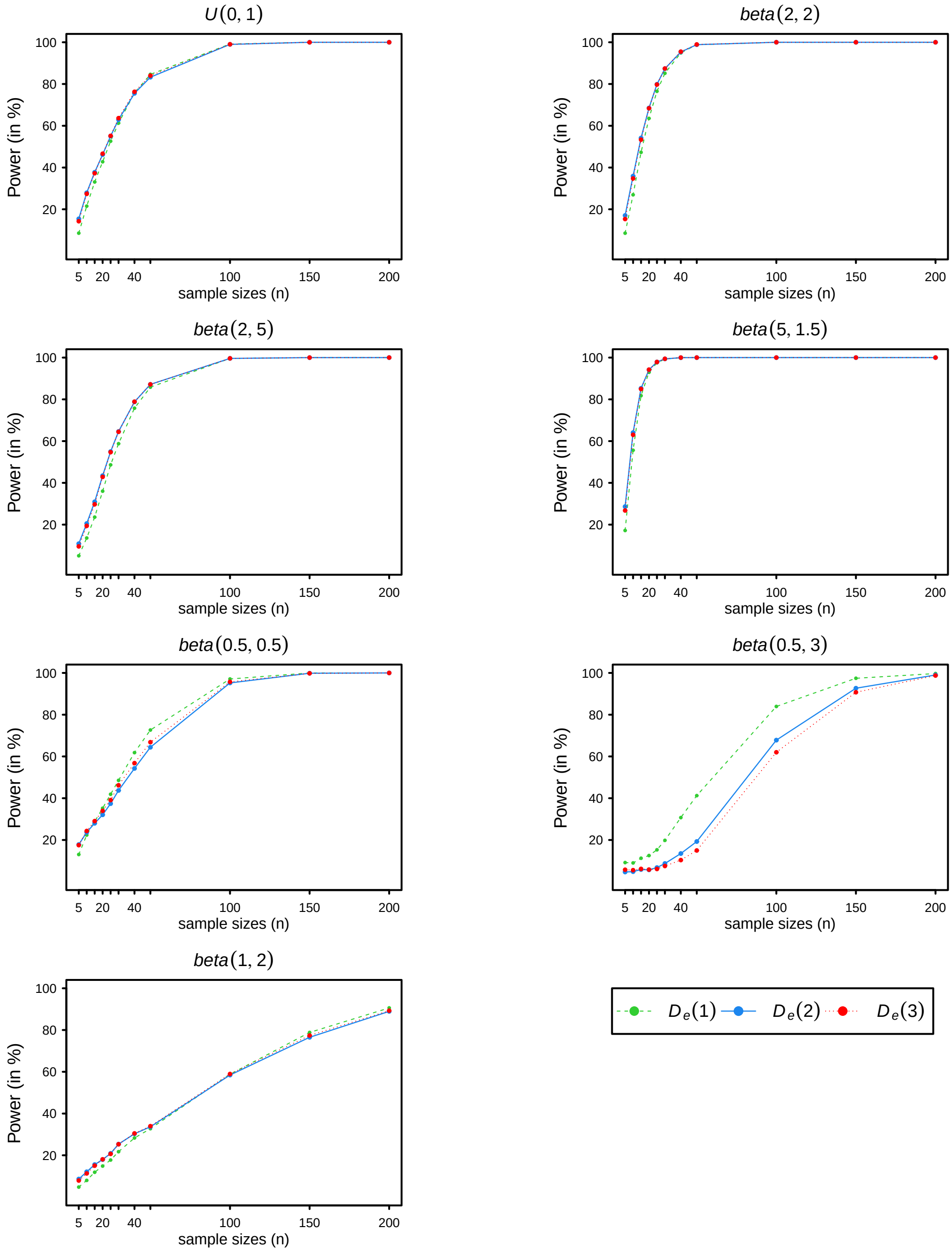

Figure 5.8: Power comparison of the D_e statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

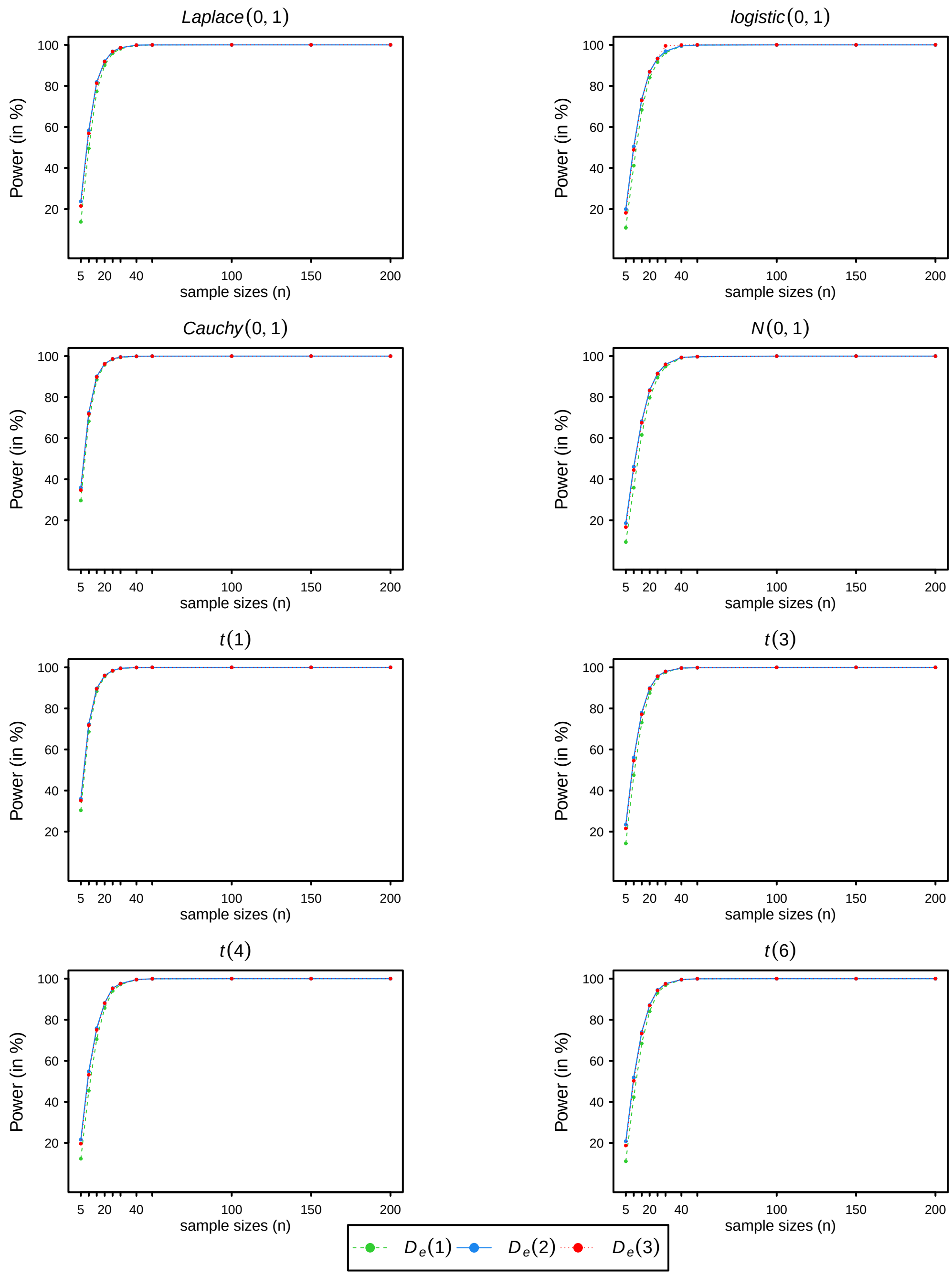

Figure 5.9: Power comparison of the D_e statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

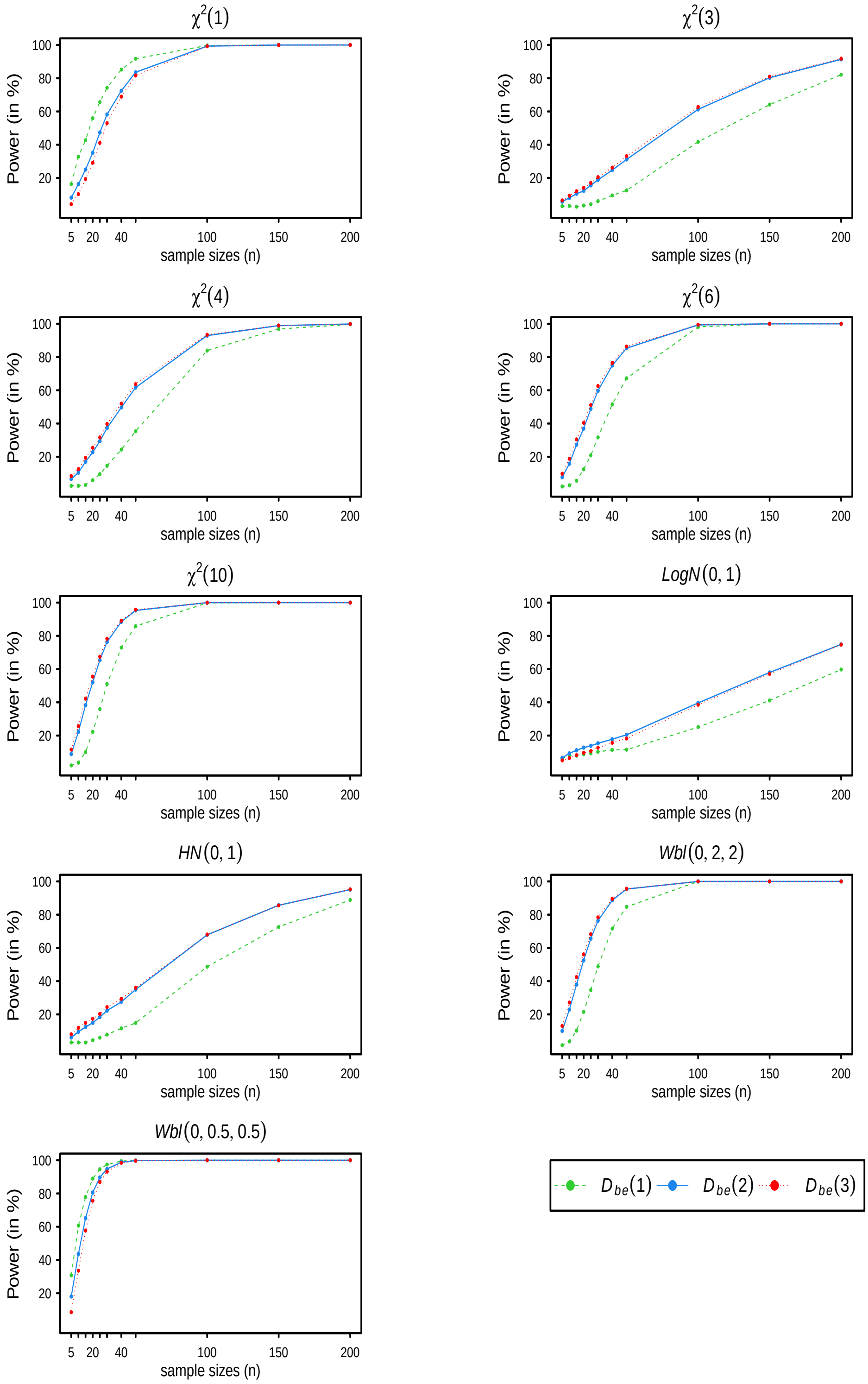

Figure 5.10: Power comparison of the D_{be} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

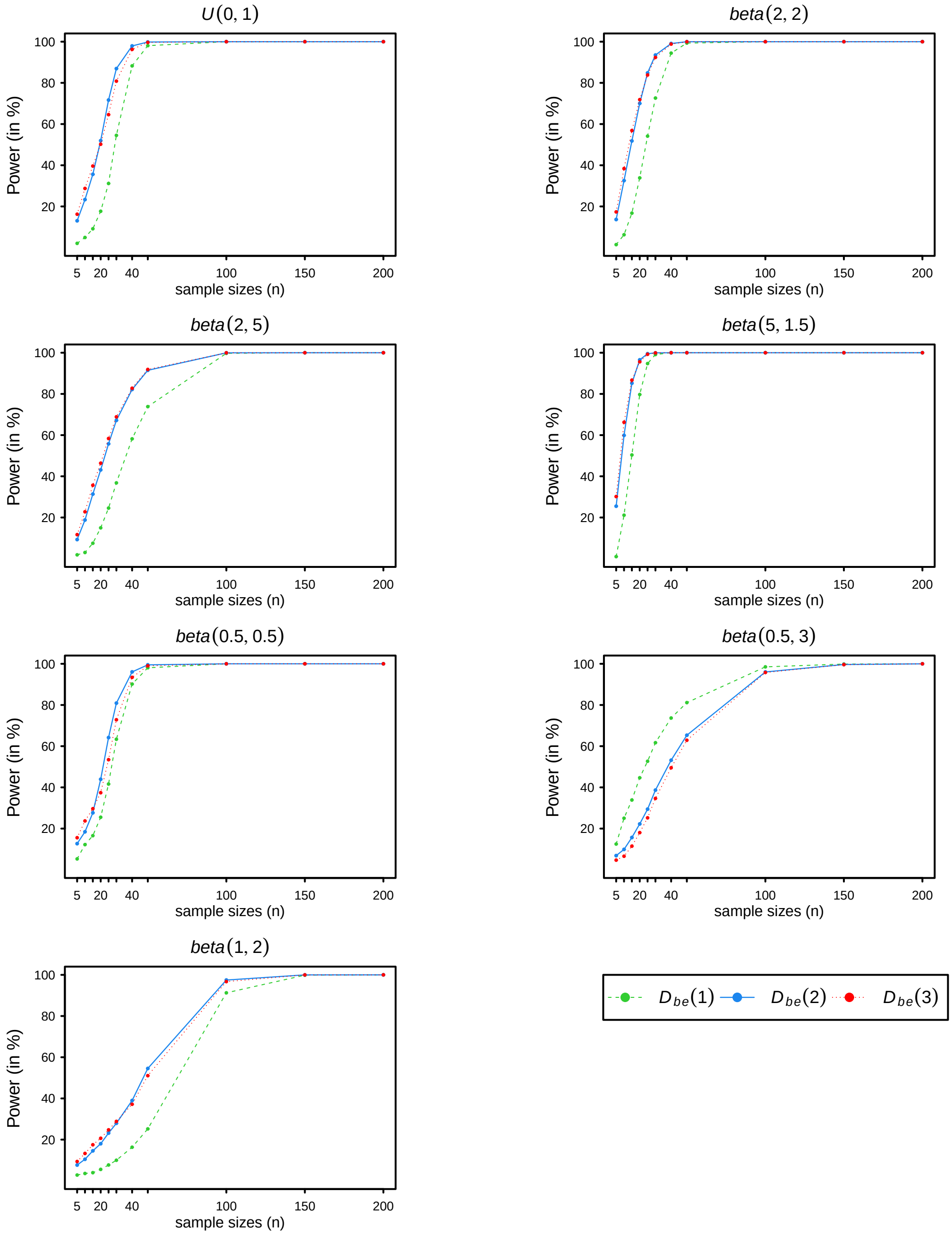

Figure 5.11: Power comparison of the D_{be} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

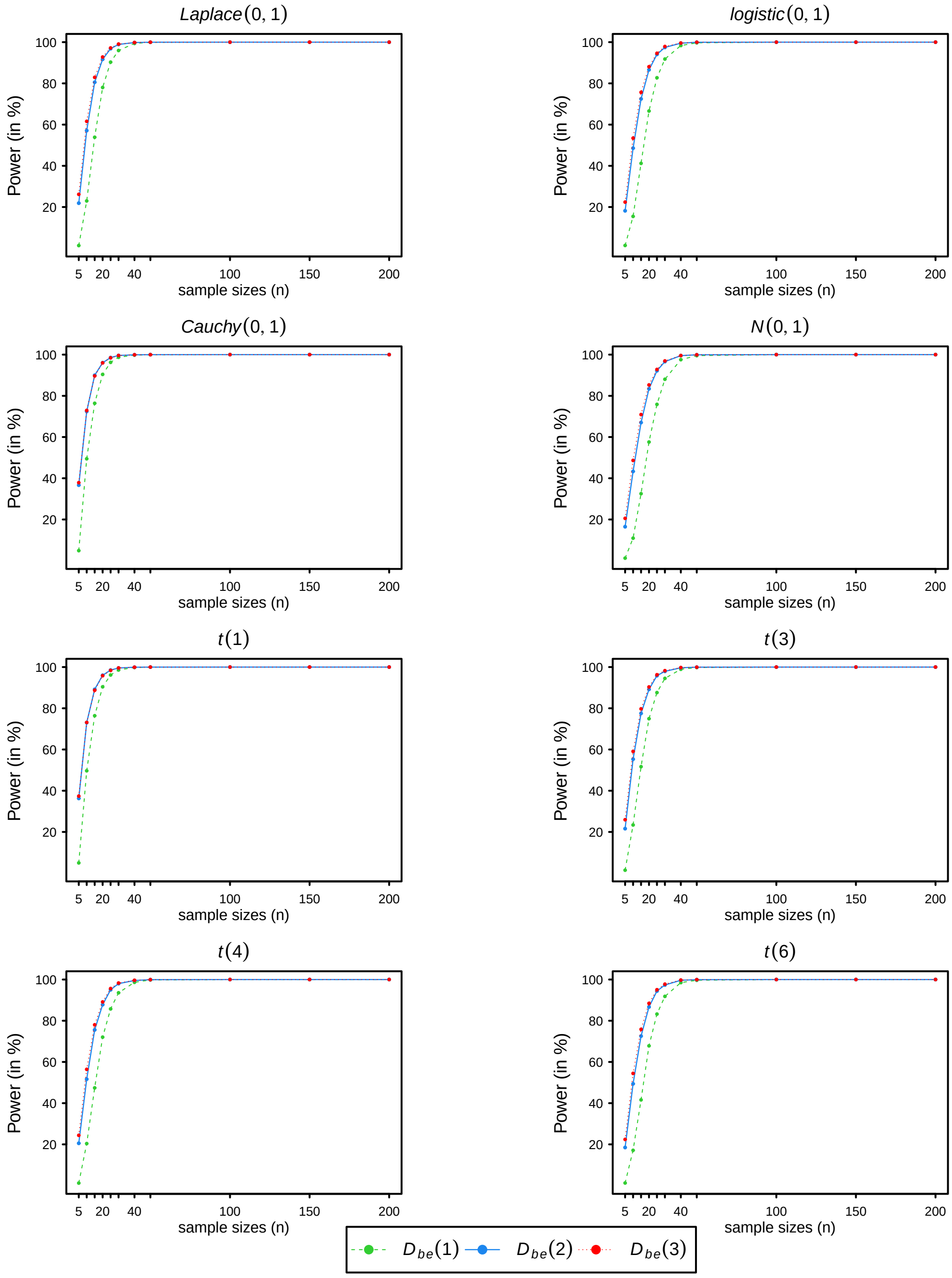

Figure 5.12: Power comparison of the D_{be} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

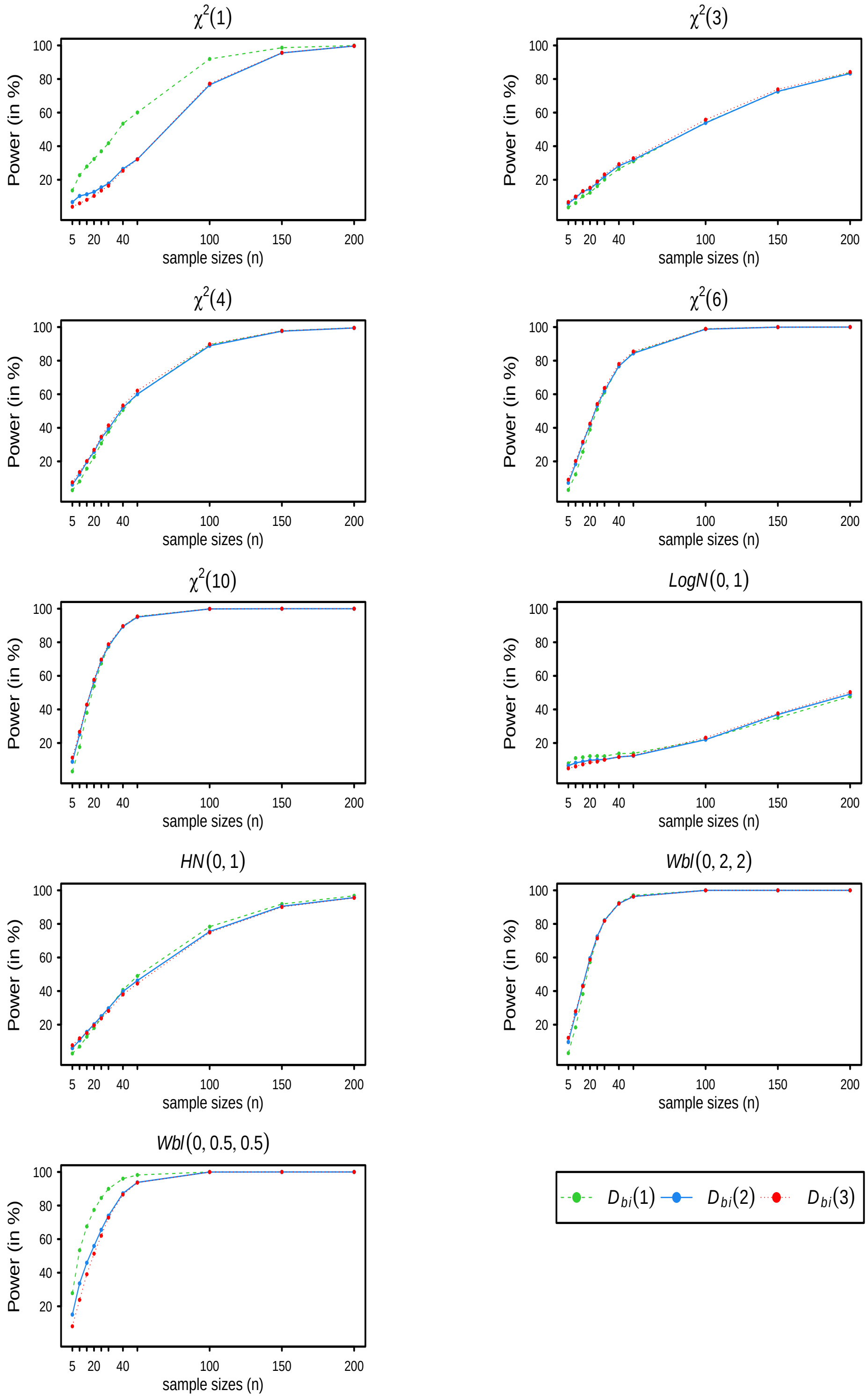

Figure 5.13: Power comparison of the D_{bi} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

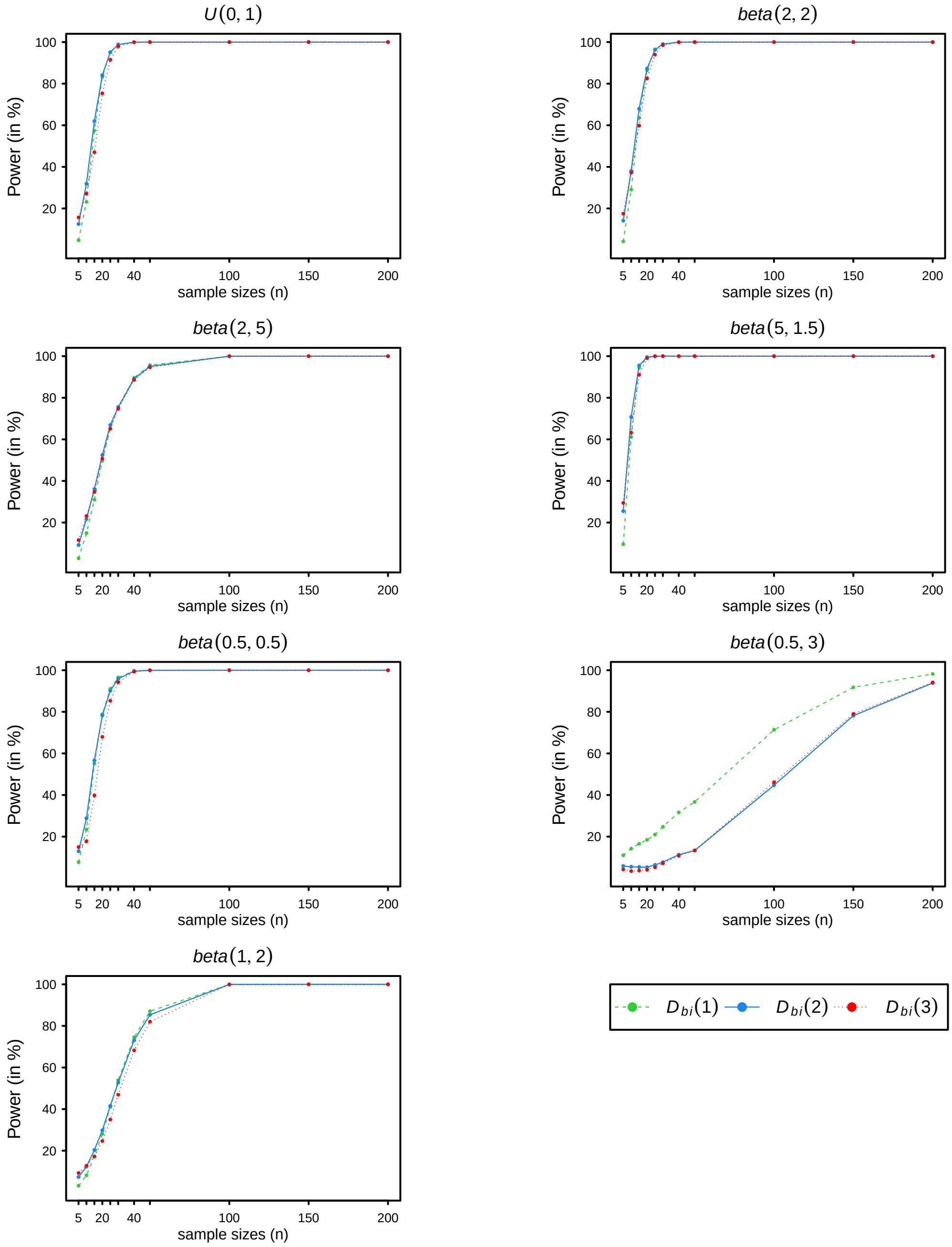

Figure 5.14: Power comparison of the D_{bi} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

