

On dimensions of groups with cocompact classifying spaces for proper actions

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Abstract

We construct groups G that are virtually torsion-free and have virtual cohomological dimension strictly less than the minimal dimension for any model for $\underline{E}G$, the classifying space for proper actions of G . They are the first examples that have these properties and also admit cocompact models for $\underline{E}G$. We exhibit groups G whose virtual cohomological dimension and Bredon cohomological dimension are two that do not admit any 2-dimensional contractible proper G -CW-complex.

1 Introduction

If G is a virtually torsion-free group, the virtual cohomological dimension $\mathrm{vcd}G$, is defined to be the cohomological dimension of a torsion-free finite-index subgroup $H \leq G$; a lemma due to Serre shows that this is well defined [7, VIII.3.1]. Now suppose that X is a contractible G -CW-complex that is *proper*, in the sense that all cell stabilizers are finite. In this case any torsion-free subgroup H will act freely on X and so X/H is a classifying space or Eilenberg-Mac Lane space BH for H . In particular, $\mathrm{vcd}G$ provides a lower bound for the dimension of any such X . K. S. Brown asked whether this lower bound is always attained [6, ch. 2] or [7, VIII.11]:

Does every virtually torsion-free group G admit a contractible proper G -CW-complex of dimension $\mathrm{vcd}G$?

Until now, this form of Brown's question has remained unanswered. We give examples of groups G with $\mathrm{vcd}G = 2$ that do not admit any 2-dimensional contractible proper G -CW-complex in Theorem 1.3 below.

One reason why this question has been so elusive is that there are many different equivariant homotopy types of contractible proper G -CW-complexes. The most natural example is the classifying space for proper G -actions, $\underline{E}G$, which plays the same role in the homotopy category of proper G -CW-complexes as EG plays for free G -CW-complexes. A model for $\underline{E}G$ is a proper G -CW complex X such that for any finite $F \leq G$, the F -fixed point set X^F is contractible. Such an X always exists, and is unique up to equivariant homotopy equivalence. Let $\mathrm{gd}G$ denote the minimal dimension of any model for $\underline{E}G$.

The version of Brown's question that concerns $\underline{E}G$ [6, ch. 2] or [7, VIII.11] is usually asked in the form: does $\mathrm{gd}G = \mathrm{vcd}G$ for every virtually torsion-free G ? We prefer to split this into two separate questions. There is an algebraic

dimension $\underline{\text{cd}}G$ that bears a close relationship to $\underline{\text{gd}}G$, analogous to the relationship between cohomological dimension and the minimal dimension of an Eilenberg-Mac Lane space. It can be shown that $\underline{\text{cd}}G = \underline{\text{gd}}G$ except that there may exist G for which $\underline{\text{cd}}G = 2$ and $\underline{\text{gd}}G = 3$, and $\underline{\text{cd}}G$ is an upper bound for the cohomological dimension of any torsion-free subgroup of G [22]. In view of this we may split Brown's question into two parts, one geometric and one algebraic.

Does there exist G for which $\underline{\text{gd}}G \neq \underline{\text{cd}}G$?

Does there exist virtually torsion-free G for which $\underline{\text{cd}}G > \text{vcd}G$?

Examples of virtually torsion-free groups G for which $\underline{\text{gd}}G = 3$ and $\underline{\text{cd}}G = 2$ were given in [2]. These groups G are Coxeter groups. Examples of G for which $\underline{\text{cd}}G > \text{vcd}G$ were given in [17], and more recently in [23, 12]. The advantage of the examples in [23, 12] is that in some sense they have the least possible torsion. For any virtually torsion-free G , it can be shown that $\underline{\text{cd}}G$ is bounded by the sum $\text{vcd}G + \ell(G)$, where $\ell(G)$ is the maximal length of a chain of non-trivial finite subgroups of G [19, 6.4]. This bound is attained for the examples in [23, 12] but not for the examples in [17]. To date, all constructions of groups G for which $\underline{\text{cd}}G > \text{vcd}G$ have used finite extensions of Bestvina-Brady groups [1], and none of these groups G admit a cocompact model for \underline{EG} . One of our main results is the construction of virtually torsion-free G admitting a cocompact \underline{EG} for which $\underline{\text{cd}}G > \text{vcd}G$. Amongst our examples, the easiest to describe are extensions of a right-angled Coxeter group by a cyclic group of prime order. By taking instead a cyclic extension of a torsion-free finite index subgroup of the same Coxeter group we obtain examples with cocompact \underline{EG} and for which $\underline{\text{cd}}G = \text{vcd}G + \ell(G)$.

Many of the terms used in the statements of our main theorems will be defined below.

Theorem 1.1. *Let L be a finite n -dimensional acyclic flag complex with an admissible simplicial action of a finite group Q , and let W_L be the corresponding right-angled Coxeter group so that Q acts as a group of automorphisms of W_L . Let N be any finite-index normal subgroup of W_L such that N is normalized by Q , and let G be the semidirect product $N \rtimes Q$. This G admits a cocompact model for \underline{EG} . Let L^{sing} denote the subcomplex of L consisting of points with non-trivial stabilizer.*

If $H^n(L, L^{\text{sing}}) \neq 0$, then $\underline{\text{cd}}G = n + 1 > \text{vcd}G = n$.

Now suppose that L_i, Q_i, n_i, N_i, W_i and G_i are as above for $i = 1, \dots, m$, and let $\Gamma := G_1 \times \dots \times G_m$. As before, there is a cocompact model for $\underline{E}\Gamma$, and

if $\bigotimes_{i=1}^m H^{n_i}(L_i, L_i^{\text{sing}}) \neq 0$, then $\underline{\text{cd}}\Gamma = m + \sum_{i=1}^m n_i = m + \text{vcd}\Gamma$.

