UNIVERSITY OF SOUTHAMPTON

A Geometric Approach To Fault Detection and Isolation in Multidimensional Systems

by

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ABSTRACT

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In this thesis the problem of fault detection and isolation for two subclasses of multidimensional systems, i.e., 3-D systems and repetitive processes is investigated by extending the geometric approach and notions of input containing conditioned invariants developed and introduced for standard 1-D linear systems to be applicable in multidimensional systems.

The problem is investigated by designing an asymptotic observer that asymptotically reconstructs the system state. In case of a failure, the observer continues to reconstruct the state normally, however, the system produces a wrong state resulting in deviation of the system state from the estimated state in the observer. Comparing the magnitude of this deviation against a predefined threshold indicates whether a failure has occurred in the system or not.

The fault detection and isolation problem for the aforementioned systems is formulated in a geometric language and necessary and sufficient conditions are developed for the solvability of this problem, and constructive methods to design observers that uniquely can isolate the failure by exploiting the subspaces that the error lies onto. Finally, the efficiency of the developed technique is examined by using examples for each system.

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Contents

Chapter 1

Introduction

Modern life is becoming more and more complex that the use of autonomous machinery for the sake of reliability, cost efficiency, availability and safety is becoming increasingly inevitable. Relying on such machinery requires a robust fault diagnosis algorithm that is capable of detecting and isolating a failure once it occurs. Developing such algorithms has received much attention in the past decades and much research effort have been focused on this task (see for example [1, 2, 3, 4, 5]).

Fault Detection and Isolation (FDI) and in general dealing with failures is a challenging design-element in building reliable systems. Especially where it is required for these systems to operate in environments with limited or no access such as space or remote areas for several years. Therefore, in case of a failure, it must be detected and isolated immediately to enable appropriate measures to be taken. One of these measures is failure accommodation and system reconfiguration so that the system continues to operate at a reduced level. This has been examined by two different approaches, namely, the multiple-model approach [6, 7, 8] and the adaptive control approach [9, 10, 11, 12].

FDI for the standard 1-D linear systems has been investigated (see for example [1, 13, 14]) previously. However, the topic of multidimensional systems where information propagates in more than one independent direction (rather than in one direction, normally taken to be the time axis) continues to provide challenging problems which arise in the continuously expanding domain of applications. For example, recent advances in technology have given rise to applications where three dimensions are involved in the process. These applications range from a three-dimensional task-specific robotic arm for facilitating stroke rehabilitation [15] to new methods for distributed information processing in Grid Sensor Networks (GSN) using the 3-D Fornasini-Marchesini (FM) model [16, 17]. These are important applications where handling a failure upon occurrence is crucial. For instance, if an actuator that moves the robotic arm in a certain direction breaks down or a node in a grid sensor network dies so that local information updating becomes impossible, these failures prevent the whole system from operating and can cause considerable damage.

An important class of multidimensional systems that represent many industrial applications is linear repetitive processes where a material or work-piece is processed by a sequence of passes of the processing tool. During each pass, a profile is generated which contributes to the following passes. These processes are considered to be a sub-class of 2-D systems since two indeterminates are required to specify each point - the time or the distance along the pass, and the pass number. Industrial examples of these processes include metal rolling, long wall coal cutting and bench mining operations, and for algorithmic examples one can refer to classes of iterative learning control schemes. A failure in these processes, can be very costly since if not detected and fixed immediately, it will contribute to the next passes and damages the whole process.

FDI methods are mainly based on the concept of redundancy, which can be either a hardware redundancy or analytical redundancy. The main idea behind hardware redundancy is to compare duplicative signals generated by various hardware, such as multiple measurements of the same signal by a number of sensors. The common techniques used in this approach are the Cross Channel Monitoring (CCM) method, residual generation using parity generation (e.g., based on sensor geometry or signal pattern), and signal processing methods such as wavelet transformation, etc [18].

Conversely, analytical redundancy uses a mathematical model of the system together with estimation techniques for FDI. As the analytical redundancy approach generally does not require extra hardware, it is comparatively a more cost-effective approach than the hardware redundancy. However, the analytical redundancy is more challenging due to the need to ensure its robustness in the presence of disturbances.

Generally, the analytical redundancy approach can be divided into quantitative model-based methods and qualitative model-based methods. The quantitative model-based methods, such as the observer-based methods, use explicit mathematical models and control theories to generate residuals. Conversely, the qualitative model-based methods use Artificial Intelligence (AI) techniques, such as pattern recognition, to capture discrepancies between observed behaviour and that predicted by a model [18].

In this thesis, a quantitative model-based approach is used to address the FDI problem. This method is inspired by the geometric approach for FDI in 1-D systems originally introduced by Massoumnia [1] wherein a geometric formulation of Beard's failure detection filter [14] is developed using concept of (A, C)-invariant and unobservability subspaces.

In what follows a brief overview of FDI for 1-D systems and Massoumnia's approach to address this problem is given. But before proceeding, the most common terminologies in the field of model-based fault diagnosis, defined by IFAC SAFEPROCESS Technical Committee [19], are stated.

Definition 1.1. *Fault* is an unpermitted deviation of at least one characteristic property or parameter of the system from the acceptable/usual/standard condition.

Definition 1.2. *Failure* is a permanent interruption of a system's ability to perform a required function under specified operating conditions.

Definition 1.3. *Error* is a deviation between a measured or computed value (of an output variable) and the true, specified or theoretically correct value.

Definition 1.4. *Residual* is a fault indicator, based on a deviation between measurements and model-equation-based computations.

Definition 1.5. *Fault detection* is the determination of the faults present in a system and the time of detection.

Definition 1.6. *Fault isolation* is the determination of the kind, location and time of detection of a fault which follows from fault detection.

1.1 FDI in 1-D Systems

An FDI process essentially consists of two stages. The first stage is residual generation and the second involves using the residuals to make the appropriate decisions. Here, the focus is on the residual generation aspect only which is visualised as in Figure 1.1.

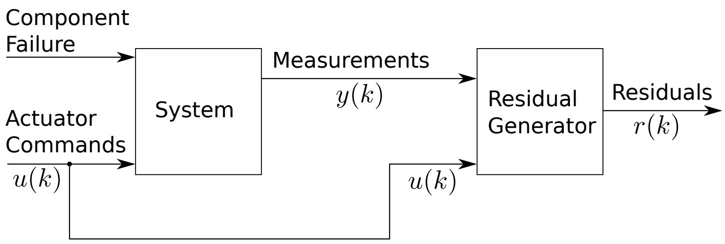

FIGURE 1.1: Block diagram of a residual generator

A residual generator takes the measurements and the commanded actuation signals as inputs and generates a residual vector which is nominally zero or close to zero when no failure is present, but is distinguishably different from zero when a component of the system fails. The residual generation process is simply subtracting the outputs of two identical sensors measuring the same quantity. A failure of either sensor corrupts the residual and this can be used to detect the failure. The process of generating the residuals from relationships among instantaneous outputs of sensors is usually called direct redundancy [20].

It is also possible to generate the residuals using temporal redundancy, which is the process of exploiting the relationship among the histories of sensor outputs and actuator inputs [20]. This is usually done by using a hypothesised model of the dynamics of the system to relate sensor outputs and actuator inputs at different instants of time. Unfortunately, most of the existing techniques exploiting temporal redundancy are tuned to hypotheses about the mode of component failure, e.g., the actuator failure is assumed to be a bias failure of an unknown magnitude. However, in most cases it is not possible to enumerate a comprehensive list of possible failure modes and characterise the behaviour of the component following each of these failures. Therefore, an approach that does not rely on prior assumptions about the mode of component failure is highly desirable.

Based on these requirements, Beard [14] and [13] took a fundamentally different approach to the residual generation problem. This work proposed a systematic procedure for designing a special observer that accentuates the effect of failure on the innovation (or prediction error) of the observer. The observer is designed so that in the absence of component failures, modelling errors, and system disturbances, the innovation vector dies away, while if the system suffers a failure, the innovation starts to grow. Moreover, the observer gain is chosen so that the direction of the innovation vector in the output space can be used to identify the failed component.

Massoumnia uses a geometric approach to reformulate and solve the Beard-Jones detection filter problem (BJDFP) [14].

Consider the following equations to describe the nominal system model:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \sum_{i=1}^{k} L_i m_i(k) \ , \\ y(k) &= Cx(k) \ , \end{aligned} \tag{1.1}$$

where the arbitrary functions $m_i(k) \in \mathcal{M}_i$ are the unknown failure modes and the maps $L_i : \mathcal{M}_i \longrightarrow \mathcal{X}$ are the failure signatures. Obviously where no faults has occurred, $m_i(k) = 0$. The failure modes together with the failure signatures can be used to model the effect of actuator failures on the dynamics of the system. For example, the effect of failure on the dynamics in the ith actuator can be represented by $L_1 = B_i$ where B_i is the ith column of B; if the actuator is dead, then $m_1(k) = -u_i(k)$ where $u_i(k)$ is the ith component of $u(k)$, while for a bias failure of the same actuator, $m_1(k)$ is some non-zero constant. It is clear that the failure signature $L_1 = B_i$ can model the effect of a wide variety of failures in the ith actuator, since the failure mode $m_i(k)$ is not restricted in any way. Also changes in the system dynamics, i.e., changes in the A matrix, can be modelled through appropriate choice of the failure signatures $L_,$. Moreover, the failure signatures are not restricted to be column vectors. The process of choosing the failure signatures is referred to as *failure modelling*. A detailed discussion of failure modelling can be found in [13, 21].

Now consider the following full-order observer for the system (1.1):

$$\hat{x}(k+1) = (A+GC)\hat{x}(k) - Gy(k) + Bu(k) \ , \tag{1.2}$$

where $G : \mathcal{Y} \longrightarrow \mathcal{X}$ is the observer gain or the output injection matrix. Defining the difference between the system state and the estimated state by the observer as the error, gives:

$$\begin{aligned} e(k+1) &= (A+GC)e(k) - \sum_{i=1}^{k} L_i m_i(k) \ , \\ r(k) &= Ce(k) \ , \end{aligned} \tag{1.3}$$

where $r(k) := C\hat{x}(k) - y(k)$ is the generated residual or the innovation vector. Moreover, assume $e(0) = 0$. Now when the ith actuator fails, $m_i(k) \neq 0$, $e(k) \in \mathcal{V}_i :=< A+GC|\mathcal{L}_i >$, where $< A+GC|\mathcal{L}_i >= \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}$ is the infimal A-invariant subspace containing $\mathcal{B}$, i.e., the reachable subspace of (A, B) and $\mathcal{L}_i :=$ Im L_i. Furthermore, $r(k) \in C\mathcal{V}_i$. Now the FDI problem reduces to finding a map $G : \mathcal{Y} \longrightarrow \mathcal{X}$, such that the family of subspaces $\{C\mathcal{V}_i, i \in k\}$ is independent; consequently the innovation generated due to each different actuator failure can be identified by finding the projection of $r(k)$ onto each of the independent subspaces $C\mathcal{V}_i$ and comparing the magnitude of this projection to a threshold.

Note that this identification procedure has almost no dependency on on the functional behaviour of the failure modes $m_i(k)$, and this is the feature that distinguishes Massoumnia's approach to the FDI problem from many others reported in the literature [18].

1.2 FDI in Multidimensional Systems

Advances in technology have raised a major interest in more complex multidimensional systems and its theory applications in general disciplines of circuits, signal processing and control areas. Therefore, FDI in these systems is of particular importance. One approach to address this problem, is to generalise the geometric framework used by Massoumnia for residual generation in the 1-D counterpart [1]. nD systems as in 1-D systems can be studied either in the transform domain or the state-space form. In the first case, the polynomial aspect is the most important and in the state-space approach, the notions of observability, controllability and minimality play an important part. There are certain difficulties in generalising 1-D to nD ($n \geq 2$) some of which are fundamentally algebraic in nature, e.g., the lack of a Euclidean algorithm or the distinction in the multidimensional case between zero primeness, factor primeness and minor primeness. For example, in 2-D systems theory, it becomes important to distinguish between factor primeness and zero primeness [22]. Other problems include the apparent absence of relationships between strongly related 1-D concepts in the nD case. For instance, concepts of controllability and observability are already generalised to the nD case [23] but for none of these generalisations, controllability plus observability equals minimality.

At an abstract level, multidimensional systems theory sets forth to investigate the basic concepts as 1-D theory, e.g., observability, controllability, causality, stability and etc. Given the difficulties in generalising 1-D systems theory, it is concluded that much of this development must start from a basic level. For this purpose, there are a "rich" variety of dynamical models in "n" indeterminates to start with. For instance, the FM model [24], which was originally proposed for 2-D systems, where the propagation of the dynamics occurs in 2 independent directions.

1.3 Problem Formulation

In this thesis, using a geometric approach it is aimed to address the FDI problem in two different classes of multidimensional systems, namely, 3-D systems and discrete linear repetitive processes. It should be mentioned that although these two subclasses are considered in this thesis, the developed approach can be readily used to address the problem in the general nD case.

For the first part of this thesis, a generalisation of FM state-space representation to the 3-D case is used to introduce a model incorporating signatures of possible failures together with failure modes. In the second part, a discrete unit memory linear repetitive process state-space representation is introduced based on a general abstract model developed in [25] that incorporates signatures of possible failures together with failure modes.

Regardless of the mathematical description of these models, the aim is to design an asymptotic observer $\hat{\Sigma}$, which is considered to be another system identical to the original system that is intended to be monitored, for the purpose of FDI in 3-D systems linear repetitive processes. Figures 1.2 and 1.3 illustrate the block diagram of these systems together with the asymptotic observer.

The logic behind this approach is to provide the same input of the system to the observer and compare its output with that of the system. Then by feeding back the result through some output-injection gains to the observer, the goal is to steer this difference to zero. Once this is achieved, since both the system and the observer have the same input, output and dynamics, the system's state must have been reconstructed successfully. Now defining the difference between the actual and the estimated state as the error signal, if there are no failures in the system this error goes to zero. Once a failure occurs, the system produces a different output while the observer keeps producing the output that originally the system was meant to produce. Hence, the actual state deviates significantly from the estimated state and the error signal will not be zero anymore and passes a pre-defined threshold indicating presence of a failure.

To isolate the detected failure, one approach is to use Massoumnia's method by projecting the innovation vectors to independent subspaces $\mathcal{V}_i$ and the subspace that this projection has a non-zero norm determines the failed component. However, since the generated residuals are dimensionally dependent on the system's output, there is no chance to isolate the fault in the case where the output has a low dimension. For example, if the output is a scalar, then there is no chance to be able to detect more than one fault concluding fault isolation by means of residual generation is rather confining for systems with low dimensional output.

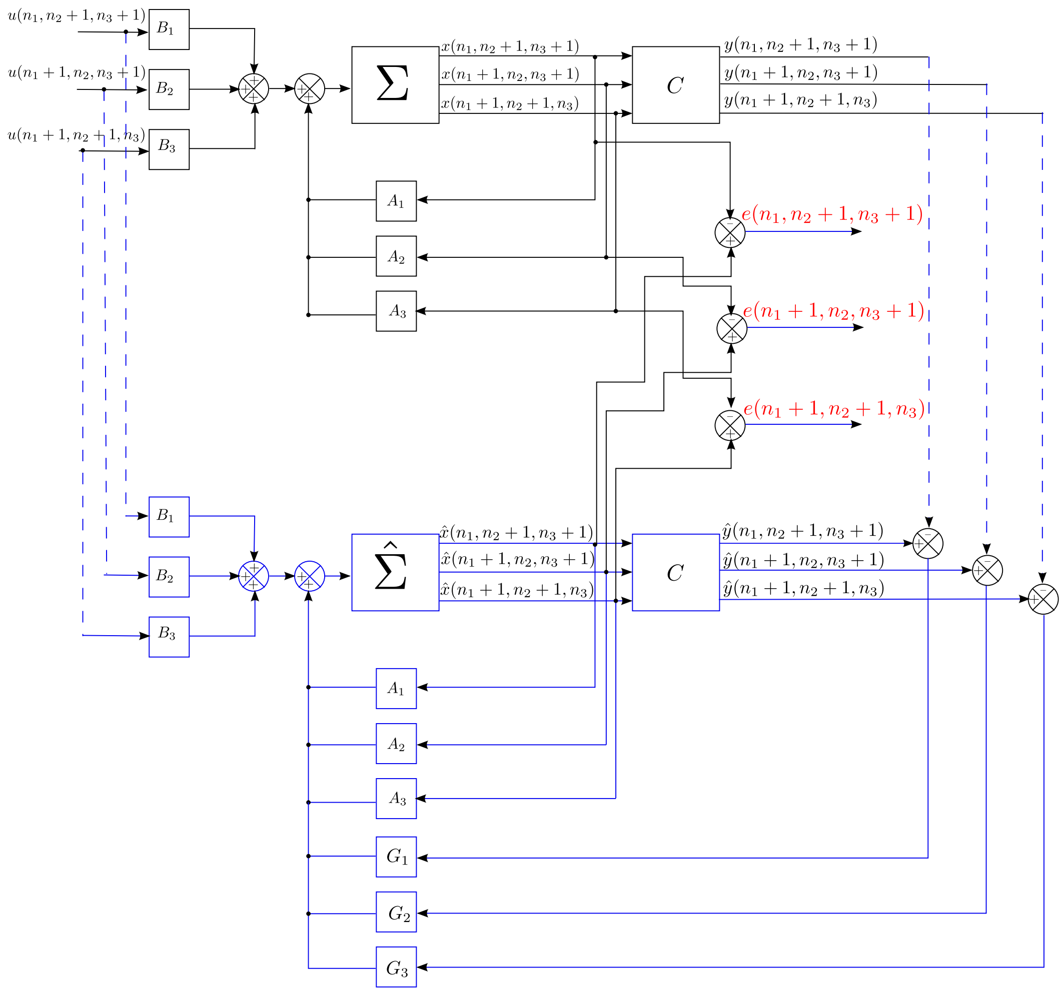

FIGURE 1.2: Block diagram of a 3-D system and an asymptotic observer for FDI.

The solution developed in this thesis proceeds by projecting the error vector onto the subspaces spanned by the failure signatures rather than the residuals. This provides more freedom for FDI regardless of the output dimension.

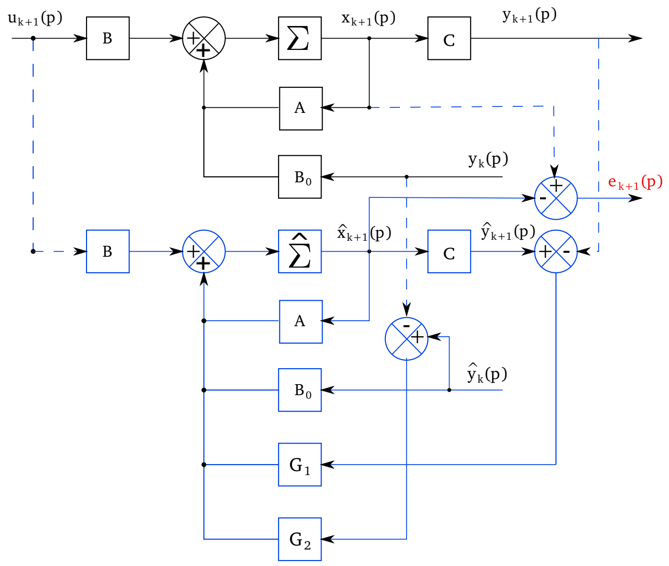

FIGURE 1.3: Block diagram of a linear repetitive process and an asymptotic observer for FDI.

1.4 Thesis Overview

In Chapter 2, the most commonly used classes of models for representing nD systems ($n \geq 2$) (i.e., Roesser and Fornasini-Marchesini models) are introduced and the specific case of 3-D systems is considered based on these models. The relevant structural properties of these systems such as reachability and observability are also reviewed.

In Chapter 3, a connection between basic concepts of linear algebra (like invariants, changes of basis) and properties of linear dynamic systems is established. A brief overview of conditioned invariant subspaces and their characteristics is given and a particular family of conditioned invariant subspaces which is of our interest for the purpose of fault identification, namely, the input-containing conditioned invariants is introduced. Moreover, the unobservability subspaces which originally were introduced in [1] are generalised to the 3-D case. Finally, the chapter is closed by establishing an LMI approach for construction of the internally and externally stabilising output injection gains.

In Chapter 4, a geometric technique to address the FDI problem in 3-D systems is developed.

In Chapter 5, linear repetitive processes are introduced and a state-space model along with the abstract model in the Banach space which the stability theory is based on for these processes is introduced. Also the concept of stability in these processes is reviewed.

Chapter 6 investigates the fault detection and isolation (FDI) problem for discrete-time linear repetitive processes (DLRP) using a geometric approach. A 2-D model for these systems that incorporates the failure description is proposed. Based on this model, a formulation of the FDI problem in geometric language is given and sufficient conditions for solvability of the problem are developed. Finally, a FDI procedure based on an asymptotic observer of the state is established.

1.4.1 Contributions

I. Designing an asymptotic observer that can isolate more faults as compared to previous observers.

II. Fault detection and isolation for 3-D systems with applications in grid sensor networks.

III. Fornasini-Marchesini representation of linear repetitive processes that enables applying a great number of already available results for these processes.

IV. Designing a stabilising gain for repetitive processes that prevents the error to travel from pass to pass.

V. Fault detection and isolation for linear repetitive processes with industrial and algorithmic applications.

VI. A fault detection and isolation algorithm in presence of noise.

Chapter 2

Multidimensional Systems and Their Structural Properties

2.1 Introduction

The field of multidimensional systems has been a subject of intense research since the early 70's when a number of researchers such as Attasi [26], Givone and Rosser [27], and Fronasini and Marchesini [28] introduced a two dimensional state-space representation of linear dynamical systems where discrete signals are modelled as a function of two independent variables. Interestingly, these models can be generalised to represent 'n-dimensional' (nD) systems where $n \geq 2$ in a straightforward manner.

At first sight, comparing these models appears to be a hard task since they are based on state-space representations having distinct structures. However, considering these differences from the realisation perspective, it turns out that the mentioned models realise different transfer functions [24]. The recursiveness of the state equation signifies the rationality of the transfer function; nontheless the realisation of a strictly causal rational transfer function cannot be achieved by every model. For Example, the model represented by Attasi in [26] can realise only the subclass of "separable filter". However, both Roesser and Fornasini-Marchesini models can realise the whole class of causal rational functions in two indeterminates [28, 29].

Structural properties of two dimensional systems such as controllability and observability have also attracted research efforts more recently [30, 31]. In the 2-D setting, reachability and observability are introduced in two forms of local and global which refers to single "local states" and the infinite set of local states lying on a "separation set" respectively [32].

In this chapter the most commonly used class of models for representing nD systems ($n \geq 2$) (i.e., Roesser and Fornasini-Marchesini models) is first introduced and then these models are extended to represent a three dimensional system which is used in Chapter 4 for the purpose of fault detection and isolation in three dimensional systems. The structural properties of three dimensional systems such as observability and reachability are also investigated and the general response formula for three dimensional systems is derived.

2.2 2-D Roesser Model

In 1972 Givone and Roesser introduced a state-space representation known as Givone-Roesser model for linear iterative circuits having more than one spatial dimension [27]. They used transition matrices that allowed treatment of such models in a relatively straightforfard manner. Roesser then extended this model to introduce the Roesser model for linear image processing where a real field is assumed rather than a finite field which was assumed in the case of iterative circuits [33].

The intrinsic feature of Roesser model is that the partial state vector is partionioned into n sub-vectors for nD systems. In the case of a 2-D system, these partitions are called the horizontal and the vertical state sub-vectors. For a 2-D system, the Roesser model is of the form

$$\begin{aligned} \begin{bmatrix} x^h(n_1+1, n_2) \\ x^v(n_1, n_2+1) \end{bmatrix} &= \mathbf{A} \begin{bmatrix} x^h(n_1, n_2) \\ x^v n_1 i, n_2) \end{bmatrix} + \mathbf{B}\ u(n_1, n_2) \\ y(n_1, n_2) &= \mathbf{C} \begin{bmatrix} x^h(n_1, n_2) \\ x^v(n_1, n_2) \end{bmatrix} + \mathbf{D}\ u(n_1, n_2)\ , \end{aligned} \tag{2.1}$$

where $n_1, n_2 \in \mathbb{N}$, $x^h(n_1+1, n_2) \in \mathbb{R}^a$, $x^v(n_1, n_2+1) \in \mathbb{R}^b$, x, y and u denote the state, output and the input of the system respectively. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the matrices with the appropriate dimensions and real constant entries.

The Roesser model presented above can be generalised to represent an nD system:

$$\begin{bmatrix} x^1(n_1+1, n_2, \dots, n_n) \\ x^2(n_1, n_2+1, \dots, n_n) \\ \vdots \\ x^n(n_1, n_2, \dots, n_n+1) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x^1(n_1, n_2, \dots, n_n) \\ x^2(n_1, n_2, \dots, n_n) \\ \vdots \\ x^n(n_1, n_2, \dots, n_n) \end{bmatrix} + \mathbf{B}\ u(n_1, n_2, \dots, n_n)\ ,$$

$$y(n_1, n_2, \dots, n_n) = \mathbf{C} \begin{bmatrix} x^1(n_1, n_2, \dots, n_n) \\ x^2(n_1, n_2, \dots, n_n) \\ \vdots \\ x^n(n_1, n_2, \dots, n_n) \end{bmatrix} + \mathbf{D}\ u(n_1, n_2, \dots, n_n)\ . \tag{2.2}$$

2.3 2-D Fornasini–Marchesini Model

Fornasini–Marchesini [24] is another class of models that is commonly used for realisation of multidimensional systems. The model's approach to address algebraic realisation problem of nD systems is to use input–output maps to obtain state-space representation by Nerode equivalence of inputs. However such representations are of infinite dimensions, thus imposing difficulties in expressing the dynamics of systems in terms of a recursive updating equation [34]. Fornasini and Marchesini were the first to overcome these difficulties by introducing the notion of *local state* and *global state* (Nerode state) in the 2-D setting [34].

The local state contains the information that is used to compute the state at each step of recursive computations while global state which is generally of infinite dimension and provides all the past information.

Fornasini and Marchesini propose two models for the local state space updating scheme in the 2-D setting. The most basic model has the form

$$\begin{aligned} x(n_1+1, n_2+1) =& \mathbf{A_1}\ x(n_1+1, n_2) + \mathbf{A_2}\ x(n_1, n_2+1) + \mathbf{A}\ x(n_1, n_2) + \mathbf{B}\ u(n_1, n_2) \\ &+ \mathbf{B_1}\ u(n_1+1, n_2) + \mathbf{B_2}\ u(n_1, n_2+1), \end{aligned} \tag{2.3}$$

$$y(n_1, n_2) = \mathbf{C}\ x(n_1, n_2) + \mathbf{D}\ u(n_1, n_2), \tag{2.4}$$

where $x(n_1, n_2) \in \mathbb{R}^n$ is the partial state vector, $u(n_1, n_2)$ the input vector at (n_1, n_2), $y(n_1, n_2)$ the output vector at (n_1, n_2) and $\mathbf{A_i}$, $\mathbf{B}$, $\mathbf{B_j}$, $\mathbf{C}$, $\mathbf{D}$, $i = 1, 2, 3$, $j = 1, 2$; are compatibly dimensioned matrices with real constant entries. The initial conditions of such model are assigned along each 1-D propagation front as $x(0, j) = \hat{x}(j)$, and

$x(i,0) = \hat{x}(i),\ i,j \in \mathbb{Z}^+$. If $\mathbf{A_3}$ and $\mathbf{B}$ in (2.3) are equal to zero, the model is said to be of second order and if $\mathbf{B_1}$ and $\mathbf{B_2}$ are equal to zero, the model is termed as first order [35].

To demonstrate the concepts of local and global states in an updating structure, define a partially ordered set to represent the concept of *past*, *present* and *future* as follows:

Definition. Let $\mathcal{P}$ be a partially ordered set. A *cross-cut* $\mathcal{C} \subset \mathcal{P}$ is a set of incomparable points such that if $i \in \mathcal{P}$ exactly one of the following is true [36]:

I. $i \in \mathcal{C}$

II. $i > j$ for some $j \in \mathcal{C}$

III. $i < j$ for some $j \in \mathcal{C}$

The cross-cut $\mathcal{C}$ partitions the set $\mathcal{P}$ into three sections according to I,II and III which are termed present, future and past respectively (see Figure 2.1). A finite dimensional local state x is assigned to each point (n_1, n_2) of the $\mathbb{Z} \times \mathbb{Z}$ plane and the global state $\mathcal{X}_r$ is defined as follows:

$$\mathcal{X}_r = \{x(n_1,n_2) : x(n_1,n_2) \in X, (n_1,n_2) \in \mathcal{C}_r\} . \tag{2.5}$$

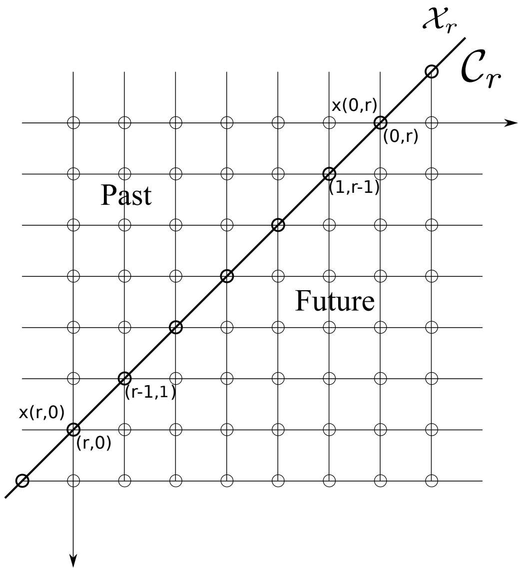

FIGURE 2.1: Crosscut $\mathcal{C}_r$ partitioning $\mathcal{P}$ into past, present and future

Given a cross-cut $\mathcal{C}_r \subset \mathbb{Z} \times \mathbb{Z}$, the solution of (2.3) in the future is uniquely determined by the global state $\mathcal{X}_r$ and the input values on cross-cut $\mathcal{C}_r$ and on the future set with respect to $\mathcal{C}_r$.

Fornasini–Marchesini and Roesser models are not independent of each other. In fact, introducing

$$\xi(h,k) = x(h,k+1) - A_1 x(h,k) \tag{2.6}$$

into the second model of Fornasini–Marchesini results in the following Roesser model

$$\begin{bmatrix} \xi(h+1,k) \\ x(h,k+1) \end{bmatrix} = \begin{bmatrix} A_2 & A_3 + A_2 A_1 \\ I_n & A_1 \end{bmatrix} \begin{bmatrix} \xi(h,k) \\ x(h,k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(h,k), \tag{2.7}$$

where now horizontal and vertical states are $\xi(h,k)$ and $x(h,k)$ respectively. It should be noted that fitting a Fornasini–Marchesini model into Roesser model will result in a higher state space dimension [37].

2.4 3-D Fornasini-Marchesini Model

The second order Fornasini–Marchesini model can straightforwardly be extended to higher dimensions. For instance, a 3-D system is described by the following first order description:

$$\begin{aligned} x(n_1+1,n_2+1,n_3+1) = {} & \mathbf{A_1}\ x(n_1,n_2+1,n_3+1) + \mathbf{A_2}\ x(n_1+1,n_2,n_3+1) \\ & + \mathbf{A_3}\ x(n_1+1,n_2+1,n_3) + \mathbf{B_1}\ u(n_1,n_2+1,n_3+1) \\ & + \mathbf{B_2}\ u(n_1+1,n_2,n_3+1) + \mathbf{B_3}\ u(n_1+1,n_2+1,n_3) \end{aligned} \tag{2.8}$$

$$y(n_1,n_2,n_3) = \mathbf{C}\ x(n_1,n_2,n_3) + \mathbf{D}\ u(n_1,n_2,n_3)\ , \tag{2.9}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are respectively local state, input and the output vectors. Then $\mathbf{C} \in \mathbb{R}^{q\times n}$, $\mathbf{D} \in \mathbb{R}^{q\times p}$, $\mathbf{A_i} \in \mathbb{R}^{n\times n}$, $\mathbf{B_i} \in \mathbb{R}^{n\times p}$ and $n_i \in \mathbb{Z}$, $i = 1,2,3$.

For the sake of notational simplicity, from now on, $x(n_1,n_2,n_3)$, $u(n_1,n_2,n_3)$ and $y(n_1,n_2,n_3)$, are denoted by x, u and y respectively. Moreover the *forward shift operator* is defined as follows.

Definition 2.1. Denote by $\mathbb{W}^{\mathbb{T}}$ the set consisting of all trajectories from $\mathbb{T}$ to $\mathbb{W}$. σ_i, $i = 1,2,3$ are the *forward shift operators* $\sigma_i : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}} \to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}$, $i = 1,2,3$ defined by

$$\begin{aligned}(\sigma_1 w)(k_1,k_2,k_3) &:= w(k_1+1,k_2,k_3)\\ (\sigma_2 w)(k_1,k_2,k_3) &:= w(k_1,k_2+1,k_3)\\ (\sigma_3 w)(k_1,k_2,k_3) &:= w(k_1,k_2,k_3+1)\ .\end{aligned}$$

The composition of i times the first shift, j times the second, and k times the third will be denoted by $\sigma_1^i\sigma_2^j\sigma_3^k$.

Consequently the model (2.8), equivalently can be written as:

$$\begin{aligned}\sigma_1\sigma_2\sigma_3 x &= A_1\sigma_2\sigma_3 x + A_2\sigma_1\sigma_3 x + A_3\sigma_1\sigma_2 x + B_1\sigma_2\sigma_3 u + B_2\sigma_1\sigma_3 u + B_3\sigma_1\sigma_2 u\ ,\\ y &= Cx + Du\ .\end{aligned} \tag{2.10}$$

In what follows this 3-D system will be denoted by $\Sigma = (A_1, A_2, A_3, B_1, B_2, B_3)$ or simply Σ.

2.4.1 Structural Properties of 3-D FM Model

In contrast to 1-D systems, a feature of 3-D system dynamics is the increasing number of initial local states on a separation set, which in this case is the plane $C_\ell := \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1+n_2+n_3 = \ell\}$, that has to be processed as the computation evolves. Considering the first local state that needs to be computed in the first iteration of the system, only a finite number (in fact, the least number) of initial states is required for this computation. However, as the system evolves and reaches the computation of states on the next separation sets, the number of required initial states increases. It is immediate that in general this computation cannot be performed without the knowledge of initial conditions along their respective 1-D "propagation fronts".