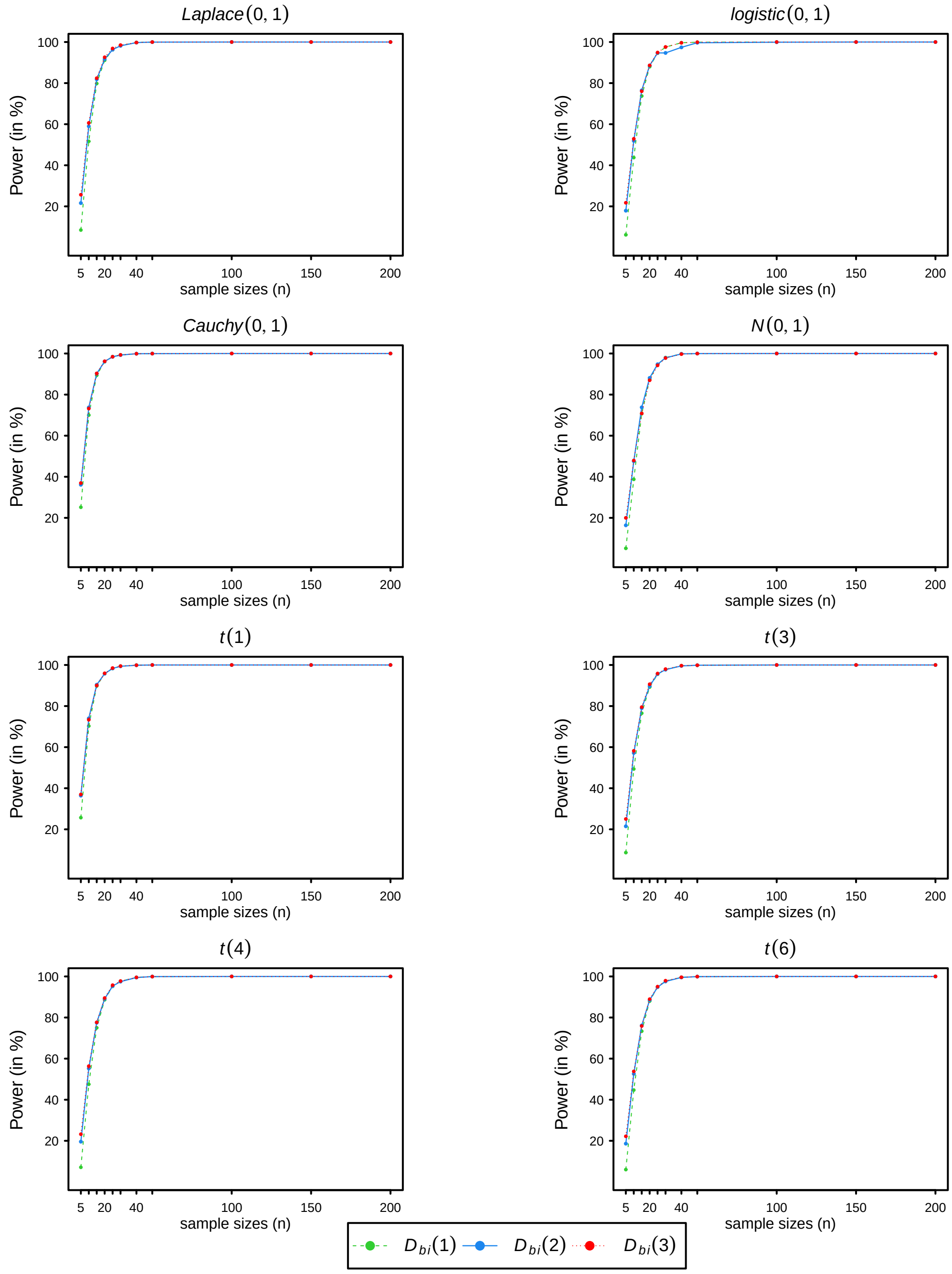

Figure 5.15: Power comparison of the D_{bi} statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

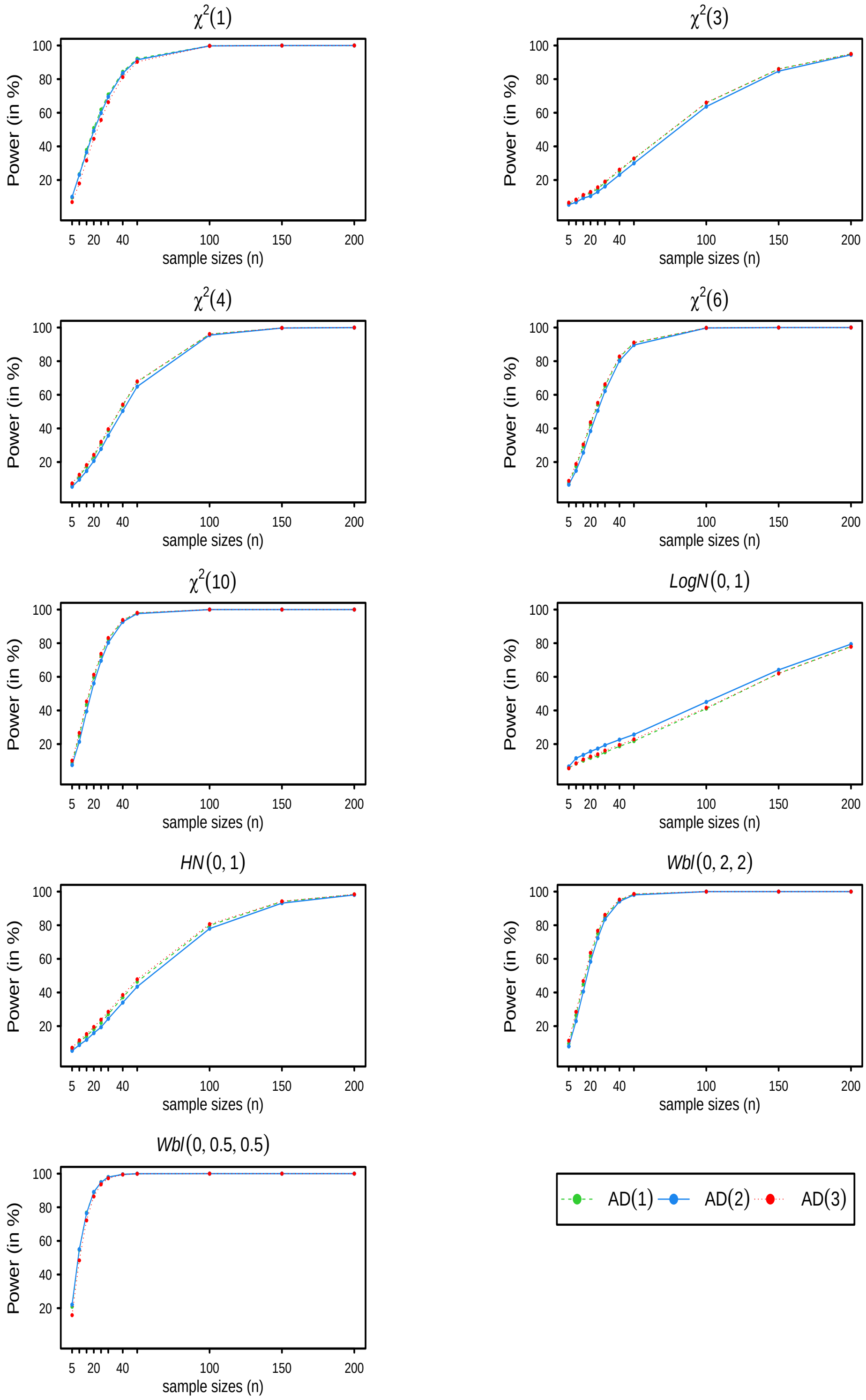

Figure 5.16: Power comparison of the AD statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

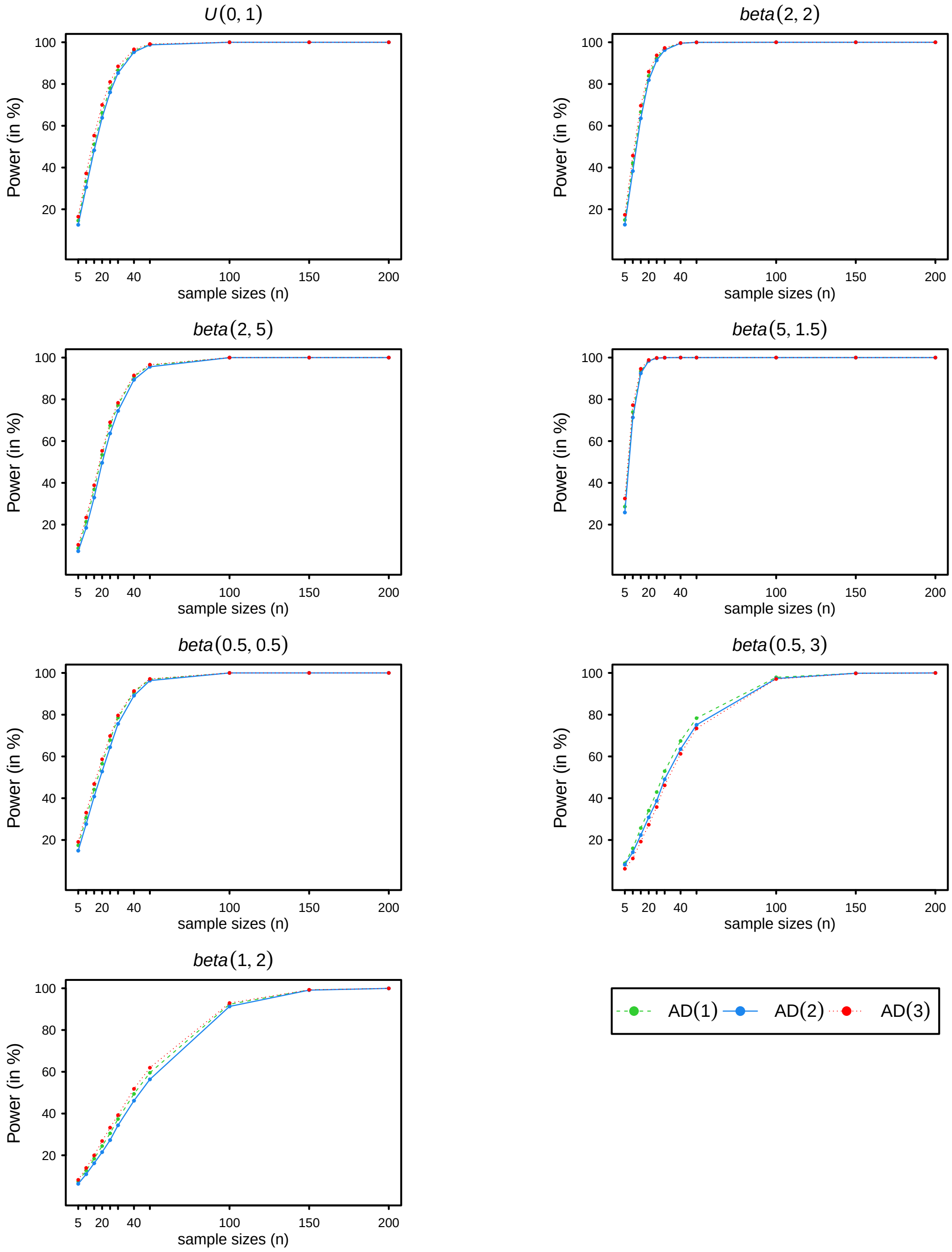

Figure 5.17: Power comparison of the AD statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

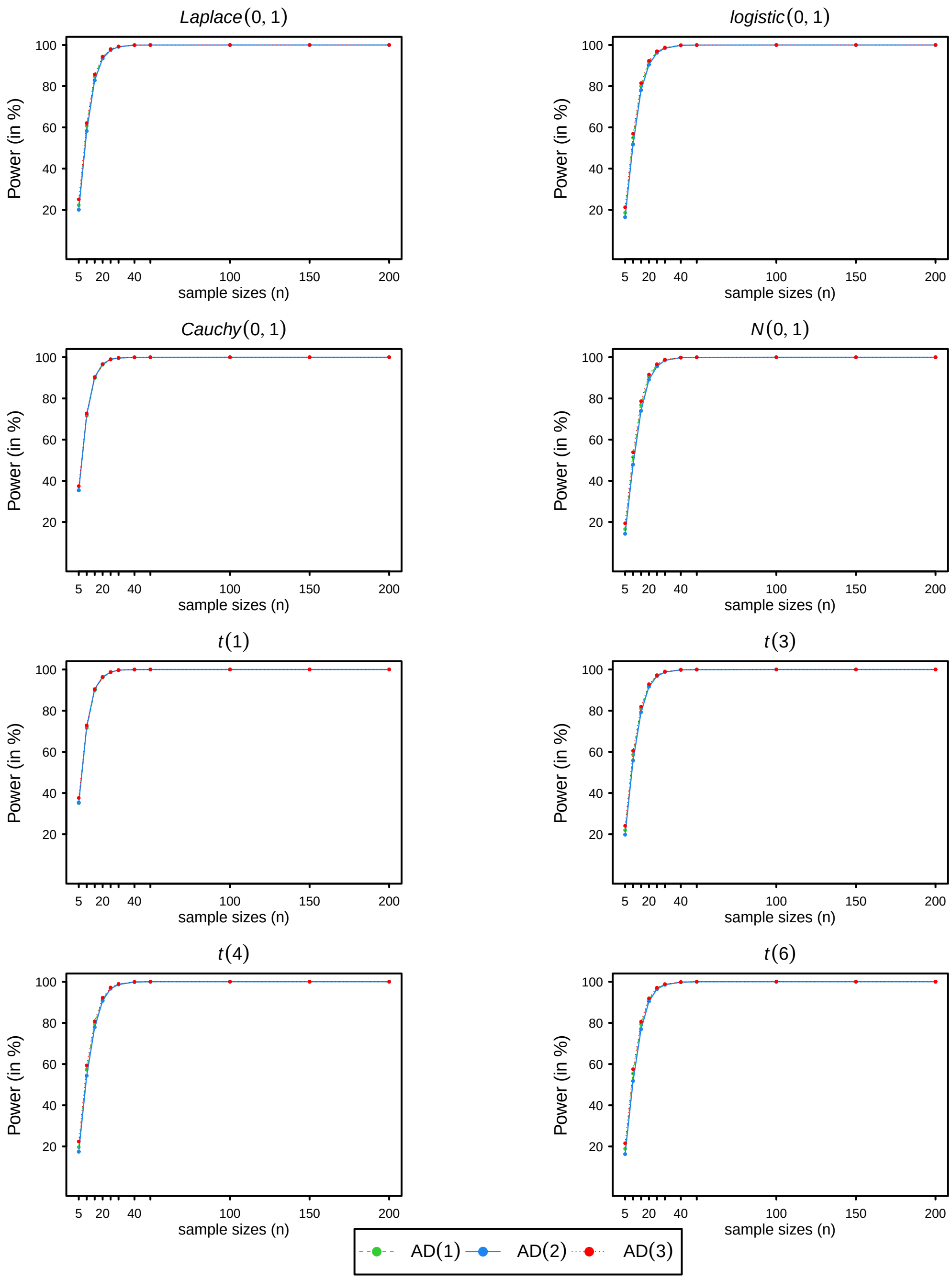

Figure 5.18: Power comparison of the AD statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

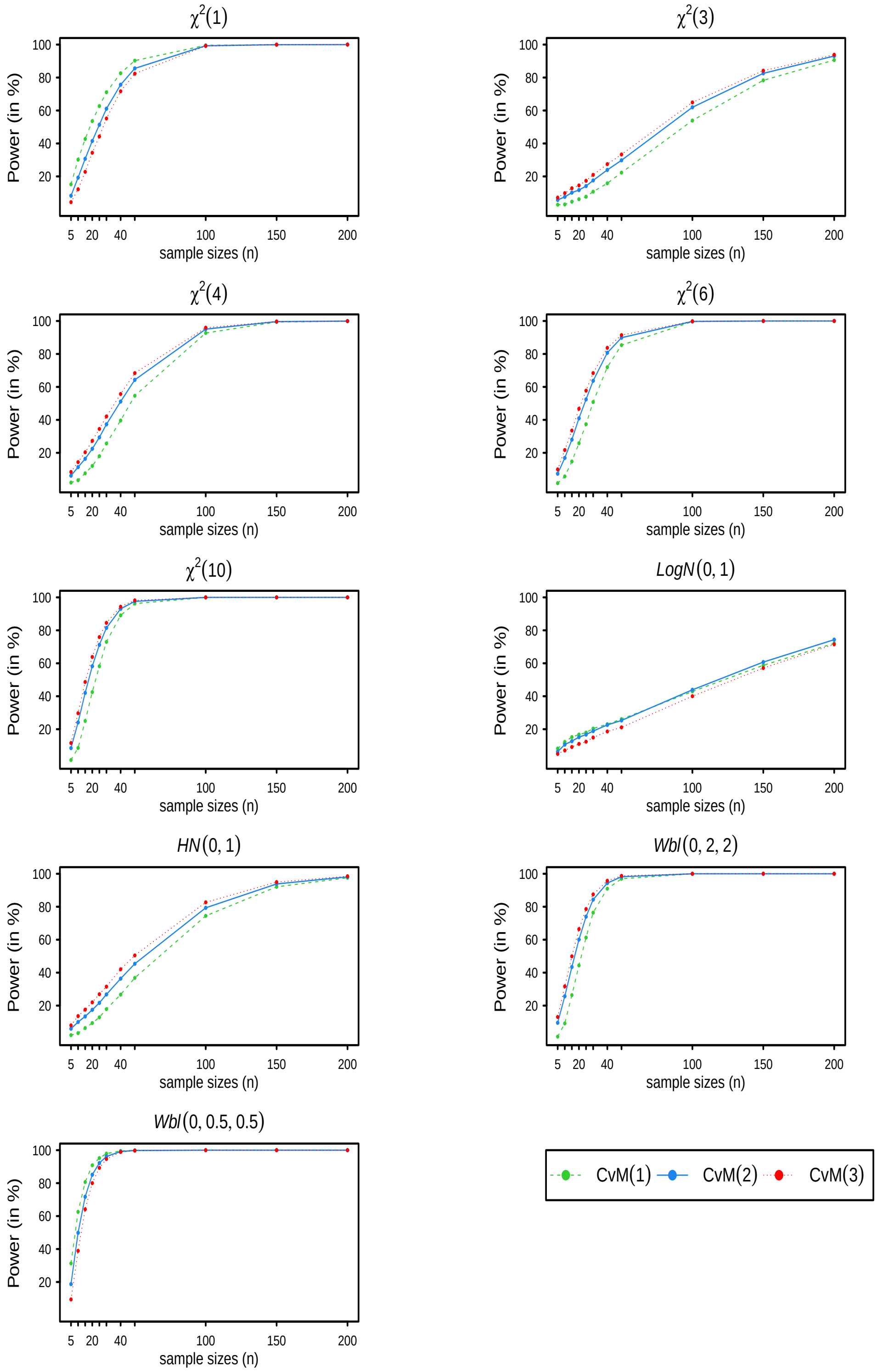

Figure 5.19: Power comparison of the CvM statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

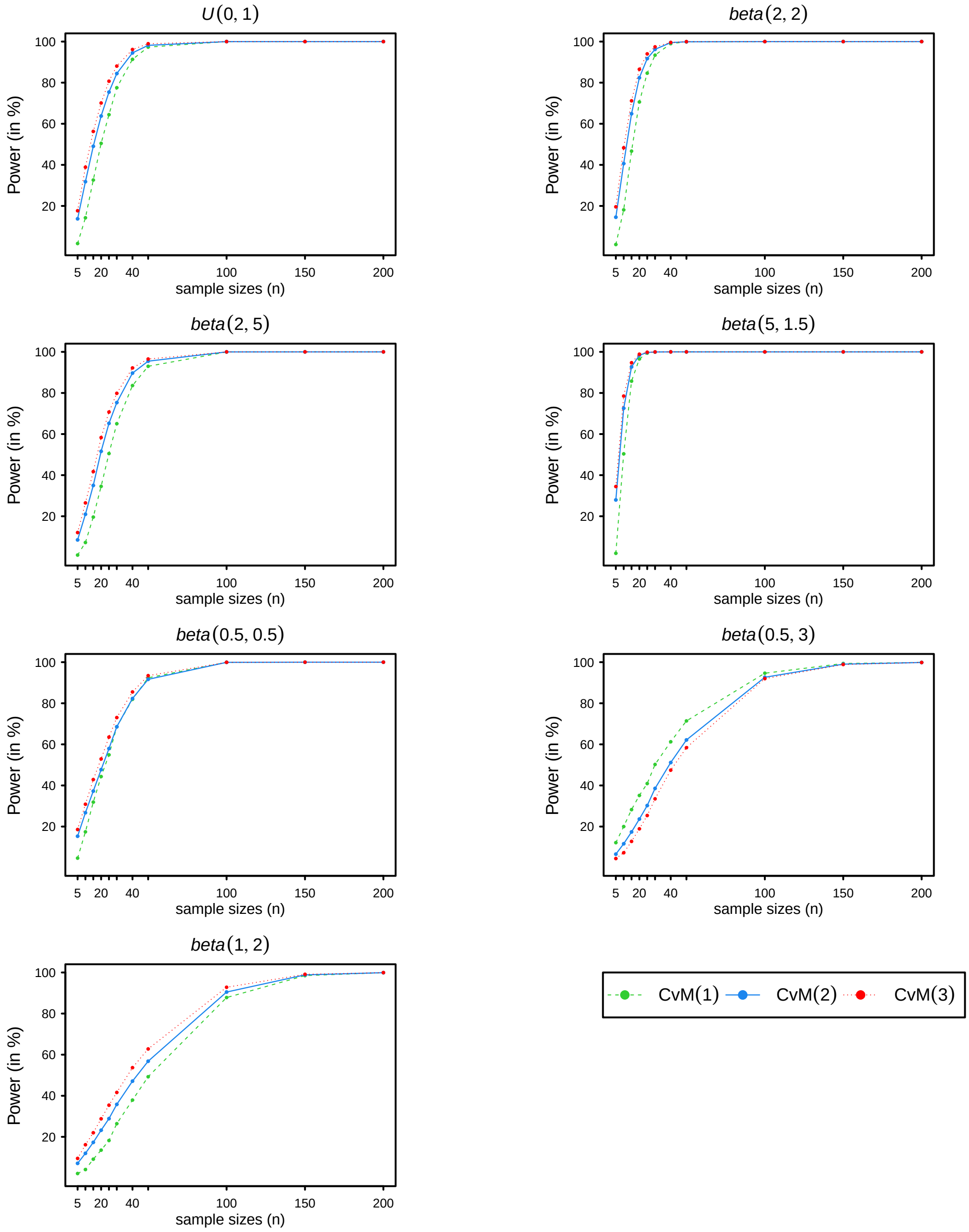

Figure 5.20: Power comparison of the CvM statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

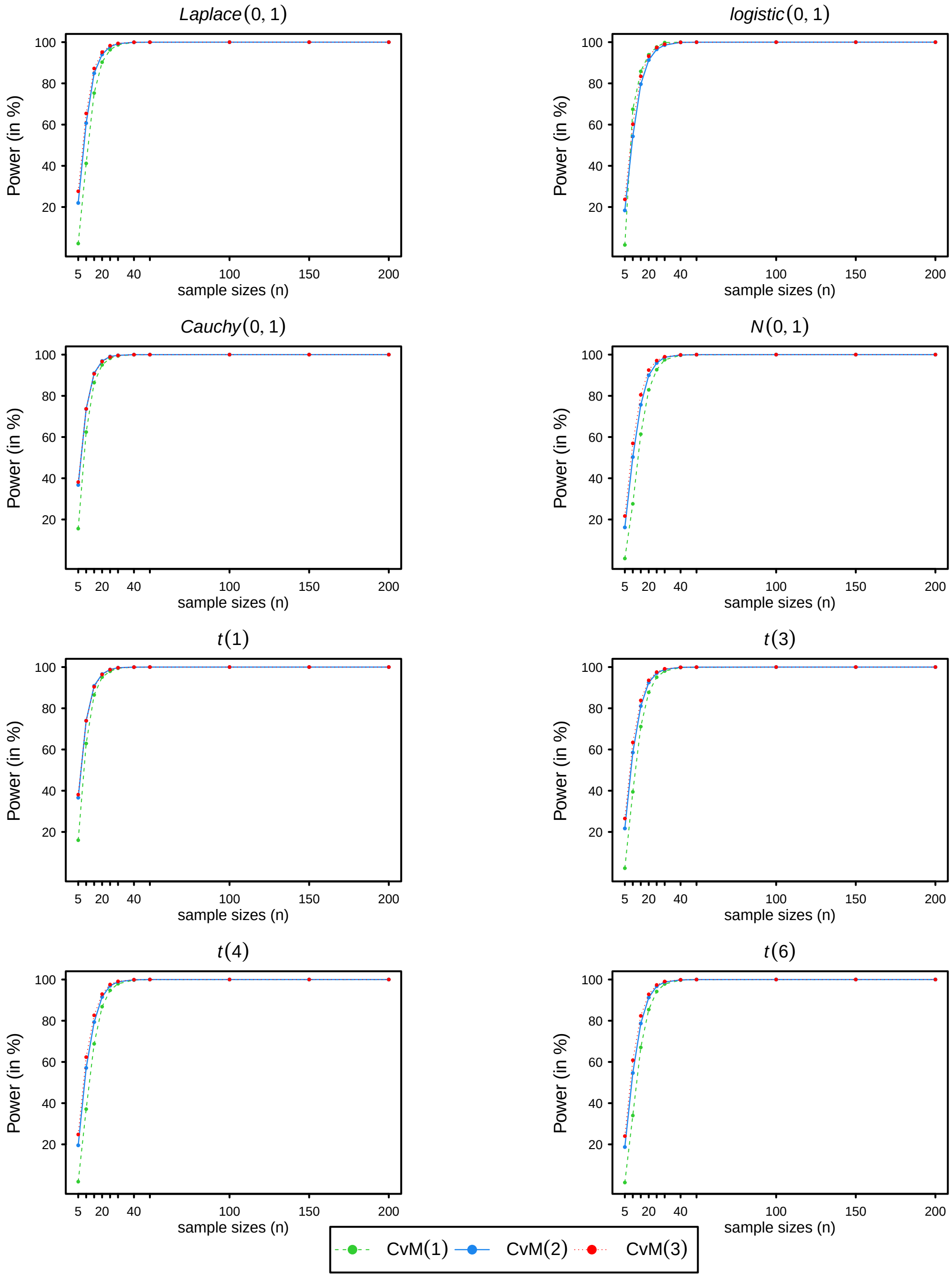

Figure 5.21: Power comparison of the CvM statistic when using three different estimators (MLE, BLUE, BLIE) at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

5.5.2 Results of power

After we choose BLIE as the estimators of the location and scale parameters, the powers of seven tests are compared. From the power results given in Table 5.11 and Figure 5.22 for the first group of alternative distributions, the following observations can be made. The two non–graphical AD and CvM tests are the most powerful against all alternative distributions with the exception of $LogN(0,1)$. Interestingly, the powers of the D test are as good as those of the others, even though D'Agostino and Stephens (1986) stated that " the D test is only a historical curiosity; It should never be used because it has poor power in comparison". Moreover, the D_e test is the best choice against $LogN(0,1)$. On the other hand, it has low powers in comparison with the other tests in this group. Also, the D_{bi} test is the best choice against $HN(0,1)$; however, it has the least power among $\chi^2(1)$, $LogN(0,1)$ and $Wbl(0,0.5,0.5)$. Although the powers of the D_{sp} and D_{be} tests are not as high as the non-graphical AD and CvM tests, we can see that the D_{sp} and D_{be} tests are much better than the D_e and D_{bi} tests. Overall, it is recommended that the D_{sp} and D_{be} tests are the most powerful for testing the Weibull distribution if one wants to use the graphical tests.

