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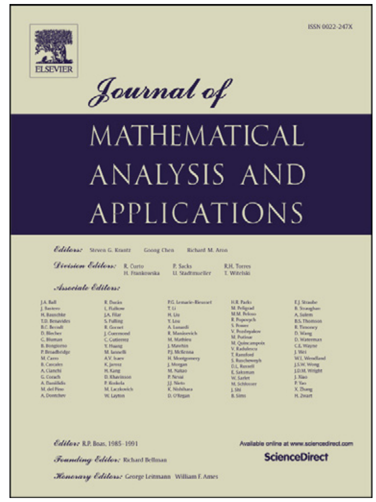

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Steady-state bifurcation analysis of a fully nonlinear quasi-geostrophic vorticity equation

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Abstract

The quasi-geostrophic vorticity equation studied in the present paper is a simplified form of the atmospheric circulation model introduced by Charney and DeVore [J. Atmos. Sci. 36(1979), 1205–1216] on the existence of multiple steady states to the understanding of the persistence of atmospheric blocking. The fluid motion defined by the equation is driven by a zonal thermal forcing and an Ekman friction forcing measured by κ. It is proved that the steady-state solution is globally unique for large κ values while multiple steady-state solutions branch off the basic steady-state solution for $\kappa < \kappa_{\rm crit}$ where the critical value $\kappa_{\rm crit}$ is less than one. Without involvement of viscosity, the equation has fully non-linear property as its non-linear part contains the highest order derivative term. Steady-state bifurcation analysis is essentially based on the compactness, which can be simply obtained for semilinear equations such as the Navier–Stokes equations but is not available for the fully nonlinear quasi-geostrophic vorticity equation in the Euler formulation. Therefore the Lagrangian formulation of the equation is employed to gain the required compactness.

Keywords: quasi-geostrophic vorticity equation, steady-state bifurcation, Lagrange formulation, fully non-linear equation

Mathematics Subject Classification: 35B32, 35B35, 35Q35, 86A10, 76B03

1. Introduction

In an effort to describe the mechanism of atmospheric blocking phenomena, Charney and DeVore [4] introduced a two-dimensional quasi-geostrophic vorticity equation and used a three mode truncation model to show heuristically the existence of multiple steady-state solutions due to non-linear interaction of zonal thermal forcing, Ekman layer energy dissipation and topography wave. Amongst them, a stable steady state with weak zonal disturbance describes the blocking phenomena. The numerical simulations of the multiple steady-state solutions of the quasi-geostrophic vorticity equations originated from Charney and DeVore [4] and have been extensively studied (see, for example, Eert [12], Ierley and Sheremet [13], Jiang et al. [17], Pedlosky [22]), Pierrehumbert and P. Malguzzi [23], Primeau [24], Tung and Rosenthal [29], Holloway and Yoden [33, 34]) in the area of atmospheric science. However, the rigorous analysis supporting the multiple steady-state phenomenon is still lacking.

In the present paper, we are interested in the following quasi-geostrophic vorticity equation simplified from Charney and DeVore [3, 4]

$$\frac{\partial \Delta\psi}{\partial t} + (\nabla \times \psi)\cdot\nabla(\Delta\psi) = -\kappa\Delta(\psi - \psi^*) \tag{1}$$

with a flat topography and the absence of the Coriolis force. Here ∇ is the gradient operator, Δ is the Laplacian, ψ is an unknown stream function, κ is an Ekman dissipative number, $\kappa\Delta\psi^*$ is an external thermal forcing and the vortex $\nabla\times\psi = (-\partial_{x_2}\psi,\, \partial_{x_1}\psi)$.

This is a fully nonlinear third-order partial differential equation. If ω is employed to represent the vorticity $\Delta\psi$, the equation (2) can be rewritten in the non-local form

$$\frac{\partial \omega}{\partial t} + (\nabla \times \psi)\cdot\nabla\omega = -\kappa(\omega - \omega^*)$$

due to in involvement of the integral equation $\psi = \Delta^{-1}\omega$. For the existence and singularities of evolutionary solutions to related non-local equations, one may consult Córdoba et al. [9] and Dong [10].

When $\kappa = 0$, the equation (2) reduces to the Euler equation. Thus the equation (2) is the Euler equation with dissipation (see, for example, [14]). The existence of a steady state and the uniqueness of small steady state for the equation (2) were obtained by Wolansky [32] and Ilyin [14]. A more general form of the equation (2) is known as the Stommel–Charney model [1, 2, 11, 27], when the fluid motion involves the Coriolis force represented the β plane approximation on the mid latitude. The existence of a steady state and the uniqueness of small steady state for the Stommel–Charney model were obtained by Barcilon et al. [1] and Hauk [11]. One may also refer to Ilyin et al. [15] for rigorous analysis on dynamical behaviours of a semilinear quasi-geostrophic equation which is mainly controlled by viscosity coefficient.

However, the uniqueness may no long valid for large forcing and multiple steady states may coexist. The purpose of present paper is to show the existence of multiple steady-state solutions of (2) with respect to a parameter range of κ and the zonal thermal forcing

$$\kappa\Delta\psi^* = -\kappa\cos x_2 \quad \text{with } \psi^* = \cos x_2, \tag{2}$$

employed in [4]. The fluid motion is in the domain $\Omega_a = [0, 2\pi/a]\times[0,2\pi]$ and satisfies the spatially periodic boundary condition [4]

$$\psi(2\pi/a, x_2) = \psi(0, x_2),\;\; \psi(x_1, 0) = \psi(x_1, 2\pi),\;\; \mathbf{x} = (x_1, x_2)\in\Omega_a. \tag{3}$$

The averaging condition

$$\int_{\Omega_a}\psi dx_1 dx_2 = 0 \tag{4}$$

is applied to rule out non-zero constants being solutions of the problem described by (2)–(3). Note that $\psi = \psi^*$ is a steady-state solution with respect to any κ. The solution multiplicity is thus obtained if there exists a family of solutions ψ_κ branching off ψ^* from a critical value $\kappa_{\text{crit}} > 0$.

The main result of the present paper reads as follows:

Theorem 1.1. *For $1/\sqrt{2} \leq a < 1$, the equations (2)–(4) admit a positive critical value*

$$\kappa_a < a\sqrt{\frac{1-a^2}{2(1+a^2)}} \tag{5}$$

and a continuous family of classical steady-state solutions (ψ_κ, κ) branching off the bifurcation point (ψ^*, κ_a) when κ varies across κ_a.

This result shows mechanism behind the existence of a basic steady-state solution bifurcating into two steady-state solutions under the single zonal forcing (2). With the thermal forcing (2), the small κ value implies that the acceleration nonlinearity dominates the circulation flow and then gives rise to multiple steady-state solutions, whereas the increment of the κ value enlarges the linear Ekman layer dissipation and then eventually eliminates the bifurcation phenomenon.

Thus (2) is quite similar to Navier-Stokes equations that the Ekman force $\kappa\Delta\psi$ plays the same roll as the Reynolds viscous force $\frac{1}{\mathrm{Re}}\Delta^2\psi$ to control the solution uniqueness and bifurcation behaviours. For the connection to the Euler equations, the Ekman dissipation force $\kappa\Delta\psi$ was recently unitized by the author [5, 6] to form a dissipative potential flow and then to produce dissipative free-surface Green functions for the cancelation of wave integral singularity in numerical simulations of body motions in free water waves.

The equation (2) is a third-order fully nonlinear partial differential equation and is quite different to traditional semilinear fluid motion equations such as the Navier–Stokes equations discussed in Temam [28] and the quasi-geostrophic equations discussed in Chen et al. [7] and Chen and Price [8]. The semilinearity indicates that the non-linear term can be controlled by the linear term. Therefore the a priori estimates and compactness analysis of Navier–Stokes type equations, available due to the presence of viscous force (see, for example, [7, 8, 28]), are not applicable to the fully non-linear equation (2). Actually, the non-linear term of (2) is the total derivative of fluid velocity along a particle trajectory and hence it is beneficial to use the Lagrangian formulation instead of the Euler formulation (2) to control the nonlinearity of (2).

For the equation (2) with the Dirichlet boundary condition, when the external forcing is changed into multiple ones the existence of multiple steady-state responses was discussed by Wolansky [31]. The present state-state bifurcation analysis is applicable to the Dirichlet boundary value problem. However, for the quasi-geostrophic vorticity equation driven by a single forcing, it was unknown whether the basic solution branches into multiple steady-state solutions when the Ekman dissipation force varies. Moreover the steady-state bifurcation analysis of the present paper, using the Krasnoselskii bifurcation theorem [18] and the linear spectral technique developed from Meshalkin and Sinai [20], Iudovich [16] and Chen et al. [7] and Chen and Price [8], is quite different to the multiple solution technique of Wolansky [31] although the Lagrangian formulation is developed from Wolansky [32].

