EQUIVARIANT VECTOR BUNDLES OVER CLASSIFYING SPACES FOR PROPER ACTIONS

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Abstract. Let $G$ be an infinite discrete group and let $EG$ be a classifying space for proper actions of $G$. Every $G$-equivariant vector bundle over $EG$ gives rise to a compatible collection of representations of the finite subgroups of $G$. We give the first examples of groups $G$ with a cocompact classifying space for proper actions $EG$ admitting a compatible collection of representations of the finite subgroups of $G$ that does not come from a $G$-equivariant (virtual) vector bundle over $EG$. This implies that the Atiyah-Hirzebruch spectral sequence computing the $G$-equivariant topological $K$-theory of $EG$ has non-zero differentials. On the other hand, we show that for right angled Coxeter groups this spectral sequence always collapses at the second page and compute the $K$-theory of the classifying space of a right angled Coxeter group.

1. Introduction

Let $G$ be an infinite discrete group and $\mathcal{F}$ be the family of finite subgroups of $G$. Recall that the orbit category $\mathcal{O}_G$ is a category whose objects are the transitive $G$-sets $G/H$, for all $H \in \mathcal{F}$, and whose morphism are all $G$-equivariant maps between the objects. A classifying space for proper actions of $G$, denoted by $EG$, is a proper $G$-CW-complex such that the fixed point set $EG^H$ is contractible for every $H \in \mathcal{F}$. The space $EG$ is said to be cocompact if the orbit space $G \setminus EG = BG$ is compact. Many interesting classes of groups $G$ have cocompact models for $EG$, for example cocompact lattices in Lie groups, mapping class groups of surfaces, $Out(F_n)$, CAT(0)-groups and word-hyperbolic groups. We refer the reader to [9] for more examples and details.

Now assume $G$ is an infinite discrete group admitting a cocompact classifying space for proper actions $EG$. If

$$\xi: E \to EG$$

is a $G$-equivariant complex vector bundle over $EG$ (see Definition 2.3) and $x$ is a point of $EG$, then the fiber $\xi^{-1}(x)$ is a complex representation of the finite isotropy group $G_x$. The connectivity of the fixed point sets of $EG$ ensures that these representations are compatible (see Definition 2.1) with one another as $x$ and hence $G_x$ varies. Therefore, every $G$-equivariant complex vector bundle over $EG$ gives rise to a compatible collection of complex representations of the finite subgroups of $G$, and hence to an element of

$$\lim_{G/H \in \mathcal{O}_G} R(H).$$

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Here, \( \lim_{G/H \in \mathcal{O}_F G} R(H) \) is the limit over the orbit category \( \mathcal{O}_F G \) of the contravariant representation ring functor

\[
R(-): \mathcal{O}_F G \to \text{Ab} \quad G/H \mapsto R(H).
\]

Denoting the Grothendieck group of the abelian monoid of isomorphism classes of complex \( G \)-vector bundles over \( EG \) by \( K^0_G(EG) \), one obtains a map

\[
\varepsilon_G: K^0_G(EG) \to \lim_{G/H \in \mathcal{O}_F G} R(H)
\]

that maps a formal difference of (isomorphism classes) vector bundles (i.e. a virtual vector bundle) to a formal difference of (isomorphism classes) of compatible collections of representations of the finite subgroups of \( G \). We say a compatible collection of representations of the finite subgroups of \( G \) can be realized as a (virtual) \( G \)-equivariant vector bundle over \( EG \) if there exists a (virtual) \( G \)-equivariant vector bundle over \( EG \) that maps to this collection under \( \varepsilon_G \). One can also look at the corresponding situation for real (orthogonal) vector bundles and real (orthogonal) representations and obtain the map

\[
\varepsilon_G: KO^0_G(EG) \to \lim_{G/H \in \mathcal{O}_F G} RO(H).
\]

The maps \( \varepsilon_G \) are equal to the edge homomorphisms of certain Atiyah-Hirzebruch spectral sequences converging to \( K^*_G(EG) \) and \( KO^*_G(EG) \) (see (1) and (2)). Lück and Oliver proved that (see Proposition 2.5) the map \( \varepsilon_G \) (real or complex) is rationally surjective, meaning that a high enough multiple of every element in the target of \( \varepsilon_G \) is contained in the image of \( \varepsilon_G \). In the last paragraph of [12, p. 596] Lück and Oliver ask for an example of a group \( G \) admitting a cocompact classifying space for proper actions \( EG \) such that \( \varepsilon_G \) is not surjective. In Section 3 of this paper we give the first example of such a group in the complex case. In Section 4 we give the first example of such a group in the real case. We also construct examples of groups \( G \) admitting a cocompact \( EG \) with the following weaker property: \( G \) admits a compatible collection of representations for its finite subgroups that cannot be realized as a \( G \)-vector bundle over \( EG \). However, for these examples we cannot exclude the possibility that there exists a virtual vector bundle that maps to this collection of representations under \( \varepsilon_G \). On the other hand, these examples are more explicit and lower dimensional.

In the final section we show that for a right angled Coxeter group \( W \), every compatible collection of representations of the finite subgroups of \( W \) can be realized as a \( W \)-equivariant vector bundle over \( EW \), so that the map

\[
\varepsilon_W: K^0_W( EW ) \to \lim_{W/H \in \mathcal{O}_F W} R(H).
\]

is always surjective. Moreover, we show that this map is actually an isomorphism and that (see Theorem 2.4)

\[
K^1_W( EW ) = 0.
\]

Using a version of the Atiyah-Segal completion theorem for infinite discrete groups proven by Lück and Oliver, we use these results to compute the complex K-theory of \( BW \), the classifying space of \( W \) (see Corollary 5.6).

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2. $G$-VECTOR BUNDLES AND ISOTROPY REPRESENTATIONS

Let $G$ be a discrete group and let $\Gamma$ be a Lie group. Let $S$ be a family of finite subgroups of $G$, i.e. any collection of finite subgroups of $G$ that is closed under conjugation and passing to subgroups. The orbit category $\mathcal{O}_SG$ is a category whose objects are the transitive $G$-sets $G/H$, for all $H \in S$, and whose morphism are all $G$-equivariant maps between the objects.

**Definition 2.1.** [12, p. 590] Let $X$ be a $G$-CW-complex. A $(G, \Gamma)$-bundle over $X$ is a $\Gamma$-principal bundle $p: E \to X$, where $E$ is a left $G$-space such that $p$ is $G$-equivariant and such that the left $G$-action and the right $\Gamma$-action on $E$ commute. We denote the set of isomorphism classes of $(G, \Gamma)$-bundles over $X$ by $\text{Bdl}_{(G, \Gamma)}(X)$. For $H \in \mathcal{F}$, let

$$\text{Rep}_G(H) = \text{Hom}(H, \Gamma)/\text{Inn}(\Gamma).$$

One can consider $\text{Rep}_G(-)$ as a contravariant functor from $\mathcal{O}_SG$ to Sets. An element of the limit

$$A = (\{\alpha_H\})_{H \in S} \in \lim_{G/H \in \mathcal{O}_SG} \text{Rep}_G(H)$$

is a called an $\mathcal{S}$-compatible collection of $\Gamma$-representations. Given such an element $A$, let $S_A$ be the family of subgroups of $G \times \Gamma$ consisting of conjugates of the subgroups of the form

$$\{(h, \alpha_H(h)) \mid h \in H\}$$

for all $H \in \mathcal{S}$ and let $E_S(G, A)$ be the universal $G \times \Gamma$-CW-complex for the family $S_A$.

**Lemma 2.2.** [12, Lemma 2.4] For every $\mathcal{S}$-compatible collection of $\Gamma$-representations $A = (\{\alpha_H\})_{H \in \mathcal{S}}$ there exists a $G$-CW-complex $B_S(G, A)$ with isotropy in $\mathcal{S}$ satisfying the following properties.

- The quotient map
  $$\pi: E_S(G, A) \to \Gamma \setminus E_S(G, A) = B_S(G, A)$$
  is a $(G, \Gamma)$-bundle over the $G$-CW-complex $B_S(G, A)$.
- The $(G, \Gamma)$-bundle $\pi: E_S(G, A) \to B_S(G, A)$ is universal in the sense that for every $G$-CW-complex $X$ with isotropy in $\mathcal{S}$ there is an isomorphism
  $$[X, B_S(G, A)]_G \cong \text{Bdl}_{(G, \Gamma)}(X)$$
  given by pulling back the universal bundle $\pi$ along a $G$-map $X \to B_S(G, A)$.
- For every $S \in \mathcal{S}$, the fixed point set $B_{\mathcal{F}}(G, A)^H$ is homotopy equivalent to $B\text{C}_G(\alpha_H)$, the classifying space of the centralizer of the image of $\alpha_H$ in $\Gamma$.