From the power results given in Table 5.12 and Figure 5.23 for the second group of the alternative distributions, we can observe that the D_{bi} test shows good power although it seem to have poor power against $beta(0.5,3)$. In addition, the D_{bi} test has greater powers than the non-graphical AD and CvM tests. While the D_e test has the worst power among all the tests except for $beta(0.5,3)$; it is even worse than the D test. The powers of the D_{sp} and D_{be} tests are not as high as those of the D_{bi}, AD and CvM tests in many cases, but they perform quite well over the alternative distributions from this group.

From the power results given in Tables 5.13 and 5.24 for the third group of the alternative distributions. we can reach a conclusion that the powers of all tests are very similar. Nevertheless, the CvM test is slightly more powerful than the others and the powers of all tests reach 100% when $n = 200$.

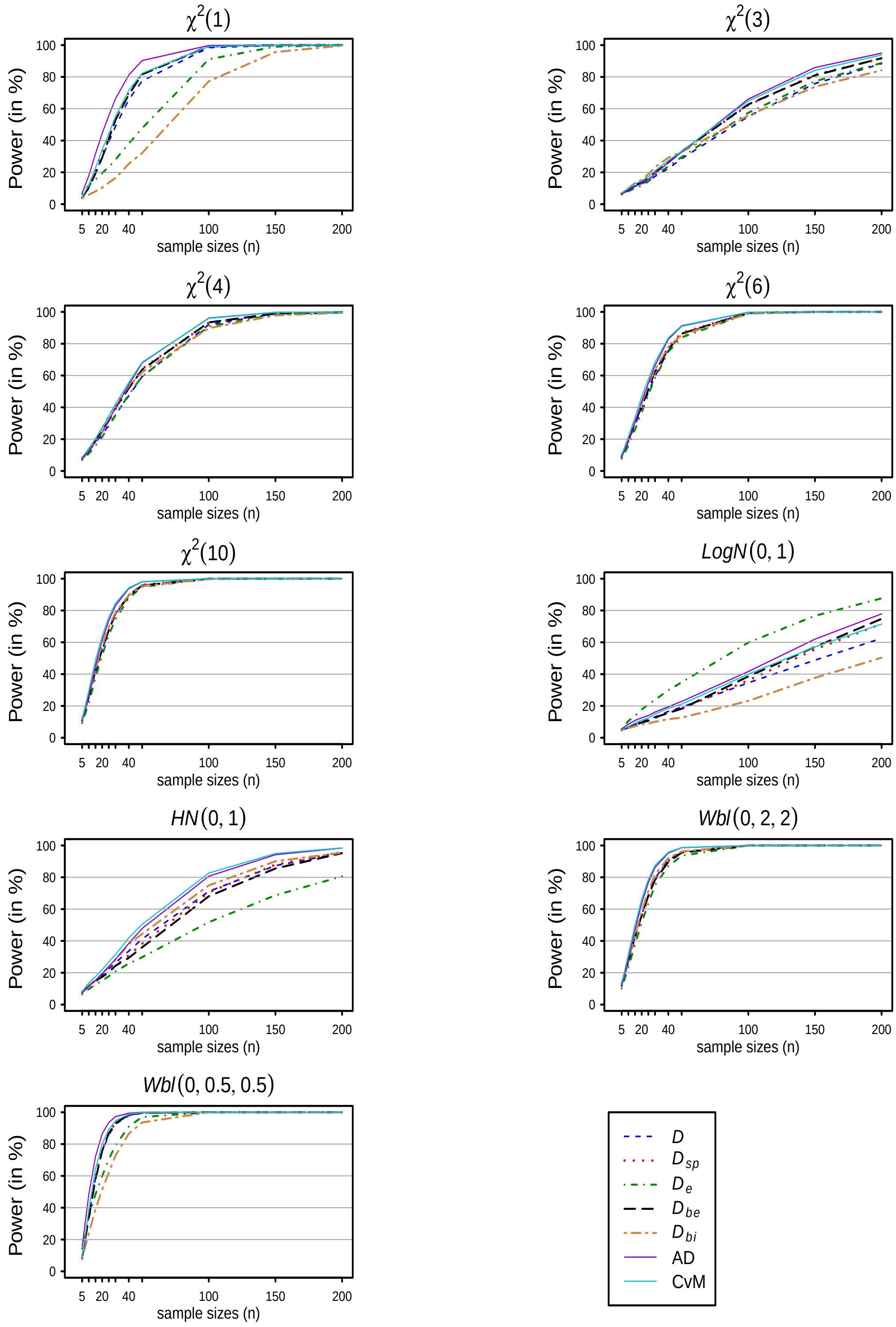

Figure 5.22: Power comparison of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group I for several sample sizes n

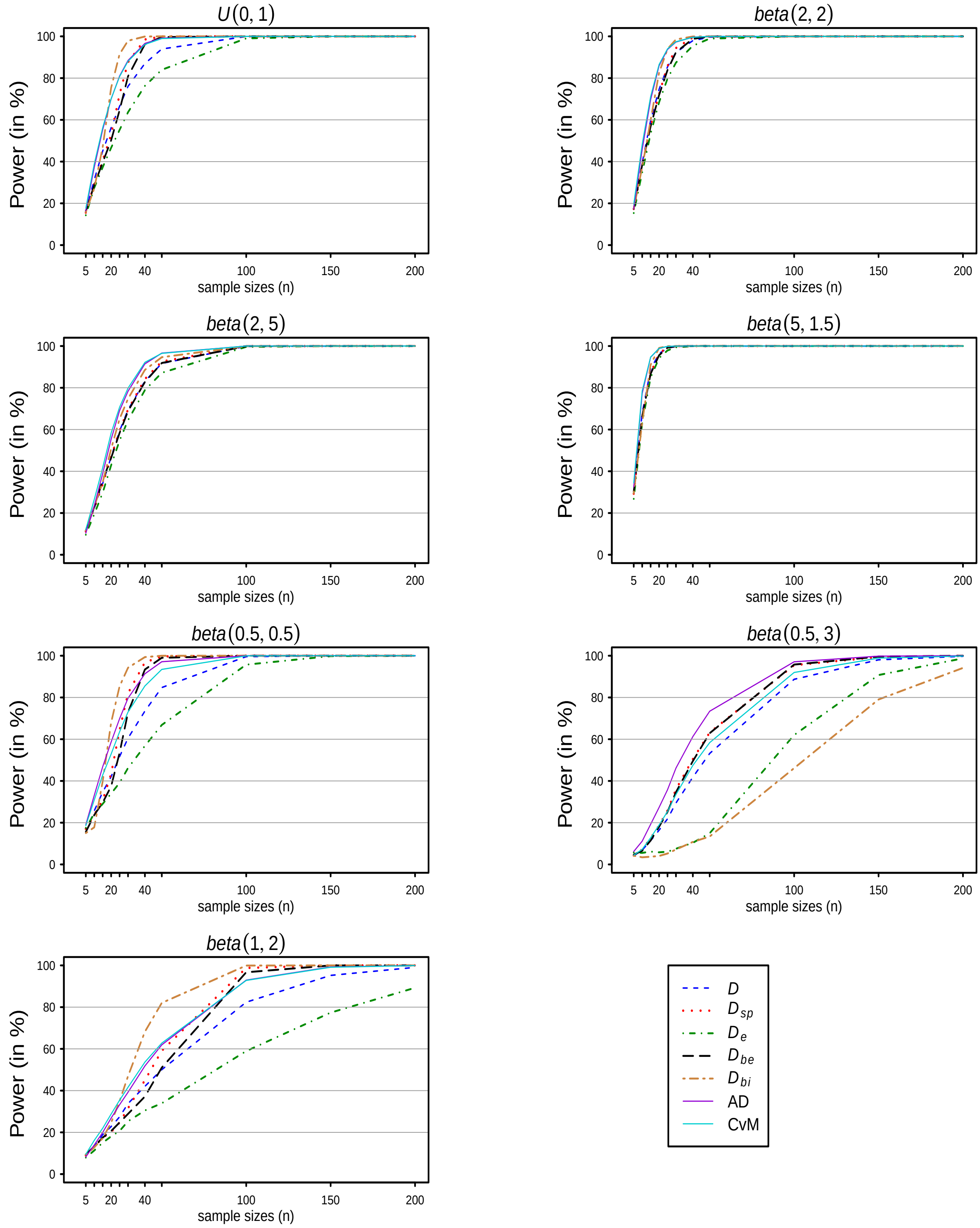

Figure 5.23: Power comparison of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group II for several sample sizes n

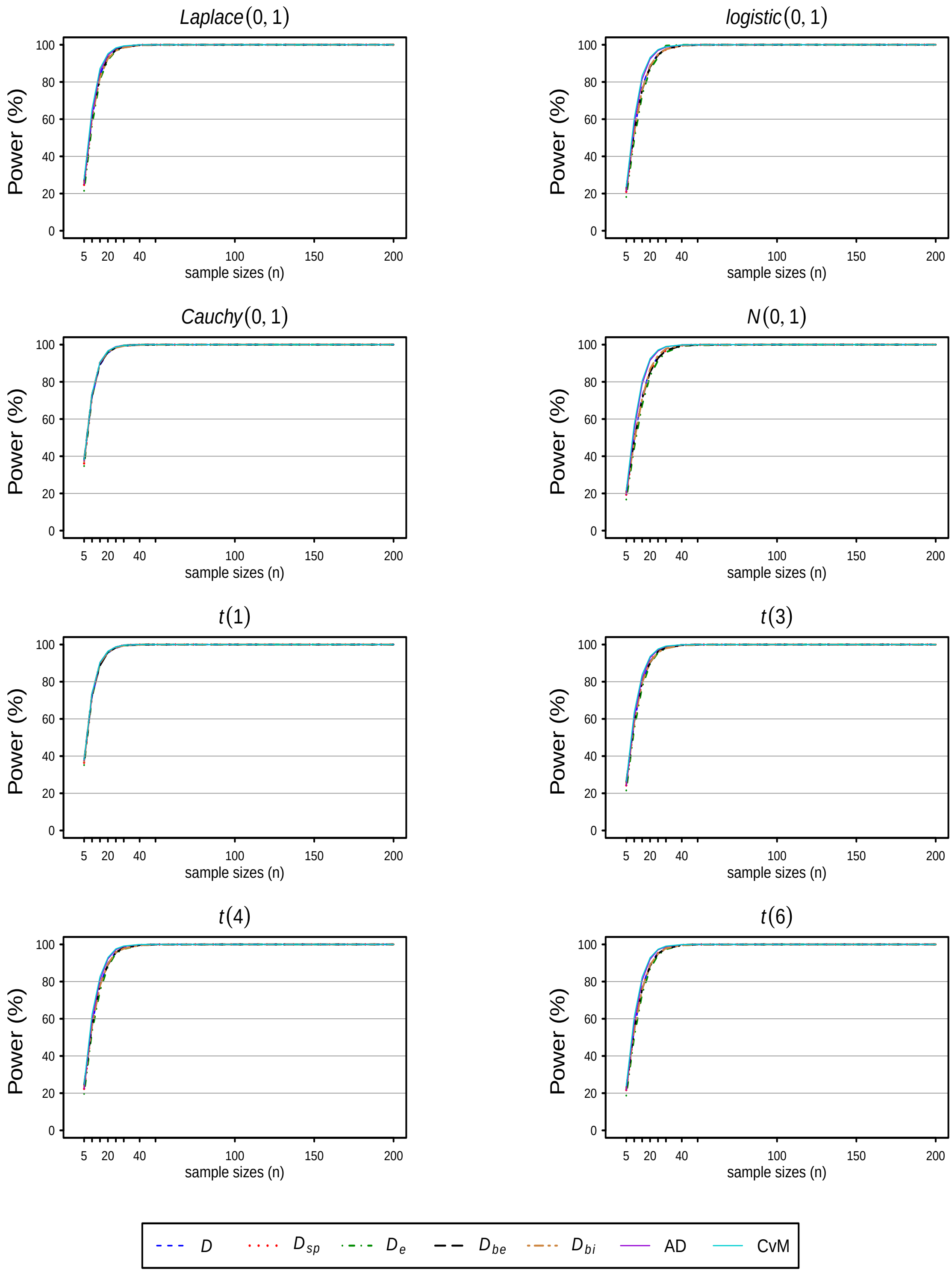

Figure 5.24: Power comparison of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.05$ against the alternative distributions from Group III for several sample sizes n

5.6 Simultaneous intervals

Similar to the normal and Weibull distributions, the null (H_0) and alternative (H_1) hypotheses for testing the exponential distribution are

H_0 : a random sample $Y_1, \ldots, Y_n$ follows the exponential distribution
H_1 : a random sample $Y_1, \ldots, Y_n$ does not follow the exponential distribution.

Any of the five graphical tests can then be used to test this H_0 in the following way.

1. Choose the significance level α and choose the graphical test.
2. Compute the BLIE of the unknown parameters μ and σ of $Exp(\mu, \sigma)$. Moreover, calculate the critical value which depends on α, sample size n and the number of simulations.
3. Sort the Y_k's in ascending order $Y_{[1]} \leq \ldots \leq Y_{[n]}$ and plot the points $(-\ln(1-p_k), Y_{[k]})$ for $k = 1, \ldots, n$ where $p_k = \frac{k-0.5}{n}$. This constructs the Q-Q plot.
4. For each k, plot a vertical interval based on a graphical test considered.
5. Join the upper limits of the k vertical intervals and the lower limits of the k vertical intervals step to obtain a band.
6. Reject the null hypothesis H_0 at level α if at least one point $(-\ln(1-p_k), Y_{[k]})$ falls outside its corresponding vertical interval.

Table 5.14: Simultaneous probability intervals for testing an exponential distribution based on the five graphical tests by using BLIE as the estimators

Graphical tests	Simultaneous intervals of $Y_{[k]}$ for $k = 1, \ldots, n$
D	$\ddot{\mu} - \ddot{\sigma} \ln(1 - \frac{k-0.5}{n} \mp c_D)$
D_{sp}	$\ddot{\mu} - \ddot{\sigma} \ln[1 - \sin^2(\arcsin\sqrt{(k-0.5)/n} \pm \frac{\pi}{2} c_m)]$
D_e	$(\ddot{\mu} + \ddot{\sigma}\mu_k) \pm \ddot{\sigma}\sigma_k c_e$
D_{be}	$(\ddot{\mu} - \ddot{\sigma} \ln(1-L),\ \ddot{\mu} - \ddot{\sigma} \ln(1-U))$
D_{bi}	$\ddot{\mu} - \ddot{\sigma} \ln\left(1 - \frac{k-0.5}{n} \mp c_{bi}\sqrt{(k-0.5)(n-k+0.5)/n^3}\right)$

Table 5.14 gives the simultaneous probability vertical intervals corresponding to each of the graphical tests D, D_{sp}, D_e, D_{be} and D_{bi}. These simultaneous probability intervals have nothing to do with the alternative distributions because they depend only on $k = 1, \ldots, n$ and their critical values of each test. In order to construct the simultaneous intervals of the above tests, we have to calculate the critical values. For this work, the critical values for the D, D_{sp}, D_e, D_{bi}, AD and CvM tests at $\alpha = 0.01, 0.05$ and 0.1 with $n = 5(5)30, 40, 50, 100, 150, 200$ are calculated as in Tables 5.2 - 5.4.

5.6.1 Illustrative examples

To illustrate the use of the graphical tests proposed in practice, the following examples are discussed.

Example 1

In order to apply the graphical tests, we generate 40 observations from $\chi^2(10)$ to be a case study as shown in Table 5.16. Also, Table 5.16 indicates the upper and lower bounds of each ordered observation from the D, D_{sp}, D_e, D_{be} and D_{bi} tests at the significance level 0.05. Figure 5.25 shows the Q-Q plots which are constructed by using the data in Table 5.16. Note that the blue triangles represent the pairs $(-\ln(1-p_k), Y_{[k]})$ and the straight line stems from plotting $\ddot{\mu} - \ddot{\sigma}\ln(1-p_k)$ against $-\ln(1-p_k)$ where $p_k = \frac{k-0.5}{n}$, $k = 1, \ldots, n$. Obviously, there is one point outside its corresponding vertical interval for the 40^{th} ordered observation based on D_{bi}. Therefore, these data do not follow an exponential distribution under the D_{bi} test. For the other plots in Figure 5.25, the sample seems to follow an exponential distribution. However, we have to check the points which are not distinguishable by the naked eye, so as to ensure that we draw the right conclusion by using the numerical results in Table 5.16.

For the non-graphical tests, the test statistics AD and CvM are 1.5627 and 0.2773, respectively. Also, the critical values at $\alpha = 0.05$ of the AD and CvM are 1.1755 and 0.2107, respectively. Then, we can conclude that the null hypothesis is rejected at $\alpha = 0.05$ under AD and CvM because the test statistics are larger than their critical values.

Table 5.15 summarises the conclusions when seven tests are applied to assess whether these data follow the exponential distribution.

Table 5.15: Test results for Example 1

Test	Inference
D	Does not reject H_0
D_{sp}	Does not reject H_0
D_e	Does not reject H_0
D_{be}	Does not reject H_0
D_{bi}	Rejects H_0
AD	Rejects H_0
CvM	Rejects H_0

Table 5.16: Data generated from $\chi^2(10)$ with sample size 40 and the corresponding probability intervals for testing the exponential distribution based on the five graphical tests using BLIE at $\alpha = 0.05$

			D		D_{sp}		D_e		D_{be}		D_{bi}	
k	$X_{[k]}$	Quantiles	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
1	3.4966	0.0126	NaN	4.6037	3.3569	3.926	3.0279	3.9653	3.324	3.996	NaN	3.768
2	3.6591	0.0382	NaN	4.8147	3.3253	4.324	3.0023	4.345	3.3275	4.3422	NaN	4.2274
3	4.2103	0.0645	NaN	5.0323	3.3594	4.6662	3.0222	4.6885	3.361	4.6665	NaN	4.6187
4	4.7391	0.0916	NaN	5.257	3.4223	4.9901	3.0669	5.0171	3.4226	4.9829	NaN	4.9878
5	4.9138	0.1193	NaN	5.4892	3.5036	5.3068	3.1283	5.3392	3.5032	5.2956	NaN	5.3483
6	5.0151	0.1479	NaN	5.7296	3.5986	5.6216	3.2027	5.6594	3.598	5.608	NaN	5.7066
7	5.7313	0.1773	3.3643	5.9786	3.7049	5.9376	3.2877	5.9806	3.7043	5.9223	NaN	6.0668
8	5.7879	0.2076	3.5401	6.2369	3.8211	6.257	3.382	6.3047	3.8207	6.2406	3.4114	6.4316
9	6.264	0.2389	3.7206	6.5053	3.9464	6.5816	3.4847	6.6336	3.9463	6.5643	3.5284	6.8033
10	6.6019	0.2712	3.9059	6.7845	4.0802	6.9127	3.5953	6.9684	4.0805	6.8948	3.6548	7.1837
11	6.6103	0.3045	4.0962	7.0754	4.2223	7.2518	3.7134	7.3106	4.2232	7.2334	3.7904	7.5746
12	6.8772	0.339	4.292	7.3792	4.3727	7.6	3.8388	7.6614	4.3741	7.5812	3.9348	7.9778
13	6.9248	0.3747	4.4936	7.697	4.5313	7.9586	3.9716	8.0219	4.5334	7.9395	4.0882	8.3951
14	7.3341	0.4117	4.7011	8.0301	4.6983	8.329	4.1116	8.3933	4.7011	8.3097	4.2506	8.8285
15	7.4817	0.4502	4.9151	8.3801	4.874	8.7124	4.259	8.7771	4.8776	8.693	4.4223	9.2801
16	7.5318	0.4902	5.136	8.7488	5.0587	9.1104	4.414	9.1745	5.0631	9.091	4.6036	9.7522
17	8.4437	0.5319	5.3641	9.1383	5.2529	9.5246	4.5768	9.587	5.2582	9.5052	4.795	10.2475
18	8.5465	0.5754	5.6001	9.551	5.457	9.9569	4.7477	10.0165	5.4632	9.9377	4.997	10.7693
19	9.2456	0.6208	5.8444	9.9901	5.6718	10.4093	4.9272	10.4648	5.679	10.3903	5.2103	11.321
20	9.6554	0.6685	6.0976	10.4589	5.898	10.8842	5.1156	10.9339	5.9062	10.8655	5.4357	11.9072
21	10.2049	0.7185	6.3605	10.9619	6.1364	11.3844	5.3136	11.4264	6.1458	11.3661	5.6742	12.5331
22	10.2092	0.7711	6.6338	11.5045	6.3881	11.913	5.5217	11.9452	6.3987	11.8952	5.9268	13.2052
23	11.2582	0.8267	6.9184	12.0934	6.6543	12.4738	5.7406	12.4934	6.6663	12.4567	6.1949	13.9317
24	11.3228	0.8855	7.2152	12.7372	6.9365	13.0714	5.9713	13.0751	6.9499	13.0551	6.4801	14.723
25	11.8595	0.948	7.5254	13.4472	7.2364	13.7111	6.2146	13.6949	7.2512	13.6958	6.7843	15.5928
26	12.0993	1.0147	7.8501	14.2388	7.556	14.3998	6.4716	14.3585	7.5724	14.3857	7.1098	16.5598
27	13.0847	1.0862	8.1908	15.1329	7.8978	15.1458	6.7437	15.0729	7.9159	15.1331	7.4593	17.6497
28	13.72	1.1632	8.5493	16.1604	8.2648	15.9596	7.0322	15.8466	8.2848	15.9487	7.8362	18.9008
29	13.7534	1.2465	8.9274	17.3679	8.6607	16.855	7.3389	16.6907	8.6827	16.8462	8.2448	20.3721
30	14.1161	1.3375	9.3274	18.8323	9.0902	17.85	7.6657	17.6193	9.1144	17.844	8.6905	22.1639
31	14.4676	1.4376	9.752	20.6932	9.5593	18.9696	8.0148	18.6512	9.5858	18.9667	9.1802	24.4674
32	14.5948	1.5488	10.2044	23.2487	10.0759	20.2485	8.3886	19.8118	10.1047	20.2495	9.7231	27.7296
33	14.7573	1.674	10.6886	27.35	10.6504	21.7383	8.7897	21.1369	10.6815	21.744	10.332	33.5368
34	15.2703	1.8171	11.2092	38.8598	11.2975	23.5193	9.2202	22.6792	11.3308	23.53	11.0247	NaN
35	16.1591	1.9841	11.7724	NaN	12.0386	25.7261	9.681	24.5199	12.0732	25.7412	11.8277	NaN
36	17.0582	2.1848	12.3856	NaN	12.9067	28.6111	10.1684	26.7945	12.9404	28.624	12.7835	NaN
37	17.2696	2.4361	13.0587	NaN	13.9568	32.731	10.6645	29.7508	13.9834	32.7065	13.9654	NaN
38	17.3136	2.7726	13.8045	NaN	15.2934	39.7535	11.1022	33.9163	15.2922	39.4435	15.5193	NaN
39	17.5179	3.2834	14.6407	NaN	17.1592	62.5058	11.1974	40.7259	17.0437	55.6012	17.8136	NaN
40	20.9488	4.382	15.5923	NaN	20.4676	40.2417	9.0037	56.7292	19.7432	∞	22.4913	NaN

NaN stands for Not a Number representing an undefined value.

