ALGEBRAIC PROOFS OF SOME FUNDAMENTAL THEOREMS IN ALGEBRAIC K-THEORY

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ABSTRACT. We present new proofs of the additivity, resolution and cofinality theorems for the algebraic K-theory of exact categories. These proofs are entirely algebraic, based on Grayson's presentation of higher algebraic K-groups via binary complexes.

INTRODUCTION

The beautiful and relatively young discipline of algebraic K-theory has seen tremendous development and far-reaching applications in many other mathematical disciplines over the last decades. This paper makes a contribution to a project (begun in [Gra12] and [Gra13]) reformulating its foundations.

The algebraic K-theory of an exact category was first described by Segal and Waldhausen, obtained by modifying Segal's construction of the K-theory of a symmetric monoidal category. Quillen's alternative Q-construction gives a very powerful tool for proving fundamental theorems in algebraic K-theory, which he exploited to prove the additivity, resolution, dévissage and localisation theorems [Qui73]. Waldhausen's later work on the $S_\cdot$-construction, in particular his version of the additivity theorem, made simpler proofs of the theorems cited above and the cofinality theorem possible [Sta89]. Common in all of these approaches is the use of some non-trivial content from homotopy theory.

Grayson [Gra12] recently gave the first presentation of the higher algebraic K-groups of an exact category by generators and relations; we take this presentation as our definition of $K_n\mathcal{N}$. The object of this paper is to present new completely algebraic proofs of the additivity, resolution and cofinality

theorems in higher algebraic K-theory of exact categories.

We assume throughout that the reader is familiar with exact categories. They are first systematically defined in [Qui73], a very nice exposition is [Büh10]. In section 1 we review the necessary details of Grayson's presentation and present a new proof of the additivity theorem, in its form concerning so-called extension categories.

THEOREM (Additivity). Let $\mathcal{B}$ be an exact category, with exact subcategories $\mathcal{A}$ and $\mathcal{C}$ closed under extensions in $\mathcal{B}$. Let $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ denote the associated extension category. Then $K_n\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \cong K_n\mathcal{A} \times K_n\mathcal{C}$, for every $n \geq 0$.

Using Grayson's presentation, the proof of the additivity theorem is rather simple. In sections 2 and 3 we present more involved proofs of the resolution and cofinality theorems.

THEOREM (Resolution). Let $\mathcal{M}$ be an exact category and let $\mathcal{P}$ be a full, additive subcategory that is closed under extensions. Suppose also that:

(1) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $\mathcal{M}$ with M and M'' in $\mathcal{P}$, then M' is in $\mathcal{P}$ as well.
(2) Given $j : M \twoheadrightarrow P$ in $\mathcal{M}$ with P in $\mathcal{P}$ there exists $j' : P' \twoheadrightarrow P$ and $f : P' \to M$ in $\mathcal{M}$ with P' in $\mathcal{P}$ such that $jf = j'$.
(3) Every object of $\mathcal{M}$ has a finite resolution by objects of $\mathcal{P}$.

Then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism $K_n(\mathcal{P}) \cong K_n(\mathcal{M})$ for every $n \geq 0$.

THEOREM (Cofinality). Let $\mathcal{M}$ be a cofinal exact subcategory of an exact category $\mathcal{N}$. Then the inclusion functor $\mathcal{M} \hookrightarrow \mathcal{N}$ induces an injection $K_0\mathcal{M} \hookrightarrow K_0\mathcal{N}$ and isomorphisms $K_n\mathcal{M} \cong K_n\mathcal{N}$ for $n > 0$.

Grayson defines the nth algebraic K-group of an exact category, denoted $K_n\mathcal{N}$, as a quotient group of the Grothendieck group of a certain related exact category $(B^{\mathrm{q}})^n\mathcal{N}$, whose objects are so-called *acyclic binary multicomplexes* (see Definition 1.5). Each of the theorems above establishes isomorphisms between the K-groups of exact categories. These theorems all have well-known algebraic folk proofs for the Grothendieck group K_0, so the general schema for our proofs is then as follows. First we verify that the hypotheses on our exact categories of interest also hold for their associated categories of acyclic binary multicomplexes. Then we apply the algebraic K_0 proof to obtain an isomorphism between their Grothendieck groups. Finally we verify that the required isomorphism still holds when we pass to the quotients defining the higher algebraic K-groups.

The remaining theorems regarded as fundamental in the algebraic K-theory of exact categories are the dévissage and localisation theorems. These theorems concern abelian categories, say $\mathcal{A}$ and $\mathcal{B}$. While the associated categories $(B^{\mathrm{q}})^n\mathcal{A}$ and $(B^{\mathrm{q}})^n\mathcal{B}$ are still exact, they will no longer be abelian, so a strategy more sophisticated than the approach of this paper will be necessary to

prove these theorems in the context of Grayson's new definition of the higher algebraic K-groups.

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1. Grayson's binary complex algebraic K-theory

In this section we recall the definitions and main result of [Gra12]. As a first application we give a simple new proof of the additivity theorem, as previously proven in [Qui73], as well as [McC93] and [Gra11], and whose version for the $S_\bullet$-construction is commonly considered to be *the* fundamental theorem in the algebraic K-theory of spaces. We shall work throughout with exact categories in the sense of Quillen [Qui73], that is, additive categories with a distinguished collection of short exact sequences that satisfies a certain set of axioms.

DEFINITION 1.1. A bounded *acyclic complex*, or *long exact sequence*, in an exact category $\mathcal{N}$ is a bounded chain complex N whose differentials *factor through short exact sequences* of $\mathcal{N}$. That is, the differentials $d_k : N_k \to N_{k-1}$ factor as $N_k \to Z_{k-1} \to N_{k-1}$ such that each $0 \to Z_k \to N_k \to Z_{k-1} \to 0$ is a short exact sequence of $\mathcal{N}$.

In an abelian category the long exact sequences defined above agree with the usual long exact sequences. Care must be taken in the case of a general exact category, as the following example shows.

