Bilinear differential forms and the Loewner framework for rational interpolation

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Take-home message

Linear theory *Leitmotiv*:

External structure **mirrored in** ***internal structure***

Linearity & time-invariance $\rightsquigarrow$ state & state equations

Take-home message

Linear theory *Leitmotiv*:

External structure* mirrored in *internal structure

Linearity & time-invariance $\leadsto$ state & state equations

***Identification/Model order reduction*:**

- **Ho-Kalman realization;**
- **Subspace identification;**
- **...**

Take-home message

IN THIS TALK:

External structure**:** **bilinear on external variables**

Internal structure**:** **state equations**

From data to state model

Take-home message

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External structure: **bilinear on external variables**

Internal structure: **state equations**

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HOW:

Loewner matrix $\rightsquigarrow$ **state trajectory** $\rightsquigarrow$ **equations**

Take-home message

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Internal structure: **state equations**

From **data** to **state model**

HOW:

Loewner matrix $\leadsto$ state trajectory $\leadsto$ equations

factorization

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factorization **linear system solution**

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Internal structure: **state equations**

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HOW:

$$\underbrace{\textbf{Loewner matrix} \rightsquigarrow \textbf{state trajectory} \rightsquigarrow \textbf{equations}}_{\textbf{Two-variable polynomial matrices}}$$

Duality: the i/s/o case

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad \begin{aligned} \frac{d}{dt}x' &= -A^\top x' + C^\top u' \\ y' &= B^\top x' - D^\top u' \end{aligned}$$

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$$\boxed{u'^\top y + y'^\top u = \tfrac{d}{dt}\left(x'^\top x\right)}$$

$$\begin{bmatrix} y' \\ u' \end{bmatrix} = \overline{w'}e^{\lambda\cdot}, \begin{bmatrix} u \\ y \end{bmatrix} = \overline{w}e^{\mu\cdot} \Longrightarrow x, x' \text{ vector-exponential}$$

$$\overline{w'}^\top \overline{w} = (\lambda + \mu)\overline{x'}^\top \overline{x}$$

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$$\begin{bmatrix} y'_i \\ u'_i \end{bmatrix} = \overline{w'_i} e^{\lambda_i \cdot}, \begin{bmatrix} u_j \\ y_j \end{bmatrix} = \overline{w_j} e^{\mu_j \cdot}, i, j = 1, \ldots, N \Longrightarrow$$

$$\overline{w'_i}^\top \overline{w_j} = (\lambda_i + \mu_j) \overline{x'_i}^\top \overline{x_j}\ ,\ i, j = 1, \ldots, N$$

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$$\boxed{\left[\frac{\overline{w'_i}^\top \overline{w_j}}{\lambda_i + \mu_j}\right]_{i,j=1,\ldots,N} = \begin{bmatrix} \overline{x'_1} & \ldots & \overline{x'_N} \end{bmatrix}^\top \begin{bmatrix} \overline{x_1} & \ldots & \overline{x_N} \end{bmatrix}}$$

Loewner matrix $\mathbb{L}$

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Factorization $\Longrightarrow$ state trajectories

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For state equations solve for A, B, C, D

$$\boxed{\begin{bmatrix}\mu_1\overline{x_1} & \ldots & \mu_N\overline{x_N} \\ \overline{y_1} & \ldots & \overline{y_N}\end{bmatrix} = \begin{bmatrix}A & B \\ C & D\end{bmatrix}\begin{bmatrix}\overline{x_1} & \ldots & \overline{x_N} \\ \overline{u_1} & \ldots & \overline{u_N}\end{bmatrix}}$$

Duality: the descriptor case-1

$$\begin{aligned} E\frac{d}{dt}x &= Ax + Bu \\ y &= Cx \end{aligned} \qquad\qquad \begin{aligned} E^\top \frac{d}{dt}x' &= -A^\top x' + C^\top u' \\ y' &= B^\top x' \end{aligned}$$

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Duality: the descriptor case-2

$$\begin{aligned} E\frac{d}{dt}x &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \begin{aligned} E^\top \frac{d}{dt}x' &= -A^\top x' + C^\top u' \\ y' &= B^\top x' \end{aligned}$$

$$u'^\top \left(\tfrac{d}{dt}y\right) - \left(\tfrac{d}{dt}y'\right)^\top u = \tfrac{d}{dt}\left(x'^\top Ax\right)$$

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Shifted Loewner matrix $\mathbb{L}_\sigma$

Duality: the descriptor case-3

For state equations, factor

$$\begin{aligned}
\begin{bmatrix} \mathbb{L} & \mathbb{L}_{\boldsymbol{\sigma}} \end{bmatrix} &= \boldsymbol{X'^*} \begin{bmatrix} \boldsymbol{EX} & \boldsymbol{AX} \end{bmatrix} \\
\begin{bmatrix} \mathbb{L} \\ \mathbb{L}_{\boldsymbol{\sigma}} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{X'^* E} \\ \boldsymbol{X'^* A} \end{bmatrix} \boldsymbol{X} \; .
\end{aligned}$$