If $\Gamma = U(n)$ ($\Gamma = O(n)$) and $\mathcal{S} = \mathcal{F}$, the family of all finite subgroups of $G$, then $\text{Rep}_G(H)$ is the set of isomorphism classes of $n$-dimensional complex (real) representations of $H$. In this case, an element of the limit

$$A = (\{\alpha_H\})_{H \in \mathcal{F}} \in \lim_{G/H \in \mathcal{O}_SF} \text{Rep}_G(H)$$

is a called is called a compatible collection of complex (real) $n$-dimensional representations of the finite subgroups of $G$. For $H \in \mathcal{F}$, let $R(H)$ ($RO(H)$) be the complex (real) representation ring of $H$, i.e. the Grothendieck group of the abelian cancellative monoid of isomorphism classes of finite dimensional complex (real) representations of $H$. Note that $\text{Rep}_{U(n)}(H)$ is
naturally a subset of $R(H)$ and $\text{Rep}_{\text{O}(n)}(H)$ is naturally a subset of $RO(H)$. One can consider $R(-)$ as a functor from $\mathcal{O}_FG$ to Ab. An element of the inverse limit

$$\alpha = ([\alpha_H])_{H \in \mathcal{F}} \in \lim_{G/H \in \mathcal{O}_FG} R(H)$$

is a called a compatible collection of complex virtual representations of the finite subgroups of $G$. One has a natural embedding

$$\lim_{G/H \in \mathcal{O}_FG} \text{Rep}_{\text{U}(n)}(H) \subset \lim_{G/H \in \mathcal{O}_FG} R(H).$$

The analogous statements for $O(n,\mathbb{R})$ and $RO$ also hold.

Now let $X$ be a proper cocompact $G$-CW-complex, i.e. $X$ has finite isotropy and the orbit space $G \setminus X$ has a finite number of cells, such that for every $H \in \mathcal{F}$, the fixed point set $X^H$ is non-empty and connected.

**Definition 2.3** ([18]). A complex (real) $G$-vector bundle over $X$ is a complex (real) vector bundle $\pi : E \to X$ such that $\pi$ is $G$-equivariant and each $g \in G$ acts on $E$ and $X$ via a bundle isomorphism. An isomorphism of $G$-vector bundles over $X$ is just an isomorphism of vector bundles that is $G$-equivariant. The set of isomorphisms classes of complex (real) $G$-vector bundles over $X$ will be denoted by $\text{Bdl}_G(X)$ ($\text{OBDl}_G(X)$). For every $x \in X$, the fiber $\pi^{-1}(x)$ is denoted by $E_x$. We refer the reader to [12, Section 1] and [20, Section I.9] for elementary properties of $G$-vector bundles over proper (cocompact) $G$-CW complexes.

**Theorem 2.4.** [12, Th. 3.2 and 3.15] There exists a 2-periodic (8-periodic) equivariant cohomology theory $K^*_G(X,A)$ ($KO^*_G(X,A)$) on the category of proper $G$-CW-pairs such that when $X$ is cocompact, $K^*_G(X)$ ($KO^*_G(X)$) is the Grothendieck group of the abelian monoid of isomorphism classes of complex (real) $G$-vector bundles over $X$. In particular, for every $H \in \mathcal{F}$, $K^*_G(G/H)$ ($KO^*_G(G/H)$) is canonically isomorphic to $R(H)$ ($RO(H)$).

As usual (see [13, Section 6] and [5, Th. 4.7]), the skeletal filtration of $X$ induces Atiyah-Hirzebruch spectral sequences

$$E^p,q_2 = H^p_G(X,K^q_G(G/-)) \Rightarrow K^{p+q}_G(X).$$

and

$$E^p,q_2 = H^p_G(X,KO^q_G(G/-)) \Rightarrow KO^{p+q}_G(X)$$

where $H^p_G(X,-)$ denotes Bredon cohomology of $X$ (see [2]).

**Proposition 2.5.** [13, Prop 5.8] If $X$ is a cocompact $G$-CW complex then the spectral sequences (1) and (2) above rationally collapse, meaning that the images of all differentials in these spectral sequences consist of torsion elements.

By our assumptions on $X$, the zeroth Bredon cohomology group $H^0_G(X,R(-))$ (resp. $H^0_G(X,RO(H))$), equals the limit of the functor $R(-)$ (resp. $RO(-)$), over the orbit category $\mathcal{O}_FG$. Consider the edge homomorphisms

$$\varepsilon_G : K^0_G(X) \to H^0_G(X,R(-))$$

and

$$\varepsilon_G : KO^0_G(X) \to H^0_G(X,RO(-))$$
of the spectral sequences (1) and (2). If \([\pi]\) is the isomorphism class of an \(n\)-dimensional complex \(G\)-vector bundle \(\pi : E \to X\), then \(\varepsilon_G([\pi])\) equals

\[
([E_{eH}])_{H \in \mathcal{F}} \in \lim_{G/H \in \mathcal{O}_G} \text{Rep}_{U(n)}(H) \subset H^0_G(X, R(-))
\]

where \([E_{eH}]\) denotes the isomorphism class in \(R(H)\) of the \(H\)-representation \(E_{eH}\). The corresponding statement for real \(G\)-vector bundles also holds. Note that it follows from Proposition 2.5 that a suitable multiple of every compatible collection of (virtual) real or complex representations of the finite subgroups of \(G\) is contained in the image of the edge homomorphism \(\varepsilon_G\).

Recall that the classifying space for proper actions \(EG\) is a terminal object in the homotopy category of proper \(G\)-CW complexes (e.g. [9, Th. 1.9]). Hence, if \(X\) is any proper cocompact \(G\)-CW complex such that \(X^H\) is non-empty and connected for each \(H \in \mathcal{F}\), then there exists a \(G\)-map \(X \to EG\) that is unique up to \(G\)-homotopy and induces commutative diagrams

\[
\begin{array}{ccc}
K^0_G(X) & \longrightarrow & \lim_{G/H \in \mathcal{O}_G} R(H) \\
\downarrow & & \downarrow \\
K^0_G(EG) & \longrightarrow & \lim_{G/H \in \mathcal{O}_G} RO(H)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
K^0_G(X) & \longrightarrow & \lim_{G/H \in \mathcal{O}_G} RO(H) \\
\downarrow & & \downarrow \\
K^0_G(EG) & \longrightarrow & \lim_{G/H \in \mathcal{O}_G} RO(EG).
\end{array}
\]

Hence, if a compatible collection \(\alpha\) of virtual representations can be realized as a virtual \(G\)-vector bundle over \(EG\), it can also be realized as a virtual \(G\)-vector bundle over \(X\).

### 3. Complex vector bundles

The purpose of this section is to construct a group \(G\) with a cocompact classifying space for proper actions \(EG\) admitting a compatible collection of complex representations of the finite subgroups of \(G\) that cannot be realized as \(G\)-equivariant virtual complex vector bundle over \(EG\), i.e. so that the edge homomorphism

\[
\varepsilon_G : K^0_G(EG) \to \lim_{G/H \in \mathcal{O}_G} R(H).
\]

is not surjective.

Let \(F = C_4 \rtimes C_2\) be the dihedral group of order 8 where \(\sigma\) is generator for \(C_4\) and \(\varepsilon\) is a generator of \(C_2\). Let \(H = \langle \sigma^2 \rangle\) be the center of \(F\), which has order two and denote the \(n\)-skeleton of the universal \(F/H\)-space \(X = E(F/H)\) by \(X^n\). We let \(F\) act on \(X\) and \(X^n\) via the projection onto \(F/H\). Consider the complex 1-dimensional representation

\[
\lambda : H = \langle \sigma^2 \rangle \to U(1) = S^1 : \sigma^2 \mapsto -1.
\]

**Lemma 3.1.** The isomorphism class \([\lambda]\) is contained in \(R(H)^{F/H}\). For \(k \in \mathbb{Z}\), the multiple \(k[\lambda]\) is contained in the image of the restriction map \(\text{res} : R(F) \to R(H)\) if and only if \(k\) is even.

**Proof.** Since \(H\) is the center of \(F\) it follows that the conjugation action of \(F/H\) on \(R(H)\) is trivial, hence \([\lambda] \in R(H)^{F/H} = R(H)\). One easily verifies that the representation

\[
\tau : F \to U(2)
\]
defined by

\[ \tau(\sigma) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \tau(\varepsilon) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

satisfies \( \text{res}(|\tau|) = 2|\lambda| \). Hence, \( k|\lambda| \) is contained in the image of \( \text{res} \) for every even \( k \in \mathbb{Z} \).