Corollary 1.2. *For each $m \geq 1$ there exists a virtually torsion-free group Γ_m admitting a cocompact $\underline{E}\Gamma_m$ and such that*

$$\underline{\text{cd}}\Gamma_m = 3m > \text{vcd}\Gamma_m = 2m.$$

For each $m \geq 1$ there exists a virtually torsion-free group Λ_m admitting a cocompact $\underline{E}\Lambda_m$ and such that

$$\underline{\text{cd}}\Lambda_m = 4m = \text{vcd}\Lambda_m + \ell(\Lambda_m) > \text{vcd}\Lambda_m = 3m.$$

Furthermore, Λ_m may be chosen so that either every finite subgroup is cyclic or, for any fixed prime q , every nontrivial finite subgroup is abelian of exponent q .

In contrast to the above results, Degrijse and Martínez-Pérez have shown that $\text{vcd}G = \text{cd}G$ for a large class of groups that contains all (finitely generated) Coxeter groups [11].

Theorem 1.3. *Suppose that L is a finite 2-dimensional acyclic flag complex such that the fundamental group of L admits a non-trivial unitary representation $\rho : \pi_1(L) \rightarrow U(n)$ for some n . Then $\text{vcd}W_L = \text{cd}W_L = 2$, there is a cocompact 3-dimensional model for $\underline{E}W_L$, and yet there exists no proper 2-dimensional contractible W_L -CW-complex.*

A different argument was used in [2] to show that $\text{cd}W_L = 2 < \text{gd}W_L = 3$ for some of the flag complexes L that appear in Theorem 1.3.

We remark that finitely generated Coxeter groups are linear over \mathbb{Z} [8], and that this property passes to subgroups and to finite extensions. Hence all of the groups appearing in the above statements are linear over \mathbb{Z} . As will be seen from the proofs, each group appearing in our statements acts properly and cocompactly on a CAT(0) cube complex; in particular they are all CAT(0) groups. A right-angled Coxeter group W_L is (Gromov) hyperbolic if and only if L satisfies the flag-no-square condition. Hyperbolicity passes to finite index subgroups and finite extensions. Since any 2- or 3-dimensional flag complex admits a flag-no-square subdivision [13, 24] it follows that the groups Γ_1 and Λ_1 in Corollary 1.2 and the groups W_L in Theorem 1.3 may be taken to be hyperbolic (and CAT(−1), a possibly stronger property) in addition to their other stated properties.

2 Classifying spaces and Bredon cohomology

The algebraic analogs of the geometric finiteness properties exhibited by classifying spaces of groups for families of subgroups are formulated using Bredon cohomology. This cohomology theory was introduced by Bredon in [3] for finite groups and was generalised to arbitrary groups by Lück (see [18]).

Let G be a discrete group. A family \mathcal{F} of subgroups of G is a non-empty set of subgroups which is closed under conjugation and taking subgroups, in the sense that if $H \in \mathcal{F}$, $g \in G$ and $K \leq H$, then $K \in \mathcal{F}$ and $gHg^{-1} \in \mathcal{F}$. The *orbit category* $\mathcal{O}_{\mathcal{F}}G$ is the category with objects the left cosets G/H for all $H \in \mathcal{F}$ and morphisms all G -equivariant functions between the objects. In $\mathcal{O}_{\mathcal{F}}G$, every morphism $\varphi : G/H \rightarrow G/K$ is completely determined by $\varphi(H)$, since $\varphi(xH) = x\varphi(H)$ for all $x \in G$. Moreover, there exists a morphism

$$G/H \rightarrow G/K : H \mapsto xK$$

if and only if $x^{-1}Hx \subseteq K$.

An $\mathcal{O}_{\mathcal{F}}G$ -module is a contravariant functor $M : \mathcal{O}_{\mathcal{F}}G \rightarrow \mathbb{Z}\text{-mod}$. The *category of $\mathcal{O}_{\mathcal{F}}G$ -modules* is denoted by $\text{Mod-}\mathcal{O}_{\mathcal{F}}G$. By definition, it has as objects all $\mathcal{O}_{\mathcal{F}}G$ -modules and as morphisms all natural transformations between these objects. The category $\text{Mod-}\mathcal{O}_{\mathcal{F}}G$ is an abelian category that contains enough projectives and so one can construct bi-functors $\text{Ext}_{\mathcal{O}_{\mathcal{F}}G}^n(-, -)$ that have all the usual properties. The n -th Bredon cohomology of G with coefficients $M \in \text{Mod-}\mathcal{O}_{\mathcal{F}}G$ is by definition

$$H_{\mathcal{F}}^n(G; M) = \text{Ext}_{\mathcal{O}_{\mathcal{F}}G}^n(\mathbb{Z}, M),$$

where $\underline{\mathbb{Z}}$ is the constant functor, which sends each object to \mathbb{Z} and each morphism to the identity map on \mathbb{Z} . There is also a notion of *Bredon cohomological dimension* of G for the family \mathcal{F} , denoted by $\text{cd}_{\mathcal{F}}(G)$ and defined by

$$\text{cd}_{\mathcal{F}}(G) = \sup\{n \in \mathbb{N} \mid \exists M \in \text{Mod-}\mathcal{O}_{\mathcal{F}}G : H_{\mathcal{F}}^n(G; M) \neq 0\}.$$

