New families of Q_B-optimal saturated two-level main effects screening designs

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Abstract:

In this paper, we study saturated two-level main effects designs which are commonly used for screening experiments. The Q_B criterion, which incorporates experimenters' prior beliefs about the probability of factors being active is used to compare designs. We show that under priors with more weight on models of small size, p-efficient designs should be recommended; when models with more parameters are of interest, A-optimal designs would be better. We identify new classes of saturated main effects designs between these two designs under different priors. The way in which the choice of designs depends on experimenters' prior beliefs is demonstrated for the cases when the number of runs $N \equiv 2 \bmod 4$. A novel method of construction of Q_B-optimal designs using conference matrices is introduced. Complete families of optimal designs are given for $N = 6, 10, 14, 18, 26, 30$.

Key words and phrases:

Conference matrix, model uncertainty, prior information, Q_B-criterion, screening, weighing design.

1 Introduction

Saturated two-level main effects designs, which allow the estimation of the main effects of $N-1$ factors in N runs, are useful for screening experiments where the goal is to identify the set of "active factors". These designs have been subject to study since Plackett and Burman (1946) gave designs for N any multiple of 4 up to 88. Since these designs allow all main effects to be estimated orthogonally with maximum efficiency, they are optimal according to any reasonable criterion. In other cases, when N is not a multiple of 4, the optimal design depends on the criterion of optimality used.

When N is not a multiple of 4, the construction of first-order designs is often based on the maximization of a design criterion, such as D- or A-efficiency, which is related to the saturated full main effects model. On the other hand, Lin (1993) discussed in detail the construction of saturated *p-efficient* two-level designs, which are efficient for fitting submodels containing only a subset of the factors.

The existing literature on saturated two-level designs for $N \equiv 2 \bmod 4$ concentrates on the choice between alphabetic-optimal designs and p-efficient designs. Using the Q_B-criterion, Tsai and Gilmour (2010) showed in one small example that there is a smooth transition from alphabetic-optimal designs to p-efficient designs as experimenters' prior beliefs about the importance of the factors change. Generalizing this idea, we now derive the explicit relations between experimenters' priors and the choice of design. A simple and effective approach to construct these Q_B-optimal designs using conference matrices is introduced and a secondary criterion is suggested to select the best among multiple Q_B-optimal designs.

In a two-level main effects design, the treatment factors, $X_1, \ldots, X_{N-1}$, have levels labeled -1 and 1, sometimes shortened to $-$ and $+$, and both factors and their levels are assumed to be exchangeable in that there is no prior knowledge about which factors are likely to be important or which level is likely to give the higher response. It is assumed that the treatment combinations will be completely randomized to the experimental units (runs). The appropriate full linear model for the data $\mathbf{y}$ is $E(\mathbf{y}) = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\mathbf{y}$ is a $N \times 1$ vector of responses, $\boldsymbol{\beta} = [\beta_0 \ \beta_1 \ \cdots \ \beta_{N-1}]^t$ is a vector of unknown parameters and $X = [\mathbf{1} \ \ X_1 \cdots X_{N-1}]$ is an $N \times N$ model matrix. Here X has all elements either -1 or 1, and we refer to it as a $(-1, 1)$ matrix of order N. A factor is said to be level-balanced if the corresponding column has the same number of 1s and -1s. We say X_i is a non-level-balanced factor if in the corresponding column the numbers of occurrences of 1s and -1s differ by 2. Without loss of generality, we require that 1 appears $N/2+1$ times and -1 appears $N/2-1$ times in the column.

The paper is organized as follows. Some known results and the Q_B-criterion for the two-level saturated screening designs are presented in Section 2. Section 3 gives the conditions for new families of Q_B-optimal designs to exist, and the construction of these designs using conference matrices is suggested in Section 4. Some concluding remarks are made in Section 5.

