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Optimisation of a Passive Vibration Isolator with Quasi-Zero-Stiffness Characteristic

A. Carrella, T.P. Waters and M.J. Brennan

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

Optimisation of a Passive Vibration Isolator with Quasi-Zero-Stiffness Characteristic

by

A. Carrella, T.P. Waters and M.J. Brennan

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Authorised for issue by
Professor M.J. Brennan
Group Chairman

Abstract

The frequency range over which a mount can isolate a mass from a vibrating base (or vice versa) is often limited by the mount stiffness required to support the weight of the mass. This compromise can be made more favourable by employing non-linear mounts with a softening spring characteristic such that small excursions about the static equilibrium position result in small dynamic spring forces and a correspondingly low natural frequency. This report concerns the force-displacement characteristic of a so-called quasi-zero-stiffness (QZS) mechanism which has an appreciable static stiffness but very small (theoretically zero) dynamic stiffness.

The mechanism studied comprises a vertical spring acting in parallel with two further springs which, when inclined at an appropriate angle, produce a cancelling negative stiffness effect. Analysis of the system shows that a QZS characteristic can be obtained if the system's parameters (angle of inclination and ratio of spring stiffnesses) are opportunely chosen. By introducing the additional criterion that the displacement of the system be largest without exceeding a desired (low) value of stiffness an optimal set of parameter values is derived. Under sufficiently large displacements the stiffness of the QZS mechanism can eventually exceed that of the simple mass-spring system and criteria for this detrimental scenario to arise are presented. Finally, it is shown that, if the system's stiffness is small, the force-displacement characteristic of the system can be approximated by a cubic equation without incurring too large an error.

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1. Introduction

Isolation of undesirable vibrations is a problem that affects most engineering structures. Better isolation is generally achieved if an isolator with small stiffness is employed. There are two reasons for this. First, since for a single degree of freedom isolation occurs at frequencies greater than $\sqrt{2}$ times the natural frequency, it is evident that a lower natural frequency results in a wider isolation region. However, the drawback of such a configuration is an increase in the static displacement of the mass. The trade-off between stiffness and static displacement was highlighted in reference [1], and in reference [2] a practical solution to the problem of vibration isolation in sea vessels where the space is limited was discussed.

The aim of this report is to investigate the possibility of achieving a very low stiffness without having the drawback of large static displacements. In particular, the analysis aims to maximise the excursions of the isolated object without the system exceeding a desired (low) stiffness.

In the literature, the isolating system studied herein is referred to as a *quasi-zero-stiffness*(QZS) mechanism [3].

1.1. Background of Quasi-Zero-Stiffness Mechanisms

If a suspension system has extremely low natural frequency, i.e. very long periods of oscillation, the frequency range of isolation becomes wider with the benefit of improved vibration isolation performance. However, these systems are of interest also in other fields, for example in geodynamics [4, 5]. The precision of instruments such as *seismographs* or *gravimeters* requires very long periods of oscillation. Probably the most well-known project was by Lacoste in the 1930's. The realisation of a spring with zero initial length, hence the name *zero-length spring*, was his patent [6]. The principle behind this device is that the physical length of the spring (once the load has been applied) is equal to its elongation. This is achieved in the manufacturing process by applying a particular pre-stress proportional to the length of the spring. This is obtained by twisting the wire of the springs as it is coiled. In practice it is found that the period of oscillation is not greater than 900 seconds.

Further configurations of QZS mechanisms for the same purpose are described in reference [7].

1.2. Non-linear Stiffness

In general, it is often assumed that the relationship between the force applied on a spring and its displacement is linear. However, in practice, either by design or physical limitations, a spring always presents some degree of non-linearity. Therefore, its static stiffness differs from its local stiffness about the equilibrium position.

Before discussing the details of an isolation system based on the principle of quasi-zero-stiffness, the concept of stiffness and the difference between *static* and *dynamic* stiffness need to be reviewed.

In this context of structural vibrations, the stiffness expresses the relationship of load and deformation of an element (e.g. springs or mountings in vibration isolation). The force exerted by a stiffness element depends on its displacement and balances the applied external force. Thus, the stiffness is given by the slope of the force-displacement characteristic (first derivative of the force with respect to the displacement). Furthermore, this is a conservative mechanism because no energy is dissipated.

The *static* stiffness represents the capability of an isolator to resist a static load. Conversely, the *dynamic* (or also *local*) stiffness is related to the force produced by the mounting when oscillations about one specific position occur. In a *linear spring* these two quantities are the same. Fig.1 depicts a non-linear force-displacement characteristic (namely, a *softening* spring). If a static load f_{st} is applied, a displacement x_0 occurs (point A of the graph). The static stiffness is given by the slope of the line $\overline{OA}$. However, around the point A the slope of the curve is almost zero, which implies that for small deviations from x_0 the local (dynamic) stiffness is very small.

Such a spring, however, is not suitable for isolation purposes. Above a certain displacement (mainly dictated by the amount of available space) it is desirable to have a steep rise in stiffness (e.g., this is achieved, for example, by bump-stops in a car suspension system), as shown in Fig.2. In so doing, the excursion range is limited even if a large force is applied.

1.3. Isolators with QZS

The force-displacement characteristic shown in Fig.2 has some desirable features. The properties of a mounting with such a load-deflection response can be exploited for isolation purposes

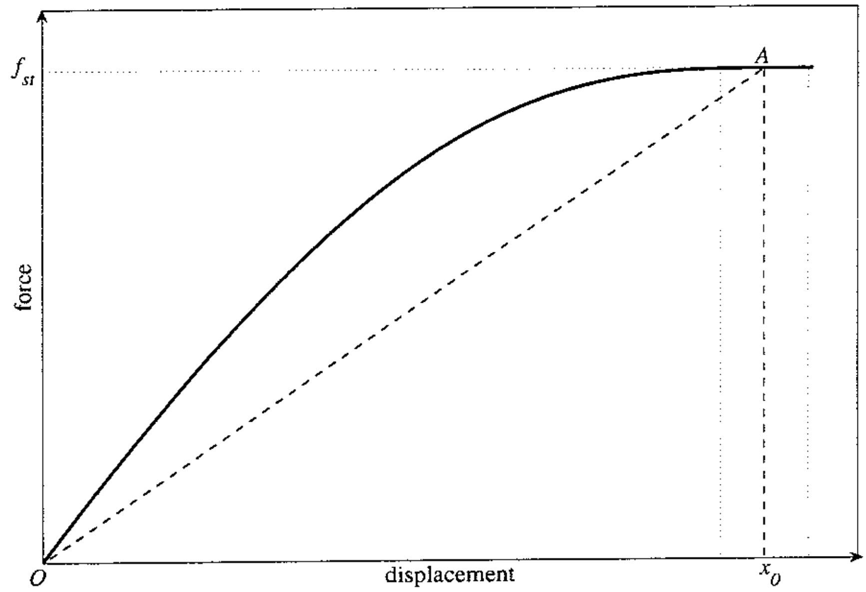

Fig. *1: Force-displacement characteristic of a spring. The curve shows a non linear stiffness characteristic (softening spring). If a static load f_{st} is applied a displacement x_0 occurs. The static stiffness at the point A is given by the slope of the line $\overline{OA}$; the dynamic stiffness (for small oscillations about x_0) is zero*

if the system designed to oscillate about x_0. As can be seen, the force, i.e. the disturbance transmitted to/from the base, is constant for small oscillations about x_0. If the oscillations becomes large the increase in stiffness has the beneficial effect of preventing too large excursions from occurring. Systems that present this kind of behaviour are ideal for vibration isolation purposes [3].

Applications of mechanisms with a quasi-zero-stiffness characteristic range from space research (e.g. to simulate zero gravity, [8]) to isolation of high precision machinery [9].

A passive isolator based on a mechanism that yields quasi-zero-stiffness is commercialised by Minus K Technology. Platus has described the principles behind this isolator [10]. However the description lacks rigorous analysis of the system. Zhang *et al* [11] provided experimental support for an analytical model of a similar mechanism. In particular, the analysis was based on two parameters: the prestress of the *correction springs* (which provide negative stiffness) and the length of the connection between these springs and the suspended mass. The variables chosen do not offer an in-depth view of the mechanism and no optimisation analysis is pre-

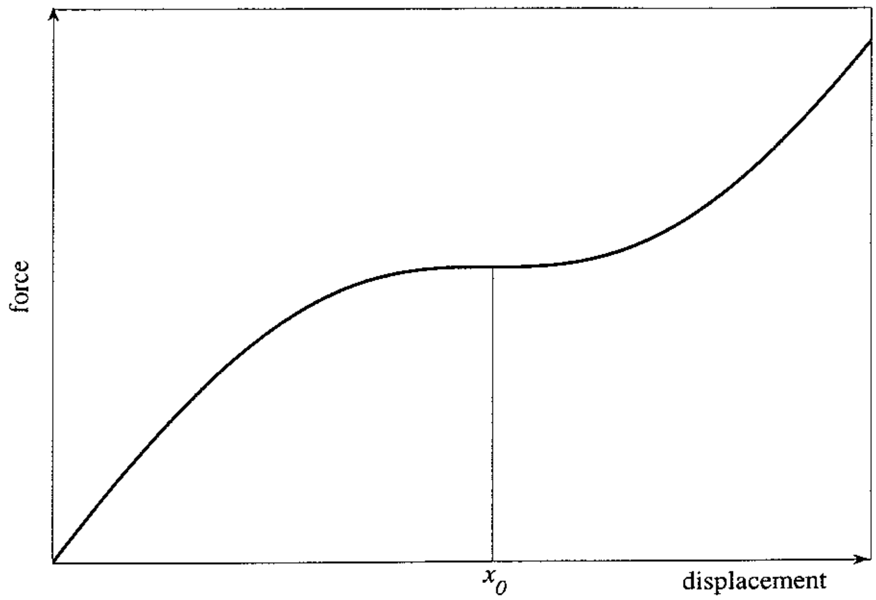

Fig. 2: *Force-displacement characteristic of an isolator with softening, quasi-zero and hardening stiffness regions*

sented. However, their experimental results prove the effectiveness of such a mechanism.
As far as the authors are aware, no attempt has been made to find an optimum configuration of a QZS isolator with the aim of maximising the oscillation range over which the system has an almost-zero stiffness. Therefore, this report focusses on this issue.