As in 2-D case, this last fact leads to the definition of the state at two different levels: the *local* state, which is defined point-wise and the *global* state, which consists of all the values of the local state on a propagation front. Using these facts, the system properties like reachability and observability are also introduced in forms of weak (local) which refers to single "local states" and strong (global) which refers to the infinite set of local states lying on a separation set.

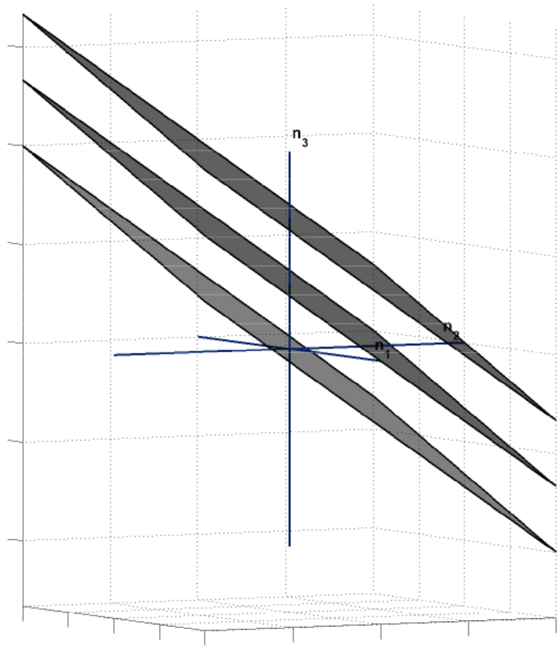

FIGURE 2.2: Representation of separation sets $\mathcal{C}_\ell$

Before proceeding to the structural properties of 3-D systems, the concepts of the separation plane, global state, global input, and *3-D shuffle product* are defined as follows.

Definition 2.2. For $\ell \in \mathbb{Z}$ the *separation plane*, *global state* and *global input* are defined, respectively, as follows

$$\begin{aligned} \mathcal{C}_\ell &:= \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 = \ell\} \,, \\ \mathcal{X}_\ell &:= \{x(n_1, n_2, n_3) : (n_1, n_2, n_3) \in \mathcal{C}_\ell\} \,, \\ \mathcal{U}_\ell &:= \{u(n_1, n_2, n_3) : (n_1, n_2, n_3) \in \mathcal{C}_\ell\} \,. \end{aligned} \tag{2.11}$$

The global state and input can be identified with formal power series as follows:

$$\mathcal{X}_\ell := \sum_{i,j=0}^{i+j\le\ell} x(\ell - i - j, i, j) z_1^{l-i-j} z_2^i z_3^j \tag{2.12}$$

$$\mathcal{U}_\ell := \sum_{i,j=0}^{i+j\le\ell} u(\ell - i - j, i, j) z_1^{l-i-j} z_2^i z_3^j \tag{2.13}$$

Shuffle products were defined for the the 2-D case in [38], the analogous definition to be used in the 3-D case is defined below.

Definition 2.3. For matrices A_i, $i = 1, 2, 3$, the *shuffle product* is defined as follows:

$$
\begin{aligned}
& A_i^{n_i} \sqcup\!\sqcup\ {}^{0}A_j := A_i^{n_i} \ , \\
& A_p^{0} \sqcup\!\sqcup\ {}^{n_j}A_j := A_j^{n_j} \ , \\
& A_i^{n_i} \sqcup\!\sqcup\ {}^{n_j}A_j := A_i(A_i^{n_i-1} \sqcup\!\sqcup\ {}^{n_j}A_j) + A_j(A_i^{n_i} \sqcup\!\sqcup\ {}^{n_j-1}A_j) \ , \\
& A_1^{0} \underset{A_2}{\overset{0}{\sqcup\!\sqcup}} {}^{0}A_3 := I \text{ (the identity matrix)} \ , \\
& A_1^{n_1} \underset{A_2}{\overset{0}{\sqcup\!\sqcup}} {}^{0}A_3 := A_1^{n_1} \ , \quad A_1^{0} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{0}A_3 := A_2^{n_2} \ , \quad A_1^{0} \underset{A_2}{\overset{0}{\sqcup\!\sqcup}} {}^{n_3}A_3 := A_3^{n_3} \ , \\
& A_1^{0} \underset{A_2}{\overset{1}{\sqcup\!\sqcup}} {}^{1}A_3 := A_2^{1} \sqcup\!\sqcup\ {}^{1}A_3 \ , \\
& A_1^{1} \underset{A_2}{\overset{0}{\sqcup\!\sqcup}} {}^{1}A_3 := A_1^{1} \sqcup\!\sqcup\ {}^{1}A_3 \ , \\
& A_1^{1} \underset{A_2}{\overset{1}{\sqcup\!\sqcup}} {}^{0}A_3 := A_1^{1} \sqcup\!\sqcup\ {}^{1}A_2 \ , \\
& A_1^{n_1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3}A_3 := A_1(A_1^{n_1-1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3}A_3) + A_2(A_1^{n_1} \underset{A_2}{\overset{n_2-1}{\sqcup\!\sqcup}} {}^{n_3}A_3) + A_3(A_1^{n_1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3-1}A_3) \ , \\
& A_1^{n_1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3}A_3 := 0 \text{ (the zero matrix)} \qquad \text{for } n_i < 0,
\end{aligned}
\tag{2.14}
$$

where $i, j \in \{1, 2, 3\}$, $i \neq j$ and $A_i \in \mathbf{R}^{n \times n}$.

2.4.2 Reachability

Reachability for doubly-indexed linear, stationary, finite-dimensional, dynamical systems have been introduced and discussed in [28, 33, 34, 39]. Analogous to the 2-D counterpart, for 3-D systems the state is defined in forms of *local state* and *global state* which necessitates introduction of *local reachability* and *global reachability* forms of this property consequently.

Suppose the sequence of all states denoted by $\mathcal{X}_d$ on the separation plane $\mathcal{C}_d$ is given. The question that naturally rises here is whether exists a control sequence such that starting from zero initial conditions, i.e., $\mathcal{X}_0$, the state sequence reaches the desired state $\mathcal{X}_d$ at "instance" d. This property is referred to as global (strong) reachability. In contrast, local reachability is somewhat weaker since only the existence of a control sequence such that the system produces the desired local state, starting from zero initial conditions, is required.

Following [39], the 3-D state space model (2.10) is said to be

- **locally reachable** if, upon assuming the initial state condition $\mathcal{X}_0 = \{0\}$, for every desired state $x_d \in \mathbb{R}^n$, there exists $(n_1, n_2, n_3) \in \mathbb{Z}^3$, with $\ell > 0$, and an input sequence $u(.,.,.)$ such that $x(n_1, n_2, n_3) = x_d$. In this case the desired state x_d is said to be reachable in $\ell = \big(n_1 + n_2 + n_3\big)$ steps.

- **globally reachable** if, upon assuming the initial state condition $\mathcal{X}_0 = \{0\}$, for every global sequence $\mathcal{X}_d$, there exists $\ell \in \mathbb{Z}^+$ and a global input sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{\ell-1}$ such that the global state $\mathcal{X}_\ell$ coincides with $\mathcal{X}_d$. In this case the desired global state $\mathcal{X}_d$ is said to be reachable in ℓ steps.

It is obvious that global reachability, a much stronger property, implies local reachability.

Let $\mathcal{X}_0 = \{0\}$. The following rational power series represents the output function of Σ corresponding to the input function $u = \sum_{n_1,n_2,n_3} u(n_1, n_2, n_3) z_1^{n_1} z_2^{n_2} z_3^{n_3} = 1$:

$$s_\Sigma = C(I - A_1 z_1 - A_2 z_2 - A_3 z_3)^{-1}(B_1 z_1 + B_2 z_2 + B_3 z_3) \ .$$

Definition 2.4. The system Σ is said to be *reachable* if it is locally reachable for any desired state x_d in $d = n_1 + n_2 + n_3$ steps.

Let $\mathcal{X}_0 = 0$. Then the following rational power series:

$$s_{_\Sigma} = C(I - A_1 z_1 - A_2 z_2 - A_3 z_3)^{-1}(B_1 z_1 + B_2 z_2 + B_3 z_3)$$

represents the transfer function of Σ. This corresponds to the system response

$$u = \sum_{n_1,n_2,n_3} u(n_1, n_2, n_3) z_1^{n_1} z_2^{n_2} z_3^{n_3} = 1 \ .$$

A local state $x_d \in \mathcal{X}$ is said to be *reachable* (from zero initial states) [39] if exists an input sequence u in the ring of formal power series in three indeterminates z_1, z_2, z_3 over $\mathbb{R}^n$ and integers $n_1 > 0$, $n_2 > 0$, and $n_3 > 0$ such that $x(n_1, n_2, n_3) = x_d$.

Definition 2.5. A state $x \in \mathcal{X}$ is *reachable* for some $u \in \mathbb{R}^n$ if

$$x = \big((I - A_1 z_1 - A_2 z_2 - A_3 z_3)^{-1}(B_1 z_1 - B_2 z_2 - B_3 z_3)u, 1\big) \ .$$

Based on this, the reachable local state-space is

$$\mathcal{X}^R = \Bigg\{x : x = \bigg(\Big((I - (A_1)z_1 - (A_2)z_2 - (A_3)z_3\Big)^{-1}\Big((B_1)z_1 - (B_2)z_2 - (B_3)z_3\Big)u, 1\bigg), \\ u \in \mathbb{R}^n\Bigg\} \ .$$

Also defining the matrices $M_{n_1,n_2,n_3} \in \mathbb{R}^{n\times n}$ as

$$M_{n_1,n_2,n_3} := \text{coeff}((I_n - (A_1)z_1 - (A_2)z_2 - (A_3)z_3)^{-1}, z_1^{n_1} z_2^{n_2} z_3^{n_3}) \,, \tag{2.15}$$

where M_{n_1,n_2,n_3} is the matrix coefficient of $z_1^{n_1} z_2^{n_2} z_3^{n_3}$ in the series expansion of $(I_n - (A_1)z_1 - (A_2)z_2 - (A_3)z_3)^{-1}$, (i.e.,$M_{0,0,0} = 0,\ M_{1,0,0} = A_1,\ M_{0,1,0} = A_2,\ M_{0,0,1} = A_3,\ M_{2,0,0} = A_1^2,\ M_{0,2,0} = A_2^2,\ M_{0,0,2} = A_3^2,\ M_{1,1,0} = A_1A_2 + A_2A_1, \dots$) , the columns of the matrix

$$R_L := \begin{bmatrix} B_1 & B_2 & B_3 & M_{1,0,0}B_1 & R' & \dots & R'' & \dots & R''' & \dots & R'''' & \dots \end{bmatrix} , \tag{2.16}$$

span $\mathcal{X}^R$, where the sub-matrices $R',\ R'',\ R''',\ R''''$ are defined as:

$$\begin{aligned}
R' &:= \begin{bmatrix} M_{0,1,0}B_1 + M_{1,0,0}B_2 & \dots & M_{n_1-1,n_2,n_3}B_1 + M_{n_1,n_2-1,n_3}B_2 \end{bmatrix} \\
R'' &:= \begin{bmatrix} M_{0,0,1}B_1 + M_{1,0,0}B_3 & \dots & M_{n_1-1,n_2,n_3}B_1 + M_{n_1,n_2,n_3-1}B_3 \end{bmatrix} \\
R''' &:= \begin{bmatrix} M_{0,0,1}B_2 + M_{0,1,0}B_3 & \dots & M_{n_1,n_2-1,n_3}B_2 + M_{n_1,n_2,n_3-1}B_3 \end{bmatrix} \\
R'''' &:= \Big[M_{0,1,1}B_1 + M_{1,0,1}B_2 + M_{1,1,0}B_3 \quad \dots \\
&\qquad \dots \quad M_{n_1-1,n_2,n_3}B_1 + M_{n_1,n_2-1,n_3}B_2 + M_{n_1,n_2,n_3-1}B_3 \Big] \,.
\end{aligned}$$

Moreover, by using the "shuffle products" defined in Definition 2.3, M_{n_1,n_2,n_3} can also be expressed in terms of shuffle products of the matrices A_i, $i = 1, 2, 3$:

$$M_{n_1,n_2,n_3} = A_1^{n_1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3}A_3.$$

Therefore the matrix R_L equivalently can be described with the following shuffle product matrix:

$$R_L := \begin{bmatrix} (A_1^{n_1-1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3}A_3)B_1 + (A_1^{n_1} \underset{A_2}{\overset{n_2-1}{\sqcup\!\sqcup}} {}^{n_3}A_3)B_2 + (A_1^{n_1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3-1}A_3)B_3 \end{bmatrix}_{n_1,n_2,n_3 \geq 0}^{0<n_1+n_2+n_3\leq \ell} . \tag{2.17}$$

The respective realisation is said to be locally reachable, or simply reachable, if the global state $\mathcal{X} = \mathcal{X}^R$, that is, the system Σ is locally reachable if R_L has full rank.

Next the stronger notion of global reachability is considered where the 3-D system is *globally reachable* if any desired global state $\mathcal{X}_d \in \mathcal{X}$ is reachable.

Recall the system model (2.10), which is rewritten in the following form:

$$\sigma_1 x = (A_1 + A_2\sigma_1\sigma_2^{-1} + A_3\sigma_1\sigma_3^{-1})x + (B_1 + B_2\sigma_1\sigma_2^{-1} + B_3\sigma_1\sigma_3^{-1})u \; . \tag{2.18}$$

The initial conditions are assigned by values of the local state on the separation set $\mathcal{C}_0$ where $\mathcal{C}_\ell$ is defined in (2.11). Equivalently, by assigning all local states of the initial global state $\mathcal{X}_0 := \{x(n_1, n_2, n_3) : (n_1, n_2, n_3) \in \mathcal{C}_0\}$.

The action of $\sigma_i, \; i = 1, 2, 3$ on the local state $x(n_1, n_2, n_3) \in \mathcal{X}_\ell$ is a shift to the next state with respect to its corresponding propagation front. Moreover, action of σ_i on the global state $\mathcal{X}_\ell$ is a shift to the next global state $\mathcal{X}_{\ell+1}$.

Denote by $\mathbb{R}[[z_1, z_1^{-1}, z_2, , z_2^{-1}, z_3, , z_3^{-1}]]$ the set of three-variable Laurent formal power series in the indeterminates z_1, z_2, z_3 with coefficients in $\mathbb{R}$. Applying the $3 - D$ z-transform $\mathcal{Z} : (\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}} \longrightarrow \mathbb{R}[[z_1, z_1^{-1}, z_2, z_2^{-1}, z_3, z_3^{-1}]]$ on (2.18) yields:

$$z_1 X(z_1, z_2, z_3) = A(z_1, z_2, z_3)X(z_1, z_2, z_3) + B(z_1, z_2, z_3)U(z_1, z_2, z_3) \; , \tag{2.19}$$

where, $A(z_1, z_2, z_3) := (A_1 + A_2 z_1 z_2^{-1} + A_3 z_1 z_3^{-1}),\; B(z_1, z_2, z_3) := (B_1 + B_2 z_1 z_2^{-1} + B_3 z_1 z_3^{-1})$. Note that action of z_1 on $X(z_1, z_2, z_3)$ is a shift to the next global state.

Iterating (2.19) along the propagation front corresponding to z_1, gives:

$$\begin{aligned}
X_1(z_1, z_2, z_3) =& A(z_1, z_2, z_3)X_0(z_1, z_2, z_3) + B(z_1, z_2, z_3)U_0(z_1, z_2, z_3) \; , \\
X_2(z_1, z_2, z_3) =& A^2(z_1, z_2, z_3)X_0(z_1, z_2, z_3) + A(z_1, z_2, z_3)B(z_1, z_2, z_3)U_0(z_1, z_2, z_3) \\
& + B(z_1, z_2, z_3)U_1(z_1, z_2, z_3) \; , \\
& \vdots \\
X_\ell(z_1, z_2, z_3) =& A^\ell(z_1, z_2, z_3)X_0(z_1, z_2, z_3) + A^{\ell-1}(z_1, z_2, z_3)B(z_1, z_2, z_3)U_0(z_1, z_2, z_3) \\
& + \ldots + B(z_1, z_2, z_3)U_{\ell-1}(z_1, z_2, z_3) \; ,
\end{aligned}$$

which by considering $X_0 = 0$, can be written as:

$$X_\ell(z_1, z_2, z_3) = \sum_{k=0}^{\ell-1} A^{\ell-1-k}(z_1, z_2, z_3)B(z_1, z_2, z_3)U_k(z_1, z_2, z_3) \; ,$$

or the matrix representation:

$$X_\ell(z_1,z_2,z_3) = \begin{bmatrix} B(z_1,z_2,z_3) & A(z_1,z_2,z_3)B(z_1,z_2,z_3) & \dots & A^{\ell-1}(z_1,z_2,z_3)B(z_1,z_2,z_3) \end{bmatrix} \begin{bmatrix} U_{\ell-1}(z_1,z_2,z_3) \\ U_{\ell-2}(z_1,z_2,z_3) \\ \vdots \\ U_0(z_1,z_2,z_3) \end{bmatrix} . \tag{2.20}$$

Now define the matrix

$$R^\ell := \begin{bmatrix} B(z_1,z_2,z_3) & A(z_1,z_2,z_3)B(z_1,z_2,z_3) & \dots & A^{\ell-1}(z_1,z_2,z_3)B(z_1,z_2,z_3) \end{bmatrix} , \tag{2.21}$$

as the global reachability matrix of the 3-D system Σ and consider how the choice of inputs may affect the state of a given system. That is, how can the input sequence $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_{\ell-1}$, be chosen such that the system moves from the global state $\mathcal{X}_0$ to a desired global state $\mathcal{X}_d$ in ℓ steps. It is evident that the global state $\mathcal{X}_d$ is reachable in ℓ steps if and only if

$$X_d(z_1,z_2,z_3) = \mathcal{Z}[\mathcal{X}_d] \in \text{Im } R^\ell ,$$

or the matrix R^ℓ has full column rank.

2.4.3 Observability

Observability of a system addresses the question of having knowledge of the output sequence, to what extent it is possible to reconstruct the state. In this section *global observability* and *local observability* are considered.

Definition 2.6. The 3-D state-space model (2.10) is said to be

- **locally observable** if, upon assuming that the initial global state $\mathcal{X}_0$ consists of a single non-zero local state $x(0,0,0)$, knowledge of the non-zero pattern of the free output evolution $y_\ell(n_1,n_2,n_3)$ in every point $(n_1,n_2,n_3) \in \mathbb{Z}^3_+$ enables the unique determination of $x(0,0,0)$ non-zero pattern.
- **globally observable** if the knowledge of the non-zero pattern of the free output evolution $y_\ell(n_1,n_2,n_3)$ in every point $(n_1,n_2,n_3) \in \mathbb{Z}^3$, $n_1+n_2+n_3 \geq 0$, allows unique determination of the non-zero pattern of the initial global state $\mathcal{X}_0$.

It is immediate that the global observability implies local observability.

Definition 2.7. The system Σ is said to be *observable* if it is locally observable.

Definition 2.8. A state $x \in \mathcal{X}$ is *indistinguishable* from the state $0 \in \mathcal{X}$ if

$$C(I - A_1 z_1 - A_2 z_2 - A_3 z_3)^{-1} x = 0 \; .$$

The *indistinguishable local state space* $\mathcal{X}^I$ is defined as:

$$\mathcal{X}^I := \left\{ x : x \in \mathcal{X}, \; C(I - A_1 z_1 - A_2 z_2 - A_3 z_3)^{-1} x = 0 \right\}$$

The subspace $\mathcal{X}^I$ coincides with the null space of the following matrix

$$O_\ell = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_3 \\ CA_1^2 \\ C(A_1^1 \underset{A_2}{\overset{1}{\sqcup\!\sqcup}} {}^1 A_3) \\ \vdots \\ CA_3^{\ell-1} \end{bmatrix} = \left[C(A_1^{n_1} \underset{A_2}{\overset{n_2}{\sqcup\!\sqcup}} {}^{n_3} A_3) \right]_{n_1, n_2, n_3 \geq 0}^{0 \leq n_1 + n_2 + n_3 < \ell} , \tag{2.22}$$

which is termed the observability matrix in $\ell \in \mathbb{Z}^+$ steps. The realisation Σ is said to be locally observable if $\mathcal{X}^I = \{0\} \subset \mathcal{X}$ or in another words O_ℓ has full rank.

Now the concept of global observability is considered. Assume initial global condition is given by $\mathcal{X}_0 = \bar{\mathcal{X}}$, where $\bar{\mathcal{X}}$ is a sequence of unknown states $x(n_1, n_2, n_3)$ such that $n_1 + n_2 + n_3 = 0$.

Then to address the global observability, the free output evolution on separation set $\mathcal{C}_\ell, (n_1 + n_2 + n_3 = \ell)$, is described by means of the following power series:

$$Y_\ell(z_1, z_2, z_3) = \sum_{i,j \in \mathbb{Z}} y(\ell - i - j, i, j) \; z_1^{\ell - i - j} \; z_2^i \; z_3^j \; ,$$

which relates it to the initial conditions on the separation set $\mathcal{C}_0$ as follows:

$$\begin{aligned}
Y_\ell(z_1,z_2,z_3) &= \sum_{i,j\in\mathbb{Z}} C\ x(\ell-i-j,i,j)\ z_1^{\ell-i-j}\ z_2^i\ z_3^j \\
&= \sum_{i,j\in\mathbb{Z}} C \sum_{p=0,q=0}^{p+q\leq\ell} (A_1^{k-p-q} \underset{A_2}{\overset{p}{\sqcup\!\sqcup}}\ {}^q A_3) x(p+q-i-j,i-p,j-q)\ z_1^{\ell-i-j}\ z_2^i\ z_3^j \\
&= C \sum_{p=0,q=0}^{p+q\leq\ell} (A_1^{k-p-q} \underset{A_2}{\overset{p}{\sqcup\!\sqcup}}\ {}^q A_3)\Big(\sum_{i,j\in\mathbb{Z}} x(p+q-i-j,i-p,j-q) \\
&\quad z_1^{p+q-i-j}\ z_2^{i-p}\ z_3^{j-q}\Big) z_1^{\ell-p-q}\ z_2^p\ z_3^q \\
&= C \sum_{p=0,q=0}^{p+q\leq\ell} (A_1^{k-p-q} \underset{A_2}{\overset{p}{\sqcup\!\sqcup}}\ {}^q A_3)\ z_1^{\ell-p-q}\ z_2^p\ z_3^q\ X_0(z_1,z_2,z_3) \\
&= C(A_1z_1+A_2z_2+A_3z_3)^\ell\ X_0(z_1,z_2,z_3)\ .
\end{aligned}$$

Consequently

$$\begin{bmatrix} Y_0(z_1,z_2,z_3) \\ \vdots \\ Y_{\ell-1}(z_1,z_2,z_3) \end{bmatrix} = O_\ell(z_1,z_2,z_3)X_0(z_1,z_2,z_3)\ ,$$

where,

$$O_\ell = \begin{bmatrix} C \\ C(A_1z_1+A_2z_2+A_3z_3) \\ \vdots \\ C(A_1z_1+A_2z_2+A_3z_3)^{\ell-1} \end{bmatrix}\ ,$$

is the global observability matrix.

2.5 General response for 3-D linear systems

In this section, the definition of the transition matrix for a 3-D system model is given together with the solution to the second Fornasini-Marchesini model. Consider the linear system model (2.10), the problem can be formulated as follows: Given matrices $A_1,\ A_2,\ A_3,\ B_1,\ B_2$, and B_3, and the input u, find a general response for (2.10).

For this purpose, define the state-transition matrix T^{n_1,n_2,n_3} for 3-D linear systems with variable coefficients as follows:

I. $T^{0,0,0} = I$ (the identity matrix) .

II. $T^{n_1,n_2,n_3} = A_1T^{n_1-1,n_2,n_3} + A_2T^{n_1,n_2-1,n_3} + A_3T^{n_1,n_2,n_3-1}$, for $n_1, n_2, n_3 = 0, 1, ...$.

III. $T^{n_1,n_2,n_3} = 0$ (the zero matrix), for $n_1 < 0$, or $n_2 < 0$, or $n_3 < 0$.

Theorem 2.9. *Consider the following boundary conditions*

$$x(n_1, n_2, n_3^0) \quad and \quad x(n_1, n_2^0, n_3) \quad and \quad x(n_1^0, n_2, n_3) \text{ for } \begin{cases} n_1 = n_1^0, n_1^0+1, n_1^0+2, ... \\ n_2 = n_2^0, n_2^0+1, n_2^0+2, ... \\ n_3 = n_3^0, n_3^0+1, n_3^0+2, ... \end{cases} .$$

$$(n_1^0, n_2^0, n_3^0 = 0, 1, ...) \tag{2.23}$$

Then a solution to the second Fornasini-Marchesini model (2.10) is given by:

$$\begin{aligned} x(\bar{n}_1, \bar{n}_2, \bar{n}_3) = & \sum_{n_1=n_1^0+1}^{\bar{n}_1} \sum_{n_2=n_2^0+1}^{\bar{n}_2} T^{\bar{n}_1-n_1, \bar{n}_2-n_2, \bar{n}_3-n_3^0-1}[A_3x(n_1, n_2, n_3^0) \\ & + B_3u(n_1, n_2, n_3^0)] \\ & + \sum_{n_1=n_1^0+1}^{\bar{n}_1} \sum_{n_3=n_1^0+1}^{\bar{n}_3} T^{\bar{n}_1-n_1, \bar{n}_2-n_2^0-1, \bar{n}_3-n_3}[A_2x(n_1, n_2^0, n_3) \\ & + B_2u(n_1, n_2^0, n_3)] \\ & + \sum_{n_2=n_2^0+1}^{\bar{n}_2} \sum_{n_3=n_3^0+1}^{\bar{n}_3} T^{\bar{n}_1-n_1^0-1, \bar{n}_2-n_2, \bar{n}_3-n_3}[A_1x(n_1^0, n_2, n_3) \\ & + B_1u(n_1^0, n_2, n_3)] \\ & + \sum_{n_1=n_1^0+1}^{\bar{n}_1} \sum_{n_2=n_2^0+1}^{\bar{n}_2} \sum_{n_3=n_3^0+1}^{\bar{n}_3} [T^{\bar{n}_1-n_1-1, \bar{n}_2-n_2, \bar{n}_3-n_3}B_1 \\ & + T^{\bar{n}_1-n_1, \bar{n}_2-n_2-1, \bar{n}_3-n_3}B_2 + T^{\bar{n}_1-n_1, \bar{n}_2-n_2, \bar{n}_3-n_3-1}. \\ & B_3]u(n_1, n_2, n_3) \end{aligned} \tag{2.24}$$

Proof. The proof follows by induction. Let $n_1 = n_2 = n_3 = 0$, and then from (2.10) it follows that:

$$\begin{aligned} x(1,1,1) = & A_1x(0,1,1) + A_2x(1,0,1) + A_3x(1,1,0) \\ & + B_1u(0,1,1) + B_2u(1,0,1) + B_3u(1,1,0) \ . \end{aligned} \tag{2.25}$$

This is identical to the result in (2.24) once $n_1 = n_2 = n_3 = 1$ and without loss of generality the boundary conditions are taken as $n_1^0 = n_2^0 = n_3^0 = 0$.

Now let $n_1 = 1$ and $n_2 = n_3 = 0$. Then from (2.10) and (2.25)

$$\begin{aligned} x(2,1,1) = & A_1A_1x(0,1,1) + A_1A_2x(1,0,1) \\ & + A_1A_3x(1,1,0) + A_1B_1u(0,1,1) \\ & + A_1B_2u(1,0,1) + A_1B_3u(1,1,0) \\ & + A_2x(2,0,1) + A_3x(2,1,0) + B_1u(1,1,1) \\ & + B_2u(2,0,1) + B_3u(2,1,) \ . \end{aligned} \tag{2.26}$$

The same result now also follows from (2.24). Similarly, the hypothesis can be proven for $n_1 = n_3 = 0, \ n_2 = 1$ and $n_1 = n_2 = 0, \ n_3 = 1$. Assuming the hypothesis is true for (n_1, n_2, n_3), (n_1+1, n_2, n_3), (n_1, n_2+1, n_3), and (n_1, n_2, n_3+1), it is next shown that the hypothesis for (n_1+1, n_2+1, n_3+1) is also true. From equations (2.10), (2.23) and (2.24) it follows that

$$x(n_1+1, n_2+1, n_3+1) = A_1\Gamma_1 + A_2\Gamma_2 + A_3\Gamma_3$$

where

$$\begin{aligned}
\Gamma_1 := &\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1} T^{\bar{n}_1-n_1,\bar{n}_2-n_2+1,\bar{n}_3}[A_3x(n_1,n_2,0)+B_3u(n_1,n_2,0)] \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1,\bar{n}_2,\bar{n}_3-n_3+1}[A_2x(n_1,0,n_3)+B_2u(n_1,0,n_3)] \\
&+\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-1,\bar{n}_2-n_2+1,\bar{n}_3-n_3+1}[A_1x(0,n_2,n_3)+B_1u(0,n_2,n_3)] \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1-1,\bar{n}_2-n_2+1,\bar{n}_3-n_3+1}B_1u(n_1,n_2,n_3) \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1,\bar{n}_2-n_2,\bar{n}_3-n_3+1}B_2u(n_1,n_2,n_3) \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1,\bar{n}_2-n_2+1,\bar{n}_3-n_3}B_3u(n_1,n_2,n_3)\ ,
\end{aligned}$$

$$\begin{aligned}
\Gamma_2 := &\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2-1,\bar{n}_3}[A_3x(n_1,n_2,0)+B_3u(n_1,n_2,0)] \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1,\bar{n}_2-n_2,\bar{n}_3-n_3+1}[A_2x(n_1,0,n_3)+B_2u(n_1,0,n_3)] \\
&+\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1,\bar{n}_2-n_2,\bar{n}_3-n_3+1}[A_1x(0,n_2,n_3)+B_1u(0,n_2,n_3)] \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1,\bar{n}_2-n_2,\bar{n}_3-n_3+1}B_1u(n_1,n_2,n_3) \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2-1,\bar{n}_3-n_3+1}B_2u(n_1,n_2,n_3) \\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2,\bar{n}_3-n_3}B_3u(n_1,n_2,n_3)\ ,
\end{aligned}$$

$$
\begin{aligned}
\Gamma_3 := &\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2,\bar{n}_3-1}[A_3x(n_1,n_2,0)+B_3u(n_1,n_2,0)]\\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1+1,\bar{n}_2,\bar{n}_3-n_3}[A_2x(n_1,0,n_3)+B_2u(n_1,0,n_3)]\\
&+\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1,\bar{n}_2-n_2+1,\bar{n}_3-n_3}[A_1x(0,n_2,n_3)+B_1u(0,n_2,n_3)]\\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1,\bar{n}_2-n_2+1,\bar{n}_3-n_3}B_1u(n_1,n_2,n_3)\\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2,\bar{n}_3-n_3}B_2u(n_1,n_2,n_3)\\
&+\sum_{n_1=1}^{\bar{n}_1+1}\sum_{n_2=1}^{\bar{n}_2+1}\sum_{n_3=1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2+1,\bar{n}_3-n_3-1}B_3u(n_1,n_2,n_3)\ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&x(n_1+1,n_2+1,n_3+1) =\\
&\sum_{n_1=n_1^0+1}^{\bar{n}_1+1}\ \sum_{n_2=n_2^0+1}^{\bar{n}_2+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2+1,\bar{n}_3-n_3^0}[A_3x(n_1,n_2,n_3^0)+B_3u(n_1,n_2,n_3^0)]\\
&+\sum_{n_1=n_1^0+1}^{\bar{n}_1+1}\ \sum_{n_3=n_1^0+1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2^0,\bar{n}_3-n_3+1}[A_2x(n_1,n_2^0,n_3)+B_2u(n_1,n_2^0,n_3)]\\
&+\sum_{n_2=n_2^0+1}^{\bar{n}_2+1}\ \sum_{n_3=n_3^0+1}^{\bar{n}_3+1} T^{\bar{n}_1-n_1^0,\bar{n}_2-n_2+1,\bar{n}_3-n_3+1}[A_1x(n_1^0,n_2,n_3)+B_1u(n_1^0,n_2,n_3)]\\
&+\sum_{n_1=n_1^0+1}^{\bar{n}_1+1}\ \sum_{n_2=n_2^0+1}^{\bar{n}_2+1}\ \sum_{n_3=n_3^0+1}^{\bar{n}_3+1} [T^{\bar{n}_1-n_1,\bar{n}_2-n_2+1,\bar{n}_3-n_3+1}B_1\\
&+T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2,\bar{n}_3-n_3+1}B_2+T^{\bar{n}_1-n_1+1,\bar{n}_2-n_2+1,\bar{n}_3-n_3}B_3]u(n_1,n_2,n_3)\ ,
\end{aligned}
\tag{2.27}
$$

and the proof is complete. □

2.6 Summary

This chapter has introduced the nD systems models used in this research together with relevant systems theoretic properties. In this work particular use will be made of models of 3-D dynamics and their reachability and observability properties, which are distinct concepts for discrete dynamics. Unlike the 1-D case, there is more than one version of these properties, e.g., local and global. However, significantly enough, their characterisations express rigorous relationships both with the reachability and observability characterisations for the standard 1-D and also the 2-D case.

Moreover, a general response formula for 3-D dynamics is developed.

Chapter 3

Geometric Approach

3.1 Introduction

The geometric approach for linear systems was first appeared in literature in the late 60's when Basile and Marro [40, 41], Wonham and Morse [42], introduced the concepts of controlled invariant and conditioned invariant subspaces. These invariant subspaces play a key role in characterising important properties of linear systems such as observability and controllability.

The essence of the geometric approach is to develop most of the mathematical support in coordinate-free form, to take advantage of simpler and more elegant results, which facilitate insight into the actual meaning of statements and procedures; the computational aspects are considered independently of the theory and handled by the standard methods of matrix algebra, once a suitable coordinate system is defined. The cornerstone of the approach is the concept of invariance of a subspace with respect to a linear transformation [43].

