EQUIVARIANT VECTOR BUNDLES OVER CLASSIFYING SPACES FOR PROPER ACTIONS

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Abstract. Let G be an infinite discrete group and let $\underline{E}G$ be a classifying space for proper actions of G. Every G-equivariant vector bundle over $\underline{E}G$ gives rise to a compatible collection of representations of the finite subgroups of G. We give the first examples of groups G with a cocompact classifying space for proper actions $\underline{E}G$ admitting a compatible collection of representations of the finite subgroups of G that does not come from a G-equivariant (virtual) vector bundle over $\underline{E}G$. This implies that the Atiyah-Hirzeburch spectral sequence computing the G-equivariant topological K-theory of $\underline{E}G$ has non-zero differentials. On the other hand, we show that for right angled Coxeter groups this spectral sequence always collapses at the second page and compute the K-theory of the classifying space of a right angled Coxeter group.

1. Introduction

Let G be an infinite discrete group and $\mathcal{F}$ be the family of finite subgroups of G. Recall that the orbit category $\mathcal{O}_{\mathcal{F}}G$ is a category whose objects are the transitive G-sets G/H, for all $H \in \mathcal{F}$, and whose morphism are all G-equivariant maps between the objects. A classifying space for proper actions of G, denoted by $\underline{E}G$, is a proper G-CW-complex such that the fixed point set $\underline{E}G^H$ is contractible for every $H \in \mathcal{F}$. The space $\underline{E}G$ is said to be cocompact if the orbit space $G \setminus \underline{E}G = \underline{B}G$ is compact. Many interesting classes of groups G have cocompact models for $\underline{E}G$, for example cocompact lattices in Lie groups, mapping class groups of surfaces, $\mathrm{Out}(F_n)$, CAT(0)-groups and word-hyperbolic groups. We refer the reader to [9] for more examples and details.

Now assume G is an infinite discrete group admitting a cocompact classifying space for proper actions $\underline{E}G$. If

$$\xi\colon\ E \to \underline{E}G$$

is a G-equivariant complex vector bundle over $\underline{E}G$ (see Definition 2.3) and x is a point of $\underline{E}G$, then the fiber $\xi^{-1}(x)$ is a complex representation of the finite isotropy group G_x. The connectivity of the fixed point sets of $\underline{E}G$ ensures that these representations are compatible (see Definition 2.1) with one another as x and hence G_x varies. Therefore, every G-equivariant complex vector bundle over $\underline{E}G$ gives rise to a compatible collection of complex representations of the finite subgroups of G, and hence to an element of

$$\lim_{G/H \in \mathcal{O}_{\mathcal{F}}G} R(H).$$

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Here, $\lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} R(H)$ is the limit over the orbit category $\mathcal{O}_{\mathcal{F}}G$ of the contravariant representation ring functor

$$R(-)\colon \mathcal{O}_{\mathcal{F}}G \to \mathrm{Ab} \qquad G/H \mapsto R(H).$$

Denoting the Grothendieck group of the abelian monoid of isomorphism classes of complex G-vector bundles over $\underline{E}G$ by $\mathrm{K}^0_G(\underline{E}G)$, one obtains a map

$$\varepsilon_G\colon \mathrm{K}^0_G(\underline{E}G) \to \lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} R(H)$$

that maps a formal difference of (isomorphism classes) vector bundles (i.e. a virtual vector bundle) to a formal difference of (isomorphism classes) of compatible collections of representations of the finite subgroups of G. We say a compatible collection of represesentations of the finite subgroups of G can be realized as a (virtual) G-equivariant vector bundle over $\underline{E}G$ if there exists a (virtual) G-equivariant vector bundle over $\underline{E}G$ that maps to this collection under ε_G. One can also look at the corresponding situation for real (orthogonal) vector bundles and real (orthogonal) representations and obtain the map

$$\varepsilon_G\colon \mathrm{KO}^0_G(\underline{E}G) \to \lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} RO(H).$$

The maps ε_G are equal to the edge homomorphisms of certain Atiyah-Hirzebruch spectral sequences converging to $\mathrm{K}^*_G(\underline{E}G)$ and $\mathrm{KO}^*_G(\underline{E}G)$ (see (1) and (2)). Lück and Oliver proved that (see Proposition 2.5) the map ε_G (real or complex) is rationally surjective, meaning that a high enough multiple of every element in the target of ε_G is contained in the image of ε_G. In the last paragraph of [12, p. 596] Lück and Oliver ask for an example of a group G admitting a cocompact classifying space for proper actions $\underline{E}G$ such that ε_G is not surjective. In Section 3 of this paper we give the first example of such a group in the complex case. In Section 4 we give the first example of such a group in the real case. We also construct examples of groups G admitting a cocompact $\underline{E}G$ with the following weaker property: G admits a compatible collection of representations for its finite subgroups that cannot be realized as a G-vector bundle over $\underline{E}G$. However, for these examples we cannot exclude the possibility that there exists a virtual vector bundle that maps to this collection of representations under ε_G. On the other hand, these examples are more explicit and lower dimensional.

In the final section we show that for a right angled Coxeter group W, every compatible collection of representations of the finite subgroups of W can be realized as a W-equivariant vector bundle over $\underline{E}W$, so that the map

$$\varepsilon_W\colon \mathrm{K}^0_W(\underline{E}W) \to \lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} R(H).$$

is always surjective. Morever, we show that this map is actually an isomorphism and that (see Theorem 2.4)

$$\mathrm{K}^1_W(\underline{E}W) = 0.$$

Using a version of the Atiyah-Segal completion theorem for infinite discrete groups proven by Lück and Oliver, we use these results to compute the complex K-theory of BW, the classifying space of W (see Corollary 5.6).

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2. G-vector bundles and isotropy representations

Let G be a discrete group and let Γ be a Lie group. Let $\mathcal{S}$ be a family of finite subgroups of G, i.e. any collection of finite subgroups of G that is closed under conjugation and passing to subgroups. The orbit category $\mathcal{O}_{\mathcal{S}}G$ is a category whose objects are the transitive G-sets G/H, for all $H \in \mathcal{S}$, and whose morphism are all G-equivariant maps between the objects.

Definition 2.1. [12, p. 590] Let X be a G-CW-complex. A (G, Γ)-bundle over X is a Γ-principal bundle $p\colon E \to X$, where E is a left G-space such that p is G-equivariant and such that the left G-action and the right Γ-action on E commute. We denote the set of isomorphism classes of (G, Γ)-bundles over X by $\mathrm{Bdl}_{(G,\Gamma)}(X)$. For $H \in \mathcal{F}$, let

$$\mathrm{Rep}_\Gamma(H) = \mathrm{Hom}(H, \Gamma)/\mathrm{Inn}(\Gamma).$$

One can consider $\mathrm{Rep}_\Gamma(-)$ as a contravariant functor from $\mathcal{O}_{\mathcal{S}}G$ to Sets. An element of the limit

$$A = ([\alpha_H])_{H\in\mathcal{S}} \in \lim_{G/H\in\mathcal{O}_{\mathcal{S}}G} \mathrm{Rep}_\Gamma(H)$$

is a called an *$\mathcal{S}$-compatible collection of Γ-representations*. Given such an element A, let $\mathcal{S}_A$ be the family of subgroups of $G \times \Gamma$ consisting of conjugates of the subgroups of the form

$$\{(h, \alpha_H(h)) \mid h \in H\}$$

for all $H \in \mathcal{S}$ and let $E_{\mathcal{S}}(G, A)$ be the universal $G \times \Gamma$-CW-complex for the family $\mathcal{S}_A$.

Lemma 2.2. [12, Lemma 2.4] *For every $\mathcal{S}$-compatible collection of Γ-representations $A = ([\alpha_H])_{H\in\mathcal{S}}$ there exists a G-CW-complex $B_{\mathcal{S}}(G, A)$ with isotropy in $\mathcal{S}$ satisfying the following properties.*

- *The quotient map*

$$\pi\colon\ E_{\mathcal{S}}(G, A) \to \Gamma \setminus E_{\mathcal{S}}(G, A) = B_{\mathcal{S}}(G, A)$$

  is a (G, Γ)-bundle over the G-CW-complex $B_{\mathcal{S}}(G, A)$.
- *The (G, Γ)-bundle $\pi\colon E_{\mathcal{S}}(G, A) \to B_{\mathcal{S}}(G, A)$ is universal in the sense that for every G-CW-complex X with isotropy in $\mathcal{S}$ there is an isomorphism*

$$[X, \mathcal{B}_{\mathcal{S}}(G, A)]_G \xrightarrow{\cong} \mathrm{Bld}_{(G,\Gamma)}(X)$$

  given by pulling back the univeral bundle π along a G-map $X \to \mathcal{B}_{\mathcal{S}}(G, A)$.
- *For every $S \in \mathcal{S}$, the fixed point set $B_{\mathcal{F}}(G, A)^H$ is homotopy equivalent to $BC_\Gamma(\alpha_H)$, the classifying space of the centralizer of the image of α_H in Γ.*