2 Known results and designs

For $N \equiv 2 \bmod 4$, Ehlich (1964) showed that if there exists a $N \times N$ (-1, 1) matrix such that

$$X^t X = \begin{bmatrix} M_1 & 0 \\ 0 & M_1 \end{bmatrix}, \tag{2.1}$$

where $M_1 = (N-2)I_{N/2} + 2\ J_{N/2}$, then X maximizes $|X^t X|$. We modify these matrices to be in a standard format with the first column being 1s and note that the resulting designs have $\frac{N}{2}$ level-balanced factors and $\frac{N}{2}-1$ non-level-balanced factors. Jacroux et al. (1983) showed that these designs, when they exist, are also optimal over all $N \times N$ $(-1,1)$ designs with respect to a wide class of Φ_p-optimality criteria, such as the A-, D- and E-criteria, for the saturated first-order model. Cheng (2014) showed that such designs are also A_s-optimal taking account of all the parameters except the intercept.

In saturated designs, we often assume that some factors' main effects are negligible, based on factor sparsity. Thus it is important to look at the performance of a design when it is projected onto lower dimensions. To study the projection efficiencies of saturated first-order designs, Lin (1993) discussed p-efficient designs by finding designs which minimize $E(s^2)$ among designs in which all factors are level-balanced when N is even (or near-level-balanced when N is odd). Lin provided a list of p-efficient designs for $3 \leq N \leq 30$. Dean and Draper (1999) used a computer search to construct saturated designs from cyclic generators for the cases $N \equiv 2 \bmod 4$ for $N = 6, \ldots, 30$ runs, which are similar to, or an improvement over, Lin's designs for the full main effects model and for the projected main effects models. Here we restrict attention to p-efficient designs which achieve the form of information matrix

$$M = \begin{bmatrix} N & \mathbf{0}^T \\ \mathbf{0} & (N \mp 2)I_{N-1} \pm 2J_{N-1} \end{bmatrix}. \tag{2.2}$$

In this setting, the first-order model is the maximal linear model of interest. It is assumed that some factors' main effects are negligible and thus one of the submodels of the first-order model will end up being fitted, but we do not in advance know which one. To incorporate experimenters' prior knowledge about the model or about the importance of each factor into the design selection procedure, Tsai, Gilmour and Mead (2007) suggested that minimizing the weighted average of the A_s-criterion functions (taking account of all parameters except the intercept) over all possible candidate models of the maximal model is useful. If a model is more likely to be the best model, then the model has more weight. They further defined a criterion, called Q_B, which is the minimization of an approximation to the weighted average of the A_s-criterion. Tsai and Gilmour (2010) showed that Q_B converges to A_s when the prior probability of each main effect being active tends to 1, and to $E(s^2)$ when the prior probability of each main effect being active tends to zero.

Letting $||\Delta||$ be the number of possible submodels of the maximal model in an N-run two-level design, $\mathcal{M}_l$ denote the lth candidate submodel and w_l be the prior probability of that model being the best model, the Q_B-criterion function is defined as

$$Q_B(d) \quad = \quad \sum_{l=1}^{||\Delta||} w_l \sum_{i=1}^{N-1} \widetilde{V}_l(\hat{\beta}_i), \tag{2.3}$$

where $\widetilde{V}_l(\hat{\beta}_i)$ is the approximate variance (Tsai, Gilmour and Mead (2000)) for the estimation of β_i under model $\mathcal{M}_l$, taken to be zero for a model that does not include β_i. For a two-level design the full first-order model is the maximal model, and the estimation for the intercept β_0 is excluded from the criterion. Let $a_{i,j}$, $i, j = 0, \ldots, (N-1)$, denote the (i, j)th element of the X^tX matrix for the first-order model, which is a measure of non-orthogonality between terms i and j. Using the same arguments as in Tsai *et al.* (2000), we have

$$\widetilde{V}_l(\hat{\beta}_i) = \sum_{j=0}^{N-1} a_{i,j}^2 \ \mathbb{I}_{\{(i,j)\in\mathcal{M}_l\}} \Big/ N^3,$$