The functions in the present paper are in the Hölder spaces $C^{k+\alpha}(\Omega_a)$ for integer $k \geq 0$ and real $\alpha \in [0,1)$. Here $C^0(\Omega_a)$ is the Banach space of all continuous functions over Ω_a under the norm

$$\|\phi\|_{C^0} = \max_{\mathbf{x}\in\Omega_a} |\phi(\mathbf{x})|.$$

The C^k and $C^{k+\alpha}$ function spaces are defined as

$$C^k(\Omega_a) = \{\phi \in C^0(\Omega_a);\ \nabla^k\phi \in C^0(\Omega_a)\}$$

with the norm

$$\|\phi\|_{C^k} = \|\phi\|_{C^0} + \|\nabla^k\phi\|_{C^0},$$

$$C^{k+\alpha}(\Omega_a) = \{\phi \in C^k(\Omega_a);\ \|\phi\|_{C^{k+\alpha}} = \|\phi\|_{C^k} + [\nabla^k\phi]_{C^\alpha}\},\ \ 0 < \alpha < 0,$$

with the semi-norm

$$[\phi]_{C^\alpha} = \sup_{\mathbf{x},\mathbf{y}\in\Omega_a,\,\mathbf{x}\neq\mathbf{y}} \frac{|\phi(\mathbf{x}) - \phi(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}.$$

We use the function space

$$C^{k+\alpha}_{\text{per}}(\Omega_a) = \left\{\phi \in C^{k+\alpha}(\Omega_a);\ \ \phi \text{ satisfies the conditions (3) and (4)}\right\}.$$

A steady-state solution ψ of (2)–(4) is said to be regular if $\psi \in C^2_{\text{per}}(\Omega_a)$ and $\Delta\psi \in C^1_{\text{per}}(\Omega_a)$.

This paper is organized as follows. Section 2 exhibits a Lagrangian formulation approach to the quasi-geostrophic flow in a neighborhood of the basic flow ψ^* so that the compactness required by the bifurcation analysis is obtained. Section 3 is devoted to the linear spectral analysis of the quasi-geostrophic vorticity equation in the Lagrangian formulation. The spectral analysis technique is essentially developed from [7, 8, 16, 20]. With the preparations of the compactness and the spectral results, Section 4 is devoted to the verification of the conditions ensuring the occurrence of the steady-state bifurcation phenomenon in Krasnoselskii's theorem. The proof of Theorem 1.1 is finally completed in Section 4.

2. Lagrangian formulation of the fluid motion

For the velocity $\mathbf{u} = (u_1, u_2) = \nabla\times\psi$ of the fluid flow in the domain Ω_a and a trajectory $\mathbf{y} = (y_1, y_2)$ initiating from a particle $\mathbf{x} = (x_1, x_2)$, the fluid motion is described by the Lagrangian formulation

$$\begin{cases} -\dfrac{\partial}{\partial t}\mathbf{y}(\mathbf{x},t) &= \mathbf{u}(\mathbf{y}(\mathbf{x},t)), \quad t>0, \\ \mathbf{y}(\mathbf{x},0) &= \mathbf{x}\in\Omega_a. \end{cases} \tag{6}$$

Thus for the operators

$$\nabla=(\partial_{x_1},\partial_{x_2}), \quad \nabla_{\mathbf{y}}=(\partial_{y_1},\partial_{y_2}), \quad \nabla\mathbf{y}\cdot\nabla_{\mathbf{y}}=(\nabla y_1)\partial_{y_1}+(\nabla y_2)\partial_{y_2}$$

and the 2×2 identity matrix I, we have

$$-\frac{\partial}{\partial t}\nabla\mathbf{y}(\mathbf{x},t) = \nabla\mathbf{y}\cdot\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x},t)), \;\; t>0, \tag{7}$$

$$\nabla\mathbf{y}(\mathbf{x},0) = I. \tag{8}$$

This system implies the Euler identity

$$-\frac{\partial}{\partial t}\det(\nabla\mathbf{y})=\det(\nabla\mathbf{y})\nabla_{\mathbf{y}}\cdot\mathbf{u}(\mathbf{y})$$

and hence the incompressible flow transformation property

$$\det(\nabla\mathbf{y})=1. \tag{9}$$

It follows from (7) that

$$\frac{\partial}{\partial t}|\partial_{x_i}y_j|\le|\partial_{x_i}y_1|\,|\partial_{y_1}u_j(\mathbf{y})|+|\partial_{x_i}y_2|\,|\partial_{y_2}u_j(\mathbf{y})|, \;\; i,\, j=1,\,2. \tag{10}$$

Here the time derivative $\partial_t|f|$ is in the sense of $\limsup_{\delta t\to 0}\frac{|f(t+\delta t)|-|f(t)|}{\delta t}$. We thus have

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}|\nabla\mathbf{y}|^2 \\ &\le \left(|\partial_{x_1}y_1|^2+|\partial_{x_2}y_1|^2\right)|\partial_{y_1}u_1(\mathbf{y})|+\left(|\partial_{x_1}y_2|^2+|\partial_{x_2}y_2|^2\right)|\partial_{y_2}u_2(\mathbf{y})| \\ &\quad +\frac{1}{2}\left(|\partial_{x_1}y_2|^2+|\partial_{x_1}y_1|^2+|\partial_{x_2}y_2|^2+|\partial_{x_2}y_1|^2\right)\left(|\partial_{y_1}u_2(\mathbf{y})|+|\partial_{y_2}u_1(\mathbf{y})|\right) \\ &\le \frac{5}{4}|\nabla\mathbf{y}|^2\|\nabla_{\mathbf{y}}\mathbf{u}\|_{C^0}. \end{aligned}$$

This together with (8) gives the flow estimate expressed as

$$|\nabla\mathbf{y}(\mathbf{x},t)| \le \sqrt{2}\mathrm{e}^{\frac{5}{4}t\|\nabla_{\mathbf{y}}\mathbf{u}\|_{C^0}}. \tag{11}$$

On the other hand, the study of the uniqueness and the multiplicity of the classical solutions around the basic solution ψ^* is based on the flow estimate expressed as

$$|\nabla\mathbf{y}(\mathbf{x},t)|\le(\sqrt{2}+\sqrt{5}t)\mathrm{e}^{2t\sqrt{\|\nabla_{\mathbf{y}}\mathbf{u}-\nabla_{\mathbf{y}}\mathbf{u}^*\|_{C^0}}} \tag{12}$$

for $\mathbf{u}^* = \nabla \times \psi^*$. Hence, for convenience, we may assume that the inequality

$$\|\nabla_{\mathbf{y}}\mathbf{u} - \nabla_{\mathbf{y}}\mathbf{u}^*\|_{C^0} \leq \frac{1}{2} \tag{13}$$

is always true since the present investigation aims at the uniqueness and bifurcation around the basic flow ψ^*.

To show the validity of (12), we set $\epsilon = \|\nabla_{\mathbf{y}}\mathbf{u} - \nabla_{\mathbf{y}}\mathbf{u}^*\|_{C^0}$ or

$$\epsilon = \left\| \sqrt{(\partial_{y_1}u_1)^2 + (\partial_{y_2}u_1 - \cos y_2)^2 + (\partial_{y_1}u_2)^2 + (\partial_{y_2}u_2)^2} \right\|_{C^0}.$$

With the use of the matrix inequality notation

$$(a_{i,j}) \leq (b_{i,j}) \text{ whenever } a_{i,j} \leq b_{i,j} \text{ for all } i \text{ and } j,$$

the equation (10) can be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} |\partial_{x_1}y_1| & |\partial_{x_2}y_1| \\ |\partial_{x_1}y_2| & |\partial_{x_2}y_2| \end{pmatrix} &\leq \begin{pmatrix} |\partial_{y_1}u_1| & |\partial_{y_2}u_1| \\ |\partial_{y_1}u_2| & |\partial_{y_2}u_2| \end{pmatrix}\begin{pmatrix} |\partial_{x_1}y_1| & |\partial_{x_2}y_1| \\ |\partial_{x_1}y_2| & |\partial_{x_2}y_2| \end{pmatrix} \\
&\leq \begin{pmatrix} \epsilon & 1+\epsilon \\ \epsilon & \epsilon \end{pmatrix}\begin{pmatrix} |\partial_{x_1}y_1| & |\partial_{x_2}y_1| \\ |\partial_{x_1}y_2| & |\partial_{x_2}y_2| \end{pmatrix}
\end{aligned}$$

