On dimensions of groups with cocompact classifying spaces for proper actions

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Abstract

We construct groups G that are virtually torsion-free and have virtual cohomological dimension strictly less than the minimal dimension for any model for $\underline{E}G$, the classifying space for proper actions of G. They are the first examples that have these properties and also admit cocompact models for $\underline{E}G$. We exhibit groups G whose virtual cohomological dimension and Bredon cohomological dimension are two that do not admit any 2-dimensional contractible proper G-CW-complex.

1 Introduction

If G is a virtually torsion-free group, the virtual cohomological dimension vcd$G$, is defined to be the cohomological dimension of a torsion-free finite-index subgroup $H \leq G$; a lemma due to Serre shows that this is well defined [7, VIII.3.1]. Now suppose that X is a contractible G-CW-complex that is *proper*, in the sense that all cell stabilizers are finite. In this case any torsion-free subgroup H will act freely on X and so X/H is a classifying space or Eilenberg-Mac Lane space BH for H. In particular, vcd$G$ provides a lower bound for the dimension of any such X. K. S. Brown asked whether this lower bound is always attained [6, ch. 2] or [7, VIII.11]:

Brown's Question (Weak Form)**.** Does every virtually torsion-free group G admit a contractible proper G-CW-complex of dimension vcd$G$?

Until now, this form of Brown's question has remained unanswered. We give examples of groups G with vcd$G = 2$ that do not admit any 2-dimensional contractible proper G-CW-complex in Theorem 1.3 below.

One reason why this question has been so elusive is that there are many different equivariant homotopy types of contractible proper G-CW-complexes. The most natural example is the classifying space for proper G-actions, $\underline{E}G$, which plays the same role in the homotopy category of proper G-CW-complexes as EG plays for free G-CW-complexes. A model for $\underline{E}G$ is a proper G-CW complex X such that for any finite $F \leq G$, the F-fixed point set X^F is contractible. Such an X always exists, and is unique up to equivariant homotopy equivalence. Let $\underline{\mathrm{gd}}G$ denote the minimal dimension of any model for $\underline{E}G$.

The version of Brown's question that concerns $\underline{E}G$ [6, ch. 2] or [7, VIII.11] is usually asked in the form:

Brown's Question (Strong Form). Does $\underline{\mathrm{gd}}G = \mathrm{vcd}G$ for every virtually torsion-free G?

We prefer to split this question into two separate questions. There is an algebraic dimension $\underline{\mathrm{cd}}G$ that bears a close relationship to $\underline{\mathrm{gd}}G$, analogous to the relationship between cohomological dimension and the minimal dimension of an Eilenberg-Mac Lane space. It can be shown that $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G$ except that there may exist G for which $\underline{\mathrm{cd}}G = 2$ and $\underline{\mathrm{gd}}G = 3$, and $\underline{\mathrm{cd}}G$ is an upper bound for the cohomological dimension of any torsion-free subgroup of G [23]. In view of this we may split the strong form of Brown's question into two parts, one geometric and one algebraic.

Does there exist G for which $\underline{\mathrm{gd}}G \neq \underline{\mathrm{cd}}G$?

Does there exist virtually torsion-free G for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$?

Examples of virtually torsion-free groups G for which $\underline{\mathrm{gd}}G = 3$ and $\underline{\mathrm{cd}}G = 2$ were given in [2]. These groups G are Coxeter groups. Examples of G for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$ were given in [18], and more recently in [24, 13]. The advantage of the examples in [24, 13] is that in some sense they have the least possible torsion. For any virtually torsion-free G, it can be shown that $\underline{\mathrm{cd}}G$ is bounded by the sum $\mathrm{vcd}G + \ell(G)$, where $\ell(G)$ is the maximal length of a chain of non-trivial finite subgroups of G [20, 6.4]. This bound is attained for the examples in [24, 13] but not for the examples in [18]. To date, all constructions of groups G for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$ have used finite extensions of Bestvina-Brady groups [1], and none of these groups G admit a cocompact model for $\underline{E}G$. One of our main results is the construction of virtually torsion-free G admitting a cocompact $\underline{E}G$ for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$. Amongst our examples, the easiest to describe are extensions of a right-angled Coxeter group by a cyclic group of prime order. By taking instead a cyclic extension of a torsion-free finite index subgroup of the same Coxeter group we obtain examples with cocompact $\underline{E}G$ and for which $\underline{\mathrm{cd}}G = \mathrm{vcd}G + \ell(G)$.

We say that a simplicial action of a group on a simplicial complex is *admissible* if the setwise stabilizer of each simplex equals to its pointwise stabilizer. Many of the other terms used in the statements of our main theorems will be defined below.

Theorem 1.1. *Let L be a finite n-dimensional acyclic flag complex with an admissible simplicial action of a finite group Q, and let W_L be the corresponding right-angled Coxeter group so that Q acts as a group of automorphisms of W_L. Let N be any finite-index normal subgroup of W_L such that N is normalized by Q, and let G be the semidirect product $N \rtimes Q$. This G admits a cocompact model for $\underline{E}G$. Let L^{sing} denote the subcomplex of L consisting of points with non-trivial stabilizer in Q.*

$$\text{If } H^n(L, L^{\mathrm{sing}}) \neq 0, \text{ then } \underline{\mathrm{cd}}G = n+1 \text{ and } \mathrm{vcd}G \leq n.$$

Now suppose that L_i, Q_i, n_i, N_i, W_i and G_i are as above for $i = 1, \dots, m$, and let $\Gamma := G_1 \times \cdots \times G_m$. As before, there is a cocompact model for $\underline{E}\Gamma$, and

$$\text{if } \bigotimes_{i=1}^{m} H^{n_i}(L_i, L_i^{\mathrm{sing}}) \neq 0, \text{ then } \underline{\mathrm{cd}}\Gamma = m + \sum_{i=1}^{m} n_i \text{ and } \mathrm{vcd}\Gamma \leq \sum_{i=1}^{m} n_i.$$

Furthermore $\mathrm{vcd}G = n$ *if either* L *is a barycentric subdivision or* L^{sing} *is a full subcomplex of* L. *Similarly,* $\mathrm{vcd}\Gamma = \sum_i n_i$ *if for each* i, *either* L_i *is a barycentric subdivision or* L_i^{sing} *is a full subcomplex of* L_i.

Corollary 1.2. *For each* $m \geq 1$ *there exists a virtually torsion-free group* Γ_m *admitting a cocompact* $\underline{E}\Gamma_m$ *and such that*

$$\underline{\mathrm{cd}}\Gamma_m = 3m > \mathrm{vcd}\Gamma_m = 2m.$$

For each $m \geq 1$ *there exists a virtually torsion-free group* Λ_m *admitting a cocompact* $\underline{E}\Lambda_m$ *and such that*

$$\underline{\mathrm{cd}}\Lambda_m = 4m = \mathrm{vcd}\Lambda_m + \ell(\Lambda_m) > \mathrm{vcd}\Lambda_m = 3m.$$

Furthermore, Λ_m *may be chosen so that either every finite subgroup is cyclic or, for any fixed prime* q, *every nontrivial finite subgroup is abelian of exponent* q.