Note that, as a free abelian group, \( R(H) \) is generated by \( |\lambda| \) and the isomorphism class of the 1-dimensional complex trivial representation \([|\tau|]\) (e.g. see [19]). Now suppose \( k \) is odd and there exists an element \( [\mu] - [\rho] \in R(F) \) such that \( \text{res}([\mu] - [\rho]) = k|\lambda| \). There are integers \( l, m, n \) and such that \( \text{res}([\mu]) = l|\text{tr}| + m|\lambda| \), \( \text{res}([\rho]) = l|\text{tr}| + n|\lambda| \) and \( m - n = k \). By changing the representative of \([\mu]\), we may also assume that

\[ \mu: F \to U(l + m) \]

where \( \mu(\sigma) \) is a diagonal matrix. Since \( \mu(\sigma^2) \) has an \( m \)-dimensional eigenspace with eigenvalues \(-1\) and an \( l \)-dimensional eigenspace with eigenvalue \( 1 \), it follows that \( \mu(\sigma) \) has an \( s \)-dimensional eigenspace with eigenvalue \( i \) and a \( t \)-dimensional eigenspace with eigenvalue \(-i\) such that \( s + t = m \). Moreover, \( \mu(\sigma^3) \) has an \( s \)-dimensional eigenspace with eigenvalue \(-i\) and a \( t \)-dimensional eigenspace with eigenvalue \( i \). Since \( \sigma \) and \( \sigma^3 \) are conjugate in \( F \), it follows that \( s = t \) proving that \( m \) is even. A similar argument shows that \( n \) is also even. But this contradicts the fact that \( k = m - n \) is odd. Hence, there does not exist an element \( [\mu] - [\rho] \in R(F) \) such that \( \text{res}([\mu] - [\rho]) = k|\lambda| \), if \( k \) is odd.

\( \square \)

The following lemma uses the notation introduced above and will be cited in the next section.

**Lemma 3.2.** Every \( F \)-equivariant complex line bundle over \( X^3 \) is isomorphic to the pullback of an \( F \)-equivariant complex line bundle over \( E(F/H) \) along the inclusion \( i: X^3 \to E(F/H) \).

**Proof.** Let \( S \) be the family of subgroups of \( F \) containing only \( H \) and the trivial subgroup. Note that isomorphism classes of \( F \)-equivariant complex line bundles correspond to isomorphism classes of \( (F, S^1 = U(1)) \)-bundles. Let \( \pi: E \to X^3 \) be an \( F \)-equivariant complex line bundle over and let \( [\alpha_H]: H \to U(1) = S^1 \) be the isomorphism class in \( \text{Rep}_{S^1}(H) \) of the \( H \)-representation induced on the fibers of \( \pi \). If we set \( \alpha_{(e)}: \{e\} \to S^1 \), then \( A = ([\alpha_K])_{K \in S} \in \text{lim}_{K \in S} \text{Rep}_{S^1}(K) \). It follows from Lemma 2.2 for \( \Gamma = S^1 \), that in order to show that \( \pi \) is the pullback of an \( F \)-equivariant complex line bundle over \( E(F/H) \) along the inclusion \( i: X^3 \to E(F/H) \), it suffices to show that every \( F \)-map from \( X^3 \) to \( B_S(F, A) \) can be extended to an \( F \)-map from \( E(F/H) \) to \( B_S(F, A) \). Here \( B_S(F, A) \) is homotopy equivalent to \( BS^1 = \mathbb{CP}^\infty \) for all \( K \in S \), again by Lemma 2.2. It follows from Bredon’s equivariant obstruction theory (see [2, Section II.1],[15, Th. I.5.1]) that the potential obstructions for extending such a map lie in the relative Bredon cohomology groups \( H_{F}^{n+1}(E(F/H), X^3; \pi_n(B_S(F, A)^-)) \) for \( n \geq 3 \). Since \( \pi_n(\mathbb{CP}^\infty) \) is zero unless \( n = 2 \), the lemma is proven. \( \square \)

The idea for the following lemma is contained in [12, p 596].

**Lemma 3.3.** There exists an \( n \geq 1 \) such that \([\lambda]\) is not contained in the image of the edge homomorphism

\[ K_{F}^{0}(X^n) \to R(H)^{F/H}. \]

**Proof.** By [7, Theorem 5.1] for \( X = \{\ast\}, \mathcal{F} = \{e, H\} \) and \( E\mathcal{F} = E(F/H) \), there are maps

\[ \alpha_n: R(F)/I^n \to K_{F}^{0}(X^n) \]
that induce a map of inverse systems from \( \{ R(F)/I^n \}_{n \geq 0} \) to \( \{ K_F^0(X^n) \}_{n \geq 0} \) that induces an isomorphism of pro-rings. Here \( I \) is the kernel of the restriction map \( R(F) \to R(H) \). This implies that for sufficiently large \( n \geq 1 \) there exists a map \( \beta_1 : K_F^0(X^n) \to R(F)/I \) making the following diagram commute

\[
\begin{array}{ccc}
R(F)/I^n & \alpha_n & K_F^0(X^n) \\
\downarrow & & \downarrow \\
R(F)/I & \alpha_1 & K_F^0(X^1) \\
\end{array}
\]

This shows that the image of the restriction map

\[ R(F) \to R(H)^{F/H} \]

coincides with the image of the edge homomorphism

\[ K_F^0(X^n) \to R(H)^{F/H}. \]

Since \([\lambda] \) does not lie in the image of \( R(F) \to R(H)^{F/H} \) by Lemma 3.1, the lemma follows. \( \square \)

Let \( n \geq 3 \). By [8, Th. A & Th. 8.3] there exists a compact \( n \)-dimensional locally CAT(0)-cubical complex \( T_{X^n} \) equipped with a free cellular \( F/H \)-action and an \( F/H \)-equivariant map \( t_{X^n} : T_{X^n} \to X^n \) that induces an isomorphism

\[
\mathcal{H}_F^*(X^n) \cong \mathcal{H}_F^*(T_{X^n})
\]

for any equivariant cohomology theory \( \mathcal{H}_F(\cdot) \) (e.g. see [11, section 1]). (We remark that [8, Th. 8.3] is stated for equivariant homology theories, but the analogous statement holds for equivariant cohomology theories by essentially the same proof.) The action of \( F \) on \( T_{X^n} \) in the above is via the projection \( F \to F/H \). Now let \( Y^n \) be the universal cover of \( T_{X^n} \) and let \( \Gamma_n \) be the group of self-homeomorphisms of \( Y^n \) that lift the action of \( F/H \) on \( T_{X^n} \). Since \( F/H \) acts freely on \( T_{X^n} \), \( \Gamma_n \) acts freely on \( Y^n \). We conclude that \( Y^n \) is an \( n \)-dimensional CAT(0)-cubical complex on which \( \Gamma_n \) acts freely, co-compactly and cellularly. Since \( Y_n \) is contractible, this implies that \( \Gamma_n \) is torsion-free. By construction there is a surjection \( \Gamma_n \to F/H \) whose kernel \( N_n \) is the torsion-free group of deck transformation of the covering \( Y^n \to T_{X^n} \). Now define the group \( G_n \) to be the pullback of \( \pi_n : \Gamma_n \to F/H \) along \( F \to F/H \). Then \( G_n \) acts on \( Y^n \) via the quotient map \( G_n \to G_n/H = \Gamma_n \) and fits into the short exact sequence

\[ 1 \to N_n \to G_n \xrightarrow{p_n} F \to 1. \]

Note that the only non-trivial finite subgroup of \( G_n \) is \( H \cong C_2 \) and that since \( N_n \) acts freely on \( Y^n \), the \( G_n \)-equivariant quotient map \( Y^n \to N_n \setminus Y^n = T_{X^n} \) induces an isomorphism ([12, Lemma 3.5])

\[
K_F^*(T_{X^n}) \cong K_F^*(Y^n).
\]
Applying (3) and (4) to the composition $Y^n \rightarrow T_{X^n} \rightarrow X^n$ and the equivariant cohomology theories $K_\ast^0(\cdot)$ and $H^\ast_\ast(\cdot, R(-))$ with $\ast = 0$, we obtain a commutative diagram

$$
\begin{array}{ccc}
K_0^0(X^n) & \xrightarrow{\cong} & K^0_{G^n}(Y^n) \\
\downarrow \varepsilon_F & & \downarrow \varepsilon_{G^n} \\
R(H)_F^{F/H} & \cong & \lim_{G^n/S \in \mathcal{O}_{G^n}} R(S).
\end{array}
$$

The fact that this diagram commutes can be seen as follows. Using equivariant cellular approximation, we may assume that the map $X^n \rightarrow Y^n$ is cellular. By considering the inclusion of zero-skeleta in $n$-skeleta, naturality yields a commutative diagram

$$
\begin{array}{ccc}
K_0^0(X^n) & \xrightarrow{\cong} & K^0_{G^n}(Y^n) \\
\downarrow \varepsilon_F & & \downarrow \varepsilon_{G^n} \\
K_0^0(X^0) & \xrightarrow{\cong} & K^0_{G^n}(Y^0).
\end{array}
$$

The edge homomorphism $\varepsilon_F: K_0^0(X^n) \rightarrow R(H)_F^{F/H} \subseteq K_0^0(X^0)$ coincides by construction with $K_0^0(X^n) \rightarrow K_0^0(X^0)$ once we restrict the codomain, and similarly for $\varepsilon_{G^n}$. Therefore, commutativity follows.

