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Inverse Force Determination: Refinements of Matrix Regularization and Sensor location Selection Methods

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

Inverse Force Determination: Refinements of Matrix Regularization and Sensor Location Selection Methods

by

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ISVR Technical Memorandum No: 924

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Authorised for issue by
Professor M.J. Brennan
Group Chairman

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ACKNOWLEDGEMENT

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1. INTRODUCTION

The prediction of operational forces is of importance for the analysis of structure-borne sound from installed machinery. Matrix-inversion can be used to reconstruct the operational forces from a set of measured operational responses and corresponding FRF's. Matrix ill-conditioning can be overcome using methods such as pseudo-inversion, singular value rejection, and Tikhonov regularization. In previous research [1–3], it has been found that Tikhonov regularization is more effective than other methods to reduce the errors in forces reconstructed. Tikhonov regularization generally gives better results over the frequency range as a whole; however, pseudo-inversion performs better than Tikhonov regularization at some frequencies (with low condition number). Also, the estimate of the operational forces varies according to the response locations used, irrespective of the force reconstruction method.

The need for regularization is greatest when the matrix is ill-conditioned. In [4, 5] it is shown that the errors in operational responses are magnified to an extent that depends on the matrix condition number. Also models were proposed to predict the influence of various errors in statistical estimates of FRF's and response spectra [4]. In this study, the error amplification between force errors and response errors is investigated in order to establish a criterion for when Tikhonov regularization should be used and when pseudo-inversion is preferable. Based on results of the error amplification factor, some typical confidence levels are determined, and a threshold called the cross-over condition number is proposed from various simulations of different system matrix size.

Another approach to improve the inverse force determination is the optimal selection of sensor locations from many possible measurement positions. This approach only focuses on the errors due to selecting sensor locations with no consideration of the inversion method. The selection of sensor locations is based on the average condition number over all frequencies. In this study, two methods are used; one is based on the 'amplification factor' (or composite condition number) and the other uses the genetic algorithms. The amplification factor is proposed by Thite [1] in order to avoid an excessive amount of calculations in obtaining the average condition numbers for all possible combinations of sensor locations. However, the optimal set of sensor locations determined using the amplification factor does not always correspond to the best set in terms of the average condition number. Therefore, in this study, a refinement of this method is proposed to select an appropriate small range of possible combinations on the basis of the amplification factor and then to calculate the average condition numbers of the selected sets.

Genetic algorithms are search algorithms based on an analogy with natural selection and are used in many applications [6–10]. Lieven [6] used genetic algorithms in order to refine a finite element model with respect to the measurements. Baek and Elliott [7] applied genetic algorithms to choose the optimal locations of the secondary sources in active structural acoustic control systems and compared the performance of different modification operations. Genetic algorithms are used in many structural dynamics problems but have not been used for sensor location selection to improve the inverse force determination. In this study, genetic algorithms are used to search for the best set of response locations. Some simulations are carried out to investigate the effects of changing the number of individuals in each generation and the corresponding number of generations on the performance of the genetic algorithms. Some modification algorithms such as the elitist model and the change of the mutation probability are proposed. Validation of the use of genetic algorithms is conducted by comparing the results obtained by the genetic algorithms with the results of a random search with the same amount of calculation.

2. EVALUATION OF TIKHONOV REGULARIZATION USING ERROR AMPLIFICATION FACTOR

2.1. Introduction to the error amplification factor

In previous research [1–3], Tikhonov regularization based on ordinary cross validation has been used in the matrix inversion used to reconstruct the operational forces. This was found to give better results than other methods, i.e., the pseudo-inverse method, or singular value rejection. However, these evaluations are based on comparing the average errors over the whole frequency range. In fact, Tikhonov regularization does not give better results at all frequencies. At some frequencies, the errors in the forces reconstructed by the pseudo-inverse method can be less than those by Tikhonov regularization. In particular, it is found that for low condition numbers Tikhonov regularization may degrade the result compared with pseudo-inversion. Therefore it is necessary to investigate the errors in the forces reconstructed by Tikhonov regularization at each frequency and for a large range of noise levels.

In order to compare the performance of Tikhonov regularization with that of the pseudo-inverse method, an 'error amplification factor' is proposed and defined as the difference between the errors in forces reconstructed and the errors in the operational responses at each frequency. The error amplification factor is written as below

$$\Delta\varepsilon = \varepsilon_f - \varepsilon_r, \tag{1}$$

$$\varepsilon_f = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(20\log_{10}\left|\tilde{F}_i\right| - 20\log_{10}\left|F_i\right|\right)^2}, \tag{2}$$

and

$$\varepsilon_r = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(20\log_{10}\left|\hat{a}_i\right| - 20\log_{10}\left|a_i\right|\right)^2}, \tag{3}$$

where ε_f is the root mean square error between the forces reconstructed $\tilde{F}_i$ and the original forces F_i in dB, ε_r is also the root mean square error in the measured operational responses $\hat{a}_i$ in dB, a_i is the operational responses without noise, and m and n are the numbers of responses and forces, respectively. This is a measure of how much errors are amplified by the matrix inversion. Only errors in the amplitudes are considered.

Various noise levels corresponding to a series of signal-to-noise ratios between –40 dB and –10dB with a step of 1 dB are used to make a large number of 'measured' operational responses and FRF's. The procedure is described in more detail in reference [2].

The analysis to reconstruct the forces is carried out under four conditions:

1) low noise level in FRF's, variable noise levels in operational responses,
2) high noise level in FRF's, variable noise levels in operational responses,
3) low noise level in operational responses, variable noise levels in FRF's, and
4) high noise level in operational responses, variable noise levels in FRF's,

where low noise level is –40 dB S/N ratio, and high noise level is –10 dB.

2.2. Simulations

The analysis object is an analytical model of a simply supported rectangular plate with dimensions of 600 *mm* × 500 *mm* × 1.5 *mm* made of steel (Young's modulus: 2.07×10^{11} N/m^2, Poisson's ratio: 0.3, density: 7850 kg/m^3, and damping loss factor: 0.03), and the locations of the forces and responses are shown in Table 1. The methods used to reconstruct the forces are the pseudo-inverse method and Tikhonov regularization with ordinary and generalized cross validation. The frequency range considered is 10 to 500 Hz.

First, a set of 5 response and 4 force locations is used to estimate the error amplification factor at each frequency. The FRF's and operational responses are made using the four noise conditions mentioned above. In all, 120 combinations of noise levels in the FRF's and the operational responses are used here, so that it takes a very long time to compute the error amplification factors. Thus, to reduce the amount of calculation, a smaller set of 8 typical combinations of noise levels are proposed instead of the 120 combinations, and the results obtained by using these 8 combinations are compared with those using all combinations. The 8 typical combinations of the noise levels are as follows:

Table 1. Non-dimensional positions of forces and responses.

Force positions and rms amplitude				Response positions		
No	*x*/*a*	*y*/*b*	Force [N]	No	*x*/*a*	*y*/*b*
1	0.41	0.43	19	1	0.55	0.40
2	0.51	0.63	10	2	0.90	0.80
3	0.62	0.41	27	3	0.60	0.50
4	0.31	0.72	6	4	0.70	0.71
5	0.83	0.25	35	5	0.61	0.31
6	0.71	0.89	16	6	0.20	0.30
7	0.11	0.26	23	7	0.30	0.70
8	0.38	0.15	8	8	0.80	0.20
9	0.19	0.87	13	9	0.50	0.90
10	0.91	0.50	30	10	0.23	0.89
11	0.84	0.75	25	11	0.26	0.46
12	0.47	0.88	10	12	0.35	0.07
13	0.14	0.66	12	13	0.59	0.11
14	0.58	0.23	21	14	0.41	0.31
				15	0.11	0.53

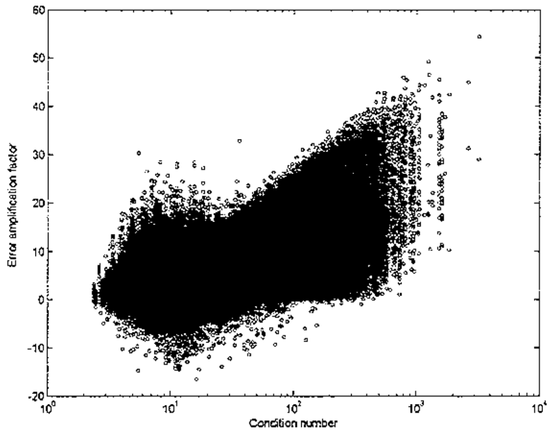
1) low noise level in FRF's and low, medium and high noise levels in operational responses (3 combinations),
2) medium noise level in FRF's and low and high noise levels in operational responses (2 combinations), and
3) high noise level in FRF's and low, medium and high noise levels in operational responses (3 combinations),

where low noise level is a S/N ratio of –40 dB, medium noise level is –25 dB and high noise level is –10 dB.

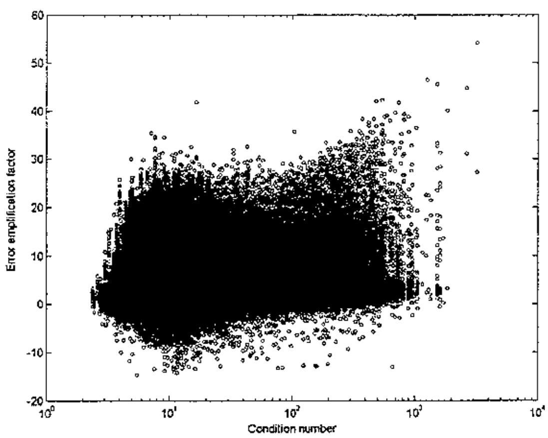
For the case of 5 response and 4 force positions, the amplification factors obtained by using the all noise levels in FRF's and operational responses are shown in Figure 1, and the amplification factors obtained using the three noise levels are shown in Figure 2. The plots in the left-hand column in these figures show the distributions of the error amplification factors with respect to the matrix condition numbers. Each point corresponds to one frequency and one combination of S/N ratios. In order to simplify the presentation, the plots in the right-hand column in the these figures show curves corresponding to the 5 %, 50 % and 95 % confidence limits. To achieve this, the points are first grouped into '1/3 octave bands' of condition number, i.e. all values between 8.9 and 11.2 are grouped together, all values between 11.2 and 14.4, etc. Then the error amplification factors in each band are sorted in ascending order with no consideration of their corresponding condition numbers, and the error amplification factors corresponding to the 5 %, 50 % and 95 % points in each band are selected and plotted. From Figures 1 and 2, it can be seen that the distribution and the fitted curves of the error amplification factors obtained by using the three noise levels in FRF's and operational responses show similar results to those found by using all the noise levels, except in the high condition numbers. The differences at high condition numbers are due to the small number of points in each band. However, this is not important because the trend of the difference between the pseudo-inverse method and Tikhonov regularization is unaffected. Consequently, the reduced set based on the three noise levels can be used to estimate the error amplification factor in other cases instead of the full set of noise levels.