If $\Gamma = \mathrm{U}(n)$ ($\Gamma = \mathrm{O}(n)$) and $\mathcal{S} = \mathcal{F}$, the family of all finite subgroups of G, then $\mathrm{Rep}_\Gamma(H)$ is the set of isomorphism classes of n-dimensional complex (real) representations of H. In this case, an element of the limit

$$A = ([\alpha_H])_{H\in\mathcal{F}} \in \lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} \mathrm{Rep}_\Gamma(H)$$

is a called is called a *compatible collection of complex (real) n-dimensional representations* of the finite subgroups of G. For $H \in \mathcal{F}$, let $R(H)$ ($RO(H)$) be the complex (real) representation ring of H, i.e. the Grothendieck group of the abelian cancellative monoid of isomorphism classes of finite dimensional complex (real) representations of H. Note that $\mathrm{Rep}_{\mathrm{U(n)}}(H)$ is naturally a subset of $R(H)$ and $\mathrm{Rep}_{\mathrm{O(n)}}(H)$ is naturally a subset of $RO(H)$. One can consider $R(-)$ as a functor from $\mathcal{O}_{\mathcal{F}}G$ to Ab. An element of the inverse limit

$$\alpha = ([\alpha_H])_{H\in\mathcal{F}} \in \lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} R(H)$$

is a called a *compatible collection of complex virtual representations* of the finite subgroups of G. One has a natural embedding

$$\lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} \mathrm{Rep}_{\mathrm{U}(n)}(H) \subset \lim_{G/H\in\mathcal{O}_{\mathcal{F}}G} R(H).$$

The analogous statements for $\mathrm{O}(n,\mathbb{R})$ and RO also hold.

Now let X be a proper cocompact G-CW-complex, i.e. X has finite isotropy and the orbit space $G\setminus X$ has a finite number of cells, such that for every $H\in\mathcal{F}$, the fixed point set X^H is non-empty and connected.

Definition 2.3 ([18]). A complex (real) G-vector bundle over X is a complex (real) vector bundle $\pi\colon E\to X$ such that π is G-equivariant and each $g\in G$ acts on E and X via a bundle isomorphism. An isomorphism of G-vector bundles over X is just an isomorphism of vector bundles that is G-equivariant. The set of isomorphims classes of complex (real) G-vector bundles over X will be denoted by $\mathrm{Bdl}_G(X)$ ($\mathrm{OBdl}_G(X)$). For every $x\in X$, the fiber $\pi^{-1}(x)$ is denoted by E_x. We refer the reader to [12, Section 1] and [20, Section I.9] for elementary properties of G-vector bundles over proper (cocompact) G-CW complexes.

Theorem 2.4. [12, Th. 3.2 and 3.15] *There exists a 2-periodic (8-periodic) equivariant cohomology theory $\mathrm{K}^*_G(X,A)$ ($\mathrm{KO}^*_G(X,A)$) on the category of proper G-CW-pairs such that when X is cocompact, $\mathrm{K}^0_G(X)$ ($\mathrm{KO}^0_G(X)$) is the Grothendieck group of the abelian monoid of isomorphism classes of complex (real) G-vector bundles over X. In particular, for every $H\in\mathcal{F}$, $\mathrm{K}^0_G(G/H)$ ($\mathrm{KO}^0_G(G/H)$) is canonically isomorphic to $R(H)$ ($RO(H)$).*

As usual (see [13, Section 6] and [5, Th. 4.7]), the skeletal filtration of X induces Atiyah-Hirzebruch spectral sequences

$$E_2^{p,q} = \mathrm{H}^p_G(X,\mathrm{K}^q_G(G/-)) \Longrightarrow \mathrm{K}^{p+q}_G(X). \tag{1}$$

and

$$E_2^{p,q} = \mathrm{H}^p_G(X,\mathrm{KO}^q_G(G/-)) \Longrightarrow \mathrm{KO}^{p+q}_G(X) \tag{2}$$

where $\mathrm{H}^p_G(X,-)$ denotes Bredon cohomology of X (see [2]).

Proposition 2.5. [13, Prop 5.8] *If X is a cocompact G-CW complex then the spectral sequences (1) and (2) above rationally collapse, meaning that the images of all differentials in these spectral sequences consist of torsion elements.*

By our assumptions on X, the zeroth Bredon cohomology group $\mathrm{H}^0_G(X,R(-))$ (resp. $\mathrm{H}^0_G(X,RO(H))$), equals the limit of the functor $R(-)$ (resp. $RO(-)$), over the orbit category $\mathcal{O}_{\mathcal{F}}G$. Consider the edge homomorphisms

$$\varepsilon_G\colon\ \mathrm{K}^0_G(X)\to \mathrm{H}^0_G(X,R(-))$$

and

$$\varepsilon_G\colon\ \mathrm{KO}^0_G(X)\to \mathrm{H}^0_G(X,RO(-))$$

of the spectral sequences (1) and (2). If $[\pi]$ is the isomorphism class of an n-dimensional complex G-vector bundle $\pi\colon E \to X$, then $\varepsilon_G([\pi])$ equals

$$([E_{e_H}])_{H\in\mathcal{F}} \in \lim_{G/H\in\mathcal{O}_\mathcal{F}G} \mathrm{Rep}_{U(n)}(H) \subset \mathrm{H}^0_G(X, R(-))$$

where $[E_{e_H}]$ denotes the isomorphism class in $R(H)$ of the H-representation E_{e_H}. The corresponding statement for real G-vector bundles also holds. Note that it follows from Proposition 2.5 that a suitable multiple of every compatible collection of (virtual) real or complex representations of the finite subgroups of G is contained in the image of the edge homomorphism ε_G.

Recall that the classifying space for proper actions $\underline{E}G$ is a terminal object in the homotopy category of proper G-CW complexes (e.g. [9, Th. 1.9]). Hence, if X is any proper cocompact G-CW complex such that X^H is non-empty and connected for each $H \in \mathcal{F}$, then there exists a G-map $X \to \underline{E}G$ that is unique up to G-homotopy and induces commutative diagrams

$$\begin{array}{ccc} \mathrm{K}^0_G(X) & \longrightarrow & \lim_{G/H\in\mathcal{O}_\mathcal{F}G} R(H) \\ \uparrow & \nearrow & \\ \mathrm{K}^0_G(\underline{E}G) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{KO}^0_G(X) & \longrightarrow & \lim_{G/H\in\mathcal{O}_\mathcal{F}G} RO(H) \\ \uparrow & \nearrow & \\ \mathrm{KO}^0_G(\underline{E}G). & & \end{array}$$

Hence, if a compatible collection α of virtual representations can be realized as a virtual G-vector bundle over $\underline{E}G$, it can also be realized as a virtual G-vector bundle over X.

3. Complex vector bundles

The purpose of this section is to construct a group G with a cocompact classifying space for proper actions $\underline{E}G$ admitting a compatible collection of complex representations of the finite subgroups of G that cannot be realized as G-equivariant virtual complex vector bundle over $\underline{E}G$, i.e. so that the edge homomorphism

$$\varepsilon_G\colon \mathrm{K}^0_G(\underline{E}G) \to \lim_{G/H\in\mathcal{O}_\mathcal{F}G} R(H).$$

is not surjective.

Let $F = C_4 \rtimes C_2$ be the dihedral group of order 8 where σ is generator for C_4 and ε is a generator of C_2. Let $H = \langle\sigma^2\rangle$ be the center of F, which has order two and denote the n-skeleton of the universal F/H-space $X = E(F/H)$ by X^n. We let F act on X and X^n via the projection onto F/H. Consider the complex 1-dimensional representation

$$\lambda\colon H = \langle\sigma^2\rangle \to \mathrm{U}(1) = S^1 : \sigma^2 \mapsto -1.$$

Lemma 3.1. *The isomorphism class* $[\lambda]$ *is contained in* $R(H)^{F/H}$*. For* $k \in \mathbb{Z}$*, the multiple* $k[\lambda]$ *is contained in the image of the restriction map* $\mathrm{res}\colon R(F) \to R(H)$ *if and only if* k *is even.*

Proof. Since H is the center of F it follows that the conjugation action of F/H on $R(H)$ is trivial, hence $[\lambda] \in R(H)^{F/H} = R(H)$. One easily verifies that the representation

$$\tau\colon F \to U(2)$$

defined by

$$\tau(\sigma) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \tau(\varepsilon) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisifies $\mathrm{res}([\tau]) = 2[\lambda]$. Hence, $k[\lambda]$ is contained in the image of res for every even $k \in \mathbb{Z}$. Note that, as a free abelian group, $R(H)$ is generated by $[\lambda]$ and the isomorphism class of the 1-dimensional complex trivial representation [tr] (e.g. see [19]). Now suppose k is odd and there exists an element $[\mu] - [\rho] \in R(F)$ such that $\mathrm{res}([\mu] - [\rho]) = k[\lambda]$. There are integers l, m, n and such that $\mathrm{res}([\mu]) = l[\mathrm{tr}] + m[\lambda]$, $\mathrm{res}([\rho]) = l[\mathrm{tr}] + n[\lambda]$ and $m - n = k$. By changing the representative of $[\mu]$, we may also assume that

$$\mu\colon\ F \to \mathrm{U}(l+m)$$

where $\mu(\sigma)$ is a diagonal matrix. Since $\mu(\sigma^2)$ has an m-dimensional eigenspace with eigenvalues -1 and an l-dimensional eigenspace with eigenvalue 1, it follows that $\mu(\sigma)$ has an s-dimensional eigenspace with eigenvalue i and a t-dimensional eigenspace with eigenvalue $-i$ such that $s + t = m$. Morever, $\mu(\sigma^3)$ has an s-dimensional eigenspace with eigenvalue $-i$ and a t-dimensional eigenspace with eigenvalue i. Since σ and σ^3 are conjugate in F, it follows that $s = t$ proving that m is even. A similiar argument shows that n is also even. But this contradicts the fact that $k = m - n$ is odd. Hence, there does not exist an element $[\mu] - [\rho] \in R(F)$ such that $\mathrm{res}([\mu] - [\rho]) = k[\lambda]$, if k is odd.