A passive isolator that has a QZS characteristic, in its simplest form, is composed of a set of two oblique linear springs (henceforth referred to as *correction springs*) connected in parallel to a third vertical resilient element (also called *supporting spring*) [3, 12] as shown in Fig.3.

If this system is subject to a force f, the force-displacement characteristic is non-linear and similar to that depicted in Fig.2. In particular, three regions can be defined: softening, quasi-zero and hardening.
In this report it is shown that there exists one optimal geometrical disposition of the springs and their coefficients that allows the largest displacement to be achieved without the system reaching a stiffness greater than a small desired value.
In the first instance, the separate effects of the two sets of springs (corrective and supportive) are studied and then combined. Since the supporting spring (vertical mount) produces a linear

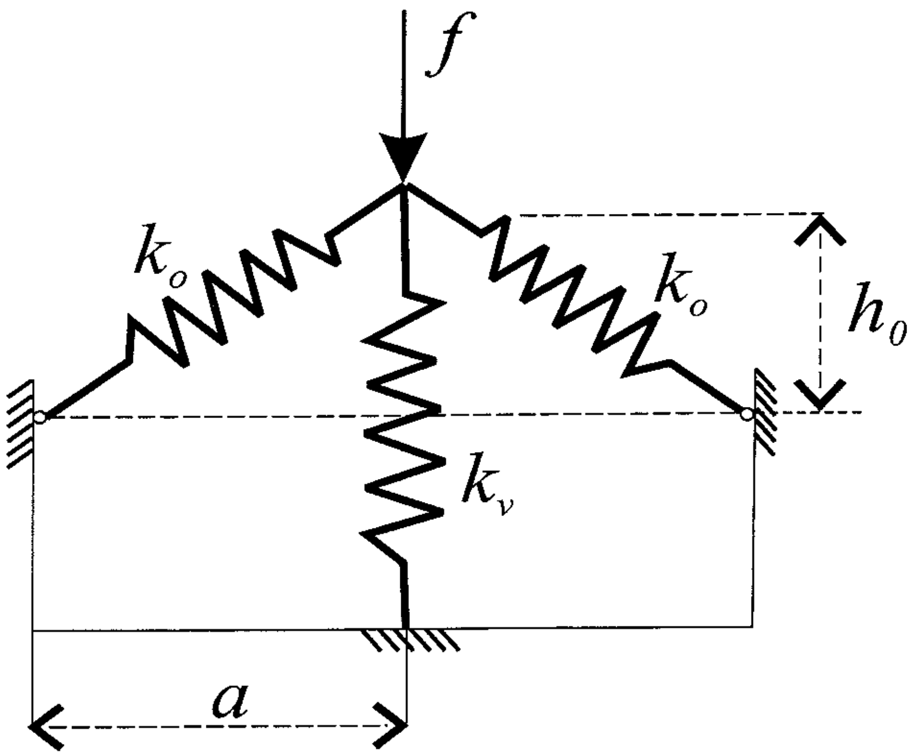

Fig. *3: Schematic representation of the simplest system with quasi-zero stiffness mechanism*

positive stiffness, its behaviour does not need further discussion. Conversely, one section focusses on the force-displacement characteristic of a system with two oblique springs of identical stiffness for an in-depth analysis of the non-linear characteristic and negative stiffness. In section 3 the system with three springs is discussed together with an optimisation analysis based on two variables: the geometry and ratio of the supporting spring's coefficient to the coefficient of the corrective springs. Finally, conclusions are drawn.

2. Force-displacement characteristic of a system with two oblique springs

The system in Fig.4 has two linear springs with identical stiffness k_o and initial length L_0. A force f is applied as shown in the figure. The springs are hinged at M and N. The point P (application point of the force) is a horizontal distance a from M and N and initially at height h_0 from the horizontal line. The springs are at an angle θ_0 from the horizontal.

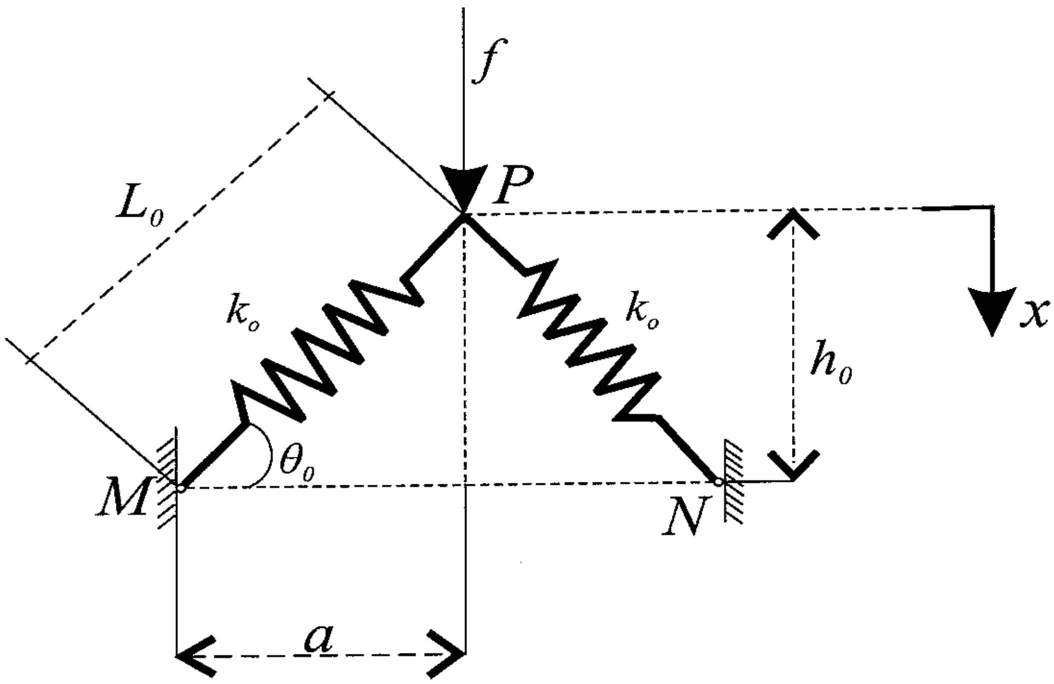

Fig. *4: Schematic representation of an isolator with two oblique springs. The springs oppose the applied load f, and the geometry gives rise to a non-linear force-displacement characteristic*

Following the application of the force f after a vertical displacement x the length of the springs becomes L. The force exerted by each of the springs is therefore

$$f_o = k_o(L - L_0) \tag{1}$$

However, only the vertical component resists the applied force f. Hence

$$f = 2k_o(L_0 - L)\sin\theta \tag{2}$$

where $\sin\theta = \frac{(h_0 - x)}{L}$.

Thus, the force-displacement relationship can be written as

$$f = 2k_o(h_0 - x)\left(\frac{L_0}{L} - 1\right) \tag{3}$$

From Fig.4 it can also be seen that

$$L_0 = \sqrt{h_0^2 + a^2} \tag{4a}$$

and

$$L = \sqrt{(h_0 - x)^2 + a^2} \tag{4b}$$

Combining Eqn.(3) with Eqns.(4) gives

$$f = 2k_o(h_0 - x)\left(\frac{\sqrt{h_0^2 + a^2}}{\sqrt{(h_0 - x)^2 + a^2}} - 1\right) \tag{5}$$

This expression can be non-dimensionalised by dividing the displacement x by L_0 and the force f by k_oL_0 . Thus, Eqn.(5) can be written as

$$\hat{f} = \frac{f}{k_oL_0} = 2(\sqrt{1-\gamma^2} - \hat{x})\left\{\left[\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1\right]^{-1/2} - 1\right\} \tag{6}$$

where $\hat{x} = x/L_0$ is the *non-dimensional displacement* and

$$\gamma = \frac{a}{L_0} = \cos\theta_0 \tag{7}$$

is a *geometrical parameter.* When $\gamma = 0$ the springs are initially vertical and when$\gamma = 1$ the springs, initially, lie horizontally.