In contrast to the above results, Degrijse and Martínez-Pérez have shown that $\mathrm{vcd}G = \underline{\mathrm{cd}}G$ for a large class of groups that contains all (finitely generated) Coxeter groups [12].

Theorem 1.3. *Suppose that* L *is a finite 2-dimensional acyclic flag complex such that the fundamental group of* L *admits a non-trivial unitary representation* $\rho : \pi_1(L) \to U(n)$ *for some* n. *Then* $\mathrm{vcd}W_L = \underline{\mathrm{cd}}W_L = 2$, *there is a cocompact 3-dimensional model for* $\underline{E}W_L$, *and yet there exists no proper 2-dimensional contractible* W_L*-CW-complex.*

Theorem 1.3 strengthens a result from [2], and gives the first negative answer to the weak form of Brown's question. A different argument was used in [2] to show that $\underline{\mathrm{cd}}W_L = 2 < \underline{\mathrm{gd}}W_L = 3$ for some of the flag complexes L that appear in Theorem 1.3.

We remark that finitely generated Coxeter groups are linear over $\mathbb{Z}$ [8], and that this property passes to subgroups and to finite extensions. Hence all of the groups appearing in the above statements are linear over $\mathbb{Z}$. As will be seen from the proofs, each group appearing in our statements acts properly and cocompactly on a CAT(0) cube complex; in particular they are all CAT(0) groups. A right-angled Coxeter group W_L is (Gromov) hyperbolic if and only if L satisfies the flag-no-square condition. Hyperbolicity passes to finite index subgroups and finite extensions. Since any 2- or 3-dimensional flag complex admits a flag-no-square subdivision [14, 25] it follows that the groups Γ_1 and Λ_1 in Corollary 1.2 and the groups W_L in Theorem 1.3 may be taken to be hyperbolic (and CAT(-1), a possibly stronger property) in addition to their other stated properties.

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2 Classifying spaces and Bredon cohomology

The algebraic analogs of the geometric finiteness properties exhibited by classifying spaces of groups for families of subgroups are formulated using Bredon cohomology. This cohomology theory was introduced by Bredon in [3] for finite groups and was generalised to arbitrary groups by Lück (see [19]).

Let G be a discrete group. A family $\mathcal{F}$ of subgroups of G is a non-empty set of subgroups which is closed under conjugation and taking subgroups, in the sense that if $H \in \mathcal{F}$, $g \in G$ and $K \leq H$, then $K \in \mathcal{F}$ and $gHg^{-1} \in \mathcal{F}$. The *orbit category* $\mathcal{O}_{\mathcal{F}}G$ is the category with objects the left cosets G/H for all $H \in \mathcal{F}$ and morphisms all G-equivariant functions between the objects. In $\mathcal{O}_{\mathcal{F}}G$, every morphism $\varphi : G/H \to G/K$ is completely determined by $\varphi(H)$, since $\varphi(xH) = x\varphi(H)$ for all $x \in G$. Moreover, there exists a morphism

$$G/H \to G/K : H \mapsto xK$$

if and only if $x^{-1}Hx \subseteq K$.

An *$\mathcal{O}_{\mathcal{F}}G$-module* is a contravariant functor $M : \mathcal{O}_{\mathcal{F}}G \to \mathbb{Z}$-mod. The *category of $\mathcal{O}_{\mathcal{F}}G$-modules* is denoted by Mod-$\mathcal{O}_{\mathcal{F}}G$. By definition, it has as objects all $\mathcal{O}_{\mathcal{F}}G$-modules and as morphisms all natural transformations between these objects. The category Mod-$\mathcal{O}_{\mathcal{F}}G$ is an abelian category that contains enough projectives and so one can construct bi-functors $\mathrm{Ext}^n_{\mathcal{O}_{\mathcal{F}}G}(-,-)$ that have all the usual properties. The *n-th Bredon cohomology of G* with coefficients $M \in$ Mod-$\mathcal{O}_{\mathcal{F}}G$ is by definition

$$\mathrm{H}^n_{\mathcal{F}}(G; M) = \mathrm{Ext}^n_{\mathcal{O}_{\mathcal{F}}G}(\underline{\mathbb{Z}}, M),$$

where $\underline{\mathbb{Z}}$ is the constant functor, which sends each object to $\mathbb{Z}$ and each morphism to the identity map on $\mathbb{Z}$. There is also a notion of *Bredon cohomological dimension* of G for the family $\mathcal{F}$, denoted by $\mathrm{cd}_{\mathcal{F}}(G)$ and defined by

$$\mathrm{cd}_{\mathcal{F}}(G) = \sup\{n \in \mathbb{N} \mid \exists M \in \text{Mod-}\mathcal{O}_{\mathcal{F}}G : \mathrm{H}^n_{\mathcal{F}}(G; M) \neq 0\}.$$

When $\mathcal{F}$ is the family of finite subgroups, then $\mathrm{H}^*_{\mathcal{F}}(G, M)$ and $\mathrm{cd}_{\mathcal{F}}G$ are denoted by $\underline{\mathrm{H}}^*(G, M)$ and $\underline{\mathrm{cd}}G$, respectively. Since the augmented cellular chain complex of any model for $E_{\mathcal{F}}G$ yields a projective resolution of $\underline{\mathbb{Z}}$ that can be used to compute $\mathrm{H}^*_{\mathcal{F}}(G; -)$, it follows that $\mathrm{cd}_{\mathcal{F}}(G) \leq \mathrm{gd}_{\mathcal{F}}(G)$. Moreover, it is known (see for example [23, 0.1]) that

$$\mathrm{cd}_{\mathcal{F}}(G) \leq \mathrm{gd}_{\mathcal{F}}(G) \leq \max\{3, \mathrm{cd}_{\mathcal{F}}(G)\}.$$

For any $\mathbb{Z}G$-module M, one may define an $\mathcal{O}_{\mathcal{F}}G$-module $\underline{M}$ by

$$\underline{M}(G/H) = \mathrm{Hom}_G(\mathbb{Z}[G/H], M);$$

note that this is compatible with the notation $\underline{\mathbb{Z}}$ introduced earlier and that this functor is isomorphic to the fixed-point functor

$$M^- : \mathcal{O}_{\mathcal{F}}G \to \mathbb{Z}\text{-mod} : G/H \mapsto M^H.$$

For any G-CW-complex X with stabilizers in $\mathcal{F}$, it can be shown that Bredon cohomology with coefficients in $\underline{M}$ is naturally isomorphic to the ordinary equivariant cohomology of X with coefficients in M: $H^*_{\mathcal{F}}(X; \underline{M}) \cong H^*_G(X; M)$. This follows because the adjointness of the restriction and coinduction functors between $\mathbb{Z}G$-mod and Mod-$\mathcal{O}_{\mathcal{F}}G$ associated to the functor

$$G = \mathcal{O}_{\{e\}}G \to \mathcal{O}_{\mathcal{F}}G : G/\{e\} \mapsto G/\{e\}$$

gives an isomorphism of cochain complexes

$$\mathrm{Hom}_{\mathcal{F}}(C^{\mathcal{F}}_*(X), \underline{M}) \cong \mathrm{Hom}_G(C_*(X), M).$$

A *subfamily* of a family $\mathcal{F}$ of subgroups of G is another family $\mathcal{G} \subseteq \mathcal{F}$. For a subfamily $\mathcal{G}$ and a G-CW-complex X with stabilizers in $\mathcal{F}$, the $\mathcal{G}$-singular set $X^{\mathcal{G}\text{-sing}}$ is the subcomplex consisting of points of X whose stabilizer is not contained in $\mathcal{G}$. When $\mathcal{G}$ consists of just the trivial subgroup this is the usual singular set and we write X^{sing} for $X^{\mathcal{G}\text{-sing}}$.