Multiplying this inequality by the matrix

$$\exp\left(-t\begin{pmatrix} \epsilon & 1+\epsilon \\ \epsilon & \epsilon \end{pmatrix}\right)$$

and using the initial condition $\nabla\mathbf{y}(\mathbf{x}, 0) = I$, we have

$$\begin{aligned}
&\begin{pmatrix} |\partial_{x_1}y_1| & |\partial_{x_2}y_1| \\ |\partial_{x_1}y_2| & |\partial_{x_2}y_2| \end{pmatrix} \\
&\leq \exp\left(t\begin{pmatrix} \epsilon & 1+\epsilon \\ \epsilon & \epsilon \end{pmatrix}\right) \\
&= \begin{pmatrix} \frac{\sqrt{\epsilon^2+\epsilon}}{2\epsilon} & \frac{\sqrt{\epsilon^2+\epsilon}}{2\epsilon} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{t(\epsilon+\sqrt{\epsilon^2+\epsilon})} & 0 \\ 0 & \mathrm{e}^{t(\epsilon-\sqrt{\epsilon^2+\epsilon})} \end{pmatrix}\begin{pmatrix} \frac{\epsilon}{\sqrt{\epsilon^2+\epsilon}} & -1 \\ \frac{\epsilon}{\sqrt{\epsilon^2+\epsilon}} & 1 \end{pmatrix}
\end{aligned}$$

and hence, for $\lambda_1 = \epsilon + \sqrt{\epsilon^2+\epsilon}$ and $\lambda_2 = \epsilon - \sqrt{\epsilon^2+\epsilon}$,

$$\begin{aligned}
|\nabla\mathbf{y}(\mathbf{x}, t)|^2 &\leq \frac{\left(\mathrm{e}^{\lambda_1 t} + \mathrm{e}^{\lambda_2 t}\right)^2 + \left(\frac{\epsilon+1}{\epsilon} + \frac{\epsilon}{\epsilon+1}\right)\left(\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t}\right)^2}{2} \\
&\leq \left[2 + \frac{1}{2}\left(\frac{\epsilon+1}{\epsilon} + \frac{\epsilon}{\epsilon+1}\right)(\lambda_1 t - \lambda_2 t)^2\right]\mathrm{e}^{2t(\epsilon+\sqrt{\epsilon^2+\epsilon})} \\
&\leq \left[2 + 2t^2[(\epsilon+1)^2 + \epsilon^2]\right]\mathrm{e}^{2t(\epsilon+\sqrt{\epsilon^2+\epsilon})} \\
&\leq (2 + 5t^2)\mathrm{e}^{2t(\epsilon+\sqrt{\epsilon^2+\epsilon})}.
\end{aligned}$$

Here we have used equation (13). The validity of (12) is thus demonstrated.

The following lemma shows the well-posedness of the fluid motion in the Lagrangian formulation:

Lemma 2.1. *Assume that* $\kappa > 0$ *and* $\psi \in C^2_{\mathrm{per}}(\Omega_a)$ *such that*

$$\|\nabla^2\psi - \nabla^2\psi^*\|_{C^0} < \frac{\kappa^2}{4}. \tag{14}$$

Then the operator $\kappa + (\nabla \times \psi) \cdot \nabla$ *is a bijection mapping the space*

$$D = \{f \in C^1_{\mathrm{per}}(\Omega_a);\ (\kappa + (\nabla \times \psi) \cdot \nabla) f \in C^1_{\mathrm{per}}(\Omega_a)\}$$

onto $C^1_{\mathrm{per}}(\Omega_a)$ *and*

$$\begin{aligned} & \|[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1} f\|_{C^1} \\ \leq \quad & \left(\frac{1}{\kappa} + \frac{\sqrt{2}(\kappa - 2\sqrt{\|\nabla^2\psi - \nabla^2\psi^*\|_{C^0}}) + \sqrt{5}}{(\kappa - 2\sqrt{\|\nabla^2\psi - \nabla^2\psi^*\|_{C^0}})^2} \right) \|f\|_{C^1}. \end{aligned}$$

Proof. For the injection assertion, we see that the equation

$$(\kappa + (\nabla \times \psi) \cdot \nabla) f = 0$$

implies, with the use of integration by parts,

$$\begin{aligned} \kappa \int_{\Omega_a} f^2 dx_1 dx_2 &= -\int_{\Omega_a} f (\nabla \times \psi) \cdot \nabla f dx_1 dx_2 \\ &= \int_{\Omega_a} f (\nabla \times \psi) \cdot \nabla f dx_1 dx_2 \\ &= -\kappa \int_{\Omega_a} f^2 dx_1 dx_2, \end{aligned}$$

which shows $f = 0$.

For the surjection assertion, we consult [32] to define the operator

$$T_\psi f(\mathbf{x}) = \int_0^\infty e^{-\kappa s} f(\mathbf{y}(\mathbf{x}, s)) ds,$$

which is utilized to show the required conditions

$$T_\psi f \in D \quad \text{and} \quad (\kappa + (\nabla \times \psi) \cdot \nabla) T_\psi f = f.$$

Indeed, upon the observation of the equation

$$\nabla T_\psi f(\mathbf{x}) = \int_0^\infty e^{-\kappa s} \nabla \mathbf{y}(\mathbf{x}, s) \cdot \nabla_{\mathbf{y}} f(\mathbf{y}(\mathbf{x}, s)) ds \tag{15}$$

and the quantity $\epsilon = \|\nabla^2\psi - \nabla^2\psi^*\|_{C^0}$, it follows from (12) that

$$\begin{aligned}|T_\psi f(\mathbf{x})| + |\nabla T_\psi f(\mathbf{x})| &\leq \int_0^\infty e^{-\kappa s}\|f\|_{C^0}ds + \int_0^\infty e^{-\kappa s}\|\nabla f\|_{C^0}\|\nabla \mathbf{y}\|_{C^0}ds \\ &\leq \frac{1}{\kappa}\|f\|_{C^0} + \int_0^\infty (\sqrt{2}+\sqrt{5}s)e^{-\kappa s+2\sqrt{\epsilon}s}\|\nabla f\|_{C^0}ds \\ &\leq \left(\frac{1}{\kappa} + \frac{\sqrt{2}}{\kappa - 2\sqrt{\epsilon}} + \frac{\sqrt{5}}{(\kappa-2\sqrt{\epsilon})^2}\right)\|f\|_{C^1},\end{aligned}$$

which gives the estimate of the operator T_ψ.

To verify the continuity of the function $\nabla T_\psi f$, we employ (7) and (12) to produce

$$|\mathbf{y}(\mathbf{x},t) - \mathbf{y}(\mathbf{x}',t)| \leq (\sqrt{2}+\sqrt{5}t)e^{2\sqrt{\epsilon}t}|\mathbf{x}-\mathbf{x}'|, \quad \mathbf{x}, \mathbf{x}' \in \Omega_a, \tag{16}$$

and

$$\begin{aligned}-\frac{\partial}{\partial t}(\nabla\mathbf{y}(\mathbf{x},t) - \nabla\mathbf{y}(\mathbf{x}',t)) &= (\nabla\mathbf{y}(\mathbf{x},t) - \nabla\mathbf{y}(\mathbf{x}',t))\cdot\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x},t)) \\ &\quad +\nabla\mathbf{y}(\mathbf{x}',t)\cdot(\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x},t)) - \nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x}',t))).\end{aligned}$$

Hence the derivation of (12) implies

$$\begin{aligned}&|\nabla\mathbf{y}(\mathbf{x},t) - \nabla\mathbf{y}(\mathbf{x}',t))| \\ &\leq \int_0^t (\sqrt{2}+\sqrt{5}t)e^{2\sqrt{\epsilon}(t-s)}|\nabla\mathbf{y}(\mathbf{x}',s)|\,|\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x},s)) - \nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x}',s))|ds \\ &\leq \mathrm{e}^{2t\sqrt{\epsilon}}(\sqrt{2}+\sqrt{5}t)^2\int_0^t |\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x},s)) - \nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y}(\mathbf{x}',s))|ds. \qquad (17)\end{aligned}$$