Fig.5 shows the non-dimensional force plotted against the non-dimensional displacement for different values of γ.

It can be seen that the system shows a highly non-linear characteristic and regions with negative stiffness. From Fig.5 it can also be observed that each curve has a maximum and a minimum. The peak represents a point of unstable equilibrium. After the peak the stiffness of the system becomes negative. If too large a force is applied the system snaps through the static equilibrium position until a new equilibrium position is reached. The maximum non-dimensional force that the system can accept before it snaps through can be expressed as a function of the geometrical parameter γ and is given by

$$\hat{f}_{max} = 2\left[1 - \left(1 - \sqrt{1-\gamma^2}\right)^{1/3}\right]^{3/2} \tag{8}$$

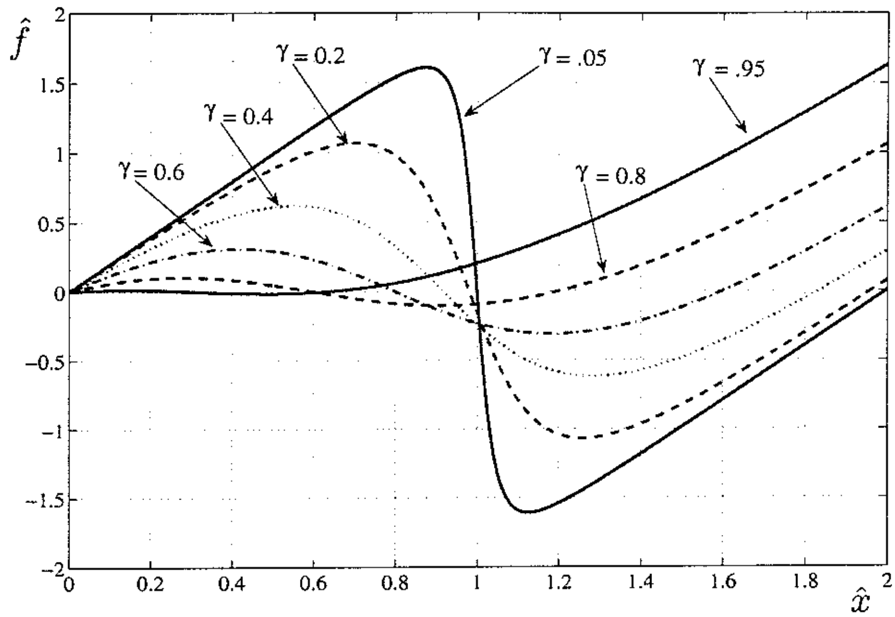

Fig. 5: *Force-deflection characteristic of the system represented in Fig.4. When $\gamma = 0$ the springs are vertical and when $\gamma = 1$ they are horizontal. After the maximum the curves have a region with negative stiffness*

which occurs at

$$\hat{x}_{max} = \sqrt{1-\gamma^2} - \gamma\sqrt{\gamma^{-2/3} - 1} \tag{9}$$

Finally, the stiffness of this system can be calculated by differentiating the force with respect to the displacement, and is given by

$$\hat{K} = \frac{d}{d\hat{x}}(\hat{f}(\hat{x})) = 2\left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1\right)^{3/2}}\right] \tag{10}$$

Following the application of the load f the springs deform. The curves in the plot represent the restoring force as the springs deform. When the springs lie in the horizontal position ($x = h_0$, that is $\hat{x} = \sqrt{1-\gamma^2}$) the restoring force is zero and this is due to the fact that there is not a vertical component of the spring force able to resist the applied load f. At this point, any perturbation (e.g. a small displacement) would bring the system to a different position of equilibrium, thus the horizontal position is a position of unstable equilibrium.

The instability described makes this system unsuitable for isolation purposes. However, the geometric non-linearity and negative stiffness of this system can be exploited in order to obtain a system that has a region with quasi-zero-stiffness (QZS).

In Fig.6 the stiffness $\hat{K}$ is plotted against the displacement $\hat{x}$, for $\gamma = 0.8$.

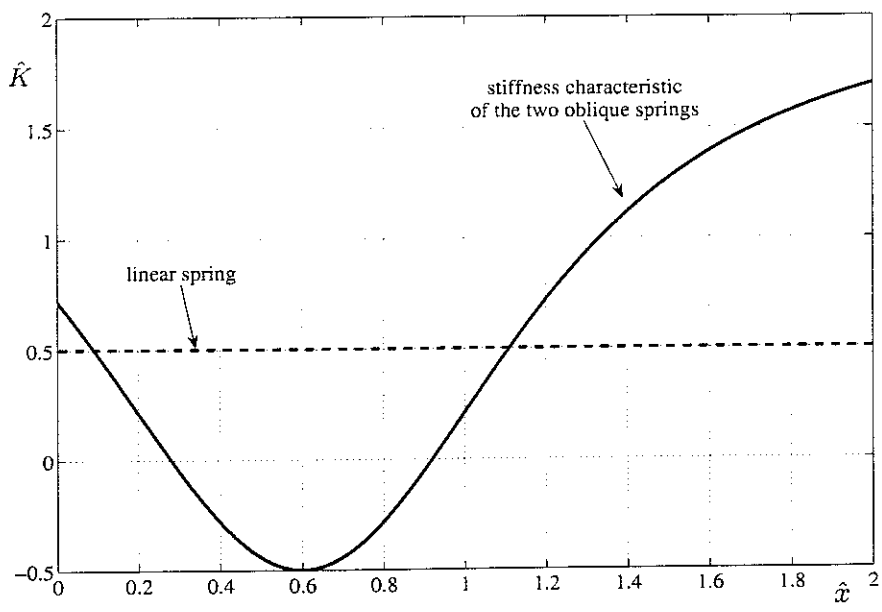

Fig. *6: Representation of the non-dimensional stiffness of the stiffness of a system with two oblique springs against the non-dimensional displacement; the dashed lines is the constant stiffness of a linear spring*

The dotted curve represents the non-dimensional stiffness of the system depicted in Fig.4 as a function of non-dimensional displacement. It is positive for small values of $\hat{x}$, goes through zero then reaches a minimum before increasing and becoming positive again. In the same figure the dashed line represents the stiffness characteristic of a linear spring with non-dimensionalised stiffness $\hat{K} = 0.5$.

The two characteristics can be combined so that the stiffness becomes zero but never negative. If carefully designed it is possible to have a very small stiffness for a relatively wide range of displacements, with beneficial consequences for the isolation performance. In [12] several systems with QZS are presented and discussed. Of particular interest to the authors is the system shown in Fig.7. It represents a parallel combination of two oblique (correcting) springs and a vertical (supporting) spring.

Unlike the system with only oblique springs, there is always a force that opposes the applied load f. When the correcting springs lie horizontal the supporting spring resists the force f. In

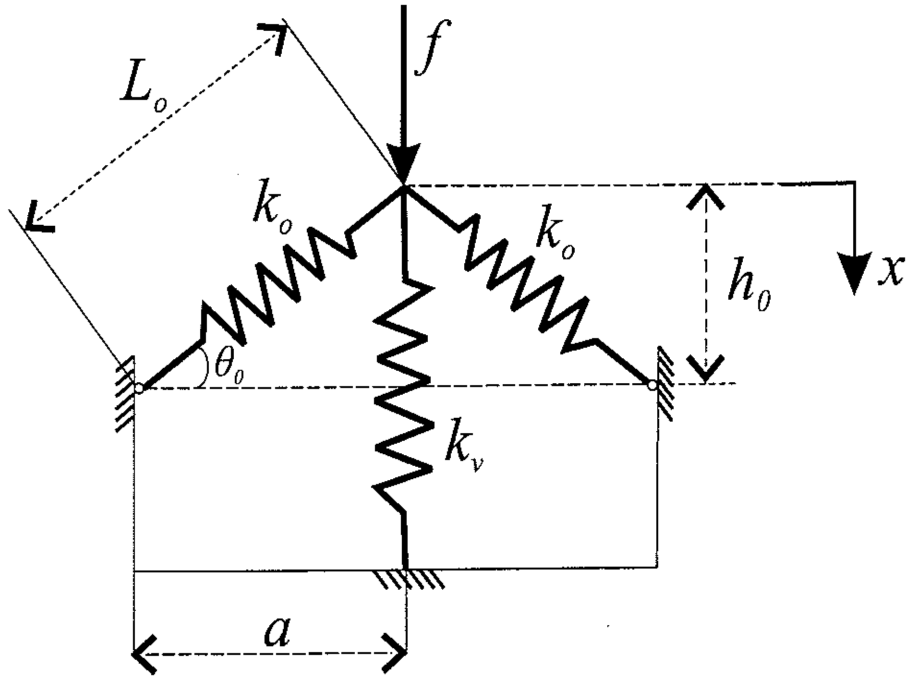

Fig. 7: *Schematic representation of an isolator which has quasi-zero stiffness*

practice, the vertical component of the correcting springs' force acts in parallel with the force of the vertical spring. By choosing the stiffness of the vertical mounting (positive) to be equal to the minimum stiffness of the oblique springs (negative) the system can have zero stiffness at one particular displacement. The stiffness of such a system is plotted in Fig.8 as function of the displacement.