EXAMPLE 1.2. Let R be a ring with a finitely-generated, stably-free, non-free projective module P (so $P \oplus R^m \cong R^n$ for some m and n). We have short exact sequences of R-modules $0 \to P \xrightarrow{i} R^n \xrightarrow{p} R^m \to 0$ and $0 \to R^m \xrightarrow{j} R^n \xrightarrow{q} P \to 0$, where i, j, p and q are the obvious inclusions and projections. The sequence

$$0 \to R^m \xrightarrow{j} R^n \xrightarrow{iq} R^n \xrightarrow{p} R^n \to 0$$

is a chain complex in the exact category of finitely-generated free modules, $\mathbf{Free}(R)$, and it is exact as a sequence of R modules, but, in the sense of the definition above, it is not long exact in $\mathbf{Free}(R)$.

The category $C^{\mathsf{q}}\mathcal{N}$ of bounded acyclic[1] complexes in an exact category $\mathcal{N}$ is itself an exact category ([Gra12], §6); a sequence of chain maps $0 \to N' \xrightarrow{\phi} N \xrightarrow{\psi}$

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[1] The q in $C^{\mathsf{q}}\mathcal{N}$ stands for "quasi-isomorphic to the zero complex". We will not make use of the notion of a quasi-isomorphism—the q is a reminder that we are dealing only with the *acyclic* complexes in $\mathcal{N}$.

$N'' \to 0$ is said to be a short exact sequence if and only if $0 \to N'_k \xrightarrow{\phi_k} N_k \xrightarrow{\psi_k} N''_k \to 0$ is short exact for all k. A word of warning here: a morphism of acyclic complexes $N \xrightarrow{\psi} N''$ such that every $N_k \xrightarrow{\psi_k} N''_k$ is an admissible epimorphism of $\mathcal{N}$ is not necessarily an admissible epimorphism in $C^q\mathcal{N}$.

EXAMPLE 1.3. Let i, j, p and q be the morphisms in Example 1.2, and note that $iq + jp = 1$. The diagram below has exact rows, and is in fact a morphism of $C^q\mathbf{Free}(R)$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R^m & \overset{\left[\begin{smallmatrix}j\\1\end{smallmatrix}\right]}{\rightarrowtail} & R^n \oplus R^m & \xrightarrow{\left[\begin{smallmatrix}1 & -j\\-p & 1\end{smallmatrix}\right]} & R^n \oplus R^m & \overset{[\,p\ 1\,]}{\twoheadrightarrow} & R^m & \longrightarrow & 0\\
& & \downarrow & & {\scriptstyle [-p\ 1]}\downarrow & & \downarrow{\scriptstyle [0\ 1]} & & \downarrow & & \\
& & 0 & \rightarrowtail & R^m & \xrightarrow{\ \ 1\ \ } & R^m & \twoheadrightarrow & 0 & &
\end{array}
$$

Each vertical arrow is an admissible epimorphism of $\mathbf{Free}(R)$, but the diagram is not an admissible epimorphism of $C^q\mathbf{Free}(R)$—its kernel is the complex discussed in Example 1.2, which is not acyclic in $\mathbf{Free}(R)$.

DEFINITION 1.4. A *binary complex* in $\mathcal{N}$ is a chain complex with two independent differentials. More precisely, a binary complex is a triple $(N, d, \tilde{d})$ such that (N, d) and $(N, \tilde{d})$ are chain complexes in $\mathcal{N}$. We call a binary chain complex *acyclic* if each of the complexes (N, d) and $(N, \tilde{d})$ is acyclic in $\mathcal{N}$. A morphism between binary complexes is a morphism between the underlying graded objects that commutes with both differentials. A short exact sequence is a composable pair of such morphisms that is short exact in every degree.

Since $C^q\mathcal{N}$ is an exact category, the reader may easily check that the category $B^q\mathcal{N}$ of bounded acyclic binary complexes in $\mathcal{N}$ is also an exact category. There is a diagonal functor $\Delta : C^q\mathcal{N} \to B^q\mathcal{N}$, sending (N, d) to (N, d, d). A binary complex that is in the image of Δ is also called *diagonal*. The diagonal functor is split by the *top* and *bottom* functors $\top, \bot : B^q\mathcal{N} \to C^q\mathcal{N}$; it is clear that $\Delta, \top$ and $\bot$ are all exact.

Taking the category of acyclic binary complexes behaves well with respect to subcategories closed under extensions. If $\mathcal{M}$ is a full subcategory closed under extension in $\mathcal{N}$ (later just called an *exact subcategory*), then $B^q\mathcal{M}$ is a subcategory closed under extensions in $B^q\mathcal{N}$. It is important here that the binary complexes in $B^q\mathcal{N}$ are bounded. Starting at the final non-zero term, one argues by induction that the objects that the extension factors through are actually in $\mathcal{M}$, using the hypothesis on $\mathcal{M}$ and $\mathcal{N}$, and the 3×3 Lemma ([Büh10], Corollary 3.6).

Since $B^q\mathcal{N}$ is an exact category, we can iteratively define an exact category $(B^q)^n\mathcal{N} = B^qB^q\cdots B^q\mathcal{N}$ for each $n \geq 0$. The objects of this category are bounded acyclic binary complexes of bounded acyclic binary complexes ... of

objects of $\mathcal{N}$. Happily, this may be neatly unwrapped: the following is an equivalent definition of $(B^{\mathrm{q}})^n\mathcal{N}$.

DEFINITION 1.5. The exact category $B^{\mathrm{q}}\mathcal{N}$ of bounded acyclic binary multi-complexes of dimension n in $\mathcal{N}$ is defined as follows. A *bounded acyclic binary multicomplex of dimension* n is a $\mathbb{Z}^n$-graded collection of objects of $\mathcal{N}$, only finitely many of which are non-zero, together with a pair of acyclic differentials d^i and $\tilde{d}^i$ in each direction $1 \le i \le n$ such that, for $i \ne j$,

$$
\begin{aligned}
d^i d^j &= d^j d^i \\
d^i \tilde{d}^j &= \tilde{d}^j d^i \\
\tilde{d}^i d^j &= d^j \tilde{d}^i \\
\tilde{d}^i \tilde{d}^j &= \tilde{d}^j \tilde{d}^i.
\end{aligned}
$$

In other words, any pair of differentials in different directions commute. A morphism $\phi : N \to N'$ between such binary multicomplexes is a $\mathbb{Z}^n$-graded collection of morphisms of $\mathcal{N}$ that commutes with all of the differentials of N and N'. A short exact sequence in $(B^{\mathrm{q}})^n\mathcal{N}$ is a composable pair of such morphisms that is short exact in every degree.