Since we proved in Lemma 3.3 that, for $n$ large enough, the isomorphism class of $\lambda$ does not lie in the image of the edge homomorphism $K_0^0(X^n) \rightarrow R(H)_F^{F/H}$ it follows from the commutative diagram above that the compatible system of representations

$$(\lambda \circ p_n|_S)_{S \in \mathcal{F}} \in \lim_{G^n/S \in \mathcal{O}_{G^n}} R(S) = H^0_\mathcal{F}(G^n, R(-))$$

does not lie in the image of the edge homomorphism $\varepsilon_{G^n}: K^0_{G^n}(Y^n) \rightarrow \lim_{G^n/S \in \mathcal{O}_{G^n}} R(S)$.

Recall from [3] that non-empty CAT(0)-cube complexes are contractible and that the fixed point set for a finite group action on a CAT(0)-cube complex is contractible. Since $G_n$ acts cellularly properly and cocompactly on the CAT(0)-cube complex $Y_n$, we deduce that $Y_n$ is a cocompact model for $EG_n$. To summarize, we have constructed a group $G = G_n$ with a cocompact classifying space for proper actions $EG$ admitting a compatible collection of complex representations of the finite subgroups of $G$ that cannot be realized as $G$-equivariant virtual complex vector bundle over $EG$.

We remark that Wolfgang Lück has shown us another quite different way to find a finite group $F$ and an $F$-CW-complex $X$ that satisfy Lemma 3.3; any such pair could be used to construct a group with similar properties to the group $G = G_n$.

4. Real vector bundles

One could apply the techniques of the previous section in the real setting to obtain a group $G$ with cocompact classifying space for proper actions $EG$ so that the edge homomorphism $\varepsilon_G: \text{KO}_G^0(EG) \rightarrow \lim_{G/H \in \mathcal{O}_F} RO(H)$
Proof. Recall that the irreducible real representations of $C_4$ are up to isomorphism the one-dimensional trivial representation, the one dimensional sign representation of $F/H = C_2$ and one 2-dimensional faithful representation in which the elements of order four act as rotations by $\pm \pi$. The restriction of the first two of the representations to $H$ gives the trivial one dimensional representation of $H$, while the restriction to $H$ of the third is $\lambda \oplus \lambda$. We therefore conclude that the image of $RO(F) \to RO(H)^{F/H}$ consists of element of the form $2n[\lambda] + m[tr]$, where $tr$ is the trivial one dimensional representation of $H$ and $n, m \in \mathbb{Z}$.

Lemma 4.1. The isomorphism class $k[\lambda]$ is contained in the image of the restriction map $RO(F) \to RO(H)^{F/H}$.

if and only if $k$ is even.

Proof. Recall that the irreducible real representations of $C_4$ are up to isomorphism the one-dimensional trivial representation, the one dimensional sign representation of $F/H = C_2$ and one 2-dimensional faithful representation in which the elements of order four act as rotations by $\pm \pi$. The restriction of the first two of the representations to $H$ gives the trivial one dimensional representation of $H$, while the restriction to $H$ of the third is $\lambda \oplus \lambda$. We therefore conclude that the image of $RO(F) \to RO(H)^{F/H}$ consists of element of the form $2n[\lambda] + m[tr]$, where $tr$ is the trivial one dimensional representation of $H$ and $n, m \in \mathbb{Z}$.
This shows that $k[\lambda]$ is contained in the image of the restriction map $RO(F) \to RO(H)^{F/H}$ if and only if $k$ is even.

\[\Box\]

**Lemma 4.2.** Let $F$ act on the infinite dimensional sphere $S^\infty$ by first projecting onto $F/H = C_2$ and then acting via the antipodal map. View $S^2$ as the 2-skeleton of $S^\infty$. Every $F$-equivariant orthogonal real line bundle over $S^2$ is isomorphic to the pullback of an $F$-equivariant orthogonal real line bundle over $S^\infty$ along the inclusion $S^2 \to S^\infty$.

**Proof.** Let $S$ be the family of subgroups of $F$ containing $H$ and the trivial subgroup. Note that isomorphism classes of $F$-equivariant orthogonal real line bundles correspond to isomorphism classes of $(F,C_2)$-bundles. Now let $[\xi]$ be an $(F,C_2)$-bundle over $S^2$ with fibers $A = (\xi_S) \in \lim_{S \in S} \text{Rep}_{C_2}(S)$. By Lemma 2.2, it suffices to show that every $F$-map $f : S^2 \to B_S(F,A)$ can be extended to an $F$-map $\tilde{f} : S^\infty \to B_S(F,A)$. Again by Lemma 2.2, $B_S(F,A)^S \cong BC_2 = \mathbb{R}P^\infty$ for all $S \in S$. It follows from Bredon’s equivariant obstruction theory (see [2, Section II.1],[15, Th. I.5.1]) that the potential obstructions for extending such a map lie in the relative Bredon cohomology groups $H^{n+1}_F(S^\infty,S^2;\pi_n(B_S(F,A)^-))$ for $n \geq 2$. Since $\pi_n(\mathbb{R}P^\infty)$ is zero unless $n = 1$, the lemma is proven.

\[\Box\]

**Lemma 4.3.** Let $F$ act on $S^2$ by first projecting onto $F/H = C_2$ and then acting via the antipodal map. There does not exist a real $F$-vector bundle $\xi : E \to S^2$ such that the representation of $H$ on the fibers of $\xi$ is isomorphic to $\lambda$.

**Proof.** Consider the infinite dimensional sphere $S^\infty$ as the universal $C_2$-space $EC_2$, where $C_2$ acts via the antipodal map and let $F$ act on $S^\infty$ via first projection onto $F/H = C_2$ and then acting via $C_2$. Now assume that there exists a real $F$-vector bundle $\xi : E \to S^2$ such that the representation of $H$ on the fibers of $\xi$ is isomorphic to $\lambda$. By Lemma 4.2 there exists a real $F$-vector bundle $\xi : E \to S^\infty$ such that the representation of $H$ on the fibers of $\xi$ is isomorphic to $\lambda$. By pulling back this bundle along the inclusion $S^n \to S^\infty$, there also exists a real $F$-vector bundle $\xi : E \to S^n$ such that the representation of $H$ on the fibers of $\xi$ is isomorphic to $\lambda$, for every $n \geq 2$. 

\[\Box\]
By the real version of [7, Theorem 5.1] (see comments below [7, Theorem 5.1]), there are maps
\[ \alpha_n : RO(F)/I^n \to KO^0_F(S^n) \]
that induce a map of inverse systems from \( \{RO(F)/I^n\}_{n \geq 0} \) to \( \{KO^0_F(S^n)\}_{n \geq 0} \) that in turn induces an isomorphism of pro-rings. Here \( I \) is the kernel of the restriction map \( RO(F) \to RO(H) \). This implies that for sufficiently large \( n \geq 1 \) there exists a map \( \beta_1 : KO^0_F(S^n) \to R(F)/I \) making the following diagram commute
\[
\begin{array}{ccc}
RO(F)/I^n & \xrightarrow{\alpha_n} & KO^0_F(S^n) \\
\downarrow & & \downarrow \\
RO(F)/I & \xrightarrow{\alpha_1} & KO^0_F(S^1).
\end{array}
\]
This shows that the image of the restriction map
\[ RO(F) \to RO(H)^{F/H} \]
coincides with the image of the edge homomorphism
\[ KO^0_F(S^n) \to RO(H)^{F/H}, \]
implying that the \( H \)-representations coming from the fibers of any real \( F \)-vector bundle over \( S^n \) can be extended to virtual \( F \)-representations. However, since \( \lambda \) does not lie in the image of \( RO(F) \to RO(H) \) by Lemma 4.1 we arrive at a contradiction and conclude that there does not exist a real \( F \)-vector bundle \( \xi : E \to S^2 \) such that the representation of \( H \) on the fibers of \( \xi \) is isomorphic to \( \lambda \).