The mechanism depicted in Fig.7 allows a very low stiffness to be realised for a range of displacements and has zero stiffness when

$$\hat{x} = \sqrt{1-\gamma^2} = \hat{x}_e \tag{11}$$

If the system oscillates with a small amplitude about this point, the stiffness will be very small (*quasi-zero*). Herein the point $\hat{x}_e$ is referred to as the **static equilibrium position.** This is the nature of a *quasi-zero-stiffness* (*QZS*) mechanism.

In the following sections a QZS system is investigated to determine the displacement range over which there is a very small stiffness.

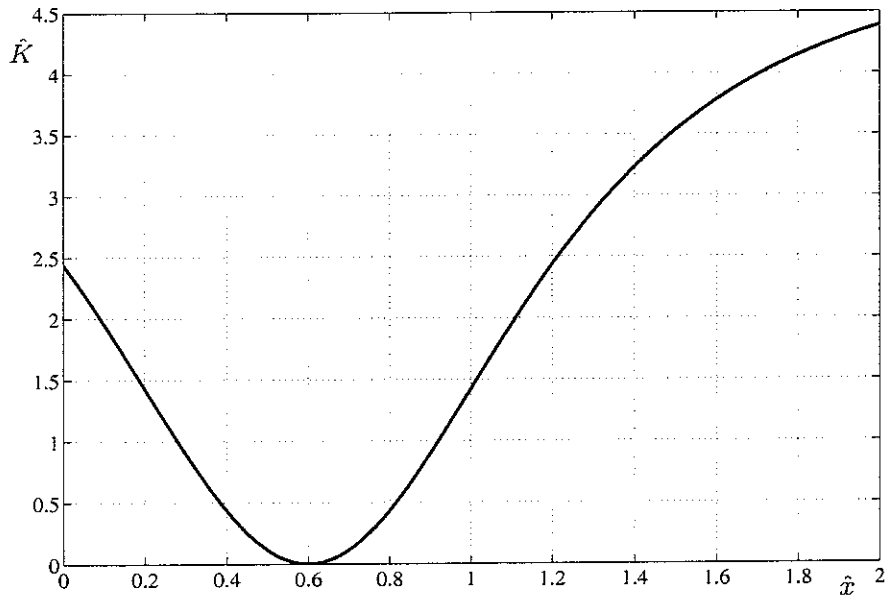

Fig. *8: Stiffness of the QZS mechanism depicted in Fig.7 with* $\gamma = 0.8$ *as function of displacement. The stiffness reaches zero at one point and is low over a displacement range*

3. A MECHANISM WITH QZS

One of the simplest system with a QZS characteristic is depicted in Fig.7. This system has a third vertical linear spring added to the two oblique springs studied in the previous section. The three springs act as if connected in parallel, thus the restoring force of the system is the sum of the stiffness of the vertical mounting and the vertical component of the restoring force exerted by the two oblique springs. If k_v denotes the spring coefficient of the vertical element, the restoring force is given by [1]:

$$f = L_0 \left\{ k_v \hat{x} + 2k_o(\sqrt{1-\gamma^2} - \hat{x}) \left[\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1 \right)^{-1/2} - 1 \right] \right\} \tag{12}$$

where again $\gamma = a/L_0$.

By introducing the ratio of spring coefficients $\alpha = \frac{k_0}{k_v}$, Eqn.(12) can be re-written in non-dimensional form as

$$\hat{f} = \frac{f}{k_v L_0} = \hat{x} + 2\alpha(\sqrt{1-\gamma^2} - \hat{x}) \left\{ \left[\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1 \right]^{-1/2} - 1 \right\} \tag{13}$$

The non-dimensional force as a function of the non-dimensional displacement is plotted in Fig.9 for several values of γ when $\alpha = 1$. As aforementioned, a system with a QZS characteristic has a force-displacement curve that has quasi-zero slope about the equilibrium point (flat region in the curve). As can be seen in Fig.9 this occurs only at a one particular value of the geometrical parameter, denoted in the figure as γ_{opt}. This will be discussed in detail later.

A physical explanation for the QZS mechanism has been provided at the end of the previous section: the stiffness of the supporting spring has to be equal and opposite to the effective stiffness of the correction springs when these are in the horizontal position.

There exists a unique relationship between the geometrical parameter $\gamma = a/L_0$ and the spring coefficient ratio $\alpha = k_o/k_v$ that yields the desired characteristic. The stiffness of the system can be found by differentiating Eqn.(13) with respect to the displacement

$$\hat{K} = \frac{d}{d\hat{x}}(f(\hat{x})) = 1 + 2\alpha \left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1 \right)^{3/2}} \right] \tag{14}$$

[1]The sign of the restoring force is upwards: thus, according to the notation in Fig.4 it is a negative force

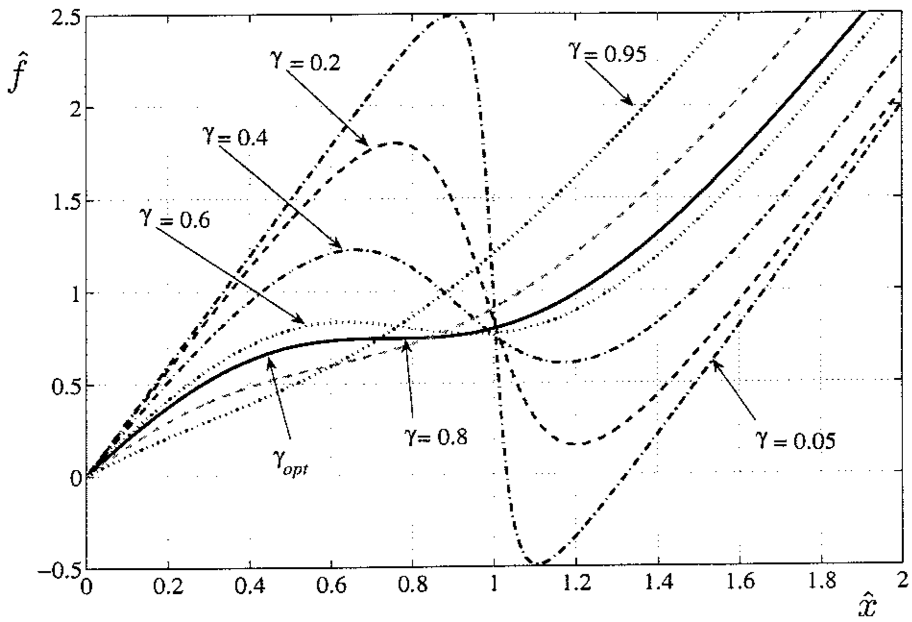

Fig. *9: Force-displacement characteristic of a QZS mechanism when $\alpha = 1$: the solid line presents the flat region that define the quasi-zero stiffness*

If this is evaluated at the static equilibrium position $\hat{x}_e = \sqrt{1-\gamma^2}$, set to zero and solved for γ, the optimum value is found to be

$$\gamma_{opt} = \frac{2\alpha}{2\alpha + 1} \tag{15a}$$

or equivalently

$$\alpha_{opt} = \frac{\gamma}{2(1-\gamma)} \tag{15b}$$

In Fig.10 the non-dimensional stiffness is plotted as function of the non-dimensional displacement for different values of γ when the spring coefficient ratio is $\alpha = 1$.

When $\gamma \neq \gamma_{opt}$ the stiffness is either always positive or becomes negative. However, when $\gamma = \gamma_{opt}$ the stiffness does not become negative (stable system) and becomes zero at the equilibrium position $\hat{x}_e = \sqrt{1-\gamma^2}$.
By designing the isolator so that the oscillations occur around this point only a small force is transmitted through the springs, which is beneficial for vibration isolation.
The variables that govern the behaviour of the QZS region have been identified. An investigation is now conducted on how to optimise these parameters to achieve the maximum displacement

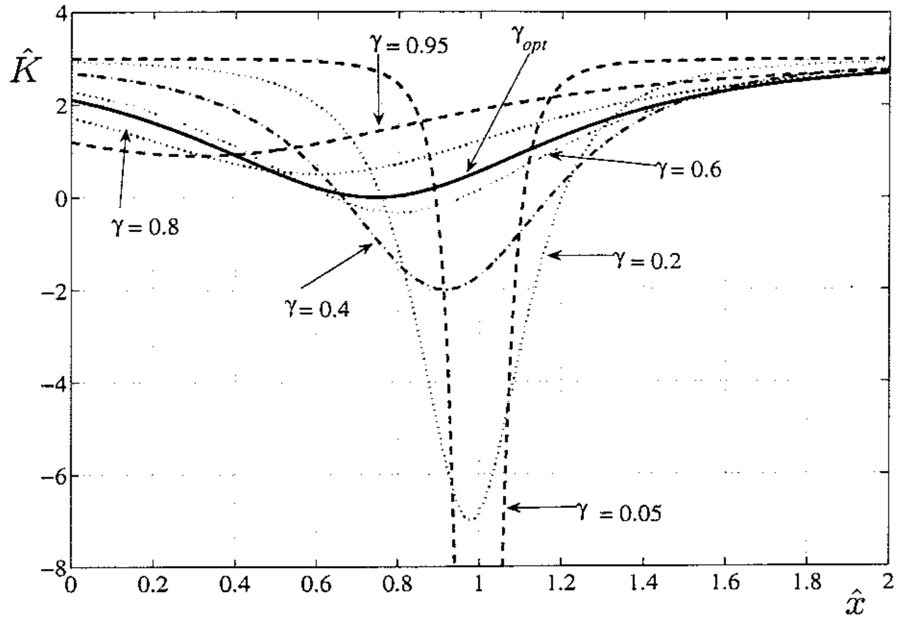

Fig. *10: Non-dimensional stiffness of a QZS mechanism when* $\alpha = 1$*: the solid line is representative of a stable system (always positive stiffness) with zero stiffness at the static equilibrium position. The stiffness is very small (quasi-zero) for a small deviation from this position*

from the operative static equilibrium position without the system exceeding a specified low stiffness value.