When \mathcal{F} is the family of finite subgroups, then $H_{\mathcal{F}}^*(G, M)$ and $\text{cd}_{\mathcal{F}}G$ are denoted by $\underline{H}^*(G, M)$ and $\underline{\text{cd}}G$, respectively. Since the augmented cellular chain complex of any model for $E_{\mathcal{F}}G$ yields a projective resolution of $\underline{\mathbb{Z}}$ that can be used to compute $H_{\mathcal{F}}^*(G; -)$, it follows that $\text{cd}_{\mathcal{F}}(G) \leq \text{gd}_{\mathcal{F}}(G)$. Moreover, it is known (see for example [22, 0.1]) that

$$\text{cd}_{\mathcal{F}}(G) \leq \text{gd}_{\mathcal{F}}(G) \leq \max\{3, \text{cd}_{\mathcal{F}}(G)\}.$$

For any $\mathbb{Z}G$ -module M , one may define an $\mathcal{O}_{\mathcal{F}}G$ -module \underline{M} by $\underline{M}(G/H) = \text{Hom}_G(\mathbb{Z}[G/H], M)$; note that this is compatible with the notation $\underline{\mathbb{Z}}$ introduced earlier. For any G -CW-complex X with stabilizers in \mathcal{F} , it can be shown that Bredon cohomology with coefficients in \underline{M} is naturally isomorphic to the ordinary equivariant cohomology of X with coefficients in M : $H_{\mathcal{F}}^*(X; \underline{M}) \cong H_G^*(X; M)$. This follows because there is an isomorphism of cochain complexes

$$\text{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}) \cong \text{Hom}_G(C_*(X), M).$$

A *subfamily* of a family \mathcal{F} of subgroups of G is another family $\mathcal{G} \subseteq \mathcal{F}$. For a subfamily \mathcal{G} and a G -CW-complex X with stabilizers in \mathcal{F} , the \mathcal{G} -singular set $X^{\mathcal{G}\text{-sing}}$ is the subcomplex consisting of points of X whose stabilizer is not contained in \mathcal{G} . When \mathcal{G} consists of just the trivial subgroup this is the usual singular set and we write X^{sing} for $X^{\mathcal{G}\text{-sing}}$.

Given a $\mathbb{Z}G$ -module M and a subfamily \mathcal{G} of \mathcal{F} , we define two further $\mathcal{O}_{\mathcal{F}}G$ -modules: a submodule $\underline{M}_{>\mathcal{G}}$ of \underline{M} , and the corresponding quotient module $\underline{M}_{\leq\mathcal{G}}$. These are defined by

$$\begin{aligned} \underline{M}_{>\mathcal{G}} : G/H &\mapsto \begin{cases} \text{Hom}_G(\mathbb{Z}[G/H], M) & \text{if } H \notin \mathcal{G}, \\ 0 & \text{if } H \in \mathcal{G}, \end{cases} \\ \underline{M}_{\leq\mathcal{G}} : G/H &\mapsto \begin{cases} 0 & \text{if } H \notin \mathcal{G}, \\ \text{Hom}_G(\mathbb{Z}[G/H], M) & \text{if } H \in \mathcal{G}. \end{cases} \end{aligned}$$

By construction there is a short exact sequence of $\mathcal{O}_{\mathcal{F}}G$ -modules

$$\underline{M}_{>\mathcal{G}} \hookrightarrow \underline{M} \twoheadrightarrow \underline{M}_{\leq\mathcal{G}}.$$

Hence there is a short exact sequence of cochain complexes.

$$0 \rightarrow \text{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}_{>\mathcal{G}}) \rightarrow \text{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}) \rightarrow \text{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}_{\leq\mathcal{G}}) \rightarrow 0$$

This gives rise to a long exact sequence in Bredon cohomology. Since there is a natural identification between Bredon cohomology with coefficients in $\underline{M}_{>\mathcal{G}}$ and the equivariant cohomology of $X^{\mathcal{G}\text{-sing}}$ with coefficients in M :

$$H_{\mathcal{F}}^*(X; \underline{M}_{>\mathcal{G}}) \cong H_G^*(X^{\mathcal{G}\text{-sing}}; M),$$

we deduce that the Bredon cohomology with coefficients in $\underline{M}_{\leq\mathcal{G}}$ is isomorphic to the equivariant cohomology of the pair $(X, X^{\mathcal{G}\text{-sing}})$ with coefficients in M :

$$H_{\mathcal{F}}^*(X; \underline{M}_{\leq\mathcal{G}}) \cong H_G^*(X, X^{\mathcal{G}\text{-sing}}; M).$$

Hence we obtain the following.

Proposition 2.1. *Let \mathcal{F} be a family of subgroups of G , with \mathcal{G} a subfamily, let X be any model for $E_{\mathcal{F}}G$, and let H be a finite-index subgroup of G . There exists a Bredon module \mathcal{C} such that the Bredon cohomology of the group G with coefficients in \mathcal{C} computes the ordinary cohomology of the pair $(X/H, X^{\mathcal{G}\text{-sing}}/H)$:*

$$H_{\mathcal{F}}^*(G; \mathcal{C}) \cong H^n(X/H, X^{\mathcal{G}\text{-sing}}/H; \mathbb{Z}).$$

Furthermore, each abelian group $\mathcal{C}(G/K)$ is finitely generated.