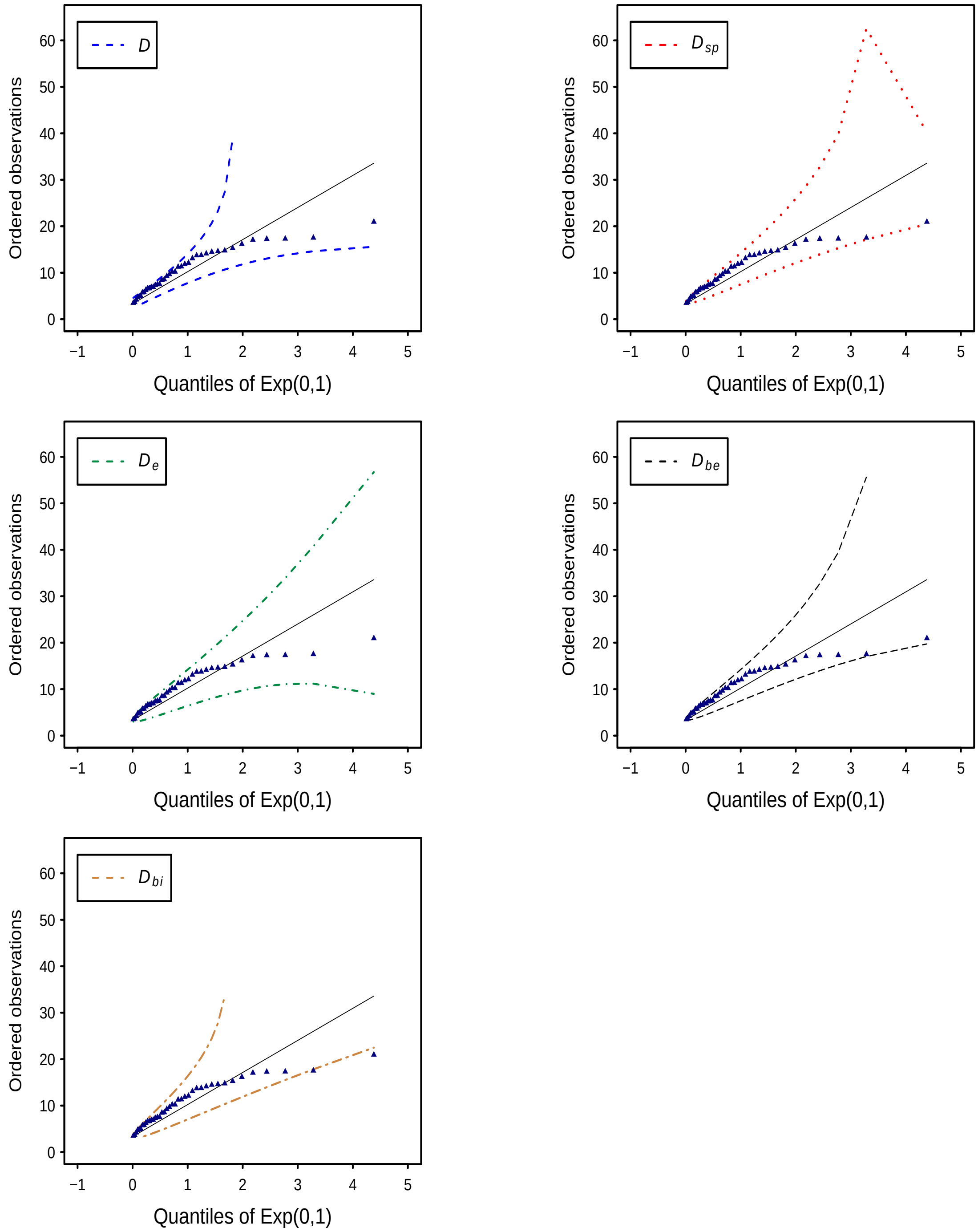

Figure 5.25: Simultaneous probability intervals of the data generated from $\chi^2(10)$ with $n = 40$ for testing the exponential distribution based on the five graphical tests using BLIE ($D(3)$, $D_{sp}(3)$, $D_e(3)$, $D_{be}(3)$ and $D_{bi}(3)$) at $\alpha = 0.05$

Example 2

The data given here arose in tests on endurance of deep–groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test (see Lawless, 1982) and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40. These data were originally discussed by Lieblein and Zelen (1956). Table 5.18 presents the data above in ascending order and the upper and lower bounds (LB and UB) of the ordered observations for the five graphical tests at $\alpha = 0.05$. Figure 5.26 illustrates the Q-Q plots which are constructed by using the data in Table 5.18. From Figure 5.26, the first four points are not distinguishable by eye, as to whether they are outside their corresponding intervals. Then, we use the numerical results in Table 5.18 to check them. The results which can be noted are that the second ordered observation, $Y_{[2]} = 28.92$, falls outside the corresponding interval based on the D_{sp}, D_{be} and D_{bi} tests; the third ordered observation, $Y_{[3]} = 33$, lies outside the interval based on the D_{bi} test and the fourth order observation, $Y_{[4]} = 41.52$, is outside the interval based on all graphical tests.

For the non-graphical tests, the test statistics AD and CvM are 1.3514 and 0.2546, respectively. The corresponding critical values ($\alpha = 0.05$ and $n = 23$) of AD and CvM tests are 1.1055 and 0.2025, respectively. Thus, we can conclude that the null hypothesis is rejected at $\alpha = 0.05$ under AD and CvM tests because the test statistics are larger than their critical values.

Table 5.17 summarises the conclusions when the seven tests are applied to assess whether these data follow an exponential distribution.

Table 5.17: Test results for Example 2

Test	Inference
D	Rejects H_0
D_{sp}	Rejects H_0
D_e	Rejects H_0
D_{be}	Rejects H_0
D_{bi}	Rejects H_0
AD	Rejects H_0
CvM	Rejects H_0

Table 5.18: The number of million revolutions before failure for each of the 23 ball bearings in the life test and their corresponding probability intervals for testing the exponential distribution at $\alpha = 0.05$ based on the five graphical tests using BLIE

			D		D_{sp}		D_e		D_{be}		D_{bi}	
k	$X_{[k]}$	Quantiles	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
1	17.88	0.022	NaN	29.1039	15.8251	23.1876	11.7589	24.0011	15.5166	24.0247	NaN	21.1948
2	28.92	0.0674	NaN	32.2264	15.5831	28.6392	11.4953	29.2063	15.5887	28.7788	NaN	27.3987
3	33.00	0.1151	NaN	35.5393	16.2085	33.4857	11.8324	34.0463	16.1582	33.412	NaN	32.9224
4	41.52	0.1651	NaN	39.0673	17.2375	38.2186	12.5075	38.807	17.1543	38.0753	NaN	38.3455
5	42.12	0.2177	NaN	42.8402	18.5444	42.9963	13.4245	43.6119	18.446	42.8309	NaN	43.8612
6	45.60	0.2733	17.7188	46.8947	20.0787	47.9073	14.5383	48.5379	19.9758	47.7418	16.2055	49.5849
7	48.80	0.3321	20.2374	51.2761	21.8191	53.0187	15.8259	53.6454	21.7184	52.8662	17.9329	55.6113
8	51.84	0.3947	22.8783	56.0419	23.759	58.3915	17.2763	58.9897	23.6657	58.2622	19.8794	62.0355
9	51.96	0.4613	25.6542	61.2661	25.9014	64.0894	18.8857	64.6281	25.8196	63.9925	22.048	68.9654
10	54.12	0.5328	28.5795	67.0462	28.2567	70.1842	20.6553	70.624	28.1901	70.1291	24.4498	76.5347
11	55.56	0.6098	31.6711	73.515	30.8419	76.7614	22.5906	77.0516	30.7941	76.7592	27.1043	84.9188
12	67.80	0.6931	34.9493	80.8588	33.6818	83.928	24.7006	84.0013	33.6565	83.9918	30.0395	94.3624
13	68.64	0.7841	38.438	89.352	36.8097	91.8218	26.9978	91.5873	36.8107	91.9688	33.2936	105.2264
14	68.64	0.8842	42.166	99.4226	40.2702	100.6277	29.4984	99.9584	40.302	100.8806	36.9187	118.0831
15	68.88	0.9954	46.1686	111.7929	44.1235	110.6029	32.2214	109.315	44.1907	110.9932	40.9857	133.9336
16	84.12	1.1206	50.4895	127.8344	48.452	122.1219	35.1891	119.9369	48.5599	122.6953	45.593	154.8008
17	93.12	1.2637	55.1838	150.6909	53.3717	135.7605	38.4238	132.2331	53.5255	136.5863	50.8813	185.9577
18	98.64	1.4307	60.3222	190.8937	59.0527	152.4713	41.9418	146.8345	59.2566	153.6612	57.0618	253.3089
19	105.12	1.6314	65.9974	NaN	65.7593	173.9975	45.733	164.7866	66.0116	175.7437	64.4704	NaN
20	105.84	1.8827	72.335	NaN	73.9356	204.06	49.6957	188.003	74.2131	206.7132	73.6916	NaN
21	127.92	2.2192	79.5102	NaN	84.4254	253.0773	53.3954	220.5422	84.614	257.1515	85.8902	NaN
22	128.04	2.73	87.7787	NaN	99.2032	380.1168	54.8792	273.4166	98.6951	375.6861	104.0047	NaN
23	173.40	3.8286	97.5346	NaN	125.7487	296.7996	40.2923	396.7201	120.5266	∞	141.1244	NaN

NaN stands for Not a Number representing an undefined value.

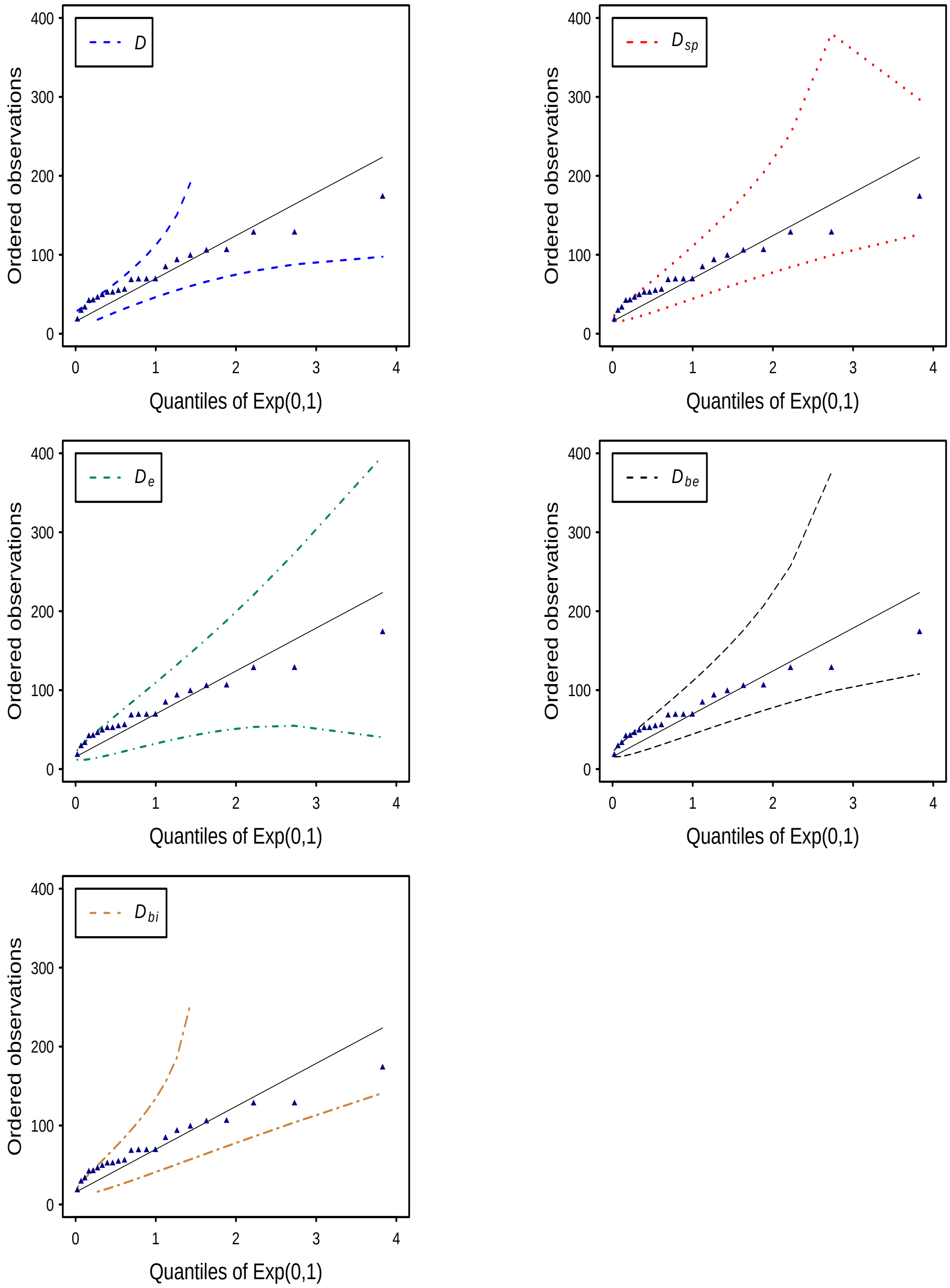

Figure 5.26: Simultaneous probability intervals of the number of million revolutions before failure for each of the 23 ball bearings in the life test for testing the exponential distribution based on the five graphical tests using BLIE ($D(3)$, $D_{sp}(3)$, $D_e(3)$, $D_{be}(3)$ and $D_{bi}(3)$) at $\alpha = 0.05$

Table 5.19: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.01$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$\chi^2(1)$	5	4.89	4.9	2.8	4.72	4.51	4.46	**5.17**
	10	11.75	**15.92**	6.05	13.33	7.82	13.66	14.35
	15	20.03	**25.62**	8.19	20.61	8.02	23.72	23.06
	20	27.39	**35.56**	10.38	31.91	8.07	34.89	32.77
	25	34.76	43.33	12.15	41.58	7.98	**44.54**	40.9
	30	42.43	51.64	13.5	50.34	9	**54.19**	50.1
	40	57.15	70.41	19.37	69.15	10.15	**72.15**	65.63
	50	70.01	80.08	23.46	77.89	11.2	**83.04**	77.37
	100	96.09	98.78	65.2	98.32	29.37	**99.31**	98.14
	150	99.5	99.91	92.28	99.96	63.84	**99.95**	99.79
	200	**100**	**100**	99.3	100	80.62	**100**	**100**
$\chi^2(3)$	5	0.54	0.6	**0.96**	0.59	0.54	0.84	0.53
	10	0.43	0.43	1.13	0.53	**1.42**	1.39	0.34
	15	0.97	0.54	1.91	0.36	**3.01**	2.1	0.57
	20	1.09	0.84	2.11	0.4	**3.68**	2.96	0.83
	25	1.76	1.28	2.95	0.51	**4.75**	3.96	1.4
	30	2.64	2.13	3.84	0.64	**6.69**	5.54	2.54
	40	3.79	4.3	6.15	1.85	**8.89**	8.56	4.17
	50	6.58	6.23	8.71	2.62	11.03	**12.72**	7.33
	100	21.29	24.76	25.38	14	21.54	**39.26**	27.31
	150	44.68	49.15	45.55	35.16	35.46	**68.32**	55.51
	200	63.42	66.91	63.06	60.16	43.38	**84.1**	74.37
$\chi^2(4)$	5	0.22	0.42	**0.96**	0.42	0.15	0.69	0.16
	10	0.61	0.33	1.71	0.42	**2.37**	2.15	0.38
	15	1.47	0.86	3.12	0.29	**5.18**	4.73	1.06
	20	2.87	1.96	5.27	0.55	**8.21**	7.56	2.64
	25	4.66	3.44	7.47	1.32	**11.52**	10.99	4.77
	30	7.24	6.25	10.45	2.23	**16.31**	15.76	8.04
	40	13.8	14.56	19.28	6.91	22.92	**28.12**	16.51
	50	23.23	22.92	27.79	11.76	29.79	**40.53**	28.6
	100	67.41	69.51	68.97	54.9	59.74	**87.1**	78.18
	150	92.93	93.01	90.95	87.38	83.94	**98.54**	96.98
	200	98.87	98.52	97.75	95.54	92.13	**99.8**	99.64
$\chi^2(6)$	5	0.25	0.35	**1.43**	0.44	0.24	0.94	0.2
	10	0.89	0.45	3.08	0.28	4.1	**4.19**	0.52
	15	3.27	1.96	6.61	0.42	**10.3**	10.14	2.59
	20	7.41	5.24	12.23	1.57	**18.24**	17.77	7.4
	25	13.26	10.33	18.34	4.25	25.46	**27.78**	14.69
	30	19.85	16.66	25.64	7.74	34.7	**38.16**	24.08
	40	37.78	37.58	44.09	23.69	48.84	**60.62**	45.69
	50	56.09	53.34	58.6	36.69	60.62	**76.31**	66.28
	100	95.45	95.32	94.53	90.12	91.45	**99.17**	98.22
	150	99.89	99.81	99.62	99.39	99.06	**99.99**	99.98
	200	**100**	99.98	99.96	99.64	99.84	**100**	**100**
$\chi^2(10)$	5	0.16	0.3	**1.56**	0.31	0.13	1.04	0.15
	10	1.63	0.62	4.82	0.2	5.92	**6.94**	0.99
	15	6.77	3.73	11.95	0.64	17.15	**18.84**	6.38
	20	14.98	11.27	22.31	3.99	30.31	**32.55**	17.05
	25	26.72	21.36	33.4	10.39	42.28	**47.75**	31.57
	30	38.57	33.72	43.88	18.5	54.31	**61.06**	46.58
	40	63.24	60.77	66.47	45.74	70.93	**81.97**	72.24
	50	80.92	76.95	80.26	62.02	81.85	**92.63**	88.45
	100	99.42	99.37	99.16	98.34	98.77	**99.94**	99.87
	150	**100**	**100**	**100**	99.98	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	2.08	1.69	1.67	1.22	2.2	1.62	**2.46**
	10	3.81	2.85	**4.8**	1.91	3.61	3.01	4.78
	15	5.04	2.92	**7.65**	1.63	3.51	4	6.31
	20	6.09	3.43	**10.44**	2.24	3.68	5.05	7.58
	25	6.98	3.61	**12.07**	2.48	3.22	5.81	8.6
	30	7.84	4.08	**14.04**	2.53	3.22	6.39	9.49
	40	9.71	5.26	**18.47**	3.65	2.91	8.24	12.08
	50	11.07	5.33	**21.03**	3.5	2.54	9.62	13.81
	100	19.94	11.4	**38.86**	7.28	3.97	20.87	24.45
	150	31.03	23.15	**55.22**	16.47	9.56	36.85	37.48
	200	40.72	35.26	**67.69**	28.46	13.92	52.51	49.48
$HN(0,1)$	5	0.32	0.44	**0.97**	0.5	0.19	0.67	0.17
	10	0.43	0.32	1.57	0.45	1.9	**1.93**	0.23
	15	1.05	0.54	2.09	0.39	**3.46**	3.42	0.72
	20	1.89	0.96	2.82	0.57	**5.73**	4.42	1.57
	25	2.96	1.44	3.74	0.69	**8.5**	6.32	2.66
	30	4.26	2.29	4.49	0.91	**11.32**	8.66	4.58
	40	7.29	5.32	6.53	2.27	**16.43**	14.06	8.34
	50	11.13	7.11	7.94	2.71	**21.45**	19.32	13.76
	100	34.06	30.96	18.33	12.68	47.19	**54.15**	46.73
	150	60.65	60.07	30.98	35.12	69.11	**80.56**	76.08
	200	78.66	77.64	43.54	60.13	79.57	**92.43**	90.42
$Wbl(0,2,2)$	5	0.09	0.24	**1.64**	0.26	0.08	1.13	0.07
	10	1.75	0.51	5.27	0.19	6.41	**7.27**	0.99
	15	7.2	3.84	12.11	0.73	17.63	**19.33**	6.41
	20	14.7	10.34	20.98	3.41	31.56	**33.1**	17.08
	25	25.38	17.41	29.76	8.57	44.55	**49.3**	31.99
	30	38.83	30.85	40.99	15.8	57.85	**63.48**	48.22
	40	61.27	57.17	61.24	39.65	73.84	**83.52**	74.12
	50	78.71	73.2	75.04	56.27	84.49	**93.09**	88.8
	100	99.69	99.66	99.17	98.36	99.53	**99.99**	99.98
	150	**100**	**100**	99.99	99.97	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	14.76	14.15	9.54	12.53	13.87	13.54	**15.03**
	10	37.8	41.4	23.32	37.02	29.93	41.65	**43.39**
	15	58.54	63.32	35.89	56.92	39.39	65.14	**65.15**
	20	73.37	77.77	45.87	74.32	46.46	**80.77**	79.47
	25	82.99	86.08	54.01	84.53	51.72	**89.62**	88.19
	30	89.93	92.64	62.09	91.25	60.14	**95.03**	93.41
	40	96.95	98.31	77.89	97.89	70.38	**98.87**	98.36
	50	98.99	99.48	88.17	99.41	80.26	**99.76**	99.58
	100	**100**	**100**	99.92	**100**	98.55	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.20: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.01$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$U(0,1)$	5	0.21	0.31	**2.95**	0.38	0.12	2.34	0.08
	10	2.45	1.01	6.56	0.38	7.1	**10.72**	1.86
	15	7.55	2.95	10.6	0.75	**29.27**	22.4	8.38
	20	14.5	7.49	15.88	2.35	**60.74**	35.43	19.6
	25	23.32	18.95	21.18	4.6	**82.66**	48.92	33.56
	30	32.36	41.33	25.87	7.68	**93.95**	60.82	46.35
	40	50.54	86.96	38.36	41.34	**99.55**	81.36	69.81
	50	68.24	97.96	47.86	74.42	**99.98**	91.92	86.33
	100	98.24	**100**	87.8	**100**	**100**	99.98	99.9
	150	99.99	**100**	98.47	**100**	**100**	**100**	**100**
	200	**100**	**100**	99.91	**100**	**100**	**100**	**100**
$beta(2,2)$	5	0.09	0.21	**2.7**	0.27	0.03	1.9	0.05
	10	3.92	1.47	9.74	0.27	11.3	**15.22**	2.81
	15	14.36	7.61	20.32	1.47	34.95	**36.98**	16.02
	20	29	19	34.28	7.11	**65.09**	59.11	38.28
	25	46.52	34.83	47.57	16.43	**84.2**	75.82	59.6
	30	60.23	56.37	57.92	26.77	**95.01**	86.37	76.44
	40	82.79	90.95	78.7	65.49	**99.6**	97.18	93.29
	50	94.31	98.68	89.47	86.91	**99.97**	99.52	98.82
	100	99.99	**100**	99.92	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	0.17	0.27	**1.39**	0.32	0.11	0.92	0.13
	10	1.31	0.47	4.08	0.2	4.93	**5.51**	0.74
	15	4.96	2.59	8.29	0.42	12.48	**13.86**	4.01
	20	9.28	5.98	14.23	1.95	**24.49**	24.19	10.52
	25	17.73	12.06	21.65	4.87	36.31	**37.74**	22.1
	30	26.93	20.8	29.63	8.55	48.33	**50.05**	34.37
	40	45.84	42.8	47.03	26.22	68.22	**72.79**	58.87
	50	65.06	60.37	60.75	38.12	80.71	**86.37**	78.24
	100	98.1	98.9	95.86	94.37	99.5	**99.88**	99.76
	150	99.96	**100**	99.82	99.95	**100**	**100**	**100**
	200	**100**	**100**	99.99	**100**	**100**	100	**100**
$beta(5,1.5)$	5	0.07	0.09	**7.41**	0.23	0.03	5.54	0.03
	10	16.84	7.38	29.85	0.93	34.89	**46**	17.4
	15	49.53	32.51	56.78	13.27	**82.77**	79.72	59.6
	20	74.5	66.24	77.18	40.19	**97.86**	93.57	86.12
	25	89.49	90.09	88.4	65.66	**99.78**	98.35	96.01
	30	96.3	98.43	94.77	84.62	**99.99**	99.61	99.05
	40	99.63	**100**	99.13	99.59	**100**	100	99.98
	50	99.99	**100**	99.93	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	0.73	1.16	4.51	1.18	0.54	**4.83**	0.6
	10	3.04	2.97	5.85	3.11	7.4	**10.35**	2.73
	15	6.84	5.4	7.52	4.29	**30.79**	17.39	7.53
	20	10.56	11.14	9.49	6.66	**57.96**	25.51	14.19
	25	15.09	24.86	10.33	9.64	**76.88**	34.97	21.44
	30	20.94	48.27	11.98	14.94	**89.29**	45.66	31.05
	40	33.71	85.31	18.4	52.5	**98**	65.94	49.42
	50	49.77	96.74	24.68	77.2	**99.8**	81.77	68.98
	100	92.98	**100**	65.41	99.99	**100**	99.86	98.84
	150	99.84	**100**	92.93	**100**	**100**	**100**	99.99
	200	**100**	**100**	99.07	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	4.02	**4.35**	1.98	3.55	3.27	3.44	3.54
	10	6.62	**10.24**	1.72	9.79	3.86	7.61	7.41
	15	10.9	**17.41**	1.88	15.26	3.33	14.21	11.96
	20	14.69	**23.8**	1.73	23.43	2.85	20.27	16.81
	25	18.27	**29.64**	1.59	29.4	2.59	26.54	21.16
	30	24.2	**37.92**	1.7	36.68	3.26	34.85	27.46
	40	33.85	**54.34**	2.29	53.69	3.06	50.26	39.13
	50	44.08	**63.56**	2.73	62.74	3.19	62.08	50.31
	100	80.06	**94.35**	23.45	93.58	8.25	94.14	84.87
	150	95.61	99.27	63.89	99.21	27.95	**99.35**	97.06
	200	99.09	99.9	88.65		41.2	**99.94**	99.5
$beta(1,2)$	5	0.26	0.43	**1.06**	0.57	0.22	0.84	0.15
	10	0.69	0.55	1.85	0.47	2.33	**2.57**	0.29
	15	1.69	0.89	2.97	0.4	**5.29**	5.06	1.27
	20	2.76	1.33	3.66	0.56	**9.82**	6.85	2.54
	25	4.3	2.04	4.43	0.9	**17**	9.51	4.54
	30	6.26	3.17	5.48	0.96	**26.87**	13.36	7.46
	40	10.69	8.91	7.64	2.7	**45.5**	21.37	13.65
	50	15.81	17.41	9.39	3.72	**61.9**	30.07	22.48
	100	46.13	87.77	21.47	45.63	**99.19**	73.2	65.2
	150	75.91	99.88	37.33	95.2	**100**	94.74	91.62
	200	90.69	**100**	52.56	99.98	**100**	99.26	98.36

The bold number is the highest power among the seven tests for each sample size.