3.1. Optimisation of the QZS mechanism

In the previous section it was shown that in order to obtain a system with a QZS characteristic the geometrical parameter, γ, and the spring coefficient ratio, α, have to satisfy Eqn.(15). Since there is an infinite number of possible combinations of the aforementioned parameters it is pertinent to question the existence of an 'optimum' combination.

A suitable criterion is that the displacement about the static equilibrium position is maximised without the system exceeding a very low specified stiffness. The goal set here is to determine a particular combination of γ and α that results in the largest oscillation with very low stiffness in this displacement range.

Fig.11 depicts the non-dimensional force as a function of the non-dimensional displacement when Eqn.(15a) or (15b) holds. In this case the force is only dependent on the variables γ and $\hat{x}$.

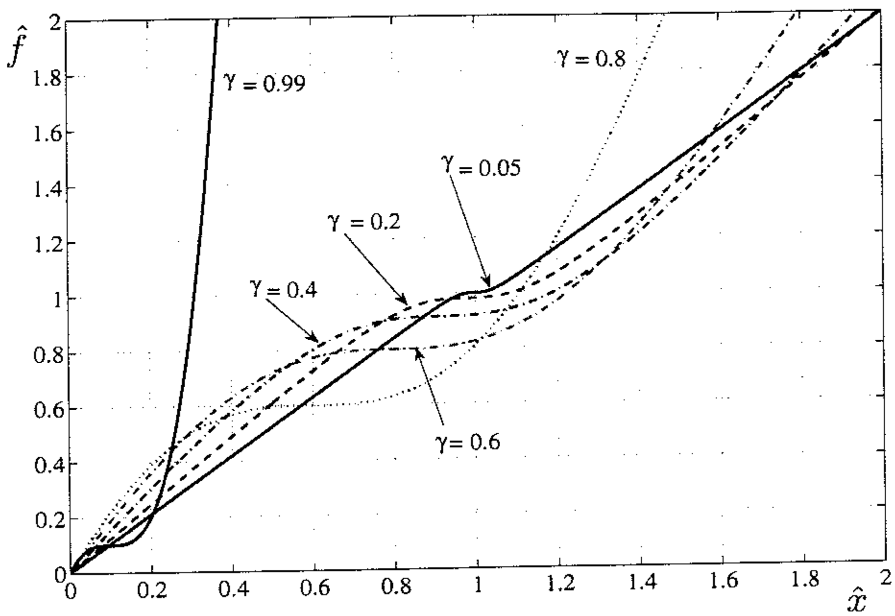

Fig. *11: Plot of non-dimensional force for different values of geometrical parameter γ and stiffness α determined according to the relationship given in Eqn.(15b)*

In Fig.11 all the curves have a quasi-zero-stiffness characteristic (a flat region can be observed). However, the flat region of the curves narrow as $\gamma \to 0$ or $\gamma \to 1$. It is apparent that the optimum value of γ that maximises the displacement range over which the stiffness is almost zero lies in between 0.6 and 0.8.

By substituting the relationship between γ and α given by Eqn.(15b) into Eqn.(14), the *optimum non-dimensional stiffness* as a function $\hat{x}$ is given by

$$\hat{K} = 1 + \frac{\gamma}{(1-\gamma)}\left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1\right)^{3/2}}\right] \tag{16}$$

In Fig.12 the non-dimensional stiffness as function of the non-dimensional displacement is plotted for several values of γ.

Examining Fig.12 it can be seen that at the static equilibrium position $\hat{x}_e$ the stiffness is zero and the largest displacement range over which there is a small stiffness it appears to happen at a value of γ in between 0.6 and 0.8.

In the design of such a isolator is of primary interest to estimate the displacement range over

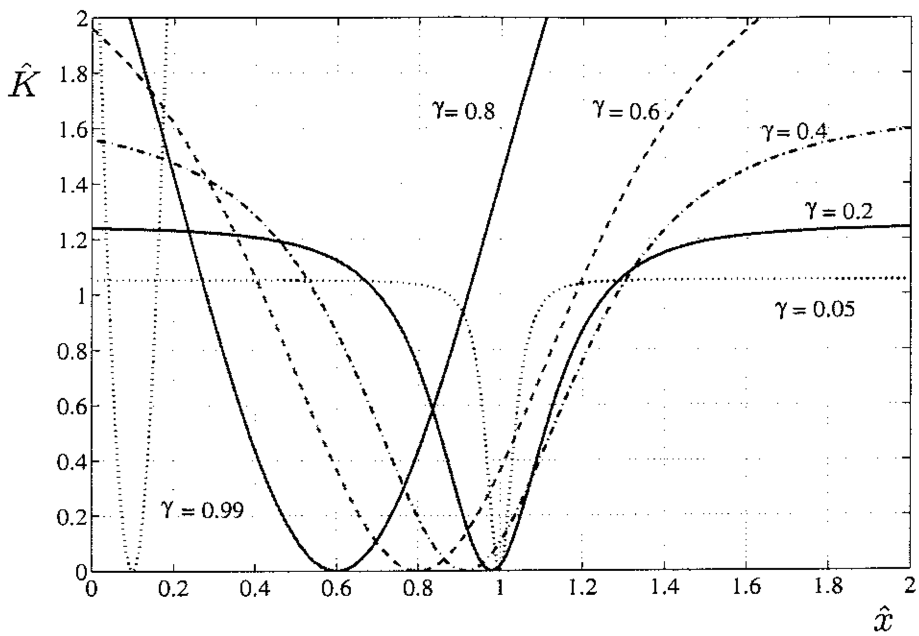

Fig. *12: Plot of non-dimensional stiffness for different values of geometrical parameter γ and α determined according to the relationship given in Eqn.(15b)*

which the system has a stiffness less than some desirable value $\hat{K}_0 \leq 1$. A value of $\hat{K}_0 = 1$ means that the system has the same stiffness as if it had only the vertical spring. Since the purpose of the corrective springs is to reduce the stiffness compared to the classic mass-spring system, the range of interest is $0 < \hat{K}_0 \leq 1$.

The displacement over which the stiffness is less than a desired value $\hat{K}_0$ can be found by setting $\hat{K} = \hat{K}_0$ and solving for $\hat{x}$, which yields

$$\hat{x} = \sqrt{1-\gamma^2} \pm \gamma \sqrt{\left[\frac{1}{1-\hat{K}_0(1-\gamma)}\right]^{2/3} - 1} \tag{17}$$

Note that the second term of the RHS represents the deviation of the system from the static equilibrium position.

The geometry that yields the maximum excursion for a desired low stiffness over this range is now determined.

Let $\hat{d}$ be the second term of the RHS of Eqn.(17) so that

$$\hat{d} = \gamma \sqrt{\left[\frac{1}{1-\hat{K}_0(1-\gamma)}\right]^{2/3} - 1} \tag{18}$$

Of great significance is the maximum value of $\hat{d}$ (for a given $\hat{K}_0$) as γ varies from 0 to 1. Although a numerical solution can be readily found, a closed form expression can be also sought. To evaluate the maximum, Eqn.(18) is differentiated with respect to γ, set to zero and solved for γ. However, this leads to a very complicated result. To derive a simpler relationship it is necessary to make the simplifying assumption that $\hat{K}_0 \ll 1$. Using a binomial expansion, Eqn.(18) approximates to

$$\hat{d} \approx \gamma\sqrt{\frac{2}{3}\hat{K}_0(1-\gamma)} \tag{19}$$

The exact and the approximate expressions for $\hat{d}$ are shown in Fig.13. It can be seen that as $\hat{K}_0$ becomes smaller the two curves become very similar.