Proof. Let M be the permutation module $\mathbb{Z}[G/H]$, and let $\mathcal{C} := \underline{M}_{\leq \mathcal{G}}$. For each $K \in \mathcal{F}$, $\mathcal{C}(G/K)$ is isomorphic to a subgroup of $\mathbb{Z}[G/H]$ and so is finitely generated, and

$$\begin{aligned} H_{\mathcal{F}}^*(G; \mathcal{C}) &\cong H_{\mathcal{F}}^*(X; \mathcal{C}) \cong H_G^*(X, X^{\mathcal{G}\text{-sing}}; \mathbb{Z}[G/H]) \\ &\cong H_H^*(X, X^{\mathcal{G}\text{-sing}}; \mathbb{Z}) \cong H^*(X/H, X^{\mathcal{G}\text{-sing}}/H; \mathbb{Z}), \end{aligned}$$

where the first two isomorphisms follow from the discussion above and the third because H has finite index in G . \square

3 Right-angled Coxeter groups

A *right-angled Coxeter system* consists of a group W called a *right-angled Coxeter group* together with a set S of involutions that generate W , subject to only the relations that certain pairs of the generators commute. We will always assume that S is finite. The right angled Coxeter system is determined by the graph $L^1(W, S)$ with vertex set S and edges those pairs of vertices that commute. Equivalently, the right-angled Coxeter system is determined by the flag complex $L(W, S)$ with vertex set S and simplices the cliques in the graph $L^1(W, S)$.

Given a right-angled Coxeter system (W, S) , the *Davis complex* $\Sigma(W, S)$ can be realized as either a cubical complex, or as a simplicial complex which is the barycentric subdivision of the cubical complex. For more details concerning the Davis complex, we refer the reader to [10] or [9]. The simplicial structure is easier to describe, so we consider this first. A *spherical* subset T of S is a subset whose members all commute; equivalently T is either the empty set, or a subset of S that spans a simplex of $L(W, S)$. A *special parabolic* subgroup of W is the subgroup of W generated by a spherical subset T . We denote the special parabolic subgroup generated by T by W_T . A *parabolic* subgroup of W is a conjugate of a special parabolic subgroup. The set of cosets of all special parabolic subgroups forms a poset, ordered by inclusion, and the simplicial complex $\Sigma(W, S)$ is the realization of this poset. By construction, W acts admissibly simplicially on $\Sigma(W, S)$ in such a way that each stabilizer subgroup is parabolic.

If T is a spherical subset of S , then the subposet of cosets contained in W_T is equivariantly isomorphic to the poset of faces of the standard $|T|$ -cube $[-1, 1]^T$, with the group $W_T \cong C_2^T$ acting via reflections in the coordinate hyperplanes. In this way we obtain a cubical structure on $\Sigma(W, S)$, in which the n -dimensional subcubes correspond to cosets wW_T with $|T| = n$. The setwise stabilizer of the cube wW_T is the parabolic subgroup wW_Tw^{-1} , which acts on the cube in such a way that the natural generators wtw^{-1} act as reflections in the coordinate hyperplanes. The simplicial complex described above is the barycentric subdivision of this cubical complex.

If we view every simplicial complex as containing a unique -1 -simplex corresponding to the empty subset of its vertex set, then we get a natural bijective correspondence between the W -orbits of cubes in $\Sigma(W, S)$ and the simplices of $L(W, S)$ which preserves incidence (the empty simplex corresponds to the 0 -cubes). Hence we obtain:

Proposition 3.1. *There is a natural bijection between subcomplexes of the simplicial complex $L(W, S)$ and non-empty W -invariant subcomplexes of the cubical complex $\Sigma(W, S)$.*

To show that $\Sigma(W, S)$ is a model for \underline{EW} , metric techniques are helpful. There is a natural $\text{CAT}(0)$ -metric on $\Sigma(W, S)$, which is best understood in terms of the cubical structure. The length of a piecewise linear path in $\Sigma(W, S)$ is defined using the standard Euclidean metric on each cube, and the distance between two points of $\Sigma(W, S)$ is the infimum of the lengths of PL-paths connecting them. According to Gromov's criterion [15], $\Sigma(W, S)$ is locally $\text{CAT}(0)$ because the link of every vertex is isomorphic to $L(W, S)$ which is a flag complex (see [10, 15]). It is easy to see that $\Sigma(W, S)$ is simply connected (for example, because its 2-skeleton is a version of the Cayley complex for W), and it follows that $\Sigma(W, S)$ is $\text{CAT}(0)$ [5, Theorem II.4.1]. Given that W acts isometrically with finite stabilizers on $\Sigma(W, S)$ it follows that $\Sigma(W, S)$ is a model for \underline{EW} via the Bruhat-Tits fixed point theorem [5, p. 179] or [2, Prop. 3].

Lemma 3.2. *Every finite subgroup of W is a subgroup of a parabolic subgroup of W . In particular, there are finitely many conjugacy classes of finite subgroups of W and every finite subgroup is isomorphic to a direct product $(\mathbb{Z}/2)^k$ for some $0 \leq k \leq n$ where n is the dimension of $\Sigma(W, S)$.*

Proof. Let F be a finite subgroup of W . By the Bruhat-Tits fixed point theorem F fixes some point of Σ , and hence F is a subgroup of a point stabilizer. Every such subgroup is parabolic, and each is conjugate to one of the finitely many special parabolics. \square

Corollary 3.3. *The commutator subgroup W' of W is a finite-index torsion-free subgroup of type F .*

Proof. The abelianization of W is naturally isomorphic to C_2^S . Every parabolic subgroup of W maps injectively into C_2^S . It follows that W' acts freely on the finite-dimensional contractible space Σ . Hence Σ/W' is a compact $K(W', 1)$, from which it follows that W' is both type F and torsion-free. \square

Lemma 3.4 ([2]). *The quotient of the pair $(\Sigma, \Sigma^{\text{sing}})$ by W is isomorphic to the pair (CL', L') , i.e., the pair consisting of the cone on the barycentric subdivision of L and its base. This isomorphism is natural for automorphisms of L . If L is acyclic then so is Σ^{sing} . If L is simply-connected, then so is Σ^{sing} .*

Proof. The first part is clear from the simplicial description of Σ . Now let V be the unique free W -orbit of vertices in the simplicial description of Σ . The star of each $v \in V$ is a copy of the cone CL' , with v as its apex. The subcomplex of Σ consisting of all simplices not containing any vertex of V is Σ^{sing} . Hence Σ is obtained from Σ^{sing} by attaching cones to countably many subcomplexes isomorphic to L' .