Table 5.21: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.01$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	AD(1)	CvM(1)
$Laplace(0,1)$	5	0.21	0.22	**5.86**	0.25	0.21	4.22	0.18
	10	18.53	10.03	30.58	2.07	32.29	**37.51**	16.49
	15	51.66	40.47	59.13	20.78	62.73	**68.79**	53.05
	20	74.5	65.27	77.71	49.39	80.01	**85.76**	77.37
	25	87.3	81.3	87.82	72	88.86	**93.7**	89.75
	30	94.35	91.41	94.2	85.32	94.46	**97.35**	95.89
	40	99.21	98.54	99.01	97.38	98.46	**99.7**	99.54
	50	99.89	99.76	99.8	99.46	99.67	**99.96**	99.96
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	0.15	0.16	**4.18**	0.24	0.11	2.79	0.09
	10	11.47	5.78	21.6	0.73	23.9	**29.67**	9.92
	15	38.55	28.11	46.86	11.77	54.12	**60.31**	41.03
	20	60.86	51.65	66.33	34.98	73.76	**79.25**	67.36
	25	77.74	69.79	79.35	56.85	85.09	**89.68**	83.53
	30	88.73	83.93	88.49	73.43	92.32	**95.51**	92.79
	40	97.66	96.42	97.07	93.05	97.7	**99.45**	98.97
	50	99.6	99.18	99.37	97.87	99.35	**99.92**	99.83
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	3.45	2.46	**19.33**	1.9	3.77	17.27	3.69
	10	45.52	36.21	54.57	21.54	**56.42**	56.22	44.01
	15	74.92	69.18	79.13	55.31	80.74	**81.07**	74.98
	20	87.95	85.35	90.57	77.87	90.92	**91.8**	89.01
	25	94.52	93.02	95.59	89.92	95.32	**96.6**	95.42
	30	97.35	96.95	98.04	95.5	97.81	**98.44**	98.03
	40	99.63	99.58	99.76	99.28	99.49	**99.86**	99.77
	50	99.92	99.9	99.91	99.88	99.88	**99.96**	99.95
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	0.07	0.22	**3.49**	0.26	0.05	2.26	0.08
	10	7.33	3.21	16.37	0.32	18.7	**24.51**	6.29
	15	28.74	19.53	37.07	6.38	47.32	**53.14**	31.52
	20	51.78	41.01	57.55	23.35	70.84	**75.23**	59.92
	25	70.7	61.07	72.24	43.63	83.55	**87.82**	79.4
	30	83.99	77.42	83.11	61.36	92.4	**94.55**	90.71
	40	96.11	94.72	94.88	88.18	98.06	**99.34**	98.52
	50	99.17	98.86	98.6	96.5	99.6	**99.85**	99.75
	100	99.99	**100**	**100**	99.99	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	3.93	2.83	**19.56**	1.92	4.2	17.22	4.28
	10	45.48	36.01	54.15	21.69	56.42	**56.56**	44.05
	15	74.53	69.1	78.74	55.66	80.53	**81.37**	74.72
	20	87.82	85.52	90.67	78.23	91.01	**91.77**	89.12
	25	94.45	93.24	95.68	89.43	95.42	**96.55**	95.61
	30	97.57	97.32	98.25	95.27	98.05	**98.84**	98.3
	40	99.61	99.65	99.73	99.31	99.6	**99.83**	99.74
	50	99.98	99.96	99.98	99.86	99.93	**100**	**99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	0.19	0.25	**6.26**	0.19	0.17	4.32	0.2
	10	18.8	11.45	29.12	3.27	31.43	**36.4**	17.46
	15	48.12	39.56	55.3	22.13	60.29	**64.84**	50.19
	20	71.33	64.7	75.37	49.52	78.45	**83.22**	74.71
	25	85.36	80.42	86.83	70.16	88.22	**92.53**	88.44
	30	92.4	89.55	92.78	82.89	93.52	**96.6**	94.74
	40	98.58	97.93	98.45	96.35	98.07	**99.59**	99.21
	50	99.67	99.41	99.55	98.84	99.31	**99.88**	99.82
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	0.11	0.19	**5.17**	0.21	0.12	3.73	0.14
	10	15.41	8.88	26.29	1.97	28.99	**33.97**	13.89
	15	43.69	33.81	51.5	17.62	57.37	**62.5**	45.55
	20	67.64	60.11	72.44	44	77.17	**81.88**	72.66
	25	82.51	76.49	84.06	65.29	86.9	**91.93**	86.88
	30	90.84	86.9	90.95	79.42	92.52	**96.11**	93.85
	40	98.19	97.28	97.97	95.04	97.52	**99.27**	98.88
	50	99.69	99.35	99.46	98.75	99.23	**99.96**	99.9
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	0.07	0.2	**4.1**	0.21	0.06	3.02	0.06
	10	12.6	6.58	22.67	0.89	25.36	**30.99**	11.32
	15	38.85	29.32	46.44	13.2	53.88	**59.91**	41.43
	20	62.03	54.01	67.3	36.69	73.91	**79.35**	67.92
	25	80.03	72.33	81.26	58.98	85.54	**90.51**	85
	30	89.2	84.9	89.38	74.46	92.13	**95.67**	93.13
	40	97.89	96.82	97.48	93.64	97.66	**99.42**	98.95
	50	99.67	99.29	99.39	98	99.34	**99.96**	99.92
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.22: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.01$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$\chi^2(1)$	5	2.63	2.63	0.49	2.88	2.63	**3.71**	3.2
	10	7.47	7.3	3.54	7.44	2.84	**11.93**	9.14
	15	12.83	12.69	5.33	12.92	2.38	**21.42**	16.06
	20	19.37	18.8	6.8	19.51	2.03	**31.41**	24.21
	25	26.03	24.68	8.17	27.24	1.85	**41.38**	33.09
	30	34.1	33.19	9.42	36.04	2.18	**50.93**	41.73
	40	49.3	51.14	14.11	53.23	2.78	**69.12**	58.22
	50	62.68	64.44	16.84	67.61	3.24	**81.44**	70.78
	100	94.62	96.56	47.14	97.59	13.64	**99.15**	97.25
	150	99.38	99.74	81.44	99.94	44.11	**99.95**	99.74
	200	**100**	**100**	96.99	**100**	67.06	**100**	99.99
$\chi^2(3)$	5	0.9	0.9	**1.44**	1.01	0.9	0.83	0.98
	10	1.41	1.72	2.1	1.71	**2.3**	1.05	1.41
	15	2.14	2.87	3.03	2.57	**3.48**	1.82	2.23
	20	2.41	3.19	3.17	3.25	**3.98**	2.27	3.04
	25	3.18	4.13	3.95	4.02	**4.93**	3.39	4.3
	30	4.62	6.05	5.26	5.27	**6.7**	4.68	5.72
	40	6.34	9.14	8.26	8.28	**9.2**	7.29	8.16
	50	9.5	**12.46**	10.85	11.51	11.11	11.33	12.2
	100	26.33	33.8	28.42	34.88	22.75	**35.72**	35.03
	150	50.03	57.85	47.95	57.5	36.7	**66.21**	62.74
	200	68.01	74.7	64.59	73.98	45.41	**82.44**	79.23
$\chi^2(4)$	5	0.83	0.83	**1.51**	1.1	0.83	0.61	0.84
	10	2.35	2.76	3.35	2.29	**3.64**	1.64	2.31
	15	3.99	4.89	5.33	4.55	**6.02**	3.81	4.88
	20	6.03	7.35	7.34	7.44	**8.41**	6.22	7.64
	25	8.72	10.3	9.81	10.42	**11.27**	9.33	11.22
	30	12.21	15.23	13.59	13.6	**16.4**	13.63	15.79
	40	20.42	25.61	23.37	22.57	23.09	24.67	**26.64**
	50	30.41	35.97	32.25	33.82	30.06	37.95	**38.72**
	100	73.5	78.52	71.33	78.46	60.75	**85.24**	84.1
	150	94.58	95.15	91.32	95.2	83.7	**98.36**	97.93
	200	99.13	99.08	97.61	99.02	92.35	**99.78**	99.76
$\chi^2(6)$	5	1.24	1.24	**2.22**	1.39	1.24	0.84	1.08
	10	3.98	4.76	5.66	4.16	**6.08**	3.3	4.54
	15	7.99	9.51	10.07	9.36	**11.49**	8.51	10
	20	13.58	16.06	16.07	15.93	**18.51**	14.97	17.65
	25	21.16	22.96	22.05	22.52	24.6	24.64	**27.88**
	30	29.21	33.21	30.28	30.66	34.16	34.73	**38.43**
	40	46.89	52.8	48.86	49.22	48.71	56.75	**58.92**
	50	64.47	66.83	61.82	65.19	59.94	74.38	**75.48**
	100	96.73	96.97	95.03	96.99	91.26	**98.97**	98.89
	150	99.93	99.89	99.56	99.94	98.96	**99.99**	**99.99**
	200	**100**	99.99	99.96	**100**	99.84	**100**	**100**
$\chi^2(10)$	5	1.37	1.37	**2.62**	1.62	1.37	1.02	1.29
	10	6.03	6.95	8.33	7.1	**8.57**	5.54	7.24
	15	14.03	16.15	17.05	16.38	**18.78**	16.44	18.6
	20	25	27.2	27.12	26.94	30.66	28.56	**32.45**
	25	37.24	39.01	37.55	37.99	40.64	44.3	**47.9**
	30	49.82	52.28	48.47	49.16	52.93	57.73	**61.55**
	40	70.94	73.31	70.01	70.59	69.85	79.57	**81.13**
	50	85.56	85.85	81.93	84.3	80.81	91.92	**92.43**
	100	99.59	99.65	99.24	99.68	98.66	99.94	**99.94**
	150	**100**	**100**	**100**	**100**	99.98	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	1.75	1.75	0.66	1.84	1.75	**2.18**	1.93
	10	3.32	3.07	3.23	3.17	2.62	**4.33**	3.74
	15	4.38	3.73	5.88	3.92	2.25	**5.99**	5.42
	20	5.52	4.32	**8.04**	4.57	2.35	6.96	6.68
	25	6.27	4.46	**9.65**	4.88	2.21	7.74	7.81
	30	7.17	5.2	**11.45**	4.97	2.58	8.59	8.93
	40	9.42	6.47	**16.2**	6.36	2.39	10.73	11.32
	50	10.69	7.06	**19.06**	7.31	2.58	12.36	13.1
	100	19.71	15.12	**37.2**	17.14	5.4	23.65	24.61
	150	31.41	28.96	**53.85**	31.71	12.22	39.61	38.6
	200	41.08	43.14	**67.54**	45.79	17.79	54.54	51.17
$HN(0,1)$	5	0.87	0.87	**1.49**	0.97	0.87	0.58	0.85
	10	1.83	2.14	2.68	2.04	**2.75**	1.54	2.07
	15	2.76	3.18	3.4	3.03	**3.96**	2.85	3.61
	20	3.86	3.74	3.69	3.66	**5.86**	3.41	4.61
	25	5.62	5.27	4.87	5.09	**7.67**	5.44	7
	30	7.19	6.91	5.69	5.83	**10.41**	7.46	9.35
	40	10.78	10.4	7.85	8.7	**15.22**	12.33	14.41
	50	14.88	13.48	9.36	11.53	19.89	17.84	**20.62**
	100	39.24	39.4	18.7	34.48	45.67	51.11	**54.1**
	150	64.46	65.13	29.7	59.18	67.16	79.16	**80.71**
	200	81.1	81.7	41	76.3	78.15	91.53	**92.26**
$Wbl(0,2,2)$	5	1.46	1.46	**2.72**	1.78	1.46	1.03	1.45
	10	6.52	7.38	8.85	7.4	**9.24**	5.82	7.61
	15	14.38	16.13	16.97	16.36	18.98	16.83	**19.19**
	20	24.46	25.78	25.51	25.88	30.49	29.05	**32.88**
	25	36.48	36.04	33.88	36.32	41.73	45.79	**49.6**
	30	49.46	50.89	45.27	46.55	56.11	60.15	**63.87**
	40	70.12	71.33	64.6	67.89	72.45	81.51	**82.99**
	50	83.75	83.96	76.74	81.62	83.23	92.44	**92.98**
	100	99.8	99.81	99.11	99.81	99.41	99.99	**99.98**
	150	**100**	**100**	99.99	**100**	99.98	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	8.79	8.79	0.26	9.1	8.79	**12.05**	10.5
	10	28.62	28.85	16.31	28.62	16.37	**39.69**	34.4
	15	48.81	48.32	28.7	48.03	20.05	**63.81**	56.69
	20	65.16	64.07	37.8	66.03	24.57	**78.78**	73.02
	25	76.98	74.9	45.17	78.09	29	**88.43**	83.72
	30	86.02	85.52	53.42	86.72	36.09	**94.24**	91.02
	40	95.27	95.43	69.89	96.12	48.33	**98.73**	97.57
	50	98.55	98.53	81.08	99.02	60.75	**99.7**	99.41
	100	**100**	**100**	99.66	**100**	95.93	**100**	**100**
	150	**100**	**100**	**100**	**100**	99.97	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.23: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.01$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$U(0,1)$	5	2.7	2.7	**4.7**	2.85	2.7	2.29	2.84
	10	7.7	7.5	9.74	7.54	10.2	8.91	**10.97**
	15	13.87	13.04	13.79	12.89	**29.74**	20.4	21.88
	20	22.36	24.38	17.91	19.09	**58.48**	32.39	35.16
	25	31.68	42.54	22	28.81	**79.82**	46.39	48.93
	30	40.94	66.37	26.74	44.6	**92.66**	58.68	60.92
	40	58.27	93.8	38.89	80.89	**99.41**	79.39	79.61
	50	72.98	99.25	46.57	96.03	**99.98**	91.17	90.78
	100	98.59	**100**	85.06	**100**	**100**	99.97	99.93
	150	99.99	**100**	97.63	**100**	**100**	**100**	**100**
	200	**100**	**100**	99.83	**100**	**100**	**100**	**100**
$beta(2,2)$	5	2.41	2.41	**4.3**	2.5	2.41	1.85	2.39
	10	11.76	12.34	14.72	12.13	15.23	12.92	**15.55**
	15	24.85	25.02	26.32	24.68	**35.98**	33.78	35.95
	20	41.68	41.58	39.01	38.46	**63.34**	55.56	58.33
	25	57.24	59.27	49.9	53.93	**81.76**	73.34	75.52
	30	69.67	77.98	60.42	66.78	**93.77**	84.87	86.5
	40	87.75	96.04	79.87	91.19	**99.48**	96.52	96.64
	50	96.19	99.55	90	98.55	**99.96**	99.46	99.49
	100	**100**	**100**	99.92	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	1.2	1.2	**2.33**	1.48	1.2	0.84	1.18
	10	5.07	5.59	6.64	5.49	**6.93**	4.47	5.88
	15	10.19	11.17	11.74	11.93	**13.53**	11.75	13.41
	20	17.21	18.5	18.24	18.2	**24.4**	20.74	24.05
	25	26.7	27.01	25.41	25.94	34.17	34.56	**37.76**
	30	36.65	38.34	33.54	34.08	46.62	46.43	**50.58**
	40	55.55	59.37	50.87	54.12	66.21	69.59	**71.41**
	50	72.22	74.3	63.14	69.8	79.08	84.91	**85.66**
	100	98.72	99.4	95.72	99.15	99.44	**99.85**	99.83
	150	99.98	**100**	99.77	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	6.75	6.75	**10.9**	7.59	6.75	5.52	7.24
	10	33.08	32.79	37.53	32.8	41.57	42.79	**46.39**
	15	62.85	62.67	62.35	60.29	**82.93**	78.18	79.25
	20	82.53	87.08	79.3	82.01	**97.54**	92.78	93.4
	25	92.97	97.24	88.95	94.23	**99.69**	98.12	98.38
	30	97.65	99.71	94.87	98.59	**99.99**	99.53	99.63
	40	99.79	**100**	99.12	99.99	**100**	**100**	**100**
	50	99.99	**100**	99.92	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	4.01	4.01	**5.77**	4.19	4.01	4.35	5.08
	10	6.14	5.48	7.24	5.86	**10.57**	8.48	9.05
	15	9.72	9.36	8.06	7.58	**30.84**	15.22	14.82
	20	13.94	21.31	8.24	12.39	**55.64**	21.76	20.87
	25	18.17	38.63	7.99	22.96	**74.28**	30.79	28.48
	30	24.23	62.05	9.52	39.43	**87.96**	41.01	36.94
	40	36.81	89.2	14.56	75.68	**97.74**	61.27	52.79
	50	51.82	97.95	18.86	93.17	**99.7**	78.86	69.95
	100	93.46	**100**	54.95	**100**	**100**	99.81	98.85
	150	99.88	**100**	87.3	**100**	**100**	**100**	99.99
	200	**100**	**100**	98	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	1.88	1.88	0.55	1.98	1.88	**2.53**	2.2
	10	3.74	3.75	0.89	3.5	1.14	**5.78**	4.04
	15	6.25	6.3	0.87	6.36	0.85	**10.73**	7.44
	20	9.01	9.03	0.85	10.66	0.47	**16.04**	10.87
	25	12.26	12.25	0.64	14.72	0.29	**22.16**	14.91
	30	17.5	18.98	0.87	20.25	0.39	**29.68**	20.64
	40	25.79	31	1.11	33.09	0.59	**44.51**	30.57
	50	36.11	42.67	0.84	45.96	0.58	**57.72**	41.78
	100	75.13	85.88	8.72	89.65	2.35	**92.48**	80.92
	150	94.02	97.83	39.86	98.56	12.61	**99.17**	96.02
	200	98.78	99.72	73.59	**99.89**	24.27	99.87	99.28
$beta(1,2)$	5	0.98	0.98	**1.81**	1.19	0.98	0.7	0.89
	10	2.45	2.81	3.42	2.69	**3.63**	1.96	2.67
	15	3.88	4.21	4.55	3.89	**5.89**	4.02	4.95
	20	5.37	4.87	4.8	5.36	**9.29**	5.8	7.13
	25	7.18	6.3	5.29	6.44	**14.62**	8.34	10.22
	30	10.03	9.78	6.67	7.1	**23.97**	11.84	14.37
	40	14.87	17.76	8.67	11.13	**42.46**	18.84	21.4
	50	20.24	28.56	10.16	18.08	**58.63**	28.23	30.28
	100	50.42	91.28	20.69	79.72	**98.93**	70.75	71.09
	150	78.55	99.93	33.87	99.15	**100**	94.08	93.5
	200	91.78	**100**	48.04	**100**	**100**	99.11	98.8

The bold number is the highest power among the seven tests for each sample size.