where $\mathbb{I}_{\{(i,j)\in\mathcal{M}_l\}}$ is an indicator variable that is 1 if model $\mathcal{M}_l$ contains both terms i and j, indicating that $a_{i,j}$ appears in the approximate variance for the estimation of β_i in model $\mathcal{M}_l$, and is 0 otherwise. Thus, the Q_B-criterion selects a design that minimizes

$$Q_B(d) \quad = \quad \sum_{l=1}^{||\Delta||} w_l \sum_{i=1}^{N-1} \sum_{j=0}^{N-1} a_{i,j}^2 \ \mathbb{I}_{\{(i,j)\in\mathcal{M}_l\}} \Big/ N^3. \tag{2.4}$$

In practice it might not be easy to specify directly the prior probability of each model being the best, but simplification is possible if we assume that the prior probability of each factor being in the best model can be specified. Although the Q_B criterion is more flexible, here we have assumed exchangability among the factors as is usual in screening experiments, so that this prior probability is the same for each factor and is denoted by π. Then the prior probability for model $\mathcal{M}_l$ being the best depends only on the number of factors included in the model. For models containing the same number of factors' main effects, say k factors, the probability for each of these models being the best is $\pi^k(1-\pi)^{N-1-k}$, $1 \leq k \leq (N-1)$. We then re-group and summarize the $a_{i,j}$s in (2.4) by the number of factors in each model. Then the $a_{i,j}$s can be divided into two groups with $a_{i,0}$ being a measure of non-orthogonality between a factor and the intercept and $a_{i,j}$, $i, j \neq 0$, being a measure of pairwise non-orthogonality between factors i and j. Using the idea of balanced incomplete block designs, we see that $a_{i,0}$ and $a_{i,j}$ appear $\binom{N-2}{k-1}$ and $\binom{N-3}{k-2}$ times over models with k factors, respectively. Thus, the Q_B-criterion in (2.4) is rewritten as

$$\sum_{k=1}^{N-1} \pi^k (1-\pi)^{N-1-k} \left[\sum_{i=1}^{N-1} \binom{N-2}{k-1} \frac{a_{i,0}^2}{N^3} \right] + \sum_{k=2}^{N-1} \pi^k (1-\pi)^{N-1-k} \left[\sum_{i=1}^{N-1} \sum_{\substack{j=1 \\ j \neq i}}^{N-1} \binom{N-3}{k-2} \frac{a_{i,j}^2}{N^3} \right]$$

$$= \pi \sum_{i=1}^{N-1} \frac{a_{i,0}^2}{N^3} + \pi^2 \sum_{i=1}^{N-1} \sum_{\substack{j=1 \\ j \neq i}}^{N-1} \frac{a_{i,j}^2}{N^3}, \tag{2.5}$$

which is a linear combination of the overall measures of non-orthogonality between a factor and the intercept and between every pair of factors.

For A_s-optimal designs with $N/2$ level-balanced factors, $N/2-1$ non-level-balanced factors, and an information matrix of the form in (2.1), the value of the Q_B-criterion function is

$$\frac{2(N-2)\pi + 2(N-2)^2\pi^2}{N^2}.$$

For p-efficient designs with $N-1$ level-balanced factors and an information matrix of the form in (2.2), the value of the Q_B-criterion function is

$$\frac{4(N-1)(N-2)\pi^2}{N^2}.$$

When comparing the two, whenever the experimenters' prior probability of the importance of each factor π is less than $1/N$ we should use the p-efficient design and when $\pi > 1/N$ the A_s-optimal design is better. That is, when experimenters' prior beliefs lead to models with few parameters, one should use a p-efficient design since it provides better projection efficiencies but, if we expect to use models with more factors, then A_s-optimal designs should be preferred.

3 New classes of Q_B-optimal designs

In considering A_s-optimal designs and p-efficient designs, different designs should be recommended depending on the prior probability of each factor being in the best model. It is reasonable to conjecture that there might be designs between these two which might be better for less extreme priors.