These fitted curves of the error amplification factors are more useful than the raw data because comparisons between the performance of the pseudo-inverse method and Tikhonov regularization are made easier. As shown in Figure 2, it can be easily seen that the error amplification factor obtained by the pseudo-inverse method increases as the condition number increases, but the error amplification factor obtained by Tikhonov regularization remains at a similar level without regard to the change of the condition number. At high condition number Tikhonov regularization gives better results, whereas at low condition numbers it can give worse results in some cases.

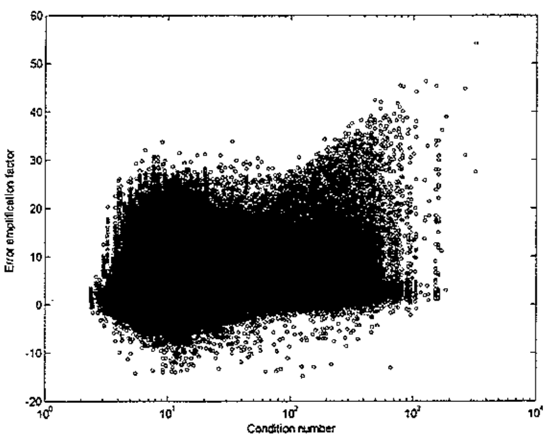
Therefore, by comparing the fitted curves of the error amplification factors obtained by each method, a value of the condition number can be determined at which the performance of Tikhonov regularization and that of the pseudo-inverse method are equal. However, to obtain generalized results, more results using different matrix dimensions are needed and these are considered first.



(a) Pseudo-inverse method

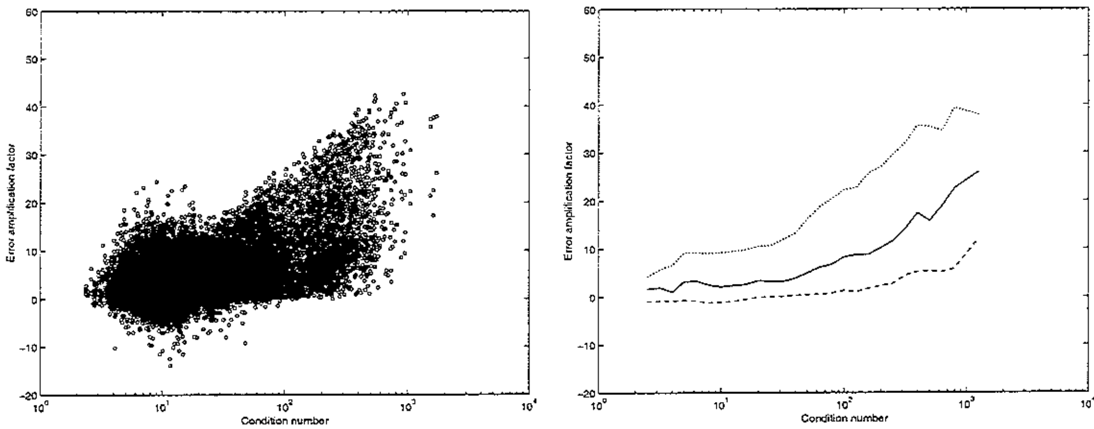


(b) Ordinary cross validation

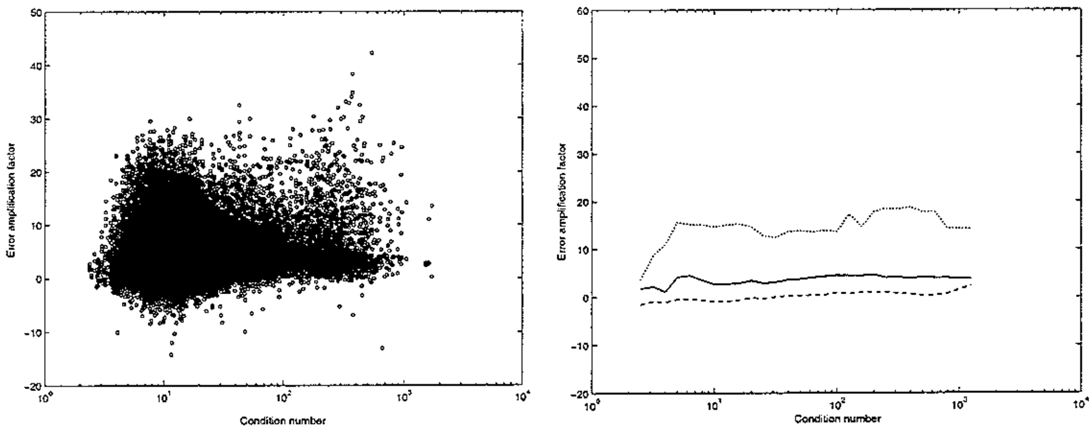


(c) Generalized cross validation

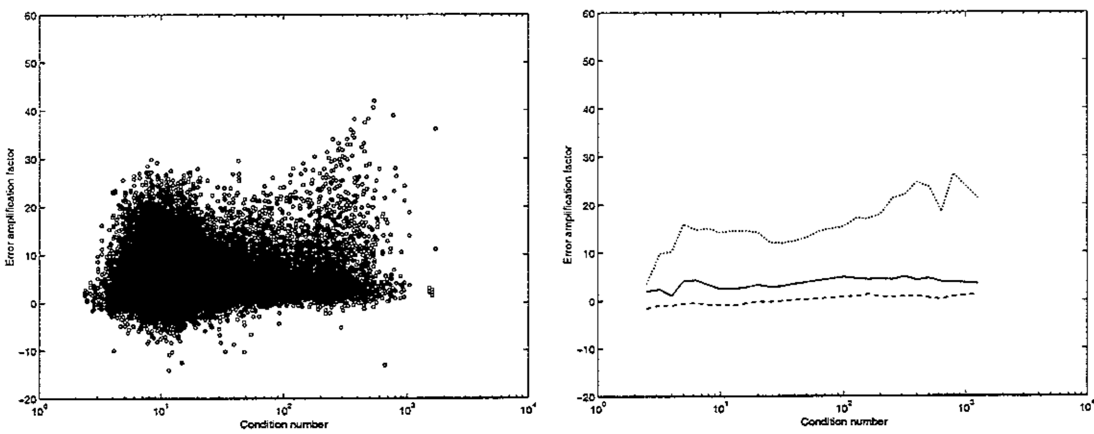
Figure 1. Correspondence between the error amplification factor and the condition number for 5 responses and 4 forces with all noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method



(b) Ordinary cross validation



(c) Generalized cross validation

Figure 2. Correspondence between the error amplification factor and the condition number for 5 responses and 4 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.

The typical value of the condition number to be used to distinguish the Tikhonov regularization from the pseudo-inverse method is proposed as the condition number corresponding to the cross-over between the 95 % fitted curves obtained by using the two methods. This is called the cross-over condition number in this study. Therefore if a condition number is greater than or equal to this cross-over condition number, the Tikhonov regularization gives better results than the pseudo-inverse method at the 95 % confidence level.

It can be expected that when the condition number is sufficiently large, Tikhonov regularization gives better results than the pseudo-inverse method. However, there is no standard means of determining when Tikhonov regularization performs better, in terms of the condition number. This problem is related with the condition number and this is also connected to the system matrix size. Consequently the selection of the Tikhonov regularization based on the condition number has to be considered together with the system matrix size.

Two groups of can be considered to investigate the error amplification factor and the cross-over condition number. The first group is formed of $n\times(n-1)$ systems, where n responses and $(n-1)$ forces are used (here, n is from 3 to 15). This group is used to find the relationship between the full matrix size and the cross-over condition number. The second group is the $m\times n$ systems, where m is a fixed number of the response locations (here, m is 10) and n is the number of the force locations from 2 to $(m-1)$ (here, n is from 2 to 9).

The cross-over condition number has thus been found for many combinations of response and force locations as follows:

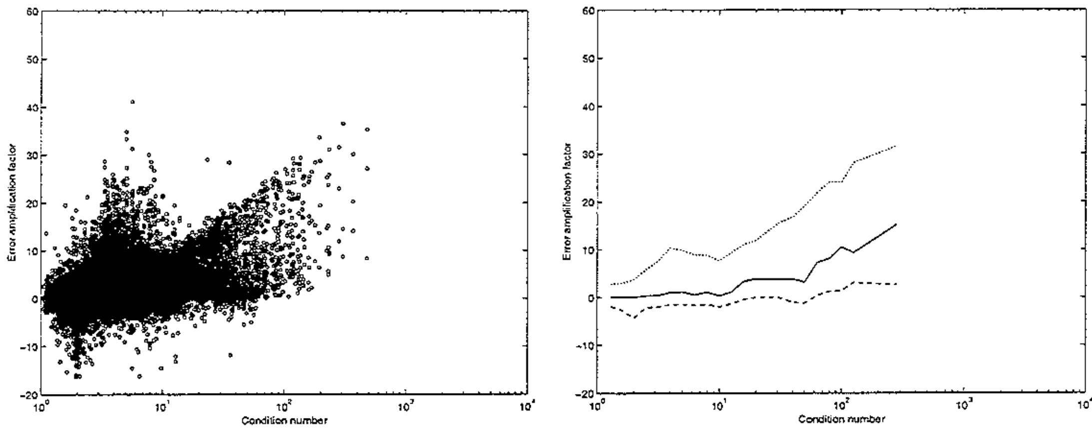
3×2, 4×3, 5×3 and 5×4 (already used), 6×5, 7×6, 8×5 and 8×7, 9×8, 10×2, 10×3, 10×4, 10×5, 10×6, 10×7, 10×8 and 10×9, 11×10, 12×11, 13×12, 14×13, and 15×14, where the first number is the number of response positions, and the second is the number of force positions.

Figures 3 to 23 show the distributions of the error amplification factors and their fitted curves for the cases mentioned above. As shown in all figures, as expected, the pseudo-inverse method gives larger error amplification factors than Tikhonov regularization at high condition numbers. Conversely, Tikhonov regularization gives larger error amplification factors than the pseudo-inverse method at low condition numbers.