Consider the projection \( p : G \to F \) and the compatible system of real orthogonal representations
\[ ([\lambda \circ p_{|S}])_{S \in F} \in \lim_{\rightarrow_{G \subseteq O_{\mathbb{Z}} G}} RO(S) = H^0_O(EG, RO(-)), \]
and assume that there exists a real \( G \)-vector bundle \( \xi : E \to \mathbb{R}^2 \) that realizes it. Since the kernel of \( p : G \to F \) is \( N \), it follows from the lemma below and our observations above that \( N \setminus \xi : N \setminus E \to N \setminus X \) is an \( F \)-vector bundle over \( S^2 \), where \( F \) acts on \( S^2 \) via projection onto \( F/H = C_2 \), followed by the antipodal map. Moreover, the representation of \( H \) on the fibers of \( N \setminus \xi \) is by construction exactly \( \lambda \). This however contradicts Lemma 4.3, so we conclude that there does not exist a real \( G \)-vector bundle \( \xi : E \to \mathbb{R}^2 \) that realizes the compatible system of real orthogonal representations \( (\lambda \circ p_{|S})_{S \in F} \).

**Lemma 4.4.** Let \( G \) be any discrete group with normal subgroup \( N \) and let \( X \) be a proper \( G \)-CW-complex. If \( \xi : E \to X \) is a \( G \)-vector bundle over \( X \) such that \( N \cap G_x \) acts trivially on \( \xi^{-1}(x) \) for every \( x \in X \), then
\[ N \setminus \xi : N \setminus E \to N \setminus X \]
is a \( G/N \)-vector bundle over \( N \setminus X \).
Proof. Denote the projection \( G \to G/N = Q \) by \( \pi \). Let us first consider the case where \( \xi \) is trivial (trivial in the sense of [10, Section 6.1.]), i.e. assume \( \xi \) is a pullback

\[
\begin{array}{ccc}
G \times_H V & \longrightarrow & G/H \\
\downarrow r & & \downarrow p \\
E & \longrightarrow & X
\end{array}
\]

of the \( G \)-vector bundle \( G \times_H V \to G/H \) along the \( G \)-map \( p: X \to G/H \) where \( H \) is some finite subgroup of \( G \) and \( V \) is a finite dimensional real \( H \)-representation such that \( H \cap N \) acts trivially on \( V \). Consider the pullback diagram

\[
\begin{array}{ccc}
Q \times_{\pi(H)} V & \longrightarrow & Q/\pi(H) \\
\downarrow w & & \downarrow q \\
P & \longrightarrow & N \ \backslash \ \xi
\end{array}
\]

of the \( Q \)-vector bundle \( Q \times_{\pi(H)} V \to Q/\pi(H) \) along the \( Q \)-map \( N \ \backslash \ \pi: N \ \backslash \ \xi \to N \ \backslash \ X \).

We define the map

\[
\psi: N \ \backslash \ E \to P: (g,v,x) \mapsto (\pi(g),v,\bar{x}).
\]

It is easy to check that \( \psi \) yields a well-defined morphism of \( Q \)-equivariant bundles over \( N \ \backslash \ \xi \). Moreover, since \( \psi \) is a fiberwise linear map of \( Q \)-vector bundles that is a fiberwise isomorphism, it follows that \( \psi \) is a homeomorphism.

Now consider the general case. Let \( \xi \in N \ \backslash \ X \). Since \( \xi: E \to X \) is locally trivial, \( x \in X \) has an open \( G \)-neighbourhood \( U \) such that there is a \( G \)-map \( p: U \to G/H \) where \( H \) is finite subgroup of \( G \) and \( \xi|_U \) is (homeomorphic to) the pullback

\[
\begin{array}{ccc}
G \times_H V & \longrightarrow & G/H \\
\downarrow \xi|_U & & \downarrow \xi \\
U & \longrightarrow & U
\end{array}
\]

of the \( G \)-vector bundle \( G \times_H V \to G/H \) along the \( G \)-map \( p: U \to G/H \). By the above, the quotient diagram

\[
\begin{array}{ccc}
Q \times_{\pi(H)} V & \longrightarrow & Q/\pi(H) \\
\downarrow N \ \backslash \ \xi|_U & & \downarrow N \ \backslash \ \xi \\
N \ \backslash \ N \ \backslash \ U & \longrightarrow & N \ \backslash \ U
\end{array}
\]

is a pullback diagram. Since \( N \ \backslash \ U \) is an open \( Q \)-neighbourhood of \( \xi \), it follows that \( N \ \backslash \ \xi: N \ \backslash \ E \to N \ \backslash \ X \) is a \( Q \)-vector bundle. \( \square \)

We finish this section by explaining how a similar approach to the one above can be used to produce a group \( G \) admitting a three dimensional cocompact model for \( EG \) that has a compatible system of one-dimensional complex representations that cannot be realized as a complex \( G \)-vector bundle over \( EG \). As in Section 3, let \( F = C_4 \rtimes C_2 \) be the dihedral group of order 8 where \( \sigma \) is a generator for \( C_4 \). Let \( H = \langle \sigma^2 \rangle \) be the center of \( F \), which has order two.
and denote the 3-skeleton of the universal $F/H$-space $X = E(F/H)$ by $X^3$. We let $F$ act on $X$ and $X^3$ via the projection onto $F/H$. Consider the complex 1-dimensional representation
\[
\lambda: H = \langle \sigma^2 \rangle \to U(1) = S^1: \sigma^2 \mapsto -1.
\]
By [8, Th. A & Th. 8.3] there exists a compact 3-dimensional locally CAT(0)-cubical complex $T_{X^3}$ equipped with a free cellular $F/H$-action, an $F/H$-equivariant map $t_{X^3}: T_{X^3} \to X^3$ and an isometric cellular involution $\tau$ on $T_{X^3}$ that commutes with the $F/H$-action on $T_{X^3}$ and the map $t_{X^3}$ such the induced $F/H$-equivariant map
\[
\langle \tau \rangle \setminus T_{X^3} \to X^3
\]
is a homotopy equivalence. Note that $F/H$ acts freely on $\langle \tau \rangle \setminus T_{X^3}$ since it acts freely on $X^3$. Hence $T_{X^3}$ is also the 3-skeleton of a universal $F/H$-space $Z$. So we may continue assuming that $Z = X$ and $\langle \tau \rangle \setminus T_{X^3} = X^3$.

Now let $Y$ be the universal cover of $T_{X^3}$ and let $\Gamma$ be the group of self-homeomorphism of $Y$ that lifts the action of $F/H \oplus \langle \tau \rangle$ on $T_{X^3}$. Then $Y$ is a 3-dimensional CAT(0)-cubical complex on which $\Gamma$ acts properly, compactly and cellularly. By construction there is a surjection $\alpha: \Gamma \to F/H \oplus \langle \tau \rangle$ whose kernel $\text{Ker}(\alpha)$ is the torsion-free group of deck transformations of $Y \to T_{X^3}$. Let $\pi$ denote the composition of $\alpha$ with the projection of $F/H \oplus \langle \tau \rangle$ onto $F/H$. Since $F/H$ acts freely on $T_{X^3}$ and every finite subgroup of $\Gamma$ must fix a point of $Y$ since $Y$ is CAT(0), it follows that every finite subgroup of $\Gamma$ is contained in the kernel of $\pi$, which we denote by $N$. Now define the group $G$ to be the pullback of $\pi: \Gamma \to F/H$ along $F \to F/H$.

Then $G$ acts on $Y$ via the quotient map $G \to G/H = \Gamma$ that fits into the short exact sequence
\[
1 \to N \to G \xrightarrow{p} F \to 1.
\]
such that $p$ maps all the finite subgroup of $G$ onto a finite subgroup of $H$ and $N \setminus Y = X^3$.

Let $\mathcal{F}$ be the family of finite subgroups of $G$, note that $Y$ is a three dimensional cocompact model for $EG$ and suppose that there exists a $G$-vector $\xi: E \to Y$ whose fibers give rise to the compatible system of representations
\[
([\lambda \circ p(s)])_{s \in \mathcal{F}} \in \lim_{G/S \in \mathcal{O}_F} R(S).
\]

Applying Lemma 4.4, we obtain an $F$-equivariant complex line bundle $N \setminus \xi: N \setminus E \to X$ such that the representation of $H$ on the fibers of $N \setminus \xi$ is isomorphic to $\lambda$. By Lemma 3.2, this bundle can be extended to an $F$-equivariant complex line bundle over $X = E(F/H)$. We now continue in a similar fashion as in the proof of Lemma 4.3 to conclude that $[\lambda]$ is contained in the image of the restriction map $R(F) \to R(H)^{F/H}$, which contradicts Lemma 3.1. We conclude that the bundle $\xi$ cannot exist.