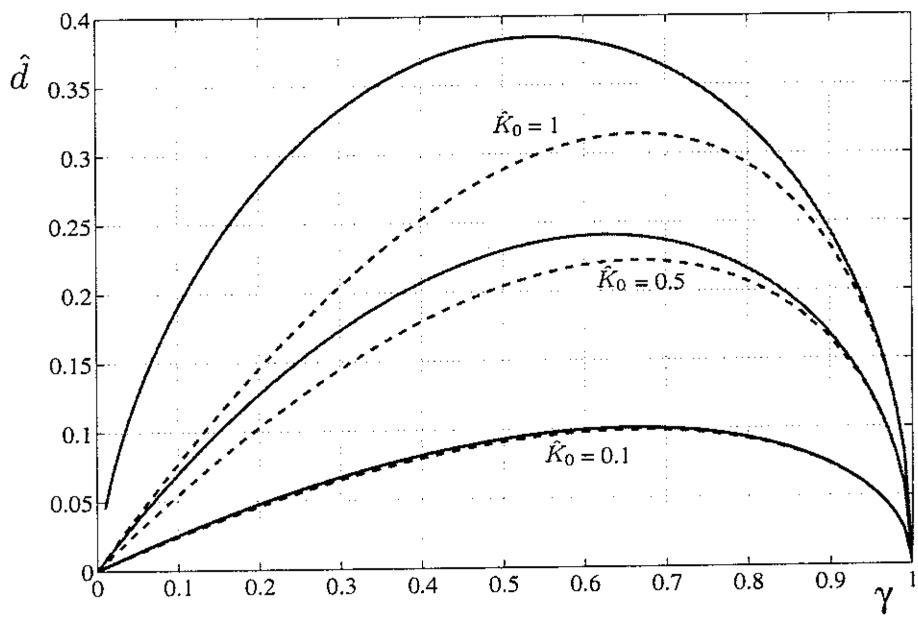

Fig. *13: Non-dimensional displacement from the static equilibrium position as function of* γ*: the* solid line *is the exact solution given by Eqn.(18); the* dashed line *is the approximate solution given by Eqn.(19)*

Provided $\hat{K}_0 \ll 1$, the displacement from the static equilibrium position can be expressed by Eqn.(19) and the maximum displacement is obtained by solving

$$\frac{d}{d\gamma}(\hat{d}) = \frac{\sqrt{6\hat{K}_0}}{6}\frac{(2-3\gamma)}{\sqrt{1-\gamma}} = 0 \tag{20}$$

It can be seen that the value of γ that yields the largest deviation from the static equilibrium position at the desired low stiffness $\hat{K}_0$ is

$$\gamma_{\hat{d}_{max}} = \frac{2}{3}, \qquad \hat{K}_0 \ll 1 \tag{21}$$

From Eqn.(15) follows that the optimum value of the spring coefficient ratio is $\alpha_{max} = 1$, i.e. when all the springs have the same stiffness.
By substituting the value of γ given by Eqn.(21) it is possible to determine the maximum excursion (from the static equilibrium) without exceeding the desired stiffness $\hat{K}_0$:

$$\left(\hat{d}_{max}\right)_{\hat{K}_0 \ll 1} = \frac{2}{9}\sqrt{2\hat{K}_0} \tag{22}$$

In the previous chapter it was shown that if only the oblique springs were acting there are two position at which the system has zero stiffness. These can be seen in Fig.5 where the maximum and minimum of the force correspond to zero stiffness. An expression of the position at which this occurs is also provided by Eqn.(9). In between these two positions the stiffness is negative. When connected in parallel with a vertical mount (which has positive stiffness) the addition of the negative and positive contributions generate the sought softening effect, with the possibility to achieve zero stiffness at one particular point (QZS mechanism). Therefore the range of displacement over which the system represented in Fig.7 has almost-zero stiffness can be found by looking at the distance between the maximum and the minimum in the force-displacement characteristic of the oblique springs. This can be found by differentiating Eqn.(9) setting it to zero and solving for γ. It is found that the largest distance between a maximum and a minimum occurs for $\gamma = (2/3)^{3/2}$. Furthermore, since at the two points in which the oblique springs have zero stiffness the system has the same stiffness of the vertical spring, i.e. $\hat{K}_0 = 1$, it follows that

$$\gamma_{\hat{d}_{max}} = \left(\frac{2}{3}\right)^{3/2}, \qquad \hat{K}_0 = 1 \tag{23}$$

which when substituted in Eqn.(18) gives

$$\left(\hat{d}_{max}\right)_{\hat{K}_0 = 1} = \left(\frac{2}{3}\right)^{3/2} \sqrt{\frac{1}{\left\{1 - \hat{K}_0\left[1 - \left(\frac{2}{3}\right)^{3/2}\right]\right\}^{2/3}} - 1} \tag{24}$$

So far it was shown that

- if $\hat{K}_0 \ll 1$ the largest displacement over which the stiffness is smaller than $\hat{K}_0$ occurs when $\gamma = 2/3$, Eqn.(21);

- conversely if $\hat{K}_0 = 1$ the value of γ that yields the maximum displacement is $\gamma = (2/3)^{3/2}$, Eqn.(23).

A general expression that relates γ and $\hat{K}_0$ with the maximum achievable displacement can be sought by assuming, for example, a relationship of the type

$$\gamma_{\hat{d}_{max}} = \left(\frac{2}{3}\right)^{c_1 \hat{K}_0 + c_2} \tag{25}$$

The constants c_1 and c_2 can be found by combining Eqn.(25) with Eqns.(23) and (21). It is found that

$$\gamma_{\hat{d}_{max}} = \left(\frac{2}{3}\right)^{\frac{\hat{K}_0}{2}+1} \tag{26}$$

By substituting Eqn.(26) into Eqn.(18) the relationship between the maximum displacement and the desired stiffness is given by

$$\hat{d}_{max} = \left(\frac{2}{3}\right)^{\frac{\hat{K}_0}{2}+1} \sqrt{\frac{1}{\left\{1 - \hat{K}_0\left[1 - \left(\frac{2}{3}\right)^{\frac{\hat{K}_0}{2}+1}\right]\right\}^{2/3}} - 1} \tag{27}$$

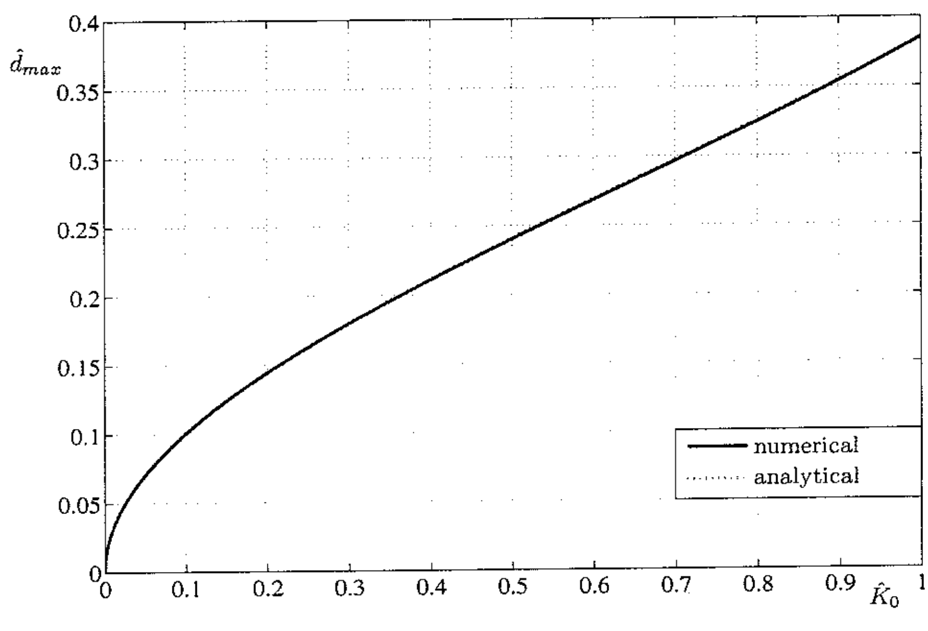

Fig. *14: Numerical and analytical representation of the maximum displacement range, $\hat{d}_{max}$, over which the system has a stiffness smaller than $\hat{K}_0$.*

Fig.14 shows the largest excursion from the static equilibrium position that can be achieved without the system having a stiffness bigger than $\hat{K}_0$. The solid line has been obtained by using the exact solution, Eqn.(18), and numerically evaluating the maximum displacement as $\hat{K}_0$ and γ vary from 0 to 1 whereas the dotted curve depicts Eqn.(27).

From Fig.14 it is also possible to observe that, for $\hat{K}_0 \ll 1$ the curve resembles a square root function. Using a binomial expansion Eqn.(27) simplifies to Eqn.(22). From a value of $\hat{K}_0$ of about 0.4 the curve suggests a linear relationship between $\hat{d}_{max}$ and $\hat{K}_0$ in this range. By manipulating Eqn.(27) it is found that

$$\hat{d}_{max} \approx \frac{1}{10}\left(1 + 2\sqrt{2}\,\hat{K}_0\right)\,, \qquad 0.4 \approx \hat{K}_0 \leq 1 \tag{28}$$

Once that the maximum displacement for the desired stiffness has been estimated from Fig.14 the relative geometry can be calculated with Eqn.(26) which is also depicted in Fig.15. As $\hat{K}_0$ goes from 0 to 1 the geometrical parameter varies between $(2/3)^{3/2}$ and $2/3$. Because of the definition of γ as given in Eqn.(4) the angles at which the oblique springs have to be placed are within the range $47° - 57°$. Similarly according to Eqn.(15b) the stiffness ratio α is constrained between 0.6 and 1.