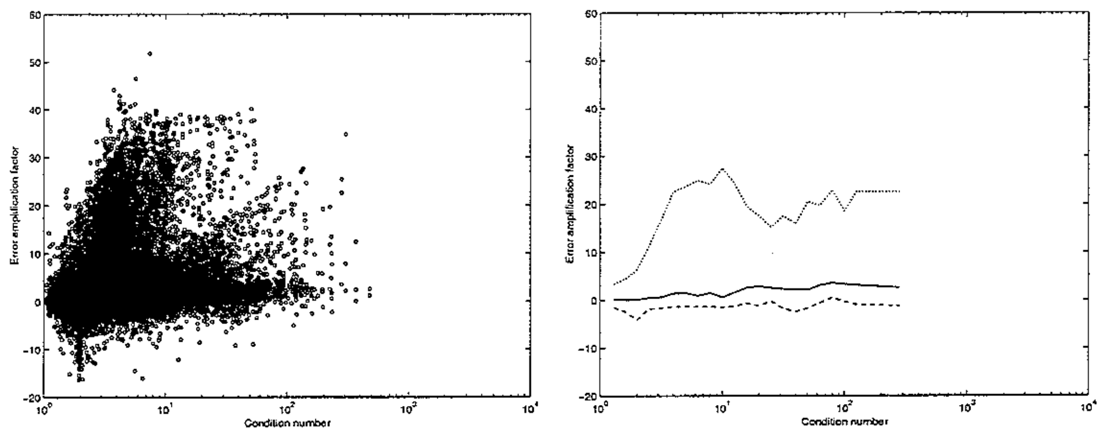
For the case of the first group, $n \times (n-1)$ (here n is 3 to 15, see Figures 2 – 4, 6, 7, 9, 10, and 18 – 23), as the number of responses (and simultaneously forces) increases, the maximum condition numbers increase and the distribution of the error amplification factor with respect to the condition number obtained by the pseudo-inverse method moves to the right (in the direction of increasing condition number). The shape of the distribution from pseudo-inversion is maintained but that obtained by Tikhonov regularization becomes more concentrated and its fitted curves become smoother and the curves move upward at low condition numbers and downward at high condition numbers.

For the case of only changing the number of the force locations with a fixed number of response locations (here 10, see Figures 11 – 18), it can be seen especially that the error amplification factors at low condition numbers become smaller as the number of the force locations increases for all methods. Thus the fitted curves also become smooth and flat as the number of force positions increases.

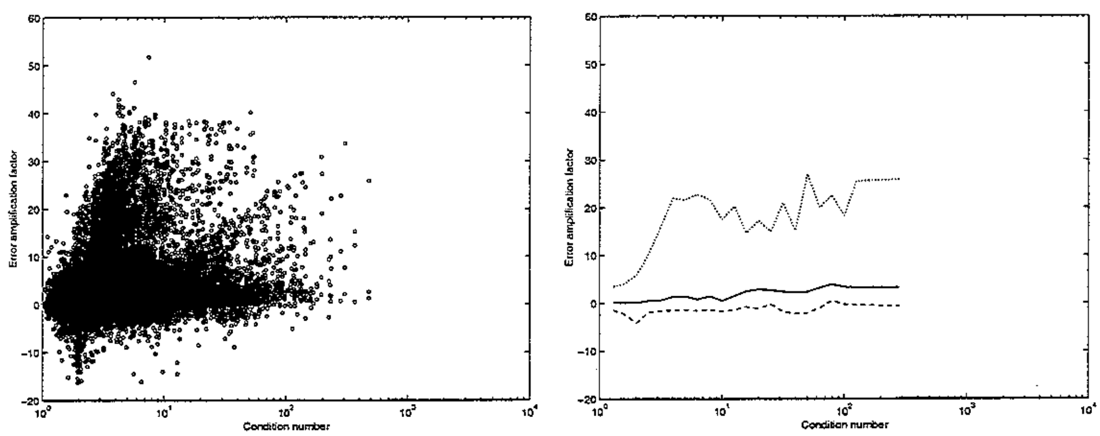
Next, the cross-over condition number is considered.



(a) Pseudo-inverse method

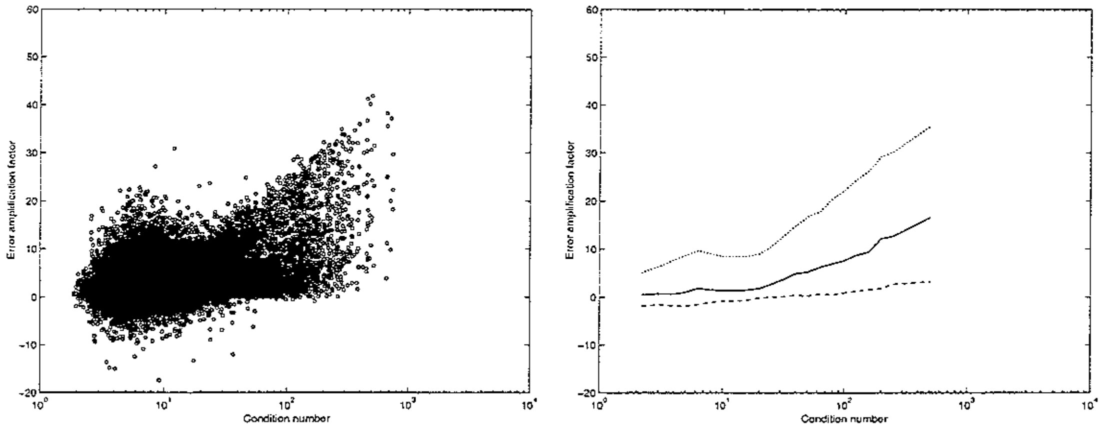


(b) Ordinary cross validation

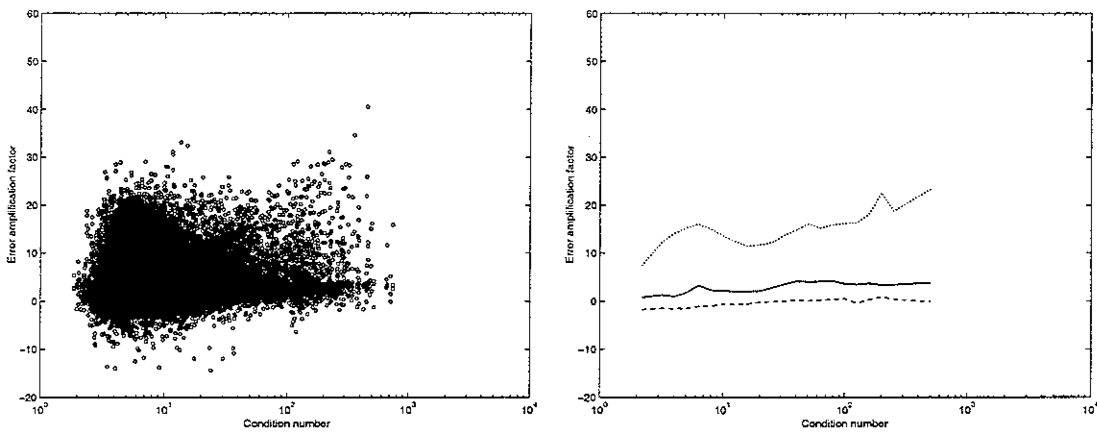


(c) Generalized cross validation

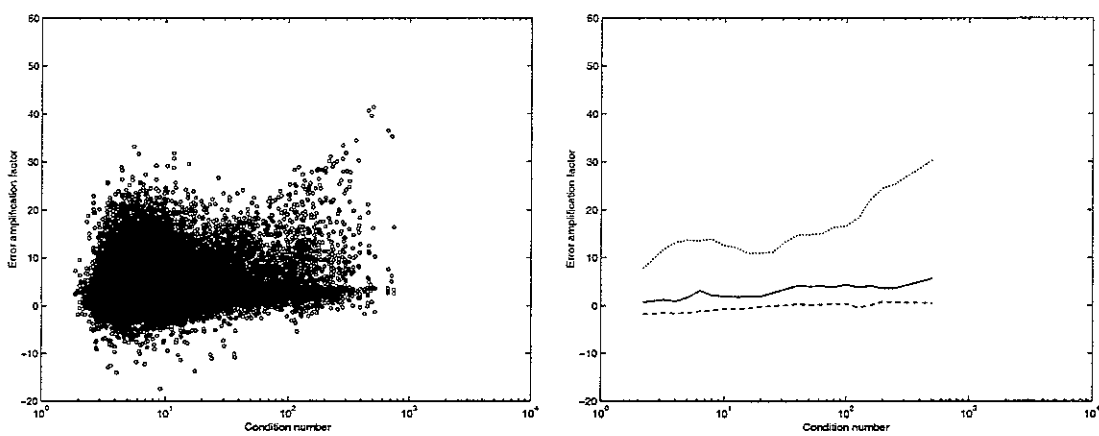
Figure 3. Correspondence between the error amplification factor and the condition number for 3 responses and 2 forces with three noise levels. – – – 5 % confidence level, —— 50 %, --------- 95 %.