Moreover, for any constant $\tau > 0$, it follows from (12) and (15) that

$$\begin{aligned}&|\nabla T_\psi f(\mathbf{x}) - \nabla T_\psi f(\mathbf{x}')| \\ &\leq \int_0^\infty e^{-\kappa s+2s\sqrt{\epsilon}}(\sqrt{2}+\sqrt{5}s)|\nabla_{\mathbf{y}} f(\mathbf{y}(\mathbf{x},s)) - \nabla_{\mathbf{y}} f(\mathbf{y}(\mathbf{x}',s))|ds \\ &\quad +\|\nabla f\|_{C^0}\int_0^\infty e^{-\kappa s}\,|\nabla\mathbf{y}(\mathbf{x},s) - \nabla\mathbf{y}(\mathbf{x}',s)|ds \\ &\leq 3\|\nabla f\|_{C^0}\int_\tau^\infty e^{-\kappa s+2s\sqrt{\epsilon}}(\sqrt{2}+\sqrt{5}s)ds \qquad (18) \\ &\quad +\int_0^\tau e^{-\kappa s+2s\sqrt{\epsilon}}(\sqrt{2}+\sqrt{5}s)|\nabla_{\mathbf{y}} f(\mathbf{y}(\mathbf{x},s)) - \nabla_{\mathbf{y}} f(\mathbf{y}(\mathbf{x}',s))|ds \qquad (19) \\ &\quad +\|\nabla f\|_{C^0}\int_0^\tau e^{-\kappa s}\,|\nabla\mathbf{y}(\mathbf{x},s) - \nabla\mathbf{y}(\mathbf{x}',s)|ds. \qquad (20)\end{aligned}$$

Therefore, for any $\varepsilon > 0$, we can use (12), (14), (16) , (17) and the continuity of ∇f and $\nabla\mathbf{u}$ to demonstrate that each of the items (18)–(20) is bounded by

$\varepsilon/3$, provided that $\tau > 0$ is sufficiently large and $|\mathbf{x} - \mathbf{x}'|$ is sufficiently small. Hence $T_\psi f \in C^1_{\text{per}}(\Omega_a)$.

The surjection is due to the validity of the identity

$$(\kappa + (\nabla \times \psi) \cdot \nabla) T_\psi f = f,$$

which is demonstrated as follows:

$$\begin{aligned}
(\nabla \times \psi) \cdot \nabla T_\psi f(\mathbf{x}) &= \lim_{t \to 0+} (\nabla_{\mathbf{y}} \times \psi(\mathbf{y}(\mathbf{x}, t))) \cdot \nabla_{\mathbf{y}} T_\psi f(\mathbf{y}(\mathbf{x}, t)) \\
&= -\lim_{t \to 0+} \frac{\partial \mathbf{y}(\mathbf{x}, t)}{\partial t} \cdot \nabla_{\mathbf{y}} T_\psi f(\mathbf{y}(\mathbf{x}, t)) \ , \quad \text{by (6)}, \\
&= -\lim_{t \to 0+} \frac{\partial}{\partial t} T_\psi f(\mathbf{y}(\mathbf{x}, t)) \\
&= -\lim_{t \to 0+} \frac{\partial}{\partial t} \int_0^\infty e^{-\kappa s} f(\mathbf{y}(\mathbf{x}, t + s)) ds \\
&= -\lim_{t \to 0+} \frac{\partial}{\partial t} \int_t^\infty e^{-\kappa(s-t)} f(\mathbf{y}(\mathbf{x}, s)) ds = -\kappa T_\psi f(\mathbf{x}) + f(\mathbf{x}).
\end{aligned}$$

The proof is completed.

As a consequence of Lemma 2.1, the steady-state problem of the Euler formulation (2)–(4) becomes the Lagrangian formulation problem

$$-\Delta \psi = \kappa [\kappa + (\nabla \times \psi) \cdot \nabla]^{-1} \psi^*$$

or

$$-\Delta \psi(\mathbf{x}) = \kappa \int_0^\infty \mathrm{e}^{-\kappa s} \psi^*(\mathbf{y}(\mathbf{x}, s)) ds, \tag{21}$$

provided that $\psi \in C^2_{\text{per}}(\Omega_a)$ satisfies the condition (14).

It is readily seen that the proof of Lemma 2.1 remains true if we utilize the estimate (11) instead of the estimate (12). More precisely, the proof of Lemma 2.1 implies the following regularity criterion.

Lemma 2.2. *For $0 < \alpha < 1$ and $\kappa > 0$, let $\psi \in C^2_{\text{per}}(\Omega_a)$ be a solution of (21) satisfying either the condition (14) or the condition*

$$\|\nabla^2 \psi\|_{C^0} < \frac{4}{5} \kappa. \tag{22}$$

Then $\psi \in C^{2+\alpha}_{\text{per}}(\Omega_a)$ and $\Delta \psi \in C^1_{\text{per}}(\Omega_a)$. That is, ψ is a regular solution of the problem described by (2)–(4).

The uniqueness assertion of Theorem 1.1 is implied from the following.

Theorem 2.1. *Let $\kappa \geq \frac{1}{a}$ and $\psi \in C^2_{\text{per}}(\Omega_a)$ be a solution of the Lagrange formulation problem (21) or the Euler formulation problem (2)–(4) satisfying the condition (14). Then ψ is regular and $\psi = \psi^*$ holds true.*

The uniqueness was discussed [1, 11, 14, 32] in the vicinity of a small steady-state solution. In contrast, Theorem 2.1 is on the uniqueness in the vicinity of the basic steady-state solution ψ^*, which is not small.

Proof. We employ Lemma 2.2 to obtain the regularity of ψ, which is a steady-state solution of (2)–(4). The observation

$$-\Delta\psi^* = \kappa[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^* \tag{23}$$

and the application of the L_2 norm

$$\|\phi\|_{L_2} = \left(\int_{\Omega_a} |\phi(\mathbf{x})|^2 dx_1 dx_2\right)^{1/2}$$

yield that

$$\begin{aligned}
&\|\Delta\psi - \Delta\psi^*\|_{L_2} \\
&\quad = \kappa\|[\kappa + (\nabla\times\psi)\cdot\nabla]^{-1}\psi^* - [\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^*\|_{L_2} \\
&\quad = \kappa\|[\kappa + (\nabla\times\psi)\cdot\nabla]^{-1}(\nabla\times\psi - \nabla\times\psi^*)\cdot\nabla[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^*\|_{L_2} \\
&\quad \le \|(\nabla\times\psi - \nabla\times\psi^*)\cdot\nabla[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^*\|_{L_2},
\end{aligned}$$

where we have used the variable transformation property (9) and the integral formulation (21). By (23), we thus have

$$\begin{aligned}
\|\Delta\psi - \Delta\psi^*\|_{L_2} &\le \frac{1}{\kappa}\|(\nabla\times\psi - \nabla\times\psi^*)\cdot\nabla\psi^*\|_{L_2} \\
&\le \frac{1}{\kappa}\|\nabla\times\Delta^{-1}\Delta(\psi - \psi^*)\|_{L_2} \\
&< \frac{1}{\kappa a}\|\Delta\psi - \Delta\psi^*\|_{L_2}.
\end{aligned}$$

whenever $\psi \ne \psi^*$. This leads to a contradiction since $\kappa a \ge 1$. Hence $\psi = \psi^*$.

The proof of Theorem 2.1 and hence the proof of Theorem 1.1 (i) are completed.

3. Linear spectral analysis

For steady-state solutions branching off the basic solution $\psi^* = \cos x_2$ or the existence of steady-state solutions in a vicinity of ψ^*, it follows from Lemma 2.1 that the steady-state Euler formulation problem (2)–(4) is equivalent to the Lagrangian formulation problem

$$\psi + \kappa\Delta^{-1}[\kappa + (\nabla\times\psi)\cdot\nabla]^{-1}\psi^* = 0, \quad \psi \in C^2_{\text{per}}(\Omega_a),\ \Delta\psi \in C^1(\Omega_a). \tag{24}$$

However the bifurcation phenomenon of (24) results from the nonlinearity and linear spectral analysis of the problem (24). This section is contributed to the spectral analysis of the operator L_κ linearized from the non-linear operator

$F(\psi, \kappa)$, the left-hand side term of (24), around the basic flow ψ^*. By an elementary manipulation, the operator L_κ can be linearized as

$$\begin{aligned} L_\kappa \psi &= \lim_{s\to 0} \frac{F(\psi^* + s\psi, \kappa) - F(\psi^*, \kappa)}{s} \\ &= \psi + \lim_{s\to 0}\left(\frac{\kappa\Delta^{-1}[\kappa + (\nabla\times\psi^* + s\nabla\times\psi)\cdot\nabla]^{-1}\psi^*}{s} \right. \\ &\quad \left. - \frac{\kappa\Delta^{-1}[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^*}{s}\right) \\ &= \psi - \kappa\Delta^{-1}[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi)\cdot\nabla[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^* \\ &= \psi - \Delta^{-1}[\kappa + (\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi)\cdot\nabla\psi^* \\ &= \psi + \Delta^{-1}[\kappa + \sin x_2\partial_{x_1}]^{-1}(\sin x_2\partial_{x_1}\psi), \end{aligned} \tag{25}$$

where we have used the solution property (23).