In addition to $(B^{\mathrm{q}})^n\mathcal{N}$, for each $n \ge 1$ we have exact categories $(B^{\mathrm{q}})^{i-1}C^{\mathrm{q}}(B^{\mathrm{q}})^{n-i}\mathcal{N}$ for $i = 1,\dots n$, each of which is equivalent to $C^{\mathrm{q}}(B^{\mathrm{q}})^{n-1}\mathcal{N}$, the category of acyclic complexes of objects of $(B^{\mathrm{q}})^{n-1}\mathcal{N}$. For each i with $1 \le i \le n$ there is a diagonal functor $\Delta_i : (B^{\mathrm{q}})^{i-1}C^{\mathrm{q}}(B^{\mathrm{q}})^{n-i}\mathcal{N} \to (B^{\mathrm{q}})^n\mathcal{N}$ that consists of 'doubling up' the differential of the (non-binary) acyclic complex and regarding it as providing both differentials in the direction i in the resulting acyclic binary multicomplex. Any object of $(B^{\mathrm{q}})^n\mathcal{N}$ that is in the image of one of these Δ_i is called *diagonal*. The diagonal binary multicomplexes are those that have $d^i = \tilde{d}^i$ for *at least one* i.

We can now formulate Grayson's presentation of the algebraic K-theory groups of $\mathcal{N}$, which we shall take to be their definition for the remainder of this paper.

THEOREM / DEFINITION 1.6 ([Gra12], Corollary 7.4). *For $\mathcal{N}$ an exact category and $n \ge 0$, the abelian group $K_n\mathcal{N}$ is presented as follows. There is one generator for each bounded acyclic binary multicomplex of dimension n, and there are relations $[N'] + [N''] = [N]$ if there is a short exact sequence $0 \to N' \to N \to N'' \to 0$ in $(B^{\mathrm{q}})^n\mathcal{N}$, and $[T] = 0$ if T is a diagonal acyclic binary multicomplex.*

Observe that if we disregard the second relation that diagonal binary multicomplexes vanish, then we obtain the Grothendieck group $K_0(B^{\mathrm{q}})^n\mathcal{N}$. Another way to say this is that $K_n\mathcal{N}$ is a quotient group of the Grothendieck group of the exact category $(B^{\mathrm{q}})^n\mathcal{N}$. Denote by $T^n_{\mathcal{N}}$ the subgroup of $K_0(B^{\mathrm{q}})^n\mathcal{N}$ generated by the classes of the diagonal binary multicomplexes in $(B^{\mathrm{q}})^n\mathcal{N}$. Then we may write $K_n\mathcal{N} \cong (K_0(B^{\mathrm{q}})^n\mathcal{N})/T^n_{\mathcal{N}}$.

Using the inductive viewpoint explained before Definition 1.5, we also obtain the following description of $K_n\mathcal{N}$.

LEMMA 1.7. *For each $n \geq 1$ we have a split short exact sequence:*

$$0 \longrightarrow K_{n-1}C^{\mathsf{q}}\mathcal{N} \xrightarrow{\Delta} K_{n-1}B^{\mathsf{q}}\mathcal{N} \longrightarrow K_n\mathcal{N} \longrightarrow 0,$$

which is functorial in $\mathcal{N}$.

Proof. This is proven already in [Gra12], but a direct proof from Theorem / Definition 1.6 is instructive, so we include it here. We have defined $K_n\mathcal{N}$ to be $K_0(B^{\mathsf{q}})^n\mathcal{N}$ modulo the sum of the images of the various diagonal maps

$$\Delta_i : K_0(B^{\mathsf{q}})^{i-1}C^{\mathsf{q}}(B^{\mathsf{q}})^{n-i}\mathcal{N} \longrightarrow K_0(B^{\mathsf{q}})^n\mathcal{N} \quad (i = 1, \ldots, n).$$

Similarly, $K_{n-1}B^{\mathsf{q}}\mathcal{N}$ is defined as the factor group of $K_0(B^{\mathsf{q}})^{n-1}B^{\mathsf{q}} = K_0(B^{\mathsf{q}})^n\mathcal{N}$ modulo the sum of the images of Δ_i for $i = 1, \ldots, n-1$. Therefore, the canonical epimorphism $K_0(B^{\mathsf{q}})^n\mathcal{N}$ naturally factorises as

$$K_0(B^{\mathsf{q}})^n\mathcal{N} \longrightarrow K_{n-1}B^{\mathsf{q}}\mathcal{N} \longrightarrow K_n\mathcal{N},$$

and $K_n\mathcal{N}$ is the factor group of $K_{n-1}B^{\mathsf{q}}\mathcal{N}$ modulo the image of the composition

$$K_0(B^{\mathsf{q}})^{n-1}C^{\mathsf{q}}\mathcal{N} \xrightarrow{\Delta_n} K_0(B^{\mathsf{q}})^n\mathcal{N} \longrightarrow K_{n-1}B^{\mathsf{q}}\mathcal{N}.$$

Furthermore, the epimorphism $K_0(B^{\mathsf{q}})^{n-1} \to K_{n-1}$ is a natural transformation of functors from the category of exact categories to abelian groups. Summarising, we obtain the following commutative diagram, whose bottom row is exact:

$$
\begin{array}{ccc}
K_0(B^{\mathsf{q}})^{n-1}C^{\mathsf{q}}\mathcal{N} & \xrightarrow{\ \Delta_n\ } & K_0(B^{\mathsf{q}})^n\mathcal{N} \\
\downarrow & & \downarrow \\
K_{n-1}C^{\mathsf{q}}\mathcal{N} & \xrightarrow{\ K_{n-1}(\Delta)\ } & K_{n-1}B^{\mathsf{q}}\mathcal{N} \twoheadrightarrow K_n\mathcal{N} \longrightarrow 0.
\end{array}
$$

The top and bottom functors ($\top$ and $\bot$) each split Δ, so $K_{n-1}(\Delta)$ is split by either of $K_{n-1}(\top)$ or $K_{n-1}(\bot)$, so the bottom row of the diagram is our required split short exact sequence. $\square$

We now present a new, elementary proof of the additivity theorem. Let $\mathcal{A}$ and $\mathcal{C}$ be exact subcategories of an exact category $\mathcal{B}$.