□

The following lemma uses the notation introduced above and will be cited in the next section.

Lemma 3.2. *Every F-equivariant complex line bundle over X^3 is isomorphic to the pullback of an F-equivariant complex line bundle over $E(F/H)$ along the inclusion $i\colon X^3 \to E(F/H)$.*

Proof. Let $\mathcal{S}$ be the family of subgroups of F containing only H and the trivial subgroup. Note that isomorphism classes of F-equivariant complex line bundles correspond to isomorphism classes of $(F, S^1 = \mathrm{U}(1))$-bundles. Let $\pi\colon E \to X^3$ be an F-equivariant complex line bundle over and let $[\alpha_H\colon H \to \mathrm{U}(1) = S^1]$ be the isomorphism class in $\mathrm{Rep}_{S^1}(H)$ of the H-representation induced on the fibers of π. If we set $\alpha_{\{e\}}\colon \{e\} \to S^1$, then $A = ([\alpha_K])_{K\in\mathcal{S}} \in \lim_{K\in\mathcal{S}} \mathrm{Rep}_{S^1}(K)$. It follows from Lemma 2.2 for $\Gamma = S^1$, that in order to show that π is the pullback of an F-equivariant complex line bundle over $E(F/H)$ along the inclusion $i\colon X^3 \to E(F/H)$, it suffices to show that every F-map from X^3 to $B_{\mathcal{S}}(F, A)$ can be extended to an F-map from $E(F/H)$ to $B_{\mathcal{S}}(F, A)$. Here $B_{\mathcal{S}}(F, A)$ is homotopy equivalent to $BS^1 = \mathbb{C}P^\infty$ for all $K \in \mathcal{S}$, again by Lemma 2.2. It follows from Bredon's equivariant obstruction theory (see [2, Section II.1],[15, Th. I.5.1]) that the potential obstructions for extending such a map lie in the relative Bredon cohomology groups $\mathrm{H}^{n+1}_F(E(F/H), X^3; \pi_n(\mathrm{B}_{\mathcal{S}}(F, A)^-))$ for $n \geq 3$. Since $\pi_n(\mathbb{C}P^\infty)$ is zero unless $n = 2$, the lemma is proven. □

The idea for the following lemma is contained in [12, p 596].

Lemma 3.3. *There exists an $n \geq 1$ such that $[\lambda]$ is not contained in the image of the edge homomorphism*

$$\mathrm{K}^0_F(X^n) \to R(H)^{F/H}.$$

Proof. By [7, Theorem 5.1] for $X = \{*\}$, $\mathcal{F} = \{e, H\}$ and $E\mathcal{F} = E(F/H)$, there are maps

$$\alpha_n\colon\ R(F)/I^n \to \mathrm{K}^0_F(X^n)$$

that induce a map of inverse systems from $\{R(F)/I^n\}_{n\geq 0}$ to $\{\mathrm{K}_F^0(X^n)\}_{n\geq 0}$ that induces an isomorphism of pro-rings. Here I is the kernel of the restriction map $R(F) \to R(H)$. This implies that for sufficiently large $n \geq 1$ there exists a map $\beta_1\colon \mathrm{K}_F^0(X^n) \to R(F)/I$ making the following diagram commute

$$\begin{array}{ccccc} R(F)/I^n & \xrightarrow{\alpha_n} & \mathrm{K}_F^0(X^n) & & \\ \downarrow & \swarrow^{\beta_1} & \downarrow & \searrow^{\varepsilon_F} & \\ & & & & R(H)^{F/H} \\ & & & \nearrow_{\varepsilon_F} & \\ R(F)/I & \xrightarrow{\alpha_1} & \mathrm{K}_F^0(X^1). & & \end{array}$$

This shows that the image of the restriction map

$$R(F) \to R(H)^{F/H}$$

coincides with the image of the edge homomorphism

$$\mathrm{K}_F^0(X^n) \to R(H)^{F/H}.$$

Since $[\lambda]$ does not lie in the image of $\mathrm{R}(F) \to \mathrm{R}(H)^{F/H}$ by Lemma 3.1, the lemma follows. $\square$

Let $n \geq 3$. By [8, Th. A & Th. 8.3] there exists a compact n-dimensional locally CAT(0)-cubical complex T_{X^n} equipped with a free cellular F/H-action and an F/H-equivariant map $t_{X^n}\colon T_{X^n} \to X^n$ that induces an isomorphism

$$\mathcal{H}_F^*(X^n) \xrightarrow{\cong} \mathcal{H}_F^*(T_{X^n}) \tag{3}$$

for any equivariant cohomology theory $\mathcal{H}_?^*(\cdot)$ (e.g. see [11, section 1]). (We remark that [8, Th. 8.3] is stated for equivariant *homology* theories, but the analogous statement holds for equivariant *cohomology* theories by essentially the same proof.) The action of F on T_{X^n} in the above is via the projection $F \to F/H$. Now let Y^n be the universal cover of T_{X^n} and let Γ_n be the group of self-homeomorphisms of Y^n that lift the action of F/H on T_{X^n}. Since F/H acts freely on T_{X^n}, Γ_n acts freely on Y^n. We conclude that Y^n is an n-dimensional CAT(0)-cubical complex on which Γ_n acts freely, cocompactly and cellularly. Since Y_n is contractible, this implies that Γ_n is torsion-free. By construction there is a surjection $\Gamma_n \to F/H$ whose kernel N_n is the torsion-free group of deck transformation of the covering $Y^n \to T_{X^n}$. Now define the group G_n to be the pullback of $\pi_n\colon \Gamma_n \to F/H$ along $F \to F/H$. Then G_n acts on Y^n via the quotient map $G_n \to G_n/H = \Gamma_n$ and fits into the short exact sequence

$$1 \to N_n \to G_n \xrightarrow{p_n} F \to 1.$$

Note that the only non-trivial finite subgroup of G_n is $H \cong C_2$ and that since N_n acts freely on Y^n, the G_n-equivariant quotient map $Y^n \to N_n \setminus Y^n = T_{X^n}$ induces an isomorphism ([12, Lemma 3.5])

$$\mathrm{K}_F^*(T_{X^n}) \xrightarrow{\cong} \mathrm{K}_{G_n}^*(Y^n). \tag{4}$$

Applying (3) and (4) to the compostion $Y^n \to T_{X^n} \to X^n$ and the equivariant cohomology theories $\mathrm{K}^*_?(\cdot)$ and $\mathrm{H}^*_?(\cdot, R(-))$ with $* = 0$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{K}^0_F(X^n) & \xrightarrow{\cong} & \mathrm{K}^0_{G_n}(Y^n) \\ \downarrow{\scriptstyle \varepsilon_F} & & \downarrow{\scriptstyle \varepsilon_{G_n}} \\ R(H)^{F/H} & \xrightarrow{\cong} & \lim_{G_n/S \in \mathcal{O}_{\mathcal{F}} G_n} R(S). \end{array}$$

The fact that this diagram commutes can be seen as follows. Using equivariant cellular approximation, we may assume that the map $X^n \to Y^n$ is cellular. By considering the inclusion of zero-skeleta in n-skeleta, naturality yields a commutative diagram

$$\begin{array}{ccc} \mathrm{K}^0_F(X^n) & \xrightarrow{\cong} & \mathrm{K}^0_{G_n}(Y^n) \\ \downarrow & & \downarrow \\ \mathrm{K}^0_F(X^0) & \longrightarrow & \mathrm{K}^0_{G_n}(Y^0). \end{array}$$

The edge homomorphism $\varepsilon_F\colon \mathrm{K}^0_F(X^n) \to R(H)^{F/H} \subseteq \mathrm{K}^0_F(X^0)$ coincides by construction with $\mathrm{K}^0_F(X^n) \to \mathrm{K}^0_F(X^0)$ once we restrict the codomain, and similarly for ε_{G_n}. Therefore, commutativity follows.

Since we proved in Lemma 3.3 that, for n large enough, the isomorphism class of λ does not lie in the image of the edge homomorphism

$$\mathrm{K}^0_F(X^n) \to R(H)^{F/H}$$

it follows from the commutative diagram above that the compatible system of representations

$$(\lambda \circ p_{n|S})_{S \in \mathcal{F}} \in \lim_{G_n/S \in \mathcal{O}_{\mathcal{F}} G_n} R(S) = \mathrm{H}^0_{\mathcal{F}}(G_n, R(-)).$$

does not lie in the image of the edge homomorphism

$$\varepsilon_{G_n}\colon \mathrm{K}^0_{G_n}(Y^n) \to \lim_{G_n/S \in \mathcal{O}_{\mathcal{F}} G_n} R(S).$$

Recall from [3] that non-empty CAT(0)-cube complexes are contractible and that the fixed point set for a finite group action on a CAT(0)-cube complex is contractible. Since G_n acts cellularly properly and cocompactly on the CAT(0)-cube complex Y_n, we deduce that Y_n is a cocompact model for $\underline{E}G_n$. To summarize, we have constructed a group $G = G_n$ with a cocompact classifying space for proper actions $\underline{E}G$ admitting a compatible collection of complex representations of the finite subgroups of G that cannot be realized as G-equivariant virtual complex vector bundle over $\underline{E}G$.