This chapter explains a connection between basic concepts of linear algebra (like invariants, changes of basis) and properties of linear dynamic systems. Also the conditioned invariant subspaces and their characteristics together with a particular family of conditioned invariant subspaces, namely, the input-containing conditioned invariants are introduced which is then used for the purpose of fault identification. Moreover, the unobservability subspaces which originally were introduced in [1] are generalised to the 3-D case. Finally, an LMI based approach is developed for construction of internally and externally stabilising output injection gains.

The following notations are used throughout this chapter:

Notation

$\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ matrices with entries in $\mathbb{R}$. $\mathbb{R}^{\bullet\times n}$ denotes the set of matrices with n columns and an unspecified (but finite) number of rows. If $A_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, 3$, the matrix A_H is defined by

$$A_H := \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \in \mathbb{R}^{n\times 3n} \ ,$$

and A_D by

$$A_D := \begin{bmatrix} A_1 & 0_{m\times n} & 0_{m\times n} \\ 0_{m\times n} & A_2 & 0_{m\times n} \\ 0_{m\times n} & 0_{m\times n} & A_3 \end{bmatrix} \in \mathbb{R}^{3m\times 3n} \ .$$

Given $A \in \mathbb{R}^{m\times n}$, its Moore-Penrose pseudo-inverse is denoted by $A^\dagger$.

Given a subspace $\mathcal{V} \subseteq \mathbb{R}^n$, the notation $\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}$ denotes the subspace of $\mathbb{R}^{3n}$ defined by

$$\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V} := \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mid v_i \in \mathcal{V}, i = 1, 2, 3 \right\} \ ,$$

and $\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathbb{R}^m$ denotes the subspace of $\mathbb{R}^{3n+m}$ defined by

$$\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathbb{R}^m := \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ x \end{bmatrix} \mid v_i \in \mathcal{V}, i = 1, 2, 3, \text{ and } x \in \mathbb{R}^m \right\} \ .$$

3.2 Invariants

First some of preliminary geometric concepts are introduced. The basic foundation of the geometric approach, is an *invariant subspace* which is defined as follows.

Definition 3.1. Consider a linear transformation $A : \mathcal{X} \longrightarrow \mathcal{X}$, with $\mathcal{X} := \mathcal{F}^n$. An *A-invariant* is a subspace $\mathcal{V} \subseteq \mathcal{X}$ such that

$$A\mathcal{V} \subseteq \mathcal{V} \ . \tag{3.1}$$

Proposition 3.2. *A subspace $\mathcal{V}$ spanned by the columns of V is an A-invariant if and only if there exists a matrix X such that*

$$AV = VX \ . \tag{3.2}$$

Proof. $\mathcal{V}$ is an A-invariant subspace if and only if each transformed column of V is a linear combination of all its columns; i.e., let $v_i(i = 1, 2, ..., r)$ form the columns of the matrix V, then

$$Av_i = VX \ ,$$

which proves the claim. □

3.2.1 Invariants and Change of Basis

A basis of a vector space of dimension n is a sequence of n vectors $\{v_1, ..., v_n\}$ by which any vector in the space can be described uniquely as a linear combination of the basis vectors. Since for a particular problem it often occurs that working with one basis is comparatively easier than another, it is essential to be able to easily transform coordinate-wise representations of vectors and operators taken with respect to one basis to their equivalent representations in another basis.

Consider a linear map $A : \mathcal{F}^n \longrightarrow \mathcal{F}^m$. A change of basis in $\mathcal{F}^n$ and $\mathcal{F}^m$ is defined by two non-singular matrices P, Q whose columns are the vectors of the new bases expressed with respect to the original ones. Suppose the new coordinates are denoted by the pair (ζ, η) and the old ones by (x, y) such that $x = P\zeta$ and $y = Q\eta$. Considering $y = Ax$, for the new basis it follows

$$\eta = Q^{-1}AP\zeta = A'\zeta \ ,$$

where $A' := Q^{-1}AP$. As a special case, if $A : \mathcal{F}^n \longrightarrow \mathcal{F}^n$, such that it can be assumed that a *unique* change of basis $T := P = Q$ exists, it yields,

$$\eta = T^{-1}AT\zeta = A'\zeta \ ,$$

where $A' := T^{-1}AT$.

It is straightforward to show that the invariance is coordinate-free. Let the columns of V span the A-invariant subspace $\mathcal{V} \subseteq \mathbb{R}^n$ and W be the new basis matrix transformed by T to the new coordinates such that $W = T^{-1}V$. Multiplying (3.2) on the left hand side yields

$$T^{-1}AV = T^{-1}VX$$

and since $V = TT^{-1}V$, a simple manipulation gives

$$(T^{-1}AT)(T^{-1}V) = (T^{-1}V)X$$

or

$$A'W = WX \; .$$

Theorem 3.3. *[51] Let $A : \mathcal{X} \longrightarrow \mathcal{X}, \mathcal{X} := \mathcal{F}^n$ be a linear map and $\mathcal{V} \subseteq \mathcal{X}$ be an A-invariant subspace of dimension r. There exists a similarity transformation T such that*

$$A' := T^{-1}AT = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix} \; , \tag{3.3}$$

where A'_{11}, A'_{12}, and A'_{22} are $r \times r$ matrices.

Proof. Consider a linear transformation $T := \begin{bmatrix} V & V' \end{bmatrix}$ such that columns of V span the invariant subspace $\mathcal{V}$. It follows that

$$W := T^{-1}V = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \; .$$

$$\begin{bmatrix} A'_{11} \\ A'_{21} \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = A'T^{-1}V = T^{-1}AV.$$

Using Proposition 3.2, it follows that

$$\begin{bmatrix} A'_{11} \\ A'_{21} \end{bmatrix} = \underbrace{T^{-1}V}_{\begin{bmatrix} I_r \\ 0 \end{bmatrix}} X = \begin{bmatrix} X \\ 0 \end{bmatrix}$$

which implies $A'_{21} = 0$. □

The following lemma establishes the effect of similarity transformation on eigenvalues of the linear matrix $A \in \mathbb{R}^{n\times n}$.

Lemma 3.4. *The similarity transformation $T \in \mathbb{R}^{n\times n}$ which transforms the matrix $A \in \mathbb{R}^{n\times n}$ to A', i.e., $A' := T^{-1}AT$, does not affect the eigenvalues of A.*

Proof.

$$|\lambda I - T^{-1}AT| = |T^{-1}\lambda IT - T^{-1}AT| = |T^{-1}||\lambda I - A||T| = |\lambda I - A| \; .$$

□

3.2.2 Invariants and System Structure

Let $\mathcal{V} \subseteq \mathcal{X}$, then vectors $v_1, v_2 \in \mathcal{X}$ are *equivalent mod* $\mathcal{V}$, if $v_1 - v_2 \in \mathcal{V}$. Each vector $v_1 \in \mathcal{X}$ has its respective *equivalence class* w defined as

$$w := \{v_2 : v_2 \in \mathcal{X}, v_2 - v_1 \in \mathcal{V}\} \, . \tag{3.4}$$

Consider two equivalent classes w_1 and w_2. If the elements of w_1 are added with arbitrary elements of w_2, then all the sums belong to one and the same class $w_1 + w_2$. Moreover, products obtained from multiplying all the elements of w_1 with an arbitrary scalar $\alpha \in \mathbb{R}$ belong to the αw class. Therefore the set of all the equivalent classes $w_i, (i = 1, 2, ...)$ together with the two operations defined above (addition and multiplication by a scalar) form the linear *quotient space* $\mathcal{X}/\mathcal{V}$.

Definition 3.5. The linear transformation $P : \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{V}$, which takes an element to its equivalence class under a given equivalence relation such that $w = Px$, is said to be the *canonical projection* of $\mathcal{X}$ on $\mathcal{X}/\mathcal{V}$.

3.2.3 Restriction of a Linear Transformation

Definition 3.6. Let $A : \mathcal{X} \longrightarrow \mathcal{X}$, $\mathcal{X} := \mathcal{F}^n$ be a linear transformation and $\mathcal{V} \subseteq \mathcal{X}$ an A-invariant subspace. The transformation $A|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}$ defined by $A|_{\mathcal{V}}(x) = A(x), \ x \in \mathcal{V}$ is termed as the restriction of the linear transformation A to the subspace $\mathcal{V}$ (see Fig. 3.1).

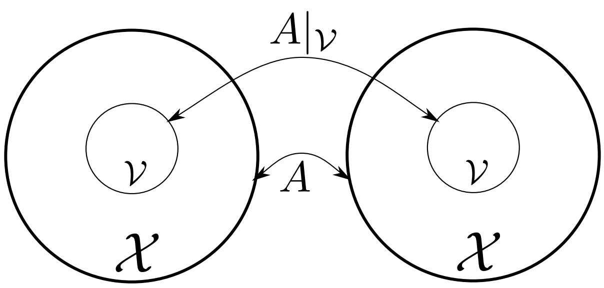

FIGURE 3.1: $A|_{\mathcal{V}}$ is the restriction of the linear map A to the A-invariant subspace $\mathcal{V}$

Following Theorem 3.3, if $\mathcal{V}$ is r dimensional, then $A|_{\mathcal{V}}$ can be represented in a basis by an $r \times r$ matrix.

It might appear that $A|_{\mathcal{V}}$ would take on exactly the same values as A. However, $A|_{\mathcal{V}}$ differs from A in the choice of domain and co-domain. In restricting a linear transformation, the rule of the function to a smaller subspace is also restricted.

3.2.4 Induced Map on a Quotient Space

Let $A : \mathcal{X} \longrightarrow \mathcal{X}$, $\mathcal{X} := \mathcal{F}^n$ be a linear transformation and $\mathcal{V} \subseteq \mathcal{X}$ an A-invariant subspace. The map induced by A on the quotient space $\mathcal{X}/\mathcal{V}$ is the map $\sigma := \mathcal{X}/\mathcal{V} \longrightarrow \mathcal{X}/\mathcal{V}$ defined by

$$\sigma(\{x\} + \mathcal{V}) = \{A(x)\} + \mathcal{V} \quad \forall \ \{x\} + \mathcal{V} \in \mathcal{X}/\mathcal{V} \ ,$$

and denoted by $A|_{\mathcal{X}/\mathcal{V}}$.

Following Theorem 3.3, if $\mathcal{V}$ and $\mathcal{X}$ are respectively r and n dimensional, then $A|_{\mathcal{V}}$ can be represented in a basis by an $(n-r) \times (n-r)$ matrix.

Corollary 3.7. *[51] Let $A := \mathcal{X} \longrightarrow \mathcal{X}$, $\mathcal{X} := \mathcal{F}^n$ be a linear transformation and $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{X}$ be a pair of A-invariant subspaces such that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{X}$. There exist a similarity transformation T such that*

$$A' := T^{-1}AT = \begin{bmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{bmatrix} , \tag{3.5}$$

where A'_{11} is an $r \times r$ matrix ($r := dim\mathcal{V}$), the restriction of A to V_1 and A'_{22} an $(n-r) \times (n-r)$ matrix, the restriction of A to V_2.

Proof. Assume $T = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, with Im $V_1 = \mathcal{V}_1$ and Im $V_2 = \mathcal{V}_2$. Hence,

$$T^{-1}V_1 = \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} , \ T^{-1}V_2 = \begin{bmatrix} 0_{r\times(n-r)} \\ I_{(n-r)} \end{bmatrix} .$$

The rest identically follows from the proof of Theorem 3.3. □

Any pair of A-invariant subspaces $\mathcal{V}_1$, $\mathcal{V}_2$, such that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{X}$, is said to decompose linear transformation $A : \mathcal{X} \longrightarrow \mathcal{X}$ into two restrictions $A|_{\mathcal{V}_1}$ and $A|_{\mathcal{V}_2}$.

3.3 Invariant Subspaces for 3-D FM Models

Following Definition 3.1 and analogous to the 2-D counterpart [44], the concept of (A_1, A_2, A_3)-invariance is first introduced and then an overview of their characterisations is given.

Definition 3.8. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is (A_1, A_2, A_3)*-invariant* if $A_i\mathcal{V} \subseteq \mathcal{V}$, $i \in \{1, 2, 3\}$.

The following result gives several characterizations of (A_1, A_2, A_3)-invariance.

Proposition 3.9. *Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace of dimension r, and let $Q \in \mathbb{R}^{(n-r)\times n}$ and $V \in \mathbb{R}^{n\times r}$, respectively, be full row-rank and full column-rank matrices such that $Im\ (V) = \ker(Q) = \mathcal{V}$.*

The following statements are equivalent:

I. $\mathcal{V}$ is (A_1, A_2, A_3)-invariant;

II. There exist matrices $X_i \in \mathbb{R}^{r\times r}$, $i = 1, 2, 3$, such that $A_iV = VX_i$, or equivalently

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} V = \begin{bmatrix} V & 0_{n\times r} & 0_{n\times r} \\ 0_{n\times r} & V & 0_{n\times r} \\ 0_{n\times r} & 0_{n\times r} & V \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}. \tag{3.6}$$

III. $A_H(\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}) \subseteq \mathcal{V}$.

IV. There exist $L_i \in \mathbb{R}^{(n-r)\times(n-r)}$ such that $QA_i = L_iQ$, $i = 1, 2, 3$.

Proof. $((I) \implies (II))$ Following from the 1-D counterpart [40], $\mathcal{V}$ is an $(A_1, A_2, A3)$-invariant subspace if and only if each transformed column of V is a linear combination of all its columns; i.e., let $v_i (i = 1, 2, ..., r)$ form the columns of the matrix V, then

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} v_i = VX.$$

$((II) \longrightarrow (I))$ Trivial.

$((I) \implies (III))$ Follows from Definition 3.8.

$((III) \implies (I))$ By contradiction and without loss of generality assume that there exists $v \in \mathcal{V}$ such that $A_1v \notin \mathcal{V}$. Then $A_H \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \notin \mathcal{V}$, which is a contradiction.

$((II) \implies (IV))$ From (3.6) it follows that $QA_iV = QVX_i = 0$, $i = 1, 2, 3$; consequently the rows of QA_i are a linear combination of the rows of Q.

$((IV) \implies (I))$ From (IV) it follows that $(QA_i)\mathcal{V} = Q(A_i\mathcal{V}) = \{0\}$, and consequently $A_i\mathcal{V} \subseteq \mathcal{V}$, $i = 1, 2, 3$. □

Given statement III of Proposition 3.9, the term A_H-*invariance* is used rather than (A_1, A_2, A_3)-invariance. As for the 2-D counterpart [44], the following is another characterisation of A_H-invariant that will be used later in this thesis.

Theorem 3.10. *Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace of dimension r. The following statements are equivalent:*

I. $\mathcal{V}$ is A_H-invariant;

II. For any basis of $\mathbb{R}^n$ whose first r columns form a basis of $\mathcal{V}$, there exists a similarity transform $T \in \mathbb{R}^n$, such that linear maps represented by $\hat{A}_i = T^{-1}A_iT$, $i = 1, 2, 3$ have the following matrix representation:

$$\begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0_{(n-r)\times r} & \hat{A}_i^{22} \end{bmatrix} , i = 1, 2, 3 \; . \tag{3.7}$$

Proof. Let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix whose first r columns span $\mathcal{V}$, i.e. $T = \begin{bmatrix} V & V' \end{bmatrix}$ where Im $(V) = \mathcal{V}$ and $V' \in \mathbb{R}^{n\times(n-r)}$.

$((I) \implies (II))$ Since $\mathcal{V}$ is (A_1, A_2, A_3)-invariant,

$$A_iT = A_i \begin{bmatrix} V & V' \end{bmatrix} = \begin{bmatrix} A_iV & A_iV' \end{bmatrix} = \begin{bmatrix} VX_i & A_iV' \end{bmatrix} , i = 1, 2, 3 \; ,$$

where $X_i \in \mathbb{R}^{r\times r}$. Multiplying the last equality by T^{-1} on the left we obtain

$$\hat{A}_i := T^{-1}A_iT = T^{-1} \begin{bmatrix} VX_i & A_iV' \end{bmatrix} = \begin{bmatrix} T^{-1}VX_i & T^{-1}A_iV' \end{bmatrix} =: \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ \hat{A}_i^{21} & \hat{A}_i^{22} \end{bmatrix} .$$

Since $T^{-1}V = \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix}$, necessarily $\hat{A}_i^{21} = 0$ for $i = 1, 2, 3$.

$((II) \implies (I))$ Consider a non-singular matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\hat{A}_i \triangleq T^{-1}A_iT = \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0_{(n-r)\times r} & \hat{A}_i^{22} \end{bmatrix} ,$$

holds. Then,

$$\hat{A}_i \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = \begin{bmatrix} X_i \\ 0_{(n-r)\times r} \end{bmatrix} ,$$

holds for $X_i = \hat{A}_i^{11}$. Multiplying by T on the left yields

$$A_i T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} X_i \ ,$$

which shows that $T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = V$, is the basis of an (A_1, A_2, A_3)-invariant subspace.

□

3.4 Invariants and Stability

Consider the free response system with the following state-space description:

$$\sigma_1\sigma_2\sigma_3 x = A_1\sigma_2\sigma_3 x + A_2\sigma_1\sigma_3 x + A_3\sigma_1\sigma_2 x \ . \tag{3.8}$$

Stability of such system is determined by the eigenvalues of the matrices $A_i,\ i = 1, 2, 3$. More precisely, if all eigenvalues of A_i have values inside unit circle. It follows from Theorem 3.10 that if $\mathcal{V}$ is A_H-invariant, by the similarity transformation $T := \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, where Im $T_1 = \mathcal{V}$, the Fornasini-Marchesini model (3.8) can be decomposed in the following form

$$\begin{aligned} \begin{bmatrix} \sigma_1\sigma_2\sigma_3 x' \\ \sigma_1\sigma_2\sigma_3 x'' \end{bmatrix} &= \begin{bmatrix} \hat{A}_1^{11} & \hat{A}_1^{12} \\ 0_{(n-r)\times r} & \hat{A}_1^{22} \end{bmatrix} \begin{bmatrix} \sigma_2\sigma_3 x' \\ \sigma_2\sigma_3 x'' \end{bmatrix} + \begin{bmatrix} \hat{A}_2^{11} & \hat{A}_2^{12} \\ 0_{(n-r)\times r} & \hat{A}_2^{22} \end{bmatrix} \begin{bmatrix} \sigma_1\sigma_3 x' \\ \sigma_1\sigma_3 x'' \end{bmatrix} \\ &+ \begin{bmatrix} \hat{A}_3^{11} & \hat{A}_3^{12} \\ 0_{(n-r)\times r} & \hat{A}_3^{22} \end{bmatrix} \begin{bmatrix} \sigma_1\sigma_2 x' \\ \sigma_1\sigma_2 x'' \end{bmatrix} \ . \end{aligned} \tag{3.9}$$

Representation (3.9) is used to introduce and study the concept of *internal* and *external stability*, first introduced for the 2-D case in [45].

If $\mathcal{V}$ is an (A_1, A_2, A_3)-invariant subspace and the boundary conditions $\{x(n_1, n_2, n_3) \mid (n_1, n_2, n_3) \in \mathcal{C}_0\} \subset \mathcal{V}$ and $u(n_1, n_2, n_3) := 0$ for all $(n_1, n_2, n_3) \in \mathbb{Z}^3$, then the sequence x compatible with the equations (3.9) satisfies $x(n_1, n_2, n_3) \in \mathcal{V}$ for all $(n_1, n_2, n_3) \in \mathbb{Z}^3_+$, i.e., the respective state transformed by the similarity map T decomposes into

$$x(n_1, n_2, n_3) = \begin{bmatrix} x'(n_1, n_2, n_3) \\ 0_{(n-r)\times 1} \end{bmatrix} . \tag{3.10}$$

and the motion on $\mathcal{V}$ is expressed by

$$\sigma_1\sigma_2\sigma_3 x' = \hat{A}_1^{11}\sigma_2\sigma_3 x' + \hat{A}_2^{11}\sigma_1\sigma_3 x' + \hat{A}_3^{11}\sigma_1\sigma_2 x' \ .$$

Hence, it is only the stability of the sub-matrices $\hat{A}_i^{11}, i = 1, 2, 3$ that determines the stability of the motion on $\mathcal{V}$.

Moreover, if $\mathcal{V}$ is an (A_1, A_2, A_3)-invariant subspace and we assign boundary conditions $\{x(n_1, n_2, n_3) \mid (n_1, n_2, n_3) \in \mathcal{C}_0\} \not\subset \mathcal{V}$ and $u(n_1, n_2, n_3) := 0$ for all $(n_1, n_2, n_3) \in \mathbb{Z}^3$, so that $x''(n_1, n_2, n_3) \neq 0$, i.e.,

$$\sigma_1\sigma_2\sigma_3 x'' = \hat{A}_1^{22}\sigma_2\sigma_3 x'' + \hat{A}_2^{22}\sigma_1\sigma_3 x'' + \hat{A}_3^{22}\sigma_1\sigma_2 x'' \ . \tag{3.11}$$

This means the projection of the state along $\mathcal{V}$ over any complement of $\mathcal{V}$ has a stable behaviour if and only if the sub-matrices $\hat{A}_i^{22}$ are stable. That is, in this case the canonical projection of the state on the quotient space $\mathcal{X}/\mathcal{V}$ tends to the origin as the system evolves.

Definition 3.11. An (A_1, A_2, A_3)-invariant subspace $\mathcal{V}$ is *internally stable* if

$$\begin{aligned}&[\{x(n_1, n_2, n_3) \mid (n_1, n_2, n_3) \in \mathcal{C}_0\} \subset \mathcal{V}] \ \text{and} \ \left[u(n_1, n_2, n_3) := 0 \text{ for all } (n_1, n_2, n_3) \in \mathbb{Z}^3\right]\\ &\Longrightarrow \lim_{n_1, n_2, n_3 \to \infty} \|x'(n_1, n_2, n_3)\| = 0 \ .\end{aligned}$$

It follows from standard results in nD systems theory (see for example [46]) that $\mathcal{V}$ is internally stable if and only if the matrices A_i^{11}, $i = 1, 2, 3$ satisfy

$$\begin{aligned}&\det(I_n - A_1^{11}\lambda - A_2^{11}\mu - A_3^{11}\nu) \neq 0\\ &\text{for all } (\lambda, \mu, \nu) \in \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_i| \leq 1, i = 1, 2, 3\} \ .\end{aligned} \tag{3.12}$$

Definition 3.12. An (A_1, A_2, A_3)-invariant subspace $\mathcal{V}$ is *externally stable* if

$$\begin{aligned}&[\{x(n_1, n_2, n_3) \mid (n_1, n_2, n_3) \in \mathcal{S}_0\} \not\subseteq \mathcal{V}] \ \text{and} \ \left[u(n_1, n_2, n_3) := 0, (n_1, n_2, n_3) \in \mathbb{Z}^3\right]\\ &\Longrightarrow \lim_{n_1, n_2, n_3 \to \infty} x''(n_1, n_2, n_3) \in \mathcal{V} \ .\end{aligned}$$

It is a matter of straightforward verification to check that $\mathcal{V}$ is externally stable if and only if the triple $(A_1^{22}, A_2^{22}, A_3^{22})$ is asymptotically stable in the sense of (3.12).

The condition (3.12) is rather difficult to check, and does not lend itself to be used for the synthesis of stabilising controllers. These issues have spurred research activity in the use of LMIs, see for example [47, 48]. The following result is a restatement for the 3-D case of the main result of [48].

Proposition 3.13. *If there exist positive-definite matrices $P_i = P_i^\top \in \mathbb{R}^n$, $i = 1, 2, 3$, such that the following LMI holds:*

$$\begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} - \begin{bmatrix} A_1^\top \\ A_2^\top \\ A_3^\top \end{bmatrix} (P_1 + P_2 + P_3) \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} > 0 , \tag{3.13}$$

then the 3-D system described by

$$\sigma_1\sigma_2\sigma_3 x = A_1\sigma_2\sigma_3 x + A_2\sigma_1\sigma_3 x + A_3\sigma_1\sigma_2 x \tag{3.14}$$

is asymptotically stable.

Proof. Assume by contradiction that (λ, μ, ν) exists in $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_i| \leq 1, i = 1, 2, 3\}$ such that $\det(I_n - A_1\lambda - A_2\mu - A_3\nu) = 0$; then there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that $(I_n - A_1\lambda - A_2\mu - A_3\nu)x = 0$, equivalently, $\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \lambda I_n \\ \mu I_n \\ \nu I_n \end{bmatrix} x = x$. Using the last equality and multiplying (3.13) on the right by $\begin{bmatrix} \lambda I_n \\ \mu I_n \\ \nu I_n \end{bmatrix} x$ and on the left by $\overline{x}^\top \begin{bmatrix} \overline{\lambda} I_n & \overline{\mu} I_n & \overline{\nu} I_n \end{bmatrix}$, it follows that

$$\begin{aligned} &\overline{x}^\top \begin{bmatrix} \overline{\lambda} I_n & \overline{\mu} I_n & \overline{\nu} I_n \end{bmatrix} \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \begin{bmatrix} \lambda I_n \\ \mu I_n \\ \nu I_n \end{bmatrix} x - \overline{x}^\top (P_1 + P_2 + P_3) x \\ &= \overline{x}^\top \left[(\overline{\lambda}\lambda - 1)P_1 + (\overline{\mu}\mu - 1)P_2 + (\overline{\nu}\nu - 1)P_3 \right] x > 0 . \end{aligned}$$

This inequality however is in contradiction with $(\lambda, \mu, \nu) \in \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_i| \leq 1, i = 1, 2, 3\}$. □

Remark 3.14. By using the well-known Schur complement, (3.13) can be written as:

$$\left[\begin{array}{c|c} \begin{matrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{matrix} & \begin{bmatrix} A_1^\top \\ A_2^\top \\ A_3^\top \end{bmatrix} (P_1 + P_2 + P_3) \\ \hline (P_1 + P_2 + P_3) \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} & P_1 + P_2 + P_3 \end{array} \right] > 0 ,$$

which is a linear matrix inequality and thus computationally tractable [49].

3.5 Conditioned Invariant Subspaces

Conditioned invariant (also known as (A, C)-invariant) subspaces were originally introduced by Basile and Marro [40] for 1-D systems as duals for controlled invariant (also (A, B)-invariant) subspaces. These subspaces play an important role in the geometric theory of linear systems. The role of such subspaces for solving problems of state estimation in presence of unknown input signals was later investigated in [41].

3.5.1 Definition and Characterizations

Definition 3.15. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is (A_H, C_D)-*invariant* for (2.10), if

$$\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \left(\mathcal{V} \dot{\oplus} \mathcal{V} \dot{\oplus} \mathcal{V} \bigcap \ker \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix} \right) \subseteq \mathcal{V} \ .$$

The set of conditioned invariants is closed under intersection. A number of characterizations of (A_H, C_D)-invariance are stated next.

Proposition 3.16. *Let $\mathcal{V}$ be a r-dimensional subspace of $\mathbb{R}^n$, and let $Q \in \mathbb{R}^{(n-r)\times n}$ and $V \in \mathbb{R}^{n\times r}$ be full row-, respectively full column-rank matrices such that Im $(V) =$* $\ker(Q) = \mathcal{V}$*. The following statements are equivalent:*

I. $\mathcal{V}$ *is (A_H, C_D)-conditioned invariant;*

II. There exist $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ *and* $\Lambda \in \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3p}$ *such that*

$$QA_H = \Gamma Q_D + \Lambda C_D \ , \tag{3.15}$$

or equivalently there exist $\Gamma_i \in \mathbb{R}^{(n-r)\times(n-r)}$ *and* $\Lambda_i \in \mathbb{R}^{(n-r)\times p}$, $i = 1, 2, 3$ *such that*

$$QA_i = \Gamma_i Q + \Lambda_i C \ \ i = 1, 2, 3 \ .$$

III. There exists $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{R}^{n\times 3p}$ *such that*

$$(A_H + GC_D)\, \mathcal{V} \dot{\oplus} \mathcal{V} \dot{\oplus} \mathcal{V} \subseteq \mathcal{V} \ , \tag{3.16}$$

or equivalently there exist $G_i \in \mathbb{R}^{n\times p}$ *such that*

$$(A_i + G_i C)\mathcal{V} \subseteq \mathcal{V} \quad \text{for } i = 1, 2, 3 \ .$$

Proof. $((I) \implies (II))$: Condition (I) is equivalent with

$$A_H \ \ker \begin{bmatrix} Q_D \\ C_D \end{bmatrix} \subseteq \ \ker Q. \tag{3.17}$$

Lemma 3.17. *Let $X : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let $\mathcal{V} \subseteq \mathbb{R}^n$, $\mathcal{V}' \subseteq \mathbb{R}^m$ be subspaces of dimension r and r', respectively. Let $Z \in \mathbb{R}^{n-r \times n-r}$, $Y \in \mathbb{R}^{m-r' \times m-r'}$ be such that* $\ker(Z) = \mathcal{V}$ *and* $\ker(Y) = \mathcal{V}'$*. Then*

$$X\mathcal{V} \subseteq \mathcal{V}' \iff \exists \ L \text{ such that } YX = LZ.$$

Applying Lemma 3.17 to (3.17) gives

$$QA_H = L \begin{bmatrix} Q_D \\ \\ C_D \end{bmatrix}$$

for some matrix L. Now partition L conformably with $\begin{bmatrix} Q_D \\ \\ C_D \end{bmatrix}$ as $L =: \begin{bmatrix} \Gamma & \Lambda \end{bmatrix}$, where $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ and $\Lambda \in \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3p}$, to conclude that (3.15) holds.

$((II) \implies (III))$: Let $x \in \mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}$, then $QA_Hx = \Lambda C_D x + \Gamma\underbrace{Q_D x}_{=0}$. Consequently, $(QA_H - \Lambda C_D)x = 0$. Let $G' \in \mathbb{R}^{n\times n-r}$ be a right-inverse of Q; it follows $Q\Big(A_H - G'\Lambda C_D\Big)x = 0$. Now define $G := G'\Lambda$.

$((III) \implies (I))$: Let $x \in \mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V} \bigcap \ker C_D$. Then $\Big(A_H + GC_D\Big)x = A_H x \in \mathcal{V}$. □

3.5.2 Unobservability Subspaces

Unobservability subspaces were introduced in the 1-D case in [1]; they provide maximal freedom when choosing the dynamics of an asymptotic observer, and consequently are useful also in the 1-D fault isolation problem. Next the analogous 3-D concept is introduced which will be used later on in the thesis to obtain necessary conditions for fault detection and isolation.

Recall from [50, pp. 350], the definition of non-observable subspace.

Definition 3.18. The *non-observable subspace* of (A_H, C_D) is the limiting subspace of the sequence $\{\mathcal{N}_i\}_{i=0,\ldots}$ defined by

$$\mathcal{N}_i := \begin{cases} \ker\, C & \text{if } i = 0 \\ \bigcap_{j=1,2,3} A_j^{-1}\mathcal{N}_{i-1} \cap \ker\, C & \text{if } i > 0\ , \end{cases}$$

where A_j^{-1} is the inverse image of A_j. It follows from the definition that the non-observable subspace is (A_1, A_2, A_3)-invariant in the sense of Definition 3.8; indeed, it is the largest A_H-invariant subspace contained in ker C. It follows *a fortiori* that it is also an (A_H, C_D)-invariant. In what follows the non-observability subspace of (A_H, C_D) is denoted by $\mathcal{N}(A_H, C_D)$, or simply $\mathcal{N}$ when it is clear which matrices A_H, C_D it corresponds to.

A 3-D unobservability subspace is defined as follows.

Definition 3.19. A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is a (A_H, C_D)-*unobservability subspace* for (2.10), if there exist $H \in \mathbb{R}^{\bullet\times 3p}$ and $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{R}^{n\times 3p}$ such that $\mathcal{S}$ is the non-observable subspace of $(A_H + GC_D, HC_D)$.

Proposition 3.20. *Let $\mathcal{S}$ be a (A_H, C_D)-unobservability subspace. Then $\mathcal{S}$ is an (A_H, C_D)-conditioned invariant.*

Proof. The non-observable subspace of $(A_H + GC_D, HC_D)$ is a $(A_H + GC_D, HC_D)$-invariant. From the equivalence of statement I and statement III of Proposition 3.16, it is also an (A_H, HC_D)-conditioned invariant. Since for all H the inclusion ker $HC_D \supseteq$ ker C_D holds, it also follows that each unobservability subspace is also an (A_H, C_D)-conditioned invariant. □

Let $\mathcal{L} \subset \mathbb{R}^n$ be a subspace. Then the set of all unobservability subspaces containing $\mathcal{L}$ is closed under intersection (see also the discussion at the end of section 3.6), and consequently there exists a smallest unobservability subspace containing $\mathcal{L}$; denoted by $\mathcal{S}(\mathcal{L})^\star$.

Proposition 3.21. *Let $\mathcal{L} \subset \mathbb{R}^n$ be a subspace. Denote by $\mathcal{W}(\mathcal{L})^\star$ the smallest (A_H, C_D)-conditioned invariant containing $\mathcal{L}$, and by $\mathcal{S}(\mathcal{L})^\star$ the smallest unobservability subspace containing $\mathcal{L}$. Then $\mathcal{S}(\mathcal{L})^\star = \mathcal{W}(\mathcal{L})^\star + \mathcal{N}$, with $\mathcal{N}$ the non-observability subspace of (A_H, C_D).*

Proof. The first step is to show that $\mathcal{W}(\mathcal{L})^\star + \mathcal{N}$ is a (A_H, C_D)-conditioned invariant. $\mathcal{W}(\mathcal{L})^\star$ is such a subspace, and consequently (see Proposition 3.16) there exists $G \in \mathbb{R}^{n\times 3p}$ such that

$$(A_H + GC_D)\mathcal{W}(\mathcal{L})^\star \dot{\oplus} \mathcal{W}(\mathcal{L})^\star \dot{\oplus} \mathcal{W}(\mathcal{L})^\star \subseteq \mathcal{W}(\mathcal{L})^\star\ ; \tag{3.18}$$

now since $\mathcal{N} \subset \ker C$ and since $\mathcal{N}$ is an A_H-invariant, it also holds that

$$(A_H + GC)\,(\mathcal{W}(\mathcal{L})^\star + \mathcal{N}) \dot{\oplus} (\mathcal{W}(\mathcal{L})^\star + \mathcal{N}) \dot{\oplus} (\mathcal{W}(\mathcal{L})^\star + \mathcal{N}) \subseteq (\mathcal{W}(\mathcal{L})^\star + \mathcal{N}) \ ;$$

now apply statement *III* of Proposition 3.16 to prove the claim.