DEFINITION 1.8. The *extension category* $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is the exact category whose objects are short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{B}$, with A in $\mathcal{A}$ and C in $\mathcal{C}$, and whose morphisms are commuting rectangles.

THEOREM 1.9 (Additivity). $K_n\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \cong K_n\mathcal{A} \times K_n\mathcal{C}$, *for every $n \geq 0$.*

Proof. The exact functors

$$
\begin{array}{rcl}
\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) & \to & \mathcal{A} \times \mathcal{C} \\
(0 \to A \to B \to C \to 0) & \mapsto & (A, C)
\end{array}
$$

and

$$
\begin{array}{rcl}
\mathcal{A} \times \mathcal{C} & \to & \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \\
(A, C) & \mapsto & (0 \to A \to A \oplus C \to C \to 0)
\end{array}
$$

induce mutually inverse isomorphisms between $K_0\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ and $K_0\mathcal{A} \times K_0\mathcal{C}$, (see, e.g., [Wei13], II.9.3). Fix an $n > 0$. The categories $(B^{\mathrm{q}})^n\mathcal{A}$ and $(B^{\mathrm{q}})^n\mathcal{C}$ are exact subcategories of $(B^{\mathrm{q}})^n\mathcal{B}$, so we can define the extension category

$$\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C}) := \mathcal{E}((B^{\mathrm{q}})^n\mathcal{A}, (B^{\mathrm{q}})^n\mathcal{B}, (B^{\mathrm{q}})^n\mathcal{C}).$$

From the above, the induced map

$$K_0\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C}) \to K_0(B^{\mathrm{q}})^n\mathcal{A} \times K_0(B^{\mathrm{q}})^n\mathcal{C}$$

is an isomorphism. But a short exact sequence of binary multicomplexes is exactly the same thing as a binary multicomplex of short exact sequences, so the categories $\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C})$ and $(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ are isomorphic, so we therefore have an isomorphism

$$K_0(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \cong K_0(B^{\mathrm{q}})^n\mathcal{A} \times K_0(B^{\mathrm{q}})^n\mathcal{C}.$$

Identifying the categories $\mathcal{E}^n(\mathcal{A},\mathcal{B},\mathcal{C})$ and $(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$, a binary multicomplex in $(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ is diagonal in direction i if and only if its constituent binary multicomplexes in $(B^{\mathrm{q}})^n\mathcal{A}, (B^{\mathrm{q}})^n\mathcal{B}$ and $(B^{\mathrm{q}})^n\mathcal{C}$ are also diagonal in direction i. Similarly, if A in $(B^{\mathrm{q}})^n\mathcal{A}$ and C in $(B^{\mathrm{q}})^n\mathcal{C}$ are diagonal, then the binary multicomplexes corresponding to $(0 \to A \to A \to 0 \to 0)$ and $(0 \to 0 \to C \to C \to 0)$ are diagonal as well, so the isomorphism $K_0(B^{\mathrm{q}})^n\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C}) \cong K_0(B^{\mathrm{q}})^n\mathcal{A} \times K_0(B^{\mathrm{q}})^n\mathcal{C}$ restricts to an isomorphism $T^n_{\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})} \cong T^n_{\mathcal{A}} \times T^n_{\mathcal{B}}$. Passing to the quotients yields the result. $\square$

2. THE RESOLUTION THEOREM

The resolution theorem relates the K-theory of an exact category to that of a larger exact category all of whose objects have a finite resolution by objects of the first category. Its most well-known application states that the K-theory of a regular ring is isomorphic to its G-theory (the K-theory of the exact category of *all* finitely-generated R-modules). As in the proof of the additivity theorem, we adapt a simple proof for K_0 to work for all K_n. The main difficulty in this proof is verifying that the hypotheses of the theorem pass to exact categories of acyclic binary multicomplexes.

The general resolution theorem for exact categories ([Qui73], §4 Corollary 2) is a formal consequence of the following theorem, which is Theorem 3 of [Qui73].

THEOREM 2.1. *Let $\mathcal{P}$ be a full, additive subcategory of an exact category $\mathcal{M}$ that is closed under extensions and satisfies:*

  (1) *For any exact sequence $0 \to P' \to P \to M \to 0$ in $\mathcal{M}$, if P is in $\mathcal{P}$, then P' is in $\mathcal{P}$.*
  (2) *For any M in $\mathcal{M}$ there exists a P in $\mathcal{P}$ and an admissible epimorphism $P \twoheadrightarrow M$.*

Then the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an isomorphism $K_n\mathcal{P} \cong K_n\mathcal{M}$ for all $n \geq 0$.

Proof. For K_0 the inverse to the induced homomorphism $K_0\mathcal{P} \to K_0\mathcal{M}$ is given by the map

$$\phi: \begin{array}{ccc} K_0\mathcal{M} & \to & K_0\mathcal{P} \\ [M] & \mapsto & [P] - [P'], \end{array}$$

where $0 \to P' \to P \to M \to 0$ is a short exact sequence of $\mathcal{M}$. The proof of Theorem 2.1 for $n = 0$ is the simple exercise of checking that ϕ is well-defined. We noted earlier that if $\mathcal{P}$ is closed under extensions in $\mathcal{M}$, then $(B^{\mathrm{q}})^n\mathcal{P}$ is closed under extensions in $(B^{\mathrm{q}})^n\mathcal{M}$ for each n, and by the same reasoning, one easily sees that if $\mathcal{P}$ and $\mathcal{M}$ satisfy hypothesis (1) of the theorem, then so do $(B^{\mathrm{q}})^n\mathcal{P}$ and $(B^{\mathrm{q}})^n\mathcal{M}$ for each n. The following proposition is about hypothesis (2).