Given a $\mathbb{Z}G$-module M and a subfamily $\mathcal{G}$ of $\mathcal{F}$, we define two further $\mathcal{O}_{\mathcal{F}}G$-modules: a submodule $\underline{M}_{\leq \mathcal{G}}$ of $\underline{M}$, and the corresponding quotient module $\underline{M}_{>\mathcal{G}}$. These are defined by

$$\underline{M}_{>\mathcal{G}} : G/H \mapsto \begin{cases} \mathrm{Hom}_G(\mathbb{Z}[G/H], M) & \text{if } \ H \notin \mathcal{G}, \\ 0 & \text{if } \ H \in \mathcal{G}, \end{cases}$$

$$\underline{M}_{\leq \mathcal{G}} : G/H \mapsto \begin{cases} 0 & \text{if } \ H \notin \mathcal{G}, \\ \mathrm{Hom}_G(\mathbb{Z}[G/H], M) & \text{if } \ H \in \mathcal{G}. \end{cases}$$

By construction there is a short exact sequence of $\mathcal{O}_{\mathcal{F}}G$-modules

$$\underline{M}_{\leq \mathcal{G}} \rightarrowtail \underline{M} \twoheadrightarrow \underline{M}_{>\mathcal{G}}.$$

Hence, there is a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}_{\leq \mathcal{G}}) \to \mathrm{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}) \to \mathrm{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}_{>\mathcal{G}}) \to 0$$

which gives rise to a long exact sequence in Bredon cohomology. By construction

$$\mathrm{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X), \underline{M}_{>\mathcal{G}}) \cong \mathrm{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X^{\mathcal{G}\text{-sing}}), \underline{M}),$$

and by adjointness isomorphism noted earlier

$$\mathrm{Hom}_{\mathcal{F}}(C_*^{\mathcal{F}}(X^{\mathcal{G}\text{-sing}}), \underline{M}) \cong \mathrm{Hom}_G(C_*(X^{\mathcal{G}\text{-sing}}), M).$$

It follows that there is a natural identification between Bredon cohomology with coefficients in $\underline{M}_{>\mathcal{G}}$ and the equivariant cohomology of $X^{\mathcal{G}\text{-sing}}$ with coefficients in M:

$$H^*_{\mathcal{F}}(X; \underline{M}_{>\mathcal{G}}) \cong H^*_G(X^{\mathcal{G}\text{-sing}}; M),$$

we deduce that the Bredon cohomology with coefficients in $\underline{M}_{\leq \mathcal{G}}$ is isomorphic to the equivariant cohomology of the pair $(X, X^{\mathcal{G}\text{-sing}})$ with coefficients in M:

$$H^*_{\mathcal{F}}(X; \underline{M}_{\leq \mathcal{G}}) \cong H^*_G(X, X^{\mathcal{G}\text{-sing}}; M).$$

Hence we obtain the following.

Proposition 2.1. *Let $\mathcal{F}$ be a family of subgroups of G, with $\mathcal{G}$ a subfamily, let X be any model for $E_{\mathcal{F}}G$, and let H be a finite-index subgroup of G. There exists an $\mathcal{O}_{\mathcal{F}}G$-module module $\mathcal{C}$ such that the Bredon cohomology of the group G with coefficients in $\mathcal{C}$ computes the ordinary cohomology of the pair $(X/H, X^{\mathcal{G}\text{-sing}}/H)$:*

$$H^*_{\mathcal{F}}(G; \mathcal{C}) \cong H^n(X/H, X^{\mathcal{G}\text{-sing}}/H; \mathbb{Z}).$$

Furthermore, each abelian group $\mathcal{C}(G/K)$ is finitely generated.

Proof. Let M be the permutation module $\mathbb{Z}[G/H]$, and let $\mathcal{C} := \underline{M}_{\leq \mathcal{G}}$. Then

$$\begin{aligned} H^*_{\mathcal{F}}(G; \mathcal{C}) &\cong H^*_{\mathcal{F}}(X; \mathcal{C}) \cong H^*_G(X, X^{\mathcal{G}\text{-sing}}; \mathbb{Z}[G/H]) \\ &\cong H^*_H(X, X^{\mathcal{G}\text{-sing}}; \mathbb{Z}) \cong H^*(X/H, X^{\mathcal{G}\text{-sing}}/H; \mathbb{Z}), \end{aligned}$$

where the first two isomorphisms follow from the discussion above and the third because H has finite index in G. □

3 Right-angled Coxeter groups

In this section we describe the results that we require concerning right-angled Coxeter groups and the Davis complex. The material up to and including Corollary 3.6 is standard; for more details we refer the reader to [11] or [10].

A *right-angled Coxeter system* consists of a group W called a *right-angled Coxeter group* together with a set S of involutions that generate W, subject to only the relations that certain pairs of the generators commute. We will always assume that S is finite. The defining relators have the forms $s^2 = 1$ and $stst = 1$ where $s \in S$ and t ranges over some subset of S depending on s. Since each relator has even length as a word in the elements of S, one may define a group homomorphism from W to the cyclic group $\mathbb{Z}/2$ by $w \mapsto [l(w)] \in \mathbb{Z}/2$, where $l(w) \in \mathbb{Z}$ denotes the length of w as a word in S, and $[l(w)]$ its image in $\mathbb{Z}/2$. The kernel of this homomorphism will be denoted W^{ev}, and consists of the elements of W that are expressible as words of even length in the elements of S. The right angled Coxeter system (W, S) is determined by the graph $L^1(W, S)$ with vertex set S and edges those pairs of vertices that commute. Equivalently, the right-angled Coxeter system is determined by the flag complex $L(W, S)$ with vertex set S and simplices the cliques in the graph $L^1(W, S)$.

Given a right-angled Coxeter system (W, S), the *Davis complex* $\Sigma(W, S)$ can be realized as either a cubical complex, or as a simplicial complex which is the barycentric subdivision of the cubical complex. The simplicial structure is easier to describe, so we consider this first. A *spherical* subset T of S is a subset whose members all commute; equivalently T is either the empty set, or a subset of S that spans a simplex of $L(W, S)$. A *special parabolic* subgroup of W is the subgroup of W generated by a spherical subset T. We denote the special parabolic subgroup generated by T by W_T. A *parabolic* subgroup of W is a conjugate of a special parabolic subgroup. The set of cosets of all special parabolic subgroups forms a poset, ordered by inclusion, and the simplicial complex $\Sigma(W, S)$ is the realization of this poset. By construction, W acts admissibly simplicially on $\Sigma(W, S)$ in such a way that each stabilizer subgroup is parabolic.

