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Fatigue life from sine-on-random excitations

Frédéric Kihm[a]*, Andrew Halfpenny[a], Neil S. Ferguson[b]

[a]HBM-nCode Products Division, Technology Center, Brunel Way, Catcliffe, Rotherham, S60 5WG, UK
[b]Institute of Sound and Vibration Research, University of Southampton, Southampton SO17 1BJ, UK

Abstract

This presentation concentrates on calculating the high cycle fatigue life from sine-on-random excitations. A spectral domain approach offers advantages over the time domain approach when computation times are prohibitive.
The proposed approach is to derive the statistical rainflow cycle histogram from a sine-on-random spectrum of stress data and then to use the appropriate material fatigue curve to obtain the estimated life.
A case study illustrates the application of the method on a component attached to a helicopter. Comparisons with traditional time-domain approaches are presented and show excellent agreement.

1. Introduction

Sine-on-random excitations are typically generated by rotating machinery. Sine-on-random excitations are specified in international standards such as in MIL STD 810G [1] or in RTCA DO 160G [2].

So far, the only way of estimating fatigue life from a sine-on-random excitation is to perform a transient analysis using time domain realizations of the required PSD and harmonic tones as the excitation. Such an analysis can be accelerated by using modal superposition but is still very demanding in terms of CPU time. It also raises the question of how long the excitation signal should be to ensure convergence on fatigue life. The time domain approach is therefore impractical and this is the main reason why a spectral approach is relevant.

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* Corresponding author. Tel.: +33 130182020; fax: +33 130182019.
  E-mail address: Frederic.kihm@hbmncode.com

Fig. 1 shows example of rainflow cycle histograms obtained from various time domain realizations of a sine-on-random spectrum. The first time series was 20 seconds long (red) and the later was 250 seconds long (blue). The obtained rainflow histograms are scaled to 2000 seconds for comparison purposes. A statistical distribution (dashed black) was obtained using the approach presented in this paper and superimposed with the time domain simulations.
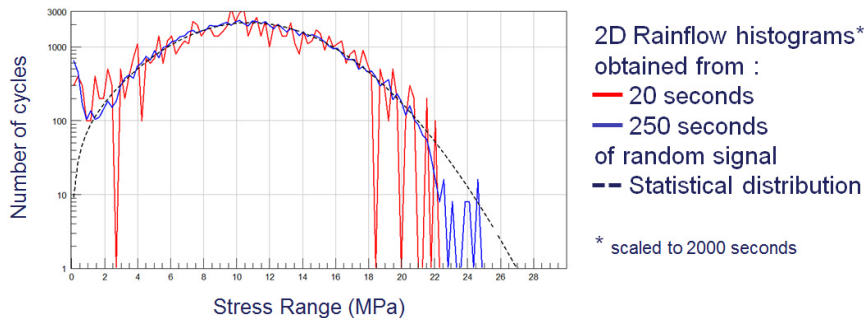

Fig. 1. Rainflow cycle histograms for various time domain realizations vs. spectral approach.

Several observations can be made from Fig. 1

- The longer the time domain simulation, the more chances there are to find cycles with high ranges.
- Longer time domain simulations seem to produce rainflow histogram that converge towards the statistical cycle distribution.
- The tail of the statistical cycle distribution on the high-range end is well defined, allowing a more robust and accurate fatigue life prediction from long term sine-on-random excitation.

2. Methodology

Evaluating the fatigue damage due to a sine-on-random excitation requires a list of stress cycles together with a fatigue curve for the material considered. The various steps involved to obtain the stress cycles are described below. The noise is assumed to be confined to a narrow band of frequencies, meaning that one mode is predominant. The sine tone frequencies are supposed to be within the same frequency range as the noise. Finally, the sine tones frequencies are supposed to be incommensurable. In this case, the sine waves frequencies must be irrational relative to each other. This means they can be considered independent and the relative phase has no importance.

Generally, a sine-on-random signal can be represented as:

$$x(t) = r(t) + \sum_{i=1}^{N} b_i . \cos\left(2\pi f_i t + \varphi_i\right) \tag{1}$$

with $r(t)$ some random noise and b_i, f_i and φ_i the amplitude, frequency and phase of the N sine tones.