We first consider the properties of the proposed new designs and then, in the next section, come back to consider how to find them when they exist. The simple result used above to compare the Q_B-efficiencies of A_s-optimal and p-efficient designs can be generalized and extended to show the global Q_B-optimality of designs in the new classes. First we require the following result.

Lemma 3.1 *For $N \equiv 2 \bmod 4$, let X be a $(-1,1)$-matrix of order N and, without loss of generality, suppose that all the entries in the first column are* 1. *Consider the class of designs such that each of the following $N-1-n_1$ columns has an even number of* 1*s, and each of the last n_1 columns has an odd number of* 1*s, where $\frac{N}{2} \leq n_1 \leq (N-1)$. If there exists a matrix X such that the information matrix M has the form*

$$M = \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}, \qquad (3.1)$$

where $B = (N \mp 2)I_{N-n_1} \pm 2J_{N-n_1}$ and $D = (N \mp 2)I_{n_1} \pm 2J_{n_1}$, then X is Q_B-optimal for a given n_1 within this class of designs.

Proof of Lemma 3.1. For an $N \times N$ $(-1,1)$-matrix X as given above, write the information matrix as $\begin{bmatrix} B & C \\ C^t & D \end{bmatrix}$. From the definition of Q_B in (2.5), Q_B is the linear combination of the squares of the off-diagonal elements of the information matrix. To minimize Q_B, we would like the off-diagonal blocks C to be 0, and thus the value of Q_B for the block diagonal matrix $\begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}$ is smaller than or equal to that for the general information matrix. Additionally, Jacroux et al. (1983) showed that, for a pair of columns which both have even or both have odd numbers of 1, the degree of non-orthogonality between these two columns has absolute value greater than or equal to 2. Thus for a given n_1, designs having the pattern of the information matrix in (3.1), with the entries of the off-diagonal elements of B and D having absolute values equal to 2, are Q_B-optimal.

To obtain designs with the pattern of the information matrix in (3.1), we note first that each of the $N-1-n_1$ columns with an even number of 1s is not orthogonal to the intercept and the non-orthogonality between the factor and the intercept is ± 2. Thus these are non-level-balanced factors with $N/2+1$ entries equal to 1 and $N/2-1$ entries equal to -1. These non-level-balanced factors are not orthogonal to each other with a measure of non-orthogonality of ± 2. Also, each of the n_1 columns with an odd number of 1s is orthogonal to the intercept, so it has $N/2$ entries equal to 1 and the other $N/2$ equal to -1. These are level-balanced factors. Again, these level-balanced factors are not orthogonal to each other with a measure of non-orthogonality of ± 2, where each level-balanced factor is orthogonal to each non-level-balanced factor.

It follows that the value of the Q_B-criterion function for a design with n_1 level-balanced factors, $N-1-n_1$ non-level-balanced factors, and information matrix of the form in

(3.1) is

$$\frac{4(N-1-n_1)\pi + 4[(N-1-n_1)^2 + n_1^2 - N + 1]\pi^2}{N^2}, \quad \text{for } \frac{N}{2} \leq n_1 \leq (N-1). \tag{3.2}$$

It is easy to see that, when $\pi \leq \frac{1}{2N-4}$, designs with $(N-1)$ level-balanced factors, i.e. the p-efficient designs, are optimal among all $(-1,1)$ designs, and as we increase the prior probability of a factor being in the model then the number of level-balanced factors in the Q_B-optimal design decreases.

Theorem 3.2 *For $N \equiv 2 \bmod 4$, if there exists an $N \times N$ $(-1,1)$-matrix X such that its information matrix for the saturated first-order model has the form in (3.1),*

(a) if $n_1 = N-1$, then X is Q_B-optimal for $\pi \leq 1/(2N-4)$;

(b) if $\frac{N}{2} < n_1 < N-1$, then X is Q_B-optimal for $1/(4n_1 - 2N + 4) < \pi \leq 1/(4n_1 - 2N)$; and

(c) if $n_1 = \frac{N}{2}$, then X is Q_B-optimal for $\pi > 1/4$.