If T is a spherical subset of S, then the subposet of cosets contained in W_T is equivariantly isomorphic to the poset of faces of the standard $|T|$-cube $[-1, 1]^T$, with the group $W_T \cong C_2^T$ acting via reflections in the coordinate hyperplanes. In this way we obtain a cubical structure on $\Sigma(W, S)$, in which the n-dimensional subcubes correspond to cosets wW_T with $|T| = n$. The setwise stabilizer of the cube wW_T is the parabolic subgroup wW_Tw^{-1}, which acts on the cube in such a way that the natural generators wtw^{-1} act as reflections in the coordinate hyperplanes. The simplicial complex described above is the barycentric subdivision of this cubical complex.

If we view every simplicial complex as containing a unique -1-simplex corresponding to the empty subset of its vertex set, then we get a natural bijective correspondence between the W-orbits of cubes in $\Sigma(W, S)$ and the simplices of $L(W, S)$ which preserves incidence (the empty simplex corresponds to the 0-cubes). Hence we obtain:

Proposition 3.1. *There is a natural bijection between subcomplexes of the simplicial complex $L(W, S)$ and non-empty W-invariant subcomplexes of the cubical complex $\Sigma(W, S)$.*

To show that $\Sigma(W, S)$ is a model for $\underline{E}W$, metric techniques are helpful.

There is a natural CAT(0)-metric on $\Sigma(W,S)$, which is best understood in terms of the cubical structure. The length of a piecewise linear path in $\Sigma(W,S)$ is defined using the standard Euclidean metric on each cube, and the distance between two points of $\Sigma(W,S)$ is the infimum of the lengths of PL-paths connecting them. According to Gromov's criterion [16], $\Sigma(W,S)$ is locally CAT(0) because the link of every vertex is isomorphic to $L(W,S)$ which is a flag complex (see [11, 16]). It is easy to see that $\Sigma(W,S)$ is simply connected (for example, because its 2-skeleton is a version of the Cayley complex for W), and it follows that $\Sigma(W,S)$ is CAT(0) [5, Theorem II.4.1]. Given that W acts isometrically with finite stabilizers on $\Sigma(W,S)$ it follows that $\Sigma(W,S)$ is a model for $\underline{E}W$ via the Bruhat-Tits fixed point theorem [5, p. 179] or [2, Prop. 3].

Lemma 3.2. *Every finite subgroup of W is a subgroup of a parabolic subgroup of W. In particular, there are finitely many conjugacy classes of finite subgroups of W and every finite subgroup is isomorphic to a direct product $(\mathbb{Z}/2)^k$ for some $0 \le k \le n$ where n is the dimension of $\Sigma(W,S)$.*

Proof. Let F be a finite subgroup of W. By the Bruhat-Tits fixed point theorem F fixes some point of Σ, and hence F is a subgroup of a point stabilizer. Every such subgroup is parabolic, and each is conjugate to one of the finitely many special parabolics. □

Recall that a group is said to be of *type F* if it admits a compact classifying space.

Corollary 3.3. *The commutator subgroup W' of W is a finite-index torsion-free subgroup of type F.*

Proof. The abelianization of W is naturally isomorphic to C_2^S. Every parabolic subgroup of W maps injectively into C_2^S. It follows that W' acts freely on the finite-dimensional contractible space Σ. Hence Σ/W' is a compact $K(W',1)$, from which it follows that W' is both type F and torsion-free. □

Lemma 3.4 ([2])**.** *The quotient of the pair $(\Sigma, \Sigma^{\mathrm{sing}})$ by W is isomorphic to the pair (CL', L'), i.e., the pair consisting of the cone on the barycentric subdivision of L and its base. This isomorphism is natural for automorphisms of L. If L is acyclic then so is Σ^{sing}. If L is simply-connected, then so is Σ^{sing}.*

Proof. The first part is clear from the simplicial description of Σ. Now let V be the unique free W-orbit of vertices in the simplicial description of Σ. The star of each $v \in V$ is a copy of the cone CL', with v as its apex. The subcomplex of Σ consisting of all simplices not containing any vertex of V is Σ^{sing}. Hence Σ is obtained from Σ^{sing} by attaching cones to countably many subcomplexes isomorphic to L'.

In the case when L is acyclic, attaching a cone to a copy of L' does not change homology. It follows that Σ^{sing} must be acyclic since Σ is. Similarly, if L is simply-connected, then attaching a cone to a copy of L' does not change the fundamental group, so Σ^{sing} must be simply-connected since Σ is. □

Now suppose a finite group Q acts by automorphisms on $L(W,S)$. This defines an action of Q on W, and hence a semidirect product $G = W \rtimes Q$.

Lemma 3.5. *There is an admissible simplicial G-action on $\Sigma(W,S)$ extending the action of W, and $\Sigma(W,S)$ becomes a cocompact model for $\underline{E}G$.*

Proof. The action of Q on the poset underlying $\Sigma(W,S)$ is defined in such a way that $q \in Q$ sends the coset wW_T to the coset $q(w)W_{q(T)}$. This combines with the W-action to give an admissible G-action on $\Sigma(W,S)$. Since $\Sigma(W,S)$ is CAT(0) and the stabilizers are finite it follows that $\Sigma(W,S)$ is a model for $\underline{E}G$. □

Corollary 3.6. *Any finite-index subgroup H of G as above admits a cocompact model for $\underline{E}H$ and is virtually torsion-free.*

Remark 3.7. *For the action of G on Σ, the stabilizer of the vertex W_T is the semidirect product $W_T \rtimes Q_T$, where $Q_T := \{q \in Q : q(T) = T\}$. If Q acts admissibly on L then Q_T fixes each element of T and the stabilizer is the direct product $W_T \times Q_T$. Similarly, the stabilizer of the vertex wW_T is the direct product $wW_Tw^{-1} \times wQ_Tw^{-1}$. Note in particular that the image of the stabilizer under the quotient map $G \to G/W \cong Q$ depends only on T, and not on w.*

Lemma 3.8. *Let N be a finite index normal subgroup of W^{ev}. There is an isomorphism ψ from the relative chain complex $C_*(CL', L')$ to a direct summand of the simplicial chain complex $C_*(\Sigma/N)$. This isomorphism is natural for automorphisms of L that preserve N. It is also natural for the inclusion of subcomplexes in L and the corresponding W/N-invariant subcomplexes of the cubical structure on Σ/N.*

Proof. The cone CL' is the realization of the poset of spherical subsets of S, with cone point the empty set $\emptyset$. For σ a simplex of CL', $\psi(\sigma)$ in $C_*(\Sigma/N)$ will be the signed sum of its $|W/N|$ inverse images under the map $\Sigma/N \to \Sigma/W = CL'$. The signs will ensure that simplices of CL' that do not contain $\emptyset$ as a vertex map to zero.