(a) Pseudo-inverse method

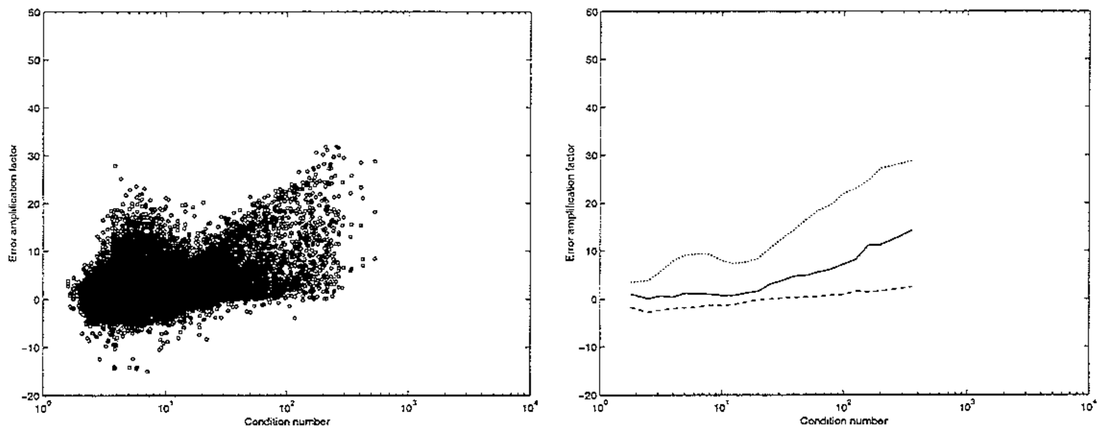


(b) Ordinary cross validation

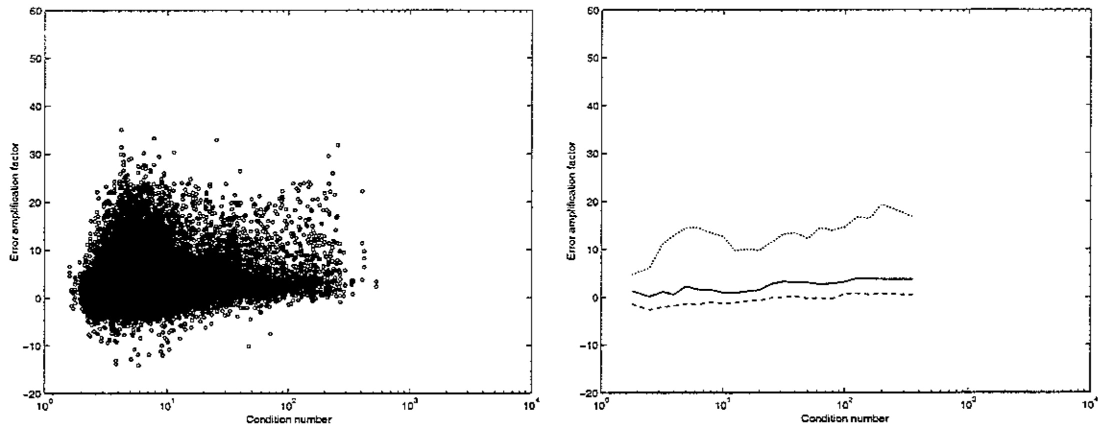


(c) Generalized cross validation

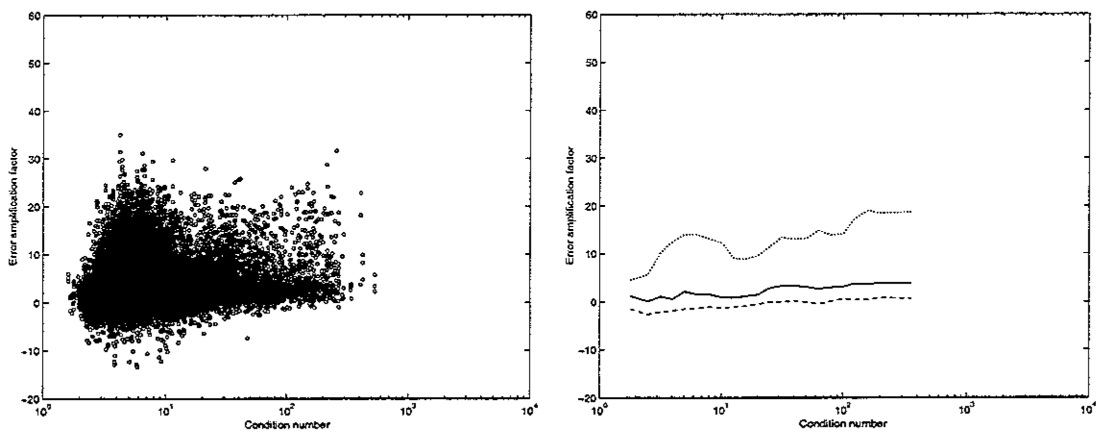
Figure 4. Correspondence between the error amplification factor and the condition number for 4 responses and 3 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

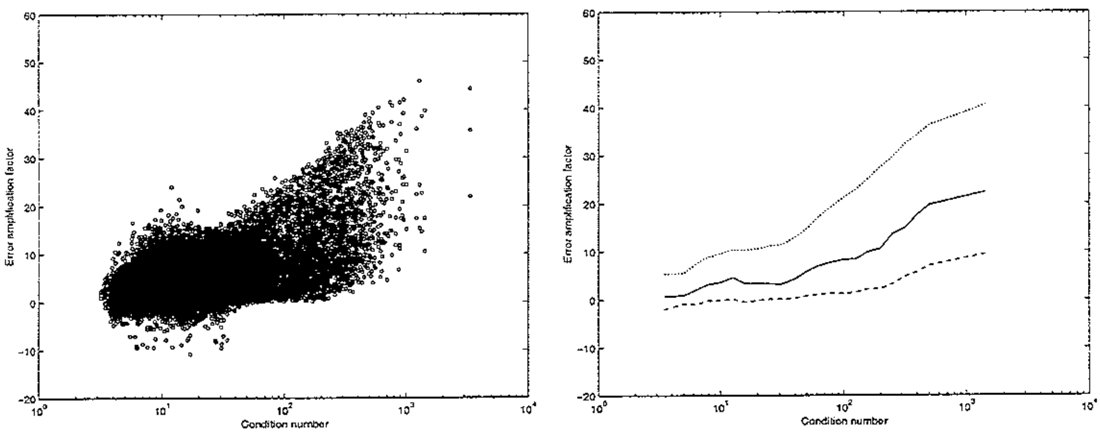


(b) Ordinary cross validation

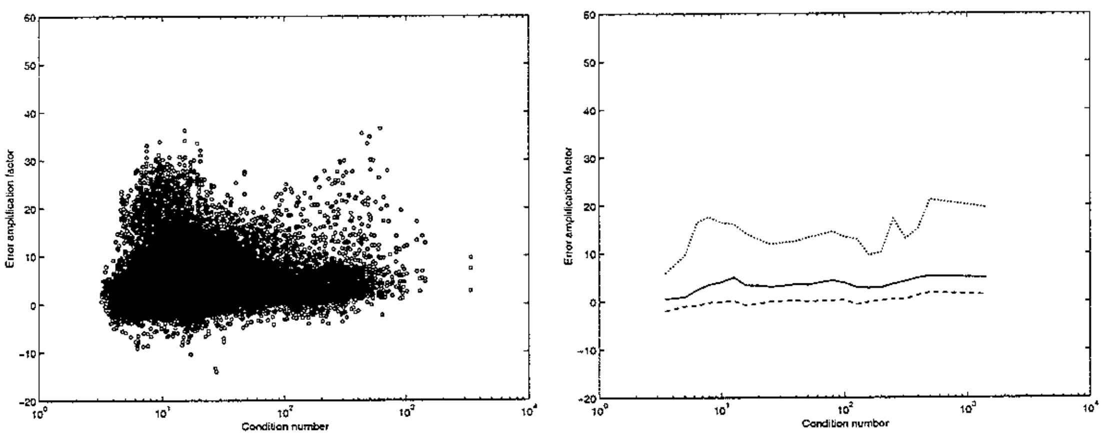


(c) Generalized cross validation

Figure 5. Correspondence between the error amplification factor and the condition number for 5 responses and 3 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method



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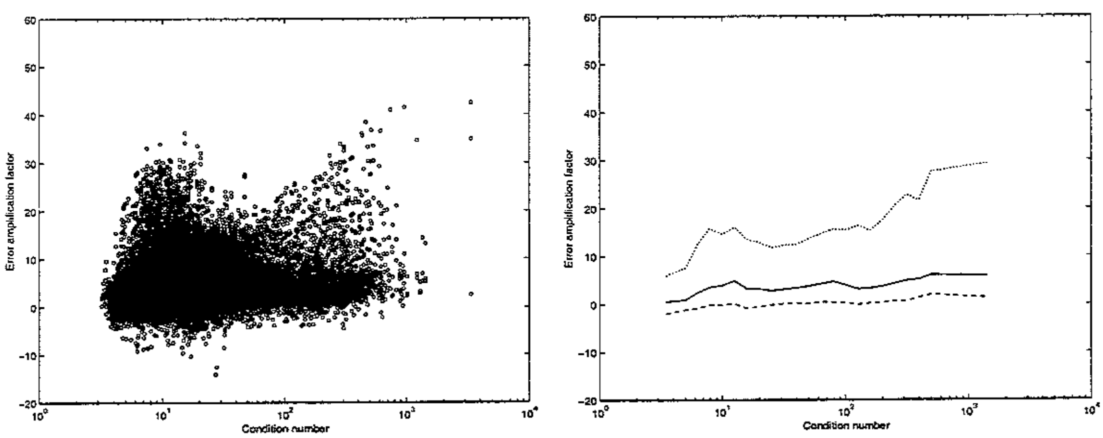
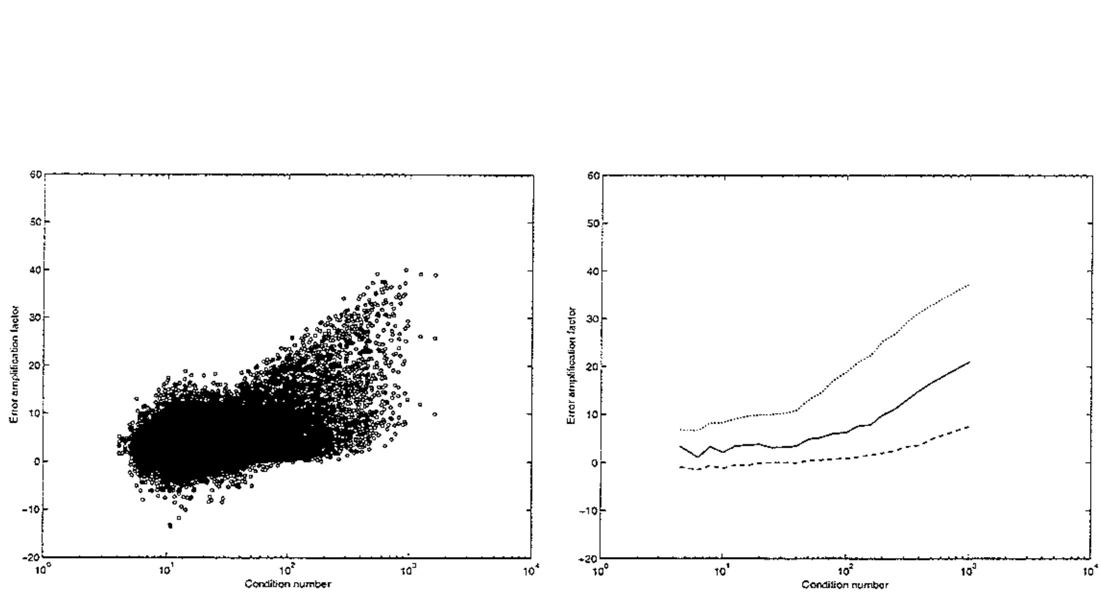
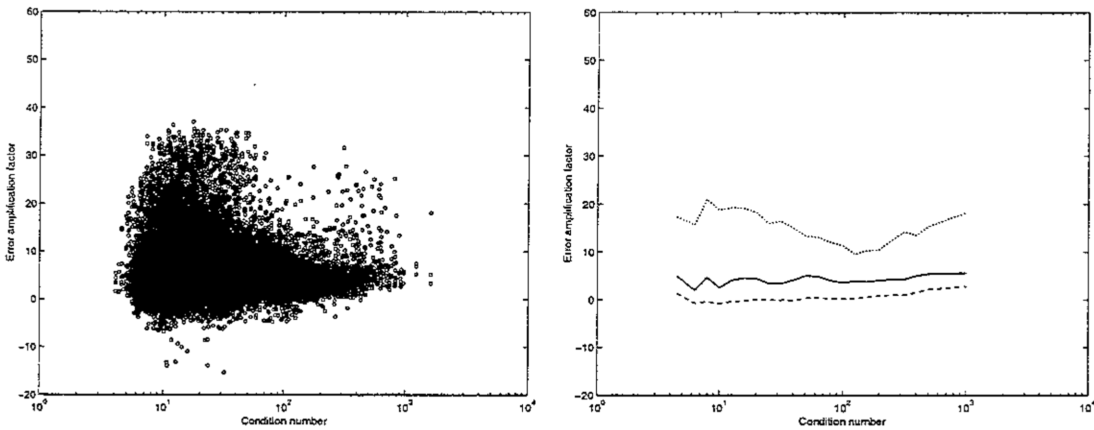


(c) Generalized cross validation

Figure 6. Correspondence between the error amplification factor and the condition number for 6 responses and 5 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method