In the case when L is acyclic, attaching a cone to a copy of L' does not change homology. It follows that Σ^{sing} must be acyclic since Σ is. Similarly, if L is simply-connected, then attaching a cone to a copy of L' does not change the fundamental group, so Σ^{sing} must be simply-connected since Σ is. \square

Now suppose a finite group Q acts by automorphisms on $L(W, S)$. This defines an action of Q on W , and hence a semidirect product $G = W \rtimes Q$.

Lemma 3.5. *There is an admissible simplicial G -action on $\Sigma(W, S)$ extending the action of W , and $\Sigma(W, S)$ becomes a cocompact model for \underline{EG} .*

Proof. The action of Q on the poset underlying $\Sigma(W, S)$ is defined in such a way that $q \in Q$ sends the coset wW_T to the coset $q(w)W_{q(T)}$. This combines with the W -action to give an admissible G -action on $\Sigma(W, S)$. Since $\Sigma(W, S)$ is CAT(0) and the stabilizers are finite it follows that $\Sigma(W, S)$ is a model for \underline{EG} . \square

Corollary 3.6. *Any finite-index subgroup H of G as above admits a cocompact model for \underline{EH} and is virtually torsion-free.*

Remark 3.7. *For the action of G on Σ , the stabilizer of the vertex W_T is the semidirect product $W_T \rtimes Q_T$, where $Q_T := \{q \in Q : q(T) = T\}$. If Q acts admissibly on L then Q_T fixes each element of T and the stabilizer is the direct product $W_T \times Q_T$. Similarly, the stabilizer of the vertex wW_T is the direct product $wW_T w^{-1} \times wQ_T w^{-1}$. Note in particular that the image of the stabilizer under the quotient map $G \rightarrow G/W \cong Q$ depends only on T , and not on w .*

Let W^{ev} denote the subgroup of elements of W expressible as words of even length in the elements of S . Thus W^{ev} is an index two subgroup of W .

Lemma 3.8. *Let N be a finite index normal subgroup of W^{ev} . There is an isomorphism ψ from the relative chain complex $C_*(CL', L')$ to a direct summand of the simplicial chain complex $C_*(\Sigma/N)$. This isomorphism is natural for automorphisms of L that preserve N . It is also natural for the inclusion of subcomplexes in L and W/N -invariant subcomplexes of the cubical structure on Σ/N .*

Proof. The cone CL' is the realization of the poset of spherical subsets of S , with cone point the empty set \emptyset . For σ a simplex of CL' , $\psi(\sigma)$ in $C_*(\Sigma/N)$ will be the signed sum of its $|W/N|$ inverse images under the map $\Sigma/N \rightarrow \Sigma/W = CL'$. The signs will ensure that simplices of CL' that do not contain \emptyset as a vertex map to zero.

In more detail, fix a transversal w_1, \dots, w_m to N in W , and for σ a simplex of CL' , viewed as a chain $\sigma = (T_0 < T_1 < \dots < T_r)$ of spherical subsets, define

$$\psi(\sigma) = \sum_{i=1}^m (-1)^{l(w_i)} w_i \sigma = \sum_{i=1}^m (-1)^{l(w_i)} (w_i W_{T_0} < \dots < w_i W_{T_r}).$$

Here $l(w)$ denotes the length of w as a word in S . For any $n \in N$, $l(wn) - l(w)$ is even, and so the sum above does not depend on the choice of transversal. The above formula clearly describes a chain map from $C_*(CL')$ to $C_*(\Sigma/N)$. Now if T is a non-empty spherical subset of S , W_T contains equal numbers of words of odd and even length, and hence so does its image $W_T/(W_T \cap N) \leq W/N$. It

follows that if $T_0 \neq \emptyset$, then $\psi(\sigma) = 0$. Hence the formula given above defines a chain map $\psi : C_*(CL', L') \rightarrow C_*(\Sigma/N)$. This clearly has the claimed naturality properties.

It remains to exhibit a splitting map $\phi : C_*(\Sigma/N) \rightarrow C_*(CL', L')$. This uses a ‘simplicial excision map’. Let v be the image of $W_\emptyset \in \Sigma$ in Σ/N , and let X be the subcomplex of Σ/N consisting of all simplices that do not have v as a vertex. There is a natural bijection between simplices of CL' containing the cone vertex and simplices of Σ/N containing v . This induces an isomorphism $C_*(\Sigma/N, X) \cong C_*(CL', L')$ and ϕ is defined as the composite of this with the map $C_*(\Sigma/N) \rightarrow C_*(\Sigma/N, X)$. \square

Corollary 3.9. *With notation as above, let K be a subcomplex of L , and let $\Sigma(K)$ be the (barycentric subdivision of the) cubical W_L -subcomplex of Σ associated to K . There is a natural isomorphism ψ from the relative chain complex $C_*(CL', L' \cup CK')$ to a direct summand of the relative simplicial chain complex $C_*(\Sigma/N, \Sigma(K)/N)$.*

Proof. $C_*(CK', K')$ is a subcomplex of $C_*(CL', L')$ and the corresponding quotient is $C_*(CL', L' \cup CK')$. Similarly, $C_*(\Sigma(K)/N)$ is a subcomplex of $C_*(\Sigma/N)$ with $C_*(\Sigma/N, \Sigma(K)/N)$ the corresponding quotient.