In more detail, fix a transversal $w_1, \dots, w_m$ to N in W, and for σ a simplex of CL', viewed as a chain $\sigma = (T_0 < T_1 < \cdots < T_r)$ of spherical subsets, define

$$\psi(\sigma) = \sum_{i=1}^{m} (-1)^{l(w_i)} w_i \sigma = \sum_{i=1}^{m} (-1)^{l(w_i)} (w_i W_{T_0} < \cdots < w_i W_{T_r}).$$

Here $l(w)$ denotes the length of w as a word in S. For any $n \in N$, $l(wn) - l(w)$ is even, and so the sum above does not depend on the choice of transversal. The above formula clearly describes a chain map from $C_*(CL')$ to $C_*(\Sigma/N)$. Now if T is a non-empty spherical subset of S, W_T contains equal numbers of words of odd and even length, and hence so does its image $W_T/(W_T \cap N) \leq W/N$. Equivalently, any transversal to $W_T \cap N$ in W_T contains equal numbers of words of odd and even length. It follows that if $T_0 \neq \emptyset$, then $\psi(\sigma) = 0$. Hence the formula given above defines a chain map $\psi : C_*(CL', L') \to C_*(\Sigma/N)$. This clearly has the claimed naturality properties.

It remains to exhibit a splitting map $\phi : C_*(\Sigma/N) \to C_*(CL', L')$. This uses a 'simplicial excision map'. Let v be the image of $W_\emptyset \in \Sigma$ in Σ/N, and let X be the subcomplex of Σ/N consisting of all simplices that do not have v as a vertex. There is a natural bijection between simplices of CL' containing the cone vertex and simplices of Σ/N containing v. This induces an isomorphism $C_*(\Sigma/N, X) \cong C_*(CL', L')$ and ϕ is defined as the composite of this with the map $C_*(\Sigma/N) \to C_*(\Sigma/N, X)$. To check that $\phi \circ \psi$ is the identity map on $C_*(CL', L')$, let $\sigma = (T_0 < T_1 < \cdots < T_r)$ be any r-simplex of CL'. If $T_0 \neq \emptyset$

then we already know that $\phi \circ \psi(\sigma) = \phi(0) = 0$, and so the given formula for $\phi \circ \psi$ does define a self-map of $C_*(CL', L')$. On the other hand, if $T_0 = \emptyset$ then $\psi(\sigma)$ contains $m = |W : N|$ distinct signed simplices, exactly one of which has $v = W_\emptyset$ as a vertex rather than some other coset of $W_\emptyset$; furthermore this simplex appears with sign $+1$. It follows that in this case $\phi \circ \psi(\sigma) = \sigma$, confirming that $\phi \circ \psi$ is the identity map of $C_*(CL', L')$. □

Corollary 3.9. *With notation as above, let K be a subcomplex of L, and let $\Sigma(K)$ be the (barycentric subdivision of the) cubical W_L-subcomplex of Σ associated to K. There is a natural isomorphism ψ from the relative chain complex $C_*(CL', L' \cup CK')$ to a direct summand of the relative simplicial chain complex $C_*(\Sigma/N, \Sigma(K)/N)$.*

Proof. $C_*(CK', K')$ is a subcomplex of $C_*(CL', L')$ and the corresponding quotient is $C_*(CL', L' \cup CK')$. Similarly, $C_*(\Sigma(K)/N)$ is a subcomplex of $C_*(\Sigma/N)$ with $C_*(\Sigma/N, \Sigma(K)/N)$ the corresponding quotient.

By naturality of ψ and ϕ we get a diagram as follows, in which the two left-hand squares with the same label on both vertical sides commute and such that the two composites labelled $\phi \circ \psi$ are equal to the relevant identity maps. A diagram chase shows that there are unique maps ψ and ϕ corresponding to the dotted vertical arrows that make the right-hand squares with the same label on both vertical sides commute, and that these maps also satisfy $\phi \circ \psi = 1$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(CK, K') & \longrightarrow & C_*(CL', L') & \longrightarrow & C_*(CL', L' \cup CK') & \longrightarrow & 0 \\
 & & \phi \uparrow \downarrow \psi & & \phi \uparrow \downarrow \psi & & \phi \uparrow \downarrow \psi & & \\
0 & \longrightarrow & C_*(\Sigma(K)/N) & \longrightarrow & C_*(\Sigma/N) & \longrightarrow & C_*(\Sigma/N, \Sigma(K)/N) & \longrightarrow & 0
\end{array}$$

□

4 Proof of Theorem 1.1

As in the statement of Theorem 1.1, let L be a finite n-dimensional flag complex equipped with an admissible simplicial action of a finite group Q, let $(W, S) = (W_L, S_L)$ be the associated right-angled Coxeter system, let N be a finite-index subgroup of W that is normalized by Q, and let G be the semidirect product $G = N \rtimes Q$.

Proof of Theorem 1.1. Note that G may be viewed as a finite-index subgroup of the semidirect product $W \rtimes Q$. Under these hypotheses, we already see from Corollary 3.5 that the Davis complex Σ is a cocompact $(n+1)$-dimensional model for $\underline{E}G$, and that G is virtually torsion-free.

Using the hypothesis that L is acyclic, we see that the subcomplex $\Sigma^{\mathrm{sing}(W)}$ of Σ consisting of those points whose stabilizer in W is non-trivial is acyclic by Lemma 3.4. In this case any finite-index torsion-free subgroup of G acts freely on the acyclic n-dimensional complex $\Sigma^{\mathrm{sing}(W)}$, which implies that $\mathrm{vcd} G \leq n$.

For the remainder of the proof it will be convenient to define $K := L^{\mathrm{sing}}$, the subcomplex of Q-singular points in L. (We warn the reader that our use of 'K' is different to that in [11, ch. 7–8].)

Next we show that $\underline{\mathrm{cd}} G \geq n + 1$. Since W^{ev} has index 2 in W and is clearly Q-invariant, we see that $(N \cap W^{\mathrm{ev}}) \rtimes Q$ is a subgroup of G of index at most 2.

Hence without loss of generality we may assume that $N \leq W^{\mathrm{ev}}$. Now consider the family $\mathcal{W}$ of finite subgroups of G, consisting of those finite subgroups that are contained in N, or equivalently the finite subgroups that map to the trivial subgroup under the factor map $G \to Q$. The stabilizers in $W \rtimes Q$ of vertices of Σ are described in Remark 3.7, and by intersecting with $G = N \rtimes Q$ we get a similar description of stabilizers in G: the stabilizer of the vertex wW_T is the direct product of the intersection $N \cap wW_Tw^{-1}$ and a subgroup that maps isomorphically to Q_T, the stabilizer in Q of the vertex T of CL'. It follows that $\Sigma^{\mathcal{W}\text{-sing}}$ is equal to the inverse image in Σ of the Q-singular set CK' in $\Sigma/W = CL'$. Hence $\Sigma^{\mathcal{W}\text{-sing}}$ is the W-invariant subcomplex of the cubical structure on Σ that corresponds (under the map of Proposition 3.1) to $K = L^{\mathrm{sing}}$. Using Corollary 3.9 applied in this case, we see that $H^{n+1}(\Sigma/N, \Sigma^{\mathcal{W}\text{-sing}}/N)$ admits a split surjection onto $H^{n+1}(CL', L' \cup CK')$, which is isomorphic to $H^n(L,K) = H^n(L, L^{\mathrm{sing}})$ by excision. Proposition 2.1 finishes the argument.