By using this theorem, we can seek an appropriate Q_B-optimal design with a given number of level-balanced and non-level-balanced factors to accommodate the experimenters' prior belief on how likely their factors are to be active. For example when $N = 10$, if the expected number of active factors is about 1 or 2, we would suggest a Q_B-optimal design with 6 level-balanced factors and 3 non-level balanced factors. On the other hand, if the expected number of active factors is higher than 2, then the A_s-optimal design with 5 level-balanced factors and 4 non-level balanced factors should be recommended.

Example: For the simple example of a 6-run experiment with five factors, the A_s-optimal design has 3 level-balanced columns and 2 non-level-balanced columns and the p-efficient design has 5 level-balanced columns. According to Theorem 3.2, when $\pi < 1/8$, the p-efficient design is the best, when $\pi \geq 1/4$, the A_s-optimal is the best, and there is a new design with four level-balanced columns and one non-level-balanced column which is optimal when $1/8 \leq \pi < 1/4$. This design is

$$X^{(\text{new})} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

The Q_B-criterion function for this design is $(\pi + 12\pi^2)/9$. The nature of screening suggests that π should be small and, for such a small experiment, the expected number of active factors is likely to be less than one. Figure 1 gives the Q_B efficiencies for the three designs for $\pi \in (0, 0.5]$. It can be seen that the p-efficient design is the best when the expected number of factors is less than 0.75, the new design is optimal when the expected number of factors is between 0.75 and 1.5 and the A_s-optimal design is the best when the expected number of factors is at least 1.5. The new design is worse than the A_s-optimal design when $\pi > 1/4$, but it is still much better than the p-efficient design. The new design appears to be potentially useful, as the range of values of π for which it is optimal seems very realistic in a screening experiment, and it is nearly optimal if π is somewhat outside this range. Furthermore, if an experimenter is reluctant to specify a prior probability of effects being active, or if there is disagreement amongst a team of experimenters, a design that is robust to uncertainty in π might be preferred. There are different ways to define this robustness, but the new design has advantages over the p-efficient design and, especially over the A_s-optimal design. Except for very small π its efficiency is over 85%.

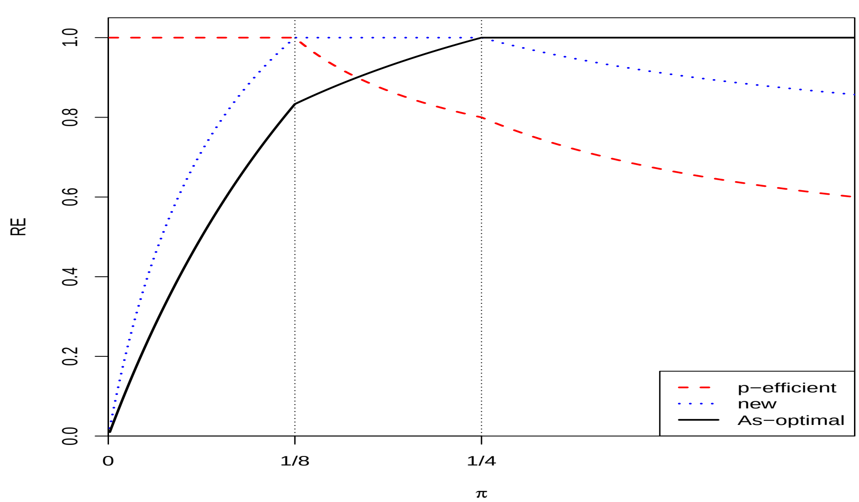

Figure 1: The Q_B-efficiency under different π for the 6-run two-level designs.

4 Construction of Q_B-optimal designs

We have shown that designs with the form of information matrix given at (3.1) are Q_B-optimal when they exist. Each such design found is Q_B-optimal for some range of π. No such designs are given in the literature and nothing is known about their existence.