We remark that Wolfgang Lück has shown us another quite different way to find a finite group F and an F-CW-complex X that satisfy Lemma 3.3; any such pair could be used to construct a group with similar properties to the group $G = G_n$.

4. Real vector bundles

One could apply the techniques of the previous section in the real setting to obtain a group G with cocompact classifying space for proper actions $\underline{E}G$ so that the edge homomorphism

$$\varepsilon_G\colon \mathrm{KO}^0_G(\underline{E}G) \to \lim_{G/H \in \mathcal{O}_{\mathcal{F}} G} RO(H)$$

is not surjective. Here one would need the real version of [7, Theorem 5.1], which also holds as explained in the paragraph below [7, Theorem 5.1].

Instead we give an explicit description of a group G that admits $\mathbb{R}^2$ as a cocompact model for $\underline{E}G$ and admits a compatible collection of real representations of its finite subgroups that cannot be realized as a real G-vector bundle over $\mathbb{R}^2$.

We start by describing a related group Γ that is a 2-dimensional crystallographic group, or wallpaper group; this group is known as $p2gg$, but we will describe it explicitly. Endow $\mathbb{R}^2$ with the CW-structure coming from the standard tesselation by unit squares with vertices at $\mathbb{Z}^2$, and let Γ be the group of automorphisms of this CW-structure that preserves the pattern shown in Figure 1. The stabilizer of a 2-cell is clearly trivial, and so the 2-cells form a single free Γ-orbit. There are two orbits of 1-cells, the vertical and horizontal edges, and again each orbit is free. There are two orbits of 0-cells, and the stabilizer of a 0-cell is cyclic of order two, generated by the rotation of order two fixing the point. Since the stabilizer of each cell acts trivially on that cell, the given CW-structure makes $\mathbb{R}^2$ into a Γ-CW-complex.

The translation subgroup T of Γ has index four, and consists of the elements $(x, y) \mapsto (x + 2m, y + 2n)$. The orientation-preserving subgroup N of Γ has index two, and consists of T together with the rotations through π about some point of $\mathbb{Z}^2$, which are of the form $(x, y) \mapsto (2m - x, 2n - y)$. Finally the elements of $\Gamma - N$ are the glide reflections whose axes bisect the sides of the 2-cells: $(x, y) \mapsto (2m + 1 - x, 2n + 1 + y)$ and $(x, y) \mapsto (2m + 1 + x, 2n + 1 - y)$. The quotients $T\backslash\mathbb{R}^2$, $N\backslash\mathbb{R}^2$ and $\Gamma\backslash\mathbb{R}^2$ are respectively a torus consisting of four squares, an S^2 obtained by identifying the boundaries of two squares, and a copy of $\mathbb{R}P^2$ obtained by identifying the edges of a square in pairs. The fact that $\Gamma - N$ contains no torsion elements is reflected in the fact that Γ/N acts freely on the sphere $N\backslash\mathbb{R}^2$.

Now let F be a copy of C_4 and let $H \cong C_2$ be the index two subgroup of F. The group G is defined as the pullback of the two maps $\Gamma \to \Gamma/N \cong C_2$ and $F \to F/H \cong C_2$. By construction the group G admits $\mathbb{R}^2$ as a cocompact model for $\underline{E}G$, and fits into a short exact sequence

$$1 \to N \to G \xrightarrow{p} F \to 1$$

such that every finite subgroup of G maps onto a subgroup of H under p.

Now let

$$\lambda : H \to O(1, \mathbb{R}) = C_2$$

be the 1-dimensional real sign representation of H, i.e. λ is the identity map. The isomorphism class $[\lambda]$ is clearly contained in $RO(H)^{F/H}$, since F is abelian.

Lemma 4.1. *The isomorphism class $k[\lambda]$ is contained in the image of the restriction map*

$$RO(F) \to RO(H)^{F/H}.$$

if and only if k is even.

Proof. Recall that the irreducible real representations of C_4 are up to isomorphism the one-dimensional trivial representation representation, the one dimensional sign representation of $F/H = C_2$ and one 2-dimensional faithful representation in which the elements of order four act as rotations by $\pm\frac{\pi}{2}$. The restriction of the first two of the representations to H gives the trivial one dimensional representation of H, while the restriction to H of the third is $\lambda \oplus \lambda$. We therefore conclude that the image of $RO(F) \to RO(H)^{F/H}$ consists of element of the form $2n[\lambda] + m[\mathrm{tr}]$, where tr is the trivial one dimensional representation of H and $n, m \in \mathbb{Z}$.

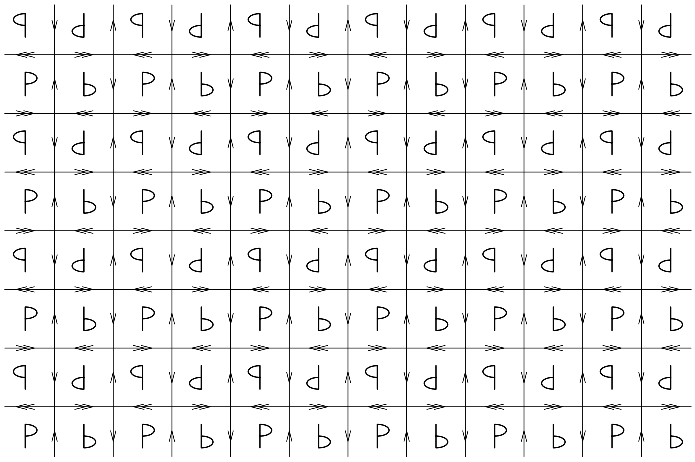

FIGURE 1. A wallpaper pattern for $\Gamma = p2gg$

This shows that $k[\lambda]$ is contained in the image of the restriction map $RO(F) \to RO(H)^{F/H}$ if and only if k is even. $\square$

Lemma 4.2. *Let F act on the infinite dimensional sphere S^∞ by first projecting onto $F/H = C_2$ and then acting via the antipodal map. View S^2 as the 2-skeleton of S^∞. Every F-equivariant orthogonal real line bundle over S^2 is isomorphic to the pullback of an F-equivariant orthogonal real line bundle over S^∞ along the inclusion $S^2 \to S^\infty$.*

Proof. Let $\mathcal{S}$ be the family of subgroups of F containing H and the trivial subgroup. Note that isomorphism classes of F-equivariant orthogonal real line bundles correspond to isomorphism classes of (F, C_2)-bundles. Now let ξ be an (F, C_2)-bundle over S^2 with fibers $A = (\xi_S) \in \lim_{S \in \mathcal{S}} \mathrm{Rep}_{C_2}(S)$. By Lemma 2.2, it suffices to show that every F-map $f\colon S^2 \to \mathrm{B}_{\mathcal{S}}(F, A)$ can be extended to an F-map $\tilde{f}\colon S^\infty \to \mathrm{B}_{\mathcal{S}}(F, A)$. Again by Lemma 2.2, $\mathrm{B}_{\mathcal{S}}(F, A)^S \cong BC_2 = \mathbb{R}P^\infty$ for all $S \in \mathcal{S}$. It follows from Bredon's equivariant obstruction theory (see [2, Section II.1],[15, Th. I.5.1]) that the potential obstructions for extending such a map lie in the relative Bredon cohomology groups $\mathrm{H}^{n+1}_F(S^\infty, S^2; \pi_n(\mathrm{B}_{\mathcal{S}}(F, A)^-))$ for $n \geq 2$. Since $\pi_n(\mathbb{R}P^\infty)$ is zero unless $n = 1$, the lemma is proven. $\square$

Lemma 4.3. *Let F act on S^2 by first projecting onto $F/H = C_2$ and then acting via the antipodal map. There does not exist a real F-vector bundle $\xi\colon E \to S^2$ such that the representation of H on the fibers of ξ is isomorphic to λ.*

Proof. Consider the infinite dimensional sphere S^∞ as a the universal C_2-space EC_2, where C_2 acts via the antipodal map and let F act on S^∞ via first projection onto $F/H = C_2$ and then acting via C_2. Now assume that there exists a real F-vector bundle $\xi\colon E \to S^2$ such that the representation of H on the fibers of ξ is isomorphic to λ. By Lemma 4.2 there exists a real F-vector bundle $\xi\colon E \to S^\infty$ such that the representation of H on the fibers of ξ is isomorphic to λ. By pulling back this bundle along the inclusion $S^n \to S^\infty$, there also exists a real F-vector bundle $\xi\colon E \to S^n$ such that the representation of H on the fibers of ξ is isomorphic to λ, for every $n \geq 2$.