PROPOSITION 2.2. *Let $\mathcal{P}$ and $\mathcal{M}$ satisfy the hypotheses of 2.1. For every object M of $(B^{\mathrm{q}})^n\mathcal{M}$ there exists a short exact sequence $0 \to P' \to P \to M \to 0$ of $(B^{\mathrm{q}})^n\mathcal{M}$ with P' and P in $(B^{\mathrm{q}})^n\mathcal{P}$. Furthermore, if M is a diagonal binary multicomplex, then we may choose P and P' to be diagonal as well.*

We shall prove Proposition 2.2 shortly. We now continue with the proof of Theorem 2.1. Together with the known isomorphism for K_0, the first part of the proposition implies that the induced map $K_0(B^{\mathrm{q}})^n\mathcal{P} \to K_0(B^{\mathrm{q}})^n\mathcal{M}$ is an isomorphism for each n. Clearly this isomorphism sends elements of $T^n_{\mathcal{P}}$ to elements of $T^n_{\mathcal{M}}$. Since the value of ϕ is independent of the choice of resolution, the second part of the proposition implies that ϕ maps elements of $T^n_{\mathcal{M}}$ to elements of $T^n_{\mathcal{P}}$. The isomorphism therefore descends to an isomorphism $K_n\mathcal{P} \to K_n\mathcal{M}$. $\qquad\qquad\square$

It remains then to prove Proposition 2.2, so for the rest of this section we assume the hypotheses of Theorem 2.1. The idea of the proof is to construct, for each M in $(B^{\mathrm{q}})^n\mathcal{M}$, a morphism of acyclic binary chain complexes $P \to M$ that is an admissible epimorphism in every degree, i.e., $P_j \twoheadrightarrow M_j$ is admissible for each j. By the assumption on $\mathcal{P}$ and $\mathcal{M}$ each of these admissible epimorphisms is part of a short exact sequence $0 \to P'_j \to P_j \to M_j \to 0$ with the P_j in $\mathcal{P}$. The P'_j form a binary complex with the induced maps, and we show that this binary complex is in fact acyclic. The result will then follow from an induction on the dimension. We shall rely on the following facts.

LEMMA 2.3. *Let $f_i : Q_i \to N$, $i = 1, \dots, m$ be a family of morphisms in an exact category, at least one of which is an admissible epimorphism. Then the induced morphism*

$$[\,f_1\ \cdots\ f_m\,] : \bigoplus_{i=1}^{m} Q_i \to N$$

is an admissible epimorphism as well.

Proof. The general case follows from the case $m = 2$. In this case, the morphism $[\,f_1\ f_2\,]$ factors as the composition

$$Q_1 \oplus Q_2 \xrightarrow{\left[\begin{smallmatrix} f_1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} N \oplus Q_2 \xrightarrow{\left[\begin{smallmatrix} 1 & f_2 \\ 0 & 1 \end{smallmatrix}\right]} N \oplus Q_2 \xrightarrow{[\,1\ 0\,]} N,$$

all of which are admissible epimorphisms. $\square$

LEMMA 2.4. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of bounded chain complexes in an exact category $\mathcal{N}$ such that B and C are acyclic. Suppose that the differentials of B and C factor through short exact sequences $0 \to Y_j \to B_j \to Y_{j-1} \to 0$ and $0 \to Z_j \to C_j \to Z_{j-1} \to 0$ of $\mathcal{N}$. Then A is acyclic if and only if each of the induced morphisms $Y_j \to Z_j$ is an admissible epimorphism in $\mathcal{N}$. In this case the differentials of A factor through the kernels X_j of $Y_j \twoheadrightarrow Z_j$.*

Proof. Without loss of generality, assume that A, B and C are all 0 in negative degrees, so that $Y_0 = B_0$ and $Z_0 = C_0$. Define X_0 to be A_0, the kernel of $Y_0 \twoheadrightarrow Z_0$. Since $Y_1 \twoheadrightarrow Z_1$ is an admissible epimorphism of $\mathcal{N}$, it has a kernel, which we shall denote by X_1. There are induced maps of kernels $X_1 \to A_1 \to X_0$, and since $0 \to Y_1 \to B_1 \to Y_0 \to 0$ and $0 \to Z_1 \to C_1 \to Z_0 \to 0$ are short exact, then so is $0 \to X_1 \to A_1 \to X_0 \to 0$ by the 3×3 lemma for exact categories. Defining X_j to be the kernel of the admissible epimorphism $Y_j \twoheadrightarrow Z_j$ for each j, this argument is repeated to show that $0 \to X_j \to A_j \to X_{j-1} \to 0$ is short exact for every j. Uniqueness of induced maps between kernels shows that the differentials of A factor through these short exact sequences, so A is acyclic. $\square$

We begin resolving binary complexes in the less involved case, in which we assume M to be diagonal.

LEMMA 2.5. *Given a diagonal bounded acyclic binary complex M in $B^q\mathcal{M}$ there exists a short exact sequence $0 \to P' \to P \to M \to 0$ where P' and P are diagonal objects of $B^q\mathcal{P}$.*

Proof. We may consider M as an object of $C^q\mathcal{N}$, as $\Delta : C^q\mathcal{N} \to B^q\mathcal{N}$ is a full embedding for any exact category $\mathcal{N}$. Represent M in $C^q\mathcal{N}$ as below. Without loss of generality we assume that M ends at place 0.

$$0 \longrightarrow M_n \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } M_k \xrightarrow{\ d\ } M_{k-1} \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } M_0 \longrightarrow 0,$$

Since $\mathcal{P}$ and $\mathcal{M}$ satisfy the hypotheses of Theorem 2.1, there exists an object Q_k of $\mathcal{P}$ and an admissible epimorphism $\epsilon_k : Q_k \twoheadrightarrow M_k$ in $\mathcal{M}$ for each $0 \le k \le n$. The diagram below is a morphism in $B^q\mathcal{M}$ with target M, and its upper row, the source of the morphism, is an object of $C^q\mathcal{P}$.