A simple and effective way to construct these designs is by using *conference matrices*. A conference matrix is an $N \times N$ $(0, \pm 1)$ matrix C with entries 0 on the diagonal and ± 1 elsewhere that satisfies $CC^t = (N-1)I_N$. It is known that for $N \equiv 2 \bmod 4$, a conference matrix is symmetric. Conference matrices were used for effect screening by Elster and Neumaier (1995) in situations where high-order interactions were thought likely and by Xiao *et al.* (2012) for screening for main effects in the presence of suspected two-factor interactions. The semi-balanced three-level designs of Tsai *et al.* (2000) can also be constructed using conference matrices. In all these cases, the conference matrices were used directly as building blocks of larger designs. Here we adapt them for use in saturated main effects designs with two-levels.

The definition of a conference matrix shows that any two rows of C are orthogonal and two columns of C are also pairwise orthogonal. Without lost of generality, we may assume that all the entries of the first row and first column, except their intersection, are equal to 1 and write the matrix C as $\begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & S \end{bmatrix}$. When $N-1$ is a odd prime power, a symmetric conference matrix may be constructed using a general method due to Paley (1933). Here the matrix S has rows and columns indexed by the finite field of order $(N-1)$, and the (i,j)th entry is $+1$ if $j-i$ is a non-zero quadratic residue in the field, -1 if $j-i$ is a quadratic nonresidue, and 0 if $i=j$. We note that any column in S has one entry of 0 and $N/2-1$ entries each of $+1$ and -1.

To obtain designs with information matrix in the form at (3.1), we replace the zero diagonal entries of C with an $N \times 1$ vector with the first element always being 1, $N-1-n_1$ entries being 1, and n_1 entries being -1. Except for the first column, those with 0s replaced by $+1$s correspond to non-level-balanced factors and those with 0s replaced by -1s correspond to level-balanced factors. Each level-balanced (non-level-balanced, respectively) factor is not orthogonal to the others with the pairwise non-orthogonality being $\pm$ 2. For any pair of columns with one 0 replaced with $+1$ and the other with -1, the two factors are orthogonal to each other. In general, by replacing the 0 diagonal entries of the conference matrices with $+1$ and -1 accordingly, we construct a complete set of Q_B-optimal designs, covering all possible values of π. For example, the conference matrix of order 6, which has $CC^t = C^tC = 5I_6$, is

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

When we replace the diagonal entries by $(1, -1, -1, -1, -1, -1)$, we obtain a design with 5 level-balanced factors. If we replace them by $(1, 1, -1, -1, -1, -1)$ we obtain a design with 4 level-balanced columns. If we replace them with $(1, 1, 1, -1, -1, -1)$, we obtain a design with 3 level-balanced factors. These designs have information matrices with the pattern at (3.1) and each of them is Q_B-optimal for a given range of π as discussed in the previous section. We note that the the resulting design with 3 level-balanced factors is the modified Ehlich's design. It is the A_s-optimal design and the A_s-criterion function value for the estimates of the main effects, excluding the intercept, of this design is 1.

The Q_B-criterion is a first-order approximation of the A_s-criterion, averaged over many models. It does not, however, fully discriminate between designs. For example, for the case of a 6-run design with 3 level-balanced factors, there are $\binom{5}{3} = 10$ ways to replace the zero diagonal with a vector with three 1s and three -1s, with the first element always being replaced by 1. All the resulting designs have the same value of the Q_B-criterion, but there are two non-isomorphic designs. In addition to the modified Ehlich's design in above paragraph, if we replace the zero diagonal elements with $(1, -1, -1, 1, -1, 1)$, then we obtain a design whose A_s-criterion function value is 1.1250. Clearly, this design is not as efficient as that for the A_s-optimal design, though they are equally good with respect to the Q_B-criterion.