Duality: the descriptor case-3

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$$\begin{aligned}\begin{bmatrix}\mathbb{L} & \mathbb{L}_{\sigma}\end{bmatrix} &= X'^{*}\begin{bmatrix}EX & AX\end{bmatrix}\\ \begin{bmatrix}\mathbb{L}\\ \mathbb{L}_{\sigma}\end{bmatrix} &= \begin{bmatrix}X'^{*}E\\ X'^{*}A\end{bmatrix}X\ .\end{aligned}$$

If X, $X'^{\top}$ have orthonormal rows (e.g. via SVD):

$E = X'\mathbb{L}X^{\top}$ **and** $A = X'\mathbb{L}_{\sigma}X^{\top}$

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If X, $X'^\top$ have orthonormal rows (e.g. via SVD):

$E = X' \mathbb{L} X^\top$ and $A = X' \mathbb{L}_\sigma X^\top$

Moreover,

$B = -X' \begin{bmatrix} \overline{u'_1} & \dots & \overline{u'_N} \end{bmatrix}$ and $C = \begin{bmatrix} \overline{y_1} & \dots \overline{y_N} \end{bmatrix} X^\top$

Duality: the higher-order case

Dual of $\mathfrak{B} = \operatorname{im} M\left(\frac{d}{dt}\right) = \texttt{ker}\, R\left(\frac{d}{dt}\right)$:

$$\mathfrak{B}^{\perp} = \left\{ w' \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{-\infty}^{+\infty} w'^{\top} w = 0 \right.$$

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$\boldsymbol{x}$ **state for** $\boldsymbol{w} \in \mathfrak{B}$**,** $\boldsymbol{x}'$ **for** $\boldsymbol{w}' \in \mathfrak{B}^{\perp}$**:**

$$\boldsymbol{w}'^{\top}\boldsymbol{w} = \frac{d}{dt}\left(\boldsymbol{x}'^{\top}\boldsymbol{x}\right)$$

Duality: the higher-order case

x state for $w \in \mathfrak{B}$, x' for $w' \in \mathfrak{B}^\perp$:

$$w'^\top w = \frac{d}{dt}\left(x'^\top x\right)$$

...same as 1st order case:

- Factorize Loewner matrix $\mathbb{L} = X'^\top X$;
- Solve for E, F, G

$$\begin{bmatrix} E & F & G \end{bmatrix} \begin{bmatrix} X\mathrm{diag}(\mu_i)_{i=1,\ldots,N} \\ X \\ W \end{bmatrix} = 0$$

- $\mathfrak{B} = \left\{ w \mid \exists\, x \text{ s.t. } E\frac{d}{dt}x + Fx + Gw = 0 \right\}$

An identifiability condition

***Theorem*: Assume $N > \mathtt{n}$, the McMillan degree of the system.**
If there are $\mathtt{n}$ linearly independent $w_i(\cdot) = \overline{w_i} e^{\mu_i \cdot}$,
and $\mathbb{L} = X'^{\top} X$ is a rank-revealing factorization, then

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Proof: Assume $\operatorname{rank} X < \mathtt{n}$, then $\exists\ \alpha_i$ s.t. w.l.o.g. $\sum_{i=1}^{\mathtt{n}} \overline{x_i} \alpha_i = 0$.

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Now $\sum_{i=1}^{\mathtt{n}} \overline{x_i} \alpha_i = x(0)$, where $x(\cdot)$ is state trajectory of $w(\cdot) := \sum_{i=1}^{\mathtt{n}} \overline{w_i} e^{\mu_i \cdot} \alpha_i$.

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$w(\cdot)$ belongs to “autonomous" (w/out inputs) subbehavior $\Longrightarrow w(\cdot) = 0 \Longrightarrow$ contradiction. □

Remarks

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- **For *J*-self-dual systems, i.e. w.r.t. $\langle \cdot, \cdot \rangle_J$,**
  $\mathbb{L} = X^\top X$ is energy matrix $\leftrightsquigarrow$ storage function

  Lossless port-Hamilt./self-adj. Hamiltonian case (w/ Birthday Boy # 1)

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- **Dual data computable from primal ones:**

$$\overline{w}e^{\mu\cdot} \in \mathfrak{B} \text{ and } \overline{v}^\top \overline{w} = 0 \Longrightarrow \overline{v}e^{-\mu\cdot} \in \mathfrak{B}^\perp$$

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- **General (non vector-exponential) discrete-time case (w/ Birthday Boy # 2) also possible.**