Considering that a periodic signal can be decomposed into its Fourier coefficients, Equation (1) can be interpreted as a representation of deterministic periodic signal on random.

The relative importance of the deterministic part in terms of power is indicated by the sine-to-random power ratio a_0^2, given in Equation (2).

$$a_0^2 = \frac{\sigma_s^2}{\sigma_r^2} = \frac{1}{2\sigma_r^2} \sum_{i=1}^{N} b_i^2 \tag{2}$$

with σ_r the RMS value of the noise and σ_s the RMS value of the combined sine waves.

2.1. From sine-on-random acceleration spectrum to sine-on-random stress spectrum

The excitation is generally in the form of a PSD of acceleration with one or several superimposed sine tones, as in MIL STD 810G [1] or in RTCA DO 160G [2]. Assuming a linear structural response, one can derive the PSD of stress from the structure's frequency response function H as per Equation (3). Similarly, the stress response to sinusoidal excitation is determined from Equation (4).

$$PSD_{stress}(f) = H(f).PSD_{excitation}(f).H^*(f) \tag{3}$$

$$Sine_{stress}(f) = H(f).Sine_{excitation}(f) \tag{4}$$

2.2. Transformation of the sine-on-random stress spectrum to the distribution of rainflow cycles

The aim is now to find a statistically derived, rainflow-like distribution of cycles corresponding to the stress PSD and sine tones. S.O. RICE [3] derived the probability density function of peaks in the case where noise and harmonic tones are within the same frequency range. The probability density function of peaks is given in Equation (5).

$$f_p(s) = s.\int_0^\infty x.e^{-\left(\frac{\sigma_r^2.x^2}{2}\right)} J_0(x.s)\prod_{i=1}^N J_0(x.b_i).dx \tag{5}$$

where s is a random variable, representing the peak value, J_0 the Bessel function of the first kind of order zero and b_i, the amplitude of the ith sine wave.

Closed form solutions exist for the integral in Equation (5) when none or one harmonic tone is considered, numerical integration is required in the case of two or more tones are present.

The rainflow cycle counting algorithm identifies stress cycles which ranges are calculated from the values of their peaks and valleys. The range of a rainflow cycle is defined as {peak – valley} of the cycle. The rainflow algorithm pairs peaks and valleys according to their value and position in the history of stress [4], for instance, the peak and valley of a cycle are not necessarily consecutive.

According to the assumptions made at the beginning of this section, the composite process made of narrowband noise and sine tones is narrowband too. This means that all peaks are positive and all valleys are negative. If the sine tones frequencies are not multiple of each other, each peak is associated with a valley of similar magnitude and the cycle's range distribution can be determined from the peak distribution given in Equation (5). Therefore, the cycle's range distribution is identical to the rainflow cycle distribution and is deduced from the peak distribution, as per Equation (6).

$$f_R(S_R) = \frac{1}{2}f_p\left(\frac{S_R}{2}\right) \tag{6}$$

When the noise corresponds to a multimodal stress response, i.e. more than one mode dominates, the noise is considered to be broadband. The bandwidth is characterized by the irregularity factor, a real scalar value calculated from the spectral moments [3,5]. In the broadband case, due to the complex nature of the rainflow counting algorithm [4], the rainflow distribution for the cycles cannot be obtained from the peak distribution alone. However, considering that the largest cycle is always made of the highest peak and the lowest valley once residuals are closed [4], the use of equation (6) to calculate the range distribution is equivalent to considering that peaks and valleys are paired in the most severe way i.e. the highest peak goes with the lowest valley, the second highest peak goes with the second lowest valley, etc. This narrowband approach is therefore expected to be conservative in the broadband case, as observed by Rychlik [6] in the case of random processes.

Other considerations like sine tones frequencies lying outside the noise frequency range or being multiples to each other and influence of relative phases are currently under investigation.

Considering random noise only, the total number of expected peaks per unit time *Np* can be derived by stating that a peak (i.e. a local maximum) is obtained from the number of zero crossings with a negative slope of the derivative of the process $\dot{r}$. *Np* is therefore calculated [3] by dividing $\sigma_{\ddot{r}}^2$ with $\sigma_{\dddot{r}}^2$, referred to as the second and fourth spectral moments of the PSD G_{rr} or the variances of the first and second derivatives of the random process respectively.