Thus in addition to using Q_B to select designs, we suggest using the A_s-criterion for the saturated main effects model as a secondary criterion to distinguish among designs with the same values of Q_B. Table 1 lists the indices for the columns in which we replace 0s by 1s (other columns having 0s replaced by -1s) to obtain Q_B-optimal designs with n_1 level-balanced (and $N-1-n_1$ non-level balanced) factors from the conference matrix with $N = 6, 10, 14, 18, 26, 30$. Each of the designs has the highest value of the A_s-criterion for the saturated main effects model among all the Q_B-optimal designs. These were found by complete enumeration. Note that for the case with one non-level-balanced factor, $N - 1 - n_1 = 1$, the choice of a column to be non-level-balanced makes no difference. We provide a list of these conference matrices in the supplemental material.

Note that there are four conference matrices for $N = 26$. The number in the brackets after the indices indicates which of the four conference matrices is used for generating the design listed in this table, being (1) if not stated. It is known that for $N \equiv 2 \bmod 4$, a conference matrix of order N exists if and only if $N - 1$ is the sum of two squares. Using conference matrices, we are able to construct a complete set of Q_B-optimal designs for $N = 6, 10, 14, 18, 26, 30$ by replacing the 0 diagonal entries of the conference matrices with +1s and −1s accordingly. However, we cannot use this method for some run-sizes, such as $N = 22$ or 34.

5 Discussion

In screening experiments, most factors are assumed to have no important effect on the response. Here we have shown that incorporating experimenters' prior beliefs about the importance of factors being in the best model into the design selection process, different designs would be recommended. This work greatly expands the available class of optimal designs and the use of conference matrices gives a simple way to obtain such optimal designs in most practically useful cases.

Since there are several Q_B-optimal designs for any given N and π, the use of a secondary criterion is helpful in making a better than random choice. Given that the Q_B criterion was originally developed as an approximation to a weighted average of A_s-efficiencies over several models, A_s-efficiency for the full model seems like a sensible secondary criterion. Then the results in Table 1, along with the conference matrices in the Supplement give all the information that is needed for experimenters to use these designs. We recommend them for practical use.

Supplementary Materials

The online supplement contains the following items:

1. the relative efficiencies for an example of 10-run designs with different numbers of level-balanced factors;
2. an example discussing the secondary criterion for Q_B-optimal designs obtained from conference matrices; and
3. the conference matrices that are required for generating the Q_B-optimal designs in Table 1 of this paper.

Table 1: The indices for the non-level-balanced columns of the conference matrices

N	n_1	$N-1-n_1$	Indices for non-level-balanced factors
10	7	2	2 3
	6	3	2 3 4
	5	4	2 3 6 8
14	11	2	2 3
	10	3	2 3 6
	9	4	2 3 6 8
	8	5	2 3 4 6 8
	7	6	2 3 4 6 7 8
18	13	2	2 3
	14	3	2 3 4
	13	4	2 3 4 9
	12	5	2 3 4 7 9
	11	6	2 3 4 5 7 9
	10	7	2 3 4 5 6 10 11
	9	8	2 3 4 5 6 10 11 14
26	23	2	2 3
	22	3	2 3 4
	21	4	2 3 4 5
	20	5	2 3 4 5 8
	19	6	2 3 4 5 6 8
	18	7	2 3 4 5 7 10 13 $^{(2)}$
	17	8	2 3 4 5 7 10 13 22 $^{(2)}$
	16	9	2 3 4 5 6 7 8 12 18
	15	10	2 3 4 5 6 7 8 16 22 25
	14	11	2 3 4 5 6 7 8 12 15 18 25
	13	12	2 3 4 5 6 7 8 9 15 16 22 25
30	27	2	2 3
	26	3	2 3 7
	25	4	2 3 7 8
	24	5	2 3 4 8 9
	23	6	2 3 4 6 8 9
	22	7	2 3 4 5 7 8 9
	21	8	2 3 4 5 7 8 9 10
	20	9	2 3 4 5 6 8 15 22 29
	19	10	2 3 4 5 6 7 8 9 29 30
	18	11	2 3 4 5 6 7 8 9 10 17 30
	17	12	2 3 4 5 6 7 10 11 15 19 23 27
	16	13	2 3 4 5 6 7 8 10 13 17 22 26 29
	15	14	2 3 4 5 6 7 8 9 13 17 21 25 29 30

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