To prove that $\mathcal{W}(\mathcal{L})^\star + \mathcal{N}$ is an unobservability subspace. Let $G \in \mathbb{R}^{n\times 3p}$ be such that (3.18) holds, and let $H \in \mathbb{R}^{\bullet\times p}$ be such that $\ker H = C\mathcal{W}(\mathcal{L})^\star$. Then straightforward verification confirms that $\mathcal{W}(\mathcal{L})^\star$ is the non-observable subspace of $(A_H + GC_D, HC)$, and is thus an unobservability subspace.

Finally, observe that $\mathcal{W}(\mathcal{L})^\star + \mathcal{N}$ also contains $\mathcal{L}$.

These considerations imply that $\mathcal{S}(\mathcal{L})^\star$, the smallest unobservability subspace containing $\mathcal{L}$, is contained in $\mathcal{W}(\mathcal{L})^\star + \mathcal{N}$. In order to prove the converse implication, assume by contradiction that there exists $x \in \mathbb{R}^n$ such that $x \in \mathcal{W}(\mathcal{L})^\star + \mathcal{N}$, but $x \notin \mathcal{S}(\mathcal{L})^\star$. Note that since $\mathcal{S}(\mathcal{L})^\star$ is an (A_H, C_D)-conditioned invariant (Proposition 3.20) containing $\mathcal{L}$, and since $\mathcal{W}(\mathcal{L})^\star$ is the smallest such subspace, it holds that $\mathcal{W}(\mathcal{L})^\star \subset \mathcal{S}(\mathcal{L})^\star$. Consequently, $x \in \mathcal{N}$. However, it is easy to see that $\mathcal{N}$, the non-observable subspace of (A_H, C_D), is contained in the unobservability subspace of $(A_H + GC_D, HC_D)$ for all $G \in \mathbb{R}^{n\times 3p}$ and $H \in \mathbb{R}^{\bullet\times 3p}$. Consequently, $\mathcal{N} \subset \mathcal{S}(\mathcal{L})^\star$, yielding a contradiction. □

3.5.3 Stabilising Gains and Their Construction

Statement *III* of Proposition 3.16 shows that analogously to the 1-D and 2-D cases, also in the 3-D case (A_H, C_D)-conditioned invariance implies that an output-feedback matrix G can be found that makes the subspace $(A_1 + G_1C, A_2 + G_2C, A_3 + G_3C)$-invariant in the sense of Def. 3.8. In the design of asymptotic observers for fault detection, it is important to ensure that external stability is also guaranteed. The purpose of this section is to show that the results of [44], stating that internal and external stability can be achieved by applying independent gain matrices, hold also for the 3-D case; and that constructive (albeit probably conservative) procedures can be stated yielding externally stabilising gain matrices.

3.5.3.1 Independence of Internal- and External Stability

Let $\mathcal{V}$ be a conditioned invariant subspace; statement *III* of Proposition 3.16 together with statement *IV* of Proposition 3.9 imply the existence of a matrix $\widetilde{\Gamma} = \begin{bmatrix}\widetilde{\Gamma_1} & \widetilde{\Gamma_2} & \widetilde{\Gamma_3}\end{bmatrix}$ such that $Q\Big(A_H + GC_D\Big) = \begin{bmatrix}\widetilde{\Gamma_1} & \widetilde{\Gamma_2} & \widetilde{\Gamma_3}\end{bmatrix} Q_D$, equivalently

$$QA_H = \begin{bmatrix}\widetilde{\Gamma} & -QG\end{bmatrix}\begin{bmatrix}Q_D \\ C_D\end{bmatrix} . \tag{3.19}$$

Denote by H any full row-rank matrix such that $\ker H = \operatorname{Im} \begin{bmatrix} Q_D \\ C_D \end{bmatrix}$; then $\begin{bmatrix} \widetilde{\Gamma} & -QG \end{bmatrix}$ is a solution of (3.19) if and only if there exists K such that

$$\begin{bmatrix} \widetilde{\Gamma} & -QG \end{bmatrix} = QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^{\dagger} + KH \ . \tag{3.20}$$

From statement II of Proposition 3.16 it follows that $\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^{\dagger} + K'H$ for some matrix K'. Comparing this expression with (3.20) confirms that $\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} - \begin{bmatrix} \widetilde{\Gamma} & -QG \end{bmatrix} = (K' - K)H$. Now partition H as $H = \begin{bmatrix} H' & H'' \end{bmatrix}$, with $H' \in \mathbb{R}^{\bullet \times 3(n-r)}$ and $H'' \in \mathbb{R}^{\bullet \times 3p}$; then

$$\Lambda = -QG + (K' - K)H'' \ .$$

From this it follows that

$$G = -Q^{\dagger}\Lambda + Q^{\dagger}\underbrace{(K' - K)}_{=:K''}H'' + \Omega\ U \ , \tag{3.21}$$

where Ω is a full column rank matrix such that $\mathcal{V} = \ker Q = \operatorname{Im} \Omega$ and U is an arbitrary matrix of suitable dimension.

Let S be a nonsingular matrix whose first r columns span $\mathcal{V}$. Since $\mathcal{V}$ an $(A_1 + G_1C, A_2 + G_2C, A_3 + G_3C)$-invariant subspace, applying Theorem 3.10 gives

$$S(A_i + G_iC)S^{-1} = \begin{bmatrix} \Delta_i^{11}(K'', U) & \Delta_i^{12}(K'', U) \\ 0 & \Delta_i^{22}(K'', U) \end{bmatrix} \ , \tag{3.22}$$

where the dependence of Δ_i^{11}, Δ_i^{12} and Δ_i^{22} in (3.22) on U and K'' arises from the fact that G itself depends on these matrices, see (3.21). U and K'' are two degrees of freedom that can be used to assign the *inner dynamics* of $\mathcal{V}$ by modifying $\Delta_i^{11}(K'', U)$ and the *external dynamics* of $\mathcal{V}$ by modifying $\Delta_i^{22}(K'', U)$. The following result shows that these dynamics can be assigned *independently* of each other.

Proposition 3.22. *For all $i \in \{1, 2, 3\}$, the matrix $\Delta_i^{22}(K'', U)$ in (3.22) does not depend on U, and the matrix $\Delta_i^{11}(K'', U)$ does not depend on K''.*

Proof. Let U_1 and U_2 be arbitrary matrices, and subtract the matrices (3.22) corresponding to the gains $-Q^{\dagger}\Lambda + Q^{\dagger}K''H'' + \Omega\ U_1$ and $-Q^{\dagger}\Lambda + Q^{\dagger}K''H'' + \Omega\ U_2$. Denoting $G' := -Q^{\dagger}\Lambda + Q^{\dagger}K''H''$, and partitioning $U_i =: \begin{bmatrix} U_i^1 & U_i^2 & U_i^3 \end{bmatrix}$, with $U_i^j \in \mathbb{R}^{r \times p}$, $i = 1, 2$,

$j = 1, 2, 3$, it is obtained:

$$\begin{bmatrix} \Delta_i^{11}(K'', U_1) - \Delta_i^{11}(K'', U_2) & \Delta_i^{12}(K'', U_1) - \Delta_i^{12}(K'', U_2) \\ 0 & \Delta_i^{22}(K'', U_1) - \Delta_i^{22}(K'', U_2) \end{bmatrix} =$$
$$S(A_i + G_i'C + \Omega U_1^i C)S^{-1} - S(A_i + G_i'C + \Omega U_2^i C)S^{-1} = S\Omega(U_1^i - U_2^i)CS^{-1} \quad (3.23)$$

Also S can without loss of generality be assumed to be structured as $S = \begin{bmatrix} V_c \\ Q \end{bmatrix}$ for some suitable matrix V_c; consequently, the second block row of (3.23) is

$$\underbrace{Q\Omega}_{=0}(U_1^i - U_2^i)CS^{-1} = 0 = \begin{bmatrix} 0 & \Delta_i^{22}(K'', U_1) - \Delta_i^{22}(K'', U_2) \end{bmatrix} .$$

It follows that $\Delta_i^{22}(K'', U_1) = \Delta_i^{22}(K'', U_2)$, which implies that the term $\Delta_i^{22}(K'', U)$ in (3.22) does not depend on U.

To show that $\Delta_i^{11}(K'', U)$ does not depend on K, proceed as follows. First, partition $H'' =: \begin{bmatrix} H_1'' & H_2'' & H_3'' \end{bmatrix}$ with $H_i'' \in \mathbb{R}^{\bullet \times p}$, $i = 1, 2, 3$. Then use (3.22) and $S = \begin{bmatrix} V_c \\ Q \end{bmatrix}$ to conclude that

$$V_c(A_i - Q^\dagger \Lambda_i C + Q^\dagger K'' H_i'' C + \Omega U C) = \Delta_i^{11}(K'', U)V_c + \Delta_i^{12}(K'', U)Q . \quad (3.24)$$

Now consider (3.24) for K_i'', $i = 1, 2$, and subtract the first equation from the second; after routine manipulations to obtain

$$\begin{aligned} V_c\Big(Q^\dagger(K_1'' - K_2'')H_i''C\Big) &= \Big(\Delta_i^{11}(K_1'', U) - \Delta_i^{11}(K_2'', U)\Big)V_c \\ &+ \Big(\Delta_i^{12}(K_1'', U) - \Delta_i^{12}(K_2'', U)\Big)Q . \end{aligned} \quad (3.25)$$

From $H \begin{bmatrix} Q_D \\ C_D \end{bmatrix} = H'Q_D + H''C_D = 0$ it follows that the subspace spanned by the rows of $H''C_D$ is a subspace of the row span of Q. Since V_c and Q have linearly independent rows, from (3.25) it follows that $\Delta_i^{11}(K_1'', U) - \Delta_i^{11}(K_2'', U) = 0$; consequently, $\Delta_i^{11}(K, U)$ in (3.22) does not depend on K. □

3.5.3.2 Construction of Stabilising Gains

Consider the problem of constructing externally stabilising gains for a given (A_i, C)-invariant subspace, $i = 1, 2, 3$. As the following result shows, external stability is equivalent to the existence of $\Gamma = \begin{bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_3\end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ and $\Lambda = \begin{bmatrix}\Lambda_1 & \Lambda_2 & \Lambda_3\end{bmatrix}$ such that Γ_i has all its eigenvalues in the open unit circle.

Proposition 3.23. *Let Γ, Λ satisfy (3.15). Then $\Gamma_i = \Delta_i^{22}(K'', U)$, the $(2,2)$-block of (3.22).*

Proof. From statement (II) of Proposition 3.16 it follows that $(QA_i - \Lambda_i C) = \Gamma_i Q$, $i = 1, 2, 3$, and hence

$$Q(A_i - (Q^\dagger + VK)\Lambda_i C) = \Gamma_i Q \ ,$$

where $G_i := -(Q^\dagger + VK)\Lambda_i$, Im $V = \mathcal{V}$ and K is an arbitrary matrix of suitable dimensions. Now consider (3.22), and partition S as $S = \begin{bmatrix} V_c \\ Q \end{bmatrix}$, as in the proof of Proposition 3.22. The second block row of (3.22) yields

$$Q(A_i + G_i C)S^{-1} = \begin{bmatrix} 0 & \Delta_i^{22}(K'', U) \end{bmatrix} = \Gamma_i Q S^{-1} \ .$$

And hence

$$\Gamma_i Q = \begin{bmatrix} 0 & \Delta_i^{22}(K'', U) \end{bmatrix} S = \Delta_i^{22}(K'', U) Q \ .$$

Since Q has full row-rank, this implies $\Gamma_i = \Delta_i^{22}(K'', U)$. □

The following is an immediate consequence of Proposition 3.23.

Corollary 3.24. *Let $\mathcal{V}$ be an (A, C)-conditioned invariant subspace of dimension r, and denote by $Q \in \mathbb{R}^{(n-r)\times n}$ a full row rank matrix such that $\ker(Q) = \mathcal{V}$. $\mathcal{V}$ is externally stabilizable if and only if there exists $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$, such that $Q(A_i + G_i C) = \Gamma_i Q$ with $\Gamma_i \in \mathbb{R}^{(n-r)\times(n-r)}$ Schur, $i = 1, 2, 3$.*

The matrix Δ_i^{11} in (3.22) is related to the *internal* stability properties of the conditioned invariant subspace, as the following result shows.

Proposition 3.25. *Let $V_c \in \mathbb{R}^{r\times n}$ be such that $\begin{bmatrix} V_c \\ Q \end{bmatrix}$ is nonsingular, and moreover $V_c V_c^\top = I_r$. Then $\Delta_i^{11}(K'', U) = V_c(A_i + \Omega U C)V_c^\top$.*

Proof. Multiply both sides of (3.22) on the right by $SV_c^\top$ to obtain

$$S(A_i + G_iC)V_c^\top = \begin{bmatrix} \Delta_i^{11}(K'', U) & \Delta_i^{12}(K'', U) \\ 0 & \Delta_i^{22}(K'', U) \end{bmatrix} \underbrace{SV_c^\top}_{=\begin{bmatrix} I \\ 0 \end{bmatrix}} .$$

Hence,

$$S(A_i + G_iC)V_c^\top = \begin{bmatrix} \Delta_i^{11}(K'', U) & \Delta_i^{12}(K'', U) \end{bmatrix} ,$$

from which it follows that $\Delta_i^{11}(K'', U) = V_c(A_i + G_iC)V_c^\top$.

From (3.21), and using the fact that the columns of $V_c^\top$ form an orthonormal basis for $\mathcal{V}$, it follows that $V_cG = \Omega U$. The claim follows. □

The following is an immediate consequence of Proposition 3.25.

Corollary 3.26. *Let $\mathcal{V}$ be an (A_i, C)-invariant subspace of dimension r, $i = 1, 2, 3$, and denote by $Q \in \mathbb{R}^{(n-r)\times n}$ a full row rank matrix such that $\ker(Q) = \mathcal{V}$. Moreover, let $V_c \in \mathbb{R}^{r\times n}$ be such that $\begin{bmatrix} V_c \\ Q \end{bmatrix}$ is nonsingular, and $V_cV_c^\top = I_r$. $\mathcal{V}$ is internally stabilisable if and only if there exist $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$, such that the triple*

$$(V_c(A_1 + G_1C)V_c^\top, V_c(A_2 + G_2C)V_c^\top, V_c(A_3 + G_3C)V_c^\top)$$

is stable.

To construct an internally stabilizing gain matrix G, the aim is to compute the matrix U with aid of the result of Proposition 3.13 and try to solve the following matrix inequality in the unknown positive-definite matrices $P_i \in \mathbb{R}^{n\times n}, i = 1, 2, 3$:

$$\begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} - \begin{bmatrix} \zeta_1^\top \\ \zeta_2^\top \\ \zeta_3^\top \end{bmatrix} (P_1 + P_2 + P_3) \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix} > 0 , \tag{3.26}$$

where $\zeta_i = V_c(A_i + \Omega UC)V_c^\top$. Note that (3.26) is not linear in U and P_i, $i = 1, 2, 3$; by introducing auxiliary variables $\Psi_1 := P_1, \Psi_2 := P_1 + P_2, \Psi_3 := P_1 + P_2 + P_3$, and using a Schur complement argument from (3.26) it is concluded that

$$\begin{bmatrix} \Psi_1 & 0 & 0 \\ 0 & \Psi_2 & 0 \\ 0 & 0 & \Psi_3 \end{bmatrix} - \begin{bmatrix} \zeta_1^\top \\ \zeta_2^\top \\ \zeta_3^\top \end{bmatrix} \Psi_3 \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix} > 0 \tag{3.27}$$

$$\iff \begin{bmatrix} \Psi_1 & 0 & 0 & \zeta_1^\top \Psi_3 \\ 0 & \Psi_2 - \Psi_1 & 0 & \zeta_2^\top \Psi_3 \\ 0 & 0 & \Psi_3 - \Psi_2 & \zeta_3^\top \Psi_3 \\ \Psi_3\zeta_1 & \Psi_3\zeta_2 & \Psi_3\zeta_3 & \Psi_3 \end{bmatrix} > 0 \ , \tag{3.28}$$

and on introducing the auxiliary variables $\Pi_i := \Psi_3 V_c \Omega U_i$, $i = 1, 2, 3$ (3.28) is equivalent to

$$\begin{aligned} &\begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} > 0 \\ &\Psi_1, \Psi_2, \Psi_3 > 0 \ , \end{aligned} \tag{3.29}$$

where

$$\Delta_1 := \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 - \Psi_1 \\ 0 & 0 \\ \Psi_3 V_c A_1 V_c^\top + \Pi_1 C V_c^\top & \Psi_3 V_c A_2 V_c^\top + \Pi_2 C V_c^\top \end{bmatrix} ,$$

$$\Delta_2 := \begin{bmatrix} 0 & (\Psi_3 V_c A_1 V_c^\top + \Pi_1 C V_c^\top)^\top \\ 0 & (\Psi_3 V_c A_2 V_c^\top + \Pi_2 C V_c^\top)^\top \\ \Psi_3 - \Psi_2 & (\Psi_3 V_c A_3 V_c^\top + \Pi_3 C V_c^\top)^\top \\ \Psi_3 V_c A_3 V_c^\top + \Pi_3 C V_c^\top & \Psi_3 \end{bmatrix} .$$

Having found solutions Ψ_i, $i = 1, 2, 3$ and Π_i to (3.29), the matrix U_i is obtained by $U_i = \Psi_3^{-1} V_c^{-1} \Omega^{-1} \Pi_i$ and finally $U = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix}$.

To construct an externally stabilizing gain matrix G, first compute, if it exists, $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ such that (3.15) holds for some $\Lambda \in \mathbb{R}^{(n-r)\times 3p}$, and moreover Γ_i is Schur, $i = 1, 2, 3$. Then compute G as a solution to

$$Q(A_i + G_i C) = \Gamma_i Q, \quad i = 1, 2, 3 \ .$$

From (3.15) it follows that

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^{\dagger} + KH \ , \tag{3.30}$$

where H is any full row-rank matrix such that $\ker(H) = \operatorname{Im} \begin{bmatrix} Q_D \\ C_D \end{bmatrix}$. Denote

$$\begin{bmatrix} V_1 & V_2 & V_3 & \bar{V} \end{bmatrix} := QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^{\dagger} ,$$

where $V_i \in \mathbb{R}^{(n-r)\times(n-r)}$, $i = 1, 2, 3$, and $\bar{V} \in \mathbb{R}^{(n-r)\times 3p}$. Partition H as

$$H =: \begin{bmatrix} H_1 & H_2 & H_3 & \bar{H} \end{bmatrix} ,$$

and rewrite (3.30) as

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Lambda \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 & \bar{V} \end{bmatrix} + K \begin{bmatrix} H_1 & H_2 & H_3 & \bar{H} \end{bmatrix} ,$$

from which it follows that $\Gamma_i = V_i + KH_i$, $i = 1, 2, 3$, and $\Lambda = \bar{V} + K\bar{H}$.

Two cases are now possible, depending on whether or not $\begin{bmatrix} Q_D \\ C_D \end{bmatrix}$ has full row-rank, or not.

In the first case $H = 0$; consequently $\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = QA_H \begin{bmatrix} Q_D \\ C_D \end{bmatrix}^{\dagger}$, with the matrix on the right-hand side being uniquely defined. This implies that if Γ_i is Schur, $i = 1, 2, 3$, then the corresponding G makes $\mathcal{V}$ externally stable. Otherwise, no G exists that makes $\mathcal{V}$ externally stable.

If $\begin{bmatrix} Q_D \\ C_D \end{bmatrix}$ is not full row-rank, a matrix K must be found such that $\Gamma_i = V_i + KH_i$ is asymptotically stable, $i = 1, 2, 3$. In finding such a K the result of Proposition 3.13 can be used and try to solve the following matrix inequality in the unknown positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, 3$:

$$\begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} - \begin{bmatrix} \Gamma_1^\top \\ \Gamma_2^\top \\ \Gamma_3^\top \end{bmatrix} (P_1 + P_2 + P_3) \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} > 0 \, , \tag{3.31}$$

where $\Gamma_i = V_i + KH_i$. Note that (3.31) is not linear in K and P_i, $i = 1, 2, 3$ as bilinear terms P_iK appear. By introducing the auxiliary variables $\Phi_1 := P_1$, $\Phi_2 := P_1 + P_2$,

$\Phi_3 := P_1 + P_2 + P_3$, and using a Schur complement argument from (3.31) gives

$$\begin{bmatrix} \Phi_1 & 0 & 0 \\ 0 & \Phi_2 & 0 \\ 0 & 0 & \Phi_3 \end{bmatrix} - \begin{bmatrix} \Gamma_1^\top \\ \Gamma_2^\top \\ \Gamma_3^\top \end{bmatrix} \Phi_3 \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} > 0$$

$$\iff \begin{bmatrix} \Phi_1 & 0 & 0 & \Gamma_1^\top \Phi_3 \\ 0 & \Phi_2 - \Phi_1 & 0 & \Gamma_2^\top \Phi_3 \\ 0 & 0 & \Phi_3 - \Phi_2 & \Gamma_3^\top \Phi_3 \\ \Phi_3 \Gamma_1 & \Phi_3 \Gamma_2 & \Phi_3 \Gamma_3 & \Phi_3 \end{bmatrix} > 0 \, , \tag{3.32}$$

and on introducing the auxiliary variable $\Theta := \Phi_3 K$, (3.43) is equivalent to

$$\begin{bmatrix} \Phi_1 & 0 & 0 & (\Phi_3 V_1 + \Theta H_1)^\top \\ 0 & \Phi_2 - \Phi_1 & 0 & (\Phi_3 V_2 + \Theta H_2)^\top \\ 0 & 0 & \Phi_3 - \Phi_2 & (\Phi_3 V_3 + \Theta H_3)^\top \\ \Phi_3 V_1 + \Theta H_1 & \Phi_3 V_2 + \Theta H_2 & \Phi_3 V_3 + \Theta H_3 & \Phi_3 \end{bmatrix} > 0$$
$$\Phi_1, \Phi_2, \Phi_3 > 0 \, . \tag{3.33}$$

Given the solutions Φ_i, $i = 1, 2, 3$ and Θ to (3.33), the matrix K is obtained as $K = \Phi_3^{-1}\Theta$.

3.6 Input-Containing (A_H, C_D)-Invariants

In the fault detection and isolation analysis later in this thesis, the concept of an input-containing (A_H, C_D)-conditioned invariant will play an important role in describing the fault dynamics of a 3-D plant. The definition is as follows.

Definition 3.27. $\mathcal{V} \subset \mathbb{R}^n$ is an *input-containing conditioned invariant* subspace for (2.10) if

$$\begin{bmatrix} A_H & B_H \end{bmatrix} ((\mathcal{V} \dot{\oplus} \mathcal{V} \dot{\oplus} \mathcal{V} \dot{\oplus} \mathbb{R}^{3m}) \cap \ker \begin{bmatrix} C_D & 0_{3p \times 3m} \end{bmatrix}) \subseteq \mathcal{V} \, .$$

The following characterisations of input-containing subspaces hold.

Proposition 3.28. *Let $\mathcal{V}$ be a r-dimensional subspace of $\mathbb{R}^n$, and let $Q \in \mathbb{R}^{(n-r)\times n}$ and $V \in \mathbb{R}^{n \times r}$ be full row-, respectively full column-rank matrices such that Im $(V) = \ker(Q) = \mathcal{V}$. The following statements are equivalent:*

I. The subspace $\mathcal{V}$ is an input-containing conditioned invariant for (2.10);

II. There exist $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ and $\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3p}$ such that

$$Q \begin{bmatrix} A_H & B_H \end{bmatrix} = \Gamma \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \end{bmatrix} + \Lambda \begin{bmatrix} C_D & 0_{3p \times 3m} \end{bmatrix} \, , \tag{3.34}$$

or equivalently there exist $\Gamma_i \in \mathbb{R}^{(n-r)\times(n-r)}$ *and* $\Lambda_i \in \mathbb{R}^{(n-r)\times p}$, $i = 1,2,3$ *such that*

$$Q \begin{bmatrix} A_i & B_i \end{bmatrix} = \Gamma_i \begin{bmatrix} Q & 0_{(n-r)\times m} \end{bmatrix} + \Lambda_i \begin{bmatrix} C & 0_{p\times m} \end{bmatrix} \quad i = 1,2,3 \ .$$

III. There exists $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{R}^{n\times 3p}$ *such that*

$$\begin{bmatrix} A_H + GC_D & B_H \end{bmatrix} \left(\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathbb{R}^{3m}\right) \subseteq \mathcal{V} \ , \tag{3.35}$$

or equivalently there exist $G_i \in \mathbb{R}^{n\times p}$ *such that*

$$\begin{bmatrix} A_i + G_iC & B_i \end{bmatrix} \left(\mathcal{V}\dot{\oplus}\mathbb{R}^{m}\right) \subseteq \mathcal{V} \quad \textit{for } i = 1,2,3 \ .$$

Proof. $((I) \Longrightarrow (II))$: Condition (I) is equivalent to

$$\begin{bmatrix} A_H & B_H \end{bmatrix} \ \ker \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix} \subseteq \ \ker Q. \tag{3.36}$$

As a consequence of Lemma 3.17 it follows

$$Q \begin{bmatrix} A_H & B_H \end{bmatrix} = L \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}$$

for some $L \in \mathbb{R}^{(n-r)\times 3(n-r)+3p}$. Now partition L conformably with $\begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}$ as $L := \begin{bmatrix} \Gamma & \Lambda \end{bmatrix}$ where $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ and $\Lambda \in \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3p}$. Then it is immediate to verify that (3.34) holds.

$((II) \Longrightarrow (III))$: Let $x \in \mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathbb{R}^{3m}$, then

$$Q \begin{bmatrix} A_H & B_H \end{bmatrix} x = \Gamma \underbrace{\begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \end{bmatrix} x}_{=0} + \Lambda \begin{bmatrix} C_D & 0_{3(n-r)\times 3m} \end{bmatrix} x \ .$$

Consequently,

$$(Q \begin{bmatrix} A_H & B_H \end{bmatrix} - \Lambda \begin{bmatrix} C_D & 0_{3(n-r)\times 3m} \end{bmatrix})x = 0 \ .$$

Let $G' \in \mathbb{R}^{n\times(n-r)}$ be a right-inverse of Q; It follows

$$Q\Big(\begin{bmatrix} A_H & B_H \end{bmatrix} - G'\Lambda \begin{bmatrix} C_D & 0_{2(n-r)\times 3m} \end{bmatrix} \Big)x = 0 \ .$$

Now define $G := -G'\Lambda$. Then

$$Q\Big(\begin{bmatrix}A_H + GC_D & B_H\end{bmatrix}\Big)x = 0\ ,$$

which proves the claim.

$((III) \Longrightarrow (I))$: Let $x \in \mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathcal{V}\dot{\oplus}\mathbb{R}^{3m} \bigcap \ker \begin{bmatrix}C_D & 0_{3p\times 3m}\end{bmatrix}$. Then

$$\begin{bmatrix}A_H + GC_D & B_H\end{bmatrix} x = \begin{bmatrix}A_H & B_H\end{bmatrix} x \in \mathcal{V}\ ,$$

which proves (3.36). Hence (I) follows from (3). □

The intersection of two input-containing subspaces is also input containing; thus the smallest input-containing subspace of $(A_i, B_i, C), i = 1, 2, 3$ is the intersection of all input-containing subspaces of $(A_i, B_i, C), i = 1, 2, 3$. Denote by $\mathcal{W}(\mathcal{B})^\star$ the smallest (A_H, C_D)-conditioned invariant containing $\mathrm{im}(B_H) =: \mathcal{B}_H$. To compute $\mathcal{W}(\mathcal{B})^\star$, the recursion (see [44, Algorithm 4.1]) can be used:

$$\mathcal{W}_i := \begin{cases} \{0\}_n & \text{if } i = 0 \\ \begin{bmatrix}A_H & B_H\end{bmatrix}\Big(\mathcal{W}_{i-1}\dot{\oplus}\mathcal{W}_{i-1}\dot{\oplus}\mathcal{W}_{i-1}\dot{\oplus}\mathbb{R}^{3m} \cap \ker \begin{bmatrix}C_D & 0_{3p\times 3m}\end{bmatrix}\Big) & i > 1\ ; \end{cases} \tag{3.37}$$

and then $\mathcal{W}(\mathcal{B})^\star = \lim_{i\to\infty} \mathcal{W}_i = \mathcal{W}_n$.

It is routine to show that results analogous to those for conditioned invariant subspaces (see sections 3.5.3.1 and 3.5.3.2) hold also in the case of input-containing conditioned invariants. Therefore, it is only required to consider the construction, if it exists, of an external stabilising G with the internal stabilising G being constructed in the same manner. Let $\mathcal{V}$ be an input-containing conditioned invariant subspace; statement III of Proposition 3.28 together with statement IV of Proposition 3.9 imply the existence of a matrix $\widetilde{\Gamma} = \begin{bmatrix}\widetilde{\Gamma_1} & \widetilde{\Gamma_2} & \widetilde{\Gamma_3}\end{bmatrix}$ such that

$$Q\Big(\begin{bmatrix}A_H & B_H\end{bmatrix} + G\begin{bmatrix}C_D & 0_{3p\times 3m}\end{bmatrix}\Big) = \begin{bmatrix}\widetilde{\Gamma_1} & \widetilde{\Gamma_2} & \widetilde{\Gamma_3}\end{bmatrix}\begin{bmatrix}Q_D & 0_{3(n-r)\times 3m}\end{bmatrix}\ ,$$

equivalently,

$$Q\begin{bmatrix}A_H & B_H\end{bmatrix} = \begin{bmatrix}\widetilde{\Gamma} & -QG\end{bmatrix}\begin{bmatrix}Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m}\end{bmatrix}\ . \tag{3.38}$$

Denote by H any full row-rank matrix such that $\ker H = \text{Im} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}$. Then $\begin{bmatrix} \widetilde{\Gamma} & -QG \end{bmatrix}$ is a solution of (3.38) if and only if there exists K such that

$$\begin{bmatrix} \widetilde{\Gamma} & -QG \end{bmatrix} = Q \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}^{\dagger} + KH \,. \tag{3.39}$$

From statement II of Proposition 3.28 it follows that

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = Q \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}^{\dagger} + K'H \,,$$

for some matrix K'. Comparing this expression with (3.39) gives

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} - \begin{bmatrix} \widetilde{\Gamma} & -QG \end{bmatrix} = (K' - K)H \,.$$

Now partition H as $H = \begin{bmatrix} H' & H'' \end{bmatrix}$, with $H' \in \mathbb{R}^{\bullet\times 3(n-r)}$ and $H'' \in \mathbb{R}^{\bullet\times 3p}$; then

$$\Lambda = -QG + (K' - K)H'' \,,$$

from which it follows that

$$G = -Q^{\dagger}\Lambda + Q^{\dagger}\underbrace{(K' - K)}_{=:K''}H'' + \Omega\, U \,, \tag{3.40}$$

where Ω is a full column rank matrix such that $\mathcal{V} = \ker Q = \text{Im}\,\Omega$ and U is an arbitrary matrix of suitable dimension.

To construct an externally stabilising gain matrix G, first compute, if it exists, $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \in \mathbb{R}^{(n-r)\times 3(n-r)}$ such that (3.34) holds for some $\Lambda \in \mathbb{R}^{(n-r)\times 3p}$, and moreover Γ_i is Schur, $i = 1, 2, 3$. Then compute G as a solution to $Q(A_i + G_i C) = \Gamma_i Q$, $i = 1, 2, 3$. Note that from (3.34) it follows that

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = Q \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}^{\dagger} + KH \,, \tag{3.41}$$

where H is any full row-rank matrix such that $\ker(H) = \text{Im} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}$. Denote

$$\begin{bmatrix} V_1 & V_2 & V_3 & \bar{V} \end{bmatrix} := Q \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}^{\dagger} ,$$

where $V_i \in \mathbb{R}^{(n-r)\times(n-r)}$, $i = 1,2,3$, and $\bar{V} \in \mathbb{R}^{(n-r)\times 3p}$. Partition H as $H := \begin{bmatrix} H_1 & H_2 & H_3 & \bar{H} \end{bmatrix}$; now rewrite (3.41) as

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Lambda \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 & \bar{V} \end{bmatrix} + K \begin{bmatrix} H_1 & H_2 & H_3 & \bar{H} \end{bmatrix} ,$$

from which it follows that $\Gamma_i = V_i + KH_i$, $i = 1,2,3$, and $\Lambda = \bar{V} + K\bar{H}$.

Two cases are now possible, depending on whether $\begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}$ has full row-rank, or not.

In the first case $H = 0$; consequently

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}^{\dagger} ,$$

with the matrix on the right-hand side is uniquely defined. This implies that if Γ_i is Schur, $i = 1,2,3$, then the corresponding G makes $\mathcal{V}$ externally stable. Otherwise, no G exists that makes $\mathcal{V}$ externally stable.

If $\begin{bmatrix} Q_D & 0_{3(n-r)\times 3m} \\ C_D & 0_{3p\times 3m} \end{bmatrix}$ is not full row-rank, a matrix K must be found such that $\Gamma_i = V_i + KH_i$ is asymptotically stable, $i = 1,2,3$. To find such a K the result of Proposition 3.13 is used to solve the matrix inequality in the unknown positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,2,3$:

$$\begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} - \begin{bmatrix} \Gamma_1^\top \\ \Gamma_2^\top \\ \Gamma_3^\top \end{bmatrix} (P_1 + P_2 + P_3) \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} > 0 , \tag{3.42}$$

where $\Gamma_i = V_i + KH_i$. Note that (3.42) is not linear in K and P_i, $i = 1,2,3$; bilinear terms P_iK appear. By introducing the auxiliary variables $\Phi_1 := P_1$, $\Phi_2 := P_1 + P_2$, $\Phi_3 := P_1 + P_2 + P_3$, and using the Schur complement argument from (3.31) it follows

$$\begin{aligned} &\begin{bmatrix} \Phi_1 & 0 & 0 \\ 0 & \Phi_2 & 0 \\ 0 & 0 & \Phi_3 \end{bmatrix} - \begin{bmatrix} \Gamma_1^\top \\ \Gamma_2^\top \\ \Gamma_3^\top \end{bmatrix} \Phi_3 \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} > 0 \\ \iff &\begin{bmatrix} \Phi_1 & 0 & 0 & \Gamma_1^\top \Phi_3 \\ 0 & \Phi_2 - \Phi_1 & 0 & \Gamma_2^\top \Phi_3 \\ 0 & 0 & \Phi_3 - \Phi_2 & \Gamma_3^\top \Phi_3 \\ \Phi_3\Gamma_1 & \Phi_3\Gamma_2 & \Phi_3\Gamma_3 & \Phi_3 \end{bmatrix} > 0 , \end{aligned} \tag{3.43}$$

and introducing the auxiliary variable $\Theta := \Phi_3 K$, (3.43) is equivalent to

$$\begin{bmatrix} \Phi_1 & 0 & 0 & (\Phi_3 V_1 + \Theta H_1)^\top \\ 0 & \Phi_2 - \Phi_1 & 0 & (\Phi_3 V_2 + \Theta H_2)^\top \\ 0 & 0 & \Phi_3 - \Phi_2 & (\Phi_3 V_3 + \Theta H_3)^\top \\ \Phi_3 V_1 + \Theta H_1 & \Phi_3 V_2 + \Theta H_2 & \Phi_3 V_3 + \Theta H_3 & \Phi_3 \end{bmatrix} > 0$$
$$\Phi_1, \Phi_2, \Phi_3 > 0 \; . \tag{3.44}$$

Given the solutions Ψ_i, $i = 1, 2, 3$ and Θ to (3.44), the matrix K is obtained by $K = \Phi_3^{-1}\Theta$.