To show that $\mathrm{vcd}G = n$ when L is a barycentric subdivision, we use the calculation of the cohomology of W with free coefficients as described in [11, section 8.5]. If v is a vertex of L that corresponds to the barycentre of an n-dimensional cell, then $L - v$ is homotopy equivalent to the subcomplex obtained from L by removing the (interior of the) n-dimensional cell. Hence we see that $H^{n-1}(L-v) \cong \mathbb{Z}$, and so by [11, cor. 8.5.3], $H^n(W;\mathbb{Z}W)$ contains a free abelian summand.

Now we show that $\mathrm{vcd}G = n$ in the case when $K = L^{\mathrm{sing}}$ is a full subcomplex of L. From the long exact sequence for the pair (L,K) we see that $H^{n-1}(K) \neq 0$, and hence $H^n(CK, K) \neq 0$. Lemma 3.8 applied to the Coxeter group W_K and its finite-index torsion-free subgroup W'_K shows that $H^n(W'_K;\mathbb{Z}) = H^n(\Sigma_K/W'_K)$ contains a summand isomorphic to $H^n(CK,K)$ and so is not zero. (Here we use Σ_K to denote $\Sigma(W_K, S_K)$ since we reserve Σ to stand for $\Sigma(W_L, S_L)$.) Since K is a full subcomplex of L, the Coxeter group W_K is naturally a subgroup of $W_L \leq G$, and hence $\mathrm{vcd}G \geq \mathrm{vcd}W_K \geq n$.

The general case of Theorem 1.1 follows from the case described above, but since we will make extensive use of the Künneth theorem it is helpful to work with cohomology with coefficients in a finite field rather than integral cohomology. Since L is finite and n-dimensional, the hypothesis that $H^n(L, L^{\mathrm{sing}}) \neq 0$ is equivalent to the existence of a prime p for which the mod-p cohomology group $H^n(L, L^{\mathrm{sing}}; \mathbb{F}_p) \neq 0$. Similarly, the hypothesis that $\bigotimes_{i=1}^m H^{n_i}(L_i, L_i^{\mathrm{sing}}) \neq 0$ is equivalent to the existence of a single prime p such that for each i, $H^{n_i}(L_i, L_i^{\mathrm{sing}}; \mathbb{F}_p) \neq 0$. For the remainder of the proof, we fix such a prime. The mod-p analogues of Proposition 2.1 and Corollary 3.9 are easily deduced from the integral versions.

Now let each G_i be defined as above in terms of L_i, Q_i, n_i and $N_i \leq W_i$, and define $\Gamma := G_1 \times \cdots \times G_m$, $Q = Q_1 \times \cdots \times Q_m$, and $W := W_1 \times \cdots \times W_m$. Finally, let $n := \sum_{i=1}^m n_i$. The direct product $\Sigma := \Sigma_1 \times \cdots \times \Sigma_m$ is a cocompact model for $\underline{E}\Gamma$ of dimension $m+n$, and so $\underline{\mathrm{cd}}\Gamma \leq m+n$. Also the direct product $\Sigma_1^{\mathrm{sing}} \times \cdots \times \Sigma_m^{\mathrm{sing}}$ is an acyclic n-dimensional simplicial complex admitting a proper Γ-action, which implies that $\mathrm{vcd}\Gamma \leq n$.

The lower bounds also work just as in the case $m = 1$; first we consider $\underline{\mathrm{cd}}\Gamma$. If we define $\mathcal{W}$ to be the family of finite subgroups of Γ that are contained in W, then a point $x = (x_1, \ldots, x_m) \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_m$ is in $\Sigma^{\mathcal{W}\text{-sing}}$ if and only if there is an i so that $x_i \in \Sigma_i^{\mathcal{W}_i\text{-sing}}$. Hence we see that if we define

$N := N_1 \times \cdots \times N_m$, then

$$\Sigma^{\mathcal{W}\text{-sing}} = \bigcup_{i=1}^{m} \Sigma_1 \times \cdots \times \Sigma^{\mathcal{W}_i\text{-sing}} \times \cdots \times \Sigma_m,$$

and by the relative Künneth Formula $H^{n+m}(\Sigma/N, \Sigma^{\mathcal{W}\text{-sing}}/N; \mathbb{F}_p)$ contains a direct summand isomorphic to $\bigotimes_{i=1}^{m} H^{n_i+1}(\Sigma_i/N_i, \Sigma_i^{\mathcal{W}_i\text{-sing}}/N_i; \mathbb{F}_p)$, which is non-zero since it contains a summand isomorphic to

$$\bigotimes_{i=1}^{m} H^{n_i+1}(CL_i, L_i \cup CK_i; \mathbb{F}_p) \cong \bigotimes_{i=1}^{m} H^{n_i}(L_i, K_i; \mathbb{F}_p) = \bigotimes_{i=1}^{m} H^{n_i}(L_i, L_i^{\text{sing}}; \mathbb{F}_p).$$

To give a lower bound for $\text{vcd}\Gamma$, start by considering the two extra hypotheses separately for each i. If L_i is a barycentric subdivision, then as above $H^{n_i}(W_i; \mathbb{Z}W_i)$ contains a free abelian summand, and so by the universal coefficient theorem $H^{n_i}(W_i; \mathbb{F}_pW_i) \neq 0$. If instead $K_i := L_i^{\text{sing}}$ is a full subcomplex of L_i, then $H^{n_i}(W_i'; \mathbb{F}_p) \neq 0$ as above. There is a surjective homomorphism from $\mathbb{F}_pW_i'$ onto $\mathbb{F}_p$ and hence a short exact sequence of $\mathbb{F}_pW_i'$ modules

$$0 \to I \to \mathbb{F}_pW_i' \to \mathbb{F}_p \to 0$$

for suitable I. The corresponding long exact sequence in cohomology implies that $H^{n_i}(W_i'; \mathbb{F}_pW_i') \to H^{n_i}(W_i'; \mathbb{F}_p)$ is surjective, since its cokernel is contained in $H^{n_i+1}(W_i'; I) = 0$. It follows that $H^{n_i}(W_i; \mathbb{F}_pW_i) \cong H^{n_i}(W_i'; \mathbb{F}_pW_i') \neq 0$. Since W acts cocompactly on $\Sigma = \Sigma_1 \times \cdots \times \Sigma_m$, the universal coefficient theorem for cohomology with compact supports may be applied [11, 8.5.9]. Hence

$$H^n(W; \mathbb{F}_pW) \cong \bigotimes_{i=1}^{m} H^{n_i}(W_i; \mathbb{F}_pW_i) \neq 0,$$

showing that $\text{vcd}\Gamma \geq n$ as required. □

5 Examples

In this section we construct sufficiently many examples of finite groups Q and Q-CW-complexes L to establish Corollary 1.2. First we collect some results concerning triangulations.