By naturality of ψ and ϕ we get a diagram as follows, in which the two left-hand squares with the same label on both vertical sides commute and such that the two composites labelled $\phi \circ \psi$ are equal to the relevant identity maps. A diagram chase shows that there are unique maps ψ and ϕ corresponding to the dotted vertical arrows that make the right-hand squares with the same label on both vertical sides commute, and that these maps also satisfy $\phi \circ \psi = 1$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_*(CK', K') & \longrightarrow & C_*(CL', L') & \longrightarrow & C_*(CL', L' \cup CK') \longrightarrow 0 \\
& & \phi \uparrow & & \phi \uparrow & & \phi \uparrow \\
& & \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\
0 & \longrightarrow & C_*(\Sigma(K)/N) & \longrightarrow & C_*(\Sigma/N) & \longrightarrow & C_*(\Sigma/N, \Sigma(K)/N) \longrightarrow 0
\end{array}$$

\square

4 Proof of Theorem 1.1

As in the statement of Theorem 1.1, let L be a finite n -dimensional flag complex equipped with an admissible simplicial action of a finite group Q , let $(W, S) = (W_L, S_L)$ be the associated right-angled Coxeter system, let N be a finite-index subgroup of W that is normalized by Q , and let G be the semidirect product $G = N \rtimes Q$.

Proof of Theorem 1.1. Note that G may be viewed as a finite-index subgroup of the semidirect product $W \rtimes Q$. Under these hypotheses, we already see from Corollary 3.5 that the Davis complex Σ is a cocompact $(n+1)$ -dimensional model for \underline{EG} , and that G is virtually torsion-free.

Using the hypothesis that L is acyclic, we see that the subcomplex $\Sigma^{\text{sing}(W)}$ of Σ consisting of those points whose stabilizer in W is non-trivial is acyclic by Lemma 3.4. In this case any finite-index torsion-free subgroup of G acts freely on the acyclic n -dimensional complex $\Sigma^{\text{sing}(W)}$, which implies that $\text{vcd} G \leq n$. The fact that equality holds can be deduced from the calculation of the

cohomology of W with free coefficients in Theorem [10, 8.5.2], but we also give an alternative proof. Let $K := L^{\text{sing}}$, the Q -singular points in L . From the long exact sequence for the pair (L, K) we see that $H^{n-1}(K) \neq 0$, and hence $H^n(CK, K) \neq 0$. Lemma 3.8 applied to the Coxeter group W_K and its finite-index torsion-free subgroup W'_K shows that $H^n(W'_K; \mathbb{Z}) = H^n(\Sigma_K/W'_K)$ contains a summand isomorphic to $H^n(CK, K)$ and so is not zero. (Here we use Σ_K to denote $\Sigma(W_K, S_K)$ since we reserve Σ to stand for $\Sigma(W_L, S_L)$.) Hence $\text{vcd} G \geq \text{vcd} W_K \geq n$.

It remains to show that $\text{cd} G \geq n + 1$. Since W^{ev} has index 2 in W and is clearly Q -invariant, we see that $(N \cap W^{\text{ev}}) \rtimes Q$ is a subgroup of G of index at most 2. Hence without loss of generality we may assume that $N \leq W^{\text{ev}}$. Now consider the family \mathcal{W} of finite subgroups of G , consisting of those finite subgroups that are contained in N , or equivalently the finite subgroups that map to the trivial subgroup under the factor map $G \rightarrow Q$. The stabilizers in $W \rtimes Q$ of vertices of Σ are described in Remark 3.7, and by intersecting with $G = N \rtimes Q$ we get a similar description of stabilizers in G : the stabilizer of the vertex wW_T is the direct product of the intersection $N \cap wW_T w^{-1}$ and a subgroup that maps isomorphically to Q_T , the stabilizer in Q of the vertex T of CL' . It follows that $\Sigma^{\mathcal{W}\text{-sing}}$ is equal to the inverse image in Σ of the Q -singular set CK' in $\Sigma/W = CL'$. Hence $\Sigma^{\mathcal{W}\text{-sing}}$ is the W -invariant subcomplex of the cubical structure on Σ that corresponds (under the map of Proposition 3.1) to $K := L^{\text{sing}}$. Using Corollary 3.9 applied in this case, we see that $H^{n+1}(\Sigma/N, \Sigma^{\mathcal{W}\text{-sing}}/N)$ admits a split surjection onto $H^{n+1}(CL', L' \cup CK')$, which is isomorphic to $H^n(L, K) = H^n(L, L^{\text{sing}})$ since L is acyclic. Proposition 2.1 finishes the argument.

The general case of Theorem 1.1 follows from the case described above. Let each G_i be defined as above in terms of L_i , Q_i , n_i and $N_i \leq W_i$, and define $\Gamma := G_1 \times \cdots \times G_m$, $Q = Q_1 \times \cdots \times Q_m$, and $W := W_1 \times \cdots \times W_m$. Finally, let $n := \sum_{i=1}^m n_i$. The direct product $\Sigma := \Sigma_1 \times \cdots \times \Sigma_m$ is a cocompact model for $\underline{E}\Gamma$ of dimension $m + n$, and so $\text{cd} \Gamma \leq m + n$. Also the direct product $\Sigma_1^{\text{sing}} \times \cdots \times \Sigma_m^{\text{sing}}$ is an acyclic n -dimensional simplicial complex admitting a proper Γ -action, which implies that $\text{vcd} \Gamma \leq n$. The lower bounds also work just as in the case $m = 1$. To give a lower bound for $\text{vcd} \Gamma$, let $K_i := L_i^{\text{sing}}$, and note that $H^{n_i-1}(K_i) \cong H^{n_i}(L_i, K_i)$. The group Γ contains a subgroup Γ' that is isomorphic to a finite-index subgroup of the product $W' := W'_1 \times \cdots \times W'_m$ of the commutator subgroups. By the universal coefficient theorem $H^n(W'; \mathbb{Z})$ is non-zero, since it contains a summand isomorphic to $\bigotimes_{i=1}^m H^{n_i}(CK_i, K_i) \neq \{0\}$. If we define \mathcal{W} to be the family of subgroups of Γ that are contained in W , then a point $x = (x_1, \dots, x_m) \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_m$ is in $\Sigma^{\mathcal{W}\text{-sing}}$ if and only if there is an i so that $x_i \in \Sigma_i^{\mathcal{W}_i\text{-sing}}$. Hence we see that if we define $N := N_1 \times \cdots \times N_m$, then