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & Q_k & = & Q_k & \longrightarrow & \cdots & \longrightarrow & Q & \longrightarrow & Q \\
 & & \downarrow & & & & \downarrow{\scriptstyle\epsilon_k} & & \downarrow{\scriptstyle d\epsilon_k} & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_n & \xrightarrow{\ d\ } & \cdots & \xrightarrow{\ d\ } & M_k & \xrightarrow{\ d\ } & M_{k-1} & \xrightarrow{\ d\ } & \cdots & \xrightarrow{\ d\ } & M_0 & \longrightarrow & 0
\end{array}
$$

We denote the top row by P^k and the morphism of complexes by $\zeta^k : P^k \to M$. We do this for each $k \in \{0, \ldots, n\}$ and form the sum

$$\zeta := [\,\zeta^n \ \ldots \ \zeta^0\,] : \bigoplus P^k \to M.$$

Call the direct sum P. Each of the complexes P^k is acyclic, so $\mathcal{P}$ is as well. Consider the morphism $\zeta_j : P_j \to M_j$, the part of ζ from the jth term of

P_j to the jth term of M. By construction and Lemma 2.3, it is an admissible epimorphism of $\mathcal{M}$. We now have a morphism $\zeta : P \to M$ that is an admissible epimorphism in every degree. The kernels of these admissible epimorphisms are all in $\mathcal{P}$ by the hypotheses on $\mathcal{P}$ and $\mathcal{M}$. These kernels form a (as we have seen in Example 1.3, not *a priori* acyclic) bounded chain complex P' in $\mathcal{P}$ under the induced maps between them.

It remains to show that this chain complex is acyclic (i.e., that P' is in $B^q\mathcal{P}$), for then $0 \to P' \to P \to M \to 0$ will be a short exact sequence of acyclic complexes. Suppose that the differential on M factors through objects Z_j of $\mathcal{M}$, and that P factors through objects Y_j of $\mathcal{P}$. By Lemma 2.4, to show that P' is acyclic in $\mathcal{M}$ it is enough to show that each $Y_j \to Z_j$ is an admissible epimorphism of $\mathcal{M}$. Since each P^k is concentrated in degrees k and $k-1$, we have $Y_j = Q_j$ and the induced map $Q_j \to Z_j$ is the composition $Q_j \twoheadrightarrow M_j \twoheadrightarrow Z_j$, which is an admissible epimorphism of $\mathcal{M}$, so P' is acyclic in $\mathcal{M}$ and its differentials factor through the kernels of the admissible epimorphisms $Q_j \twoheadrightarrow Z_j$ (call them X_j). But each Q_j is an object of $\mathcal{P}$, and so each X_j is an object of $\mathcal{P}$ as well, by the hypotheses on $\mathcal{P}$ and $\mathcal{M}$, so P' is acyclic in $\mathcal{P}$.

Finally we consider P', P and M as diagonal binary complexes (by applying Δ). Then $0 \to P' \to P \to M \to 0$ is the required short exact sequence of acyclic diagonal binary complexes. $\qquad\square$

A little more work is required if the binary complex M is not diagonal. The idea in this case is due to Grayson, and relies on the acyclicity of the chain complexes

$$0 \longrightarrow Q \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} Q \oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right]} Q \oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right]} Q \oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]} Q \longrightarrow 0$$

and

$$0 \longrightarrow Q \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} Q \oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 0\\1 & 0\end{smallmatrix}\right]} Q \oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 0\\1 & 0\end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix}0 & 0\\1 & 0\end{smallmatrix}\right]} Q \oplus Q \xrightarrow{\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]} Q \longrightarrow 0$$

of arbitrary length, where Q is an object of any exact category.

LEMMA 2.6. *Given an arbitrary bounded acyclic binary complex M in $B^q\mathcal{M}$ there exists a short exact sequence $0 \to P' \to P \to M \to 0$, where P' and P are objects of $B^q\mathcal{P}$.*

Proof. Let M denote the element of $B^q\mathcal{M}$ given by the binary complex below

$$0 \Longrightarrow M_n \underset{d'}{\overset{d}{\Longrightarrow}} \cdots \underset{d'}{\overset{d}{\Longrightarrow}} M_k \underset{d'}{\overset{d}{\Longrightarrow}} M_{k-1} \underset{d'}{\overset{d}{\Longrightarrow}} \cdots \underset{d'}{\overset{d}{\Longrightarrow}} M_0 \Longrightarrow 0$$

and as before, for $0 \le k \le n$, let $\epsilon_k : Q_k \twoheadrightarrow M_k$ be admissible epimorphisms in $\mathcal{M}$ with Q_k in $\mathcal{P}$. For $0 \le k \le n$ and $1 \le l \le k$, inductively define two collections of morphisms $\delta_{k,l}, \delta'_{k,l} : Q_k \to M_{k-l}$ by

$$\begin{cases} \delta_{k,1} = d \circ \epsilon_k \\ \delta'_{k,1} = d' \circ \epsilon_k \end{cases}$$

and

$$\begin{cases} \delta_{k,l+1} = d \circ \delta'_{k,l} \\ \delta'_{k,l+1} = d' \circ \delta_{k,l}. \end{cases}$$

Since each of its differentials is acyclic, the top row of the diagram below is an object of $B^{\mathrm{q}}\mathcal{P}$ for each $k \in \{0, \ldots, n\}$.

The morphisms $\delta_{k,l}$ and $\delta'_{k,l}$ have been constructed so that the vertical morphisms commute with the top and bottom differentials, so the diagram represents a morphism in $B^{\mathrm{q}}\mathcal{M}$, which we shall again denote by $\zeta^k : P^k \to M$. Following the same method of proof as of the previous lemma, each P^k is acyclic so their direct sum is acyclic as well and so

$$\zeta := [\,\zeta^n \,\ldots\, \zeta^0\,] : \bigoplus P^k \to M.$$

is a morphism in $B^{\mathrm{q}}\mathcal{M}$. By construction and Lemma 2.3 again, each morphism $\zeta^j : P_j \to M_j$ is an admissible epimorphism in $\mathcal{M}$. Each of these morphisms therefore has a kernel in $\mathcal{P}$ and these kernels form a binary complex with the induced maps.

We wish to show that both differentials of this binary complex are acyclic in $\mathcal{P}$. Consider the top differential first. Denote the objects that the top differentials of M, P and each P^k factor through by Z_j, Y_j and Y_j^k (so Y_j is the sum of all the factors Y_j^k) As in the proof of Lemma 2.5, it is enough to show that each induced morphism $Y_j \to Z_j$ is an admissible epimorphism of $\mathcal{M}$. But since $Y_j \to Z_j$ is equal to the product of induced maps $\bigoplus Y_j^k \to Z_j$, it suffices, by Lemma 2.3, to show that, for each j, one of the morphisms $Y_j^k \to Z_j$ is an admissible epimorphism of $\mathcal{M}$. Since $Y_j^j = Q_j$, taking $k = j$ yields the result, as shown by the diagram below.