Proposition 5.1. *Any finite Q-CW-complex is equivariantly homotopy equivalent to a finite simplicial complex of the same dimension with an admissible Q-action. If L is any simplicial complex with Q-action, the Q-action on the barycentric subdivision L' of L is admissible. For any admissible action of Q on L, L^{sing} is a subcomplex. If $M \leq L$ is any subcomplex of a simplicial complex L, then its barycentric subdivion M' is a full subcomplex of the flag complex L'.*

Proof. The first claim follows easily from the simplicial approximation theorem. Simplices of L' correspond to chains in the poset of simplices of L; since Q acts as automorphisms of this poset the action on L' is admissible. For an admissible action of Q on L, a simplex of L is fixed by $H \leq Q$ if and only if each of its vertices is fixed. Hence each L^H is the full subcomplex on the H-fixed vertices, and $L^{\text{sing}} = \bigcup_{1<H\leq Q} L^H$ is a subcomplex. Finally if M is any subcomplex of L, the poset of simplices of M is a subposet of the poset of simplices of L, and so M' is a full subcomplex of L'. □

Example 1. Let Q be the alternating group A_5, and define a Q-CW-complex as follows. For the 1-skeleton L^1 of L take the complete graph on five vertices, with the natural action of $Q = A_5$. In A_5, the 24 elements of order five split into two conjugacy classes of size 12, and any element g of order 5 is conjugate to g^{-1} (but is not conjugate to g^2 or g^3). Define L by using one of the two conjugacy classes of 5-cycles to describe attaching maps for six pentagonal 2-cells. By construction there is a Q-action without a global fixed point and it is easily checked that L is acyclic. In fact, $\pi_1(L)$ is isomorphic to $SL(2,5)$, the unique perfect group of order 120, and L is isomorphic to the 2-skeleton of the Poincaré homology sphere [3, I.8]. The singular set for the Q-action consists of the 1-skeleton and the five lines of symmetry of each pentagonal 2-cell. Equivalently the singular set is the 1-skeleton of the barycentric subdivision of L (i.e., the simplicial complex with 21 vertices coming from the poset of faces of L). In particular $H^2(L, L^{\text{sing}}) \neq 0$. For this 21-vertex triangulation, L^{sing} is not a full subcomplex of L. This could be rectified by taking a finer triangulation, but instead note that L is the barycentric subdivision of a polygonal complex. By taking each Q_i to be A_5 and each L_i to be this 21-vertex triangulation of L, we obtain groups Γ_m having the properties stated in the first part of Corollary 1.2.

Example 2. Fix distinct primes p and q, and let Q be cyclic of order q, generated by g. For the Q-fixed point set L^Q, take a mod-p Moore space $M(1,p)$. This space has a CW-structure with 1-skeleton a circle and one 2-cell f. The 2-cell f is attached to the circle via a map of degree p. Now define L^2 by adding on a free Q-orbit of 2-cells $f_0, \ldots, f_{q-1}$, where $f_i = g^i f_0$, so that each f_i is attached to the circle by a degree one map. L^2 is simply connected, and $H_2(L^2)$ is a free $\mathbb{Z}Q$-module of rank one, since it has a $\mathbb{Z}$-basis given by the elements

$$e_j := f - \sum_{i=0}^{p} g^i f_j = f - \sum_{i=0}^{p} g^{i+j} f_0$$

for $0 \le j < q$, and $g^j e_0 = e_j$ for each j. Make L by attaching a free Q-orbit of 3-cells to kill each e_j, so that L is acyclic (and also contractible). The long exact sequence for the pair $(L, L^Q) = (L, L^{\text{sing}})$ implies that $H^3(L, L^Q) \cong \mathbb{Z}/p$. To establish the second part of Corollary 1.2 we take each Q_i to be cyclic of order q_i, take each L_i to be a suitable triangulation of the above Q_i-CW-complex for some fixed choice of p, and take N_i to be the commutator subgroup of the Coxeter group $W_i := W_{L_i}$. For any such choice, we obtain a group Λ_m as in the statement. To ensure that Λ_m contains only cyclic finite subgroups we must take the primes q_i all distinct, whereas to ensure that Λ_m contains only abelian finite subgroups of exponent q we take $q_i = q$ for all i.

6 Contractibility and acyclicity

In [2], it was shown that certain right-angled Coxeter groups W have the property that $\text{vcd}W = \underline{\text{cd}}W = 2 < \underline{\text{gd}}W = 3$. In this section we improve this result by showing that for these same groups there is no 2-dimensional contractible proper W-CW-complex.

We will use a few subsidiary results in the proof. Results similar to Propositions 6.1 and 6.2 appear in [9], and with extra hypotheses in [26]. Proposition 6.3 is a corollary of the celebrated Gerstenhaber-Rothaus theorem [15].

Proposition 6.1. *If Y is a subcomplex of a 2-dimensional acyclic complex, then $H_2(Y) = 0$ and each $H_i(Y)$ is free abelian.*

Proof. If Y is any subcomplex of an n-dimensional acyclic complex Z, then consideration of the homology long exact sequence for the pair (Z, Y) shows that $H_n(Y)$ is trivial and that $H_{n-1}(Y)$ is free abelian. Since H_0 is always free abelian, the case $n = 2$ gives the claimed result. □

Proposition 6.2. *Let Q be a finite soluble group and let X be a 2-dimensional acyclic Q-CW-complex. Then the fixed point set X^Q is also acyclic.*

Proof. The finite soluble group Q has a normal subgroup N of prime index, the factor group Q/N acts on the N-fixed point set X^N, and the equality $X^Q = (X^N)^{Q/N}$ holds. Hence it suffices to consider the case in which Q has prime order.

By the P. A. Smith theorem, X^Q is mod-p acyclic in the case when Q has order p. By the previous proposition, $H_i(X^Q)$ is free abelian for all i. By the universal coefficient theorem, the rank of the ith mod-p homology group of X^Q is equal to the rank of $H_i(X^Q)$. Hence X^Q must be acyclic. □

Proposition 6.3. *Let Γ be a group and $\rho : \Gamma \to U(n)$ be a unitary representation of Γ. Define $\widetilde{\Gamma} := \Gamma * \langle x_1, \ldots, x_r\rangle / \langle\langle w_1, \ldots, w_r\rangle\rangle$ where each w_i is a word in elements of Γ and $x_1, \ldots, x_r$. Let d_{ij} be the total exponent of x_j in w_i and set $d = \det(d_{ij})$. If $d \neq 0$, then ρ extends to a representation $\tilde{\rho} : \widetilde{\Gamma} \to U(n)$.*

Proof. Extending ρ to a representation of $\widetilde{\Gamma}$ is equivalent to finding solutions $\overline{x}_i \in U(n)$ to the system of equations $\overline{w}_1 = \cdots = \overline{w}_r = 1$. Here $\overline{w}_i$ is the word in elements of $U(n)$ and variables $\overline{x}_1, \ldots, \overline{x}_r$ corresponding to the word w_i. In more detail, the elements of $U(n)$ appearing in $\overline{w}_i$ are obtained by applying ρ to the elements of Γ appearing in the word w_i, while each occurrence of x_i is replaced by $\overline{x}_i$. When such a solution has been found, we may define $\widetilde{\rho}(x_i) := \overline{x}_i$. The existence of a solution to this system is established in [15, theorem 1]. □

We recall that the *nerve* of a covering is the simplicial complex whose vertices are the sets in the cover, and whose simplices are the finite collections with a non-empty intersection [17, section 3.3].