We can now examine the critical real spectral problem

$$L_\kappa\psi = 0 \tag{26}$$

in the space $C^{2+\alpha}_{\rm per}(\Omega_a)$ with $0 \le \alpha \le 1$. Here κ is said to be a critical if equation (26) admits a non-zero solution or an eigenfunction $\psi \in C^{2+\alpha}_{\rm per}(\Omega_a)$. The spectral problem is restricted in the even function subspace

$$\hat{C}^{2+\alpha}_{\rm per}(\Omega_a) = \left\{\psi \in C^{2+\alpha}_{\rm per}(\Omega_a);\ \psi(-\mathbf{x}) = \psi(\mathbf{x})\right\}.$$

By Fourier expansion, the function ψ in $\hat{C}^{2+\alpha}_{\rm per}(\Omega_a)$ is generally expressed as

$$\psi = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} b_{m,n}\cos(max_1 + nx_2).$$

The spectral result is stated as follows:

Theorem 3.1. *Let $\frac{1}{\sqrt{2}} \le a < 1$ and $\kappa > 0$. Then there exists a positive critical value*

$$\kappa_a < a\sqrt{\frac{1-a^2}{2(1+a^2)}}$$

such that

$$\dim \bigcup_{i=1}^{\infty}\left\{\psi = \sum_{n=-\infty}^{\infty} b_n\cos(ax_1 + nx_2) \in \hat{C}^{2+\alpha}_{\rm per}(\Omega_a);\ L^i_{\kappa_a}\psi = 0\right\} = 1. \tag{27}$$

If $m \ne 1$ is a nonnegative integer, then it is valid that

$$\dim\left\{\psi = \sum_{n=-\infty}^{\infty} b_n\cos(max_1 + nx_2) \in \hat{C}^{2+\alpha}_{\rm per}(\Omega_a);\ L_\kappa\psi = 0\right\} = 0. \tag{28}$$

Theorem 3.1 is proved by a continued fraction technique developed from Chen et al. [7] and Chen and Price [8] and originated from Mishalkin and Sinai [20] and Iudovich [16]. However, the linear operator L_κ now involves the Lagrangian formulation aspect.

Proof. To verify the validity of (28), we use (25) to rewrite the spectral equation $L_\kappa \psi = 0$ as

$$\Delta\psi + (\kappa + \sin x_2 \partial_{x_1})^{-1}(\sin x_2 \partial_{x_1} \psi) = 0. \tag{29}$$

It is readily seen that the operator

$$(\kappa + \sin x_2 \partial_{x_1})^{-1} = [\kappa + (\nabla \times \psi^*) \cdot \nabla]^{-1}$$

maps C^1 into C^1 or $\Delta\psi \in C^1$. Thus we may apply the operator $(\kappa + \sin x_2 \partial_{x_1})$ to (29) to produce the spectral equation

$$\kappa\Delta\psi + \sin x_2(\Delta + 1)\partial_{x_1}\psi = 0. \tag{30}$$

Multiplying (30) by $(\Delta + 1)\psi$ and integrating the resultant equation over the domain Ω_a, we have the integral equation

$$0 = \int_{\Omega_a} \Delta\psi(\Delta\psi + \psi) dx_1 dx_2. \tag{31}$$

The substitution of the function

$$\psi = \sum_{n=-\infty}^{\infty} b_n \cos(amx_1 + nx_2), \quad m \neq 1.$$

into (31) simply implies $b_n \equiv 0$ and hence (28) is verified. Here we have used the average condition (4) to confirm $b_0 = 0$ whenever $m = 0$.

To show the existence of the critical number κ_a, we substitute the eigenfunction

$$\psi = \sum_{n=-\infty}^{\infty} b_n \cos(ax_1 + nx_2) \tag{32}$$

into (30) to obtain the iteration equation, for arbitrary integer n,

$$2\kappa(a^2 + n^2)b_n - a[a^2 + (n+1)^2 - 1]b_{n+1} + a[a^2 + (n-1)^2 - 1]b_{n-1} = 0 \tag{33}$$

or

$$\kappa d_n(\beta_n - 1)b_n - (\beta_{n+1} - 1)b_{n+1} + (\beta_{n-1} - 1)b_{n-1} = 0 \tag{34}$$

for

$$\beta_n = a^2 + n^2 \text{ and } d_n = \frac{2\beta_n}{a(\beta_n - 1)}. \tag{35}$$

Notice that $(\beta_n - 1)b_n \neq 0$ for any n since $b_n \equiv 0$ if and only if $b_{n_0} = 0$ for an integer n_0 (see [20]). This enables us to define the quantities

$$\gamma_n = \frac{(\beta_n - 1)b_n}{(\beta_{n-1} - 1)b_{n-1}}, \quad \gamma_{-n} = \frac{(\beta_n - 1)b_{-n}}{(\beta_{n-1} - 1)b_{-n+1}} \quad \text{for} \quad n > 0. \tag{36}$$

Thus by dividing (34) with the quntity $(\beta_n - 1)b_n$, the equation (35) is written as

$$kd_n - \gamma_{n+1} + \frac{1}{\gamma_n} = 0 \quad \text{for } n > 0, \tag{37}$$

$$kd_n - \frac{1}{\gamma_{-n}} + \gamma_{-n-1} = 0 \quad \text{for } n > 0, \tag{38}$$

$$d_0 k - \gamma_1 + \gamma_{-1} = 0 \quad \text{for } n = 0. \tag{39}$$

With the use of (37)–(38), we have

$$\gamma_{\pm n} = \frac{\mp 1}{\kappa d_n \mp \gamma_{\pm(n+1)}} = \cfrac{\mp 1}{\kappa d_n + \cfrac{1}{\kappa d_{n+1} + \cfrac{1}{\ddots}}} \quad \text{for} \quad n \geq 1. \tag{40}$$

It follows from (35), (39) and (40) that the spectral problem (26) or (34) is equivalent to the equation

$$\frac{a}{1 - a^2} = \cfrac{1}{\kappa^2 d_1 + \cfrac{1}{d_2 + \cfrac{1}{\kappa^2 d_3 + \cfrac{1}{d_4 + \cfrac{1}{\ddots}}}}}. \tag{41}$$

The function $P(\kappa)$, representing the right-hand side term of (41), is the Stieltjes continued fraction. It follows from [26] or [30, Theorem 28.1] that $P(\kappa)$ uniformly convergent to a positive value and is an analytic function of $\kappa > 0$. Upon observation of P being strictly monotone function of κ such that

$$\lim_{\kappa \to \infty} P(\kappa) = 0, \quad \lim_{\kappa \to 0} P(\kappa) = \infty,$$

there exists a unique critical value $\kappa = \kappa_a > 0$ satisfying (41). Thus for such a critical value $\kappa = \kappa_a$, the coefficients b_n of the associated eigenfunction ψ in the form of (32) and (36) are subject to the expression

$$b_n = \begin{cases} c\dfrac{a^2 - 1}{a^2 + n^2 - 1}\gamma_1 \cdots \gamma_n, & n \geq 1, \\ c, & n = 0, \\ (-1)^n b_{-n}, & n \leq -1. \end{cases} \tag{42}$$

for an arbitrary constant c. Equation (40) implies that

$$\lim_{n\to\infty}\gamma_n = \frac{-1}{\dfrac{2\kappa}{a} - \lim\limits_{n\to\infty}\gamma_n}$$

or

$$\lim_{n\to\infty}\gamma_n = \frac{\kappa}{a} - \sqrt{\frac{\kappa^2}{a^2}+1} = \frac{-1}{\frac{k}{a}+\sqrt{\frac{\kappa^2}{a^2}+1}}.$$

This gives the smoothness of the eigenfunction ψ expressed by (32) and (42) and hence $\psi \in C^{2+\alpha}_{\text{per}}(\Omega_a)$. That is,

$$\dim\left\{\psi = \sum_{n=-\infty}^{\infty} b_n\cos(ax_1+nx_2) \in \hat{C}^{2+\alpha}_{\text{per}}(\Omega_a);\ L_{\kappa_a}\psi = 0\right\} = 1. \tag{43}$$

The upper bound of the critical value κ_a is an immediate consequence of the inequality

$$\frac{a}{1-a^2} \leq \frac{1}{\kappa_a^2 d_1} = \frac{a^3}{2\kappa_a^2(a^2+1)},$$

which follows from (35) and (41).