By the real version of [7, Theorem 5.1] (see comments below [7, Theorem 5.1]), there are maps

$$\alpha_n\colon RO(F)/I^n \to \mathrm{KO}^0_F(S^n)$$

that induce a map of inverse systems from $\{RO(F)/I^n\}_{n\geq 0}$ to $\{\mathrm{KO}^0_F(S^n)\}_{n\geq 0}$ that in turn induces an isomorphism of pro-rings. Here I is the kernel of the restriction map $RO(F) \to RO(H)$. This implies that for sufficiently large $n \geq 1$ there exists a map $\beta_1\colon \mathrm{KO}^0_F(S^n) \to R(F)/I$ making the following diagram commute

$$\begin{array}{ccccc} RO(F)/I^n & \xrightarrow{\alpha_n} & \mathrm{KO}^0_F(S^n) & \xrightarrow{\varepsilon_F} & \\ \downarrow & \swarrow^{\beta_1} & \downarrow & & RO(H)^{F/H} \\ RO(F)/I & \xrightarrow{\alpha_1} & \mathrm{KO}^0_F(S^1). & \xrightarrow{\varepsilon_F} & \end{array}$$

This shows that the image of the restriction map

$$RO(F) \to RO(H)^{F/H}$$

coincides with the image of the edge homomorphism

$$KO^0_F(S^n) \to RO(H)^{F/H},$$

implying that the H-representations coming from the fibers of any real F-vector bundle over S^n can be extended to virtual F-representations. However, since λ does not lie in the image of $\mathrm{RO}(F) \to \mathrm{RO}(H)$ by Lemma 4.1 we arrive at a contradiction and conclude that there does not exist a real F-vector bundle $\xi\colon E \to S^2$ such that the representation of H on the fibers of ξ is isomorphic to λ. $\square$

Consider the projection $p\colon G \to F$ and the compatible system of real orthogonal representations

$$([\lambda \circ p_{|S}])_{S\in\mathcal{F}} \in \lim_{G/S\in\mathcal{O}_\mathcal{F}G} RO(S) = \mathrm{H}^0_G(\underline{E}G, RO(-)),$$

and assume that there exists a real G-vector bundle $\xi\colon E \to \mathbb{R}^2$ that realizes it. Since the kernel of $p\colon G \to F$ is N, it follows from the lemma below and our observations above that $N \setminus \xi\colon N \setminus E \to N \setminus X$ is an F-vector bundle over S^2, where F acts on S^2 via projection onto $F/H = C_2$, followed by the antipodal map. Morever, the representation of H on the fibers of $N \setminus \xi$ is by construction exactly λ. This however contradicts Lemma 4.3, so we conclude that there does not exist a real G-vector bundle $\xi\colon E \to \mathbb{R}^2$ that realizes the compatible system of real orthogonal representations $(\lambda \circ p_{|S})_{S\in\mathcal{F}}$.

Lemma 4.4. *Let G be any discrete group with normal subgroup N and let X be a proper G-CW-complex. If $\xi\colon E \to X$ is a G-vector bundle over X such that $N \cap G_x$ acts trivially on $\xi^{-1}(x)$ for every $x \in X$, then*

$$N \setminus \xi\colon N \setminus E \to N \setminus X$$

is a G/N-vector bundle over $N \setminus X$.

Proof. Denote the projection $G \to G/N = Q$ by π. Let us first consider the case where ξ is trivial (trivial in the sense of [10, Section 6.1.]), i.e. assume ξ is a pullback

$$\begin{array}{ccc} G \times_H V & \longrightarrow & G/H \\ {\scriptstyle r}\uparrow & & \uparrow{\scriptstyle p} \\ E & \xrightarrow{\ \xi\ } & X \end{array}$$

of the G-vector bundle $G \times_H V \to G/H$ along the G-map $p\colon X \to G/H$ where H is some finite subgroup of G and V is a finite dimensional real H-representation such that $H \cap N$ acts trivially on V. Consider the pullback diagram

$$\begin{array}{ccc} Q \times_{\pi(H)} V & \longrightarrow & Q/\pi(H) \\ {\scriptstyle w}\uparrow & & \uparrow{\scriptstyle N\backslash p} \\ P & \xrightarrow{\ q\ } & N \setminus X \end{array}$$

of the Q-vector bundle $Q \times_{\pi(H)} V \to Q/\pi(H)$ along the Q-map $N \setminus p\colon N \setminus X \to Q/\pi(H)$. We define the map

$$\psi\colon N \setminus E \to P : \overline{(g, v, x)} \mapsto (\pi(g), v, \overline{x}).$$

It is easy to check that ψ yields a well-defined morphism of Q-equivariant bundles over $N \setminus X$. Moreover, since ψ is a fiberwise linear map of Q-vector bundles that is a fiberwise isomorphism, it follows that ψ is a homeomorphism.

Now consider the general case. Let $\overline{x} \in N \setminus X$. Since $\xi\colon E \to X$ is locally trivial, $x \in X$ has an open G-neighourhood U such that there is a G-map $p\colon U \to G/H$ where H is finite subgroup of G and $\xi_{|U}$ is (homeomorphic to) the pullback

$$\begin{array}{ccc} G \times_H V & \longrightarrow & G/H \\ \uparrow & & \uparrow{\scriptstyle p} \\ \xi_{|U} & \xrightarrow{\ \xi\ } & U \end{array}$$

of the G-vector bundle $G \times_H V \to G/H$ along the G-map $p\colon U \to G/H$. By the above, the quotient diagram

$$\begin{array}{ccc} Q \times_{\pi(H)} V & \longrightarrow & Q/\pi(H) \\ \uparrow & & \uparrow{\scriptstyle N\backslash p} \\ N \setminus \xi_{|U} & \xrightarrow{\ N\backslash \xi\ } & N \setminus U \end{array}$$

is a pullback diagram. Since $N \setminus U$ is an open Q-neighbourhood of $\overline{x}$, it follows that $N \setminus \xi\colon N \setminus E \to N \setminus X$ is a Q-vector bundle. □

We finish this section by explaining how a similar approach to the one above can be used to produce a group G admitting a three dimensional cocompact model for $\underline{E}G$ that has a compatible system of one-dimensional complex representations that cannot be realized as a complex G-vector bundle over $\underline{E}G$. As in Section 3, let $F = C_4 \rtimes C_2$ be the dihedral group of order 8 where σ is a generator for C_4. Let $H = \langle \sigma^2 \rangle$ be the center of F, which has order two and denote the 3-skeleton of the universal F/H-space $X = E(F/H)$ by X^3. We let F act on X and X^3 via the projection onto F/H. Consider the complex 1-dimensional representation

$$\lambda\colon\ H = \langle \sigma^2 \rangle \to \mathrm{U}(1) = S^1 : \sigma^2 \mapsto -1.$$

By [8, Th. A & Th. 8.3] there exists a compact 3-dimensional locally CAT(0)-cubical complex T_{X^3} equipped with a free cellular F/H-action, an F/H-equivariant map $t_{X^3}\colon\ T_{X^3} \to X^3$ and an isometric cellular involution τ on T_{X^3} that commutes with the F/H-action on T_{X^3} and the map t_{X^3} such the induced F/H-equivariant map

$$\langle \tau \rangle \setminus T_{X^3} \to X^3$$

is a homotopy equivalence. Note that F/H acts freely on $\langle \tau \rangle \setminus T_{X^3}$ since it acts freely on X^3. Hence T_{X^3} is also the 3-skeleton of a universal F/H-space Z. So we may continue assuming that $Z = X$ and $\langle \tau \rangle \setminus T_{X^3} = X^3$.

Now let Y be the univeral cover of T_{X^3} and let Γ be the group of self-homeomorphism of Y that lifts the action of $F/H \oplus \langle \tau \rangle$ on T_{X^3}. Then Y is a 3-dimensional CAT(0)-cubical complex on which Γ acts properly, compactly and cellularly. By construction there is a surjection $\alpha\colon\ \Gamma \to F/H \oplus \langle \tau \rangle$ whose kernel $\mathrm{Ker}(\alpha)$ is the torsion-free group of deck transformations of $Y \to T_{X^3}$. Let π denote the composition of α with the projection of $F/H \oplus \langle \tau \rangle$ onto F/H. Since F/H acts freely on T_{X^3} and every finite subgroup of Γ must fix a point of Y since Y is CAT(0), it follows that every finite subgroup of Γ is contained in the kernel of π, which we denote by N. Now define the group G to be the pullback of $\pi\colon\ \Gamma \to F/H$ along $F \to F/H$. Then G acts on Y via the quotient map $G \to G/H = \Gamma$ that fits into the short exact sequence

$$1 \to N \to G \xrightarrow{p} F \to 1.$$

such that p maps all the finite subgroup of G onto a finite subgroup of H and $N \setminus Y = X^3$.

Let $\mathcal{F}$ be the family of finite subgroups of G, note that Y is a three dimensional cocompact model for $\underline{E}G$ and suppose that there exists a G-vector $\xi\colon\ E \to Y$ whose fibers give rise to the compatible system of representations

$$([\lambda \circ p_{|S}])_{S \in \mathcal{F}} \in \lim_{G/S \in \mathcal{O}_{\mathcal{F}} G} R(S).$$

Applying Lemma 4.4, we obtain an F-equivariant complex line bundle $N \setminus \xi\colon\ N \setminus E \to X$ such that the representation of H on the fibers of $N \setminus \xi$ is isomorphic to λ. By Lemma 3.2, this bundle can be extended to an F-equivariant complex line bundle over $X = E(F/H)$. We now continue in a similar fashion as in the proof of Lemma 4.3 to conclude that $[\lambda]$ is contained in the image of the restriction map $R(F) \to R(H)^{F/H}$, which contradicts Lemma 3.1. We conclude that the bundle ξ cannot exist.