Lemma 6.4. *Let X be a CW-complex, let S be a finite indexing set, and let $X(s)$ be a subcomplex of X such that each $X(s)$ is acyclic and each intersection of $X(s)$'s is either empty or acyclic. Define*

$$X^{\#} := \bigcup_{s \in S} X(s),$$

and let $|\mathcal{N}|$ be the realization of the nerve of the covering of $X^{\#}$ by the subcomplexes $X(s)$. There is a map $f : X^{\#} \to |\mathcal{N}|$ which is a homology isomorphism and induces a surjection of fundamental groups.

Proof. In the case when each intersection of $X(s)$'s is either contractible or empty, it is well-known that there is a homotopy equivalence $f : X^{\#} \to |\mathcal{N}|$ [17, 4.G, Ex. 4]. We use Quillen's plus construction to reduce to this case.

For $T \subseteq S$, define $X(T)$ to be the intersection $X(T) = \bigcap_{s \in T} X(s)$. Suppose that $U \subseteq S$ is such that $X(U)$ is non-empty. In this case, since $X(U)$ is acyclic we can find a set A_U of 2-cells with attaching maps from the boundary of the 2-cell to $X(U)$ so that each attaching map represents a conjugacy class of commutators in $\pi_1(X(U))$ and so that the fundamental group of the resulting complex $\widehat{X}(U)$ is trivial. Moreover, there is a set B_U of 3-cells and attaching maps from the boundary of the 3-cell to $\widehat{X}(U)$ so that the resulting complex X_U contains $X(U)$ as a subcomplex, is simply-connected, and such that the inclusion of $X(U)$ into X_U is a homology isomorphism. Define Y by attaching to X 2- and 3-cells indexed by $\coprod_U A_U$ and $\coprod_U B_U$ respectively. Define a subcomplex $Y(s)$ of Y by attaching to $X(s)$ the 2- and 3-cells indexed by $\coprod_{s\in U} A_U$ and $\coprod_{s\in U} B_U$ respectively. Finally define $Y(T) := \bigcap_{s\in T} Y(s)$, and $Y^\# := \bigcup_{s\in S} Y(s)$. The nerve of the covering of $Y^\#$ by the subcomplexes $Y(s)$ is naturally isomorphic to $\mathcal{N}$. A Mayer-Vietoris spectral sequence argument shows that the inclusion $X^\# \to Y^\#$ is a homology isomorphism, and this map induces a surjection $\pi_1(X^\#) \to \pi_1(Y^\#)$ because the 1-skeleta of $X^\#$ and $Y^\#$ are equal. □

It is also possible to prove the above result directly using the Mayer-Vietoris spectral sequence and the van Kampen theorem to keep track of the homology and fundamental group respectively.

Proof of Theorem 1.3. The Davis complex $\Sigma = \Sigma(W_L, S_L)$ is a cocompact 3-dimensional model for $\underline{E}W_L$. Since L is acyclic, Σ^{sing} is a 2-dimensional acyclic proper W_L-CW-complex in which the fixed point set for any non-trivial finite subgroup is contractible. This suffices to show that $\underline{\mathrm{cd}}W_L = 2$.

Now suppose that X is any contractible proper 2-dimensional W_L-CW-complex. Let $S = S_L$, and define $X^\#$ as the union of the fixed point sets X^s: $X^\# := \bigcup_{s\in S} X^s$. By construction, the realization of the nerve of the covering of $X^\#$ by the sets X^s is equal to L. By Propostion 6.2, for each $T \subseteq S$ that spans a simplex of L the subset

$$X(T) := \bigcap_{s\in T} X^s = X^{\langle T\rangle}$$

is acyclic, and for each T that does not span a simplex of L, $X(T)$ is empty. By Lemma 6.4, it follows that $X^\#$ is acyclic and that there is a natural surjection $\phi : \pi_1(X^\#) \to \pi_1(L)$. Define $\rho' := \rho \circ \phi : \pi_1(X^\#) \to U(n)$, a non-trivial unitary representation of $\pi_1(X^\#)$. We use this representation to obtain a contradiction.

Pick $g \in \pi_1(X^\#)$ so that $\rho'(g) \neq 1$. Since X is contractible, there exists a connected subcomplex X_1 of X with $X^\# \subseteq X_1$ such that $X_1 - X^\#$ comprises only finitely many cells, and such that g maps to the identity element of $\pi_1(X_1)$. By Proposition 6.1, $H_2(X_1) = 0$, and $H_1(X_1)$ is free abelian. In general $X_1 - X^\#$ will contain some 0-cells; by contracting some of the 1-cells in $X_1 - X^\#$ we may get rid of these extra 0-cells without changing the homotopy type. In this way we replace X_1 by a complex X_2 with the following properties: $X^\# \subseteq X_2$; $H_1(X_2)$ is free abelian and $H_2(X_2) = \{0\}$; X_2 consists of $X^\#$ with finitely many 1- and 2-cells added; g is in the kernel of the map $\pi_1(X^\#) \to \pi_1(X_2)$. Unlike X_1, X_2 is not a subcomplex of X but this is irrelevant. Since X_2 is made by attaching finitely many cells to the acyclic complex $X^\#$, note that $H_1(X_2)$ is free abelian of finite rank. Now make X_3 by attaching 2-cells to exactly kill $H_1(X_2)$. Thus X_3 is an acyclic 2-complex, obtained by attaching the same finite number, r say, of 1- and 2-cells to $X^\#$. If we write $\Gamma = \pi_1(X^\#)$

and $\widetilde{\Gamma} := \pi_1(X_3)$, then the relationship between Γ and $\widetilde{\Gamma}$ is exactly as in the hypotheses of Proposition 6.3. Here, the group generator $x_i \in \widetilde{\Gamma}$ corresponds to a based loop in X_3 that remains in $X^{\#}$ except that it travels once along the ith of the r new 1-cells, and the word w_i spells out the attaching map for the ith of the r new 2-cells as a word in the elements of Γ and the new loops x_j. Moreover, since $X^{\#}$ and X_3 are both acyclic, the relative homology groups $H_i(X_3, X^{\#})$ all vanish, which tells us that the determinant d appearing in the statement of Proposition 6.3 is equal to ± 1. Now Proposition 6.3 can be applied and tells us that the representation $\rho' : \pi_1(X^{\#}) \to U(n)$ extends to a representation $\tilde{\rho} : \pi_1(X_3) \to U(n)$. However, this contradicts the fact that $\rho'(g) \neq 1$, while g maps to the identity in $\pi_1(X_3)$. □

Remark 6.5. *As an example of a suitable L, take a flag triangulation of the 2-skeleton of the Poincaré homology sphere (which was discussed in the previous section); here there is a faithful representation $\rho : \pi_1(L) \cong SL(2,5) \to U(2)$.*

Remark 6.6. *There is a version of Brown's question that remains open: for $m > 2$, is there a virtually torsion-free group G such that $\mathrm{vcd}G = m$ but there exists no contractible m-dimensional proper G-CW-complex?*

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