$$H^{n+m}(\Sigma/N, \Sigma^{\mathcal{W}\text{-sing}}/N) \cong \bigotimes_{i=1}^m H^{n_i+1}(\Sigma_i/N_i, \Sigma_i^{\mathcal{W}_i\text{-sing}}/N_i),$$

the latter of which is non-zero since it contains a summand isomorphic to

$$\bigotimes_{i=1}^m H^{n_i+1}(CL_i, L_i \cup CK_i) \cong \bigotimes_{i=1}^m H^{n_i}(L_i, K_i) = \bigotimes_{i=1}^m H^{n_i}(L_i, L_i^{\text{sing}}).$$

□

5 Applications

In this section we describe various pairs (Q, L) consisting of a finite group Q and an admissible Q -action on a flag complex L , producing sufficiently many examples to establish Corollary 1.2. Since any Q -CW-complex is equivariantly homotopy equivalent to a flag simplicial complex with an admissible action of Q , we do not give explicit triangulations of L .

Example 1. Let Q be the alternating group A_5 , and define a Q -CW-complex as follows. For the 1-skeleton L^1 of L take the complete graph on five vertices, with the natural action of $Q = A_5$. In A_5 , the 24 elements of order five split into two conjugacy classes of size 12, and any element g of order 5 is conjugate to g^{-1} (but is not conjugate to g^2 or g^3). Define L by using one of the two conjugacy classes of 5-cycles to describe attaching maps for six pentagonal 2-cells. By construction there is a Q -action without a global fixed point and it is easily checked that L is acyclic. In fact, $\pi_1(L)$ is isomorphic to $SL(2, 5)$, the unique perfect group of order 120, and L is isomorphic to the 2-skeleton of the Poincaré homology sphere [3, I.8]. By taking each Q_i to be A_5 and each L_i to be a flag triangulation of L as defined above, we obtain groups Γ_m having the properties stated in the first part of 1.2.

Example 2. Fix distinct primes p and q , and let Q be cyclic of order q , generated by g . For the Q -fixed point set L^Q , take a mod- p Moore space $M(1, p)$. This space has a CW-structure with 1-skeleton a circle and one 2-cell f . The 2-cell f is attached to the circle via a map of degree p . Now define L^2 by adding on a free Q -orbit of 2-cells f_0, \dots, f_{q-1} , where $f_i = g^i f_0$, so that each f_i is attached to the circle by a degree one map. L^2 is simply connected, and $H_2(L^2)$ is a free $\mathbb{Z}Q$ -module of rank one, since it has a \mathbb{Z} -basis given by the elements

$$e_j := f - \sum_{i=0}^{p-1} g^i f_j = f - \sum_{i=0}^{p-1} g^{i+j} f_0$$

for $0 \leq j < q$, and $g^j e_0 = e_j$ for each j . Make L by attaching a free Q -orbit of 3-cells to kill each e_j . To establish the second part of Corollary 1.2 we take each Q_i to be cyclic of order q_i , take each L_i to be as above for some fixed choice of p , and take N_i to be the commutator subgroup of the Coxeter group $W_i := W(L_i, S_i)$. For any such choice, we obtain a group Λ_m as in the statement. To ensure that Λ_m contains only cyclic finite subgroups we must take the primes q_i all distinct, whereas to ensure that Λ_m contains only abelian finite subgroups of exponent q we take $q_i = q$ for all i .

6 Contractibility and acyclicity

In [2], it was shown that certain right-angled Coxeter groups W have the property that $\text{vcd}W = \text{cd}W = 2 < \text{gd}W = 3$. In this section we improve this result by showing that for these same groups there is no 2-dimensional contractible proper W -CW-complex.

We will use a few subsidiary results in the proof.

Proposition 6.1. *If Y is a subcomplex of a 2-dimensional acyclic complex, then $H_2(Y) = 0$ and each $H_i(Y)$ is free abelian.*

Proof. If Y is any subcomplex of an n -dimensional acyclic complex Z , then consideration of the homology long exact sequence for the pair (Z, Y) shows

that $H_n(Y)$ is trivial and that $H_{n-1}(Y)$ is free abelian. Since H_0 is always free abelian, the case $n = 2$ gives the claimed result. \square

Proposition 6.2. *Let Q be a finite soluble group and let X be a 2-dimensional acyclic Q -CW-complex. Then the fixed point set X^Q is also acyclic.*

Proof. The finite soluble group Q has a normal subgroup N of prime index, the factor group Q/N acts on the N -fixed point set X^N , and the equality $X^Q = (X^N)^{Q/N}$ holds. Hence it suffices to consider the case in which Q has prime order.