In the sine-on-random case, the total number of expected peaks per unit time *Np_sor* can be derived by extending the equation to obtain *Np*. The suggested extension consists in adding the spectral moments of the sine wave to the spectral moments of the noise. The averaged number of peaks per unit time calculated from the stress PSD and the sine waves characteristics is given in Equation (7).

$$N_{p\_sor} = \sqrt{\frac{\sigma_{\dddot{r}}^2 + 1/2\sum_{1}^{N} f_i^4 b_i^2}{\sigma_{\ddot{r}}^2 + 1/2\sum_{1}^{N} f_i^2 b_i^2}} \tag{7}$$

where f_i and b_i are the frequency and amplitude of the ith harmonic out of the N harmonics added to the noise. The distribution of fatigue cycles, comparable to a rainflow cycle histogram, can be calculated using equation (8):

$$N(S_R) = f_R(S_R).N_{p\_sor}.T \tag{8}$$

where *T* is the exposure duration.

2.3. Fatigue damage

The damage at a given stress range S_R is obtained as the ratio of the cycles counted at this stress range level, to the number of cycles required to fail the component at this stress range level, which is given by the material curve like the one illustrated in Fig. 2. The total damage is obtained by summing the damage from each individual level of stress range using Miner's rule [7].
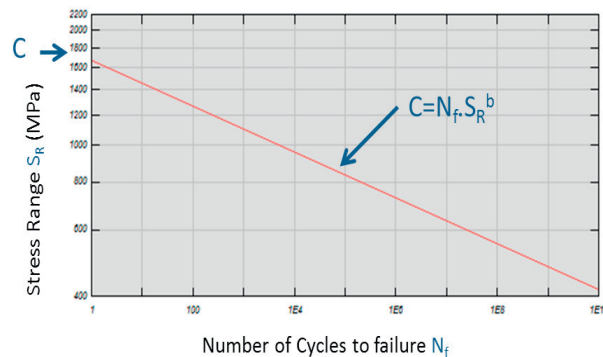

Fig. 2. Typical SN fatigue curve.

The SN curve may be made of several segments. Each segment is represented as a straight line in log space as described by the Basquin power-law relationship given in Equation (9).

$$C = N_f . S_R{}^b \tag{9}$$

where S_R is the stress range in MPa, N_f is the number of rainflow cycles to failure, C is the Basquin coefficient and b is the Basquin exponent. Fatigue damage is determined by Equation (10).

$$D = \frac{1}{N_f} = \frac{1}{C}\int_0^\infty N(S_R).S_R{}^b.dS_R \tag{10}$$

D is the fatigue damage ratio. If $D \geq 1$ then the component is likely to fail within the test duration T. If $D < 1$ then the fatigue life can be determined as T/D in seconds.

3. Industrial application

The present case study illustrates the use of the technique described in this paper to quantify vibration-induced fatigue damage on a component attached to the fuselage of a helicopter. Helicopters are particularly challenging with regards to vibration.

Standard vibration qualification tests are typically performed according to one of the commonly used military design standards such as MIL-STD-810F [2], and RTCA DO-160E [3]. The endurance test commonly uses a sine-on-random vibration profile which is representative of the actual aircraft loading profile. The main source of these sinusoidal tones is attributable to harmonics of the main rotor.

The component is flight safety-critical and the general vibration specification was considered inadequate in this case. A test tailoring exercise was therefore authorised to determine a more appropriate sine-on-random qualification test [8] using the 'GlyphWorks Accelerated Testing' package from HBM-nCode [9].

3.1. The excitation

The vibration endurance test profile was compressed into 20 hours of high cycle fatigue in each axis. It is illustrated in Fig. 3. It is made of a random profile between 10 and 500 Hz with $\sim = 2$ gRMS and sine tones at rotors blade passing frequencies and their harmonics (1nR,2nR,3nR) with amplitudes ranging between 0.1g to 2.0g. In most cases the vertical and lateral accelerations dominate the loading environment and the fore/aft axis is relatively benign. Only the vertical axis is discussed here.
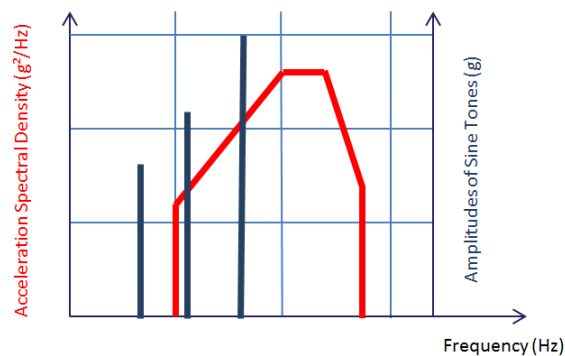

Fig. 3. Sine on random excitation spectrum used.