5. Right angled Coxeter groups

Let $\Gamma$ be a finite graph. We denote the vertex set of $\Gamma$ by $S = V(\Gamma)$ and the set edges of $\Gamma$ by $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. The right angled Coxeter group determined by $\Gamma$ is the Coxeter group $W$ with presentation
\[
W = \langle S \mid s^2 \text{ for all } s \in V(\Gamma) \text{ and } (st)^2 \text{ if } (s,t) \in E(\Gamma) \rangle.
\]
Note that $W$ fits into the short exact sequence
\[
1 \to N \to W \xrightarrow{p} F = \bigoplus_{s \in S} C_2 \to 1
\]
where \( p \) takes \( s \in S \) to the generator of the \( C_2 \)-factor corresponding to \( s \). A subset \( J \subseteq S \) is called spherical if the subgroup \( W_J = \langle J \rangle \) is finite (and hence isomorphic to \( \bigoplus_{s \in J} C_2 \)). The empty subset of \( S \) is by definition spherical. We denote the poset of spherical subsets of \( S \) ordered by inclusion by \( S \). If \( J \in S \), then \( W_J \) is called a spherical subgroup of \( W \), while a coset \( wW_J \) is called spherical coset. We denote the poset of spherical cosets, ordered by inclusion, by \( W_S \). Note that \( W \) acts on \( W_S \) by left multiplication, preserving the ordering.

The Davis complex \( \Sigma \) of \( W \) is the geometric realization of \( W_S \). One easily sees that \( \Sigma \) is a proper cocompact \( W \)-CW-complex. Since \( \Sigma \) admits a complete CAT(0)-metric such that \( W \) acts by isometries, it follows that \( \Sigma \) is a cocompact model for \( E_W \) (see [4, Th. 12.1.1 & Th. 12.3.4]). A consequence of this fact is that every finite subgroup of \( W \) is subconjugate to some spherical subgroup of \( W \). This implies that the group \( N \) defined above is torsion-free.

Since the quotient space \( W \setminus \Sigma \) is homeomorphic to the geometric realization of the poset \( S \), which is contractible since it has a minimal element, another consequence is that the quotient \( B_W = W \setminus E_W \) is contractible. We refer the reader to [4] for more details and information about these groups and the spaces on which they act.

Let \( F \) be the family of finite subgroups of \( W \). Given an abelian group \( A \), we denote by \( \mathcal{A}: O_F W \to \text{Ab} \) the trivial functor that takes all objects to \( A \) and all morphisms to the identity map. One can verify that

\[
H^*_W(E_W, A) \cong H^*(B_W, A).
\]

**Lemma 5.1.** Let \( A = ([p_H])_{H \in F} \in \lim_{W/H \in O_F W} \text{Rep}_F(H) \). For every \( k \geq 0 \), the contravariant functor

\[
\mathcal{O}_F W \to \text{Ab}: W/H \mapsto \pi_k(B_F(W, A)^H)
\]

equals the trivial functor \( \pi_k(BF) \).

**Proof.** Let \( EF \) be a contractible \( F \)-CW-complex with free \( F \)-action and consider the product space \( \mathcal{F}_W \times EF \). This space becomes a \((W \times K)\)-CW-complex by letting \((w, f) \in W \times F \) act on \((x, y) \in \mathcal{F}_W \times EF \) as

\[
(w, f) \cdot (x, y) = (w \cdot x, p(w)f \cdot y).
\]

One checks that with this action \( \mathcal{F}_W \times EF \) is a model for \( \mathcal{F}_W(W, A) \), i.e. \( (\mathcal{F}_W \times EF)^K \) is contractible when \( K \in \mathcal{F}_A \) and empty otherwise. By definition, it follows that \( \mathcal{F}_W \times BF \) is a model \( B_F(W, A) \), where \( W \) acts trivially on the second coordinate. Since \( \mathcal{F}_W \) is contractible for every \( H \in \mathcal{F} \), the lemma follows easily.

Let \( \Gamma \) be either the orthogonal group \( O(n, \mathbb{R}) \) or the unitary group \( U(n) \).

**Lemma 5.2.** Every element of

\[
\lim_{W/H \in O_F W} \text{Rep}_F(H)
\]

is of the form \( ([\lambda \circ p_H])_{H \in F} \) for some group homomorphism \( \lambda: F \to \Gamma \).

**Proof.** Every finite subgroup \( H \) of \( W \) is isomorphic to a finite direct sum of \( C_2 \)'s. Since every element of order 2 in \( \Gamma \) is conjugate in \( \Gamma \) to a diagonal matrix with \( \pm 1 \) on the diagonal and commuting matrices can be simultaneously diagonalized (e.g. see [6, Th. 1.3.12]), it follows that the image of every homomorphism \( H \to \Gamma \) is conjugate to a finite subgroup of \( \Gamma \) consisting
of diagonal matrices. Hence, every element of \( \lim_{W/H \in \mathcal{O}_F W} \text{Rep}_F(H) \) is of the form \((\alpha_H)|_{H \in F}\) where \(\alpha_H: H \rightarrow \Gamma\) is a homomorphism whose image lands in the finite abelian subgroup of \(\Gamma\) consisting of diagonal matrices. Since every finite subgroup of \(W\) is subconjugate to a spherical subgroup \(W_J\), the compatibility of the representations tells us that \((\alpha_H)|_{H \in F}\) is completely determined by the homomorphisms \(\alpha_{\langle s \rangle}: \langle s \rangle \rightarrow \Gamma\), for \(s \in S\). Since the images of the \(\alpha_{\langle s \rangle}\)'s are diagonal, they commute. Therefore, one can define the homomorphism

\[ \lambda: F = \bigoplus_{s \in S} \mathbb{C}_2 \rightarrow \Gamma: (\sigma_s)s \in S \mapsto \sum_{s \in S} \alpha_{\langle s \rangle}(\sigma_s). \]

The compatibility of the representations implies that

\[ ([\lambda \circ p_H])_{H \in F} = ([\alpha_H])_{H \in F}, \]

proving the lemma.

The following theorem applies to both complex and real representations and vector bundles.

**Theorem 5.3.** Let \(W\) be a right angled Coxeter group. Every compatible collection of representations of the finite subgroups of \(W\) can be realized as a \(W\)-equivariant vector bundle over the Davis complex \(\Sigma = EW\).

**Proof.** Consider \(A = ([p_H])_{H \in F} \in \lim_{W/H \in \mathcal{O}_F W} \text{Rep}_F(H)\). It follows from Lemma 2.2 that the existence of a \((W,A)\)-bundle over \(\Sigma\) follows from the existence a \(W\)-map \(\Sigma \rightarrow B_F(G,A)\). Since by Lemma 5.1, the contravariant functor \(\pi_k(B_F(W,A)^-): \mathcal{O}_F(W) \rightarrow \text{Ab}: W/H \mapsto \pi_k(B_F(W,A)^H)\) equals the trivial functor \(\pi_k(BF)\) for all \(k \geq 0\), it follows from (5) and the contractibility of \(BW\) that the Bredon cohomology groups

\[ H^{k+1}_W(\Sigma, \pi_k(B_F(W,A)^-)) \]

are zero for all \(k \geq 0\). Since there certainly exists a \(W\)-map from the 0-skeleton of \(\Sigma\) to \(B_F(W,A)\), it follows from Bredon’s equivariant obstruction theory that there exists a \(W\)-map \(\Sigma \rightarrow B_F(W,A)\).

Now consider a compatible collection of representations of the finite subgroups of \(W\). By Lemma 5.2, this collection is of the form

\[ ([\lambda \circ p_H])_{H \in F} = \lim_{W/H \in \mathcal{O}_F W} \text{Rep}_F(H) \]

for some group homomorphism \(\lambda: F \rightarrow \Gamma\). Letting \(A = ([p_H])_{H \in F}\), it follows from the above that there exists a \((W,A)\)-bundle \(\xi: E \rightarrow \Sigma\). If \(\Gamma = \text{O}(n, \mathbb{R})\) then

\[ \xi: E \times_F \mathbb{R}^n \rightarrow \Sigma \]

is a real \(W\)-vector bundle over \(\Sigma\) that realizes \((\lambda \circ p_H)|_{H \in F}\), and if \(\Gamma = \text{U}(n)\) then

\[ \xi: E \times_F \mathbb{C}^n \rightarrow \Sigma \]

is a complex \(W\)-vector bundle over \(\Sigma\) that realizes \((\lambda \circ p_H)|_{H \in F}\). Here \(F\) acts on \(\mathbb{R}^n\) or \(\mathbb{C}^n\) via the map \(\lambda\).