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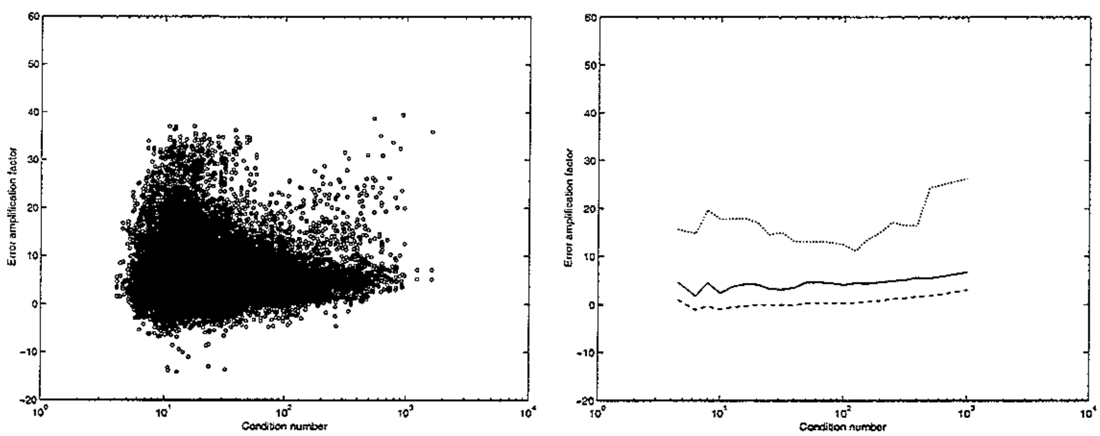
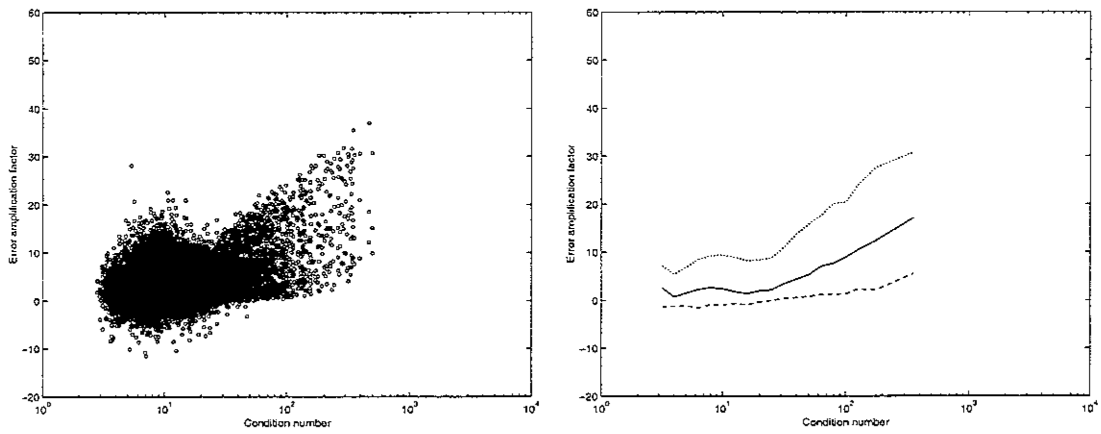
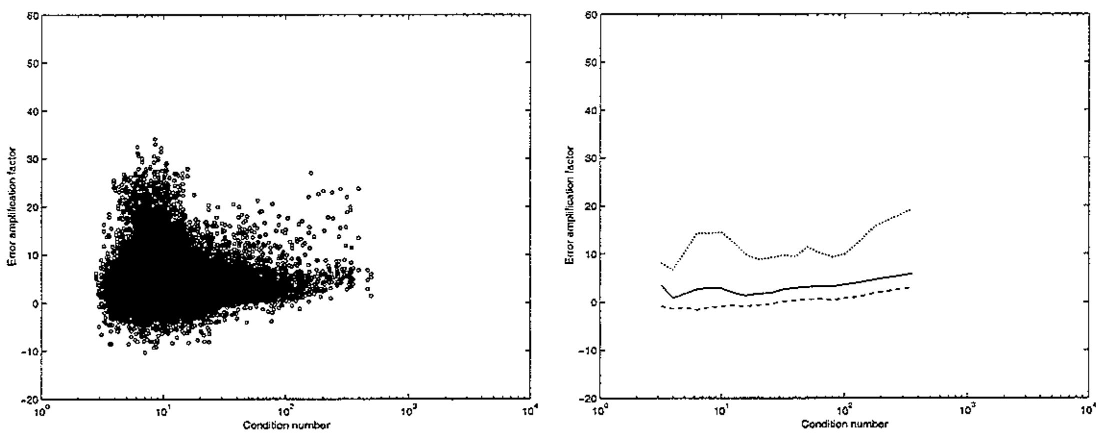


(c) Generalized cross validation

Figure 7. Correspondence between the error amplification factor and the condition number for 7 responses and 6 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method



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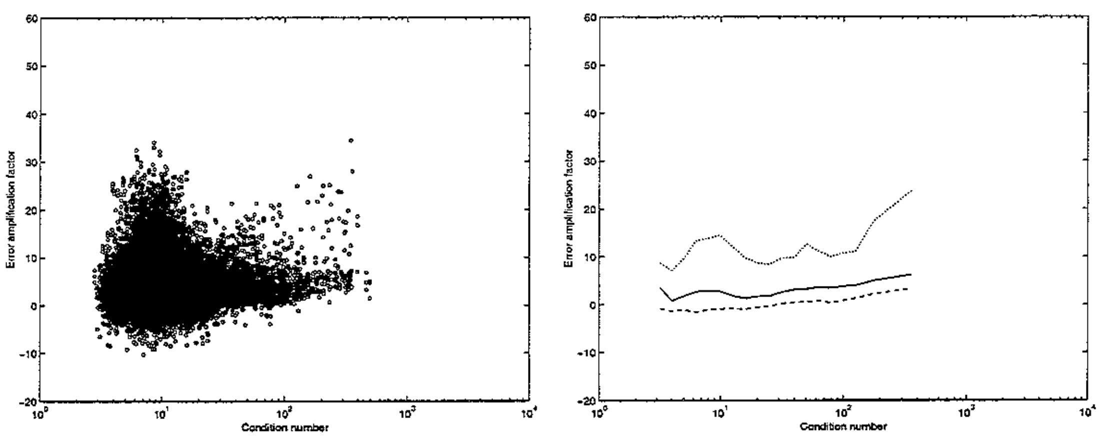
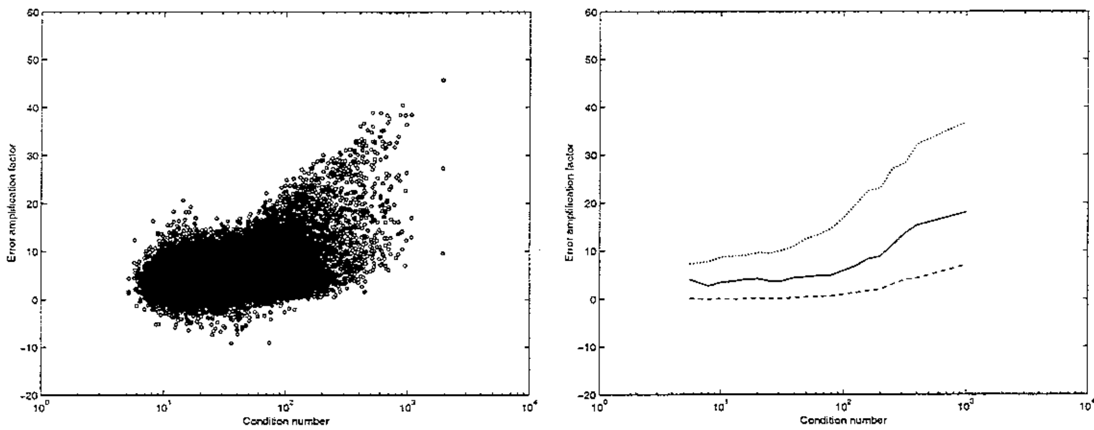
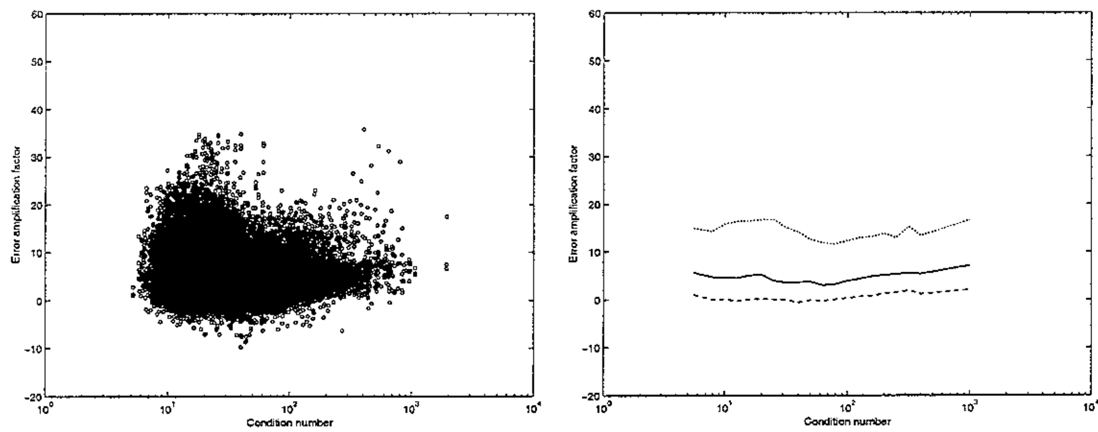


(c) Generalized cross validation

Figure 8. Correspondence between the error amplification factor and the condition number for 8 responses and 5 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method