Example 3.1. Consider the system defined by

$$A_1 = \begin{bmatrix} -2 & \frac{-1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & \frac{-15}{4} & 0 & \frac{7}{4} \\ -3 & \frac{23}{4} & \frac{5}{2} & \frac{-11}{4} \\ 0 & \frac{-7}{2} & 0 & \frac{3}{2} \end{bmatrix}, A_2 = \begin{bmatrix} \frac{-16}{3} & \frac{55}{6} & \frac{17}{3} & \frac{-29}{6} \\ 0 & \frac{-11}{6} & 0 & \frac{5}{6} \\ \frac{-34}{3} & \frac{139}{6} & \frac{35}{3} & \frac{-71}{6} \\ 0 & \frac{-5}{3} & 0 & \frac{2}{3} \end{bmatrix}, A_3 = \begin{bmatrix} \frac{-7}{5} & \frac{-217}{30} & \frac{6}{5} & \frac{109}{30} \\ 0 & \frac{-59}{6} & 0 & \frac{29}{6} \\ \frac{-12}{5} & \frac{143}{30} & \frac{11}{5} & \frac{-71}{30} \\ 0 & \frac{-29}{3} & 0 & \frac{14}{3} \end{bmatrix},$$

$$B_1 = B_2 = 0_{4\times 2} \; , \quad B_3 = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 7 \\ 2 & 2 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad D = 0_{3\times 2} \; .$$

The aim is to construct a stabilising output injection matrix G.

Step 1. Building the subspaces $\mathcal{W}_3^{1\star}$, and $\mathcal{W}_3^{2\star}$ and using the recursion Algorithm (3.37),

$$\mathcal{W}_3^{1\star} = \text{ Im } \left(\begin{bmatrix} -0.7071 \\ -0.4714 \\ -0.2357 \\ -0.4714 \end{bmatrix} \right), \quad \mathcal{W}_3^{2\star} = \text{ Im } \left(\begin{bmatrix} -0.4781 \\ -0.1195 \\ -0.8367 \\ -0.2390 \end{bmatrix} \right).$$

Step 2. Building $\mathcal{W}^\star = \mathcal{W}_3^{1\star} \dot{+} \mathcal{W}_3^{2\star}$:

$$\mathcal{W}^\star = \begin{bmatrix} -0.7071 & -0.4781 \\ -0.4714 & -0.1195 \\ -0.2357 & -0.8367 \\ -0.4714 & -0.2390 \end{bmatrix} ,$$

and the kernel of

$$Q = \begin{bmatrix} -0.7065 & 0.4837 & 0.1984 & 0.4770 \\ 0.0000 & -0.6740 & -0.1123 & 0.7302 \end{bmatrix}$$

is exactly $\mathcal{W}^\star$.

Step 3. Construct an externally and internally stabilizing output injection matrix $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix}$ such that $\mathcal{W}^\star$ is an $(A_i + G_i)$-invariant input containing subspace, $i = 1, 2, 3$:

Since

$$\begin{bmatrix} V_1 & V_2 & V_3 & \bar{V} \end{bmatrix} = Q \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0 \\ C_D & D_D \end{bmatrix}^\dagger ,$$

$$\begin{bmatrix} H_1 & H_2 & H_3 & \bar{H} \end{bmatrix} = \text{ Im } (\begin{bmatrix} Q_D & 0 \\ C_D & D_D \end{bmatrix}) ,$$

it yields

$$V_1 = \begin{bmatrix} -0.5067 & 0.3078 \\ -0.2010 & 0.0817 \end{bmatrix} , \quad V_2 = \begin{bmatrix} -0.4432 & 0.8287 \\ -0.3871 & 0.6125 \end{bmatrix} , \quad V_3 = \begin{bmatrix} -0.4397 & 0.4777 \\ -0.1709 & 0.1410 \end{bmatrix} ,$$

$$\bar{V} = \begin{bmatrix} 0.4600 & 0.8567 & 0.0312 & 1.2070 & 1.3950 & -0.3015 & 0.2024 & 1.3879 & -0.0682 \\ \\ 0.1950 & 0.2608 & -0.0368 & 0.9996 & 0.9918 & -0.1654 & 0.1489 & 0.3942 & -0.0485 \end{bmatrix} ,$$

$$H_1 = \begin{bmatrix} 0 & -0.0000 \\ -0.4200 & 0.7678 \\ 0 & 0.0000 \end{bmatrix} , H_2 = \begin{bmatrix} -0.4069 & 0.7439 \\ -0.0000 & -0.0000 \\ -0.1040 & 0.1901 \end{bmatrix} , H_3 = \begin{bmatrix} -0.1040 & 0.1901 \\ 0 & 0 \\ 0.4069 & -0.7439 \end{bmatrix} ,$$

$$\bar{H} = \begin{bmatrix} \bar{H}_1 & \bar{H}_2 & \bar{H}_3 \end{bmatrix} ,$$

where

$$\bar{H}_1 = \begin{bmatrix} -0.0000 & 0.0000 & 0.0000 \\ -0.2967 & -0.3603 & -0.1272 \\ 0.0000 & -0.0000 & 0.0000 \end{bmatrix} , \quad \bar{H}_2 = \begin{bmatrix} -0.2875 & -0.3491 & -0.1232 \\ -0.0000 & 0.0000 & -0.0000 \\ -0.0735 & -0.0892 & -0.0315 \end{bmatrix}$$

$$\bar{H}_3 = \begin{bmatrix} -0.0735 & -0.0892 & -0.0315 \\ -0.0000 & -0.0000 & -0.0000 \\ 0.2875 & 0.3491 & 0.1232 \end{bmatrix}$$

Solving the LMI (3.44) for Φ_i, Ψ_i, K and U_i, $i = 1, 2, 3$ gives

$$\Phi_1 = \begin{bmatrix} 15.0115 & -0.14006 \\ -0.14006 & 15.2356 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 22.0711 & -0.42019 \\ -0.42019 & 22.7433 \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} 28.2383 & -1.1205 \\ -1.1205 & 30.0307 \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} 16.91 & 0 \\ 0 & 16.91 \end{bmatrix}, \Psi_2 = \begin{bmatrix} 25.365 & 0 \\ 0 & 25.365 \end{bmatrix},$$

$$\Psi_3 = \begin{bmatrix} 33.82 & 0 \\ 0 & 33.82 \end{bmatrix}, \quad K = \begin{bmatrix} -1.2186 & -0.5864 & 0.4317 \\ -0.8588 & -0.1921 & 0.0231 \end{bmatrix},$$

$$U = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix},$$

where

$$U_1 = \begin{bmatrix} -0.1547 & 3.1568 & 0.6241 \\ 6.8677 & 10.0520 & 2.3804 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -43.1943 & -49.0157 & -13.9290 \\ 40.9437 & 49.2220 & 14.2804 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} 73.3271 & 96.2555 & 32.1438 \\ 1.1131 & 6.3128 & -0.0789 \end{bmatrix}.$$

Finally the stabilising gain is

$$G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix},$$

with

$$G_1 = \begin{bmatrix} -1.4980 & -4.2654 & -1.0348 \\ -3.2297 & -5.9039 & -1.3484 \\ 5.3591 & 5.8468 & 1.4753 \\ -2.5019 & -5.0023 & -1.0008 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 16.6137 & 18.0666 & 4.5725 \\ -4.8081 & -6.8264 & -2.0368 \\ 56.9213 & 66.7259 & 19.2758 \\ 2.6141 & 2.1981 & 1.1680 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} -47.1101 & -62.5076 & -20.5948 \\ -24.0331 & -34.1712 & -10.2761 \\ -41.7793 & -51.2486 & -18.7387 \\ -28.9114 & -40.0829 & -12.3430 \end{bmatrix}.$$

3.7 Summary

In this chapter a review of some preliminary results of the geometric approach developed in [51] for 1-D systems was give. The concepts and definitions were generalised for the 3-D Fornasini-Marchesini model based on the geometric concepts used in the 2-D case in [45, 50, 44] and some generalisations thereof.

The conditioned invariants for 3-D systems were introduced and their important of input-containing subclass was illustrated. Moreover, internal and external stability of 3-D systems under the invariance property was investigated.

Finally existance of an stabilising gain was studied and a method for their construction was developed. The development here results in an LMI based procedure for the synthesis of the asymptotic observer that reconstructs the local state of the FM model such that the difference between the estimated state and the original state tends to zero as the system evolves.

Significant role of this chapter in FDI becomes apparent later in Chapter 4 and 6 while addressing the problem in 3-D systems and repetitive processes respectively.

Chapter 4

Fault Detection and Isolation in 3-D Systems

4.1 Introduction

Fault detection and isolation is a crucial part of designing high reliability systems that can in some cases be required to operate for several years often out of reach, e.g., space. The topic of multidimensional systems continues to provide challenging problems which arise in the continuously expanding domain of applications. For example, recent advances in technology have given rise to applications where three dimensions are involved in the process. These applications range from a three-dimensional task-specific robotic arm for facilitating stroke rehabilitation [15] to new methods for distributed information processing in Grid Sensor Networks (GSN) using 3-D Fornasini-Marchesini (FM) model [16, 17]. These are important applications where handling a failure upon occurrence is very important. For instance, if an actuator that moves the robotic arm in a certain direction breaks down or a node in a grid sensor network dies so that local information updating becomes impossible, these failures prevent the whole system from operating and can cause considerable damage.

The need for these multidimensional systems has led to and extensive study of the subject by various researchers employing different approaches. For example, [52] considers the fault detection and isolation problems for two-dimensional state-space models using dead-beat observers. However, in reality designing a dead-beat observer is not always feasible since a large gain is required to stabilise the system in a short period of time. Moreover, there has been no research effort to address the fault detection and isolation problem specifically in 3-D systems. In this chapter the FDI problem for 3-D systems is investigated using the geometric approach developed in Chapter 3.

The nominal (i.e. fault-free) plant is assumed to be described by (2.10). To model the dynamics of the system after a sensor or actuator failure, following Massoumnia (see [1]) the nominal model is augmented with additional terms that represent the failure modes:

$$
\begin{aligned}
\sigma_1\sigma_2\sigma_3 x &= A_1\sigma_2\sigma_3 x + A_2\sigma_1\sigma_3 x + A_3\sigma_1\sigma_2 x \\
&+ B_1\sigma_2\sigma_3 u + \begin{bmatrix} L_1^1 & \dots L_1^{k_1} \end{bmatrix} \sigma_2\sigma_3 \begin{bmatrix} m_1^1 \\ \vdots \\ m_1^{k_1} \end{bmatrix} \\
&+ B_2\sigma_1\sigma_3 u + \begin{bmatrix} L_2^1 & \dots L_2^{k_2} \end{bmatrix} \sigma_1\sigma_3 \begin{bmatrix} m_2^1 \\ \vdots \\ m_2^{k_2} \end{bmatrix} \\
&+ B_3\sigma_1\sigma_2 u + \begin{bmatrix} L_3^1 & \dots & L_3^{k_3} \end{bmatrix} \sigma_1\sigma_2 \begin{bmatrix} m_3^1 \\ \vdots \\ m_3^{k_3} \end{bmatrix} \\
y &= Cx + \begin{bmatrix} J^1 & \dots & J^{p'} \end{bmatrix} \begin{bmatrix} n^1 \\ \vdots \\ n^{p'} \end{bmatrix},
\end{aligned}
\tag{4.1}
$$

where $m_i^k \in \left(\mathbb{R}^{\ell_i^k}\right)^{\mathbb{Z}^3}$, $n^j \in (\mathbb{R}^{p_j})^{\mathbb{Z}^3}$, and the matrices $L_i^k \in \mathbb{R}^{n\times\ell_i^k}$, $i = 1, 2, 3$, $k = 1, \dots, k_i$ and $J^k \in \mathbb{R}^{p\times p_k}$, $k = 1, \dots, p$ are called the *actuator*- and the *sensor failure signatures*, respectively.

Under fault-free conditions $m_i^k = 0$ and $n^h = 0$ for $1 \leq i \leq 3$, $1 \leq k \leq k_i$, and $1 \leq h \leq p'$, and the model (4.1) reduces to (2.10). To model, for example, the effect of a complete failure in the j-th actuator in the i-th independent variable, set $L_i^j = B_i^j$ where B_i^j is the j-th column of the input matrix B_i, and $m_i^j = -u_i^j$. Moreover, in the case of a bias in the j-th actuator in the i-th independent variable, m_i^j is set to a positive constant $\alpha \in \mathbb{R}^+$, and $L_i^j = B_i^j$ as in the dead actuator case. Other types of failures (possibly affecting also the dynamics of the system as represented in the matrices A_i, $i = 1, 2, 3$) can be accommodated in this framework; see section III of [1] for more details.

For simplicity of exposition, the following assumptions are made

Observability: The representation (2.10) is observable (in the sense of [24], p. 65);

Actuator-only faults: In (4.1) $J^k = 0_{p\times p_k}$, $k = 1, \dots, p'$;

Unambiguous failure modes: The failure signature matrix L_i^k has full column rank, $1 \leq i \leq 3$, $1 \leq k \leq k_i$;

No simultaneous failures: If there exist $1 \leq \bar{i} \leq 3$ and $1 \leq \bar{k} \leq k_i$ such that $m_{\bar{i}}^{\bar{k}} \neq 0$, then $m_i^h = 0$ for $i \neq \bar{i}$, and $h \neq \bar{k}$.

4.2 Failure detection and identification

To perform failure detection and identification, the aim is set to design an asymptotic observer for the nominal plant (2.10) and the failure model (4.1) that, under the assumptions stated at the end of Section 4.1, takes as inputs the input and output plant variables, and produces as output a residual which asymptotically provides information about the presence and the location of the failure. In this section this idea is formalised and necessary and sufficient conditions for the FDI problem to be solvable are developed.

The dynamics of the observer we will be designing are

$$\begin{aligned} \sigma_1\sigma_2\sigma_3\hat{x} &= A_1\sigma_2\sigma_3\hat{x} + A_2\sigma_1\sigma_3\hat{x} + A_3\sigma_1\sigma_2\hat{x} + B_1\sigma_2\sigma_3 u + B_2\sigma_1\sigma_3 u + B_3\sigma_1\sigma_2 u \\ &\quad -G_1(\sigma_2\sigma_3 y - \sigma_2\sigma_3\hat{y}) - G_2(\sigma_1\sigma_3 y - \sigma_1\sigma_3\hat{y}) - G_3(\sigma_1\sigma_2 y - \sigma_1\sigma_2\hat{y}) \\ \hat{y} &= C\hat{x} \ , \end{aligned} \tag{4.2}$$

where $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$, are the gain matrices; $\hat{x}$ is the state estimate; and $\hat{y} = C\hat{x}$ is the corresponding output. The difference $\hat{x} - x$ is termed the *error vector*, denoted by e, and $\hat{y} - y$ is the *residual vector*, denoted by $r = Ce$.

If no faults have occurred, the plant dynamics are described by (2.10), and consequently the error- and the residual dynamics are described by

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 e &= (A_1 + G_1C)\sigma_2\sigma_3 e + (A_2 + G_2C)\sigma_1\sigma_3 e + (A_3 + G_3C)\sigma_1\sigma_2 e \ ; \\ r &= Ce \ . \end{aligned} \tag{4.3}$$

In the presence of an actuator fault described as in (4.1), subtracting (2.10) from (4.1) and rearranging the resulting equations yields the following description of the error dynamics:

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 e &= (A_1 + G_1C)\sigma_2\sigma_3 e + (A_2 + G_2C)\sigma_1\sigma_3 e + (A_3 + G_3C)\sigma_1\sigma_2 e \\ &\quad -\sum_{k=1}^{k_1} L_1^k\sigma_2\sigma_3 m_1^k - \sum_{k=1}^{k_2} L_2^k\sigma_1\sigma_3 m_2^k - \sum_{k=1}^{k_3} L_3^k\sigma_1\sigma_2 m_3^k \ . \end{aligned} \tag{4.4}$$

Under the assumptions stated at the end of section 4.1, the error dynamics corresponding to a single failure in the k-th actuator of B_i is described by

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 e &= (A_1 + G_1C)\sigma_2\sigma_3 e + (A_2 + G_2C)\sigma_1\sigma_3 e + (A_3 + G_3C)\sigma_1\sigma_2 e \\ &\quad -L_i^k\sigma_p\sigma_q m_i^k \ , \quad i, p, q = 1, 2, 3, \quad i \neq p \neq q \ . \end{aligned} \tag{4.5}$$

with arbitrary boundary conditions $\{e(n_1, n_2, n_3) \mid n_1 + n_2 + n_3 = 0\}$, and m_i^k is an *unknown* input.

Disregarding for the moment the essential property of asymptotic stability for the observer (4.2), the dynamics of (4.5) are next considered under *zero boundary conditions* $\{e(n_1, n_2, n_3) = 0 \mid n_1 + n_2 + n_3 = 0\}$.

4.2.1 A special case: zero boundary conditions

Given the assumptions stated at the end of section 4.1, it is straightforward to verify that under the dynamics (4.5) with zero boundary conditions, the error vector $e(n_1, n_2, n_3)$ for $n_1 + n_2 + n_3 = \ell$ belongs to

$$R_i^k := \left[\left(A_1 + G_1 C \quad {}^{n_1 - m} \underset{A_2 + G_2 C}{\overset{n_2 - m}{\sqcup}} {}^{n_3 - m} \quad A_3 + G_3 C \right) L_i \right] \; , \; \begin{cases} m = 1 & \text{index}(n) = i \\ m = 0 & \text{otherwise} \end{cases} . \tag{4.6}$$

Moreover, define the associated (G_1, G_2, G_3)-dependent subspace by

$$\mathcal{V}_i^k := \text{Im} \ (R_i^k) \; , \tag{4.7}$$

Then the following gives a geometric characterization of $\mathcal{V}_i^k$.

Proposition 4.1. *Let $\mathcal{V}_i^k$ be defined as in (4.7), where R_i^k is defined as in (4.6). Then $\mathcal{V}_i^{k\star}$ is the smallest $(A_1 + G_1 C, A_2 + G_2 C, A_3 + G_3 C)$-invariant subspace containing im (L_i^k).*

Proof. It follows from the definition of R_i^k and the fact that the subspace spanned by R_i^k is the reachability subspace which is the smallest $(A_1 + G_1 C, A_2 + G_2 C, A_3 + G_3 C)$-invariant subspace containing im (L_i^k). □

Clearly, $\mathcal{V}_i^k$ is an input-containing (A_H, C_D)-conditioned invariant for the system described by

$$\sigma_1 \sigma_2 \sigma_3 x \quad = \quad A_1 \sigma_2 \sigma_3 x + A_2 \sigma_1 \sigma_3 x + A_3 \sigma_1 \sigma_2 x - L_i^k \sigma_p \sigma_q m_i^k \; , \quad i, p, q = 1, 2, 3, \quad i \neq p \neq q \; .$$

containing Im L_i^k.

The FDI problem can now be stated in the geometric setting as:

Find $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$ such that the family $\mathcal{V}_i^k$ defined by (4.7) are independent. i.e., they satisfy

$$\mathcal{V}_{\bar{i}}^{\bar{k}} \bigcap \left(\sum_{k\neq\bar{k}} \mathcal{V}_{\bar{i}}^{k} + \sum_{i\neq\bar{i}} \sum_{k} \mathcal{V}_i^k \right) = \{0\} \ ,$$

for all $\bar{i} = 1, 2, 3$, $\bar{k} = 1, \dots, q$. Given that $\mathcal{V}_i^k$ is (A_H, C_D)-invariant in the sense of Definition 3.15; the problem can be equivalently reformulated as follows:

FDI problem with zero boundary conditions

Find subspaces $\mathcal{V}_i^k$, $i = 1, 2, 3$, $k = 1, \dots, k_i$, such that

(a) There exist $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$ such that

$$(A_i + G_i C)\mathcal{V}_i^k \subset \mathcal{V}_i^k \qquad i = 1, 2, 3, \qquad k = 1, \dots, k_i \ ;$$

(b) $\text{im}(L_i^k) \subseteq \mathcal{V}_i^k$;

(c) $\mathcal{V}_{\bar{i}}^{\bar{k}} \bigcap \left(\sum_{k\neq\bar{k}} \mathcal{V}_{\bar{i}}^{k} + \sum_{i\neq\bar{i}} \sum_{k} \mathcal{V}_i^k \right) = \{0\}$,
for all $\bar{i} = 1, 2, 3$, $\bar{k} = 1, \dots, k_i$.

Now define $\mathcal{W}_i^{k\star}$ to be the *smallest* (A_H, C_D)*-invariant subspace containing* $\text{im}(L_i^k)$; note that $\mathcal{W}_i^{k\star}$ only depends on A_i, $i = 1, 2, 3$, C, and Im (L_i^k). $\mathcal{W}_i^{k\star}$ can be computed in a manner analogous to the recursion (3.37). Recall also from section 3.5.2 the definition of $\mathcal{S}_i^{k\star} := \mathcal{S}(\text{im}(L_i^k))^\star$, the smallest unobservability subspace containing Im (L_i^k). The following result shows that the considered problem is solvable if and only if the family $\{\mathcal{W}_i^{k\star}\}$, or equivalently the family $\{\mathcal{S}_i^{k\star}\}$, satisfies condition (c).

Theorem 4.2. *The following statements are equivalent:*

I. The residual generation problem with zero boundary conditions is solvable;

II. The family $\{\mathcal{W}_i^{k\star}\}_{\substack{i=1,2,3;\\ k=1,\dots,k_i}}$ *of smallest* (A_H, C_D)*-invariant subspaces containing* $\text{im}(L_i^k)$ *satisfies condition* (c)*;*

III. The family $\{\mathcal{S}_i^{k\star}\}_{\substack{i=1,2,3;\\ k=1,\dots,k_i}}$ *of smallest unobservability subspaces containing* $\text{im}(L_i^k)$ *satisfies condition* (c).

Proof. $((I) \implies (II))$ Follows from the minimality of the $\mathcal{W}_i^{k\star}$, that satisfy $\mathcal{W}_i^{k\star} \subseteq \mathcal{V}_i^k$, $i = 1, 2, 3$, $k = 1, \dots, q$ for any family $\{\mathcal{V}_i^k\}$ satisfying $(a) - (c)$.

$((II) \Longrightarrow (I))$ By definition the $\mathcal{W}_i^{k\star}$ are (A_H, C_D)-invariant, but it must be shown that the *same* $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$, can be found such that conditions $(a)-(c)$ are satisfied for $\mathcal{W}_i^{k\star}$. For this purpose, write each $\mathcal{W}_i^{k\star}$ as the following direct sum:

$$\mathcal{W}_i^{k\star} = \mathcal{V}_i^{k\star} \dot{\oplus} \left(\mathcal{W}_i^{k\star} \bigcap \ker\, C \right) ,$$

where $\mathcal{V}_i^{k\star}$ is some suitable subspace of $\mathbb{R}^n$, and let $W_i^{k\star} \in \mathbb{R}^{n\times\bullet}$ be a basis matrix for $\mathcal{W}_i^{k\star}$ structured according to such a decomposition, i.e. $W_i^{k\star} = \begin{bmatrix} V_i^{k\star} & W_i^{k\star\prime} \end{bmatrix}$, with the columns of $V_i^{k\star}$ spanning $\mathcal{V}_i^{k\star}$ and those of $W_i^{k\star\prime}$ spanning $\mathcal{W}_i^{k\star} \bigcap \ker C$. Moreover, condition (c) is equivalent to

$$\mathcal{V}_{\bar{i}}^{\bar{k}\star} \bigcap \left(\sum_{k\neq\bar{k}} \mathcal{V}_{\bar{i}}^{k\star} + \sum_{i\neq\bar{i}} \sum_{k} \mathcal{V}_i^{k\star} \right) = \{0\},$$

for all $\bar{i} = 1, 2, 3$, $\bar{k} = 1, \ldots, k_{\bar{i}}$. This last condition is equivalent to the matrix

$$\begin{bmatrix} V_1^1 & \ldots & V_1^{k_1} & V_2^1 & \ldots & V_2^{k_2} & V_3^1 \ldots & V_3^{k_3} \end{bmatrix}$$

having full column rank. Hence each of the equations

$$\begin{aligned} A_i \begin{bmatrix} V_1^1 & \ldots & V_1^{k_1} & V_2^1 & \ldots & V_2^{k_2} & V_3^1 \ldots & V_3^{k_3} \end{bmatrix} = \\ - G_i C \begin{bmatrix} V_1^1 & \ldots & V_1^{k_1} & V_2^1 & \ldots & V_2^{k_2} & V_3^1 \ldots & V_3^{k_3} \end{bmatrix} , \quad i = 1, 2, 3 , \end{aligned}$$

has a solution G_i.

$((II) \Longleftrightarrow (III))$ Denote the non-observable subspace of (A_H, C_D) by $\mathcal{N}$. Recall from Proposition 3.21 that $\mathcal{S}_i^{k\star} = \mathcal{W}_i^{k\star} + \mathcal{N}$. Consequently, $C\mathcal{S}_i^{k\star} = C\mathcal{W}_i^{k\star}$, thus proving that the family $\{\mathcal{S}_i^{k\star}\}$ satisfies condition (c) if and only if the family $\{\mathcal{W}_i^{k\star}\}$ also satisfies this condition. □

Remark 4.3. With reference to the proof of Theorem 4.2, note that a set of gains G_i, $i = 1, 2, 3$ can be computed as $G_i := -A_i V \left[(CV)^\top CV\right]^{-1} (CV)^\top$, where $V := \begin{bmatrix} V_1^1 & \ldots & V_1^{k_1} & V_2^1 & \ldots & V_2^{k_2} & V_3^1 \ldots & V_3^{k_3} \end{bmatrix}$, and the columns of $\begin{bmatrix} V_i^1 & \ldots & V_i^{k_i} \end{bmatrix}$ form a basis for $\mathcal{W}_i^{k\star} \setminus \left(\mathcal{W}_i^{k\star} \bigcap \ker\, C\right)$. A similar procedure yields a set of gains for the family $\{\mathcal{S}_i^{k\star}\}$.

Remark 4.4. Given the current state of the art, it is unclear whether given a subspace $\mathcal{L}$, stabilising $\mathcal{S}(\mathcal{L})^\star$ rather than $\mathcal{W}(\mathcal{L})^\star$ gives any more freedom in the assignment of the external dynamics of an asymptotic observer, as it is the case with 1-D observers. However, the result of Proposition 3.21 implies that for the former choice in the light of Theorem 4.2, the LMI to be solved (see Section 3.5.3.2) is of smaller dimension.

4.2.2 The general case: asymptotic observers for fault detection

The result of Theorem 4.2 is a necessary *structural* requirement on the system: unless the subspaces $\mathcal{W}_i^{k\star}$ (equivalently, $\mathcal{S}_i^{k\star}$) satisfy condition (c), even in the special case of zero error in the boundary conditions of the system, i.e. $x(n_1, n_2, n_3) = \hat{x}(n_1, n_2, n_3)$ for all (n_1, n_2, n_3) such that $n_1 + n_2 + n_3 = 0$, fault detection with an observer (4.2) is impossible. The case in which the boundary conditions of the observer are exactly the same as those of the plant hardly ever occurs and it is necessary to introduce stability into problem formulation.

It is easy to verify that the sum of conditioned invariants is in general not a conditioned invariant. However, the following result shows that this property holds for the subspaces $\mathcal{W}_i^{k\star}$ defined in Theorem 4.2 (see also Lemma 4 p. 842 of [1]).

Proposition 4.5. *Let $\mathcal{W}_i^{k\star}$, $i = 1, 2, 3$, $k = 1, \ldots, k_i$, be the smallest (A_H, C_D)-invariant subspace containing Im (L_i^k). Denote by $\mathcal{W}^\star$ the smallest (A_H, C_D)-invariant subspace containing $\sum_{i=1}^3 \sum_{k=1}^{k_i}$ Im (L_i^k). Assume that the family $\{\mathcal{W}_i^{k\star}\}$ satisfies condition (c); then*

$$\mathcal{W}^\star = \sum_{i=1}^{3} \sum_{k=1}^{k_i} \mathcal{W}_i^{k\star} \; . \tag{4.8}$$

Proof. It is first shown that $\mathcal{W}^\star \subseteq \sum_{i=1}^3 \sum_{k=1}^{k_i} \mathcal{W}_i^{k\star}$. By the argument of the implication $(II) \Longrightarrow (I)$ in Theorem 4.2, it follows that there exist G_i such that $\mathcal{W}_i^{k\star}$ is $(A_i + G_i C)$-invariant, $i = 1, 2, 3$. This implies that $\sum_{i=1}^3 \sum_{k=1}^{k_i} \mathcal{W}_i^{k\star}$ is also an (A_H, C_D)-invariant. Surely, this subspace contains $\sum_{i=1}^3 \sum_{k=1}^{k_i}$ Im (L_i^k), and since $\mathcal{W}^\star$ is the smallest (A_H, C_D)-invariant containing it, the inclusion follows.

In order to prove the converse inclusion, observe that for all $\bar{i} = 1, 2, 3$ and $\bar{k} = 1, \ldots, k_i$, it holds that Im $(L_{\bar{i}}^{\bar{k}}) \subset \sum_{i=1}^3 \sum_{k=1}^{k_i}$ Im (L_i^k). Since $\mathcal{W}^\star$ is (A_H, C_D)-invariant, it follows that $\mathcal{W}_{\bar{i}}^{\bar{k}\star} \subset \mathcal{W}^\star$ and consequently the required inclusion also holds. This concludes the proof. □

The observer (4.2) is said to solve the *asymptotic residual generation problem* if for arbitrary boundary conditions $\hat{x}_{|\mathcal{S}_k}$, asymptotically the residual r is either zero (if there is no fault) or (if a fault occurs) it belongs to one, and only one, of the subspaces $C\mathcal{W}_i^{k\star}$, thus allowing the unique identification of the fault. A sufficient condition is given in the following result.

Theorem 4.6. *Let $\mathcal{W}_i^{k\star}$, $i = 1, 2, 3$, $k = 1, \ldots, k_i$, be the smallest (A_H, C_D)-invariant subspace containing Im (L_i^k). Denote by $\mathcal{W}^\star$ the smallest (A_H, C_D)-invariant subspace containing $\sum_{i=1}^3 \sum_{k=1}^{k_i}$ Im (L_i^k). Assume that the family $\{\mathcal{W}_i^{k\star}\}$ satisfies condition (c), and that $\mathcal{W}^\star$ is internally and externally stabilizable. Then there exist $G_i \in \mathbb{R}^{n \times p}$, $i = 1, 2, 3$, such that the observer (4.2) solves the asymptotic residual generation problem.*

Proof. Since $\mathcal{W}^\star$ is an internally and externally (A_H, C_D)-conditioned invariant, gain matrices $G_i \in \mathbb{R}^{n\times p}$, $i = 1, 2, 3$ can be determined, to construct an observer of the form (4.2). Next, how the observer functions in the two situations when there is no fault, or when a fault has occurred is investigated.

Assume that no fault has occurred; then the dynamics of the error are described by (4.3). Since $\mathcal{W}^\star$ is externally stable, the dynamics of the error due to the component of the boundary conditions $\hat{x}_{|\mathcal{S}_0}$ lying outside of $\mathcal{W}^\star$ is asymptotically stable, and consequently tends to zero in time. Since $\mathcal{W}^\star$ is internally stable, the dynamics of the error due to the component of the boundary conditions in $\mathcal{W}^\star$ is also asymptotically stable, and consequently goes to zero. Hence, asymptotically the error vector is zero and consequently the residual is also zero.