5. Right angled Coxeter groups

Let Γ be a finite graph. We denote the vertex set of Γ by $S = V(\Gamma)$ and the set edges of Γ by $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. The right angled Coxeter group determined by Γ is the Coxeter group W with presentation

$$W = \langle S \mid s^2 \text{ for all } s \in V(\Gamma) \text{ and } (st)^2 \text{ if } (s,t) \in E(\Gamma) \rangle.$$

Note that W fits into the short exact sequence

$$1 \to N \to W \xrightarrow{p} F = \bigoplus_{s \in S} C_2 \to 1$$

where p takes $s \in S$ to the generator of the C_2-factor corresponding to s. A subset $J \subseteq S$ is called spherical if the subgroup $W_J = \langle J \rangle$ is finite (and hence isomorphic to $\bigoplus_{s\in J} C_2$). The empty subset of S is by definition spherical. We denote the poset of spherical subsets of S orderded by inclusion by $\mathcal{S}$. If $J \in \mathcal{S}$, then W_J is called a spherical subgroup of W, while a coset wW_J is called spherical coset. We denote the poset of spherical cosets, ordered by inclusion, by $W\mathcal{S}$. Note that W acts on $W\mathcal{S}$ by left multiplication, preserving the ordering. The Davis complex Σ of W is the geometric realization of $W\mathcal{S}$. One easily sees that Σ is a proper cocompact W-CW-complex. Since Σ admits a complete CAT(0)-metric such that W acts by isometries, it follows that Σ is a cocompact model for $\underline{E}W$ (see [4, Th. 12.1.1 & Th. 12.3.4]). A consequence of this fact is that every finite subgroup of W is subconjugate to some spherical subgroup of W. This implies that the group N defined above is torsion-free. Since the quotient space $W \setminus \Sigma$ is homeomorphic to the geometric realization of the poset $\mathcal{S}$, which is contractible since it has a minimal element, another consequence is that the quotient $\underline{B}W = W \setminus \underline{E}W$ is contractible. We refer the reader to [4] for more details and information about these groups and the spaces on which they act.

Let $\mathcal{F}$ be the family of finite subgroups of W. Given an abelian group A, we denote by

$$\underline{A}\colon\ \mathcal{O}_{\mathcal{F}}W \to \mathrm{Ab}$$

the trivial functor that takes all objects to A and all morphism to the identity map. One can verify that

$$\mathrm{H}^*_W(\underline{E}W, \underline{A}) \cong \mathrm{H}^*(\underline{B}W, A). \tag{5}$$

Lemma 5.1. *Let $A = ([p_{|H}])_{H\in\mathcal{F}} \in \lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} \mathrm{Rep}_F(H)$. For every $k \geq 0$, the contravariant functor*

$$\mathcal{O}_{\mathcal{F}}W \to \mathrm{Ab}\colon\ W/H \mapsto \pi_k(B_{\mathcal{F}}(W,A)^H)$$

equals the trivial functor $\underline{\pi_k(BF)}$.

Proof. Let EF be a contractible F-CW-complex with free F-action and consider the product space $\underline{E}W \times EF$. This space becomes a $(W \times K) - CW$-complex by letting $(w,f) \in W \times F$ act on $(x,y) \in \underline{E}W \times EF$ as

$$(w,f)\cdot(x,y) = (w\cdot x, p(w)f\cdot y).$$

One checks that with this action $\underline{E}W \times EF$ is a model for $E_{\mathcal{F}}(W,A)$, i.e. $(\underline{E}W \times EF)^K$ is contractible when $K \in \mathcal{F}_A$ and empty otherwise. By definition, it follows that $\underline{E}W \times BF$ is a model $B_{\mathcal{F}}(W,A)$, where W acts on trivially on the second coordinate. Since $\underline{E}W^H$ is contractible for every $H \in \mathcal{F}$, the lemma follows easily.

□

Let Γ be either the orthogonal group $\mathrm{O}(n,\mathbb{R})$ or the unitary group $\mathbf{U}(n)$.

Lemma 5.2. *Every element of*

$$\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} \mathrm{Rep}_{\Gamma}(H)$$

is of the form $([\lambda \circ p_{|H}])_{H\in\mathcal{F}}$ for some group homomorphism $\lambda\colon\ F \to \Gamma$.

Proof. Every finite subgroup H of W is isomorphic to a finite direct sum of C_2's. Since every element of order 2 in Γ is conjugate in Γ to a diagonal matrix with ± 1 on the diagonal and commuting matrices can be simultaneously diagonalized (e.g. see [6, Th. 1.3.12]), it follows that the image of every homomorphism $H \to \Gamma$ is conjugate to a finite subgroup of Γ consisting of diagonal matrices. Hence, every element of $\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W}\mathrm{Rep}_{\Gamma}(H)$ is of the form $([\alpha_H])_{H\in\mathcal{F}}$ where $\alpha_H\colon H\to\Gamma$ is a homomorphism whose image lands in the finite abelian subgroup of Γ consisting of diagonal matrices. Since every finite subgroup of W is subconjugate to a spherical subgroup W_J, the compatibility of the representations tells us that $([\alpha_H])_{H\in\mathcal{F}}$ is completely determined by the homomorphisms $\alpha_{\langle s\rangle}\colon\langle s\rangle\to\Gamma$, for $s\in S$. Since the images of the $\alpha_{\langle s\rangle}$ are diagonal, they commute. Therefore, one can define the homomorphism

$$\lambda\colon F=\bigoplus_{s\in S}C_2\to\Gamma:(\sigma_s)_{s\in S}\mapsto\sum_{s\in S}\alpha_{\langle s\rangle}(\sigma_s).$$

The compatibility of the representations implies that

$$([\lambda\circ p_{|H}])_{H\in\mathcal{F}}=([\alpha_H])_{H\in\mathcal{F}},$$

proving the lemma. □

The following theorem applies to both complex and real representations and vector bundles.

Theorem 5.3. *Let W be a right angled Coxeter group. Every compatible collection of representations of the finite subgroups of W can be realized as a W-equivariant vector bundle over the Davis complex $\Sigma=\underline{E}W$.*

Proof. Consider $A=([p_{|H}])_{H\in\mathcal{F}}\in\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W}\mathrm{Rep}_F(H)$. It follows from Lemma 2.2 that the existence of a (W,A)-bundle over Σ follows from the existence a W-map $\Sigma\to B_{\mathcal{F}}(G,A)$. Since by Lemma 5.1, the contravariant functor

$$\pi_k(B_{\mathcal{F}}(W,A)^{-})\colon\ \mathcal{O}_{\mathcal{F}}(W)\to\mathrm{Ab}:W/H\mapsto\pi_k(B_{\mathcal{F}}(W,A)^H)$$

equals the trivial functor $\underline{\pi_k(BF)}$ for all $k\geq 0$, it follows from (5) and the contractibility of $\underline{B}W$ that the Bredon cohomology groups

$$\mathrm{H}^{k+1}_W(\Sigma,\pi_k(B_{\mathcal{F}}(W,A)^{-}))$$

are zero for all $k\geq 0$. Since there certainly exists a W-map from the 0-skeleton of Σ to $B_{\mathcal{F}}(W,A)$, it follows from Bredon's equivariant obstruction theory that there exists a W-map $\Sigma\to B_{\mathcal{F}}(W,A)$.

Now consider a compatible collection of representations of the finite subgroups of W. By Lemma 5.2, this collection is of the form

$$([\lambda\circ p_{|H}])_{H\in\mathcal{F}}\in\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W}\mathrm{Rep}_{\Gamma}(H)$$

for some group homomorphism $\lambda\colon F\to\Gamma$. Letting $A=([p_{|H}])_{H\in\mathcal{F}}$, it follows from the above that there exists a (W,A)-bundle $\xi\colon E\to\Sigma$. If $\Gamma=\mathrm{O}(n,\mathbb{R})$ then

$$\xi\colon\ E\times_F\mathbb{R}^n\to\Sigma$$

is a real W-vector bundle over Σ that realizes $([\lambda\circ p_{|H}])_{H\in\mathcal{F}}$, and if $\Gamma=U(n)$ then

$$\xi\colon\ E\times_F\mathbb{C}^n\to\Sigma$$

is a complex W-vector bundle over Σ that realizes $([\lambda\circ p_{|H}])_{H\in\mathcal{F}}$. Here F acts on $\mathbb{R}^n$ or $\mathbb{C}^n$ via the map λ. □

Lemma 5.4. *If W is a right angled Coxeter group, then $\mathrm{H}^n_W(\Sigma,R(-))=0$ for all $n>0$, and $\mathrm{H}^0_W(\Sigma,R(-))$ is free abelian of rank equal to the number of spherical subgroups of W.*

Proof. This is proven in much the same way as the corresponding result for homology in [17]. In more detail, one uses the cubical structure on Σ, in which there is one orbit of n-cubes with stabilizer isomorphic to $(C_2)^n$ for each n-tuple of commuting elements of S. (For each $n \geq 0$, for each spherical subgroup $W_J \cong (C_2)^n$ and for each $w \in W$, the subposet consisting of all special cosets contained in wW_J is order isomorphic to the poset of faces of an n-cube. Furthermore this isomorphism is equivariant for the stabilizer subgroup $wW_Jw^{-1} \cong (C_2)^n$, acting on the n-cube as the group generated by reflections in its coordinate planes. The realizations of these subposets are the cubes that make up the cubical structure on Σ. For more details concerning the cubical structure on Σ see [4, Ch. 1.1-1.2 or Ch. 7].) Since the stabilizer of a cube of strictly positive dimension acts non-trivially on the cube, this cubical structure is not a W-CW-structure on Σ. However, its barycentric subdivision is a simplicial complex naturally isomorphic to the realization of the poset $W\mathcal{S}$ as described in the introduction to this section.