To prove the spectral simplicity given in (27), it is sufficient to verify the property

$$\dim\bigcup_{i=1}^{2}\left\{\psi = \sum_{n=-\infty}^{\infty} b_n\cos(ax_1+nx_2) \in \hat{C}^{2+\alpha}_{\text{per}}(\Omega_a);\ L^i_{\kappa_a}\psi = 0\right\} = 1. \tag{44}$$

We see that the equation $L^2_{\kappa_a}\psi = 0$ can be written in the form

$$L_{\kappa_a}\psi' = 0 \quad \text{and} \quad \psi' = L_{\kappa_a}\psi \tag{45}$$

or, equivalently,

$$\kappa_a\Delta\psi' + \sin x_2(\Delta+1)\partial_{x_1}\psi' = 0, \tag{46}$$
$$\kappa_a\Delta\psi + \sin x_2(\Delta+1)\partial_{x_1}\psi = (\kappa_a + \sin x_2\partial_{x_1})\Delta\psi'. \tag{47}$$

By the Fourier expansions

$$\psi = \sum_{n=-\infty}^{\infty} b_n\cos(ax_1+nx_2) \quad \text{and} \quad \psi' = \sum_{n=-\infty}^{\infty} b'_n\cos(ax_1+nx_2),$$

the equations (46)-(47) reduce respectively to the iteration equations

$$2\kappa_a\beta_n b'_n - a(\beta_{n+1}-1)b'_{n+1} + a(\beta_{n-1}-1)b'_{n-1} = 0 \tag{48}$$

and

$$\begin{aligned}&2\kappa_a\beta_n b_n - a(\beta_{n+1}-1)b_{n+1} + a(\beta_{n-1}-1)b_{n-1}\\&= 2\kappa_a\beta_n b'_n - a\beta_{n+1}b'_{n+1} + a\beta_{n-1}b'_{n-1}\end{aligned} \tag{49}$$

for any arbitrary integer n. Therefore from the demonstration of the assertion (43), it remains to prove that $\psi' = 0$ or $b'_n \equiv 0$. Due to the equivalence of (34) and (48), all the equations involving the proof of (43) hold true if b_n is replaced by b'_n therein.

Multiplying the nth equation of (48) by $(-1)^n(\beta_n - 1)b_n$ and the nth equation of (49) by $(-1)^n(\beta_n - 1)b'_n$ and then summing the resultant equations respectively, we have

$$0 = \sum_{n=-\infty}^{\infty}(-1)^n(\beta_n-1)b_n[2\kappa_a\beta_n b'_n - a(\beta_{n+1}-1)b'_{n+1}+a(\beta_{n-1}-1)b'_{n-1}] \tag{50}$$

and

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(-1)^n(\beta_n-1)b'_n[2\kappa_a\beta_n b_n - a(\beta_{n+1}-1)b_{n+1} + a(\beta_{n-1}-1)b_{n-1}]\\&\quad= \sum_{n=-\infty}^{\infty}(-1)^n(\beta_n-1)b'_n[2\kappa_a\beta_n b'_n - a\beta_{n+1}b'_{n+1} + a\beta_{n-1}b'_{n-1})].\end{aligned} \tag{51}$$

Rearranging terms in the summations, we see that the right-hand side term of (50) is identical to the left-hand side term of (51). Thus (51) becomes

$$0 = \sum_{n=-\infty}^{\infty}(-1)^n 2\kappa_a(\beta_n-1)\beta_n b'^2_n - \sum_{n=-\infty}^{\infty}(-1)^n a(\beta_n-1)b'_n[\beta_{n+1}b'_{n+1}-\beta_{n-1}b'_{n-1}]. \tag{52}$$

Therefore it remains to show that (52) leads to $b'_n \equiv 0$. To do so, we formulate the second term on the right-hand side of (52) as follows:

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(-1)^n a(\beta_n-1)b'_n[\beta_{n+1}b'_{n+1} - \beta_{n-1}b'_{n-1}]\\&= \sum_{n=0}^{\infty} a(-1)^n(\beta_n-1)\beta_{n+1}b'_n b'_{n+1} + \sum_{n=1}^{\infty} a(-1)^n(\beta_{-n}-1)\beta_{-n+1}b'_{-n}b'_{-n+1}\\&\quad - \sum_{n=1}^{\infty} a(-1)^n(\beta_n-1)\beta_{n-1}b'_n b'_{n-1} - \sum_{n=0}^{\infty} a(-1)^n(\beta_{-n}-1)\beta_{-n-1}b'_{-n}b'_{-n-1}\\&= \sum_{n=0}^{\infty} 2a(-1)^n(\beta_n-1)\beta_{n+1}b'_n b'_{n+1} - \sum_{n=1}^{\infty} 2a(-1)^n(\beta_n-1)\beta_{n-1}b'_n b'_{n-1},\end{aligned}$$

where we have used the relationship $b'_{-n} = (-1)^n b'_n$ given in (42). Moreover, it follows from (36) that

$$\sum_{n=-\infty}^{\infty}(-1)^n a(\beta_n-1)b'_n[\beta_{n+1}b'_{n+1} - \beta_{n-1}b'_{n-1}]$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} 2a(-1)^n \frac{(\beta_{n+1}-1)\beta_{n+1}b_{n+1}'^2}{\gamma_{n+1}} - \sum_{n=1}^{\infty} 2a(-1)^n(\beta_{n-1}-1)\beta_{n-1}b_{n-1}'^2\gamma_n \\
&= -\sum_{n=1}^{\infty} 2a(-1)^n \frac{(\beta_n-1)\beta_n b_n'^2}{\gamma_n} + \sum_{n=0}^{\infty} 2a(-1)^n(\beta_n-1)\beta_n b_n'^2\gamma_{n+1} \\
&= \sum_{n=1}^{\infty} 2a(-1)^n(\beta_n-1)\beta_n b_n'^2\kappa_a d_n + 2a(\beta_0-1)\beta_0 b_0'^2\gamma_1, \qquad (53)
\end{aligned}
$$

where we have used the identity

$$
\frac{1}{\gamma_n} - \gamma_{n+1} = -\kappa_a d_n
$$

defined by (40). Combining the equations (35), (52) and (53), we have

$$
\begin{aligned}
0 &= \left(\sum_{n=1}^{\infty} + \sum_{n=-1}^{-\infty} + \sum_{n=0}\right)(-1)^n 2\kappa_a(\beta_n-1)\beta_n b_n'^2 \\
&\quad - \sum_{n=1}^{\infty}(-1)^n 4(\beta_n-1)\beta_n b_n'^2\kappa_a\frac{\beta_n}{\beta_n-1} - 2a(\beta_0-1)\beta_0 b_0'^2\gamma_1 \\
&= \sum_{n=1}^{\infty}(-1)^n\left(4\kappa_a - 4\kappa_a\frac{\beta_n}{\beta_n-1}\right)\beta_n(\beta_n-1)b_n'^2 + 2\kappa_a\beta_0(\beta_0-1)b_0'^2 \\
&\quad -2a(\beta_0-1)\beta_0 b_0'^2\gamma_1 \\
&= -4\kappa_a\sum_{n=1}^{\infty}(-1)^n\beta_n b_n'^2 - 2\kappa_a\beta_0 b_0'^2, \qquad (54)
\end{aligned}
$$

since

$$
-2a(\beta_0-1)\beta_0 b_0'^2\gamma_1 = 2a(\beta_0-1)\beta_0 b_0'^2\frac{\kappa_a a}{1-a^2} = -2\kappa_a\beta_0^2 b_0'^2
$$

due to (35), (40) and (41).

On the other hand, multiplying the nth equation of (48) by $(\beta_n-1)b_n'/(4\kappa_a)$ and summing the resultant equations yield

$$
\begin{aligned}
0 &= \frac{1}{2}\sum_{n=-\infty}^{\infty}(\beta_n-1)\beta_n b_n'^2 \\
&= \sum_{n=1}^{\infty}(\beta_n-1)\beta_n b_n'^2 + \frac{1}{2}(\beta_0-1)\beta_0 b_0'^2. \qquad (55)
\end{aligned}
$$

Multiplying (54) by $(\beta_0-1)/(4\kappa_a)$ and then adding the resultant equation to (55), we have

$$
0 = \sum_{n=1}^{\infty}(\beta_n-1)\beta_n b_n'^2 - \sum_{n=1}^{\infty}(-1)^n(\beta_0-1)\beta_n b_n'^2
$$

$$\begin{aligned} &\geq \sum_{n=2}^{\infty}(\beta_n+\beta_0-2)\beta_n b_n'^2+(\beta_1-1)\beta_1 b_1'^2+(\beta_0-1)\beta_1 b_1'^2 \\ &= \sum_{n=2}^{\infty}(2a^2+n^2-2)(a^2+n^2)b_n'^2+(2a^2-1)(a^2+1)b_1'^2, \end{aligned}$$

and so, after the use of the condition $2a^2 \geq 1$,

$$0=\sum_{n=2}^{\infty}(2a^2+n^2-2)(a^2+n^2)b_n'^2.$$

This implies $b_n'=0$ for $n \geq 2$. Substitution of this finding into (48) with $n=2$ and 1 produces the result $b_1'=0$ and $b_0'=0$. Consequently, the validity of the spectral simplicity expressed by (44) is obtained due to (43).