Now consider instead the case when one fault has occurred, for example corresponding to the error signature L_i^k; then the dynamics of the error is described by (4.5), with $m_i^k \in \left(\mathbb{R}^{\ell_i^k}\right)^{\mathbb{Z}^3}$ nonzero. Asymptotically the error vector lies in $\mathcal{W}^\star$; consequently the residual corresponding to it lies in the subspace $\mathcal{W}^\star = \left(\sum_{i=1}^{3}\sum_{k=1}^{k_i}\mathcal{W}_i^{k\star}\right)$. Since condition (c) holds, it is possible by projecting the error vector onto the subspaces $\mathcal{W}_i^{k\star}$ to determine which type of fault the error corresponds to. □

Example 4.1. Consider a 3-D system is described by the following state-space model matrices:

$$A_1 = \begin{bmatrix} -\frac{92}{525} & -\frac{1}{2100} & \frac{23}{700} \\ \frac{1}{210} & -\frac{379}{2100} & \frac{23}{700} \\ -\frac{23}{315} & \frac{23}{3150} & -\frac{27}{350} \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{79}{135} & \frac{1}{54} & \frac{7}{30} \\ -\frac{5}{27} & -\frac{103}{270} & \frac{7}{30} \\ -\frac{14}{27} & \frac{7}{135} & \frac{2}{15} \end{bmatrix}, \quad A_3 = \begin{bmatrix} -\frac{35}{36} & \frac{13}{180} & \frac{2}{5} \\ -\frac{13}{18} & \frac{-8}{45} & \frac{2}{5} \\ -\frac{8}{9} & \frac{4}{45} & \frac{3}{10} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \frac{1286}{2541} & \frac{1}{5} \\ \frac{23}{5000} & \frac{2}{5} \\ \frac{885}{889} & \frac{1}{10} \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{773}{1047} & \frac{4}{5} \\ \frac{1213}{2153} & \frac{3}{10} \\ \frac{531}{625} & \frac{7}{10} \end{bmatrix} \quad B_3 = \begin{bmatrix} \frac{21}{5} & \frac{23}{10} \\ \frac{7}{2} & \frac{1}{5} \\ \frac{17}{10} & \frac{56}{5} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1944}{2021} & \frac{389}{502} & \frac{655}{754} \\ \frac{23}{5000} & \frac{1011}{1237} & \frac{211}{2500} \\ \frac{1}{10} & \frac{1}{2} & \frac{3}{10} \end{bmatrix}.$$

One possibility to obtain the observer gain $G = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix}$ such that the subspace in which the error vector lies, is $(A_i + G_C)$-invariant, is to solve the following equation:

$$G_iC\begin{bmatrix} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \end{bmatrix} = -A_i\begin{bmatrix} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \end{bmatrix}, \quad i = 1, 2, 3,$$

which yields:

$$G_1 = \begin{bmatrix} 0.2899 & 0.3699 & -1.0530 \\ 0.0221 & 0.3697 & -0.2777 \\ 0.0656 & -0.1356 & 0.1053 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1.0415 & 1.5832 & -4.2389 \\ 0.4465 & 1.5826 & -2.5160 \\ 0.8767 & 1.1237 & -3.2991 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} 1.7355 & 2.6046 & -7.0916 \\ 1.3637 & 2.6042 & -6.0147 \\ 1.5459 & 2.1450 & -6.0799 \end{bmatrix}.$$

It is clear that in this case, $A_i + G_iC = 0$. Other gains can be obtained as discussed in Section 3.6 or using the Matlab Geometric Toolbox [53] routines such as `sstar`.

Next detection and isolation of both types of failures in this system are investigated.

I. **Biased actuator**: Consider that a bias emerges in the second actuator (i.e., L_2) when the system evolution reaches the plane $n_1 + n_2 + n_3 = 36$. Figure 4.1 illustrates the error evolution corresponding to this bias.

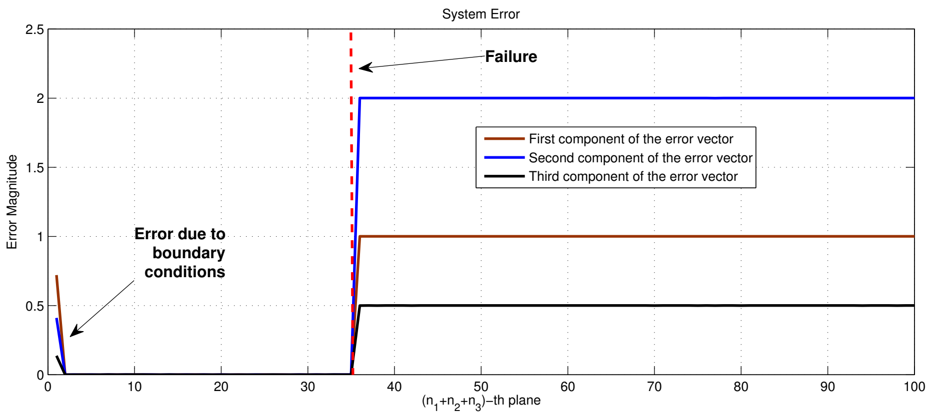

FIGURE 4.1: System's error evolution before and after a bias

It can be seen that since the eigenvalues of $A_i + G_iC$ are assigned to zero, after one iteration, the error immediately goes to zero and after the occurrence of the failure, due to modelling the bias with a positive constant as discussed previously, the error rises and maintains a constant level thereafter.

Comparing the error magnitude to a predefined threshold enables detection of the presence of failure and diagnosis of the failure type. To identify the failed component of the system, the error vector is projected onto the subspaces spanned by the failure signatures. This is illustrated in Figure 4.2.

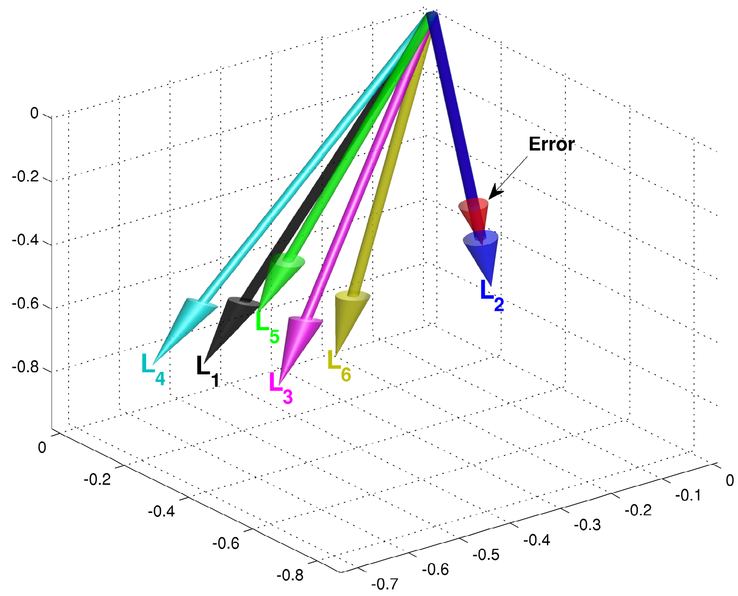

FIGURE 4.2: Error vector lying in the subspace spanned by the biased actuator

II. **Dead actuator**: Now consider the case in which the system evolution encounters a failure due to the complete failure of L_5 once the system reaches the 45th plane (i.e., $n_1+n_2+n_3 = 45$). The error evolution for this scenario is shown in Figure 4.3. Similar to the biased actuator case, since the eigenvalues of the dynamics of the observer are set to zero, the error immediately goes to zero after one iteration. Thereafter it remains constantly zero until it reaches the failure stage where it gradually rises and fluctuates randomly. Note that only by comparing the error magnitude to a threshold, it is possible to detect presence of a failure and failure type since each failure mode produces a structurally different type of error.

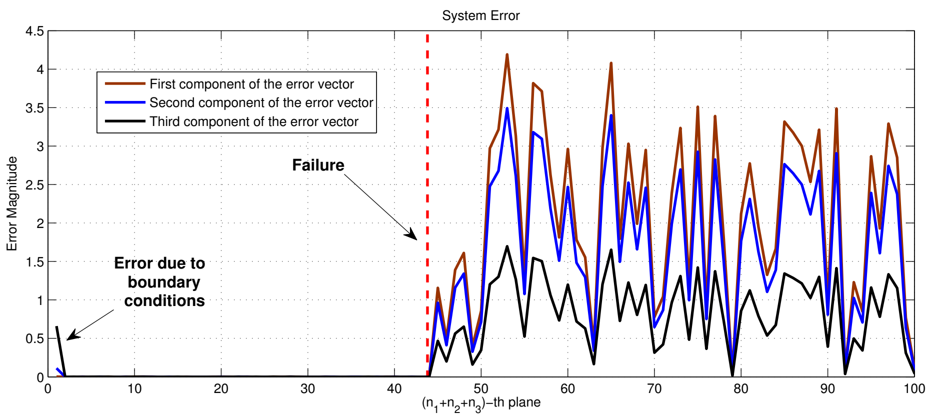

FIGURE 4.3: System's error evolution before and after a complete failure

Similar to the biased actuator scenario, by projecting the the error vector onto the subspaces spanned by the failure signatures one can identify the failed actuator. This is illustrated in Figures 4.4.

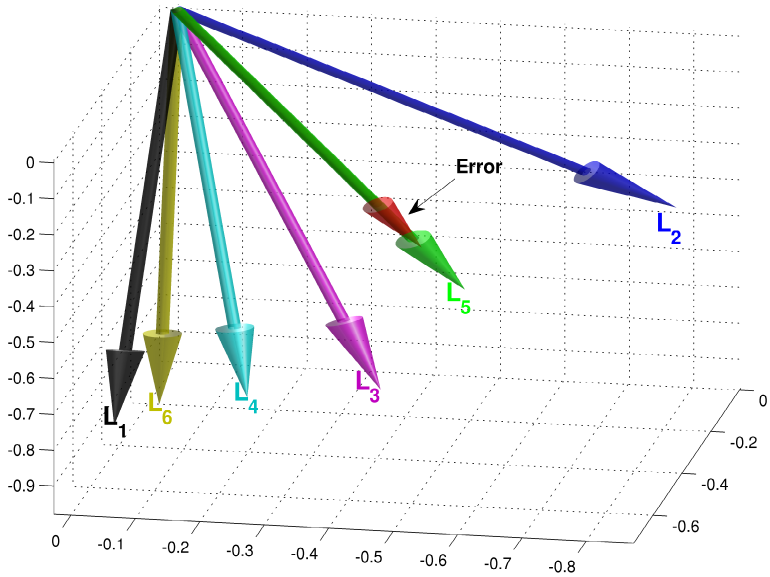

FIGURE 4.4: Error vector lying in the subspace spanned by the failed actuator

4.3 Summary

In this chapter using the geometric approach developed in Chapter 4, a framework for FDI in 3-D systems was developed. To this purpose, a 3-D Fornasini-Marchesini model that incorporates the failure signatures and modes is introduced. The major results of this chapter are Theorem 4.2, which gives structural necessary conditions for the FDI to be possible, and Theorem 4.6, which gives sufficient conditions are given for the existence of an asymptotic observer to perform FDI.

Two types of failure have been defined and modelled: 1. Biased actuator ; 2. Dead actuator, under each of which the system behaviour is unique, therefore enabling the type of the failure occurred in the system to be determined.

Additionally, an example demonstrating both types of failure considered has also been given. This illustrates how the geometric fault detection algorithms developed in this chapter can uniquely isolate the faults considered.

Chapter 5

Linear Repetitive Processes

5.1 Introduction

Linear Repetitive Processes were first introduced as multi-pass processes in the early 70's as a result of a work on modelling and control of metal rolling and long wall coal cutting operations.

Repetitive processes were defined as those involving the process of a material or a workpiece by a sequence of "sweeps" or passes of the processing tool. During each pass an output, termed as the pass profile, is generated which then contributes to the dynamics of the following passes. I.e., the output dynamics on any pass acts as a forcing function on, and hence contributes to, the dynamics of the next pass. It is this interaction between passes which leads to the unique control problem associated with these processes where oscillations can occur in the sequence of output pass profiles that increase in amplitude from pass to pass.

Linear repetitive processes are inherently 2-D in nature since two variables are required to specify each point - the time or the distance along the pass and the pass number.

These processes are especially interesting since many industrial applications such as multi-layer printing, metal rolling, long wall coal cutting and bench-mining operations can be modelled as repetitive processes. Moreover, in recent years applications have emerged where adopting a repetitive process setting for analysis has distinct advantages over the classic alternatives, including classes of Iterative Learning Control (ILC) laws [54] and iterative algorithms for solving non-linear dynamic optimal stabilisation problems based on the maximum principle [55].

In this chapter a brief introduction of linear repetitive processes is given. Moreover, a state-space representation along with the abstract Banach space [56] model, on which the stability analysis for these processes is based, is introduced. Following [57, 58, 59, 60], a review of the stability theory for these processes is also provided.

5.2 Basic Features of Linear Repetitive Processes and Their Mathematical Representation

Attempts to control repetitive processes using standard or 1-D techniques in general fail, since they ignore the inherently two-dimensional information propagation in these processes - along the pass and from pass to pass. [25, 58].

Essentially, any general model of repetitive processes should incorporate all their unique features. For example, in the most general case, a repetitive process has non-linear dynamics and a variable pass length. For this general case, these features then can be outlined as follows [58]:

- A number of passes, indexed by $k \geq 0$, through a set of dynamics.
- Each pass is characterised by a pass length α_k, and a pass profile $y_k(p)$ defined on $0 \leq p \leq \alpha_k$, where $y_k(p)$ can be a vector or a scalar quantity.
- An initial pass profile $y_0(k)$ defined on $0 \leq p \leq \alpha_0$, where α_0 is the initial pass length. The function $y_0(p)$ together with the initial conditions on each pass form the boundary conditions for the process.
- Each pass is subject to its own disturbances and control inputs.
- The process can be of unit memory, i.e., the dynamics on pass $k + 1$ explicitly depend on the independent inputs to the pass and the pass profile on the previous pass k.

Figure 5.1 illustrates some of these essential features.

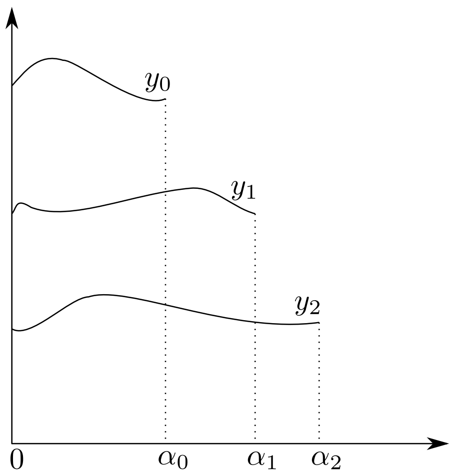

FIGURE 5.1: Graphical representation of a sequence of pass profiles

Clearly to analyse such a process would be a formidable task and thus as yet, research efforts have been limited to linear processes with a constant pass length α with the justification that the practical applications mostly fall into this category [58].

5.2.1 A General Abstract Representation

A mathematical formulation of a linear repetitive process with constant pass length α has been proposed in [25] based on an abstract model in a Banach space setting. Such models include all previously studied examples such as discrete repetitive processes as special cases and is the basis of the stability theory for these processes.

Suppose y_k is regarded as a point in a suitably chosen function space. In particular, suppose that $y_k \in E_k,\ k \geq 0$, where E_k denotes an appropriately chosen Banach space. Then a general abstract model for repetitive processes can be formulated as a recursion equation of the form

$$y_{k+1} = f_k\ y(k)\ , \quad k \geq 0 \tag{5.1}$$

where f_{k+1} is an abstract mapping of E_k into E_{k+1}.

Repetitive processes also exist where the current pass profile is a function of the independent inputs to that pass and a finite number $M > 1$ of previous pass profiles. An example for these processes is so-called bench-mining systems [61] and M is termed as the memory length of the system. These processes are designated as "non-unit memory of length 'M" or, simply, "non-unit memory", and are easily accommodated within the general structure of (5.1). Essentially, all that is required is to replace this equation by

$$y_{k+1} = \tilde{f}_{k+1}(y_k, y_{k-1}, \cdots, y_{k+1-M})\ . \quad k \geq 0\ ,$$

In the case of processes with linear dynamics, the following definition characterises a so-called unit-memory linear repetitive process in a Banach space setting and forms the basis for onward development and in particular stability theory.

Definition 5.1. [57, Def. 1.2.1] A *linear repetitive process* S of constant pass length $\alpha > 0$ consists of a Banach space E, a linear subspace $\mathcal{W}$ of E, and a bounded linear operator L mapping E into itself (also denoted by $L \in B(E, E)$). Then the system dynamics are described by the following linear recursion relation

$$y_{k+1} = Ly_k + b_{k+1}\ , \quad k \geq 0 \tag{5.2}$$

where $y_k \in E$ is the pass profile on pass k and $b_{k+1} \in \mathcal{W}$. Here the term Ly_k represents the contribution from pass k to pass $k+1$ and b_{k+1} represents initial conditions, disturbances and control input effects.

In the non-unit memory case, let $L^{(j)} \in B(E, E)$, $1 \leq j \leq M$. Then the abstract model of a non-unit memory linear repetitive process of memory length M has the following dynamics

$$y_{k+1} = L_\alpha^{(1)} y_k + \cdots + L_\alpha^{(M)} y_{k+1-M} + b_{k+1}, \quad k \geq 0\,, \tag{5.3}$$

where $y_k \in E$, $b_{k+1} \in W \subset E$. Note that with $L \equiv L^{(1)}$, this last equation reduces to (5.2) in the case when $M = 1$. Moreover, (5.3) can be regarded as a unit-memory linear repetitive process S in the product space $E^M := E \times E \times \cdots \times E$ (M times) by writing it in the following 'companion form' where I denotes the identity operator on E.

$$\begin{bmatrix} y_{k+2-M} \\ \vdots \\ \vdots \\ \vdots \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & I \\ L^{(M)} & L^{(M-1)} & L^{(M-2)} & \cdots & L^{(1)} \end{bmatrix} \begin{bmatrix} y_{k+1-M} \\ \vdots \\ \vdots \\ \vdots \\ y_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_{k+1} \end{bmatrix}, \quad k \geq 0 \tag{5.4}$$

and using the notation

$$L_\alpha := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & I \\ L^{(M)} & L^{(M-1)} & L^{(M-2)} & \cdots & L^{(1)} \end{bmatrix}.$$

Thus results derived for the unit-memory case can be readily applied to the non-unit memory generalisation.

The abstract model presented here is rather general and can be used to represent a vast number of examples [57, Example 1.2.1 - 1.2.13]. For the interest of this thesis, here the representation of discrete linear repetitive processes is investigated only.

5.2.2 A Discrete Non-unit Memory Linear Repetitive Process Representation

A state-space model of a discrete linear repetitive process with pass length α and memory length M is given by:

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + \sum_{j=1}^{M-1} B_{j-1} y_{k+1-j}(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + \sum_{j=1}^{M} D_{j-1} y_{k+1-j}(p) \ , \end{aligned} \tag{5.5}$$

where $x_k(p) \in \mathbb{R}^n$, $u_k(p) \in \mathbb{R}^l$, $y_k(p) \in \mathbb{R}^m$ are respectively the state, input and output vectors on pass k at time instant p. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{m\times n}$, $B_j \in \mathbb{R}^{n\times m}$ are the state, input, output and memory matrices respectively. k denotes the pass index while the time index along each pass is denoted by p. Each pass has a length α (i.e., $0 \le p \le \alpha$). M is the memory length.

In this case, set $E_\alpha = \ell_2^m[0, \alpha]$ - the space of all real $m \times 1$ vectors of length α (corresponding to $p = 1, 2, \cdots, \alpha$). Then it is immediate to check that this model is a special case of S over $1 \le p \le \alpha$ with

$$(L_\alpha^{(j)} y)(p) := \sum_{h=0}^{p-1} CA^{p-1-h} B_0 y(h) + D_{j-1} y(p)$$

and over $1 \le p \le \alpha$, $k \ge 0$,

$$b_{k+1} := CA^p d_{k+1} + \sum_{h=0}^{p-1} CA^{p-1-h} Bu_{k+1}(h) + Du_{k+1}(p) \ .$$

The simplest possible set of boundary conditions for model (5.5) is

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \qquad k \ge 0 \ , \\ y_{1-j}(p) &= \hat{y}_{1-j}(p), \quad 1 \le j \le M \ , \ 0 \le p \le \alpha \ , \end{aligned} \tag{5.6}$$

where d_{k+1} is a $n \times 1$ known constant vector, $\hat{y}_{1-j}(p)$, $1 \le j \le M$ is a $m \times 1$ vector whose entries are known functions of p over $0 \le p \le \alpha$.

For simplicity of discussion, here we consider the following unit-memory repetitive model since all the obtained results generalise in an analogous manner to the case where $M > 1$:

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \ , \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \ , \end{aligned} \tag{5.7}$$

where the initial conditions are

$$\begin{aligned} x_{k+1}(0) &= d_{k+1} \ , \quad k \geq 0 \ , \\ y_0(p) &= \hat{y}_1(p) \ , \quad 0 \leq p \leq \alpha \ . \end{aligned} \tag{5.8}$$

It should be pointed out that the boundary conditions of (5.6) and (5.8) are the simplest possible and cases exist (for instance see [57, Example 1.2.6]) where they are not adequate to adequately model the underlying process dynamics (even for initial simulation and/or control analysis). Instead, it is sometimes necessary to consider a state initial vector sequence which is an explicit function of (in the unit memory case for simplicity) the previous pass profile. One possible form is:

$$x_{k+1}(0) = d_{k+1} + \sum_{j=1}^{N} J_j y_k(p_j) \ , \tag{5.9}$$

where d_{k+1} is as in (5.6), $0 \leq p_1 < p_2 < \cdots < P_N \leq \alpha$ are N sampling points along the previous pass and J_j, $1 \leq j \leq N$, is an $n \times m$ matrix with constant entries. One case of particular interest (the most general) is

$$x_{k+1}(0) = d_{k+1} + \sum_{j=1}^{\alpha-1} J_j y_k(j) \ . \tag{5.10}$$

5.3 Stability

Stability theory for repetitive processes consists of two separate concepts - *asymptotic stability* and the stronger condition of *stability along the pass*. This is somehow expected since these processes, as mentioned, are governed by two independent variables in two directions of along-the-pass and pass-to-pass.

In this section, results of applying the stability theory for the Banach space to the discrete form of the linear repetitive processes are presented. Consider the unit-memory state-space model of a discrete linear repetitive process (5.7). Then the following, characterises asymptotic stability and stability along the pass for these processes.

5.3.1 Asymptotic stability

The interesting control problem for repetitive processes is the possible presence of oscillations in the output sequence of pass profiles that increase in amplitude from pass to pass. Thus, the immediate definition of asymptotic stability is to require that the sequence of pass profiles 'settles down' to a so-called limit profile as $k \longrightarrow \infty$, given any initial profile y_0 and any input sequence $\{u_{k+1}\}$ which 'settles down' to a steady disturbance u_∞ as $k \longrightarrow \infty$. This is depicted in Figure 5.2.

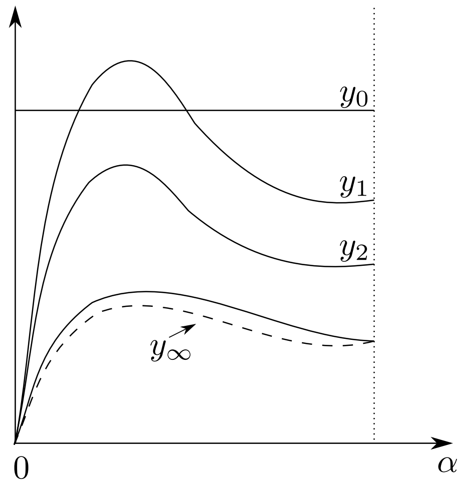

FIGURE 5.2: Schematic representation of asymptotic stability

Definition 5.2. The linear repetitive process S of constant finite pass length $\alpha > 0$ is said to be asymptotically stable if given any pass profile y_0 and any strongly convergent input sequence, the sequence generated by

$$y_{k+1}(p) = Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \ ,$$

converges strongly to a *limit profile* y_∞ as $k \longrightarrow \infty$.

The following result characterises asymptotic stability for discrete unit-memory linear repetitive processes.

Theorem 5.3. *[57, Corollary 2.1.3] The linear repetitive process generated by the discrete unit-memory linear repetitive process (5.7) is asymptotically stable, if and only if,* $r(D_0) < 1$ *where* $r(.)$ *denotes the spectral radius of its matrix argument.*

At first, it is kind of surprising to see that asymptotic stability is essentially independent of the system matrices and particularly independent of the eigenvalues of the matrix A. This is an immediate consequence of the fact that the pass length α is finite, and will change once the case of $\alpha \longrightarrow \infty$ is considered.

Asymptotic stability guarantees existence of a limit profile for the process which is defined as follows.

Definition 5.4. [57, Def. 2.1.3] Suppose that the linear repetitive process S is asymptotically stable and the input sequence applied converges strongly to u_∞ as $k \longrightarrow \infty$. Then the strong limit

$$y_\infty := \lim_{k \longrightarrow \infty} y_k$$

is termed the limit profile corresponding to the input sequence.

Corollary 5.5. *[57] Suppose that asymptotic stability holds and the input sequence applied $\{u_{k+1}\}$ converges strongly as $k \longrightarrow \infty$ to u_∞. Then the strong limit y_∞ exists corresponding to this input sequence and its state-space model is described by*

$$\begin{aligned} x_\infty(p+1) &= (A + B_0(I - D_0)^{-1}C)x_\infty(p) + (B + B_0(I - D_0)^{-1}D)u_\infty(p) \\ y_\infty(p) &= (I - D_0)^{-1}Cx_\infty(p) + (I - D_0)^{-1}Du_\infty(p) \\ x_\infty(0) &= d_\infty \end{aligned} \tag{5.11}$$

where d_∞ is the strong limit of the sequence $\{d_k\}$.

In physical terms, this implies that under asymptotic stability, the repetitive dynamics of the process can be replaced by those of a 1-D discrete linear system once a sufficiently large number of passes have elapsed.

5.3.1.1 Asymptotic Stability Under Dynamic Boundary Conditions

The following result characterises asymptotic stability of processes described by (5.7) and (5.9).

Theorem 5.6. *[62, Theorem 1] Suppose that the pair $\{A, B_0\}$ is controllable. Then the discrete linear repetitive process (with $D_0 = D_1 = 0$) is asymptotically stable if and only if all solutions $z \in \mathbb{C}$ of*

$$|zI - \sum_{j=1}^{N} J_j C(A + z^{-1}B_0C)^{p_j}| = 0 \tag{5.12}$$

have modulus strictly less than unity.

Further dimension reduction is possible in some cases such as the following:

Corollary 5.7. *[62, Corollary 1] Consider the discrete linear repetitive process described by (5.7) and (5.9) in the special case when* $J_j = JT_j$, $1 \le j \le N$, *where* J *is an* $n \times m$ *matrix with constant entries and* T_j, $1 \le j \le N$, *are* $m \times m$ *matrices with constant entries. Then these processes are asymptotically stable if and only if*

$$|zI - \sum_{j=1}^{N} T_j C(A + z^{-1}B_0C)^{p_j} J| = 0 \implies |z| < 1 \ .$$

Under conditions mentioned in Theorem 5.3, the asymptotic stability of a repetitive process, i.e., BIBO stability over the finite pass length α, guarantees the existence of a limit profile. However, the resulting limit profile could produce unacceptable along-the-pass dynamics that makes the system unstable.

Consider the following SISO discrete unit-memory linear repetitive process over $0 \le p \le \alpha$, $k \ge 0$, where β is a real scalar.

$$\begin{aligned} x_{k+1}(p+1) &= -x_{k+1}(p) + u_{k+1}(p) + (1+\beta)y_k(p) \\ y_{k+1}(p) &= x_{k+1}(p) \\ x_{k+1}(0) &= 0 \ . \end{aligned} \tag{5.13}$$

This process is asymptotically stable since $D_0 = 0$ and the resulting limit profile is described over $0 \le p \le \alpha$ by

$$\begin{aligned} y_\infty(p+1) &= \beta y_\infty(p) + u_\infty(p) \ , \\ y_\infty(0) &= 0 \ . \end{aligned} \tag{5.14}$$

It is clear to see that if $|\beta| \ge 1$, the sequence of pass profiles converges in the pass-to-pass direction (k) to an unstable 1-D discrete linear process and this occurs despite the state matrix A is stable in the 1-D discrete linear system sense. This, gives rise to the concept of stability along the pass which is discussed in the following section.

5.3.2 Stability Along The Pass

The problem illustrated by (5.14) is the finite pass that even an unstable 1-D discrete linear system can only produce a bounded output for such a length. If the limit profile is unstable, as a 1-D discrete linear system, then obviously this is unacceptable in many applications where tracking a reference signal is required (e.g., ILC).

Stability along the pass prevents this problem from arising by demanding the BIBO property uniformly with respect to the pass length, and can be analysed mathematically by letting $\alpha \longrightarrow \infty$. This leads to several sets of necessary and sufficient conditions [57] for this property, such as the following one:

Theorem 5.8. *[57, Corollary 2.2.2] Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{A, C\}$ is observable. Then a discrete linear repetitive process described (5.7) and (5.8) is stable along the pass if and only if $r(D_0) < 1$, $r(A) < 1$, and all eigenvalues of $G(z) = C(zI - A)^{-1}B_0 + D_0$ have modulus strictly less than unity $\forall\ |z| = 1$.*

These conditions can be tested by of well-known 1-D linear systems tests. Their application to the aforementioned example shows stability along the pass also applies a constraint on the state dynamics of both the current pass ($r(A) < 1$) and, in the SISO case for simplicity, the complete frequency response of the transfer function describing the contribution of the previous pass profile. Also it is easy to see that stability along the pass ensures that the resulting limit profile is stable as a 1-D discrete linear system, that is $r(A + B_0(I - D_0)^{-1}C) < 1$.

Stability along the pass requires that the signals involved are uniformly bounded when both independent variables p and k can take unbounded values. Equivalently this property should hold for any p and k in the positive quadrant of the 2-D plane, that is, $(p, k) \in P := \{(p, k) : p \geq 0, k \geq 0\}$.

In terms of design to track a given reference signal, such as in the ILC applications where the information from previous passes is used to update the control signal on the current pass to improve the performance from pass to pass by reducing the error (which is defined on each pass as the difference between a given reference signal and the process output), stability along the pass imposes the requirement that the control law must achieve the required level of attenuation over the complete frequency range. This, by comparison to the 1-D linear systems case, is most likely to result in a very difficult design problem [60].

5.3.2.1 Stability Along The Pass Under Dynamic Boundary Conditions

The following result characterises stability along the pass under dynamic boundary conditions.

Theorem 5.9. *[62, Theorem 2] Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{A, C\}$ is observable. Then the discrete linear repetitive process described by (5.7) and (5.9) is stable along the pass if and only if*

I. Theorem 5.6 holds ;

II. $r(A) < 1$ *;*

III. All eigenvalues of the transfer function matrix $G(z) := C(zI - A)^{-1}B_0$ *have modulus strictly less than unity* $\forall\ |z| = 1$.

It is worth noting that the conditions of Theorem 5.9 can be tested via well-known 1-D linear systems tests. The starting point of this approach is to drive a 1-D equivalent model of the dynamics of the process [62, Sec. 4].

5.4 Stability of Discrete Linear Repetitive Processes via 2-D Spectral Methods

Recall from Section 2.3 that a first type Fornasini-Marchesini state-space model, disregarding the output, is given by

$$\begin{aligned} x(n_1+1, n_2+1) = &\mathbf{A}'\ x(n_1+1, n_2) + \mathbf{A}''\ x(n_1, n_2+1) + \mathbf{A}'''\ x(n_1, n_2) + \mathbf{B}'\ u(n_1, n_2) \\ &+ \mathbf{B}''\ u(n_1+1, n_2) + \mathbf{B}'''\ u(n_1, n_2+1)\ . \end{aligned} \tag{5.15}$$

Considering $A' = B' = B'' = 0$ and denoting A'', A''', B''', n_1, and n_2 by A_1, A_2, B, p and k respectively, (5.15) yields the following equivalent reperesntation of (5.7):

$$x(p+1, k+1) = A_1\ x(p, k+1) + A_2\ x(p, k) + B\ u(p, k+1)\ , \tag{5.16}$$

where $A_2 := B_0C$.

Stability of the model described by (5.16) has been investigated in [38, 48, 63]. Using the results from [48] for 2-D discrete linear systems, we state conditions for stability along the pass for unit memory discrete linear repetitive processes in terms of matrices with constant entries.

Proposition 5.10. *The unit memory discrete linear repetitive process described by (5.16) is stable along the pass, if there exist* $n \times n$ *positive definite symmetric matrices* P_1 *and* P_2 *such that*

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix} (P_1 + P_2) \begin{bmatrix} A_1 & A_2 \end{bmatrix} > 0 \tag{5.17}$$

Proof. The proof follows similar to that of Proposition 3.13 and is omitted. □

Remark 5.11. By the well-known Schur complement, (5.17) can be written as:

$$\left[\begin{array}{cc|c} P_1 & 0 & \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix}(P_1+P_2) \\ 0 & P_2 & \\ \hline (P_1+P_2)\begin{bmatrix} A_1 & A_2 \end{bmatrix} & & P_1+P_2 \end{array}\right] > 0\,,$$

which is a linear matrix inequality and thus computationally tractable [49].

5.5 Application to Iterative Learning Control

Iterative learning control (ILC) was developed for systems defined over a finite duration that perform the same operation over and over again with resetting to the starting location once each operation is complete. Each execution of the task is known as a trial in the literature and the control objective takes the form of reference trajectory $y_{ref}(t)$ defined over a finite interval $0 \leq t \leq \alpha$, where $\alpha < \infty$ denotes the trial duration or length, which must be tracked, or followed, to a high precision. The novel feature is the use of information from previous trials to update the control input for the next trial and thereby sequentially improve performance from trial-to-trial.

Since the original work [64], ILC has become an established area of control systems research, both in terms of the development of control law design algorithms (see e.g. [65]) and their experimental validation and implementation. The survey papers [66, 67] are one initial source for the literature, where applications span many areas, including robotics and process/manufacturing systems. More recently, ILC algorithms first developed in the engineering domain have been used in robotic-assisted upper limb stroke rehabilitation with supporting clinical trials [68, 69].

In application, ILC can be treated as a 2-D system where one direction of information propagation is from trial-to-trial and the other is along the trial. The first work on using a 2-D systems setting for the design of linear ILC laws was reported in [70]. More recently the theory of linear repetitive processes [57] has been used to design ILC control laws with experimental verification on a gantry robot executing a pick and place operation that replicates many industrial applications to which ILC is applicable [69, 71, 72].

A brief overview of a recent study [73] that shows how repetitive process can be used to analyse ILC schemes is considered in this section. The case of interest is when the plant to be controlled can be modeled, at least for initial control-related analysis, as a discrete linear time-invariant system with state-space model defined by the matrices $\{A_c, B_c, C_c\}$. In the ILC setting this is written as

$$\begin{aligned} \dot{x}_k(t) &= A_c x_k(t) + B_c u_k(t), 0 \leq t \leq \alpha \\ y_k(t) &= C_c x_k(t) \end{aligned} \tag{5.18}$$

where on trial k, $x_k(t) \in \mathbb{R}^n$ is the state vector, $y_k(t) \in \mathbb{R}^m$ is the output vector, $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs, and $\alpha < \infty$ is the trial duration. If the signal to be tracked is denoted by $y_{ref}(t)$ then $e_k(t) = y_{ref}(t) - y_k(t)$ is the error on trial k and the most basic requirement is to force the error to converge in k.