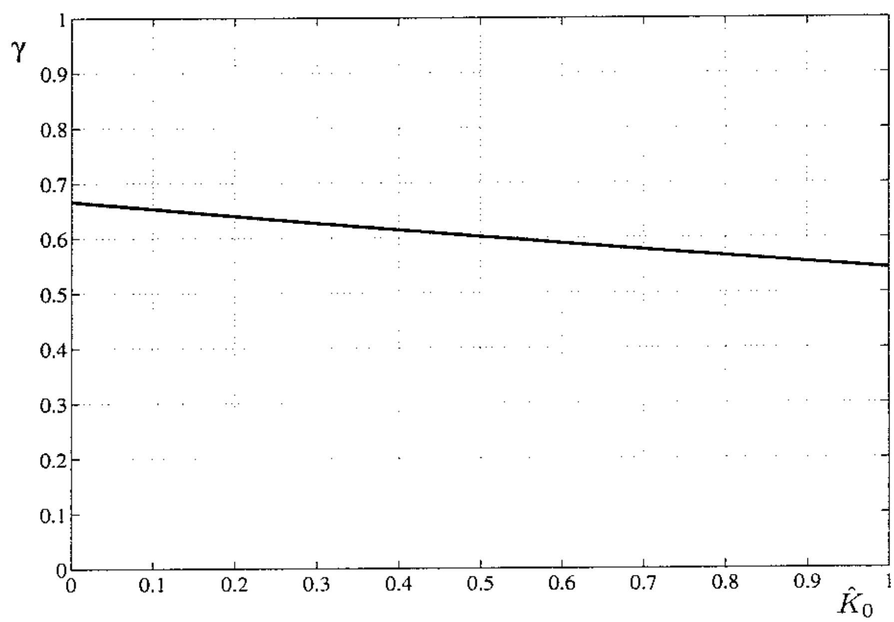

Fig. *15: Geometrical parameter γ versus system's stiffness $\hat{K}_0$. In order to obtain the maximum displacement for any $\hat{K}_0 < 1$ γ has to be comprised between $(2/3)^{3/2}$ and $2/3$*

The analysis presented so far is based on the assumption that γ and α respect Eqns.(15), that is the system parameters are optimised to achieve zero stiffness at the static equilibrium position. In practice this means that for a given geometry (i.e. for a given γ) the spring ratio has to be equal to α_{opt} as in Eqn.(15b). However, it is possible that, due to manufacturing errors for example, the actual value of α is different from the desired value. Therefore it is important to predict how the system responds should γ and α be detuned from their optimal relationship.

Let ϵ be a fractional change in α, that is $\alpha = \alpha_{opt}(1 \pm \epsilon)$. Substituting this expression of α in Eqn.(14) it becomes

$$\hat{K} = 1 + 2\alpha_{opt}(1 \pm \epsilon)\left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1\right)^{3/2}}\right] \tag{29}$$

which can be rearranged to give

$$\hat{K} = \hat{K}_{opt} \pm 2\epsilon\alpha_{opt}\left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1\right)^{3/2}}\right] \tag{30}$$

where $\hat{K}_{opt}$ is the stiffness when α and γ optimised.

Eqn.(30) expresses the stiffness of the system when the geometrical and spring parameter are not in the optimal ratio. Since oscillations occur about the static equilibrium position the stiffness is evaluated at $\hat{x} = \hat{x}_e = \sqrt{1-\gamma^2}$, when the parameters are detuned. By definition $\hat{K}_{opt} = 0$ at the equilibrium position, thus substituting $\hat{x}_e = \sqrt{1-\gamma^2}$ into Eqn. (30)it is found that

$$\left(\hat{K}\right)_{\hat{x}=\hat{x}_e} = -(\pm\epsilon) \tag{31}$$

The physical meaning of Eqn.(31) is that the stiffness of the system is equal but of opposite sign to the fraction change in the spring ratio. It has also to be noted that the stiffness is independent of the geometrical parameter γ. Of particular interest is the fractional change in stiffness compared to the optimum value (which incidentally is zero at the equilibrium position) when the system is detuned. By differentiating the stiffness with respect to ϵ it results

$$\frac{d\hat{K}}{d\epsilon} = \mp 1 \tag{32}$$

Eqn.(32) shows the robustness of such a mechanism to non-optimal parameters: the system will experience a change in stiffness equal but opposite to the fractional change of optimal spring ratio α_{opt}. Thus, for example, if the spring ratio is 1% smaller than α_{opt} the stiffness at the equilibrium position, which should be zero in optimal conditions, will increase by 1%, becoming positive. Likewise, an increase in α of ϵ% will imply a decrease in stiffness by the same amount.

It has been shown that it is possible to design an isolation system with QZS characteristic and maximise the displacement from the equilibrium position without exceeding a desired low stiffness. However, there are some negative aspects of this mechanism that need to be discussed. In the best case scenario that the system has optimally tuned parameters Fig.11 and 12 show that outside the QZS region (flat part of the force-displacement curve), the stiffness rises sharply. It is of interest to know how poorly the isolator performs should the amplitude of the oscillations become very large, that is $|(\hat{x} - \hat{x}_e)| \gg h_0$.

Fig.12 shows that for very large excursions from the static equilibrium position the stiffness tends to an asymptotic value that depends on γ: the greater γ the greater the stiffness becomes for large displacements. In the limit the stiffness becomes

$$\lim_{\hat{x}\to\pm\infty} 1 + \frac{\gamma}{(1-\gamma)}\left[1 - \frac{\gamma^2}{\left(\hat{x}^2 - 2\sqrt{1-\gamma^2}\hat{x} + 1\right)^{3/2}}\right] = \frac{1}{1-\gamma} \tag{33}$$

In order to exploit the whole displacement range over which the system has the desired low value $\hat{K}_0$, γ is constrained between $(2/3)^{3/2}$ and $2/3$. Thus, the cost of having an almost zero stiffness is that the stiffness can become between two and three times higher than the vertical spring with consequent degradation of isolation performance when compared to the simpler mass-spring system.

3.2. Approximation to the Stiffness of the QZS Isolator

In the previous sections the force-displacement characteristic of QZS isolator has been studied. Also its stiffness and an optimum configuration of the system have been discussed.

The relationship between force and the displacement is expressed by Eqn.(13) and is graphically shown in Fig.9. It can be seen that the curves have a shape similar to that of a cubic function. In this section a simplified cubic expression of the force is therefore sought and the error in the approximation is investigated.

Using a Taylor series expansion, a function can be expressed by a power series of order N. The expansion is given by [13]

$$f(x) = f(x_0) + \sum_{n=1}^{N} \frac{f^n(x_0)}{n!}(x - x_0)^n \tag{34}$$

where x_0 is the point at which the function is expanded and f^n denotes the n-th derivative of f.

Since the system is designed to oscillate about the static equilibrium position it is of interest to expand the force characteristic around this point, i.e. $x_0 = \hat{x}_e$. By expanding Eqn.(13) using Eqn.(34) an approximate expression for the force can be found. The approximate force is found to be

$$\hat{f}_{app}(\hat{x}) = \frac{\alpha}{\gamma^3}\left(\hat{x} - \sqrt{1-\gamma^2}\right)^3 + \left[1 - 2\alpha\frac{(1-\gamma)}{\gamma}\right]\left(\hat{x} - \sqrt{1-\gamma^2}\right) + \sqrt{1-\gamma^2} \tag{35}$$

or, by introducing the variable $\hat{y} = \hat{x} - \sqrt{1-\gamma^2}$

$$\hat{f}_{app}(\hat{y}) = \frac{\alpha}{\gamma^3}\hat{y}^3 + \left[1 - 2\alpha\frac{(1-\gamma)}{\gamma}\right]\hat{y} + \sqrt{1-\gamma^2} \tag{36}$$

In particular if γ and α are chosen according to one of Eqns.(15), the force-displacement relationship approximated by a third order polynomial is given by

$$\hat{f}_{app}(\hat{y}) = \left[\frac{1}{2\gamma^2(1-\gamma)}\right]\hat{y}^3 + \sqrt{1-\gamma^2} \tag{37}$$

By opportunely scaling the coordinate system Eqn.(37) can be rewritten as

$$\hat{F}_{app}(\hat{y}) = \left[\frac{1}{2\gamma^2(1-\gamma)}\right]\hat{y}^3 \tag{38}$$

where $\hat{F}_{app} = \hat{f}_{app} - \sqrt{1-\gamma^2}$.

Eqn.(38)is plotted in Fig.16. Also shown in the same figure are the curves relating to the fifth and seventh order polynomials.