**Lemma 5.4.** If \(W\) is a right angled Coxeter group, then \(H^0_W(\Sigma, R(-)) = 0\) for all \(n > 0\), and \(H^0_W(\Sigma, R(-))\) is free abelian of rank equal to the number of spherical subgroups of \(W\).
Proof. This is proven in much the same way as the corresponding result for homology in [17]. In more detail, one uses the cubical structure on $\Sigma$, in which there is one orbit of $n$-cubes with stabilizer isomorphic to $(C_2)^n$ for each $n$-tuple of commuting elements of $S$. (For each $n \geq 0$, for each spherical subgroup $W_J \cong (C_2)^n$ and for each $w \in W$, the subposet consisting of all special cosets contained in $wW_J$ is order isomorphic to the poset of faces of an $n$-cube. Furthermore this isomorphism is equivariant for the stabilizer subgroup $wW_Jw^{-1} \cong (C_2)^n$, acting on the $n$-cube as the group generated by reflections in its coordinate planes. The realizations of these subposets are the cubes that make up the cubical structure on $\Sigma$. For more details concerning the cubical structure on $\Sigma$ see [4, Ch. 1.1-1.2 or Ch. 7].) Since the stabilizer of a cube of strictly positive dimension acts non-trivially on the cube, this cubical structure is not a $W$-CW-structure on $\Sigma$. However, its barycentric subdivision is a simplicial complex naturally isomorphic to the realization of the poset $WS$ as described in the introduction to this section.

Let $\Sigma^n$ denote the $n$-skeleton of $\Sigma$ with the cubical structure. Firstly, $\Sigma^0$ consists of a single free $W$-orbit of vertices, so $H^1_W(\Sigma^0; R(-))$ is isomorphic to the ordinary cohomology of a point; since $W$ acts freely the calculation reduces to an equivariant cohomology calculation for the trivial group action.

Let $I = [-1, 1]$ be an interval, with $C_2$ acting by $x \mapsto -x$ (i.e., swapping the ends of the interval). Note that $I$ is equivalently isomorphic to the Davis complex for the Coxeter group $C_2$. Let $\partial I$ denote the two end points $\{-1, 1\}$. Make $I$ into a $C_2$-CW-complex, for example by taking three 0-cells in two orbits at the points $-1, 0$ and 1, and one free orbit of 1-cells consisting of the two intervals $[-1, 0]$ and $[0, 1]$. The cellular $C_2$-Bredon cochain complex for the pair $(I, \partial I)$ with coefficients in $R(-)$ is a cochain complex of free abelian groups in which the degree zero term has rank two, the degree one term has rank one, and all other terms are trivial. A direct computation with this cochain complex shows that $H^1_{C_2}(I, \partial I; R(-))$ is isomorphic to $\mathbb{Z}$.

Next consider $I^n$ with $C_2^n$ acting as the direct product of $n$ copies of the above action of $C_2$ on $I$. This is the Davis complex for the Coxeter group $C_2^n$. Since the representation ring of a direct product of finite groups is naturally identified with the tensor product of the representation rings [19, Ch. 3.2], the $C_2^n$-Bredon cochain complex for the pair $(I^n, \partial I^n)$ with coefficients in $R(-)$ is naturally isomorphic to the tensor product of $n$ copies of the $C_2$-Bredon cochain complex for $(I, \partial I)$ with coefficients in $R(-)$. (If one wants to think about this cochain complex geometrically, it arises from the $(C_2)^n$-CW-structure on $I^n$ in which the cells are the direct products of the cells arising in the $C_2$-CW-structure on $I$.) Since these cochain complexes consist of finitely generated free abelian groups, there is a Künneth formula as described in for example [16, Thm 60.3]. Since $H^*_{C_2}(I, \partial I; R(-))$ is free abelian the Künneth formula implies that

$$H^*_{C_2}(I^n, \partial I^n; R(-)) \cong \bigotimes_{i=1}^n H^*_{C_2}(I, \partial I; R(-)).$$

It follows that for each $n$, $H^m_{C_2}(I^n, \partial I^n; R(-))$ is isomorphic to $\mathbb{Z}$ for $m = 0$ and is zero for $m > 0$.

From these computations, it follows easily that $H^m_W(\Sigma^n, \Sigma^{n-1}; R(-))$ is zero for $m > 0$ and is isomorphic to a direct sum of copies of $\mathbb{Z}$ indexed by the $W$-orbits of $n$-cubes in $\Sigma$. By induction on $n$ one sees that $H^m_W(\Sigma^n; R(-))$ is zero for $m > 0$ and isomorphic to a direct sum of copies of $\mathbb{Z}$ indexed by the $W$-orbits of cubes of dimension at most $n$ for $m = 0$. The
claimed result follows, since the $W$-orbits of cubes in $\Sigma$ are in bijective correspondence with the spherical subgroups of $W$. \hfill \square

Before stating our theorem concerning $K^*_W(\mathbb{E}W)$, we make some remarks concerning the representation ring of a direct sum of copies of the cyclic group $C_2$, indexed by a (finite) set $S$. For any finite group $G$, the collection of all isomorphism types of 1-dimensional complex representations of $G$ is an abelian group, with product given by taking the tensor product of representations. Furthermore, this group is naturally isomorphic to the group $\text{Hom}(G, \text{U}(1))$. In the case when $G$ is abelian, every irreducible representation of $G$ is 1-dimensional, and so $\text{Hom}(G, \text{U}(1))$ forms a basis for the additive group of the representation ring. In this way the representation ring $R(G)$ is naturally isomorphic to the integral group algebra of the group $\text{Hom}(G, \text{U}(1))$. In the case when $G = \bigoplus_{s \in S} C_2$ is a direct sum of copies of $C_2$ indexed by $S$, we may view $G$ as a vector space over the field of two elements, in which case $\text{Hom}(G, \text{U}(1))$ may be identified with the dual space. For $s \in S$, let $s^*$ denote the 1-dimensional representation of $G$ with the properties that $s^*(s) = -1$ and $s^*(t) = 1$ for $t \in S - \{s\}$. Let $S^*$ denote the set of these representations: $S^* := \{s^* \mid s \in S\}$. In terms of vector spaces over the field of two elements, $S^* \subseteq \text{Hom}(G, \text{U}(1))$ is the dual basis to the set $S \subseteq G$. The set $S^*$ generates the representation ring of $G$, giving rise to the following presentation:

$$R(G) = \mathbb{Z}[S^*]/(s^{*2} - 1 \mid s \in S),$$

in which the monomials $s_1^{*} s_2^{*} \cdots s_k^{*}$ for all subsets $\{s_1, \ldots, s_k\} \subseteq S$ correspond to the irreducible representations.

Suppose now that $J$ is a subset of $S$. The inclusion $J \subseteq S$ identifies $H = \bigoplus_{s \in J} C_2$ with a subgroup of $G = \bigoplus_{s \in S} C_2$. The induced map $R(G) \rightarrow R(H)$ of representation rings is described easily in terms of the above ring presentation: for $s \in J$, $s^* \in R(G)$ restricts to $s^* \in R(H)$, while for $s \notin J$, $s^* \in R(G)$ restricts to $1 \in R(H)$.

Now suppose that $\Gamma$ is a graph with vertex set $V(\Gamma) = S$, and let $W$ be the right angled Coxeter group associated to $\Gamma$. The abelianization of $W$ is naturally identified with $G = \bigoplus_{s \in S} C_2$. There is a unique equivariant map $\alpha: \mathbb{E}W \rightarrow \ast$, from the $W$-space $\mathbb{E}W$ to a point $\ast$, viewed as a $G$-space with trivial action. If $J$ is a spherical subset of $S$ then $W_J = \bigoplus_{s \in J} C_2$ maps isomorphically to the corresponding subgroup of $G = \bigoplus_{s \in S} C_2$. If $x \in \mathbb{E}W$ is a 0-cell fixed by $W_J = \bigoplus_{s \in J} C_2$, then $\alpha(x) = \ast$, and this map is $W_J$-equivariant. The induced map $\alpha^*: K_G^*(\ast) \rightarrow K_W^*(\mathbb{E}W)$, and the composite map $K_G^*(\ast) \rightarrow K_W^*(\mathbb{E}W)$ will be used in the statement and proof of our theorem. If we identify $R(G)$ with $K_G^0(\ast)$ and $R(W_J)$ with $K_W^0(W_J)$, then the composite is identified with the restriction map.