Suppose that $||\cdot||$ is a signal norm in a suitably chosen function space with a norm-based topology. Then the construction of a sequence of input functions such that performance is gradually improving with each successive trial can be refined to a convergence condition on the input and error, that is,

$$\lim_{k\to\infty} ||e_k|| = 0, \ \lim_{k\to\infty} ||u_k - u_\infty|| = 0$$

In many applications, a digital implementation will be required and most ILC designs assume that this is done by direct digital control, that is, sample the plant model and design the control law in the digital domain. Consider, for simplicity, the SISO case, let N denote the number of samples along the trial, and introduce for trial k the following vectors

$$\begin{aligned} U_k &= \begin{bmatrix} u_k(0) & u_k(1) & \dots & u_k(N-1) \end{bmatrix}^T \\ Y_k &= \begin{bmatrix} y_k(1) & y_k(2) & \dots & y_k(N) \end{bmatrix}^T \end{aligned}$$

Then the plant dynamics can be written as

$$Y_k = HU_k \tag{5.19}$$

where

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_2 & h_1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_N & h_{N-1} & \dots & h_1 \end{bmatrix}$$

where the h_i, $1 \leq i \leq N$, are the system Markov parameters and, for simplicity, it is assumed that $h_1 \neq 0$.

The representation of the dynamics in the form (5.19) over the finite interval enables the ILC dynamics to be treated as a static system in $\mathbb{R}^N$. Now consider the phase-lead ILC law

$$u_{k+1}(i) = u_k(i) + \gamma e_k(i+1) \tag{5.20}$$

where $e_k(i+1)$ is causal information since it was generated on the previous trial and is therefore available for use on trial $k+1$. Also it is easily shown that trial-to-trial error convergence occurs under an ILC law of this form, in the SISO case for simplicity, when

$$|1 - CB\gamma| < 1 \; . \tag{5.21}$$

This condition does not involve the plant state matrix and hence ILC can converge for even unstable plants but at the possible cost of unacceptable along the trial dynamics. Consider a gantry robot executing the following set of operations in synchronization with a moving conveyor: i) collect an object from a location, ii) transfer it over a finite duration, iii) place it on a moving conveyor, iv) return to the original location and then v) repeat the previous four steps for as many objects as required. Suppose also that the object has an open top and is filled with liquid, and/or is fragile in nature. Then unwanted vibrations during the transfer time could have very detrimental effects. Hence in such cases there is also a need to control the along the trial dynamics.

The many designs for discrete systems based on lifting, see the relevant references in [66, 67] for a selection of these, would proceed in cases such as that outlined above by first designing a feedback controller to stabilize the plant and/or obtain acceptable along the trial dynamics and then apply ILC to the resulting system. An alternative is to use a 2-D systems setting, where the two directions of information are trial-to-trial and along the trial, respectively. This approach allows one step design for the trial-to-trail and the along the trial ILC process dynamics. An obvious starting point for this approach is the Roesser [33] and Fornasini-Marchesini [24] state-space models. For example, in [70] it was shown how trial-to-trial error convergence of linear ILC schemes in the discrete domain could be examined as a stability problem in terms of a Roesser state-space model interpretation of the dynamics.

Given that the trial length is finite by definition, it follows that ILC fits naturally into the class of repetitive processes [57]. Repetitive processes cannot be controlled using standard systems theory and algorithms because such an approach ignores their inherent 2-D systems structure, that is, information propagation occurs from pass-to-pass, the k direction, and along a given pass, the p direction and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed [57] based on an abstract model of the dynamics in a Banach space setting that includes a very large class of processes with linear dynamics and a constant pass length as special cases. In terms of their dynamics, it is the pass-to-pass coupling, noting again their unique feature, which is critical.

The abstract model based stability theory is defined in bounded-input bounded-output (BIBO) terms. In particular, a bounded initial pass profile is required to produce a bounded sequence of pass profiles, where boundedness is defined in terms of the norm on the underlying function space. Two forms termed asymptotic and along the pass, respectively, are possible, where the former demands this property over the finite and fixed pass length α for a given example and the latter for all possible values of the pass length.

Next it is shown how a repetitive process setting can be used to analyze ILC schemes and, in particular, how the stability theory of these processes can be employed to develop algorithms for control law design for trial-to-trial error convergence and along the trial performance. The links with the design in [70] are also discussed. Given that the repetitive process setting is used the term pass instead of trial will be used from this point onwards.

5.5.1 ILC Analysis and Control Law Design

Consider the discrete domain and assume that the state-space model (5.18) have been sampled by the zero-order hold method at a uniform rate T_s seconds to produce a discrete state-space model with matrices $\{A, B, C\}$. Also, for analysis purposes only, write the state equation of the sampled dynamics as

$$x_k(p) = Ax_k(p-1) + Bu_k(p-1) \tag{5.22}$$

and introduce

$$\begin{aligned} \eta_{k+1}(p+1) &= x_{k+1}(p) - x_k(p) \\ e_k(p) &= y_{ref}(p) - y_k(p) \end{aligned} \tag{5.23}$$

A commonly used ILC law has the structure

$$u_{k+1}(p) = u_k(p) + \Delta u_{k+1}(p) \tag{5.24}$$

where $\Delta u_{k+1}(p)$ denotes the update added to the control signal on the previous pass to form the corresponding signal on the current one. Consider also the case when

$$\Delta u_{k+1}(p) = K_1\eta_{k+1}(p+1) + K_2 e_k(p+1) \tag{5.25}$$

and introduce the notation

$$\begin{aligned} \hat{A} &= A + BK_1, & \hat{B}_0 &= BK_2 \\ \hat{C} &= -C(A + BK_1), & \hat{D}_0 &= I - CBK_2 \end{aligned} \tag{5.26}$$

Then the ILC scheme can be written as

$$\begin{aligned}\eta_{k+1}(p+1) &= \hat{A}\eta_{k+1}(p) + \hat{B}_0 e_k(p) \\ e_{k+1}(p) &= \hat{C}\eta_{k+1}(p) + \hat{D}_0 e_k(p)\end{aligned} \tag{5.27}$$

which is a discrete linear repetitive process state-space model where $\eta_{k+1}(p)$ is the state vector on the current pass, governing the along the pass dynamics and $e_{k+1}(p)$ is the error on the current pass. The terms $\hat{B}_0 e_k(p)$ and $\hat{D}_0 e_k(p)$ describe the contribution of the error on pass k to the pass state and output dynamics, respectively, on pass $k+1$.

5.6 Summary

This chapter illustrated the abstract model for representing linear repetitive processes. It was then illustrated how this abstract representation allows the analysis of discrete linear repetitive processes with the emphasis on the unit-memory type with constant pass length α that will be used later in Chapter 6 for the purpose of fault detection and isolation.

Within Section 5.3, based on the rigorous stability theory in Banach space setting developed by Rogers and Owens [58], concepts of asymptotic stability and stability along the pass for discrete linear repetitive processes was introduced. It was explained that existence of two distinct types of stability is due to inherent dependency of these processes on two independent variables. Asymptotic stability demands the BIBO stability over the pass length, whereas stability along the pass as the stronger condition, is the requirement that bounded sequences of inputs produce bounded sequence of outputs independently of pass length. It is worth noting that examples exist where it is shown that asymptotic stability is all that is achievable [74] or practically required [75].

In Section 5.4, it was illustrated that a unit memory linear repetitive process can be accommodated in the 2-D model described by Fornasini-Marchesini first type. Following [48], the stability of this model was investigated and LMI-based solutions were provided to examine the stability along the pass in repetitive processes.

Finally, Section 5.5 represented a brief overview of iterative learning processes [73] where it is shown how repetitive process can be used to analyse ILC schemes.

Chapter 6

Fault Detection and Isolation in Linear Repetitive Processes

6.1 Introduction

Repetitive processes represent an extensive class of important industrial operations such as long-wall coal cutting, metal rolling, printing, and modelling of fluid dynamics in distribution pipelines such as gas networks [76]. In the last decade, a number of applications have emerged where adopting a repetitive process setting for analysis has certain advantages over alternatives. Examples of these algorithmic applications include classes of Iterative Learning Control (ILC) schemes [77] which was briefly presented in Section 5.5 and iterative algorithms for solving non-linear dynamic optimal control problems based on the maximum principle [78]. In such applications, because of the repetitive nature of the process, a failure unless promptly detected and fixed affects not only the current process but also the following ones. Thus fault detection and isolation (FDI) is an important problem that needs to be addressed.

Although the FDI problem for linear systems has been studied intensively due to its significance, and a variety of different methods have been developed to address it, there has been no attempt to specifically investigate the problem in repetitive processes. A comprehensive survey of various FDI techniques can be found in [18] and [79]. One of these techniques is the geometric approach that was developed in Chapter 3 for 3-D systems. In this chapter, this problem is investigated specifically for repetitive processes. A 2-D model for these processes that incorporates the failure description is developed and an asymptotic observer is constructed, similar to that of Chapter 4, that by observing the output and the input of the system, asymptotically reconstructs the state. Once a failure occurs, the reconstructed state starts to deviate from the actual state space. Then by using a geometric approach, a fault detection and isolation technique is developed that under suitable assumptions, can detect and uniquely isolate a failure.

Repetetive processes were introduced in Chapter 5. Recall that repetitive processes are a distinct class of 2-D systems characterised by a series of passes through a set of dynamics defined over a fixed finite duration (pass length) [61]. At each pass, an output is produced which contributes also to the dynamics of the following passes. If it is the previous pass only which contributes to the current pass, the process is called *unit memory*, whereas if the previous M pass profiles contribute to the current one, M is the *memory length*.

In this chapter, for simplicity of discussion a unit-memory linear repetitive process is used to investigate the FDI problem and we use the model (5.16), restated below, for this purpose.

$$\begin{aligned} x(p+1,k+1) &= A_1\ x(p,k+1) + A_2\ x(p,k) + B\ u(p,k+1)\ , \\ y(p,k) &= Cx(p,k)\ , \end{aligned} \tag{6.1}$$

where $A_2 := B_0C$. The boundary conditions for this model are given by

$$\begin{aligned} x(p,0) &= d(p,1)\ , \\ x(0,k) &= \hat{y}(0,k)\ , \end{aligned}$$

where the vectors d and $\hat{y}$ are defined as in (5.6).

A geometric approach analogous to the one developed in Chapter 3 for 3-D systems can be used here as a result of this modelling. Therefore only the geometric background for these processes is given without giving the analogous proofs and the interested reader is referred to Chapter 3 and the geometric notions and results for 2-D systems presented in [45, 50, 44].

6.2 Geometric Background

Following Section 3.5, for the pair $(A_i, C), i = 1, 2$ of the discrete linear repetitive process (6.1), a conditioned-invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ is defined as follows:

Definition 6.1. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is a *conditioned invariant subspace* if

$$A_H(\mathcal{V}_D \cap \mathcal{C}_D) \subseteq \mathcal{V}\ , \tag{6.2}$$

where $\mathcal{C} := \ker C$, $\mathcal{C}_D := \mathrm{diag}(\mathcal{C}, \mathcal{C})$, $A_H := \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $\mathcal{V}_D = \mathcal{V} \oplus \mathcal{V}$.

Denote the family of conditioned invariant subspaces containing a given subspace $\mathcal{L}$ by $\underline{\mathcal{W}}(\mathcal{L})$. Clearly, the family $\underline{\mathcal{W}}(\mathcal{L})$ is closed under intersection. Therefore, there exists a smallest subspace in the family $\underline{\mathcal{W}}(\mathcal{L})$, called the *infimal* element and denoted by $\mathcal{W}^\star(\mathcal{L})$.

A recursive algorithm to find the subspace $\mathcal{W}^\star(\mathcal{L})$ is given below [1]:

$$\begin{cases} \mathcal{W}(\mathcal{L})^0 = \{0\} \ ; \\ \mathcal{W}(\mathcal{L})^{h+1} = \mathcal{L} + A_1\Big(\mathcal{W}(\mathcal{L})^h \cap \text{Ker}C\Big) + A_2\Big(\mathcal{W}(\mathcal{L})^h \cap \text{Ker}C\Big) \ . \end{cases} \tag{6.3}$$

The following proposition gives the most important properties of 2-D conditioned invariants.

Proposition 6.2. *Let $\mathcal{V}$ be a r-dimensional subspace of $\mathbb{R}^n$, and let $Q \in \mathbb{R}^{(n-r)\times n}$ be a full rank matrix such that* $\ker(Q) = \mathcal{V}$. *The following statements are equivalent:*

I. $\mathcal{V}$ is (A_H, C_D)-conditioned invariant;

II. There exist $\Gamma = \begin{bmatrix}\Gamma_1 & \Gamma_2\end{bmatrix} \in \mathbb{R}^{(n-r)\times 2(n-r)}$ and $\Lambda \in \begin{bmatrix}\Lambda_1 & \Lambda_2\end{bmatrix} \in \mathbb{R}^{(n-r)\times 2m}$ such that

$$QA_H = \Gamma Q_D + \Lambda C_D \ , \tag{6.4}$$

or equivalently there exist $\Gamma_i \in \mathbb{R}^{(n-r)\times(n-r)}$ and $\Lambda_i \in \mathbb{R}^{(n-r)\times m}$, $i = 1, 2$ such that

$$QA_i = \Gamma_i Q + \Lambda_i C \ \ i = 1, 2 \ .$$

III. There exists $G = \begin{bmatrix}G_1 & G_2\end{bmatrix} \in \mathbb{R}^{n\times 2m}$ such that

$$(A_H + GC_D)\, \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V} \subseteq \mathcal{V} \ , \tag{6.5}$$

or equivalently there exist $G_i \in \mathbb{R}^{n\times m}$ such that

$$(A_i + G_i C)\mathcal{V} \subseteq \mathcal{V} \quad \textit{for } i = 1, 2 \ .$$

Similarly to the 1-D case [40], the following theorem establishes a fundamental result for the decomposition of the system matrices with respect to an invariant subspace.

Theorem 6.3. *The following statements are equivalent:*

I. $\mathcal{V} \subseteq \mathbb{R}^n$ *is an* (A_i, C)*-invariant subspace of dimension* m, $i = 1, 2$.

II. There exists $T \in \mathbb{R}^{n\times n}$, *such that*

$$\hat{A}_i = T^{-1}(A_i + G_iC)T = \begin{bmatrix} \hat{A}_i^{11} & \hat{A}_i^{12} \\ 0_{(n-m)\times m} & \hat{A}_i^{22} \end{bmatrix} ,$$

where, $G_i \in \mathbb{R}^{n\times m}$ *is the output-injection matrix.*

Proof. Similar to that of [45, Theorem 2.1] and hence is omitted. □

From Theorem 6.3, using a similarity transformation $T \in \mathbb{R}^{n\times n}$ for a conditioned invariant subspace $\mathcal{V} \subseteq \mathcal{X}$ of the repetitive process described by (6.1), it immediately follows that

$$\begin{bmatrix} x'(p+1,k+1) \\ x''(p+1,k+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_1^{11} & \hat{A}_1^{12} \\ 0 & \hat{A}_1^{22} \end{bmatrix} \begin{bmatrix} x'(p,k+1) \\ x''(p,k+1) \end{bmatrix} + \begin{bmatrix} \hat{A}_2^{11} & \hat{A}_2^{12} \\ 0 & \hat{A}_2^{22} \end{bmatrix} \begin{bmatrix} x'(p,k) \\ x''(p,k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(p,k+1) \ . \tag{6.6}$$

Also it follows from standard results in nD systems theory (see for example [24, Prop. 3]) that $\mathcal{V}$ is internally stable if and only if the matrices $\hat{A}_i^{11}$, $i = 1, 2$ satisfy

$$\det(I_n - \hat{A}_1^{11}\lambda - \hat{A}_2^{11}\mu) \neq 0 \\ \text{for all } (\lambda, \mu) \in \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i| \leq 1, i = 1, 2\} \ , \tag{6.7}$$

and externally stable if and only if the matrices $\hat{A}_i^{22}$, $i = 1, 2$ satisfy

$$\det(I_n - \hat{A}_1^{22}\lambda - \hat{A}_2^{22}\mu) \neq 0 \\ \text{for all } (\lambda, \mu) \in \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i| \leq 1, i = 1, 2\} \ . \tag{6.8}$$

Definition 6.4. The system (6.1) (i.e., the pair (A_i, C)) is said to be *detectable* if there exists output-injection matrices G_i such that $A_i + G_iC$ is stable.

The framework for fault isolation in this chapter depends on the concept of *input-containing* conditioned invariant subspaces [44, 51].

Definition 6.5. A subspace $\mathcal{V} \subset \mathbb{R}^n$ is an *input-containing conditioned invariant* for (5.5), if

$$\begin{bmatrix} A_H & B_H \end{bmatrix} \left((\mathcal{V} \oplus \mathcal{V} \oplus \mathbb{R}^l) \cap \ker \begin{bmatrix} C_D & 0_{2(m\times l)} \end{bmatrix} \right) \subseteq \mathcal{V} \,, \tag{6.9}$$

where

$$A_H := \begin{bmatrix} A_1 & A_2 \end{bmatrix} \,, \quad B_H = \begin{bmatrix} B & 0_{n\times l} \end{bmatrix} \,, \quad C_D := \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \,.$$

Similar to conditioned invariant subspaces, the following characterisations of input-containing subspaces hold.

Proposition 6.6. *Let $\mathcal{V}$ be an r-dimensional subspace of $\mathbb{R}^n$, and let $Q \in \mathbb{R}^{(n-r)\times n}$ be a full row-rank matrix such that* $\ker\,(Q) = \mathcal{V}$*. The following statements are equivalent:*

I. $\mathcal{V}$ is an input-containing conditioned invariant for (5.5) ;

II. There exist matrices $\Gamma := \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}$ and $\Lambda := \begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix}$ with $\Gamma_i \in \mathbb{R}^{(n-r)\times(n-r)}$ and $\Lambda_i \in \mathbb{R}^{(n-r)\times m}, i = 1, 2,$ such that

$$Q \begin{bmatrix} A_H & B_H \end{bmatrix} = \Gamma \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \end{bmatrix} + \Lambda \begin{bmatrix} C_D & 0_{2(m\times l)} \end{bmatrix} \,, \tag{6.10}$$

where $Q_D := \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$;

III. There exist a matrix $G := \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ with $G_i \in \mathbb{R}^{n\times m}, i = 1, 2,$ such that

$$\begin{bmatrix} A_H + GC_D & B_H \end{bmatrix} \left(\mathcal{V} \oplus \mathcal{V} \oplus \mathbb{R}^l \right) \subseteq \mathcal{V} \,. \tag{6.11}$$

Proof. Similar to that of [44, Lemma 3.1] and hence is omitted. □

Following Proposition 6.6, for an input-containing conditioned invariant subspace $\mathcal{V}$, existence of an output-injection matrix G is guaranteed. The task hence is to construct a matrix G, if it exists, such that $\mathcal{V}$ is an internally and externally stable $(A_H + GC_D)$-invariant.

6.2.1 Construction of a Stabilising Gain G

The aim is to construct, if it exists, an output-injection $G := \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ such that ker $(Q) := \mathcal{V}$ is an internally and externally stable input-containing $(A_i + G_iC, C)$-invariant subspace. From (6.10) it follows:

$$Q \begin{bmatrix} A_H & B_H \end{bmatrix} = \begin{bmatrix} \Gamma & \Lambda \end{bmatrix} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix} . \tag{6.12}$$

The solution of (6.12) for Γ and Λ is given by:

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = Q \begin{bmatrix} A_H & B_H \end{bmatrix} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix}^{\dagger} + KH , \tag{6.13}$$

where H is a full row rank matrix such that

$$\ker\,(H) = \mathrm{Im} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix} ,$$

H has linearly independent rows, and K is an arbitrary matrix of compatible dimensions which represents a first degree of freedom in construction of G that can be exploited for external stabilisation of $\mathcal{V}$.

Using (6.10), the solutions G to the equation $\Lambda = -QG$ can be computed as $G = G_\Lambda + \Omega U$, where $G_\Lambda := -Q^\top(QQ^\top)^{-1}\Lambda$, the matrix Ω is a basis for ker (Q) and U is an arbitrary matrix of compatible dimensions which represents a second degree of freedom in construction of G that can be exploited for internal stabilisation of $\mathcal{V}$.

Following from Theorem 6.3, for $i = 1, 2$, it follows that

$$T \begin{bmatrix} A_i + G_iC \end{bmatrix} T^{-1} = \begin{bmatrix} \Delta_i^{11}(K,U) & \Delta_i^{12}(K,U) \\ 0 & \Delta_i^{22}(K,U) \end{bmatrix} , \tag{6.14}$$

where $T := \begin{bmatrix} T_c \\ Q \end{bmatrix}$, and the rows of T_c are linearly independent from those of Q. It follows from [44, Lemma 3.2] that the choice of K affects $\Delta_i^{22}(K,U)$ but not $\Delta_i^{11}(K,U)$ and the choice of U affects $\Delta_i^{11}(K,U)$ but not t $\Delta_i^{22}(K,U)$.

Proposition 6.7. *Let Γ_i, $\Lambda_i, i = 1, 2$, satisfy (6.10), Then $\Gamma_i = \Delta_i^{22}(K,U)$, the $(2,2)$-block of (6.14).*

Proof. Analogous to that of Proposition 3.23 and hence is omitted. □

Proposition 6.8. *Let Γ_i, $\Lambda_i, i = 1, 2$, satisfy (6.10). Let T_C be such that* $T := \begin{bmatrix} T_c \\ Q \end{bmatrix}$ *is non-singular. Then*

$$T_c(A_i + \Omega UC)T_c^\top = \Delta_i^{11}(K, U) \ .$$

Proof. Similar to that of Proposition 3.25 and hence is omitted. □

To construct a stabilising output-injection matrix G, write (6.13) as:

$$\begin{bmatrix} \Gamma & \Lambda \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & \bar{V} \end{bmatrix} + K \begin{bmatrix} H_1 & H_2 & \bar{H} \end{bmatrix} \ , \tag{6.15}$$

where

$$\begin{bmatrix} V_1 & V_2 & \bar{V} \end{bmatrix} := Q \begin{bmatrix} A_H & B \end{bmatrix} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix}^\dagger \ ,$$

and

$$\ker \begin{bmatrix} H_1 & H_2 & \bar{H} \end{bmatrix} = \operatorname{Im} \begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C_D & 0_{2n\times 2l} \end{bmatrix} \ ,$$

are partitioned compatibly with $\begin{bmatrix} \Gamma & \Lambda \end{bmatrix}$. Thus, $\Gamma_i = V_i + KH_i, i = 1, 2$, and $\Lambda = \bar{V} + K\bar{H}$. If $\begin{bmatrix} Q_D & 0_{2(n-r)\times 2l} \\ C & 0_{2(m\times l)} \end{bmatrix}$ has full rank, there are no degrees of freedom to exploit for stabilisation.

Computing, if it exists, an externally stabilising G for the conditioned invariant subspace $\mathcal{V}$ is now considered. From the previous discussion, the problem reduces to finding matrices K such that $\Gamma_i = V_i + KH_i$ is asymptotically stable. Using Proposition 5.10, one solution to determine K is to solve the following LMI for K:

$$\begin{bmatrix} \Phi & 0 \\ 0 & \Psi - \Phi \end{bmatrix} - \begin{bmatrix} \Gamma_1^\top \\ \Gamma_2^\top \end{bmatrix} \Psi \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix} > 0 \, , \tag{6.16}$$

for some $\Phi := P_1 > 0$ and $\Psi := P_1 + P_2 > 0$. By the Schur's complement formula and using $\Gamma_i = V_i + KH_i$, for $i = 1, 2$, (6.16) is equivalent to

$$\begin{bmatrix} \Phi & 0 & (\Psi V_1 + \Theta H_1)^\top \\ 0 & \Psi - \Phi & (\Psi V_2 + \Theta H_2)^\top \\ \Psi V_1 + \Theta H_1 & \Psi V_2 + \Theta H_2 & \Psi \end{bmatrix} > 0 \, , \tag{6.17}$$

for some $\Phi > 0, \Psi > 0$ and Θ of compatible dimensions, where $\Theta := \Psi K$.

Similarly, another LMI can be set up and by exploiting U, the second degree of freedom, internally stabilisation (if possible) of the conditioned invariant subspace $\mathcal{V}$ can be achieved.

6.3 Failure Modelling in Discrete Linear Repetitive Processes

Consider the unit-memory repetitive system (6.1). In what follows, it is assumed that the process is detectable in the sense of [80, Th. 5.15]. To model the dynamics of the system after a failure, the model is augmented with additional terms that represent the failure modes:

$$\begin{aligned} x(p+1,k+1) &= A_1\ x(p,k+1) + A_2\ x(p,k) + B\ u(p,k+1) \\ &\quad + \begin{bmatrix} L^1 & L^2 & \cdots & L^l \end{bmatrix} \begin{bmatrix} m^1(p,k+1) \\ m^2(p,k+1) \\ \vdots \\ m^l(p,k+1) \end{bmatrix}. \end{aligned} \tag{6.18}$$

where similar to the 3-D case, $m^i(p,k+1)$ and the matrices L^i, $i = 1, 2, ..., l$, are termed the *failure modes* and *signatures*, respectively. Failure modes are unknown arbitrary functions corresponding to the type of the failure in the process. In the absence of failures, these modes are identical to zero while have some non-zero value once a failure occurs.

Failure signatures together with the failure modes enable modelling a variety number of failures in the process, such as actuator failures, changes in the process dynamics and sensor failures [1]. Four specific types of failure are considered in what follows.

• **Dead actuator**: Suppose the i-th actuator is dead, then the failure signature L^i is the i-th column of the input matrix B, and the failure mode is $m^i(p,k) = -u^i(p,k)$ where $u^i(p,k)$ is the i-th component of the input $u(p,k)$.

• **Biased actuator**: If there is a bias in the i-th actuator, the failure signature L^i is the i-th column of the input matrix B, and $m^i(p,k) = b$ where $b \in \mathbb{R}$ is a non-zero constant.

• **Saturated actuator**: An actuator could saturate at one of its end points if the input is too large. This scenario can be modelled by combination of the first two cases (i.e., $m^i(p,k) = b - u^i(p,k)$).

• **Displacement of actuators**: In this case, which is the most complicated one, the i-th actuator responds to the input in a wrong way, namely the i-th column of the input matrix B, denoted by B^i, is changed to some different column vector $B^{i'}$. In this case, the failure signature is described by $L^i = \begin{bmatrix} B^i & B^{i'} \end{bmatrix}$ and is no longer a column vector, but a matrix. The corresponding failure mode is represented by $m^i(p,k) = \begin{bmatrix} -u^i(p,k) & u^i(p,k) \end{bmatrix}^T$.

For simplicity of discussion, the following assumptions are made:

- **Detectability:** The pairs (A_1, C) and (A_2, C) are detectable. This guarantees that an asymptotic observer can be designed;
- **Unambiguous failure modes:** The failure signature matrix L^i has full column rank, $i = 1, 2, ..., l$;
- **No simultaneous failures:** If there exist $1 \leq \bar{i} \leq l$ such that $m^{\bar{i}}(p,k) \neq 0$, then $m^i(p,k) = 0$ for $i \neq \bar{i}$.

Hence, the process dynamics in the i-th failure situation is modelled as:

$$\begin{aligned} x(p+1,k+1) &= A_1\ x(p,k+1) + A_2\ x(p,k) + B\ u(p,k+1) + L^i m^i(p,k+1)\ , \\ y(p,k) &= C\ x(p,k)\ . \end{aligned} \tag{6.19}$$

6.4 Fault Detection and Isolation

Consider designing a full-order observer of the following form for the nominal model (6.19):

$$\begin{aligned} \hat{x}(p+1,k+1) = {} & A_1\hat{x}(p,k+1) + A_2\hat{x}(p,k) + Bu(p,k+1) \\ & - G_1\Big(y(p,k+1) - \hat{y}(p,k+1)\Big) - G_2\Big(y(p,k) - \hat{y}(p,k)\Big)\ , \end{aligned} \tag{6.20}$$

where $G_i, i = 1, 2$, is the output-injection matrix. Moreover, define the error vector as $e(p+1, k+1) = x(p+1, k+1) - \hat{x}(p+1, k+1)$. If no failure is present, the error dynamics can be computed by subtracting (6.20) from (6.1), to give

$$e(p+1, k+1) = (A_1 + G_1 C)\ e(p, k+1) + (A_2 + G_2 C)\ e(p, k)\ , \tag{6.21}$$

which assuming a sufficiently long pass, converges asymptotically to zero as $p \longrightarrow \alpha$ if $A_i + G_i C,\ i = 1, 2$ are stable matrices.

In the presence of a failure, the error dynamics is obtained by subtracting (6.20) from (6.19):

$$e(p+1, k+1) = (A_1 + G_1 C)\ e(p, k+1) + (A_2 + G_2 C)\ e(p, k) + L^i m^i(p, k+1)\ . \tag{6.22}$$

In case of failure, the error vector does not lie on the zero subspace even if $A_i + G_i C,\ i = 1, 2$ are stable, but lies asymptotically on the reachability subspace [24] of the system (6.22) once $\alpha \longrightarrow \infty$. Note that $e(p, k+1)$ represents the error corresponding to the current pass whereas $e(p, k)$ represents the error corresponding to the previous pass. Denote by $\mathcal{L}^i := \text{Im } L^i$ and by $\mathcal{V}^\star(\mathcal{L}^i)$ the smallest conditioned invariant subspace containing $\mathcal{L}^i$ (i.e., the reachability subspace of $(A_1 + G_1 C, A_2 + G_2 C, L^i)$. G_2 can be selected as $G_2 = -B_0$ so that the error from the previous pass is cancelled; of course, this is just one possible choice. For the FDI scheme to work G_1 should be chosen so as to make $\mathcal{V}^\star(\mathcal{L}^i)$ into an externally stabilisable $(A_1 + G_1 C)$-invariant subspace. This stabilisability requirement in a fault-free situation described by (6.21), guarantees the convergence of the error to zero even if the initial error is not congruent. In the case when one fault has occurred, for example corresponding to the error signature L^i, the dynamics of the error is described by (6.22) with $m^i(p, k+1)$ non-zero and the error signature asymptotically lies in $\mathcal{V}^\star(\mathcal{L}^i)$. The internal stabilisability of the conditioned invariant is implied by the assumption that the system is detectable. One possible G_1 can be determined by solving:

$$-A_1 \begin{bmatrix} L^1 & L^2 & \cdots & L^l \end{bmatrix} = G_1 C \begin{bmatrix} L^1 & L^2 & \cdots & L^l \end{bmatrix}\ .$$

Other stabilising gains, if they exist, can be computed similar to the method detailed in Section 6.2. Additionally, the GA Toolbox for MATLAB [53] routines e.g. `sstar` can be used to compute the stabilising gains.

Having derived the error dynamics in two situations of a fault-free and faulty process, the FDI problem can be stated in geometric terms as:

Geometric FDI Problem in Linear Repetitive Processes

Find subspaces $\mathcal{V}^i$, $i = 1, 2, \ldots, l$, such that:

I. There exist stabilising gains $G_1,\ G_2 \in \mathbb{R}^{m\times n}$, such that $(A_1 + G_1C)\mathcal{V}^i \subset \mathcal{V}^i$ and $(A_2 + G_2C)\mathcal{V}^i \subset \mathcal{V}^i, i = 1, 2, \ldots, l$;

II. $\mathcal{L}^i \subseteq \mathcal{V}^i$;

III. $\mathcal{V}^i \cap (\sum_{j\neq i} \mathcal{V}^j) = \{0\}$, $i = 1, 2, \ldots, l$.

The first condition guarantees that the subspaces $\mathcal{V}^i$ are internally and externally stable and invariant under the error dynamics and hence the error due to a non-zero $m^i(p, k)$ remains inside $\mathcal{V}^i$. The second condition states that the subspaces $\mathcal{V}^i$ should contain the image of the failure signature. The last condition establishes that the subspaces have trivial intersection, which enables unique isolation of the failure.

If conditions I, II, and III are satisfied, the procedure to construct an asymptotic observer for the purposes of fault detection can be stated as follows:

Construction of an Asymptotic FDI Observer

---

1. Check the detectability of (A_1, C_1) and (A_2, C). If detectable, proceed to the next step. If not, stop ;

2. Compute the family of smallest conditioned invariant subspaces $\underline{\mathcal{W}}^\star(\mathcal{L}^i)$, $i = 1, 2, \ldots, l$ containing $\mathcal{L}^i$, $i = 1, 2, \ldots, l$ by using algorithm (3.37) ;

3. Verify condition (III) for the family $\underline{\mathcal{W}}^\star(\mathcal{L}^i)$. If not satisfied, stop ;

4. Find stabilising gains $G_i, i = 1, 2$, if they exist, such that condition (I) holds. If not, stop .

Once the matrices $G_i,\ i = 1, 2$ have been obtained, a *threshold value* $\varepsilon > 0$ can be specified. If the norm of the error $e(p, k)$ is greater than ε, it is assumed that a fault has occurred. The determination of an appropriate ε on the basis of the fault description (and in a realistic situation, also on the basis of the size of disturbances and of the noise level) is an important issue which is left as a problem for future research.

The FDI procedure essentially consists of two stages, namely, 1) fault detection that concerns comparing the error norm to a predefined threshold; 2) fault isolation that is carried out by projecting the error onto the subspaces spanned by the failure signatures. This procedure is given in the following algorithm:

Algorithm 6.4.1: FDI PROCEDURE()

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1 : **for** $k = 0$ **to** K^*
2 :   **for** $p = 1$ **to** α
3 :   **if** $||e(p,k)|| > \varepsilon$
            then
4 :   Compute $e'(p,k)$, the projection of $e(p,k)$
            onto $\mathcal{V}^\star(\mathcal{L}^i)$;
5 :   Compute
            $f \ := \ arg \ \max\{||e_i'(p,k)||\}, \ i = 1, ..., l$
6 :   **return** (f) ;
7 :   **end if**
8 :   **end for**
9 : **end for**

$^*K$ is the number of passes.

Note that due to the assumptions, the computation of f in **Step 5** is well-defined.

6.5 Example

In this section, the new fault detection and isolation technique developed in the previous sections is applied to the metal rolling process presented in [81, p. 703]. Consider a multi-roll roll system (Figure 6.1) consists of three separate pairs of rolls which are controlled by separate input signals, i.e. different rolling forces. The deformation of the workpiece takes place between these pairs of rolls with parallel axes revolving in opposite directions. The metal strip to be rolled to a pre-specified thickness (also termed the gauge or shape) through a series of rolls for successive reductions.