The proof of Theorem 3.1 is completed.

4. Bifurcation analysis

This section is contributed for the proof of the bifurcation assertion of Theorem 1.1. The following steady-state bifurcation theorem is crucial to approach the result.

Theorem 4.1. *(Krasnoselskii [18] and Nirenberg [21]) For a Banach space X, a constant value $\kappa_{\text{crit}}>0$ and an open neighborhood $\mathcal{D}$ of the point $(0,\kappa_{\text{crit}})$ in the Banach space $X\times[0,\infty)$, let M_κ, N and F be the operators with*

$$F(\psi,\kappa)=\psi+\kappa M_\kappa\psi+N(\psi,\kappa),\ \ (\psi,\kappa)\in\mathcal{D},$$

subject to the following conditions:

(i) *$F:\mathcal{D}\mapsto X$ is continuous,*

(ii) *$M_\kappa:\mathcal{D}\mapsto X$ is linear, compact and continuous,*

(iii) *$N:\mathcal{D}\mapsto X$ is nonlinear and compact,*

(iv) *$N(0,\kappa)\equiv 0$ and $N(\psi,\kappa)=o(\|\psi\|_X)$ uniformly for $(\psi,\kappa)\in\mathcal{D}$,*

(v) *the spectral simplicity condition*

$$\dim\bigcup_{n=1}^{\infty}\{\psi\in X,(Id-\kappa_{\text{crit}}L_{\kappa_{\text{crit}}})^n\psi=0\}=1$$

holds true for Id the identity operator in X.

Then there exists a continuous family $(\psi_\kappa,\kappa)\in\mathcal{D}$, different to the trivial one $(0,\kappa)$, such that

$$F(\psi_\kappa,\kappa)=0, \tag{56}$$

or the solution family of (56) branches off $(0,\kappa_{\text{crit}})$ when κ varies across the critical value κ_{crit}.

Proof of Theorem 1.1. From Lemmas 2.1 and 2.2 we see that a solution ψ bifurcating from ψ^* is regular whenever $\psi \in C^{2+\alpha}_{\rm per}(\Omega_a)$. Thus it suffices to seek bifurcating solutions in the function space $\hat{C}^{2+\alpha}_{\rm per}(\Omega_a)$ for $0 < \alpha < 1$. Recall $\psi^* = \cos x_2$, the operator

$$F(\psi, \kappa) = \psi + \kappa\Delta^{-1}[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^*,$$

set in the previous section, the operator L_κ defined by (25) and the critical number κ_a in Theorem 3.1. For a constant ϵ such that $0 < \epsilon < \kappa_a$, we introduce the symbols

$$\begin{cases} X = \hat{C}^{2+\alpha}_{\rm per}(\Omega_a), \\ \mathcal{D} = \left\{\psi \in \hat{C}^{2+\alpha}_{\rm per}(\Omega_a);\ \|\nabla^2\psi - \nabla^2\psi^*\|_{C^\alpha} < \frac{(\kappa_a-\epsilon)^2}{4}\right\} \times (\kappa_a - \epsilon, \kappa_a + \epsilon), \\ M_k\psi = \frac{1}{\kappa}(L_k\psi - \psi) = \frac{1}{\kappa}\Delta^{-1}[\kappa + (\nabla \times \psi^*) \cdot \nabla]^{-1}(\nabla \times \psi^*) \cdot \nabla\psi, \\ N(\psi, \kappa) = F(\psi, \kappa) - [\psi - \psi^* + kM_k(\psi - \psi^*)]. \end{cases}$$

To verify the bifurcation assertion now remains to demonstrate the validity of the assumptions of Krasnolselskii's theorem.

Firstly, we verify the assumptions (i, ii) of Theorem 4.1. For the even function property of $F(\psi, \kappa)(\mathbf{x})$ with $(\psi, \kappa) \in \mathcal{D}$, we see that the even function ψ implies $\nabla \times \psi$ to be an odd function and so $\mathbf{y}$. This observation implies that

$$\psi^*(\mathbf{y}(-\mathbf{x}, s)) = \cos(-y_2(\mathbf{x}, s)) = \psi^*(\mathbf{y}(\mathbf{x}, s)),$$

and hence $F(\psi, \kappa)(\mathbf{x})$ is an even function of $\mathbf{x} \in \Omega_a$.

To show the continuity of F, for (ψ, κ), $(\psi', \kappa') \in \mathcal{D}$, we note that

$$\begin{aligned} &|\Delta F(\psi', \kappa') - \Delta F(\psi, \kappa) - (\Delta\psi' - \Delta\psi)| \\ \leq\ & \left|\kappa'[\kappa' + (\nabla \times \psi') \cdot \nabla]^{-1}\psi^* - \kappa'[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^*\right| \\ &+ \left|\kappa'[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^* - \kappa[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^*\right| \\ =\ & \kappa'|[\kappa' + (\nabla \times \psi') \cdot \nabla]^{-1}(\kappa - \kappa' + [\nabla \times \psi - \nabla \times \psi'] \cdot \nabla)[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^*| \\ &+ |\kappa - \kappa'|\,|[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^*|. \end{aligned}$$

By the Lagrangian formulation

$$[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}f(\mathbf{x}) = \int_0^\infty \mathrm{e}^{-\kappa s}f(\mathbf{y}(\mathbf{x}, s))ds, \tag{57}$$

we have

$$\begin{aligned} &\|\Delta F(\psi', \kappa') - \Delta F(\psi, \kappa) - (\Delta\psi' - \Delta\psi)\|_{C^0} \\ \leq\ & \|(\kappa - \kappa' + [\nabla \times \psi - \nabla \times \psi'] \cdot \nabla)[\kappa + (\nabla \times \psi) \cdot \nabla]^{-1}\psi^*\|_{C^0} + \frac{|\kappa - \kappa'|}{\kappa}\|\psi^*\|_{C^0} \\ \leq\ & \frac{2}{\kappa}|\kappa - \kappa'| + \|\nabla \times \psi - \nabla \times \psi'\|_{C^0} \int_0^\infty e^{-\kappa s}\|\nabla\mathbf{y} \cdot \nabla_{\mathbf{y}}\psi^*(\mathbf{y})\|_{C^0}ds, \end{aligned}$$

which is bounded by, using (12) and $\sqrt{\|\nabla^2\psi - \nabla^2\psi^*\|_{C^0}} \le (\kappa_a - \epsilon)/2$,

$$\begin{aligned}
&\frac{2|\kappa-\kappa'|}{\kappa} + \frac{\left(\sqrt{2}(\kappa-\kappa_a+\epsilon)+\sqrt{5}\right)\|\nabla\psi^*\|_{C^0}\|\nabla\psi-\nabla\psi'\|_{C^0}}{(\kappa-\kappa_a+\epsilon)^2} \\
\le\ &\frac{2|\kappa-\kappa'|}{\kappa} + \frac{\left(\sqrt{2}(\kappa-\kappa_a+\epsilon)+\sqrt{5}\right)\|\nabla\psi-\nabla\psi'\|_{C^0}}{(\kappa-\kappa_a+\epsilon)^2}.
\end{aligned}$$

Additionally, by Lemma 2.1, we have

$$\begin{aligned}
\|\Delta F(\psi,\kappa)-\Delta\psi\|_{C^1} &\le \kappa\|[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}\psi^*\|_{C^1} \\
&\le 1+\frac{\sqrt{2}\kappa(\kappa-2\sqrt{\|\nabla^2\psi-\nabla^2\psi^*\|_{C^0}})+\sqrt{5}\kappa}{(\kappa-2\sqrt{\|\nabla^2\psi-\nabla^2\psi^*\|_{C^0}})^2} \\
&\le 1+\frac{\sqrt{2}(\kappa-\kappa_a+\epsilon)\kappa+\sqrt{5}\kappa}{(\kappa-\kappa_a+\epsilon)^2}.
\end{aligned}$$

With the use of the above C^0 and C^1 estimates, the required continuity of the operator F in the intermediate Hölder space is thus derived from the interpolation inequality

$$\|\Delta f\|_{C^\alpha} \quad \le \quad 2\|\Delta f\|_{C^0}^{1-\alpha}\|\Delta f\|_{C^1}^{\alpha}$$

and the Hölder inequality of the Laplace operator

$$\|\nabla^2 f\|_{C^\alpha} \le c\|\Delta f\|_{C^\alpha}.$$

For the assumption (ii) of Theorem 4.1, we rewrite the operator M_κ as

$$\begin{aligned}
M_k\psi &= \frac{1}{\kappa}\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi^*)\cdot\nabla\psi \\
&= \frac{1}{\kappa}\Delta^{-1}\psi-\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}\psi.
\end{aligned}$$

This formulation enables us to apply the argument on the continuity of the operator $F(\psi,\kappa)$ to obtain the continuity of the operator $M_\kappa : \mathcal{D} \mapsto X$ and result of $M_\kappa\psi \in C^{2+\delta}(\Omega_a)$ for any $\alpha < \delta < 1$. The compactness of the operator M_κ is due to the compact imbedding of $C^{2+\delta}(\Omega_a)$ into $C^{2+\alpha}(\Omega_a)$.