Table 5.24: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.01$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$Laplace(0,1)$	5	5.41	5.41	**8.86**	5.82	5.41	4.2	5.48
	10	33.41	34.55	37.98	34.51	37.23	34.61	**38.59**
	15	62.73	62.9	64.01	62.33	62.9	66.82	**69.67**
	20	81.56	80.13	80.06	80.55	79.08	84.12	**86.44**
	25	90.92	89.37	88.87	90.49	87.59	93.01	**94.28**
	30	96.03	95.52	94.69	95.38	93.69	97.04	**97.67**
	40	99.47	99.21	99.08	99.23	98.2	99.65	**99.74**
	50	99.94	99.86	99.82	99.93	99.59	99.96	**99.97**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	3.68	3.68	**6.55**	3.88	3.68	2.77	3.78
	10	24.45	25.85	29.06	25.3	29.02	26.49	**30.44**
	15	50.82	51.98	53.03	50.96	55.07	57.66	**60.38**
	20	70.97	69.97	69.54	69.95	72.82	77.01	**79.7**
	25	83.63	82.47	80.94	83.1	83.47	88.61	**90.08**
	30	91.83	91.43	89.46	90.59	91.1	94.93	**95.71**
	40	98.45	98.17	97.41	97.97	97.39	99.35	**99.47**
	50	99.73	99.67	99.4	99.59	99.26	99.92	**99.95**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	18.97	18.97	**20.01**	19.51	18.97	18.12	19.46
	10	56.16	56.96	58.07	56.96	58.85	56.42	**59.19**
	15	80.22	81.38	81.23	80.46	80.12	81.71	**82.95**
	20	90.88	91.68	91.38	91.6	90.07	92.05	**92.94**
	25	95.9	96	95.94	96.29	94.53	96.91	**97.24**
	30	97.9	98.35	98.18	98.54	97.41	98.65	**98.74**
	40	99.78	99.8	99.77	99.73	99.44	99.88	**99.9**
	50	99.94	99.93	99.92	99.93	99.88	99.96	**99.97**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	3.16	3.16	**5.55**	3.37	3.16	2.26	3.04
	10	19.04	20.47	23.16	20.25	23.94	21.13	**25.03**
	15	41.15	42.52	43.67	42.42	48.03	50.15	**52.84**
	20	63.28	62.69	61.37	61.88	69.58	72.22	**75.32**
	25	78.63	77.01	74.24	76.1	82.06	86.33	**87.92**
	30	88.98	88.4	84.6	86.41	91.27	93.8	**94.68**
	40	97.48	97.54	95.52	96.67	97.76	99.19	**99.29**
	50	99.44	99.43	98.66	99.29	99.48	99.84	**99.88**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	19.53	19.53	**20.06**	19.05	19.53	18.39	19.97
	10	56	56.87	57.92	57.28	58.82	56.61	**59.53**
	15	79.82	80.94	80.82	80.36	80.26	82.36	**83.46**
	20	90.97	91.49	91.16	91.2	90.24	92.03	**92.94**
	25	95.77	96.05	95.8	96.26	94.69	96.76	**97.06**
	30	98.24	98.53	98.41	98.43	97.62	98.92	**99.04**
	40	99.72	99.76	99.78	99.74	99.49	99.85	**99.87**
	50	**100**	99.99	99.99	99.98	99.88	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	5.77	5.77	**9.11**	5.73	5.77	4.32	5.87
	10	31.87	33.41	35.96	33.61	36.16	33.62	**37.13**
	15	58.3	59.82	60.7	59.67	60.79	62.93	**65.4**
	20	78.33	78.28	77.94	77.72	77.84	81.85	**83.8**
	25	89.15	88.66	87.9	88.09	87.2	91.93	**93.13**
	30	94.71	94.55	93.46	93.88	92.63	96.28	**96.91**
	40	98.97	98.89	98.58	98.72	97.79	99.54	**99.6**
	50	99.77	99.71	99.57	99.74	99.18	99.87	**99.88**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	4.62	4.62	**7.61**	5.05	4.62	3.69	4.63
	10	29.27	30.54	33.31	30.39	33.43	30.88	**35**
	15	54.85	56.11	57.17	55.85	57.96	60.13	**62.95**
	20	75.62	75.46	75.19	74.73	76.46	79.96	**82.32**
	25	87.63	86.52	85.42	86.13	85.61	91.16	**92.46**
	30	93.54	92.82	91.72	92.93	91.71	95.72	**96.41**
	40	98.7	98.5	98.18	98.34	97.21	99.17	**99.29**
	50	99.8	99.71	99.54	99.76	99.16	99.96	**99.98**
	100	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	3.75	3.75	**6.67**	4.11	3.75	2.94	3.78
	10	25.81	26.96	29.97	27.35	30.19	28.07	**31.58**
	15	50.77	51.58	52.61	51.03	54.61	57.6	**60.2**
	20	71.68	70.94	70.41	71.44	72.9	76.91	**79.47**
	25	85.34	84.23	82.89	83.78	84.1	89.48	**91.03**
	30	92.66	92.11	90.32	91.03	91.26	95.25	**96.12**
	40	98.65	98.3	97.66	98.14	97.3	99.29	**99.43**
	50	99.79	99.78	99.45	99.58	99.22	99.93	**99.95**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.25: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.01$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$\chi^2(1)$	5	1.24	1.24	0.7	1.17	1.24	**2.34**	1.28
	10	4	3.91	4.3	4.39	1.19	**9.03**	5.18
	15	8.33	8.41	5.94	9.05	1.29	**17.45**	10.86
	20	14.57	13.55	7.44	15.22	1.44	**28.05**	18.59
	25	20.56	20.02	8.75	22.48	1.5	**37.91**	26.69
	30	28.4	27.7	9.97	30.88	1.73	**47.45**	35.37
	40	44.48	46.84	14.79	50.09	2.84	**66.41**	52.76
	50	58.43	60.92	17.14	63.66	3.62	**79.31**	66.8
	100	93.73	96.3	44.48	97.32	15.57	**99.06**	96.45
	150	99.31	99.72	79.47	99.94	45.68	**99.95**	99.7
	200	100	100	96.31	100	69.16	**100**	99.99
$\chi^2(3)$	5	1.42	1.42	**1.43**	1.39	1.42	1.05	1.2
	10	2.24	2.3	1.95	2.27	**2.54**	1.79	2.29
	15	2.92	3.33	2.88	3.11	**3.61**	2.58	3.29
	20	3.38	3.62	3.02	3.89	4.25	3.48	**4.32**
	25	4.07	4.75	3.85	4.6	5.09	4.55	**5.41**
	30	5.52	6.5	5.16	5.8	6.69	5.99	**6.95**
	40	7.72	**9.94**	8.11	9.42	9.9	8.87	9.75
	50	11.36	13.28	10.65	11.97	12.05	13.2	**14.69**
	100	28.48	35.85	28.27	36.18	24.38	**39.55**	37.73
	150	52.58	59.03	48.18	58.8	38.01	**68.28**	65.22
	200	70.19	75.83	64.67	75.11	47.26	**83.85**	80.48
$\chi^2(4)$	5	1.49	1.49	1.51	**1.71**	1.49	1.14	1.34
	10	3.38	3.69	3.15	3.15	**3.98**	2.77	3.66
	15	5.27	6.19	5.06	5.63	6.25	5.58	**6.52**
	20	7.8	8.26	7.17	8.65	8.83	8.5	**9.90**
	25	10.53	11.8	9.49	11.63	11.84	12.15	**13.96**
	30	14.6	16.33	13.45	14.8	16.05	16.79	**19.16**
	40	23.74	27.18	23.07	24.79	24.77	28.91	**30.39**
	50	33.54	37.41	31.92	34.66	32.59	41.26	**43.17**
	100	75.49	80.16	71.32	79.71	63.02	**87.29**	85.64
	150	95.24	95.47	91.42	95.53	84.88	**98.56**	98.26
	200	99.24	99.19	97.67	99.08	93.07	**99.80**	99.78
$\chi^2(6)$	5	2.05	2.05	**2.20**	2.03	2.05	1.66	1.97
	10	5.74	6.21	5.35	5.51	6.56	5.17	**6.59**
	15	10.06	11.07	9.51	11.33	11.43	11.55	**13.29**
	20	17.08	17.72	15.89	17.98	18.44	19.76	**22.55**
	25	23.92	25.28	21.69	24.63	25.36	29.7	**32.26**
	30	32.91	34.73	30.17	32.45	34.15	40.03	**42.63**
	40	51.71	54.65	48.64	52.24	50.1	61.41	**63.33**
	50	67.69	67.98	61.44	65.96	61.85	77.11	**78.47**
	100	97.17	97.19	95.04	97.2	92.22	**99.17**	99.04
	150	99.93	99.91	99.6	99.93	99.05	**99.99**	99.99
	200	100	99.99	99.96	100	99.86	**100**	100
$\chi^2(10)$	5	2.49	2.49	2.61	**2.75**	2.49	1.92	2.3
	10	8.61	8.9	7.78	9.11	9.43	8.6	**10.28**
	15	17.27	18.54	16.3	19.01	18.5	20.95	**23.53**
	20	29.66	29.35	26.84	30.25	30.01	35.55	**38.54**
	25	41.2	41.63	37.11	40.65	41.42	50.23	**52.98**
	30	53.82	53.71	48.63	51.44	52.61	63.16	**65.96**
	40	74.31	74.55	69.79	72.85	71.1	82.54	**84.01**
	50	87.54	86.63	81.88	85.01	82.28	93.02	**93.64**
	100	99.64	99.68	99.24	99.69	98.84	**99.95**	99.94
	150	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	0.94	0.94	0.71	0.94	0.94	**1.42**	0.91
	10	2.11	2.02	**3.70**	2.05	1.78	3.07	2.31
	15	2.76	2.42	**6.15**	2.48	1.67	4.22	3.44
	20	3.75	2.9	**8.42**	3.08	2.03	5.52	4.68
	25	4.42	3.34	**9.91**	3.38	2.13	6.46	5.63
	30	5.21	3.86	**11.83**	3.69	2.55	7.05	6.59
	40	7.37	5.33	**16.46**	5.28	2.6	9.08	8.55
	50	8.75	5.97	**19.27**	5.85	2.99	10.3	10.53
	100	17.01	14.53	**37.39**	16.13	6.18	21.69	21.19
	150	29.09	28.38	**54.09**	31.28	12.99	37.03	35.64
	200	38.63	43	**67.55**	45.73	18.82	52.42	47.41
$HN(0,1)$	5	1.44	1.44	**1.48**	1.45	1.44	1.2	1.33
	10	2.96	2.92	2.44	2.94	3	2.62	**3.29**
	15	4.09	3.93	3.21	3.92	3.89	4.18	**4.92**
	20	5.25	4.38	3.6	4.83	4.95	5.48	**6.80**
	25	6.79	6.12	4.77	5.88	6.72	7.42	**8.78**
	30	9.06	7.34	5.62	6.61	8.67	9.68	**11.73**
	40	13.01	10.77	7.81	9.85	13.85	15.16	**17.61**
	50	17.61	13.83	9.15	12	18.74	20.75	**24.61**
	100	42.58	39.41	18.78	35.09	44.74	55.42	**57.61**
	150	67.37	64.46	30.23	59.23	66.26	81.2	**83.09**
	200	83.05	81.75	41.5	76.05	77.69	92.64	**93.28**
$Wbl(0,2,2)$	5	2.62	2.62	2.72	**2.80**	2.62	2.14	2.39
	10	9.05	9.56	8.27	9.87	9.99	9.22	**11.21**
	15	17.93	18.64	16.23	19.08	17.95	21.92	**24.44**
	20	29.31	27.98	25.29	29.17	28.4	36.55	**40.04**
	25	40.94	38.48	33.43	39.39	40.01	52.4	**55.28**
	30	54.4	51.55	45.55	48.63	53.38	66	**69.18**
	40	74.06	72.34	64.45	70.38	71.84	84.34	**85.44**
	50	86.28	84.51	76.67	82.37	83.54	93.62	**94.34**
	100	99.85	99.82	99.16	99.81	99.48	**100**	**100**
	150	**100**	**100**	99.99	**100**	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	3.96	3.96	1.45	4.1	3.96	**8.22**	4.77
	10	20.12	19.99	18.18	20.38	9.67	**34.18**	24.36
	15	40.35	39.76	30.24	40.91	14.66	**58.78**	47.7
	20	58.8	56.44	39.45	60.15	20.62	**76.34**	66.58
	25	71.5	70.39	46.53	73.52	26.42	**86.59**	79.02
	30	82.56	81.96	54.92	83.78	32.95	**93.21**	88.53
	40	94.05	94.22	70.2	95.32	48.62	**98.56**	96.82
	50	98.16	98.11	80.5	98.63	62.27	**99.65**	99.21
	100	**100**	**100**	99.55	**100**	96.45	**100**	**100**
	150	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.26: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.01$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$U(0,1)$	5	4.5	4.5	**4.71**	4.28	4.5	4.2	4.68
	10	11	10.11	9.39	10.41	8.96	14.28	**15.50**
	15	17.99	15.04	13.47	16.31	16.89	26.62	**27.81**
	20	27.71	20.76	18.09	22.44	41.8	41	**42.57**
	25	36.64	32.88	22.2	27.77	**69.66**	53.68	54.57
	30	46.61	53.59	27.32	36.88	**88.00**	65.37	66.89
	40	63.66	89.35	39.44	73.53	**99.03**	83.64	83.22
	50	77.5	98.56	47.1	93.14	**99.95**	93.24	92.98
	100	98.98	**100**	85.75	**100**	**100**	99.98	99.96
	150	99.99	**100**	97.87	**100**	**100**	**100**	**100**
	200	**100**	**100**	99.86	**100**	**100**	**100**	**100**
$beta(2,2)$	5	4.08	4.08	**4.29**	3.94	4.08	3.68	4.2
	10	16	15.51	14.07	15.4	14.91	18.91	**21.40**
	15	30.57	28.66	25.65	28.75	27.73	41.11	**43.59**
	20	48.21	42.45	39.24	43.1	51.9	63.07	**65.46**
	25	61.99	57.42	49.93	56.34	74.6	78.75	**79.83**
	30	74.27	73.04	60.8	66.48	**90.15**	88.43	89.45
	40	90.25	94.35	80.02	89.81	**99.27**	97.61	97.58
	50	97.25	99.27	90.16	97.68	**99.97**	99.62	99.64
	100	**100**	**100**	99.92	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	2.26	2.26	2.32	**2.48**	2.26	1.72	2.14
	10	7.23	7.29	6.39	7.15	7.54	7.04	**8.84**
	15	12.4	12.85	11.42	14.24	12.68	15.8	**17.80**
	20	21.42	20.28	17.98	20.98	21.84	27.29	**30.21**
	25	30.63	29.17	25.03	28.36	32.05	40.56	**43.53**
	30	41.33	39.13	33.55	36.32	43.1	52.84	**55.88**
	40	60.36	59.63	50.84	56.72	64.54	74.04	**75.57**
	50	75.4	73.92	63.13	70.14	78.4	87.17	**88.17**
	100	98.94	99.4	95.83	99.11	99.42	**99.89**	**99.88**
	150	99.99	**100**	99.78	99.99	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	10.54	10.54	10.87	10.67	10.54	10	**11.33**
	10	40.72	38.6	36.74	39.32	35.45	51.67	**53.44**
	15	68.5	64.71	62.04	65.31	69.92	83.02	**83.47**
	20	86.4	82.73	79.62	83.53	94.64	95.01	**95.14**
	25	94.6	94.94	89.14	93.07	**99.35**	98.73	98.83
	30	98.26	99.18	95.1	97.69	**99.93**	99.71	99.72
	40	99.87	**100**	99.15	99.97	**100**	**100**	**100**
	50	99.99	**100**	99.92	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	5.7	5.7	5.8	5.61	5.7	6.32	**6.66**
	10	8.71	7.45	7.34	8.06	5.08	**12.30**	12.1
	15	12.69	8.96	8.34	9.98	14.93	**19.25**	17.92
	20	18.31	12.62	9.05	12.36	**41.02**	28.16	25.24
	25	22.59	25.63	8.68	16.35	**64.54**	37.16	32.51
	30	29.22	48.06	10.55	27.56	**82.45**	46.98	41.42
	40	43.09	83.64	15.88	67.36	**96.94**	66.73	57.69
	50	57.37	96.45	20.15	88.69	**99.59**	81.97	74.16
	100	94.5	**100**	57.2	**100**	**100**	99.86	99.08
	150	99.89	**100**	88.75	**100**	**100**	**100**	99.99
	200	**100**	**100**	98.23	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	0.95	0.95	0.62	0.97	0.95	**1.72**	1.04
	10	1.84	1.93	1.15	1.95	0.54	**4.42**	2.07
	15	4.02	3.92	1.08	4.26	0.4	**8.90**	4.88
	20	6.42	6.2	0.99	8.02	0.26	**14.36**	7.86
	25	9.19	9.59	0.73	11.64	0.19	**20.12**	11.49
	30	13.89	15.31	1.02	16.73	0.3	**27.75**	16.83
	40	22.31	27.68	1.27	30.24	0.6	**42.47**	26.52
	50	32.33	39.21	0.92	42.44	0.68	**55.34**	38.02
	100	73.03	85.26	6.86	88.73	2.91	**92.20**	78.88
	150	93.41	97.68	35.54	98.53	13.86	**99.14**	95.75
	200	98.7	99.7	70.39	**99.88**	26.71	99.87	99.21
$beta(1,2)$	5	1.73	1.73	**1.81**	1.73	1.73	1.43	1.69
	10	3.73	3.72	3.19	3.53	3.87	3.37	**4.18**
	15	5.29	5.11	4.33	5.1	5	6.06	**7.08**
	20	7.33	5.64	4.84	6.88	6.72	8.42	**10.00**
	25	8.95	6.72	5.28	7.56	10.48	11.36	**13.01**
	30	12.39	9.53	6.8	8.01	**18.01**	15.07	17.65
	40	18.06	15.54	8.77	12.05	**36.47**	23.34	25.81
	50	23.51	24.15	10.16	16.35	**54.09**	32.06	35.01
	100	54.3	89.71	21.01	75.76	**98.79**	74.7	74.41
	150	81.41	99.91	34.83	99.01	**100**	95.14	94.81
	200	92.84	**100**	48.81	99.99	**100**	99.28	99.06

The bold number is the highest power among the seven tests for each sample size.

Table 5.27: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.01$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$Laplace(0,1)$	5	8.46	8.46	**8.79**	8.36	8.46	7.31	8.35
	10	39.04	39.23	37.06	39.03	37.66	41.16	**44.52**
	15	66.88	65.71	63.59	65.87	62.69	71.01	**73.79**
	20	84.32	81.82	80.1	82.73	79.17	86.99	**88.92**
	25	92.31	90.72	88.9	91.42	88.04	94.33	**95.29**
	30	96.53	95.85	94.78	95.82	93.88	97.62	**97.95**
	40	99.62	99.34	99.07	99.35	98.4	99.76	**99.79**
	50	99.95	99.87	99.83	99.92	99.69	99.97	**99.98**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	6.19	6.19	**6.52**	6.21	6.19	5.47	6.16
	10	30.06	30.17	27.98	29.87	29.62	33.5	**36.67**
	15	55.63	55.08	52.49	54.74	53.13	62.96	**65.69**
	20	75.21	71.77	69.58	73.09	71.09	81.59	**83.65**
	25	85.62	83.92	80.85	84.67	82.93	90.73	**91.70**
	30	93.18	91.61	89.6	91.43	90.86	95.98	**96.50**
	40	98.86	98.34	97.39	98.22	97.53	99.49	**99.60**
	50	99.82	99.71	99.39	99.61	99.36	99.96	**99.98**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	21.24	21.24	20.54	**21.58**	21.24	21.29	21.45
	10	57.86	58.47	57.78	58.4	57.53	58.71	**60.33**
	15	80.77	81.25	80.98	80.64	79.94	82.2	**83.06**
	20	91.1	91.3	91.37	91.4	90.38	92.55	**92.90**
	25	95.65	95.98	95.96	96.22	94.9	**97.04**	96.94
	30	97.92	98.21	98.22	98.46	97.57	98.56	**98.62**
	40	99.72	99.77	99.77	99.72	99.51	**99.88**	99.87
	50	99.93	99.92	99.92	99.94	99.87	99.96	**99.97**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	5.31	5.31	**5.54**	5.31	5.31	4.52	5.31
	10	24.39	24.26	22.34	24.3	24.12	28.5	**31.81**
	15	46.67	45.86	43.07	46.41	44.31	56.89	**59.18**
	20	68.23	64.22	61.49	65.82	65.92	78.42	**80.25**
	25	81.48	78.43	74.17	78.02	79.98	89.15	**90.24**
	30	90.62	88.44	84.82	87.27	89.65	95.21	**95.93**
	40	98.1	97.49	95.53	97	97.57	99.42	**99.48**
	50	99.55	99.46	98.67	99.28	99.53	99.91	**99.92**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	21.44	21.44	20.77	20.64	21.44	21.42	**21.78**
	10	58.1	58.04	57.73	58.77	57.09	58.95	**60.48**
	15	80.19	81.07	80.7	80.24	79.93	82.61	**83.03**
	20	91.22	91.42	91.26	91	90.39	92.53	**92.87**
	25	95.68	95.96	95.85	96.25	94.98	96.76	**96.85**
	30	98.29	98.55	98.43	98.39	97.84	98.93	**98.95**
	40	99.72	99.75	99.78	99.74	99.54	99.83	**99.84**
	50	**100**	99.99	99.99	99.99	99.92	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	8.83	8.83	**9.06**	8.4	8.83	7.85	8.87
	10	37.04	36.92	35.16	37.71	36.37	39.51	**42.39**
	15	62.16	62.15	60.14	62.98	60.23	67.05	**69.44**
	20	80.95	79.44	77.78	79.89	77.85	84.46	**86.03**
	25	90.5	89.67	87.84	89.28	87.85	93.25	**93.91**
	30	95.39	94.92	93.6	94.4	93.04	96.96	**97.33**
	40	99.14	99	98.59	98.92	98.1	99.61	**99.66**
	50	99.78	99.74	99.57	99.74	99.33	99.88	**99.90**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	7.34	7.34	7.56	**7.64**	7.34	6.41	7.24
	10	34.56	34.46	32.59	34.39	33.97	37.62	**40.67**
	15	58.97	58.85	56.58	59.43	56.85	64.94	**67.19**
	20	78.79	76.88	75.1	76.84	75.82	83.52	**85.15**
	25	89.07	87.71	85.38	87.47	85.61	92.81	**93.49**
	30	94.6	93.35	91.78	93.51	91.86	96.42	**96.94**
	40	99.02	98.66	98.18	98.63	97.65	99.35	**99.42**
	50	99.85	99.75	99.54	99.77	99.3	99.97	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	6.25	6.25	**6.64**	6.51	6.25	5.54	6.23
	10	31.12	30.96	29.09	31.37	30.56	34.6	**37.88**
	15	55.67	54.52	51.87	54.54	52.77	62.68	**65.29**
	20	75.28	72.56	70.33	74.31	71.62	81.4	**83.34**
	25	87.46	85.22	82.89	85.32	83.75	91.46	**92.52**
	30	93.75	92.51	90.44	91.67	91.28	96.15	**96.80**
	40	98.91	98.47	97.75	98.37	97.55	99.47	**99.53**
	50	99.86	99.81	99.46	99.59	99.35	99.96	**99.96**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.28: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.1$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	$AD(1)$	$CvM(1)$
$\chi^2(1)$	5	21.3	25.32	20.45	**26.32**	21.57	14.75	23.45
	10	37.96	**43.45**	31.3	43.31	34.45	30.69	40.7
	15	49.27	**57.21**	40.9	55.87	42.01	46.32	53.11
	20	59.66	**68.03**	49.57	67.24	49.38	59.42	63.74
	25	68.05	75.37	59.19	**75.71**	55.47	70.18	72.68
	30	74.6	82.33	66.5	**82.68**	61.2	77.97	78.91
	40	85.3	**91.2**	79.71	90.91	74.13	88.91	88.62
	50	91.98	**95.84**	88.01	95.3	81.64	95.15	94.31
	100	99.64	99.92	99.49	99.87	98.28	**99.93**	99.79
	150	**100**	**100**	99.96	99.99	99.86	**100**	99.99
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	7.89	6.59	8.38	6.93	8.14	**11.86**	6.67
	10	8.86	6.54	10.48	6.85	11.58	**14.77**	7.97
	15	10.38	7.16	12.9	7.06	16.39	**18.26**	10.07
	20	12.05	7.83	15.18	8.49	19.51	**21.17**	11.68
	25	14.39	9.86	18.26	10.21	24.31	**25.03**	14.69
	30	17.53	13.02	22.3	12.83	**29.58**	29.16	18.64
	40	23.41	19.39	28.9	18.14	**38.57**	37.32	26.07
	50	30.56	26.1	36.13	23.43	44.78	**45.72**	33.43
	100	59.9	58.45	66.26	55.35	71.02	**77.12**	67.1
	150	80.49	80.47	84.38	78.08	85.91	**92.06**	86.47
	200	91.71	91.88	93.44	90.52	93.69	**97.42**	95.23
$\chi^2(4)$	5	6.86	5.5	8.09	5.75	7.31	**12.92**	5.66
	10	10.33	5.88	12.43	6.14	13.89	**20.06**	8.69
	15	14.92	9.12	18.59	9.14	23.18	**28.05**	14.28
	20	21.22	12.89	25.11	13.94	32.46	**36.06**	21.03
	25	26.89	19.13	32.69	19.06	41.74	**44.77**	28.98
	30	34.54	26.82	40.3	26.5	49.38	**52.76**	37.9
	40	47.99	41.81	54.28	39.11	63.83	**66.72**	53.13
	50	61.37	55.95	67.05	51.97	73.84	**78.25**	67.63
	100	93.47	91.88	94.35	90.62	95.47	**97.96**	96.41
	150	99.34	99.1	99.39	98.87	99.5	**99.88**	99.75
	200	99.91	99.87	99.87	99.86	99.91	**100**	99.97
$\chi^2(6)$	5	6.72	4.73	8.47	5.43	7.37	**14.84**	5.1
	10	13.48	7.14	17.15	7.55	19.41	**29.15**	12.36
	15	23.88	13.58	29.01	15.02	36.1	**42.65**	24.43
	20	36.12	24.48	41.47	24.58	49.61	**56.31**	39.02
	25	46.8	36.06	53.74	35.62	62.26	**67.47**	51.69
	30	58.21	49.2	64.1	48.54	71.67	**76.51**	64.06
	40	76.22	70.13	79.77	66.96	85.32	**88.79**	81.96
	50	87.22	83.26	88.71	80.5	91.85	**95.05**	91.22
	100	99.7	99.38	99.58	99.13	99.65	**99.92**	99.83
	150	**100**	99.99	99.99	**100**	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	7.08	4.23	9.15	4.89	7.97	**17.55**	5.08
	10	18.8	8.75	23.44	10.8	26.07	**37.8**	17.48
	15	35.37	21.13	40.93	23.24	48.36	**56.99**	37.66
	20	52.18	37.36	57.17	38.71	65.28	**71.49**	56.22
	25	65.4	53.96	70.48	52.73	76.88	**82.31**	70.76
	30	76.61	68.03	79.69	67.74	85.33	**89.08**	81.82
	40	90.24	85.83	91.38	83.66	94.18	**96.46**	93.72
	50	96.45	94.79	96.61	93.18	97.97	**99.02**	98.06
	100	99.98	99.95	99.96	99.95	99.98	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	13.58	12.01	13.04	12.8	13.2	10.45	**14.09**
	10	18.28	13.92	**18.93**	14.68	17.71	13.67	18.64
	15	19.99	14.84	**23.09**	15.17	18.42	16.33	21.77
	20	22.08	15.44	**26.89**	15.69	19.83	18.17	23.64
	25	23.65	16.12	**30.34**	16.27	20.45	20.66	25.8
	30	25.32	17.44	**34.15**	17.72	20.85	22.95	27.7
	40	29.33	19.68	**40.43**	18.89	23.68	26.9	31.81
	50	32.28	22.27	**46.33**	20.33	25.05	30.79	35.1
	100	49.67	38.93	**70.46**	37.49	39.72	55.12	54.21
	150	63.21	58.77	**84.16**	57.33	55.2	74.63	70.31
	200	76.03	75.17	**92.44**	74.26	70.05	87.54	82.85
$HN(0,1)$	5	6.9	6.36	8.36	6.68	7.42	**13.13**	5.72
	10	9.83	6.48	12.24	7.03	12.65	**19.7**	8.63
	15	13.46	8.48	16.11	8.85	20.43	**24.29**	12.73
	20	18.02	10.07	19.9	10.47	27.48	**30.28**	17.95
	25	22.12	14.07	24.12	13.8	34.35	**35.73**	23.28
	30	27.07	18.14	27.92	17.26	41.05	**41.17**	29.35
	40	36.22	26.19	34.83	22.68	**54.06**	51.84	40.98
	50	45.13	35.54	41.07	28.87	**63.6**	60.14	51.42
	100	76.84	72	66.87	65.6	**89**	88.45	84.6
	150	91.15	90.57	82.66	86.77	96.58	**97.13**	96.24
	200	97.16	97.28	91.54	95.96	99.15	**99.36**	99.05
$Wbl(0,2,2)$	5	7.29	3.85	9.48	4.49	8.06	**18.86**	4.63
	10	19.82	8.93	24.65	11.06	27.41	**40.43**	19.03
	15	36.8	21.94	41.82	23.69	49.98	**59.23**	39.21
	20	53.69	37.43	57.57	37.98	68.59	**74.8**	58.91
	25	67.47	53.65	70.91	53.43	80.76	**85.06**	73.98
	30	79.28	69.28	81.24	67.3	88.97	**91.93**	85.09
	40	91.38	86.85	91.7	84.18	96.19	**97.31**	95.14
	50	96.9	95.22	96.49	93.16	98.9	**99.39**	98.65
	100	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	37.17	41.56	36.64	**43.33**	38.22	26.81	41.35
	10	67.87	**71.57**	61.41	70.63	64.76	62.03	71.37
	15	83.1	86.13	77.66	85.53	78.75	82.04	**86.4**
	20	91.85	93.59	87.6	93.61	87.35	92.45	**94.01**
	25	95.88	97.32	93.77	96.88	92.28	96.94	**97.37**
	30	98.19	**99.07**	96.89	98.8	96.03	98.93	98.83
	40	99.63	99.78	99.26	**99.81**	98.86	99.76	99.72
	50	99.94	**100**	99.86	99.95	99.63	**100**	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.29: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.1$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	$AD(1)$	$CvM(1)$
$U(0,1)$	5	9.97	4.5	14.02	5.47	10.61	**25.11**	7.43
	10	25.71	11.31	32.75	13.14	34.47	**48.92**	27.36
	15	42.16	22.61	47.12	22.66	**69.71**	66.53	49.02
	20	56.3	43.96	58.53	38.3	**89.95**	79.81	66.19
	25	66.99	70.26	68.38	59.07	**97.45**	88.46	78.67
	30	77.76	88.02	77.1	80.27	**99.5**	93.86	87.22
	40	89.67	98.96	87.28	96.73	**100**	98.52	96.3
	50	95.66	99.96	93.7	99.69	**100**	99.73	99.18
	100	99.99	**100**	99.93	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	9.71	3.47	13.74	4.1	10.87	**27.27**	6.35
	10	31.62	14.06	38.14	16.7	40.61	**57.87**	32.97
	15	55.02	33.71	59.9	34.86	75.06	**78.69**	62.34
	20	73.41	58.41	75.92	55.75	**92.53**	91.52	82.12
	25	84.83	79.93	86.26	75.38	**98.32**	96.54	92.19
	30	92.65	92.71	92.71	89.06	**99.62**	98.99	96.82
	40	98.3	99.49	98.07	98.58	**99.98**	99.84	99.58
	50	99.74	99.98	99.7	99.94	**99.99**	99.97	99.96
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	6.65	4.24	8.98	4.69	7.31	**17.49**	4.81
	10	16.47	7.7	21.27	9.19	22.87	**35.18**	14.94
	15	29.55	15.92	34.55	18.47	42.6	**50.98**	31.7
	20	44.54	28.98	49.17	29.3	62.09	**67.85**	50.35
	25	57.57	44.01	61.91	42.09	75.78	**79.36**	65.3
	30	69.11	57.88	71.97	56.14	85.05	**86.9**	76.41
	40	84.33	79.28	85.62	75.23	95.09	**95.17**	91.17
	50	92.93	91.21	92.61	87.57	98.33	**98.45**	96.38
	100	99.92	99.95	99.93	99.94	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	18.36	2.98	23.87	6.51	19.83	**43.64**	13.11
	10	60.97	35.2	66.91	40.47	73.13	**84.78**	66.32
	15	86.97	73.49	89.1	72.19	**97.09**	96.81	92.25
	20	96.34	94.8	96.68	92.55	**99.78**	99.48	98.49
	25	99.05	99.37	99.11	98.81	**100**	99.95	99.73
	30	99.79	99.95	99.82	99.89	**100**	99.99	99.96
	40	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	13.65	10.13	20.64	11.97	14.64	**28.33**	14.14
	10	27.62	20.32	35.09	22.15	35.34	**45.88**	32.14
	15	39.13	33.76	45.14	31.99	**66.77**	60.52	48.9
	20	49.91	55.81	53.32	47.62	**86.4**	73.3	62.32
	25	59.49	76.27	62.25	67.6	**95.07**	81.98	72.46
	30	68.97	90.69	70.04	84.06	**98.28**	89.82	82.26
	40	81.86	98.95	80.77	97.29	**99.89**	96.46	92.31
	50	91.36	99.96	88.66	99.68	**100**	99.18	97.26
	100	99.91	**100**	99.79	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	17.55	**21.81**	17.28	21.21	18.51	14.19	19.03
	10	27.23	**34.65**	20.41	34.37	23.95	22.71	29.65
	15	35.66	**46.86**	25.83	46.32	29.32	33.6	38.54
	20	43.01	**55.68**	29.78	55.47	32.71	43.77	46.83
	25	48.93	62.75	36.53	**64.09**	36.97	53.01	53.34
	30	56.89	71.56	44	**72.03**	42.85	62.39	60.86
	40	67.19	**82.04**	56.39	81.49	53.71	75.2	71.93
	50	76.54	**88.89**	66.87	87.71	61.68	84.32	80.59
	100	96.13	99.13	95.58	**99.27**	91.9	98.71	97.14
	150	99.51	**99.96**	99.34	99.95	98.43	99.94	99.74
	200	99.93	99.99	99.95	**100**	99.83	**100**	99.97
$beta(1,2)$	5	7.23	6.31	9.12	6.38	7.81	**14.87**	5.81
	10	11.29	7.7	14.25	8.11	14.38	**22.84**	10.45
	15	17.14	9.99	20.29	10.45	26.41	**30.73**	17.57
	20	22.39	13.14	24.29	13.33	**39.28**	38.04	24.56
	25	27.39	17.81	29.1	16.92	**53.9**	46.19	31.71
	30	35.1	26.31	35.01	22.08	**66.66**	52.83	40.02
	40	45.82	44.44	42.75	33.64	**84.07**	64.78	53.56
	50	55.65	61.8	48.45	48.02	**93.4**	74.22	64.68
	100	87.04	99.37	76.73	97.26	**99.99**	97.13	94.44
	150	97.31	**100**	91.21	99.99	**100**	99.8	99.46
	200	99.64	**100**	98.12	**100**	**100**	99.98	99.97