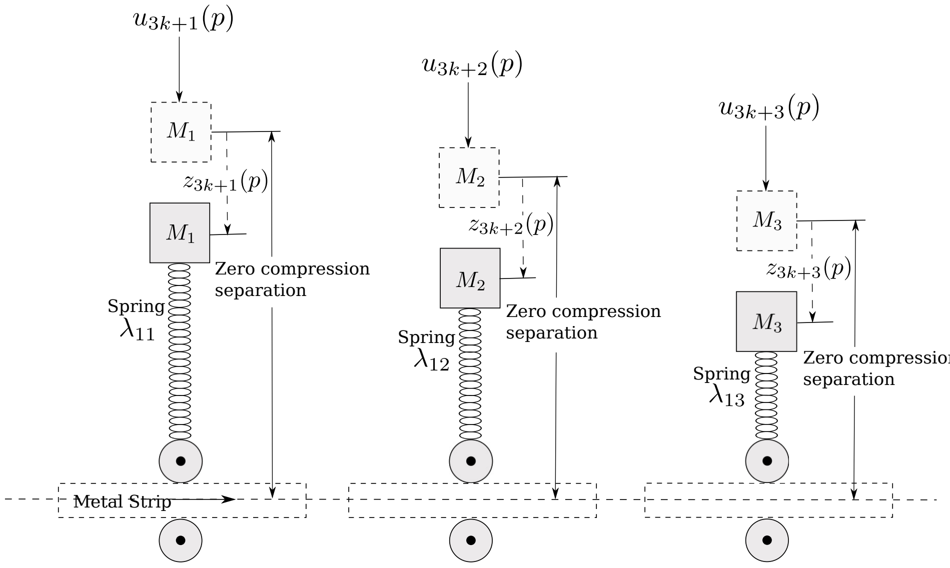


FIGURE 6.1: Multi-roll rolling machine operation

$$A = \begin{bmatrix} \frac{35}{36} & \frac{7}{72} & 0 & 0 & 0 & 0 \\ -\frac{5}{18} & \frac{35}{36} & 0 & 0 & 0 & 0 \\ \frac{35}{5216} & \frac{7}{10432} & \frac{160}{163} & \frac{16}{163} & 0 & 0 \\ \frac{175}{2608} & \frac{35}{5216} & -\frac{30}{163} & \frac{160}{163} & 0 & 0 \\ -\frac{8}{2007} & -\frac{4}{10035} & \frac{67}{10520} & -\frac{27}{42394} & -\frac{135}{137} & \frac{27}{274} \\ -\frac{80}{2007} & -\frac{8}{2007} & \frac{67}{1052} & -\frac{67}{1052} & -\frac{20}{137} & \frac{135}{137} \end{bmatrix}, \quad B_0 = \begin{bmatrix} \frac{1}{26} \\ \frac{5}{63} \\ -\frac{13}{2608} \\ -\frac{65}{1304} \\ \frac{60}{20263} \\ \frac{127}{4289} \end{bmatrix},$$

$$B = 10^{-3} \times \begin{bmatrix} -\frac{5}{18} & 0 & 0 \\ -\frac{25}{9} & 0 & 0 \\ -\frac{71}{37004} & -\frac{30}{163} & 0 \\ -\frac{93}{4847} & -\frac{300}{163} & 0 \\ \frac{47}{41236} & -\frac{11}{9214} & -\frac{20}{137} \\ \frac{72}{6317} & -\frac{55}{4607} & -\frac{200}{137} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{77}{223} & \frac{77}{2230} & -\frac{559}{1023} & -\frac{183}{3349} & \frac{135}{137} & \frac{27}{274} \end{bmatrix}.$$

which are computed using the following parameters:

$$\hat{A}_i = \frac{1}{1+a_{0i}T^2}\begin{bmatrix} 1 & T \\ -a_{0i}T & 1 \end{bmatrix}, \quad \hat{B}_i = \frac{c_{0i}T}{1+a_{0i}T^2}\begin{bmatrix} T \\ 1 \end{bmatrix},$$

$$\hat{B}_{0i} = \frac{(-b_{0i}+a_{0i}b_{2i})T}{1+a_{0i}T^2}\begin{bmatrix} T \\ 1 \end{bmatrix}, \quad C_i = \frac{1}{1+a_{0i}T^2}\begin{bmatrix} 1 & T \end{bmatrix},$$

$$a_{0i} = \frac{\lambda_{1i}\lambda_2}{M_i(\lambda_{1i}+\lambda_2)}, \quad b_{2i} = \frac{-\lambda_2}{\lambda_{1i}+\lambda_2},$$

$$b_{0i} = \frac{-\lambda_{1i}\lambda_2}{M_i(\lambda_{1i}+\lambda_2)}, \quad c_{0i} = \frac{-\lambda_{1i}}{M_i(\lambda_{1i}+\lambda_2)},$$

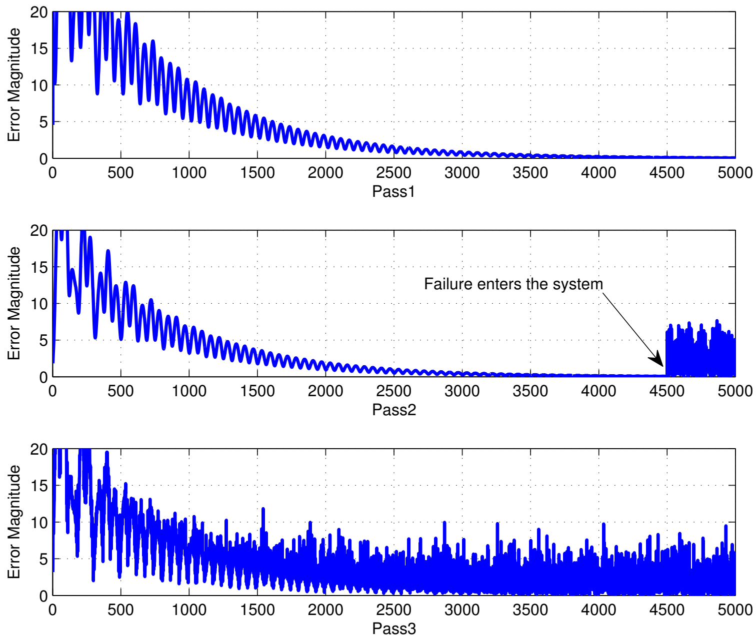
$$\lambda_{11} = 40N/m\,, \lambda_{12} = 60N/m\,, \lambda_{13} = 80N/m\,,$$
$$\lambda_2 = 100N/m\,, M_1 = 10Kg\,, M_2 = 20Kg\,, M_3 = 30Kg$$

There are three actuators in the system. The input is considered to be a decreasing force along each pass. The first 3 passes of the process each of length $\alpha = 5000$ are simulated using model (6.19) and an asymptotic observer of the form (6.20) is designed. For the gains G_1 and G_2 of the observer, G_1 is computed in a similar manner as discussed in Section 3.5.3.2, and G_2 is chosen as $G_2 := -B_0$.

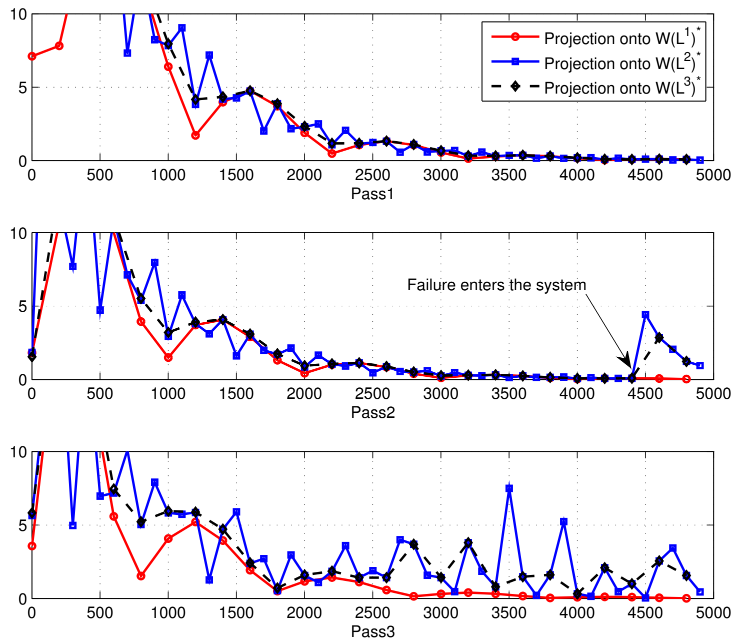
Since the simulations are carried out regardless of disturbances and noise, it is reasonable to set our threshold to 0. We consider two types of faults happening in the system:

6.5.1 Dead actuator

The first case we consider is where one of the actuators, say the first one, is dead at pass $k_0 = 4500$ at $p_0 = 2$. Figure 6.2(a) shows that the error vector goes to zero from some non zero boundary conditions at the beginning. Thereafter, the error constantly stays at zero until $k_0 = 4500$, $p_0 = 2$ is reached where the first actuator dies.



(a) Error norm for dead actuator failure in passes 1 to 3



(b) Norm of projection of the error to the subspaces containing $\mathcal{W}^\star(\mathcal{L}^i)$, $i = 1, 2, 3$ in passes 1 to 3.

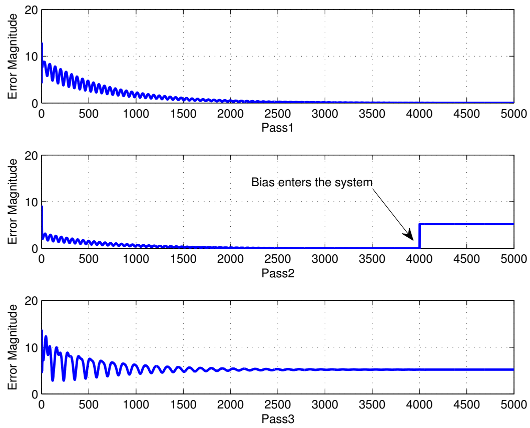
FIGURE 6.2: Dead actuator

Now that it has become obvious that a failure has occurred, we isolate the fault. This is done as discussed in Section 6.4 by projecting the the error vector onto subspaces

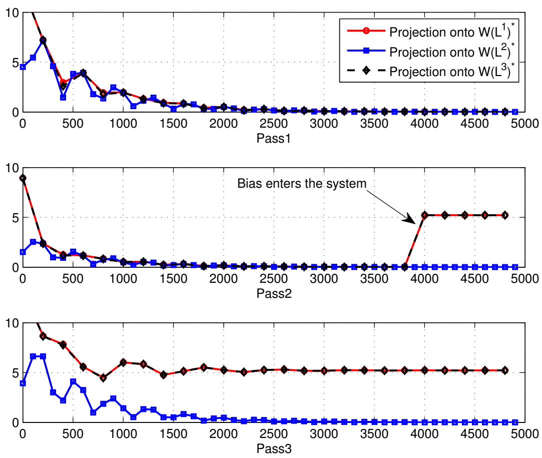
$\mathcal{W}^{\star}(\mathcal{L}^1)$, $\mathcal{W}^{\star}(\mathcal{L}^2)$, and $\mathcal{W}^{\star}(\mathcal{L}^3)$. This is depicted in Figure 6.2(b). It can be seen that at the beginning of the process the error converges to zero. After the fault happens, the error deviates from zero and lies in $\mathcal{W}^{\star}(\mathcal{L}^1)$, implying that the fault has occurred in the first actuator.

6.5.2 Biased actuator

The next case we consider is a biased actuator. Suppose one of the actuators, say the second one, is biased. The bias enters the system at $k_0 = 4000$ at $p_0 = 2$. Figure 6.3(a) illustrates this bias where it can be observed that the error due to non-zero initial conditions goes to zero and then rises and constantly stays at some constant value $b \in \mathbb{R}$ after the bias enters the system.



(a) Error norm for biased actuator failure in passes 1 to 3



(b) Norm of projection of the error to the subspaces containing $\mathcal{W}^{\star}(\mathcal{L}^i)$, $i = 1, 2, 3$ in passes 1 to 3.

FIGURE 6.3: Biased actuator

Then as shown in Figure 6.3(b), projecting the error vector onto subspaces $\mathcal{W}^\star(\mathcal{L}^1)$, $\mathcal{W}^\star(\mathcal{L}^2)$, and $\mathcal{W}^\star(\mathcal{L}^3)$, reveals that the bias fault has occurred in the second actuator.

Additionally, by looking at the error signal, one can also recognise the type of the failure (i.e., dead or biased). In case of a dead actuator as can be seen in Figure 6.2(b), the behaviour of the error signal depends on the input signal, which is a decreasing signal on each pass here, whereas in the case of biased actuator, the error signal does not depend on the input and stays constantly at $b \in \mathbb{R}$ after the bias enters the system. Refinements of this sort could have importance implications in practical applications, but we will not pursue them here.

6.6 Summary

In this chapter a geometric approach to address the FDI problem in discrete linear repetitive processes was developed. Based on the discrete unit-memory model (5.7), a model that incorporates failure signatures and modes was developed.

In our method, the whole system state is reconstructed instead of exploiting just the system output for residual generation. Thus, a wider range of failures can be detected and isolated compared to other methods used in the past (for example [1]).

At the end the effectiveness of the proposed approach was illustrated by providing an industrial example in which we detect and isolate a dead and a biased actuator for a metal rolling application.

Chapter 7

Conclusion and Future Work

7.1 Conclusion

This thesis investigated the FDI problem in two sub-classes of multidimensional systems namely, 3-D systems and linear repetitive processes. A geometric model-based approach was developed to propose a solution that upon satisfying a number of necessary and sufficient conditions, a failure can be isolated uniquely.

The technique is although inspired by its 1-D counterpart [1], it is distinctive in the sense that since the 1-D counterpart developed by Massoumnia is relied on the residual generation and the residual vectors are dimensionally dependent on the system's output which can be of low dimensions, this imposes a restriction in the number of faults that can be isolated. Consider the case where the output is of dimension one (i.e., a scalar), it is easy to see that it is not possible to isolate more than one failure due to the dimension of the output. In the proposed technique, instead, the state of the system is considered only, which offers more opportunity to address a wider class of failures.

The results developed in this thesis can only be considered preliminary for several reasons. A basic issue is the "compounded weakness" inherent in stating conditions which are only sufficient (see Theorem 4.6) for the solvability of the problem, which in turn rely only on sufficient conditions (see Proposition 3.13) for the existence of stabilising gains. However, one can always find a stabilising gain which assigns the eigenvalues of the dynamics of the observer to zero as stated in Theorem 4.2 so that the error vector immediately goes to zero if no faults are present in the system and passes a certain threshold where a fault has occurred at the stage where the system starts running. The issue of how conservative our conditions are, and consequently how robust with respect to modelling errors and disturbances a fault detection scheme based on the principles used in this report is, is of course directly linked to this problem.

7.2 Future Research

There are important issues that need to be addressed to make the presented approach more realistic for application to real-life situations. One aspect to consider is that the approach outlined in this thesis treats all independent variables to be on an equal footing, while for some applications such as failure detection in grid sensor networks, one of the independent variable is time. A "time-relevant" fault estimation framework is needed, where the distinguished role of the independent variable time in the modelling of the fault and in its estimation is recognized and exploited to provide better performance (on time-relevant systems, see also [82, 83]).

We have also assumed that the corrective actions for failure accommodation are known. However a possible future work could be the case where these corrective modes are unknown and a reconfiguration model should be sought such that the system operates at its best possible performance.

In this thesis, we assumed that the system is not affected by noise. However in real life applications, noise and disturbances are inevitable parts of a system. Therefore, one possible future work could be the case where the system is affected by noise.

The research interests in the area of multidimensional systems have been stimulated by contributions dealing with river pollution modelling [84], modelling of a single-carriageway traffic flow [85], gas absorption and water stream heating [86], etc. FDI for these systems is a very important task and can be investigated using the same approach to detect a collision in a motorway or pollution in a river. Moreover, Iterative Learning control as an algorithmic application of repetitive processes is another area that there has been no previous attempts to specifically address the FDI problem therein. What is termed as a failure in this thesis, is very general and can be expressed as any occurrence in the system that disturbs the system's dynamics such that the its behaviour changes. This expression lets our technique to be applicable in an extensive range of applications from detection and isolation of actuator failures in a plant to detection and localisation of a leakage in pipelines. Below we outline a few of these possiblities for future work.

7.2.1 FDI for Repetitive Processes in Presence of Noise

Consider a unit-memory linear repetitive process that is affected by noise. in the sequel, we attempt to shortly investigate the FDI problem in this case.

Fault detection in presence of noise is directly linked to defining a suitable threshold that the error signal is compared to. In general, thresholds are of two types, either fixed or adaptive to the input, each of which has its own advantages and disadvantages. An adaptive threshold changes according to the inputs to the system; thus, it has many advantages over the fixed threshold. In case of the fixed threshold, if the threshold is set too high, sensitivity to fault detection will decrease. In contrast, choosing the threshold too low increases the rate of false alarms. An adaptive threshold, however, does not have these problems but one downside of using this type of threshold is the high order of its dynamics [87]. Different methods of selecting a threshold in fault detection and identification problems are studied in [88, 89].

Selecting an appropriate threshold ε is directly related to the degree to which the system is affected by noise or disturbances. It is clear that in absence of noise (AWGN), setting $\varepsilon = 0$ is reasonably convincing. However, in presence of noise, an appropriate choice of threshold, if exists, in addition to fault detection (i.e., $||e_k(p) > \varepsilon||$) should also enable isolating the fault uniquely.

In practice, a known class of failures can occur in the system and detection is limited by a bounded noise signal $\begin{bmatrix} 0 & N_{max} \end{bmatrix}$, and a bounded model error. Hence, a priori knowledge of the system and its behaviour under known failure classes plays an important role in determining a reasonable threshold. It should be noted that in presence of too much noise that considerably affects the system's behaviour, FDI is not possible. In this section, we investigate finding a fixed threshold with an example using statistical and geometric analysis.

Consider the linear repetitive process Σ with the following description:

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 4 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.5 \\ 0.5 & 0.3 \end{bmatrix},$$

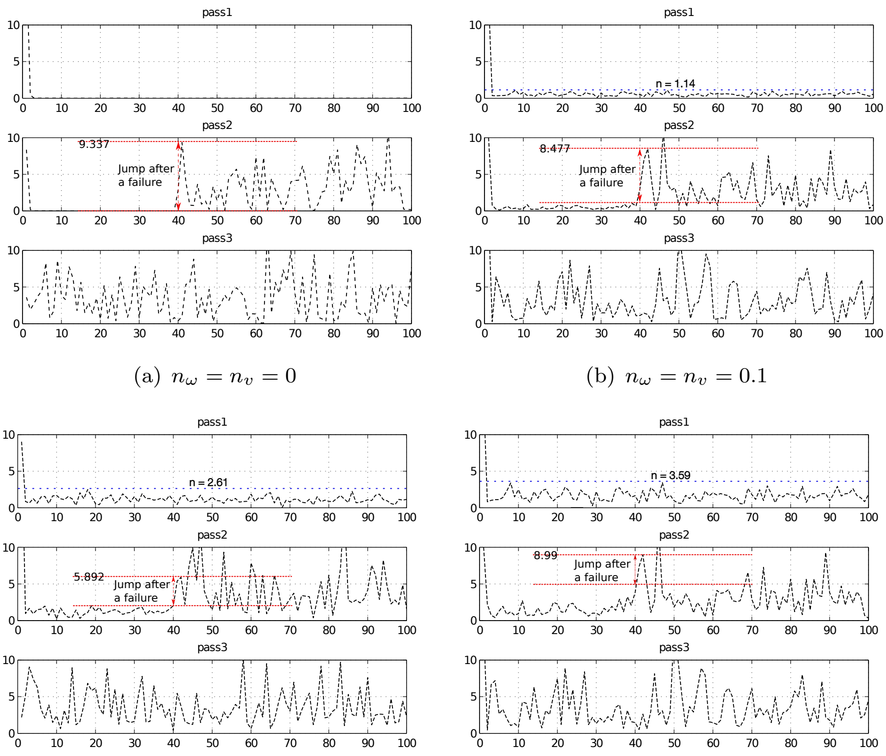
$$C = \begin{bmatrix} 0.4 & 0.1 & 0.6 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}.$$

Assume that the first actuator, i.e., $\begin{bmatrix} 1 & 1 & 4 \end{bmatrix}^\top$, dies at $k = 2,\ p = 40$. The challenge in selecting an appropriate threshold arises once the system is affected by noise. Assume the system Σ is affected by Guassian noise with zero mean (i.e., noise power is equal to its variance). The state-space model accommodating the failure signature together with noise is described by

$$\begin{aligned} x_{k+1}(p+1) &= A_1 x_{k+1}(p) + B u_{k+1}(p) + A_2 x_k(p) + L^1 m^1_{k+1}(p) + \omega_{k+1}(p)\,, \\ y_k(p) &= C x_k(p) + v_k(p)\,, \end{aligned} \tag{7.1}$$

where $\omega \sim \mathcal{N}(0, n_\omega)$ and $v \sim \mathcal{N}(0, n_v)$.

We consider four different scenarios where $n_\omega,\ n_v \in \{0,\ 0.1,\ 0.5,\ 1\}$. In the absence of noise ($n_\omega = n_v = 0$), the error initially goes to zero but after the failure occurs, the error norm increases significantly that suggests selecting a threshold $\varepsilon = 0$ is a reasonable choice. In the presence of noise however, the error due to the initial conditions does not go to zero but approaches towards a range $\begin{bmatrix} 0 & e_{init} \end{bmatrix}$, where e_{init} increases with the noise power. The performance of the error signal for the three first passes of the mentioned scenarios is depicted in Figures 7.1(a) - 7.1(d).



(a) $n_\omega = n_v = 0$ (b) $n_\omega = n_v = 0.1$

(c) $n_\omega = n_v = 0.5$ (d) $n_\omega = n_v = 1$

FIGURE 7.1: Error magnitude before and after failure of the first actuator.

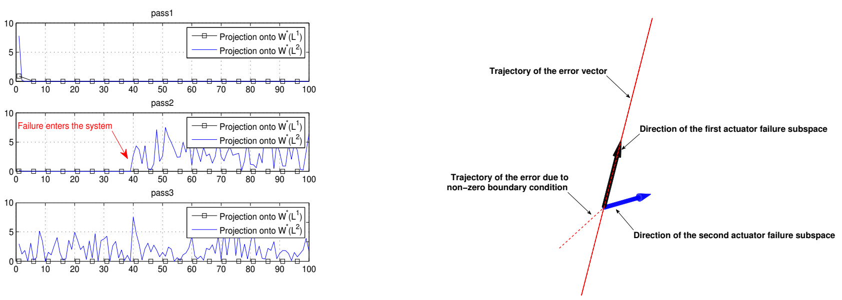
Note that immediately after the failure, the error norm jumps significantly. However, with the noise increasing, the upperbound n increases as well. Hence this significance becomes less obvious as the noise power rises. This is also shown in Table 7.2.1 where the average error norm before the failure is compared to the error norm immediately after the failure in 10000 simulations.

Inp./Out. Noise Variance	Magnitude Ratio (×Times)
0.1	31
0.5	6.96
1	3.74

TABLE 7.1: Comparison of the average error norm before the failure and the error norm after the failure in 10000 simulations.

These experimental trials help selecting a fault detection threshold for the case of the first actuator failure in various noise scenarios of this example. An identical procedure can be carried out for the second actuator failure to obtain a priori knowledge of the system corresponding to the respective failure class.

To isolate the detected failure, we plot the norm of the projection of the error to the subspaces that contain failure signatures (i.e., $\mathcal{W}^\star(\mathcal{L}^i)$, $i = 1, 2$) for the first three passes of the process (see Figure 7.2(a)). In the absence of noise, this projection goes to zero initially but after the failure, it deviates from zero and lies on $\mathcal{W}^\star(\mathcal{L}^1)$ where the fault has happened. Moreover, we plot the trajectories of the error vector on the second pass where the fault has happened. From Figure 7.2(b) it is clear that these trajectories only travel along the first actuator signature. Hence, the failed actuator can be isolated uniquely.

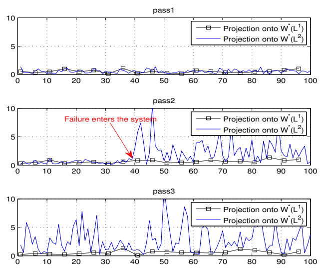


(a) Norm of projection of the error to the subspaces containing $\mathcal{W}^\star(\mathcal{L}^i)$, $i = 1, 2$ in passes 1 to 3.

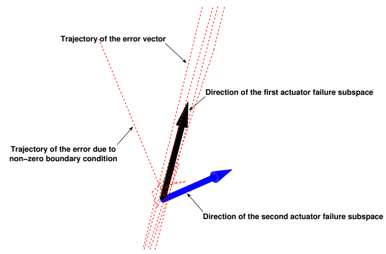
(b) Trajectories of the error vector lies on the subspace spanned by L^1 after going to zero.

FIGURE 7.2: Failure isolation for the case $n_\omega = n_v = 0$.

Now consider the three scenarios where $n_\omega, n_v \in \{0.1, 0.5, 1\}$. In the presence of noise, projection of the error neither goes to zero nor lies on the failed actuator but lies at a distance d and d' to the failed and healthy actuator, respectively. In the presence of noise, $d \ll d'$ should hold distinctively for FDI to perform efficiently. After a threshold is passed, the error trajectories no longer travel along the failed actuator signature due to noise but form a "tube" around it. This is illustrated in Figures 7.3(a) - 7.5(b). It can be seen that as the noise power increases the error trajectories scatter more across the space that fault isolation is not possible even if fault detection might be (see Figure 7.5(a)-7.5(b)).

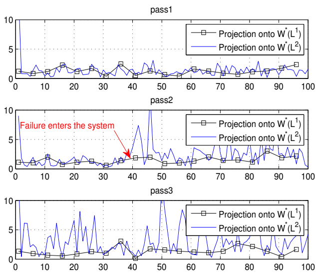


(a) Norm of projection of the error to the subspaces containing $\mathcal{W}^{\star}(\mathcal{L}^i)$, $i = 1, 2$ in passes 1 to 3.

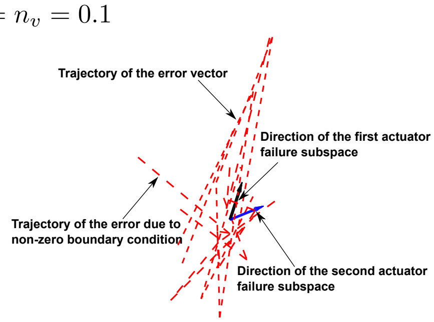


(b) Trajectory of the error after a failure.

Figure 7.3: $n_\omega = n_v = 0.1$

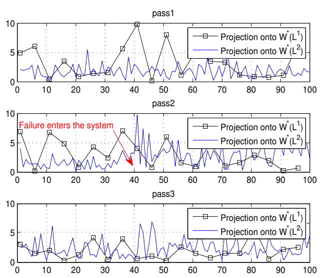


(a) Norm of projection of the error to the subspaces containing $\mathcal{W}^{\star}(\mathcal{L}^i)$, $i = 1, 2$ in passes 1 to 3.

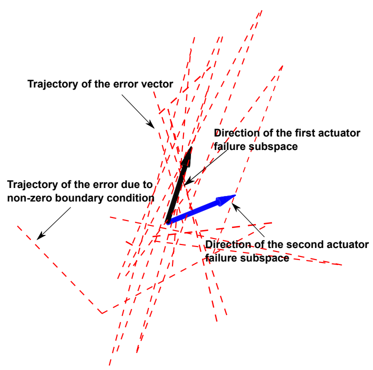


(b) Trajectory of the error after a failure.

Figure 7.4: $n_\omega = n_v = 0.5$



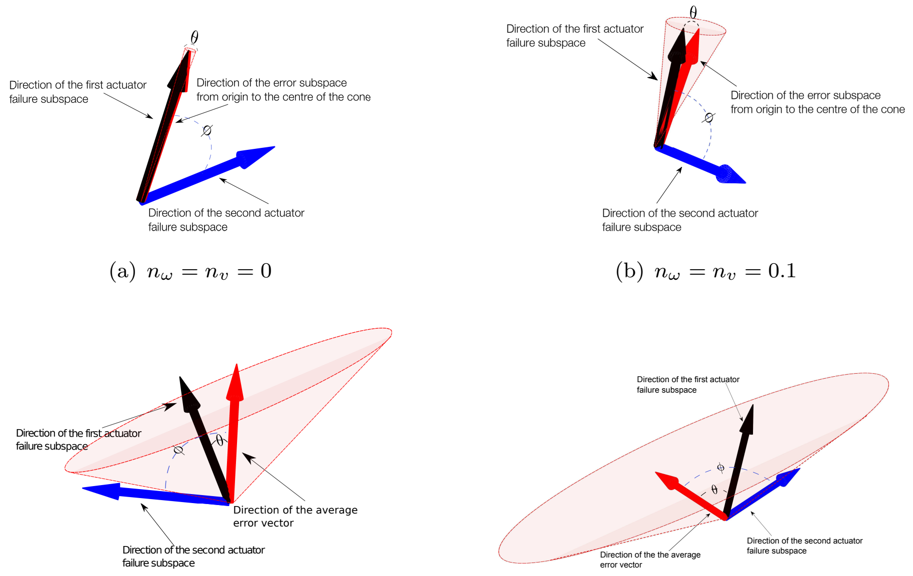
(a) Norm of projection of the error to the subspaces containing $\mathcal{W}^{\star}(\mathcal{L}^i)$, $i = 1, 2$ in passes 1 to 3.



(b) Trajectory of the error after a failure.

Figure 7.5: $n_\omega = n_v = 1$

From this example it is concluded that for FDI algorithm to work, noise should be bounded in a specific range. Moreover, in presence of noise, the error vectors do not stay in a subspace but form a "cone" with the vertex at origin. This can be seen intuitively since the zero subspace is included in all of these subspaces (that forms the vertex of the "cone"). Moreover, the axis is formed by adjoining $\{0\}$ to the centroid of the error vectos which can be obtained through well-defined algorithms (see for example [90]). Then, the error vectors with the largest distance to the centroid determine the slants of the cone. Choosing the appropriate threshold, if possible, is related to the openness of the cone vertex or the radius of the base r. Figure 7.6(a)-7.6(d) shows the directional subspaces spanned by the two actuators of the presented example together with the error cone and the average error direction.



(a) $n_\omega = n_v = 0$

(b) $n_\omega = n_v = 0.1$

(c) $n_\omega = n_v = 0.5$

(d) $n_\omega = n_v = 1$

FIGURE 7.6: Error cone for the presented example

Now we investigate the problem of finding a fixed threshold for fault isolation. For this purpose, we first define the *isolation section* for each failure signature.

Definition 7.1. The smallest subspace in which a failure can be isolated uniquely with a probability $\eta < p \leq 1$ is called an *isolation section* where η is the minimum acceptable performance of the fault isolation algorithm.

Denote the angle between the error vector and the failed actuator by θ and the angle between the error vector and the healthy actuator by ϕ. Moreover, partition the angle between the failure signatures L_q and $L_{\overline{q}}$, $q, \overline{q} = 1, 2, ..., l$, $q \neq \overline{q}$ equally into ψ_q and $\psi_{\overline{q}}$. Then the breadths formed by ψ_q's are the isolation sections corresponding to the q-th signature.

In the case of this example, these isolation sections are two "cones" generated from the rotation of a triangle with the failure signature as the axis and vertecies at origin. Slants of the cones are determined by the breadths ψ_q's.

To isolate a failure, we continuously measure the angle between a failure signature and the error vector and denote it by $\Omega_i,\ i = 1, 2, ..., l$. Then the smaller Ω_i represents a possible failure of the respective actuator. This can be used in parallel to other methods mentioned earlier to provide a better result.

In this example, $\psi_1 = \psi_2 = \frac{63.31}{2}^\circ$. Simulating the first actuator failure 10000 times with this threshold in various noise levels, results in the following performances of the fault isolation algorithm:

Inp./Out. Noise Variance	Success Rate
0	99.55%
0.1	91.70%
0.5	82.63%
1	76.51%

TABLE 7.2: Performance of the FDI algorithm with the angular threshold $\psi/2$ in 10000 simulations.

It should be noted that in the absence of noise, due to non-zero initial conditions the success rate is not 100%. In the case where more actuators are present with non-equal angle distribution, this approach is still adoptable, however the isolation section will no longer be symmetric and as the angle between failure signatures decreases, the probability of a false alarm increases.

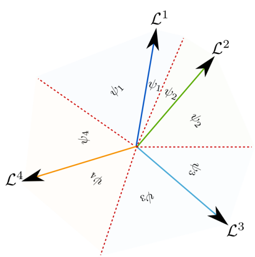


FIGURE 7.7: An example of several non-symmetric isolation sections $\psi_i,\ i = 1, ..4$ corresponding to each signature.

Our FDI procedure is as follows:

Algorithm 7.2.1: FDI PROCEDURE()

---

1 : AEN* $= 0$;
2 : **for** $k = 0$ **to** K^{**}
3 : **for** $p = 1$ **to** α
4 : $||e_k(0)|| = rand$;
5 : Compute
AEN $= (\text{AEN} + ||e_k(p-1)||)/p$;
6 : **return** (AEN) ;
7 : **if** $||e_k(p)||/\text{AEN} > \varepsilon$
then
8 : Compute $e'_k(p)$, the projection of $e_k(p)$
down onto $\mathcal{V}^\star(\mathcal{L}^i)$;
9 : Compute
$f \ := \ arg \ \max\{||e'_i(p)||\}, \ i = 1, ..., l$
10 : **return** (f) ;
11 : Compute Ω_i, $i = 1, ..., l$ the angle between the i-th
failure signature and the error vector $e_k(p)$;
12 : Compute
$g \ := \ arg \ \min\{\Omega_i\}, \ i = 1, ..., l$
13 : **return** (g) ;
14 : **end if**
15 : **end for**
16 : **end for**

---

* AEN is the Average Error Norm.

** K is the number of passes.

Note that ε is obtained from statistical analysis of the previous history of the system. Moreover, due to the assumptions, the computation of f in **Step 9** is well-defined and Steps **11-13** are used to verify the results from **Step 10**.

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