Next, to verify the assumptions (iii, iv), we notice that

$$N(\psi,\kappa) \quad = \quad \kappa\Delta^{-1}[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}\psi^*+\psi^*-\kappa M_\kappa(\psi-\psi^*).$$

Therefore the compactness of the operator N is implied in the proof of the continuity of F and the compactness of the operator M_κ. To prove the nonlinear assertion, we transform the operator N into an explicit quadratic form. That is, by the solution property of ψ^* satisfying (24),

$$\begin{aligned}
N &= F(\psi,\kappa)-\psi+\psi^*-\kappa M_\kappa(\psi-\psi^*) \\
&= \kappa\Delta^{-1}[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}\psi^* \\
&\quad +\psi^*-\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi^*)\cdot\nabla(\psi-\psi^*) \\
&= \kappa\Delta^{-1}[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}\psi^*-\kappa\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}\psi^* \\
&\quad +\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*,
\end{aligned}$$

since

$$(\nabla\times\psi^*)\cdot\nabla(\psi-\psi^*) = -(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*.$$

By elementary manipulations, we have

$$\begin{aligned}
N &= -\kappa\Delta^{-1}[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}\psi^*\\
&\quad+\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*\\
&= -\Delta^{-1}[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*\\
&\quad+\Delta^{-1}[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*\\
&= -\Delta^{-1}\left[(\kappa+(\nabla\times\psi)\cdot\nabla)^{-1}-(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}\right](\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*\\
&= \Delta^{-1}[\kappa+(\nabla\times\psi)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\\
&\quad\cdot\nabla[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*.
\end{aligned}$$

With the use of this quadratic form and (57), we have

$$\begin{aligned}
&\|\Delta N(\psi,\kappa)\|_{C^0}\\
&\le \frac{1}{\kappa}\|(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*\|_{C^0}\\
&\le \frac{1}{\kappa}\|\nabla\psi-\nabla\psi^*\|_{C^0}\,\|\nabla[\kappa+(\nabla\times\psi^*)\cdot\nabla]^{-1}(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*\|_{C^0}\\
&\le \frac{1}{\kappa}\|\nabla\psi-\nabla\psi^*\|_{C^0}\,\|\nabla[(\nabla\times\psi-\nabla\times\psi^*)\cdot\nabla\psi^*]\|_{C^0}\int_0^\infty \mathrm{e}^{-\kappa s}\|\nabla\mathbf{y}^*\|_{C^0}ds\\
&\le \frac{1}{\kappa}\|\nabla\psi-\nabla\psi^*\|_{C^0}(\|\nabla^2\psi-\nabla^2\psi^*\|_{C^0}+\|\nabla\psi-\nabla\psi^*\|_{C^0})\int_0^\infty \mathrm{e}^{-\kappa s}(2+s)ds,
\end{aligned}$$

where the flow trajectory $\mathbf{y}^*$ is defined by the velocity $\nabla\times\psi^* = (-\sin x_2, 0)$ and is in the following form

$$\mathbf{y}^*(\mathbf{x},t) = \mathbf{x}+t(\sin x_2,\,0). \tag{58}$$

Hence

$$\|\Delta N(\psi,\kappa)\|_{C^0} \le \frac{2\kappa+1}{\kappa^3}\|\psi-\psi^*\|^2_{C^2}.$$

For the estimate of the operator N in the Hölder semi-norm, we employ (12), (57) and (58) to produce the estimates

$$\begin{aligned}
[(\kappa+(\nabla\times\psi)\cdot\nabla)^{-1}f]_{C^\alpha} &\le [f]_{C^\alpha}\int_0^\infty \mathrm{e}^{-\kappa s}\|\nabla\mathbf{y}\|^\alpha_{C^0}ds\\
&\le [f]_{C^\alpha}\frac{\sqrt{2}(\kappa-2\alpha\sqrt{\|\nabla^2\psi-\nabla^2\psi^*\|_{C^0}})+\sqrt{5}}{(\kappa-2\alpha\sqrt{\|\nabla^2\psi-\nabla^2\psi^*\|_{C^0}})^2}
\end{aligned}$$

and

$$\begin{aligned}
&[\nabla(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}f]_{C^\alpha}\\
&\quad= \int_0^\infty \mathrm{e}^{-\kappa s}[\nabla\mathbf{y}^*\cdot\nabla_{\mathbf{y}^*}f(\mathbf{y}^*(\cdot,s)]_{C^\alpha}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\nabla f\|_{C^0}\int_0^\infty \mathrm{e}^{-\kappa s}[\nabla \mathbf{y}^*]_{C^\alpha}ds + [\nabla f]_{C^\alpha}\int_0^\infty \mathrm{e}^{-\kappa s}\|\nabla \mathbf{y}^*\|_{C^0}^{1+\alpha}ds \\
&\leq \|\nabla f\|_{C^0}\int_0^\infty \mathrm{e}^{-\kappa s}2sds + [\nabla f]_{C^\alpha}\int_0^\infty \mathrm{e}^{-\kappa s}(2+s)^{1+\alpha}ds \\
&\leq \frac{2}{\kappa^2}\|\nabla f\|_{C^0} + \frac{4\kappa^2+4\kappa+2}{\kappa^3}[\nabla f]_{C^\alpha}.
\end{aligned}
$$

Let c be a constant independent of ψ and κ close to κ_a and the constant may change from line to line. Hence for

$$\mathbf{w} = \nabla\times\psi - \nabla\times\psi^*,$$

the Hölder semi-norm of the operator N is estimated as

$$
\begin{aligned}
&[\Delta N(\psi,\kappa)]_{C^\alpha} \\
&= [(\kappa + (\nabla\times\psi)\cdot\nabla)^{-1}\mathbf{w}\cdot\nabla(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}\mathbf{w}\cdot\nabla\psi^*]_{C^\alpha} \\
&\leq c[\mathbf{w}\cdot\nabla(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}\mathbf{w}\cdot\nabla\psi^*]_{C^\alpha} \\
&\leq c\|\mathbf{w}\|_{C^0}[\nabla(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}\mathbf{w}\cdot\nabla\psi^*]_{C^\alpha} \\
&\quad + c[\mathbf{w}]_{C^\alpha}\|\nabla(\kappa+(\nabla\times\psi^*)\cdot\nabla)^{-1}\mathbf{w}\cdot\nabla\psi^*\|_{C^0} \\
&\leq c\left(\|\mathbf{w}\|_{C^0}\|\nabla(\mathbf{w}\cdot\nabla\psi^*)\|_{C^\alpha} + [\mathbf{w}]_{C^\alpha}\|\nabla(\mathbf{w}\cdot\nabla\psi^*)\|_{C^0}\right) \\
&\leq c\|\nabla\psi - \nabla\psi^*\|_{C^{1+\alpha}}^2.
\end{aligned}
$$

This shows that the assumptions (iii,iv) of Theorem 4.1 hold true.

Finally, for the verification of the spectral condition, we apply Theorem 3.1 to obtain the existence of critical value κ_a satisfying the simplicity condition

$$\dim\bigcup_{i=1}^\infty\left\{\psi = \sum_{n=-\infty}^\infty b_n\cos(ax_1+nx_2)\in\hat{C}^{2+\alpha}_{\mathrm{per}}(\Omega_a);\ L^i_{\kappa_a}\psi = 0\right\} = 1.$$

Therefore, this together with (28) for $m\neq 1$ produces the validity of the assumption (v) of Theorem 4.1:

$$\dim\bigcup_{i=1}^\infty\left\{\psi\in\hat{C}^{2+\alpha}_{\mathrm{per}}(\Omega_a);\ L^i_{\kappa_a}\psi = 0\right\} = 1. \tag{59}$$

The bifurcation assertion of Theorem 1.1 is thus follows from Theorem 4.1 and the proof of Theorem 1.1 is completed.

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