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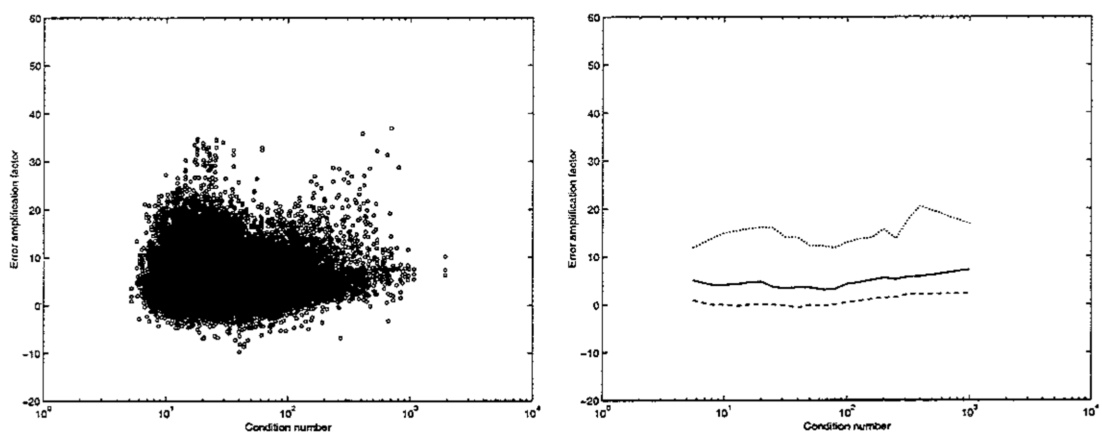
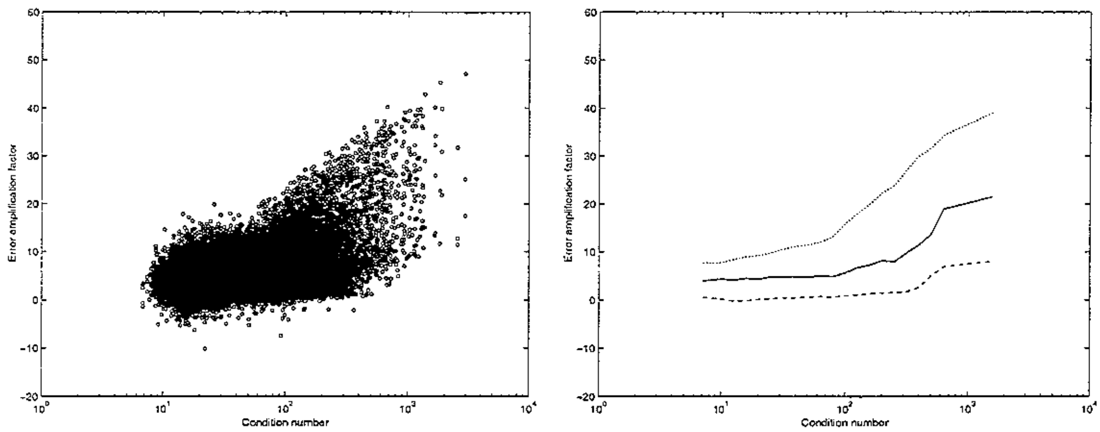
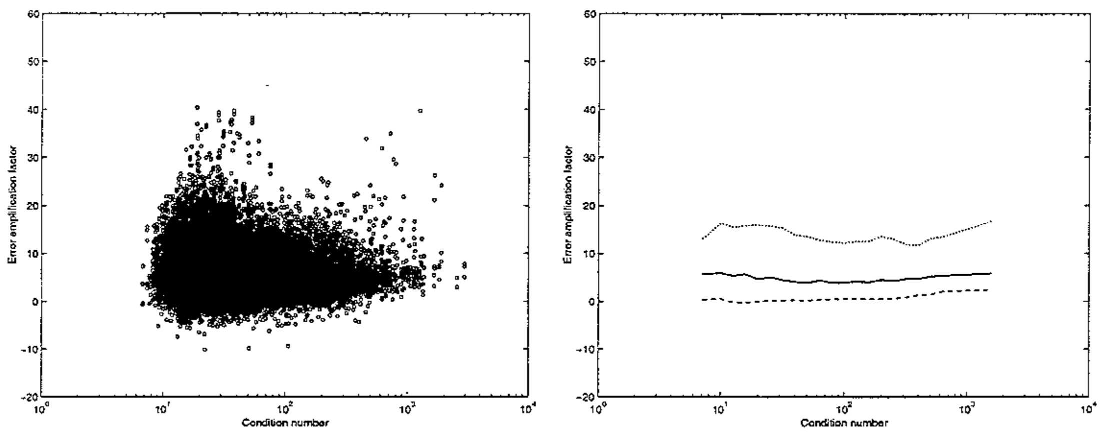


(c) Generalized cross validation

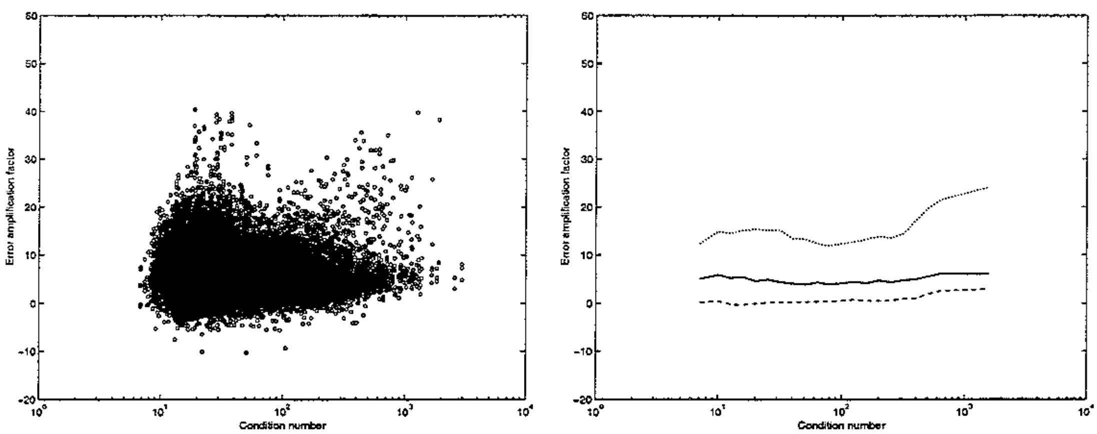
Figure 9. Correspondence between the error amplification factor and the condition number for 8 responses and 7 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

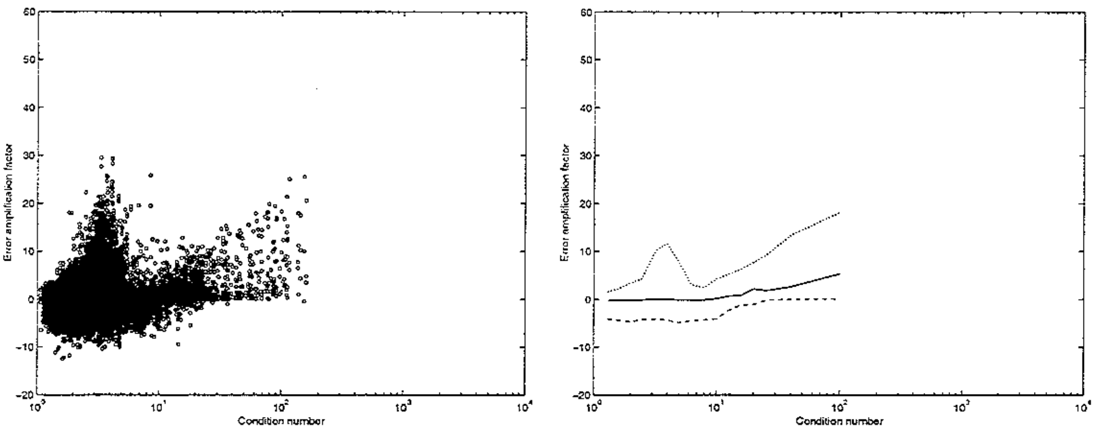


(b) Ordinary cross validation

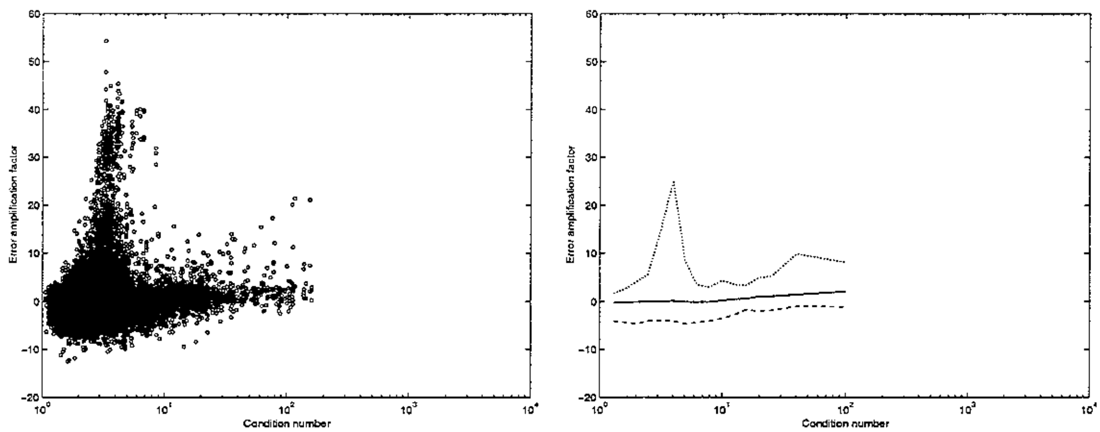


(c) Generalized cross validation

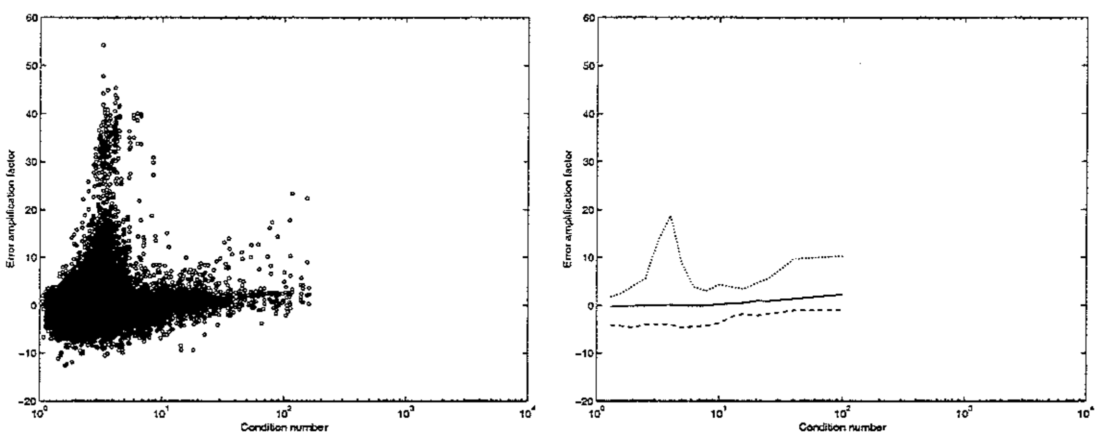
Figure 10. Correspondence between the error amplification factor and the condition number for 9 responses and 8 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