The bold number is the highest power among the seven tests for each sample size.

Table 5.30: Powers (in %) of the five graphical tests and the two non-graphical tests by using MLE at $\alpha = 0.1$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(1)$	$D_{sp}(1)$	$D_e(1)$	$D_{be}(1)$	$D_{bi}(1)$	$AD(1)$	$CvM(1)$
$Laplace(0,1)$	5	16.13	3.39	19.1	5.77	17.39	**34.4**	10.84
	10	53.16	35.46	58.08	39.35	60.61	**70.85**	53.86
	15	81.12	68.64	83.62	70.1	85.55	**89.81**	82.81
	20	92.8	86.76	93.53	86.82	94.48	**96.59**	93.8
	25	97.02	94.55	97.55	94.69	97.94	**98.8**	97.82
	30	98.96	97.71	99.06	98.32	99.11	**99.61**	99.27
	40	99.93	99.77	99.94	99.75	99.93	**99.98**	99.95
	50	99.99	99.97	99.98	99.97	99.98	**99.99**	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	12.9	3.13	15.83	4.72	13.97	**31.68**	8.15
	10	45.37	26.67	51.26	30.22	53.44	**66.88**	46.54
	15	73.75	58.66	76.85	59.58	81.19	**87.24**	77.2
	20	87.8	79.19	89.37	79.18	92.63	**95.23**	90.95
	25	94.59	90	95.37	90.61	97	**98.32**	96.65
	30	97.17	95.74	97.79	96.1	98.68	**99.35**	98.69
	40	99.77	99.45	99.73	99.28	99.83	**99.95**	99.91
	50	99.98	99.94	99.96	99.94	**100**	99.99	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	33.64	13.06	35.47	20.4	34.67	**44.58**	29.04
	10	71.08	58.97	74.04	61.76	76.01	**77.75**	70.73
	15	89.38	84.62	91.43	84.95	92.53	**93.1**	90.44
	20	96.23	94.34	97.14	94.13	97.46	**97.85**	96.81
	25	98.61	97.9	99.06	97.9	99.14	**99.37**	99.09
	30	99.41	99.12	99.71	99.45	99.74	**99.79**	99.66
	40	99.94	99.9	99.94	99.87	99.95	**100**	99.99
	50	99.98	99.97	99.98	99.98	99.98	**99.99**	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	11.23	2.98	14.45	4.36	12.19	**29.24**	7.06
	10	40.65	21.23	46.8	25.22	49.83	**64.58**	42.81
	15	68.29	50.02	71.77	51.68	78.83	**85.37**	73.36
	20	85.6	74.6	87.25	74.23	92.25	**94.55**	89.69
	25	93.45	87.95	94	86.9	97.19	**98.04**	96.05
	30	97.41	95.06	97.61	94.56	99.17	**99.38**	98.82
	40	99.59	99.34	99.6	99.12	99.96	**99.96**	99.87
	50	99.95	99.92	99.9	99.84	99.99	**99.99**	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	34.15	13.5	36.03	20.25	35.04	**45.28**	29.52
	10	71.37	58.93	74.06	62.63	76.5	**78.8**	70.79
	15	89.63	84.54	91.37	84.46	92.48	**92.96**	90.45
	20	96.15	94.35	96.98	94.14	97.2	**97.48**	96.59
	25	98.42	97.71	98.92	97.82	98.96	**99.22**	98.82
	30	99.52	99.33	99.73	99.3	99.71	**99.87**	99.71
	40	99.95	99.93	99.96	99.91	99.96	99.96	**99.97**
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	16.4	3.51	19.66	5.99	17.71	**33.91**	11.39
	10	51.13	33.91	56.37	38.44	58.74	**69.33**	51.44
	15	76.65	64.87	79.63	66.6	82.75	**87.21**	79.01
	20	90.38	84.14	91.75	84.23	92.95	**95.4**	92.03
	25	95.82	93.27	96.73	93.27	97.23	**98.18**	96.96
	30	98.52	97.17	98.7	96.94	98.75	**99.48**	98.92
	40	99.82	99.61	99.82	99.52	99.84	**99.92**	99.87
	50	99.93	99.88	99.96	99.92	99.95	**99.98**	99.96
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	14.52	3.13	17.58	5.36	15.68	**31.86**	9.54
	10	49.61	31.44	55	35.16	57.33	**68.98**	50.26
	15	74.74	61.93	78.16	63.55	81.39	**86.21**	77.49
	20	88.91	82.19	90.16	81.95	92.43	**94.68**	91.14
	25	95.78	92.44	96.51	92.14	97.13	**98.32**	96.99
	30	98.19	96.72	98.35	96.86	98.8	**99.38**	98.9
	40	99.67	99.52	99.76	99.31	99.76	**99.92**	99.88
	50	**100**	99.95	**100**	99.91	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	13.19	3.03	16.23	4.92	14.28	**31.68**	7.94
	10	46.68	27.79	52.31	31.86	54.86	**67.53**	47.29
	15	73.18	58.25	76.46	58.98	81.06	**86.22**	76.27
	20	88.14	79.74	89.5	80.27	92.24	**94.84**	90.84
	25	95.33	91.26	95.97	91.08	97.06	**98.18**	96.72
	30	98.22	96.46	98.37	96.03	98.78	**99.37**	98.88
	40	99.75	99.44	99.73	99.39	99.81	**99.92**	99.86
	50	99.99	99.96	99.99	99.91	99.99	99.99	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.31: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.1$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	$AD(2)$	$CvM(2)$
$\chi^2(1)$	5	14.53	14.76	12.48	14.87	13.35	**16.26**	14.81
	10	25.27	26.29	19.51	24.19	17.87	**30.81**	26.87
	15	36.72	37.51	25.76	33.83	21.41	**45.84**	40.15
	20	47.73	49.82	33.37	45.66	25.89	**58.63**	52.03
	25	57.26	59.56	41.55	57.39	30.15	**69.44**	62.15
	30	66.28	68.32	48.41	67.18	35.9	**77.53**	70.65
	40	79	82.02	64.08	80.56	48.77	**88.49**	83.47
	50	87.9	90.22	75.44	89.44	59.48	**94.74**	90.81
	100	99.42	99.74	98.3	99.69	94.04	**99.9**	99.67
	150	99.94	99.99	99.9	99.99	99.26	**100**	99.97
	200	**100**	**100**	**100**	**100**	99.98	**100**	**100**
$\chi^2(3)$	5	11.02	10.87	**11.66**	11.55	11.31	10.83	11.44
	10	13.68	14.16	14.7	14.57	**16.2**	13.15	14.18
	15	16.41	17.59	18.23	17.9	**21.06**	16.44	17.9
	20	18.22	20.31	20.53	21.23	**24.14**	18.88	20.26
	25	20.79	24.21	24.2	25.6	**29.12**	21.96	23.36
	30	24.63	29.21	29.22	29.35	**33.88**	26.6	27.56
	40	31.16	36.42	36.34	36.96	**42.22**	34.86	35.72
	50	38.13	43.71	43.46	45.14	**47.74**	42.59	42.86
	100	66.1	73.17	71.6	73.73	72.01	**75.04**	73.67
	150	84.26	88.48	87.03	88.45	86	**91.23**	89.64
	200	93.47	95.43	94.61	95.53	93.82	**97.25**	96.49
$\chi^2(4)$	5	11.59	11.48	12.65	**12.92**	12.18	11.49	12.06
	10	17.61	18.25	19.06	18.97	**21.3**	17.32	18.94
	15	23.88	25.59	26.46	26.99	**30.3**	24.66	27
	20	30.42	33.17	33.6	34.91	**38.74**	32.4	34.45
	25	36.66	41.31	41.27	42.09	**47.31**	40.7	41.95
	30	44.64	48.94	48.75	50.69	**53.93**	49.39	50.19
	40	57.33	62.45	62.02	63.94	**66.59**	63.56	64.43
	50	69.21	73.54	73.15	74.19	75.85	**76.05**	75.97
	100	95.35	95.87	95.31	96.49	95.48	**97.64**	97.58
	150	99.52	99.7	99.45	99.59	99.41	**99.88**	99.82
	200	99.94	99.96	99.91	99.95	99.88	**100**	**100**
$\chi^2(6)$	5	13.13	12.93	14.31	**14.91**	14.08	13.25	13.98
	10	24.46	25.42	26.47	26.55	**29.18**	24.89	27.32
	15	36.69	38.03	39.06	38.82	**43.97**	38.6	41.23
	20	48.02	50.93	50.72	50.99	**56.21**	52.42	54.23
	25	58.43	62.28	61.98	61.97	**66.47**	63.59	65.21
	30	68.7	71.75	71.02	72.18	**75.1**	73.83	74.88
	40	82.33	84.57	83.7	84.23	86.23	87.45	**87.66**
	50	90.82	91.65	91.15	92.1	92.33	94.23	94.38
	100	99.77	99.75	99.66	99.78	99.59	99.91	99.91
	150	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	15.5	15.28	16.81	**17.14**	16.35	15.04	16.52
	10	31.91	32.48	34.01	34.64	**37.03**	33.32	35.7
	15	49.22	49.97	51.32	51.46	**56.15**	53.22	55.42
	20	63.48	65.76	65.59	65.52	**70.06**	68.48	69.93
	25	74.72	76.86	76.41	77.03	79.59	79.54	**80.61**
	30	83.45	84.47	83.98	84.65	86.78	87.69	**88.13**
	40	93.24	93.65	93.13	93.23	94.49	95.9	**96.3**
	50	97.58	97.62	97.35	97.73	98.04	98.78	**98.93**
	100	99.99	99.98	99.97	99.99	99.98	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	11.88	11.87	10.38	12.13	11.77	**12.49**	12.05
	10	16.02	15.74	15.76	16.08	14.92	**17.52**	16.63
	15	18.17	17.62	20.38	17.69	15.9	**20.66**	19.85
	20	20.84	20.06	**24.47**	20.1	17.48	23.09	22.42
	25	22.71	21.55	**28.72**	21.71	17.54	25.16	24.66
	30	24.72	23.4	**32.33**	23.96	18.74	27.61	26.91
	40	28.68	27.08	**39.22**	27.51	21.73	31.8	31.38
	50	32.09	30.77	**45.54**	32.01	24.34	35.59	35.14
	100	50.39	52.23	**71.03**	54.06	39.33	58.27	55.84
	150	64.39	69.96	**85.37**	70.89	55.48	76.75	72.41
	200	77.18	83.55	**93.29**	84.74	70.36	88.42	84.32
$HN(0,1)$	5	11.73	11.44	12.63	**12.7**	12.27	11.53	12.2
	10	16.82	16.44	18.23	17.56	**18.99**	16.74	18.1
	15	20.81	20.69	21.88	21.07	**25.86**	21.46	23.52
	20	25.45	25.82	25.99	25.61	**31.91**	26.9	29.03
	25	29.39	30.14	29.79	30.03	**37.49**	31.84	33.85
	30	34.9	34.77	33.12	34.25	**43.54**	37.94	39.71
	40	43.01	44.3	39.84	42.56	**55.02**	48.76	50.91
	50	51.16	52.11	45.09	50.82	**63.33**	57.03	59.12
	100	79.84	82.19	68.82	81.51	87.53	86.91	**88.04**
	150	92.39	94.45	82.78	93.2	96.03	96.75	**97.1**
	200	97.56	98.5	91.48	98.31	98.93	**99.27**	99.23
$Wbl(0,2,2)$	5	16.73	16.31	17.85	**18.89**	17.71	16.58	17.79
	10	33.81	33.93	35.99	36.43	**38.79**	35.61	37.77
	15	50.34	50.72	51.98	52.24	**58.15**	55.19	57.53
	20	64.73	66.6	66.03	66.78	**73.35**	71.64	73.13
	25	76.34	78.2	77.15	78.17	83.04	82.56	**83.69**
	30	85.6	86.68	85.13	85.95	90.02	90.53	**90.98**
	40	94.01	94.73	93.51	94.42	96.32	97.01	**97.01**
	50	97.92	98.17	97.23	98.29	98.86	99.24	**99.27**
	100	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	26.57	26.99	22.95	26.24	24.53	**29.75**	26.92
	10	54.91	56.49	46.69	53.15	45.31	**62.7**	58.17
	15	74.78	75.47	64.41	73.07	60.27	**82.44**	78.45
	20	86.77	87.75	77.16	86.31	71.69	**92.34**	89.96
	25	92.76	93.68	86.38	93.13	80.33	**96.75**	94.97
	30	96.78	97.37	92.31	96.8	87.34	**98.76**	97.83
	40	99.2	99.44	97.79	99.43	95.11	**99.73**	99.55
	50	99.87	99.91	99.43	99.87	98.31	**100**	99.94
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.32: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.1$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	$AD(2)$	$CvM(2)$
$U(0,1)$	5	20.48	20.49	**24.08**	21.37	21.03	22.4	23.36
	10	38.03	37.26	41.27	37.68	**46.65**	44.73	45.33
	15	53.14	56.42	53.44	52.64	**75.43**	63.83	64.11
	20	65.28	76.8	63.55	71.53	**91.57**	77.47	76.72
	25	74.25	90.17	71.87	86.83	**97.86**	86.71	85.45
	30	82.87	96.8	79.16	95.25	**99.5**	92.96	91.9
	40	92.24	99.81	87.8	99.51	**100**	98.26	97.67
	50	96.81	99.98	93.88	99.99	**100**	99.63	99.49
	100	99.99	**100**	99.91	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	22.77	22.47	25.12	24.04	23.74	23.77	**25.44**
	10	46.82	46.45	49.41	48.11	53.61	53.08	**54.77**
	15	67.71	68.35	68.24	67.1	**80.37**	75.99	77.08
	20	82.16	85.43	81.2	83.79	**93.81**	90.05	90.07
	25	90.08	94.41	89.3	93.57	**98.62**	95.84	95.61
	30	95.22	98.39	94.34	97.65	**99.64**	98.65	98.58
	40	98.87	99.86	98.45	99.8	**99.98**	99.78	99.77
	50	99.88	99.99	99.71	99.99	**99.99**	99.97	99.97
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	15.73	15.38	17.13	**17.15**	16.71	15.09	16.65
	10	28.86	29.11	31.22	31.11	**33.61**	30.15	32.51
	15	42.31	43	44.09	44.88	**50.53**	46.97	49.23
	20	56.88	59	57.9	57.9	**67.38**	63.93	65.6
	25	67.6	71.11	69.13	69.85	**78.53**	75.98	76.89
	30	77.59	80.3	77.19	79.93	**86.34**	84.67	85.07
	40	88.96	91.41	88.28	90.73	**95.03**	94.4	94.66
	50	94.92	96.46	93.99	96.39	**98.23**	98.06	98.12
	100	99.96	99.98	99.94	99.99	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	34.99	34.7	38.82	37.59	35.99	40.05	**41.5**
	10	74.08	73.62	75.94	74.18	82.15	82.44	**82.76**
	15	91.76	93.79	91.9	93.3	**98.02**	96.33	96.33
	20	97.77	99.14	97.48	98.99	**99.82**	99.38	99.26
	25	99.5	99.93	99.31	99.91	**100**	99.92	99.9
	30	99.88	99.99	99.8	**100**	**100**	99.99	99.99
	40	99.99	**100**	**100**	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	22.18	22.71	**27.6**	20.22	20.44	26.49	25.55
	10	32.77	32.53	36.96	30.66	41.76	**42.01**	39.26
	15	42.84	50.53	44.37	44.92	**68.15**	57.19	52.95
	20	51.94	71.54	50.77	64.08	**86.14**	69.96	63.67
	25	61.16	86.05	58.05	79.88	**94.44**	79.01	72.65
	30	70.33	94.32	64.88	91.18	**97.86**	87.79	82.17
	40	82.12	99.28	75.51	98.72	**99.87**	95.73	91.98
	50	91.27	99.99	84.17	99.89	**100**	98.9	96.88
	100	99.92	**100**	99.46	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	13.04	13.09	11.67	12.48	12.08	**14.23**	13.01
	10	16.59	17.86	11.58	15.93	11.2	**21.16**	17.64
	15	23.82	25.16	13.85	22.45	11.94	**31.39**	25.95
	20	31.06	32.96	15.06	30.44	13.48	**40.61**	33.36
	25	36.68	41.11	18.73	38.49	15.51	**49.29**	40.25
	30	45.23	50.69	23.84	48.57	19.56	**58.88**	49.66
	40	58.02	64.77	33.9	62.83	26.96	**72.38**	62.37
	50	68.31	75.55	45.76	74.57	34.99	**82.22**	72.56
	100	94.52	97.87	87.44	97.96	75.92	**98.46**	95.66
	150	99.27	99.75	98.08	99.84	94.75	**99.92**	99.61
	200	99.88	99.98	99.82	99.99	99.25	**100**	99.92
$beta(1,2)$	5	13.28	13.23	**14.87**	14.13	13.75	12.91	13.84
	10	18.49	18.26	20.31	19.44	**20.89**	19.19	20.68
	15	25.17	23.96	25.84	24.31	**32.25**	27.43	29.15
	20	30.29	30.84	29.69	30.07	**43.96**	34.42	35.44
	25	36.18	38.98	34.24	37.2	**56.83**	42.06	42.95
	30	42.72	48.64	39.37	44.37	**68.05**	49.28	49.71
	40	52.15	64.91	46.5	58.17	**83.87**	62.14	62.5
	50	60.62	77.92	51.29	73.65	**92.96**	71.43	71.37
	100	88.93	99.77	76.8	99.47	**99.99**	96.48	95.76
	150	97.7	**100**	90.25	**100**	**100**	99.77	99.59
	200	99.7	**100**	97.46	**100**	**100**	99.98	99.98

The bold number is the highest power among the seven tests for each sample size.