**Theorem 5.5.** Let $W$ be the right angled Coxeter group determined by a finite graph $\Gamma$, with vertex set $S$, and let $G = \bigoplus_{s \in S} C_2$ be the abelianization of $W$. The map $\alpha^*: K_G^*(\ast) \rightarrow K_W^*(\mathbb{E}W)$ is surjective in each degree. In particular, $K_W^0(\mathbb{E}W) = 0$ and there is a ring isomorphism

$$K_W^0(\mathbb{E}W) \cong \mathbb{Z}[S^*]/(s^{*2} - 1, s^*t^* - s^* - t^* + 1 \mid s \in S = V(\Gamma), (s, t) \notin E(\Gamma)).$$

It follows that $K_W^0(\mathbb{E}W) \cong \mathbb{Z}^d$ as an abelian group, where $d$ is the number of spherical subgroups of $W$.

**Proof.** Consider the Atiyah-Hirzebruch spectral sequence (1)

$$E_2^{p,q} = H^p_W(\mathbb{E}W, K_W^q(W/-)) \Longrightarrow K_W^{p+q}(\mathbb{E}W)$$

for the $K$-theory of the universal bundle $\mathbb{E}W$. Since $W_J$ is a proper subgroup of $W$, the Atiyah-Hirzebruch spectral sequence becomes

$$E_2^{p,q} = H^p_W(\mathbb{E}W_J, K_W^q(W_J/-)) \Longrightarrow K_W^{p+q}(\mathbb{E}W_J)$$

for the $K$-theory of the universal bundle restricted to $W_J$. The statement of the theorem follows from the fact that the $E_2$-term of the spectral sequence is non-zero only for $p + q = 0$.\hfill \square
where $K^0_W(W/-) = R(-)$ if $q$ is even and $K^0_W(W/-) = 0$ if $q$ is odd (see [12, Th. 3.2]). In the lemma above, we proved that $H^k_W(\Sigma, R(-)) = 0$ for $k > 0$. It therefore follows that

$$K^0_W(EW) = \begin{cases} \mathbb{H}_W^0(EW, R(-)) = \lim_{W/H \in O_{F^W}} R(H) & \text{if } n = 0 \\ 0 & \text{if } n = 1. \end{cases}$$

Let $I$ be the ideal

$$(s^*^2 - 1, s^*t^* - s^* - t^* + 1 \mid s \in S, (s, t) \notin E(\Gamma))$$

in the polynomial ring $\mathbb{Z}[S^*]$. Note that as an abelian group $\mathbb{Z}[S^*]/I$ is free, with basis elements the commuting products $G_s^1 \cdots G_s^k$, for all $J = \{s_1, \ldots, s_k\} \in S$ (The case $J = \emptyset$ corresponds to the unit of $\mathbb{Z}[V(\Gamma)]/I$). This shows that

$$\mathbb{Z}[S^*]/I \cong \mathbb{Z}^d$$

as an abelian group, where $d$ is the number of spherical subgroups of $W$.

We claim there is an isomorphism of rings

$$\lim_{W/H \in O_{F^W}} R(H) \cong \mathbb{Z}[S^*]/I.$$ 

Since every finite subgroup of $W$ is subconjugate to a spherical subgroup of $W$, it follows that

$$\lim_{W/H \in O_{F^W}} R(H) \cong \lim_{J \in S} R(W_J)$$

as rings. By the remarks in the paragraph preceding the statement of the theorem, there are ring isomorphisms

$$R(W_J) = \mathbb{Z}[J^*]/(s^*^2 - 1 \mid s \in J), \quad R(G) = \mathbb{Z}[S^*]/(s^*^2 - 1 \mid s \in S),$$

which are natural for inclusions $J \subseteq J' \subseteq S$. From this it follows that the natural ring homomorphism

$$\rho: R(G) \to \lim_{W/H \in O_{F^W}} R(H)$$

is surjective, and that $\lim_{W/H \in O_{F^W}} R(H)$ is isomorphic to the ring described in the statement; in particular its additive group is free abelian of the same rank as $K^0_W(EW)$. Since $\rho$ factors through $K^0_W(EW)$, the claimed isomorphism follows. 

Before stating our corollary concerning $K^*(BG)$, we recall some facts from [1] concerning $K^i(BG)$, where as above $G = \bigoplus_{s \in S} C_2$. For any finite group $H$, Atiyah showed that $K^i(BH) = 0$ for $i$ odd, and that $K^2(BH)$ is naturally isomorphic to the completion of the representation ring $R(H)$ at its augmentation ideal. To discuss the case of $G$, it is convenient to take new generators for $R(G)$; replace the irreducible representation $s^*$ by the degree zero virtual representation $\overline{s} = s^* - 1$. With respect to these generators one obtains the presentation

$$R(G) = \mathbb{Z}[\overline{s}]/(\overline{s}(\overline{s} + 2) \mid s \in S),$$

where $\overline{s} = \{\overline{s} \mid s \in S\}$. If $H = \bigoplus_{s \in J} C_2$, then of course there is a similar description of $R(H)$, which is natural for the inclusion $J \subseteq S$. Note that if $s \notin J$, then the image of $\overline{s}$ under the restriction map $R(G) \to R(H)$ is zero.

Completing $R(G)$, as described above, with respect to its augmentation ideal gives rise to the following presentation for the ring $K^0(BG)$:

$$K^0(BG) = \mathbb{Z}[\overline{s}]/(\overline{s}(\overline{s} + 2) \mid s \in S),$$
which is natural for the inclusion $J \subseteq S$, and so also describes the induced map $K^0(BG) \to K^0(BH)$. The additive group of this ring is the direct sum of one copy of $\mathbb{Z}$, generated by 1, and for each non-empty subset $J \subseteq S$, one copy of the 2-adic integers, $\mathbb{Z}_2$, consisting of the set of power series in the element $\prod_{s \in J} \bar{s}$ with zero constant term.

**Corollary 5.6.** Let $W$ be the right angled Coxeter group determined by a finite graph $\Gamma$ with vertex set $S = V(\Gamma)$, and let $G = \bigoplus_{s \in S} C_2$ be the abelianization of $W$. The induced map $K^*(BG) \to K^*(BW)$ is surjective in each degree. In particular $K^1(BW) = 0$ and there is a ring isomorphism

$$K^0(BW) \cong \mathbb{Z}[\bar{S}]/(\bar{s} + 2), \quad \bar{s} \mid s \in S, \ (s,t) \notin E(\Gamma).$$

Here, $\mathbb{Z}[\bar{S}]$ is the formal power series ring with $\mathbb{Z}$ coefficients in the variables $\bar{S} = \{\bar{s} \mid s \in S\}$.

**Proof.** The version of the Atiyah-Segal completion theorem that is proven for infinite discrete groups admitting a cocompact model for the classifying space for proper actions in [12, Theorem 4.4.(b)] implies that

$$K^n(BW) = K^n_W(\mathbb{E}W)_j,$$

where the ideal $J$ is the kernel of the augmentation map $K^n_W(\mathbb{E}W) \to \mathbb{Z}$ that maps vector bundles to their dimension. Changing variables in the above theorem to $\bar{s} = s^* - 1$, we see that $K^i(BW) = 0$ for $i$ odd and that $K^0(BW)$ is the completion of the ring

$$\mathbb{Z}[\bar{S}]/(\bar{s} + 2), \quad \bar{s} \mid s \in S, \ (s,t) \notin E(\Gamma)$$

with respect to the ideal generated by the set $\bar{S} = \{\bar{s} \mid s \in S\}$. This completion is the ring described in the statement. \hfill \Box

There is an alternative proof of Corollary 5.6 that does not use Theorem 5.5 or results from [12]. Instead one uses a description of $W$ as a free product with amalgamation. If the graph $\Gamma$ is not a complete graph, then there is an expression $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_3 = \Gamma_1 \cap \Gamma_2$, in which each $\Gamma_i$ is a full subgraph of $\Gamma$ and has fewer vertices than $\Gamma$. This gives an expression for $W$ as a free product with amalgamation $W = W_1 *_{W_2} W_2$. From this one obtains a Mayer-Vietoris sequence that can be used to compute $K^*(BW)$. To establish Corollary 5.6, one shows by induction on $|S|$ that $K^*(BW)$ is as described and that for each $J \subseteq S$, the map $K^*(BW) \to K^*(BW_J)$ is a split surjection.

**References**