Let Σ^n denote the n-skeleton of Σ with the cubical structure. Firstly, Σ^0 consists of a single free W-orbit of vertices, so $\mathrm{H}^*_W(\Sigma^0; R(-))$ is isomorphic to the ordinary cohomology of a point; since W acts freely the calculation reduces to an equivariant cohomology calculation for the trivial group action.

Let $I = [-1, 1]$ be an interval, with C_2 acting by $x \mapsto -x$ (i.e., swapping the ends of the interval). Note that I is equivariantly isomorphic to the Davis complex for the Coxeter group C_2. Let ∂I denote the two end points $\{-1, 1\}$. Make I into a C_2-CW-complex, for example by taking three 0-cells in two orbits at the points -1, 0 and 1, and one free orbit of 1-cells consisting of the two intervals $[-1, 0]$ and $[0, 1]$. The cellular C_2-Bredon cochain complex for the pair $(I, \partial I)$ with coefficients in $R(-)$ is a cochain complex of free abelian groups in which the degree zero term has rank two, the degree one term has rank one, and all other terms are trivial. A direct computation with this cochain complex shows that $H^m_{C_2}(I, \partial I; R(-))$ is isomorphic to $\mathbb{Z}$ for $m = 0$ and is zero for $m > 0$.

Next consider I^n with C_2^n acting as the direct product of n copies of the above action of C_2 on I. This is the Davis complex for the Coxeter group C_2^n. Since the representation ring of a direct product of finite groups is naturally identified with the tensor product of the representation rings [19, Ch. 3.2], the C_2^n-Bredon cochain complex for the pair $(I^n, \partial I^n)$ with coefficients in $R(-)$ is naturally isomorphic to the tensor product of n copies of the C_2-Bredon cochain complex for $(I, \partial I)$ with coefficients in $R(-)$. (If one wants to think about this cochain complex geometrically, it arises from the $(C_2)^n$-CW-structure on I^n in which the cells are the direct products of the cells arising in the C_2-CW-structure on I.) Since these cochain complexes consist of finitely generated free abelian groups, there is a Künneth formula as described in for example [16, Thrm 60.3]. Since $H^*_{C_2}(I, \partial I; R(-))$ is free abelian the Künneth formula implies that

$$H^*_{C_2^n}(I^n, \partial I^n, R(-)) \cong \bigotimes_{i=1}^{n} H^*_{C_2}(I, \partial I; R(-)).$$

It follows that for each n, $H^m_{C_2^n}(I^n, \partial I^n; R(-))$ is isomorphic to $\mathbb{Z}$ for $m = 0$ and is zero for $m > 0$.

From these computations, it follows easily that $H^m_W(\Sigma^n, \Sigma^{n-1}; R(-))$ is zero for $m > 0$ and is isomorphic to a direct sum of copies of $\mathbb{Z}$ indexed by the W-orbits of n-cubes in Σ. By induction on n one sees that $H^m_W(\Sigma^n; R(-))$ is zero for $m > 0$ and isomorphic to a direct sum of copies of $\mathbb{Z}$ indexed by the W-orbits of cubes of dimension at most n for $m = 0$. The claimed result follows, since the W-orbits of cubes in Σ are in bijective correspondence with the spherical subgroups of W. $\square$

Before stating our theorem concerning $\mathrm{K}_W^*(\underline{E}W)$, we make some remarks concerning the representation ring of a direct sum of copies of the cyclic group C_2, indexed by a (finite) set S. For any finite group G, the collection of all isomorphism types of 1-dimensional complex representations of G is an abelian group, with product given by taking the tensor product of representations. Furthermore, this group is naturally isomorphic to the group $\mathrm{Hom}(G, \mathrm{U}(1))$. In the case when G is abelian, every irreducible representation of G is 1-dimensional, and so $\mathrm{Hom}(G, \mathrm{U}(1))$ forms a basis for the additive group of the representation ring. In this way the representation ring $R(G)$ is naturally isomorphic to the integral group algebra of the group $\mathrm{Hom}(G, \mathrm{U}(1))$. In the case when $G = \bigoplus_{s\in S} C_2$ is a direct sum of copies of C_2 indexed by S, we may view G as a vector space over the field of two elements, in which case $\mathrm{Hom}(G, \mathrm{U}(1))$ may be identified with the dual space. For $s \in S$, let s^* denote the 1-dimensional representation of G with the properties that $s^*(s) = -1$ and $s^*(t) = 1$ for $t \in S - \{s\}$. let S^* denote the set of these representations: $S^* := \{s^* \mid s \in S\}$. In terms of vector spaces over the field of two elements, $S^* \subseteq \mathrm{Hom}(G, \mathrm{U}(1))$ is the dual basis to the set $S \subseteq G$. The set S^* generates the representation ring of G, giving rise to the following presentation:

$$R(G) = \mathbb{Z}[S^*]/({s^*}^2 - 1 \mid s \in S),$$

in which the monomials $s_1^* s_2^* \cdots s_k^*$ for all subsets $\{s_1, \dots, s_k\} \subseteq S$ correspond to the irreducible representations.

Suppose now that J is a subset of S. The inclusion $J \subseteq S$ identifies $H = \bigoplus_{s\in J} C_2$ with a subgroup of $G = \bigoplus_{s\in J} C_2$. The induced map $R(G) \to R(H)$ of representation rings is described easily in terms of the above ring presentation: for $s \in J$, $s^* \in R(G)$ restricts to $s^* \in R(H)$, while for $s \notin J$, $s^* \in R(G)$ restricts to $1 \in R(H)$.

Now suppose that Γ is a graph with vertex set $V(\Gamma) = S$, and let W be the right angled Coxeter group associated to Γ. The abelianization of W is naturally identified with $G = \bigoplus_{s\in S} C_2$. There is a unique equivariant map $\alpha\colon \underline{E}W \to *$, from the W-space $\underline{E}W$ to a point $*$, viewed as a G-space with trivial action. If J is a spherical subset of S then $W_J = \bigoplus_{s\in J} C_2$ maps isomorphically to the corresponding subgroup of $G = \bigoplus_{s\in S} C_2$. If $x \in \underline{E}W$ is a 0-cell fixed by $W_J = \bigoplus_{s\in J} C_2$, then $\alpha(x) = *$, and this map is W_J-equivariant. The induced map $\alpha^*\colon \mathrm{K}_G^*(*) \to \mathrm{K}_W^*(\underline{E}W)$, and the composite map $\mathrm{K}_G^*(*) \to \mathrm{K}_{W_J}^*(\{x\})$ will be used in the statement and proof of our theorem. If we identify $R(G)$ with $\mathrm{K}_G^0(*)$ and $R(W_J)$ with $\mathrm{K}_{W_J}^0(\{x\})$, then the composite is identified with the restriction map.

Theorem 5.5. *Let W be the right angled Coxeter group determined by a finite graph Γ, with vertex set S, and let $G = \bigoplus_{s\in S}$ be the abelianization of W. The map $\alpha^*\colon \mathrm{K}_G^*(*) \to \mathrm{K}_W^*(\underline{E}W)$ is surjective in each degree. In particular, $\mathrm{K}_W^1(\underline{E}W) = 0$ and there is a ring isomorphism*

$$\mathrm{K}_W^0(\underline{E}W) \cong \mathbb{Z}[S^*]/({s^*}^2 - 1, s^*t^* - s^* - t^* + 1 \mid s \in S = V(\Gamma), (s,t) \notin E(\Gamma)).$$

It follows that $\mathrm{K}_W^0(\underline{E}W) \cong \mathbb{Z}^d$ as an abelian group, where d is the number of spherical subgroups of W.