3.2. Stress sine on random spectrum at critical locations

Both a frequency response analysis and a transient analysis were performed. The frequency response analysis allows finding the spectral response in stress, whereas the transient analysis gives the time series response in stress.

Results from two critical locations were retained for the present case study. Both results show very different sine-to-random power ratios and irregularity factors, as reported in Table 1.

Table 1. Characteristics of two responses.

Location ID	sine-to-random power ratio $a_0{}^2$	irregularity factor
Location 1	1.2	0.98
Location 2	8.2	0.87

The normal stress was calculated on multiple planes. This allows the computed damage due to the loads in X, Y and Z directions to be summed up in the same plane. The critical plane is the plane with the most predicted fatigue damage. The responses shown in Fig. 4 and 5 are given for the same (critical) plane.

Time domain realizations of 100 seconds, sampled at 1024 points per second were obtained from the excitation spectra in the various directions (see Fig. 3). The transient modal superposition analysis allows obtaining time histories of stress responses at the various nodes. An example stress response is illustrated in Fig. 4.
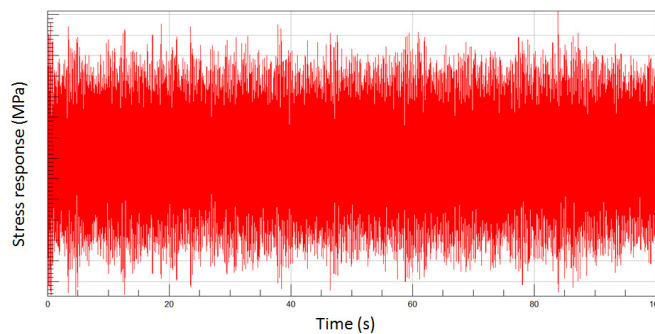

Fig. 4. Example stress response at Location 1

Applying equations (3) and (4) to the transfer function obtained from the frequency response analysis, the sine-on-random response in stress is obtained for Location 1 and 2 and illustrated in Fig. 5 (a) and Fig. 5 (b) respectively.
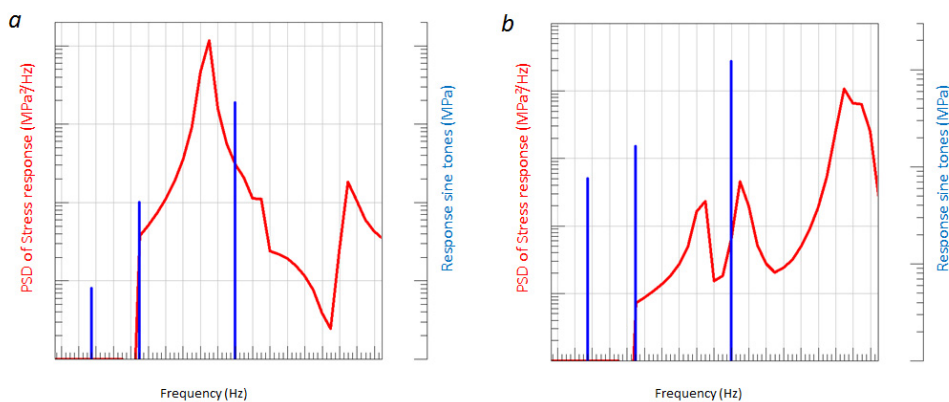

Fig. 5. Sine on random stress response spectrum at (a) Location 1, (b) Location 2

3.3. Derived statistical rainflow distributions and damage– comparison with time domain

The technique presented in this paper allows to evaluate the rainflow stress cycles distribution at every node of the FE model of a component subject to sine-on-random loading. The rainflow cycles distribution can also be obtained from deterministic counting of the cycles in the time domain stress responses.