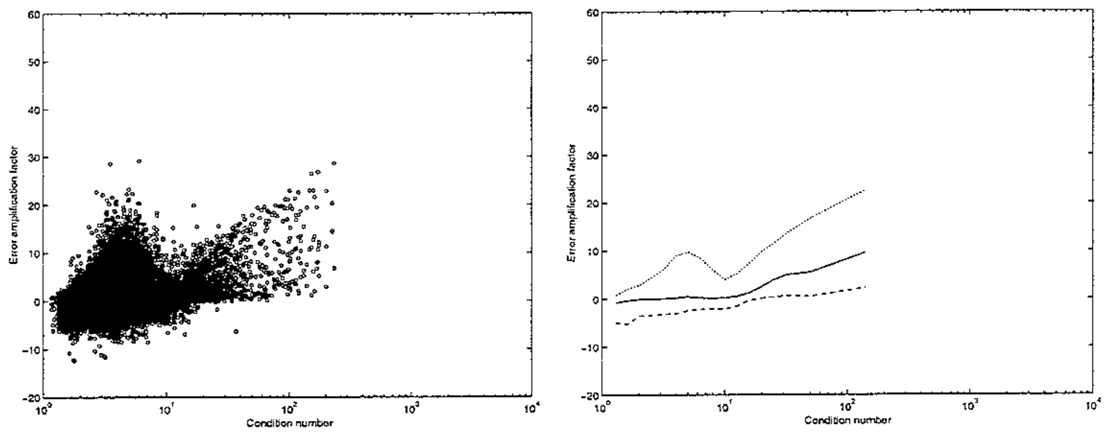


(b) Ordinary cross validation

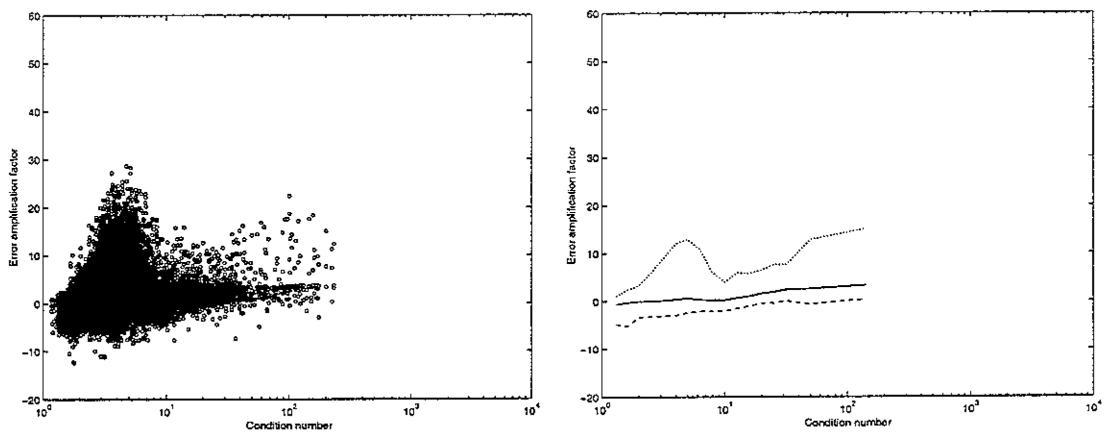


(c) Generalized cross validation

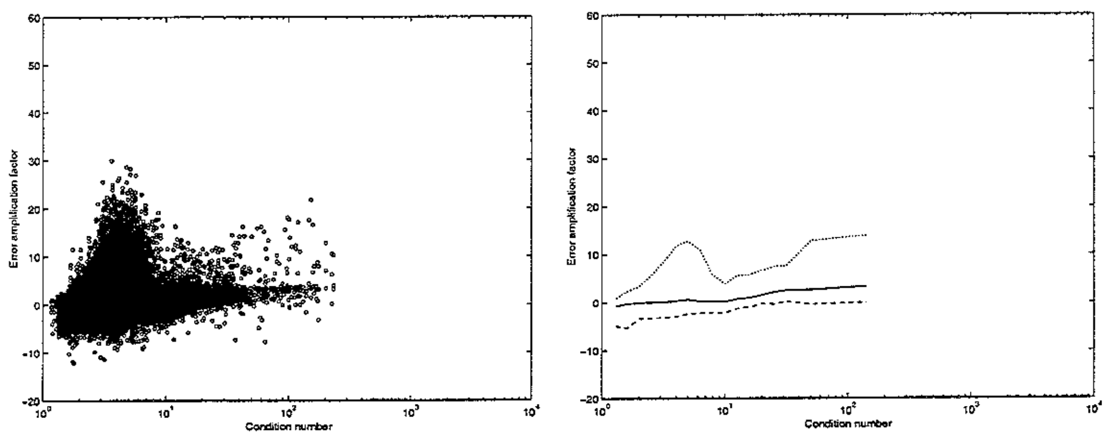
Figure 11. Correspondence between the error amplification factor and the condition number for 10 responses and 2 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

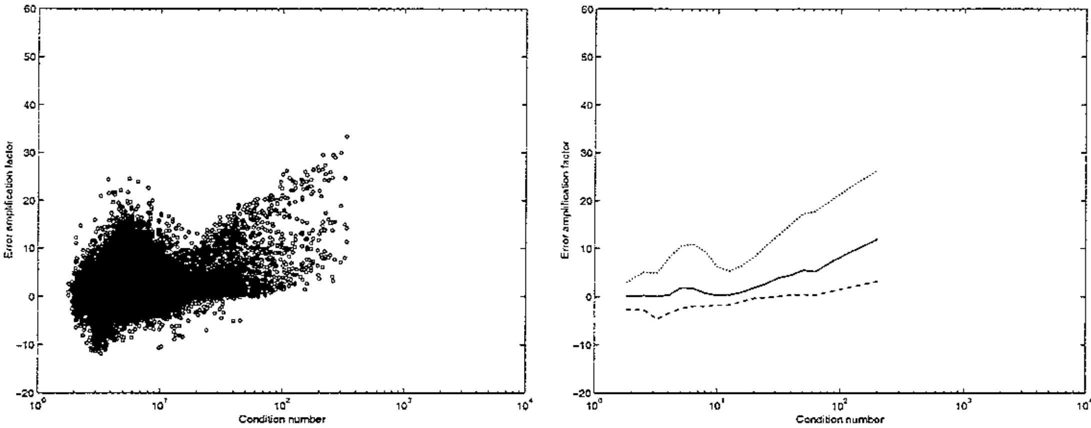


(b) Ordinary cross validation

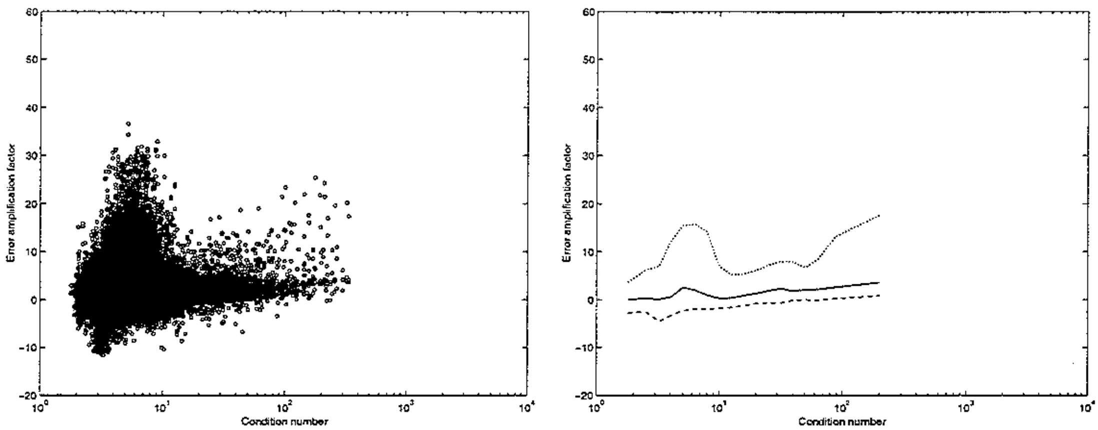


(c) Generalized cross validation

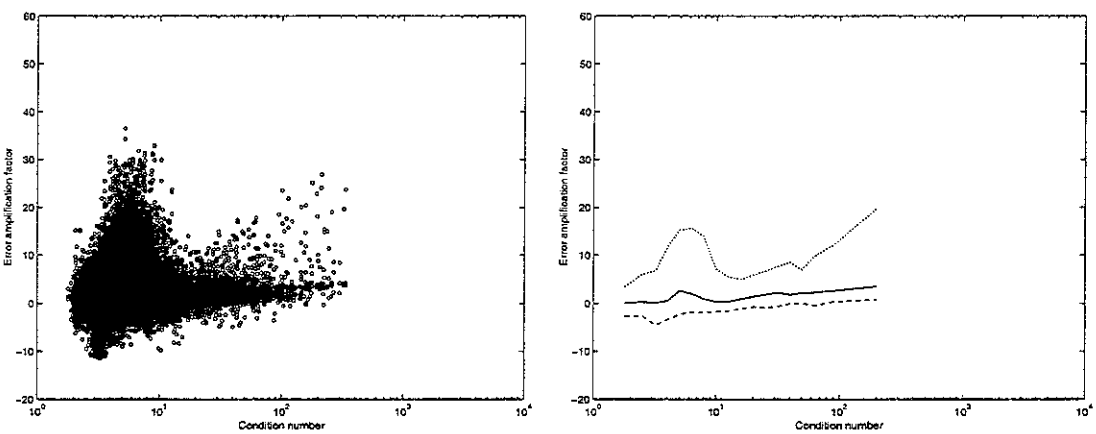
Figure 12. Correspondence between the error amplification factor and the condition number for 10 responses and 3 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