By the P. A. Smith theorem, X^Q is mod- p acyclic in the case when Q has order p . By the previous proposition, $H_i(X^Q)$ is free abelian for all i . By the universal coefficient theorem, the rank of the i th mod- p homology group of X^Q is equal to the rank of $H_i(X^Q)$. Hence X^Q must be acyclic. \square

Lemma 6.3. *Let X be a CW-complex, let S be a finite indexing set, and let $X(s)$ be a subcomplex of X such that each $X(s)$ is acyclic and each intersection of $X(s)$'s is either empty or acyclic. Define*

$$X^\# := \bigcup_{s \in S} X(s),$$

and let $|\mathcal{N}|$ be the realization of the nerve of the covering of $X^\#$ by the subcomplexes $X(s)$. There is a map $f : X^\# \rightarrow |\mathcal{N}|$ which is a homology isomorphism and induces a surjection of fundamental groups.

Proof. In the case when each intersection of $X(s)$'s is either contractible or empty, it is well-known that there is a homotopy equivalence $f : X^\# \rightarrow |\mathcal{N}|$ [16, 4.G, Ex. 4]. We use Quillen's plus construction to reduce to this case.

For $T \subseteq S$, define $X(T)$ to be the intersection $X(T) = \bigcap_{s \in T} X(s)$. Suppose that $U \subseteq S$ is such that $X(U)$ is non-empty. In this case, since $X(U)$ is acyclic we can find a set A_U of 2-cells with attaching maps from the boundary of the 2-cell to $X(U)$ so that each attaching map represents a conjugacy class of commutators in $\pi_1(X(U))$ and so that the fundamental group of the resulting complex $\hat{X}(U)$ is trivial. Moreover, there is a set B_U of 3-cells and attaching maps from the boundary of the 3-cell to $\hat{X}(U)$ so that the resulting complex X_U contains $X(U)$ as a subcomplex, is simply-connected, and such that the inclusion of $X(U)$ into X_U is a homology isomorphism. Define Y by attaching to X 2- and 3-cells indexed by $\coprod_U A_U$ and $\coprod_U B_U$ respectively. Define a subcomplex $Y(s)$ of Y by attaching to $X(s)$ the 2- and 3-cells indexed by $\coprod_{s \in U} A_U$ and $\coprod_{s \in U} B_U$ respectively. Finally define $Y(T) := \bigcap_{s \in T} Y(s)$, and $Y^\# := \bigcup_{s \in S} Y(s)$. The nerve of the covering of $Y^\#$ by the subcomplexes $Y(s)$ is naturally isomorphic to \mathcal{N} . A Mayer-Vietoris spectral sequence argument shows that the inclusion $X^\# \rightarrow Y^\#$ is a homology isomorphism, and this map induces a surjection $\pi_1(X^\#) \rightarrow \pi_1(Y^\#)$ because the 1-skeleta of $X^\#$ and $Y^\#$ are equal. \square

Proof of Theorem 1.3. The Davis complex $\Sigma = \Sigma(W_L, S_L)$ is a cocompact 3-dimensional model for $\underline{E}W_L$. Since L is acyclic, Σ^{sing} is a 2-dimensional acyclic proper W_L -CW-complex in which the fixed point set for any non-trivial finite subgroup is contractible. This suffices to show that $\text{cd}W_L = 2$.

Now suppose that X is any contractible proper 2-dimensional W_L -CW-complex. Let $S = S_L$, and define $X^\#$ as the union of the fixed point sets

X^s : $X^\# := \bigcup_{s \in S} X^s$. By construction, the nerve of the covering of $X^\#$ by the sets X^s is equal to L . By Proposition 6.2, for each $T \subseteq S$ that spans a simplex of L the subset

$$X(T) := \bigcap_{s \in T} X^s = X^{(T)}$$

is acyclic, and for each T that does not span a simplex of L , $X(T)$ is empty. By Lemma 6.3, it follows that there is a natural surjection $\phi : \pi_1(X^\#) \rightarrow \pi_1(L)$. Define $\rho' := \rho \circ \phi : \pi_1(X^\#) \rightarrow U(n)$, a non-trivial unitary representation of $\pi_1(X^\#)$. We use this representation to obtain a contradiction.

Pick $g \in \pi_1(X^\#)$ so that $\rho'(g) \neq 1$. Since X is contractible, there exists connected X_1 with $X^\# \subseteq X_1 \subseteq X$ such that $X_1 - X^\#$ comprises only finitely many cells, and such that g maps to the identity element of $\pi_1(X_1)$. By Proposition 6.1, $H_1(X_1)$ is free abelian. By contracting some of the 1-cells in $X_1 - X^\#$ we may replace X_1 by a complex X_2 , with the following properties: $X^\# \subseteq X_2$; $H_1(X_2)$ is free abelian and $H_2(X_2) = \{0\}$; X_2 consists of $X^\#$ with finitely many 1- and 2-cells added; g is in the kernel of the map $\pi_1(X^\#) \rightarrow \pi_1(X_2)$. Now make X_3 by attaching 2-cells to exactly kill $H_1(X_2)$. Thus X_3 is an acyclic 2-complex, obtained by attaching finitely many 1- and 2-cells to $X^\#$. By the Gerstenhaber-Rothaus theorem [14], the representation $\rho' : \pi_1(X^\#) \rightarrow U(n)$ extends to a representation $\tilde{\rho} : \pi_1(X_3) \rightarrow U(n)$. However, this contradicts the fact that $\rho'(g) \neq 1$, while g maps to the identity in $\pi_1(X_3)$. \square

Remark 6.4. *As an example of a suitable L , take a flag triangulation of the 2-skeleton of the Poincaré homology sphere (which was discussed in the previous section); here there is a faithful representation $\rho : \pi_1(L) \cong SL(2, 5) \rightarrow U(2)$.*

Remark 6.5. *There is a version of Brown's question that remains open: for $m > 2$, is there a virtually torsion-free group G such that $\text{vcd} G = m$ but there exists no contractible m -dimensional proper G -CW-complex?*

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