The bottom differential is dealt with entirely analogously.   □

Proof of Proposition 2.2. We proceed by induction on n. In the base case $n = 0$, there is nothing to show. For the inductive step, we view an acyclic binary

multicomplex M in $(B^{\mathrm{q}})^{n+1}\mathcal{M}$, as an acyclic binary multicomplex of objects of $(B^{\mathrm{q}})^n\mathcal{M}$, i.e., as an object of $B^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{M}$. By the inductive hypothesis, the inclusion of $(B^{\mathrm{q}})^n\mathcal{P}$ into $(B^{\mathrm{q}})^n\mathcal{M}$ satisfies the hypotheses of Theorem 2.1, so by Lemma 2.6 there exists a short exact sequence $0 \to P' \to P \to M \to 0$ in $B^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{M}$ with P' and P in $B^{\mathrm{q}}(B^{\mathrm{q}})^n\mathcal{P} = (B^{\mathrm{q}})^{n+1}\mathcal{P}$, and so the first part follows. For the second part, suppose that M is diagonal in some direction i. We consider M as a diagonal acyclic binary complex of (not necessarily diagonal) objects of $(B^{\mathrm{q}})^n\mathcal{M}$, that is, we "expand" M along the i direction. Then by Lemma 2.5 there exist diagonal acyclic binary complexes P' and P in $(B^{\mathrm{q}})^{n+1}\mathcal{P}$ that are diagonal in direction i, and an exact sequence $0 \to P' \to P \to M \to 0$, so the proof is complete. $\square$

3. The cofinality theorem

Unlike the additivity and resolution theorems, the cofinality theorem was not proved by Quillen in [Qui73]. A proof for exact categories based on work by Gersten was given in [Gra79]. More general versions can be found in [Sta89] and [TT90]. It is proven in [Gra12] that the hypotheses of the cofinality theorem are satisfied by the appropriate exact categories of acyclic binary complexes, the main work in our proof is in ensuring that the results pass to the quotients defining K_n.

Definition 3.1. An exact subcategory $\mathcal{M}$ of an exact category $\mathcal{N}$ is said to be *cofinal* in $\mathcal{N}$ if for every object N_1 of $\mathcal{N}$ there exists another object N_2 of $\mathcal{N}$ such that $N_1 \oplus N_2$ is isomorphic to an object of $\mathcal{M}$.

An obvious example of a cofinal exact subcategory is the category of free R-modules inside the category of projective R-modules, for any ring R. More generally, every exact category is cofinal in its *idempotent completion* ([TT90], Appendix A). The cofinality theorem relates the K-theory of the cofinal subcategory to the K-theory of the exact category containing it. Throughout this section $\mathcal{M} \hookrightarrow \mathcal{N}$ is the inclusion of a cofinal exact subcategory of an exact category $\mathcal{N}$.

Define an equivalence relation on the objects of $\mathcal{N}$ by declaring $N_1 \sim N_2$ if there exist objects M_1 and M_2 of $\mathcal{M}$ such that

$$N_1 \oplus M_1 \cong N_2 \oplus M_2.$$

Since $\langle M \rangle = 0$ for every M in $\mathcal{M}$, the cofinality of $\mathcal{M}$ in $\mathcal{N}$ ensures that equivalence classes of $\sim$ form a group under the natural operation $\langle N_1 \rangle + \langle N_2 \rangle = \langle N_1 \oplus N_2 \rangle$; we denote this group by $K_0(\mathcal{N}$ rel. $\mathcal{M})$. The following lemma and its corollary were first observed in the proof of Theorem 1.1 in [Gra79].

Lemma 3.2. *The sequence:*

$$0 \longrightarrow K_0\mathcal{M} \longrightarrow K_0\mathcal{N} \longrightarrow K_0(\mathcal{N} \text{ rel. } \mathcal{M}) \longrightarrow 0$$
$$[N] \longmapsto \langle N \rangle$$

is well-defined and exact. $\square$

COROLLARY 3.3. *For any pair of objects N_1, N_2 of $\mathcal{N}$ with the same class in* coker($K_0\mathcal{M} \to K_0\mathcal{N}$) *there exists a (single) object N' in $\mathcal{N}$ such that each $N_i \oplus N'$ is in $\mathcal{M}$.*

Proof. By the lemma, if N_1 and N_2 have the same class in coker($K_0\mathcal{M} \to K_0\mathcal{N}$), then $\langle N_1 \rangle = \langle N_2 \rangle$. From cofinality there exists a P in $\mathcal{N}$ such that $N_1 \oplus P$ is in $\mathcal{M}$, so $\langle 0 \rangle = \langle N_1 \oplus P \rangle = \langle N_2 \oplus P \rangle$. Hence there exist objects P_1 and P_2 of $\mathcal{M}$ such that each $(N_i \oplus P) \oplus P_i$ is an object of $\mathcal{M}$. Setting $N' = P \oplus P_1 \oplus P_2$, each $N_i \oplus N'$ is an object of $\mathcal{M}$. $\qquad\square$

We show now that cofinality of $\mathcal{M}$ in $\mathcal{N}$ passes to the associated categories of acyclic binary multicomplexes.

LEMMA 3.4. *For every object N of $B^q\mathcal{N}$ there exists a diagonal object T in $B^q\mathcal{N}$ such that $N \oplus T$ is in $B^q\mathcal{M}$.*

Proof. This is Lemma 6.2 of [Gra12]. $\qquad\square$

COROLLARY 3.5. *For each $n \geq 1$, the category $(B^q)^n\mathcal{M}$ is cofinal in $(B^q)^n\mathcal{N}$.*

Proof. This follows by straightforward induction on the previous lemma (the hypothesis that T is diagonal is not actually needed). $\qquad\square$

We can also conclude that $(C^q)^n\mathcal{M}$ is cofinal in $(C^q)^n\mathcal{N}$. This can be proved independently, but it also follows trivially from Corollary 3.5.

These facts in hand, we now proceed to the proof of the cofinality theorem.