Since the stiffness of the system is of particular interest, it is necessary to evaluate the approximate expression of the stiffness and the error between this and the exact expression.

An approximate expression for the stiffness can be obtained by differentiating Eqn.(36):

$$\hat{K}_{app} = 3\frac{\alpha}{\gamma^3}\hat{y}^2 + \left[1 - 2\alpha\frac{(1-\gamma)}{\gamma}\right] \tag{39}$$

In the case when the parameters α and γ are optimally related the previous expression of the stiffness simplifies to

$$\hat{K}_{app} = \frac{3}{2}\frac{1}{\gamma^2(1-\gamma)}\hat{y}^2 \tag{40}$$

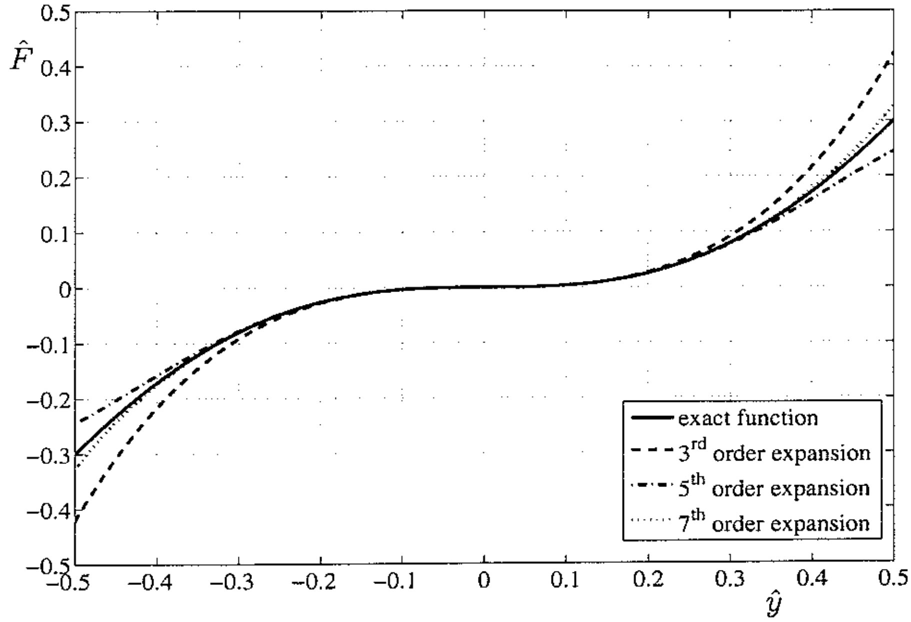

Fig. *16: Force-displacement characteristic. The solid line shows the plot of the exact expression Eqn.(13) in the new coordinate system. The other lines are for the approximate expressions of third, fifth and seventh order polynomials*

If only if α and γ are in their optimal ratio the stiffness, expressed by Eqn. (40), becomes zero at $\hat{y} = 0$.

The approximate expression leads inevitably to an error; although this can be reduced by expanding the function as polynomial of higher order, an estimation of this error is thus needed to understand to which extent an approximate function can be used without introducing too large an error in the static analysis of this system.

There are different ways to estimate the error when approximating a function with a finite power series, as opposed to an infinite power series. Some analytical expressions for the reminder have been provided by Lagrange and Cauchy [13]. However, here a numerical estimation of the error is calculated by

$$err(\%) = \left| 1 - \frac{\hat{K}_{app}}{\hat{K}_{ex}} \right| * 100 \qquad (41)$$

where $\hat{K}_{app}$ is the approximate value of the function and $\hat{K}_{ex}$ is the exact one.

A large difference between the exact and approximate solution is expected only when large displacements from the equilibrium position occur. This is a direct consequence of having expanded the function about the static equilibrium point. Focussing the interest on the maximum displacement range that allows the stiffness to be smaller than one, that is $\hat{K}_0 < 1$, the biggest error will occur at $\hat{y} = \hat{d}_{max}$.

Should the excursion become as large as $\hat{d}_{max}$ the error made in using the approximate rather than the exact expression of the stiffness can be estimated, for any value of desired stiffness $\hat{K}_0$, as follow. For any given $\hat{K}_0$ the maximum excursion from the equilibrium position $\hat{d}_{max}$ is calculated using Eqn.(27). Then both exact and approximate expressions of the stiffness are evaluated at $\hat{d}_{max}$. The error is thus determined by means of Eqn.(41). The result is shown in Fig.17. A large error is observed if the desired stiffness is high ($\hat{K}_0 \approx 1$). However, for values of $\hat{K}_0 < 0.3$ the approximation error is less than 10%.

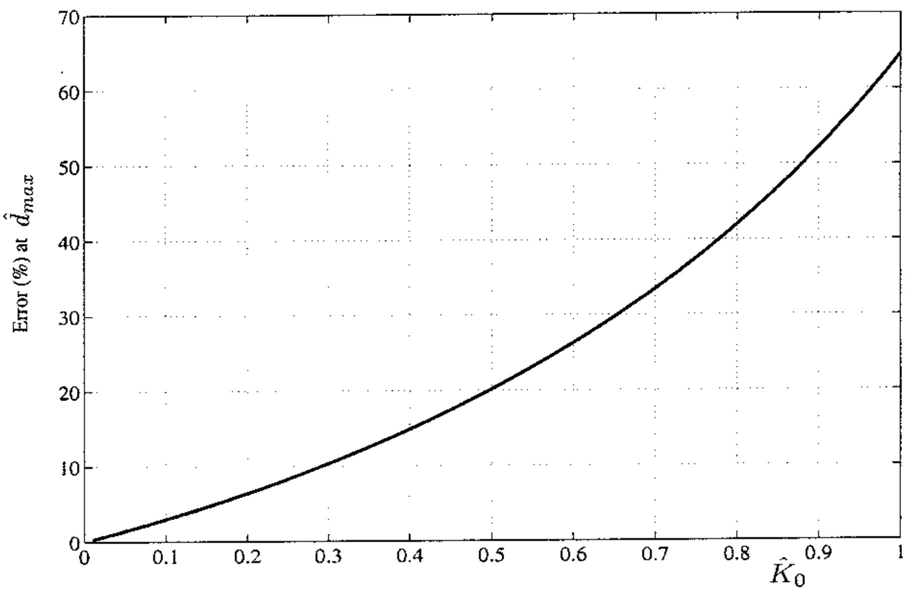

Fig. *17: Approximation error evaluated at the maximum distance achievable from the static equilibrium position for a given* $\hat{K}_0$

4. Conclusions

The performance of an isolation system with quasi zero stiffness mechanism has been discussed. The main feature of these mechanisms is the possibility of achieving a very low stiffness without paying the price of having a large static deflection. This is accomplished by combining the negative stiffness provided by two linear springs with a certain geometry and a vertical mounting with a positive linear stiffness.

The analysis had been carried out with non-dimensional parameters. In particular two parameters are of major importance: the system's stiffness, which has been non-dimensionalised with respect to the stiffness of the vertical spring, and the angle at which the oblique springs are inclined.

It has been found that in order to obtain a QZS characteristic there is a direct relationship between the geometrical parameter (namely, the initial angle at which the correction spring are set) and the coefficients of the springs that constitute the system. Furthermore, an optimisation analysis has been carried out with the aim of maximising the deviation from static equilibrium position that results in a desired low stiffness. It was found that the largest displacement achievable is a function of the desired stiffness. A closed form expression has been determined. It has also been found that the oblique spring have to be inclined at an angle that varies between approximately 47° and 57°.

Because of manufacturing tolerances, in practice the stiffness of a spring differs from the claimed value. It was therefore of interest to measure the robustness of the system should the geometry and the springs' ratio be different from the design values. The stiffness has been calculated if the spring ratio α is changed by a quantity ϵ. It was shown that the fractional change in stiffness at the equilibrium position is equal (but with opposite sign) to the fractional change in stiffness. Thus, and increase of springs' ratio of $\epsilon\%$ correspond to a decrease of $\epsilon\%$ of the system's stiffness.

For sake of completeness, the drawbacks of this mechanism were also discussed. The cost of having an almost zero stiffness for small oscillations about the static equilibrium position, is an abrupt increase in stiffness should the oscillations become large. An expression that estimates the stiffness in the limit of an infinitely large displacement has been proposed. The increase in stiffness is a function of the geometry and the smaller the angle the worse (in terms of stiffening) the system behaves for large displacements.

Finally, the exact function that describes the force-displacement relationship has been simplified to a simpler cubic function. In order to avoid the introduction of large errors in the analysis of the system a study of the approximation error has been carried out. The cubic expression derives from a series expansion about the equilibrium position. Therefore in the proximity of the equilibrium point the exact and approximate functions do not differ. However, the further away from this position the largest the difference become. The analysis has been focussed at the maximum deviation from the equilibrium that allows the desired stiffness. It has been shown that if the desired stiffness is small so is the error in using the approximate expression.

Future studies of this system will look into the dynamic response. The expression of the stiffness leads to an equation of motion that is the same as the Duffing's oscillator which has been extensively covered in the literature.

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