Fig. 6 (a) and Fig. 6 (b) compare the rainflow cycles distributions obtained from deterministic counting of the cycles on the time domain stress responses (red) vs statistical approach (blue) for both critical locations 1 and 2.
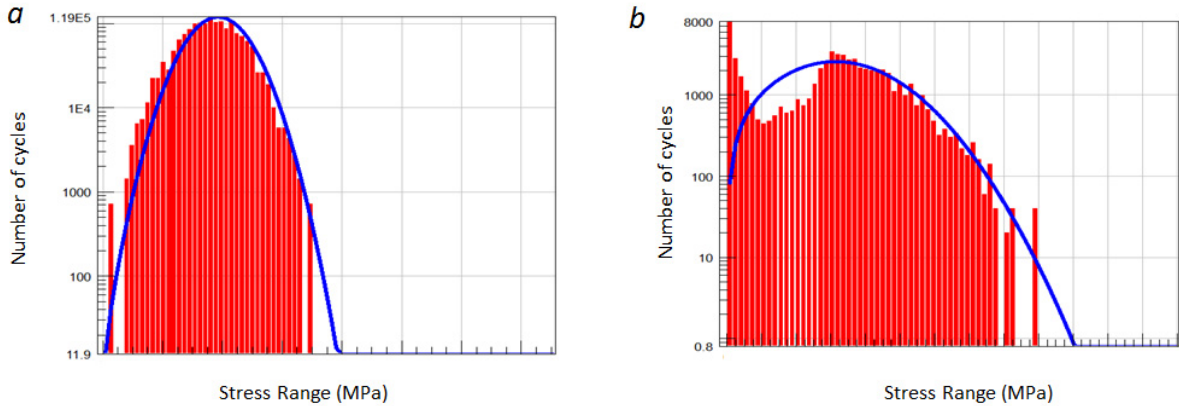

Fig. 6. Comparison between time domain (red) and statistical (blue) cycle distributions for (a) location 1 and (b) location 2

From Table 1 and Fig. 5 (a) and 5 (b), location 1 can be associated to a unimodal narrowband response with sine tones which frequencies lie within the frequency band of the noise (except the lowest frequency one but it has the lowest amplitude). The sine tones have frequencies that are multiple to each other. The response at location 1 is therefore not respecting all the required assumptions to use the technique described in this paper. The response at location 2 is even worse in that respect as its irregularity factor is smaller, meaning it cannot be considered as narrowband. However, in both cases the statistical approach is giving a good approximation of the deterministic histogram and it leads to a far better definition of the tail at the high ranges end, which is particularly important for fatigue analysis.

Fatigue damage can be assessed from the rainflow cycle distributions obtained and an SN material curve. Table 2 show the resulting damage calculated from an SN curve having a Basquin exponent of b=12.

Table 2. Fatigue Damage.

Location ID	Fatigue damage from time domain	Fatigue damage from spectral domain
Location 1	16 E-3	23 E-3
Location 2	26 E-3	36 E-3

Again, the results from the spectral domain and time domain approaches are very similar, and so although some of the assumptions required to obtain the statistical rainflow histogram are not strictly respected. The fatigue damage obtained from the spectral approach is roughly 40% higher, mostly due to the best definition of the tail of the rainflow cycle distributions. The results from the spectral approach are more reliable as they implicitly take into account the probability of high cycles appearing at long exposure durations.

4. Conclusions

A spectral domain approach to fatigue analysis offers advantages over the time domain approach when limited stress time history data are available or computation times are prohibitive. A very robust method of deriving fatigue estimates for sine-on-random vibration is presented.

A case study illustrates the use of the method to assess the fatigue damage on a component attached to the fuselage of a helicopter. Comparisons with traditional time-domain approaches show excellent agreement. The results from the spectral approach are more reliable as they implicitly take into account the probability of high cycles appearing at long exposure durations.

Other applications of spectral fatigue analysis include finding out what the margins of a successful test are or checking that a profile for random vibration test is not over accelerated. In this last case, the excessive loads introduce levels of stress above yield, which alters the load paths and change the failure mode.

So far the existing approach for high-cycle fatigue damage assessment under random vibrations was limited to Gaussian random loads. The new method extends it to multiple sine tones added to Gaussian random loads. This new method is particularly useful to take into account the vibration environment generated by rotating machinery.

Acknowledgements

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