Relations with interpolation

Left/right interpolation data:

$$\{(\mu_i, u_i'^*, y_i'^*) \in \mathbb{C} \times \mathbb{C}^p \times \mathbb{C}^m\}_{i=1,\dots,N}$$
$$\{(\lambda_i, u_i, y_i) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^p\}_{i=1,\dots,N} \;.$$

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Looking for $H \in \mathbb{R}^{p\times m}(\xi)$ such that:

$$\begin{aligned} u_i'^* H(\mu_i) &= y_i'^*\ ,\ i = 1, \ldots, N \\ H(\lambda_i) u_i &= y_i\ ,\ i = 1, \ldots, N\ . \end{aligned}$$

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LCF/RCF $H(\xi) = N(\xi)D(\xi)^{-1} = P(\xi)^{-1}Q(\xi) \Longrightarrow$
model vector-exponential trajectories

$$\begin{bmatrix} u_i \\ y_i \end{bmatrix} e^{\mu_i \cdot} \text{ and } \begin{bmatrix} u_i' \\ y_i' \end{bmatrix} e^{\lambda_i \cdot}$$

of

$$\begin{array}{l} \mathfrak{B} = \mathtt{ker} \ \left[Q\left(\frac{d}{dt}\right) \quad -P\left(\frac{d}{dt}\right) \right] \text{ and} \\ \mathfrak{B}^\perp = \mathtt{ker} \ \left[D^\top\left(-\frac{d}{dt}\right) \quad -N^\top\left(-\frac{d}{dt}\right) \right] \ , \end{array}$$

respectively.

Relations with bilinear differential forms

$$\begin{aligned} \mathfrak{B} &= \texttt{ker}\, R\left(\frac{d}{dt}\right) = \text{im}\, M\left(\frac{d}{dt}\right) \\ \mathfrak{B}^{\perp} &= \text{im}\, R^{\top}\left(-\frac{d}{dt}\right) = \texttt{ker}\, M^{\top}\left(-\frac{d}{dt}\right) \end{aligned}$$

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$R(\xi)M(\xi) = 0_{\texttt{p}\times\texttt{m}} \Longrightarrow \exists\ \Psi(\zeta, \eta)$ such that

$$R(-\zeta)M(\eta) = (\zeta + \eta)\Psi(\zeta, \eta)$$

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$$R(-\zeta)M(\eta) = (\zeta+\eta)\Psi(\zeta,\eta) = (\zeta+\eta)X'(\zeta)^\top X(\eta)$$

with $X\left(\frac{d}{dt}\right)$, $X'\left(\frac{d}{dt}\right)$ **state maps for** $\mathfrak{B}$, **resp.** $\mathfrak{B}^\perp$

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with $X\left(\frac{d}{dt}\right)$, $X'\left(\frac{d}{dt}\right)$ state maps for $\mathfrak{B}$, resp. $\mathfrak{B}^{\perp}$

$$\boxed{\frac{R(-\zeta)M(\eta)}{\zeta+\eta} = X'(\zeta)^{\top}X(\eta) \leftrightsquigarrow \left[\frac{\overline{w'_i}^{\top}\overline{w_j}}{\lambda_i+\mu_j}\right]_{i,j=1,\ldots,N} = X'^{\top}X}$$

Conclusions

- **Data $\leadsto$ state trajectories**

  $\leadsto$ state equations

  using bilinear structure on state from bilinear structure on external variables

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- Data $\rightsquigarrow$ state trajectories
  $\rightsquigarrow$ state equations

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- Model order reduction:
  Data $\rightsquigarrow$ reduced-order state trajectories
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  Lossless port-Hamilt. & self-adj. Hamilt. case

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- **Current research:**
  - **1*D* parametric modelling**

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  $\leadsto$ reduced-order state equations

  Lossless port-Hamilt. & self-adj. Hamilt. case

- **Current research:**
  - **$1D$ parametric modelling**
  - **2D case: Roesser models for $\overline{w_i} e^{\lambda_1^i \cdot_1} e^{\lambda_2^i \cdot_2}$, $i = 1, \ldots, N$**

Some related publications

- Rapisarda, P. and Trentelman, H.L. (2011) “Identification and data-driven model reduction of state-space representations of lossless and dissipative systems from noise-free data". *Automatica*, 47, (8), 1721-1728.

- van der Schaft, A.J. and Rapisarda, P. (2011) “State maps from integration by parts". *SIAM J. Contr. Opt.*, 49, (6), 2415-2439.

- Rao, S. and Rapisarda, P. (2013) “Realization of lossless systems via constant matrix factorizations". *IEEE Trans. Aut. Contr.*, 58, (10), 2632-2636.

- Rapisarda, P. and van der Schaft, A.J. (2013) “Identification and data-driven reduced-order modeling for linear conservative port- and self-adjoint Hamiltonian systems". In *Proc. 52nd IEEE CDC*, Firenze, Italy.

THANK YOU

... and happy 60th birthdays, jongens!