Proof. Consider the Atiyah-Hirzebruch spectral sequence (1)

$$E_2^{p,q} = \mathrm{H}_W^p(\underline{E}W, \mathrm{K}_W^q(W/-)) \Longrightarrow \mathrm{K}_W^{p+q}(\underline{E}W)$$

where $\mathrm{K}^q_W(W/-) = R(-)$ if q is even and $\mathrm{K}^q_W(W/-) = 0$ if q is odd (see [12, Th. 3.2]). In the lemma above, we proved that $\mathrm{H}^k_W(\Sigma, R(-)) = 0$ for $k > 0$. It therefore follows that

$$\mathrm{K}^n_W(\underline{E}W) = \begin{cases} \mathrm{H}^0_W(\underline{E}W, R(-)) = \lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} R(H) & \text{if } n = 0 \\ 0 & \text{if } n = 1. \end{cases}$$

Let I be the ideal

$$(s^{*2} - 1, s^*t^* - s^* - t^* + 1 \mid s \in S,\ (s,t) \notin E(\Gamma))$$

in the polynomial ring $\mathbb{Z}[S^*]$. Note that as an abelian group $\mathbb{Z}[S^*]/I$ is free, with basis elements the commuting products $s^*_1 \ldots s^*_k$, for all $J = \{s_1 \ldots, s_k\} \in \mathcal{S}$ (The case $J = \emptyset$ corresponds to the unit of $\mathbb{Z}[V(\Gamma)]/I$). This shows that

$$\mathbb{Z}[S^*]/I \cong \mathbb{Z}^d$$

as an abelian group, where d is the number of spherical subgroups of W.

We claim there is an isomorphism of rings

$$\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} R(H) \cong \mathbb{Z}[S^*]/I.$$

Since every finite subgroup of W is subconjugate to a spherical subgroup of W, it follows that

$$\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} R(H) \cong \lim_{J\in\mathcal{S}} R(W_J)$$

as rings. By the remarks in the paragraph preceeding the statement of the theorem, there are ring isomorphisms

$$R(W_J) = \mathbb{Z}[J^*]/(s^{*2} - 1 \mid s \in J), \qquad R(G) = \mathbb{Z}[S^*]/(s^{*2} - 1 \mid s \in S),$$

which are natural for inclusions $J \subseteq J' \subseteq S$. From this it follows that the natural ring homomorphism

$$\rho\colon\ R(G) \to \lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} R(H)$$

is surjective, and that $\lim_{W/H\in\mathcal{O}_{\mathcal{F}}W} R(H)$ is isomorphic to the ring described in the statement; in particular its additive group is free abelian of the same rank as $\mathrm{K}^0_W(\underline{E}W)$. Since ρ factors through $\mathrm{K}^0_W(\underline{E}W)$, the claimed isomorphism follows. □

Before stating our corollary concerning $\mathrm{K}^*(BW)$, we recall some facts from [1] concerning $\mathrm{K}^*(BG)$, where as above $G = \bigoplus_{s\in S} C_2$. For any finite group H, Atiyah showed that $\mathrm{K}^i(BH) = 0$ for i odd, and that $\mathrm{K}^{2i}(BH)$ is naturally isomorphic to the completion of the representation ring $R(H)$ at its augmentation ideal. To discuss the case of G, it is convenient to take new generators for $R(G)$; replace the irreducible representation s^* by the degree zero virtual representation $\overline{s} = s^* - 1$. With respect to these generators one obtains the presentation

$$R(G) = \mathbb{Z}[\overline{S}]/(\overline{s}(\overline{s} + 2) \mid s \in S),$$

where $\overline{S} = \{\overline{s} \mid s \in S\}$. If $H = \bigoplus_{s\in J} C_2$, then of course there is a similar description of $R(H)$, which is natural for the inclusion $J \subseteq S$. Note that if $s \notin J$, then the image of $\overline{s}$ under the restriction map $R(G) \to R(H)$ is zero.

Completing $R(G)$, as described above, with respect to its augmentation ideal gives rise to the following presentation for the ring $\mathrm{K}^0(BG)$:

$$\mathrm{K}^0(BG) = \mathbb{Z}[[\overline{S}]]/(\overline{s}(\overline{s} + 2) \mid s \in S),$$

which is natural for the inclusion $J \subseteq S$, and so also describes the induced map $\mathrm{K}^0(BG) \to \mathrm{K}^0(BH)$. The additive group of this ring is the direct sum of one copy of $\mathbb{Z}$, generated by 1, and for each non-empty subset $J \subseteq S$, one copy of the 2-adic integers, $\mathbb{Z}_2$, consisting of the set of power series in the element $\prod_{s\in J} \overline{s}$ with zero constant term.

Corollary 5.6. *Let W be the right angled Coxeter group determined by a finite graph Γ with vertex set $S = V(\Gamma)$, and let $G = \bigoplus_{s\in S} C_2$ be the abelianization of W. The induced map $\mathrm{K}^*(BG) \to \mathrm{K}^*(BW)$ is surjective in each degree. In particular $\mathrm{K}^1(BW) = 0$ and there is a ring isomorphism*

$$\mathrm{K}^0(BW) \cong \mathbb{Z}[[\overline{S}]]/(\overline{s}(\overline{s}+2),\ \overline{s}\overline{t} \mid s \in S,\ (s,t) \notin E(\Gamma)).$$

Here, $\mathbb{Z}[[\overline{S}]]$ is the formal power series ring with $\mathbb{Z}$ coefficients in the variables $\overline{S} = \{\overline{s} \mid s \in S\}$.

Proof. The version of the Atiyah-Segal completion theorem that is proven for infinite discrete groups admitting a cocompact model for the classifying space for proper actions in [12, Theorem 4.4.(b)] implies that

$$\mathrm{K}^n(BW) = \mathrm{K}^n_W(\underline{E}W)_{\hat{J}},$$

where the ideal J is the kernel of the augmentation map $\mathrm{K}^n_W(\underline{E}W) \to \mathbb{Z}$ that maps vector bundles to their dimension. Changing variables in the above theorem to $\overline{s} = s^* - 1$, we see that $\mathrm{K}^i(BW) = 0$ for i odd and that $\mathrm{K}^0(BW)$ is the completion of the ring

$$\mathbb{Z}[\overline{S}]/(\overline{s}(\overline{s}+2),\ \overline{s}\overline{t} \mid s \in S,\ (s,t) \notin E(\Gamma))$$

with respect to the ideal generated by the set $\overline{S} = \{\overline{s} \mid s \in S\}$. This completion is the ring described in the statement. □

There is an alternative proof of Corollary 5.6 that does not use Theorem 5.5 or results from [12]. Instead one uses a description of W as a free product with amalgamation. If the graph Γ is not a complete graph, then there is an expression $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_3 = \Gamma_1 \cap \Gamma_2$, in which each Γ_i is a full subgraph of Γ and has fewer vertices than Γ. This gives an expression for W as a free product with amalgamation $W = W_1 *_{W_3} W_2$. From this one obtains a Mayer-Vietoris sequence that can be used to compute $\mathrm{K}^*(BW)$. To establish Corollary 5.6, one shows by induction on $|S|$ that $\mathrm{K}^*(BW)$ is as described and that for each $J \subseteq S$, the map $\mathrm{K}^*(BW) \to \mathrm{K}^*(BW_J)$ is a split surjection.

References

[1] Atiyah, M. F., *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math. 9 (1961) 23–64.
[2] Bredon, G. E., *Equivariant cohomology theories* , Lecture Notes in Mathematics 34, Springer (1967).
[3] Bridson, M. R. and Haefliger, A., *Metric spaces of non-positive curvature* Springer Verlag Vol. 319 (1999).
[4] Davis, M. W., *The geometry and topology of Coxeter groups*, Princeton University Press, vol. 32 (2008).
[5] Davis, J. F. and Lück, W., *Spaces over a category and assembly maps in isomorphism conjectures in K- and L- theory*, *K*-theory 15 (1998), 201–252.
[6] Horn, R. and Johnson, C. *Matrix analysis*, Cambridge university press (2012).
[7] Jackowski, S., *Families of subgroups and completion*, Journal of Pure and Applied Algebra 37 (1985), 167–179.
[8] Leary, I. J., *A metric Kan-Thurston theorem*, Journal of Topology 6(1) (2013), 251–284.
[9] Lück, W., *Survey on classifying spaces for families of subgroups*, Infinite Groups: Geometric, Combinatorial and Dynamical Aspects, Springer (2005), 269–322.
[10] Lück, W, *The Burnside ring and equivariant stable cohomotopy for infinite groups*, Journal of Pure and Applied Mathematics Quaterly 1(3) (2005), 479–541.

[11] Lück, W, *Equivariant cohomological Chern characters*, International Journal of Algebra and Computation 15 (2005), 1025–1052.
[12] Lück, W, and Oliver, B., *The completion theorem in K-theory for proper actions of a discrete group*, Topology 40 (2001), 585–616.
[13] Lück, W, and Oliver, B., *Chern characters for the equivariant K-theory of proper G-CW-complexes*, Cohomological methods in homotopy theory (2001), 217–247.
[14] Lück, W, and Uribe, B., *Equivariant principal bundles and their classifying spaces*, Algebraic and Geometric Topology 14 (2014), 1925-1995.
[15] May, J.P., *Equivariant homotopy and cohomology theory*, CBMS regional conferences series in mathematics 91.
[16] Munkres, J. R., *Elements of Algebraic Topology*, Addison-Wesley (1984).
[17] Sánchez-García, R., *Equivariant K-homology for some Coxeter groups*, Journal of the London Mathematical Society, 75(3), (2007), 773–790.
[18] Segal, G., Equivariant K-theory, Publ. IHES 34 (1968) 129–151.
[19] Serre, J.-P., *Linear representations of finite groups*, Graduate Text in Mathematics, Springer, Vol 42 (2012).
[20] t. Dieck, T., *Transformation groups*, Studies in Math. 8, de Gruyter (1987).