Table 5.33: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLUE at $\alpha = 0.1$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(2)$	$D_{sp}(2)$	$D_e(2)$	$D_{be}(2)$	$D_{bi}(2)$	AD(2)	CvM(2)
$Laplace(0,1)$	5	30.88	30.37	31.94	**33.48**	32.7	31.38	33.41
	10	66.41	66.38	67.92	68.63	68.87	67.59	**69.81**
	15	87.22	87.09	87.77	87.22	88.22	88.68	**89.92**
	20	95.13	95.06	95.23	95.06	95.27	96.09	**96.46**
	25	98.22	98.21	98.26	98.33	98.17	98.65	**98.82**
	30	99.38	99.27	99.34	99.43	99.2	99.57	**99.61**
	40	99.94	99.95	99.96	99.88	99.9	99.98	**99.98**
	50	99.99	99.98	99.98	99.98	99.98	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	27.2	26.81	28.55	29.28	28.96	27.97	**30.16**
	10	59.42	59.74	61.36	61.86	63.72	63.09	**65.02**
	15	82.03	81.99	82.74	81.82	84.87	85.42	**86.65**
	20	91.98	92	91.95	92.1	93.82	94.5	**94.92**
	25	96.47	96.9	96.66	96.82	97.44	97.94	**98.02**
	30	98.65	98.55	98.3	98.91	98.86	99.23	**99.33**
	40	99.86	99.79	99.78	99.81	99.83	99.94	**99.94**
	50	99.98	99.98	99.96	99.98	**100**	99.98	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	44.93	44.62	44.32	v46.9	45.9	44.67	45.89
	10	77.95	78.64	78.34	79.99	79.91	78.68	**79.76**
	15	92.37	93.06	93.22	93.14	93.51	93.68	**93.94**
	20	97.46	97.78	97.81	97.6	97.67	98.03	**98.03**
	25	99.17	99.28	99.31	99.17	99.2	**99.46**	99.45
	30	99.67	99.78	99.78	99.79	99.73	**99.86**	99.85
	40	99.96	99.97	99.97	99.95	99.94	**100**	**100**
	50	**99.99**	99.98	99.98	**99.99**	99.97	**99.99**	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	25.19	24.71	26.79	27.1	26.44	25.54	**27.84**
	10	55.91	55.6	57.72	57.22	60.44	60.35	**62.44**
	15	78.04	78.02	78.32	78.16	83.1	83.38	**84.63**
	20	90.64	90.9	90.5	90.62	93.28	93.87	**94.3**
	25	95.85	96.36	95.74	96.02	97.52	97.78	**97.87**
	30	98.38	98.68	98.2	98.59	**99.27**	99.22	99.26
	40	99.75	99.82	99.74	99.81	**99.96**	99.94	99.96
	50	99.95	99.98	99.91	99.97	99.98	**99.99**	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	45.06	44.6	44.56	46.36	46.28	44.94	**46.65**
	10	78.66	79.08	78.56	80.24	**80.57**	79.32	80.56
	15	92.5	93.1	92.91	92.26	93.59	93.46	**93.93**
	20	97.22	97.53	97.43	97.58	97.44	97.72	**97.73**
	25	98.96	99.13	99.17	99.06	99.08	99.28	**99.32**
	30	99.74	99.79	99.8	99.79	99.76	**99.9**	99.89
	40	99.96	**99.98**	99.97	99.97	99.94	99.97	99.97
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	30.17	29.67	31.69	**32.86**	31.89	30.52	32.71
	10	64.4	64.33	65.75	65.99	67.71	66.19	**68.54**
	15	84.04	83.97	84.8	84.29	86.03	86.09	**87.29**
	20	93.44	93.69	93.79	93.44	94.05	94.69	**95.3**
	25	97.39	97.47	97.46	97.62	97.58	98.02	**98.22**
	30	99	98.92	98.97	98.91	98.92	99.4	**99.5**
	40	99.87	99.87	99.89	99.86	99.84	99.91	**99.91**
	50	99.95	99.96	99.95	99.96	99.93	**99.98**	99.97
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	28.11	27.81	29.87	**31.63**	29.58	28.51	30.55
	10	63.2	62.9	64.66	64.02	66.43	65.2	**67.41**
	15	82.26	82.64	82.99	83.18	84.62	84.72	**86.07**
	20	92.59	92.7	92.52	92.63	93.44	94.08	**94.78**
	25	97.17	97.36	97.36	97.23	97.52	98.03	**98.29**
	30	98.91	98.84	98.82	98.91	98.97	99.33	**99.37**
	40	99.82	99.83	99.83	99.81	99.8	99.92	**99.93**
	50	**100**	**100**	**100**	99.96	99.99	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	27.34	26.84	28.67	29.48	29	28.06	**30.33**
	10	60.82	60.59	62.65	62.24	64.75	63.81	**65.76**
	15	81.54	81.52	82	81.61	84.65	84.59	**85.84**
	20	92.18	92.2	92.18	92.16	93.54	94.09	**94.53**
	25	97.11	96.95	96.89	96.77	97.32	97.9	**98.1**
	30	98.91	98.83	98.81	98.7	98.86	99.26	**99.38**
	40	99.83	99.8	99.79	99.85	99.76	99.92	**99.92**
	50	**100**	99.99	**100**	99.98	99.99	99.99	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.34: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.1$ against the alternative distributions from Group I for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	AD(3)	CvM(3)
$\chi^2(1)$	5	8.11	8.7	**13.17**	8.64	7.39	12.15	8.17
	10	16.66	18.07	19.28	16.12	11.66	**25.22**	18.32
	15	28.05	30.07	23.96	26.66	16.39	**40.32**	31.87
	20	40.46	43.01	30	38.87	22.1	**54.07**	43.75
	25	50.05	54.57	36.89	51.86	27.57	**65.29**	55.19
	30	60.29	64.08	42.95	62.69	33.61	**74.18**	65.15
	40	75.24	79.4	58.27	78.15	47.29	**86.68**	79.72
	50	85.26	88.97	70.86	88.03	59.17	**93.84**	88.82
	100	99.27	99.65	97.75	99.63	94.07	**99.89**	99.61
	150	99.92	99.98	99.85	99.99	99.28	**100**	99.97
	200	100	100	100	100	99.98	**100**	100
$\chi^2(3)$	5	12.81	12.9	11.47	13.12	13.06	12.16	**13.13**
	10	16.41	16.14	13.83	16.62	**17.77**	15.29	16.67
	15	19.79	20.1	17.5	20.5	**22.23**	19.39	21.43
	20	21.77	22.79	19.88	24.02	**25.86**	21.76	23.42
	25	24.71	26.77	23.45	27.91	**30.83**	25.33	27.5
	30	28.59	31.3	28.36	31.87	**35.36**	29.5	31.83
	40	35.07	38.96	35.79	39.87	**42.95**	37.87	39.57
	50	41.34	46.3	43.06	47.13	**49.43**	46.16	47.11
	100	68.9	74.56	71.55	75.08	73.06	**77.27**	76.13
	150	85.51	89.28	87.2	89.19	86.74	**92.00**	90.73
	200	93.96	95.69	94.64	95.72	94.2	**97.42**	96.75
$\chi^2(4)$	5	14.66	14.73	12.16	**15.34**	15.12	13.65	14.93
	10	21.14	21.65	17.68	21.85	**23.80**	20.83	23.31
	15	28.46	29.09	25.3	30.68	**32.13**	29.16	32.08
	20	35.05	36.52	32.62	37.74	**40.61**	36.92	39.58
	25	42.1	44.55	40.67	44.92	**49.01**	45.18	47.99
	30	49.5	52.13	47.99	53.39	**55.60**	53.19	55.28
	40	61.51	64.84	61.71	66.63	67.94	67.05	**68.67**
	50	72.21	75.66	73.02	75.99	77.19	78.56	**79.05**
	100	96.23	96.24	95.35	96.8	95.75	**97.96**	97.9
	150	99.58	99.73	99.47	99.62	99.46	**99.88**	99.85
	200	99.95	99.96	99.93	99.95	99.89	**100**	100
$\chi^2(6)$	5	16.97	16.74	13.47	17.91	17.75	15.66	**17.92**
	10	29.4	30.21	24.65	30.3	32.25	30.05	**32.88**
	15	41.98	42.41	37.82	42.69	45.19	44.3	**47.35**
	20	53.14	54.16	49.79	54.58	57.86	57.17	**59.36**
	25	63.27	64.97	61.21	65.2	68.46	68.19	**70.61**
	30	72.8	73.94	70.44	74.52	76.35	77.06	**79.05**
	40	84.84	85.82	83.66	85.96	87.28	89.09	**89.99**
	50	92.2	92.7	91.08	92.88	92.9	95.21	**95.51**
	100	99.81	99.77	99.68	99.79	99.65	99.92	**99.93**
	150	**100**	**100**	**100**	**100**	99.99	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	19.63	19.57	15.73	21.27	20.4	18.49	**21.45**
	10	37.71	37.72	32.1	38.48	40.05	39.03	**42.48**
	15	55.01	54.82	50.3	55.58	57.39	58.49	**61.58**
	20	68.51	68.54	64.72	68.88	71.82	72.66	**74.74**
	25	78.82	78.94	76.15	79.06	80.87	83.05	**84.69**
	30	86.25	86	83.75	86.16	87.38	89.49	**90.82**
	40	94.67	94.07	93.05	94.16	94.83	96.65	**96.99**
	50	98.12	97.99	97.37	97.95	98.23	99.05	**99.12**
	100	99.99	99.98	99.97	99.99	99.98	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	9.22	9.07	**11.20**	9.75	9.44	10.67	9.59
	10	11.78	11.71	**16.14**	11.8	11.85	14.12	11.95
	15	14.21	13.95	**20.86**	14.09	13.99	17.05	15.13
	20	16.43	16.46	**24.71**	16.43	15.54	19.02	17.16
	25	18.09	18.56	**28.82**	18.35	16.75	21.52	19.87
	30	20.35	20.7	**32.51**	20.74	18.25	23.69	21.94
	40	24.22	24.62	**39.12**	25.12	21.4	27.8	26.2
	50	27.49	29.13	**45.68**	29.5	24.28	31.58	30.25
	100	46.57	51.03	**71.04**	53.23	39.79	55.43	51.65
	150	61.34	69.3	**85.48**	70.25	56.05	74.55	69.23
	200	74.8	83.46	**93.23**	84.58	70.7	87.46	82.27
$HN(0,1)$	5	14.64	14.74	12.3	15.06	14.82	13.7	**15.40**
	10	20.95	20.59	17.29	21.18	21.43	20.69	**22.67**
	15	25.97	24.5	21.36	25.05	25.9	26.16	**29.17**
	20	30.89	28.58	25.51	29.03	31.84	31.51	**34.40**
	25	34.74	33.23	29.65	32.81	36.7	36.89	**40.10**
	30	40.13	37.1	32.86	36.99	42.65	42.49	**46.38**
	40	48.58	45.9	40.01	45.4	53.53	53.24	**56.51**
	50	55.72	53.57	45.6	52.45	62.38	61.48	**64.73**
	100	82.63	82.51	69.3	81.8	87.05	88.92	**89.88**
	150	93.65	94.38	83.36	93.22	95.95	97.2	**97.65**
	200	97.91	98.49	91.68	98.18	98.87	99.4	**99.43**
$Wbl(0,2,2)$	5	21.21	21.19	16.73	22.78	22.1	20.01	**23.16**
	10	40.17	39.82	34.19	41.29	41.38	42.2	**45.10**
	15	56.49	55.79	50.71	56.92	58.33	61.16	**64.27**
	20	70.02	69.65	65.09	70.07	73.1	76.02	**77.91**
	25	80.46	80.06	76.88	79.91	83.14	85.78	**87.18**
	30	88.51	87.94	84.87	87.48	90	92.44	**93.34**
	40	95.44	95.29	93.65	94.87	96.26	97.5	**97.75**
	50	98.31	98.32	97.27	98.48	98.86	99.43	**99.48**
	100	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	13.59	14.33	**23.18**	14.34	12.18	22.71	14.33
	10	43.61	45.67	43.51	41.9	35.49	**56.34**	47.32
	15	67.1	68.64	60.29	65.88	53.62	**78.58**	71.88
	20	82.72	84.15	72.45	82.56	68.06	**90.56**	85.72
	25	90.4	92.05	82.66	91.29	78.46	**95.72**	93.12
	30	95.45	96.58	89.12	95.84	86.17	**98.46**	96.92
	40	98.86	99.27	96.83	99.21	94.71	**99.69**	99.34
	50	99.79	99.89	99.07	99.86	98.21	**99.98**	99.89
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

Table 5.35: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.1$ against the alternative distributions from Group II for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	$AD(3)$	$CvM(3)$
$U(0,1)$	5	26.58	27.22	23.52	27.23	25.64	27.07	**28.90**
	10	45.59	44.17	40.64	43.43	42.32	52.18	**53.25**
	15	60.54	56.76	53.49	55.47	64.41	70.33	**71.11**
	20	71.79	71.02	63.66	68.21	**86.08**	82.36	81.99
	25	80.38	85.72	72.42	81.88	**95.99**	90.19	89.63
	30	86.81	94.67	79.51	92.26	**99.13**	94.81	94.42
	40	94.64	99.51	88.59	99.18	**99.96**	98.83	98.61
	50	97.67	99.98	94.35	99.97	**100**	99.79	99.68
	100	**100**	**100**	99.91	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	28.83	28.95	23.79	29.31	29.07	29.26	**31.71**
	10	53.79	52.77	47.66	53.55	52.81	60.56	**62.54**
	15	73.79	71.28	67.67	70.99	74.92	81.32	**82.48**
	20	86.59	84.79	81.01	84.5	90.91	92.54	**92.98**
	25	92.66	93.42	89.46	92.31	**97.74**	97.02	97.15
	30	96.51	97.75	94.33	97.17	**99.48**	99.14	99.11
	40	99.22	99.81	98.53	99.7	**99.97**	99.86	99.83
	50	99.92	99.99	99.72	99.99	**99.99**	99.98	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	20.28	20.14	16.02	21	20.91	18.78	**21.23**
	10	35.08	34.61	29.59	35.78	36.42	36.79	**40.13**
	15	48.91	47.89	42.85	50.02	50.61	53.48	**56.63**
	20	63.11	61.73	57.25	61.45	66.39	69.22	**71.57**
	25	72.79	72.72	68.86	72.16	77.96	80.26	**81.94**
	30	81.32	81.09	76.85	81.11	85.61	87.61	**88.79**
	40	91.15	91.75	88.37	91.58	94.82	95.63	**95.91**
	50	95.95	96.72	94.08	96.85	98.06	98.63	**98.74**
	100	99.98	99.98	99.94	99.99	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	42.17	42.92	37.6	43.82	41.97	46.34	**48.39**
	10	79.6	78.5	74.96	78.46	77.56	86.63	**87.02**
	15	93.87	93.16	91.86	93.15	96.08	**97.64**	97.6
	20	98.4	98.61	97.54	98.53	**99.64**	99.54	99.49
	25	99.72	99.87	99.36	99.82	**99.99**	99.96	99.97
	30	99.92	99.99	99.82	99.98	**100**	99.99	99.99
	40	99.99	**100**	**100**	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	27.34	28.83	27.91	26.14	25	**30.84**	28.5
	10	38.97	38.09	38.09	35.9	31.43	**48.38**	44.74
	15	49.71	47	45.8	44.31	54.62	**63.19**	58.28
	20	58.97	63.13	52.38	57.02	**78.92**	75.13	68.34
	25	67.85	80.15	60.44	72.84	**91.26**	82.94	77.46
	30	76.25	91.18	66.9	87.14	**96.89**	90.5	85.47
	40	86.17	98.89	77.42	97.94	**99.77**	96.66	93.53
	50	93.26	99.94	86.14	99.83	**100**	99.25	97.8
	100	99.96	**100**	99.41	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	8.62	8.9	**12.02**	8.99	8.16	11.74	8.72
	10	12.21	13.06	11.18	11.45	8.05	**17.74**	12.9
	15	18.17	20.17	12.73	17.61	9.25	**27.86**	20.47
	20	25.19	28.26	12.66	25.92	11.25	**36.82**	27.28
	25	31.28	37.01	14.92	33.79	13.93	**46.13**	35.33
	30	40.26	46.66	18.61	44.31	17.84	**55.76**	45.03
	40	53.56	61.7	27.4	59.97	25.75	**70.24**	58.51
	50	64.59	73.94	38.5	72.42	35.06	**81.01**	69.79
	100	93.82	97.7	84.39	97.8	75.93	**98.39**	95.17
	150	99.15	99.75	97.7	99.8	94.83	**99.92**	99.56
	200	99.87	99.98	99.79	99.99	99.28	**100**	99.91
$beta(1,2)$	5	16.46	16.79	14.53	**17.53**	16.31	15.79	17.31
	10	23.82	22.93	19.62	23.13	22.8	24.13	**25.97**
	15	30.69	28.12	25.56	28.81	29.45	33.3	**35.43**
	20	36.13	33.01	29.62	33.2	39.32	40.09	**41.91**
	25	42.72	39.39	34.34	38.79	**51.25**	48.27	50.72
	30	48.17	47.04	39.47	44.52	**62.80**	54.93	56.79
	40	58.05	61.89	47.04	56.44	**80.24**	66.58	68.22
	50	65.31	75.29	52.07	70.89	**91.38**	75.91	76.64
	100	91.12	99.75	77.49	99.16	**99.99**	97.31	96.89
	150	98.22	**100**	90.55	**100**	**100**	99.8	99.77
	200	99.8	**100**	97.27	**100**	**100**	99.98	99.98

The bold number is the highest power among the seven tests for each sample size.

Table 5.36: Powers (in %) of the five graphical tests and the two non-graphical tests by using BLIE at $\alpha = 0.1$ against the alternative distributions from Group III for several sample sizes n

Alternatives	n	$D(3)$	$D_{sp}(3)$	$D_e(3)$	$D_{be}(3)$	$D_{bi}(3)$	$AD(3)$	$CvM(3)$
$Laplace(0,1)$	5	36.66	35.83	30.13	37.55	37.88	35.53	**39.32**
	10	70.86	70.46	66.2	72.03	71.44	71.74	**74.28**
	15	89.62	89	87.43	89.03	89.44	90.4	**91.59**
	20	96.03	95.81	95.14	95.9	95.89	96.79	**97.18**
	25	98.63	98.5	98.23	98.6	98.49	98.88	**99.04**
	30	99.53	99.42	99.31	99.53	99.37	99.68	**99.73**
	40	99.97	99.97	99.96	99.92	99.94	99.98	**99.98**
	50	99.99	99.98	99.98	99.98	99.98	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	33.02	32.51	26.33	34.13	34.07	32.86	**36.53**
	10	65.21	64.73	59.88	65.93	65.93	68.08	**70.54**
	15	85.27	84.61	82.19	84.04	85.35	88.13	**89.83**
	20	93.84	93.05	91.68	93.15	93.83	95.55	**96.04**
	25	97.41	97.23	96.6	97.25	97.57	98.38	**98.62**
	30	99.03	98.67	98.23	99.03	98.92	99.43	**99.54**
	40	99.91	99.83	99.78	99.8	99.81	99.95	**99.95**
	50	99.98	99.98	99.96	99.98	99.99	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	46.01	45.35	42.99	**47.44**	47.08	45.8	47.34
	10	77.95	78.28	77.7	79.29	**80.14**	78.54	79.03
	15	92.1	92.9	92.99	92.81	**93.53**	93.51	93.4
	20	97.19	97.53	97.68	97.54	97.78	**97.84**	97.75
	25	98.98	99.22	99.29	99.13	99.27	**99.40**	99.4
	30	99.59	99.74	99.78	99.77	99.72	**99.83**	99.79
	40	99.96	99.96	99.97	99.95	99.95	**100**	100
	50	99.98	99.98	99.98	99.99	99.97	**99.99**	99.99
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	30.88	30.4	25.18	31.92	31.83	30.61	**34.59**
	10	62.43	61.29	56.07	61.95	62.62	66.38	**68.77**
	15	82.15	80.61	77.73	81.07	82.41	86.65	**88.07**
	20	92.62	91.83	90.2	91.89	93.22	94.99	**95.59**
	25	96.88	96.66	95.79	96.52	97.4	98.18	**98.43**
	30	98.8	98.82	98.22	98.7	99.12	99.41	**99.51**
	40	99.83	99.84	99.73	99.85	99.95	99.96	**99.96**
	50	99.97	99.97	99.91	99.96	99.99	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	46.68	46.01	43.34	47.1	47.41	46.7	**48.19**
	10	78.61	79.09	77.91	79.51	**80.70**	79.53	80.12
	15	92.24	92.94	92.68	92.08	**93.72**	93.5	93.48
	20	97.09	97.36	97.43	97.43	97.58	**97.63**	97.57
	25	98.89	99.12	99.16	99.04	99.17	99.24	**99.26**
	30	99.69	99.79	99.79	99.73	99.82	**99.88**	99.84
	40	**99.97**	**99.97**	**99.97**	**99.97**	99.94	99.96	99.96
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	35.53	35.34	30.28	37.24	36.37	35.29	**38.40**
	10	68.66	68.45	64.4	68.98	69.98	70.46	**72.83**
	15	86.39	85.99	84.42	85.76	87.05	87.95	**89.19**
	20	94.54	94.41	93.66	94.24	94.6	95.58	**95.89**
	25	97.87	97.75	97.43	97.94	97.81	98.27	**98.54**
	30	99.19	99.13	98.98	99.13	99.04	99.52	**99.57**
	40	99.88	99.9	99.89	99.89	99.85	99.92	**99.92**
	50	99.96	99.97	99.95	99.96	99.94	99.98	**99.98**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	33.98	33.57	28.23	35.94	34.6	33.33	**36.74**
	10	68.12	67.43	63.18	67.87	68.52	70.14	**72.48**
	15	85.09	84.43	82.6	85.4	85.68	86.96	**88.39**
	20	93.86	93.39	92.42	93.46	93.76	95.04	**95.59**
	25	97.78	97.71	97.29	97.56	97.85	98.37	**98.59**
	30	99.19	99.08	98.78	99.01	99.12	99.41	**99.50**
	40	99.89	99.86	99.82	99.86	99.84	99.93	**99.95**
	50	**100**	**100**	**100**	99.97	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	33.05	32.8	27.02	33.72	33.9	32.99	**36.36**
	10	66.29	65.53	61.14	66.38	67.03	69	**71.44**
	15	84.69	83.74	81.55	83.84	84.86	87.12	**88.81**
	20	93.71	93.2	91.93	93.28	94	95.12	**95.88**
	25	97.76	97.41	96.91	97.2	97.57	98.38	**98.67**
	30	99.13	99	98.79	98.93	98.98	99.42	**99.54**
	40	99.88	99.81	99.79	99.91	99.83	99.93	**99.94**
	50	**100**	**100**	**100**	99.98	99.99	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bold number is the highest power among the seven tests for each sample size.

CHAPTER 6 Conclusions and future work

6.1 Conclusions

The five different sets of simultaneous intervals, associated with the D, D_{sp}, D_e, D_{be} and D_{bi} tests, were constructed in order to make an objective judgement on Q-Q plots. As stated in the introduction, one of the objectives of this research is to compare the powers among the graphical tests and some non-graphical tests. This thesis aimed to identify the tests which are most powerful under specific alternative distributions. Based on the simulation, specific recommendations (see Table 6.1) regarding the use of the graphical tests in practice are made.

Table 6.1: Recommended tests

Testing	Support and shape of alternative distributions		
	support $(0, \infty)$ and asymmetric	support $(0, 1)$	support $(-\infty, \infty)$ and symmetric
Normality			
1. based on a simple random sample	D_{bi}	D_{bi}	D_e
2. based on residuals of a linear regression	D_{sp}, D_{bi}, T_n	D_{bi}	T_n
Weibull distribution	D_{sp}, D_{be}	D_{bi}	D_{sp}, D_{be}
Exponential distribution	D_{sp}, D_{be}	D_{bi}	any test

Based on the investigations set out in the previous chapters, the Kolmogorov-Smirnov test (D test) has a very low power. Although the Shapiro-Wilk, Anderson-Darling, Craḿer-von-Mises and T_n tests are non-graphical, they may not be more powerful than the graphical tests. In general, the D_{bi} and D_e tests should be used for testing normality based on a simple random sample. The D_{bi} and D_{sp} tests should be used for testing normality based on the residuals from a linear regression model. For testing the Weibull and exponential distributions, the D_{be}, D_{bi} and D_{sp} tests should be used. Although the D_e test is one of the graphical tests recommended for testing normality when a simple random sample is considered, it is a bad choice for the other cases. Interestingly, the investigations on the Type I error rates in Chapter

3 imply that the tests for normality based on residuals from a linear regression model should be treated differently from a simple random sample. Matlab codes are provided to produce simultaneous probability intervals on Q-Q plots. We hope that the proposed graphical tests can help practitioners to use the graphical tests on Q-Q plots with confidence.

6.2 Future work

Possible extensions to our work are:

1. Censored data-: Throughout this thesis, we consider completely observed data. However, in many real problems the complete data may not be available. For example, in medical statistics, a study is often completed before the death of all patients is observed, which implies that the death times of the remaining patients are not recorded. With censored data, statistics based only on the complete observations are of no use. Thus, the construction of simultaneous probability intervals for censored data should be of interest.

2. Outliers-: Outliers in a dataset are values that are far away from the rest of the values in the data set. If some prior knowledge about possible outliers is available, it would be useful to construct special simultaneous bounds for such cases.

3. Multivariate normality-: This would assess whether a sample of d–dimensional random vector is from a d–dimensional normal distribution.

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Training, Coursework, Conferences and Publication

Training

1. $4^{th} - 8^{th}$ July 2011, Research and Presentation Skills Training, University of Southampton
2. $9^{th} - 13^{th}$ January 2012, APTS training: Statistical Inference, University of Cambridge
3. $9^{th} - 13^{th}$ January 2012, APTS training: Statistical Computing, University of Cambridge
4. $16^{th} - 20^{th}$ April 2012, APTS training: Statistical Modelling, University of Nottingham
5. $16^{th} - 20^{th}$ April 2012, APTS training: Statistical Asymptotics, University of Nottingham
6. $2^{nd} - 6^{th}$ July 2012, APTS training: Applied Stochastic Processes, University of Warwick
7. $2^{nd} - 6^{th}$ July 2012, APTS training: Computer Intensive Statistics, University of Warwick
8. $3^{rd} - 7^{th}$ September 2012, APTS training: Spatial and Longitudinal Data Analysis, University of Glasgow.
9. $3^{rd} - 7^{th}$ September 2012, APTS training: Nonparametric Smoothing, University of Glasgow
10. 30^{th} September 2013, Workshop for Postgraduate Students who teach Mathematics, Statistics, and Operational Research, University of Southampton
11. 1^{st} October 2013, Finding Information to Support your Research in MATHEMATICS, University of Southampton

12. 16^{th} October 2013, Preparing for the PhD Viva, University of Southampton

13. 8^{th} November 2013, E-theses training, University of Southampton

14. 31^{st} Mar 2014, Presenting Your Research, University of Southampton

15. 8^{th} May 2014, Invigilation training session, University of Southampton

16. 19^{th} May 2014, How to Turn a Chapter of your PhD into an Article, University of Southampton

17. 15^{th} May 2014, Endnote Bibliographic Software: basic course, University of Southampton

18. 2^{nd} June 2014, Endnote Bibliographic Software: advanced course, University of Southampton

Coursework

1. Univariate distribution theory and inference

2. Bayesian Methods

3. Survival Analysis

4. Generalized linear models

5. Design of Experiments

6. Computer Intensive Statistical Methods

7. Multivariate Analysis

Conferences

1. The 35^{th} Research Students' Conference in Probability and Statistics, 9^{th}–12^{th} July 2012, at University of Southampton, UK

2. The 36^{th} Research Students' Conference in Probability and Statistics, 25^{th}–28^{th} March 2013, at Lancaster University, UK

3. The 8^{th} International Conference on Multiple Comparison Procedures, 8^{th}–12^{th} July 2013, at University of Southampton, UK

4. The 9^{th} ICSA International Conference on Challenges of Statistical Methods for Interdisciplinary Research and Big Data, 20^{th}–23^{rd} December 2013, at the Hong Kong Baptist University, Hong Kong

5. The 21^{th} International Conference on Computational Statistics (COMPSTAT2014), 19^{th}–22^{nd} August 2014, at the International Conference Centre Geneva, Switzerland

Publications

1. Chantarangsi, W., Liu, W., Bretz, F., Kiatsupaibul, S., Hayter, A.J. and Wan, F. (2015). Normal probability plots with confidence. *Biometrical Journal*, **57(1)**, 52–63.

2. Chantarangsi, W., Liu, W., Bretz, F., Kiatsupaibul, S. and Hayter, A.J. (2015). Normal probability plots with confidence for the residuals in linear regression, submitted to *Journal of Statistical Computation and Simulation.*

3. Chantarangsi, W., Liu, W., Bretz, F., Kiatsupaibul, S. and Hayter, A.J. (2015). Q-Q plots with confidence for testing Weibull and exponential distributions, submitted to *Journal of Quality Technology.*