THEOREM 3.6 (Cofinality). *The inclusion functor $\mathcal{M} \hookrightarrow \mathcal{N}$ induces an injection $K_0\mathcal{M} \hookrightarrow K_0\mathcal{N}$ and isomorphisms $K_n\mathcal{M} \cong K_n\mathcal{N}$ for $n > 0$.*

Proof. The case $n = 0$ is part of Lemma 3.2, so we proceed directly to $n > 0$. We consider $n = 1$ first. The inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$ induces a morphism of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0C^q\mathcal{M} & \overset{\Delta}{\longrightarrow} & K_0B^q\mathcal{M} & \longrightarrow & K_1\mathcal{M} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0C^q\mathcal{N} & \overset{\Delta}{\longrightarrow} & K_0B^q\mathcal{N} & \longrightarrow & K_1\mathcal{N} & \longrightarrow & 0.
\end{array}
$$

By Lemma 1.7, these short exact sequences are split. Furthermore, the downwards maps commute with the splittings $\top : K_0B^q\mathcal{M} \to K_0C^q\mathcal{M}$ and $\top : K_0B^q\mathcal{N} \to K_0C^q\mathcal{N}$, so the morphism is a morphism of split exact sequences. The induced map $K_0B^q\mathcal{M} \to K_0B^q\mathcal{N}$ is a monomorphism, as $K_0B^q\mathcal{M}$ is cofinal in $K_0B^q\mathcal{N}$, by Lemma 3.5. The map $K_1\mathcal{M} \to K_1\mathcal{N}$ is therefore a direct summand of a monomorphism and must be a monomorphism itself.

The induced map $K_1\mathcal{M} \to K_1\mathcal{N}$ sends the class $x + K_0C^q\mathcal{M} \in K_1\mathcal{M}$ to

$x + K_0 C^{\mathrm{q}} \mathcal{N} \in K_1 \mathcal{N}$, where $x \in K_0 B^{\mathrm{q}} \mathcal{M}$. Let $x = [N_1] - [N_2]$ be a generic element of $K_0 B^{\mathrm{q}} \mathcal{N}$, where N_1 and N_2 are acyclic binary complexes with objects in $\mathcal{N}$. By Lemma 3.4, there exist diagonal acyclic binary complexes T_1 and T_2 in $B^{\mathrm{q}} \mathcal{N}$ such that $N_i \oplus T_i \in B^{\mathrm{q}} \mathcal{M}$ for $i = 1, 2$. Then

$$[N_i] + K_0 C^{\mathrm{q}} \mathcal{N} = [N_i \oplus T_i] + K_0 C^{\mathrm{q}} \mathcal{N}$$

and therefore

$$x + K_0 C^{\mathrm{q}} \mathcal{N} = [N_1] - [N_2] + K_0 C^{\mathrm{q}} \mathcal{N} = [N_1 \oplus T_1] - [N_2 \oplus T_2] + K_0 C^{\mathrm{q}} \mathcal{N}$$

is in the image of $K_1 \mathcal{M} \to K_1 \mathcal{N}$. Thus $K_1 \mathcal{M} \to K_1 \mathcal{N}$ is surjective and the case $n = 1$ of the theorem is proven.

The remaining cases now follow by an induction on n. By Lemma 1.7 again, the inclusion $\mathcal{M} \hookrightarrow \mathcal{N}$ induces a morphism of split short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_n C^{\mathrm{q}} \mathcal{M} & \xrightarrow{\Delta} & K_n B^{\mathrm{q}} \mathcal{M} & \longrightarrow & K_{n+1} \mathcal{M} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_n C^{\mathrm{q}} \mathcal{N} & \xrightarrow{\Delta} & K_n B^{\mathrm{q}} \mathcal{N} & \longrightarrow & K_{n+1} \mathcal{N} & \longrightarrow & 0.
\end{array}
$$

If $n \geq 1$, the induced map $K_n B^{\mathrm{q}} \mathcal{M} \to K_n B^{\mathrm{q}} \mathcal{N}$ is an isomorphism by the inductive hypothesis, since $B^{\mathrm{q}} \mathcal{M}$ is cofinal in $K_n B^{\mathrm{q}} \mathcal{N}$, by Lemma 3.4 again. Therefore $K_{n+1} \mathcal{M} \to K_{n+1} \mathcal{N}$ is a direct summand of an isomorphism, so it is itself an isomorphism. □

Remark. It is possible to give a proof of the cofinality theorem that does not use Lemma 1.7, although it is less natural than the above. The intermediate steps may be of independent interest, so we outline this alternative argument below. Details will be included in the author's PhD thesis [Har15].

Alternative proof of cofinality theorem (outline). Modifying Grayson's proof of Lemma 6.2 in [Gra12], one can prove the following (stronger) variant of our Lemma 3.4:

For all $n \geq 1$, if N is in $(B^{\mathrm{q}})^n \mathcal{N}$ and $i \in \{1, \ldots, n\}$ is any direction, then there exists an object T in $(B^{\mathrm{q}})^n \mathcal{N}$ that is diagonal in direction i such that $N \oplus T$ is in $(B^{\mathrm{q}})^n \mathcal{M}$. Moreover, if N is diagonal in direction $j \in \{1, \ldots, n\}$, $j \neq i$, then T may be chosen to be diagonal in both direction i and direction j.

This can then be used to prove the following statement:

Let $x + T_{\mathcal{M}}^n$ be a class in $T_{\mathcal{N}}^n / T_{\mathcal{M}}^n$, for $n \geq 1$. Then $x + T_{\mathcal{M}}^n = [t] + T_{\mathcal{M}}^n$, where $[t]$ is the class in $K_0 (B^{\mathrm{q}})^n \mathcal{N}$ of a single diagonal acyclic binary multicomplex t in $(B^{\mathrm{q}})^n \mathcal{N}$.

From this one can deduce that the canonical map

$$T_{\mathcal{N}}^n / T_{\mathcal{M}}^n \longrightarrow K_0 (B^{\mathrm{q}})^n \mathcal{N} / K_0 (B^{\mathrm{q}})^n \mathcal{M}$$

is injective. The surjectivity of this map is an evident application of the above-mentioned variant of Lemma 3.4. Finally, a straightforward argument using the snake lemma shows that induced homomorphism $K_n\mathcal{M} \to K_n\mathcal{N}$ is bijective for $n \geq 1$. $\qquad\square$

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