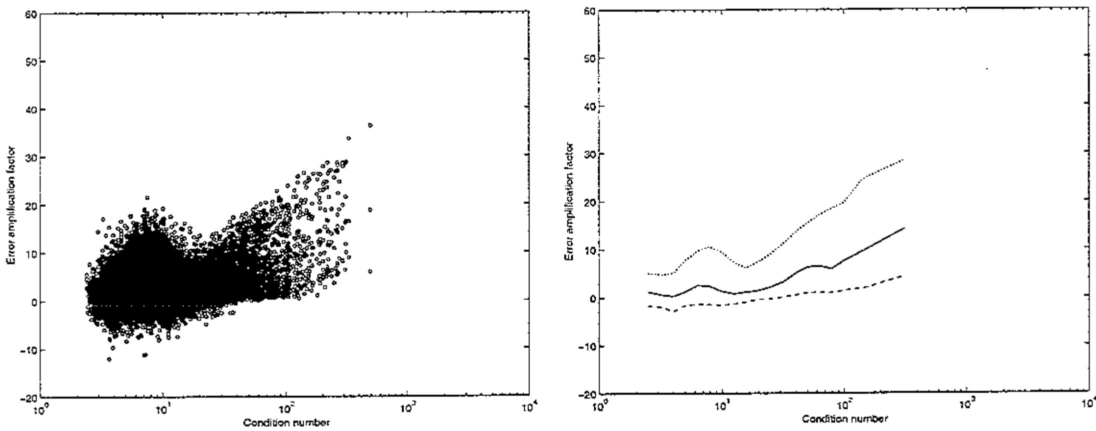


(b) Ordinary cross validation

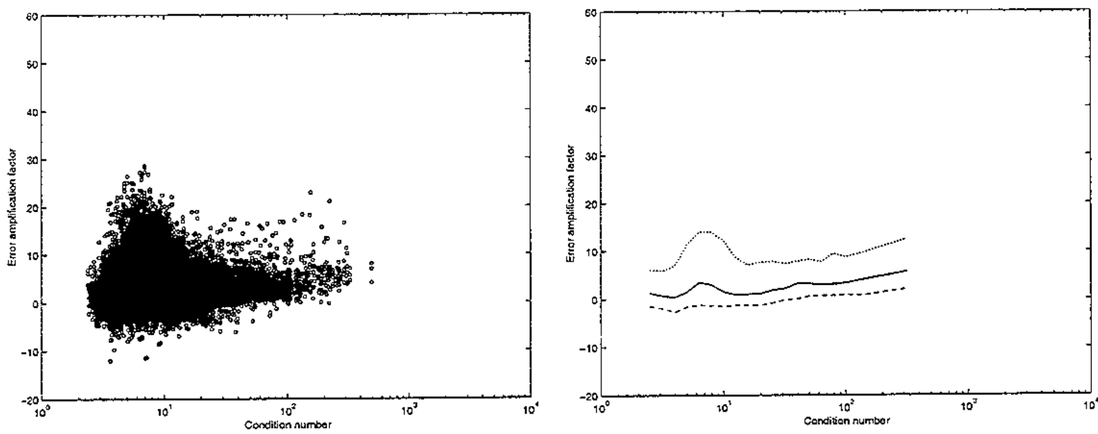


(c) Generalized cross validation

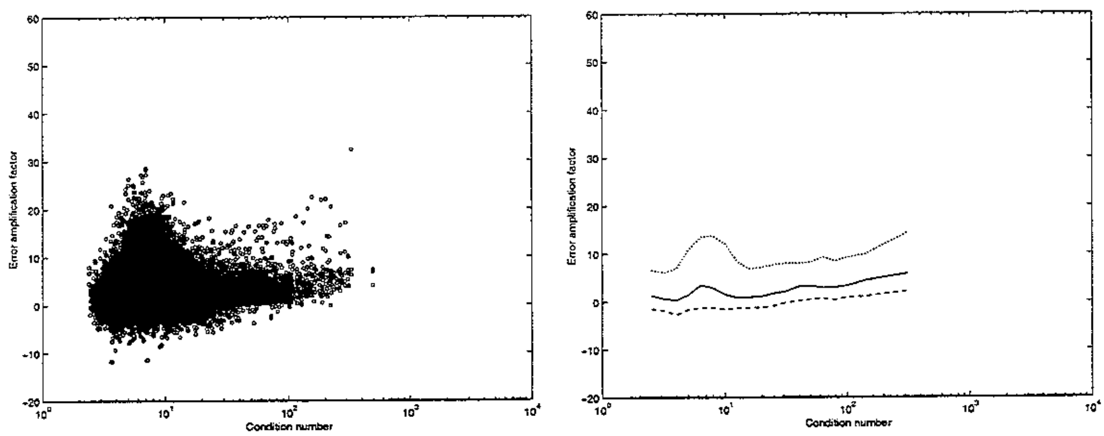
Figure 13. Correspondence between the error amplification factor and the condition number for 10 responses and 4 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

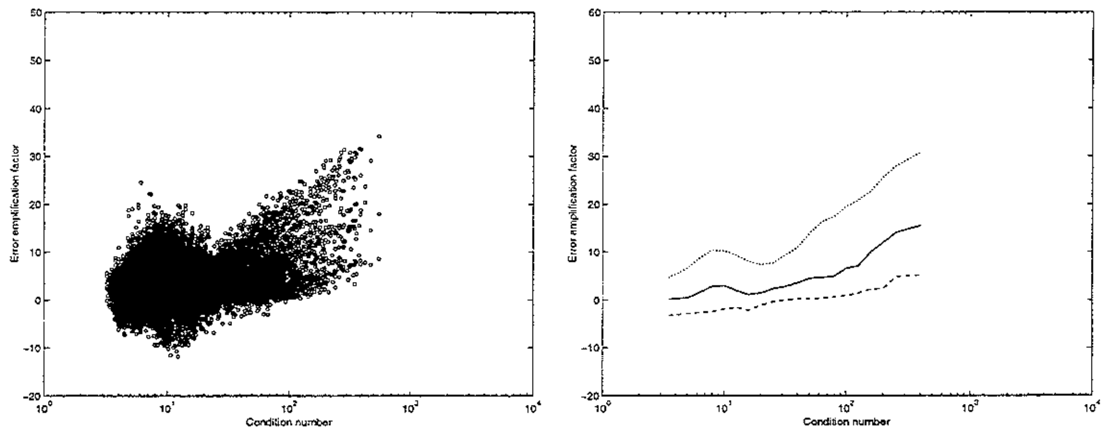


(b) Ordinary cross validation

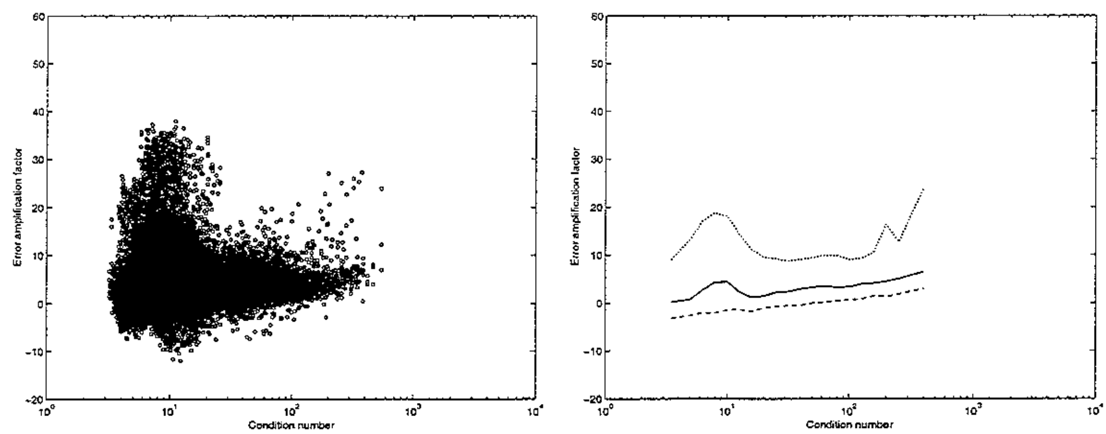


(c) Generalized cross validation

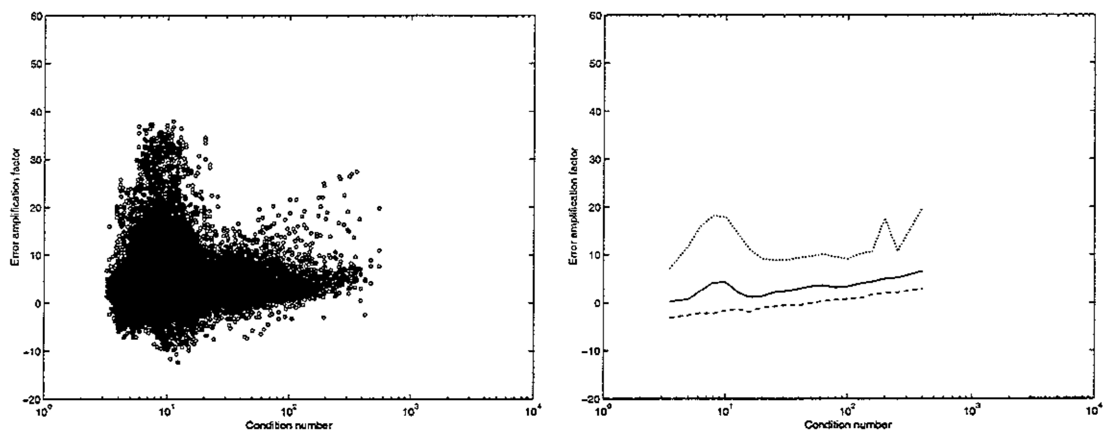
Figure 14. Correspondence between the error amplification factor and the condition number for 10 responses and 5 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

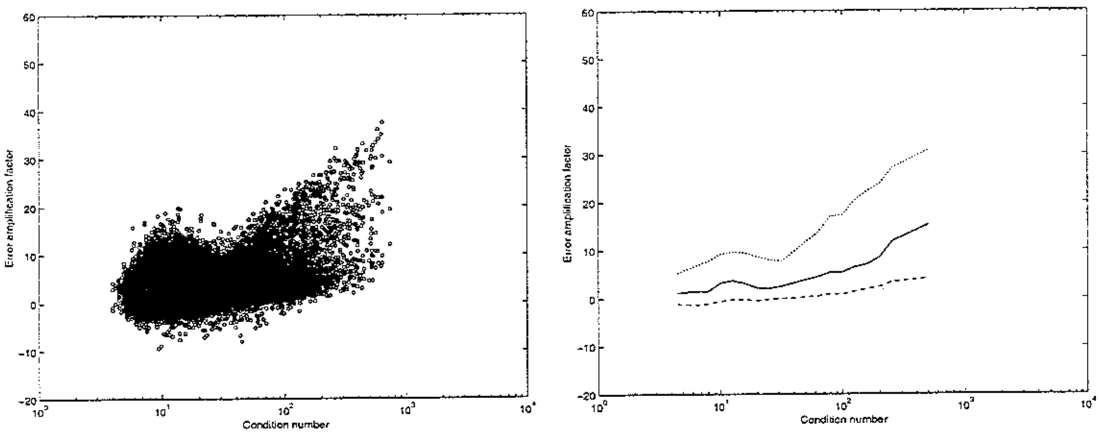


(b) Ordinary cross validation

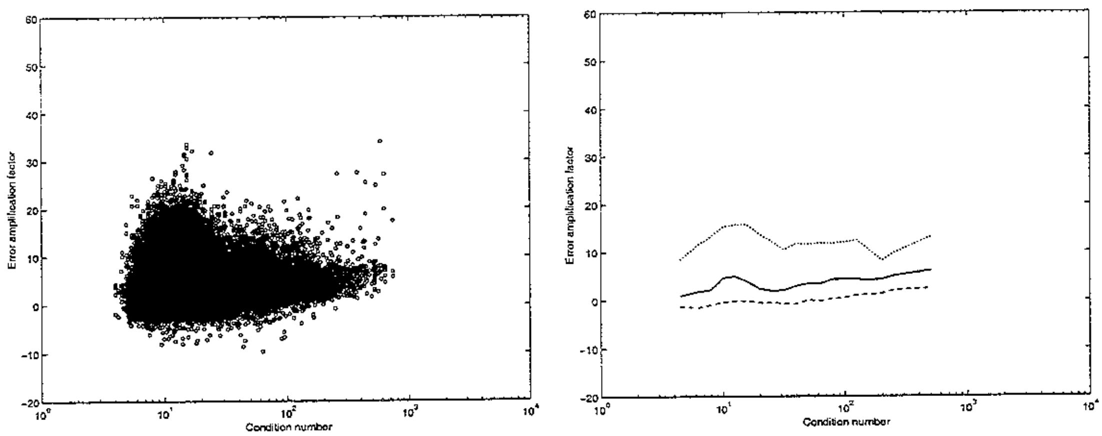


(c) Generalized cross validation

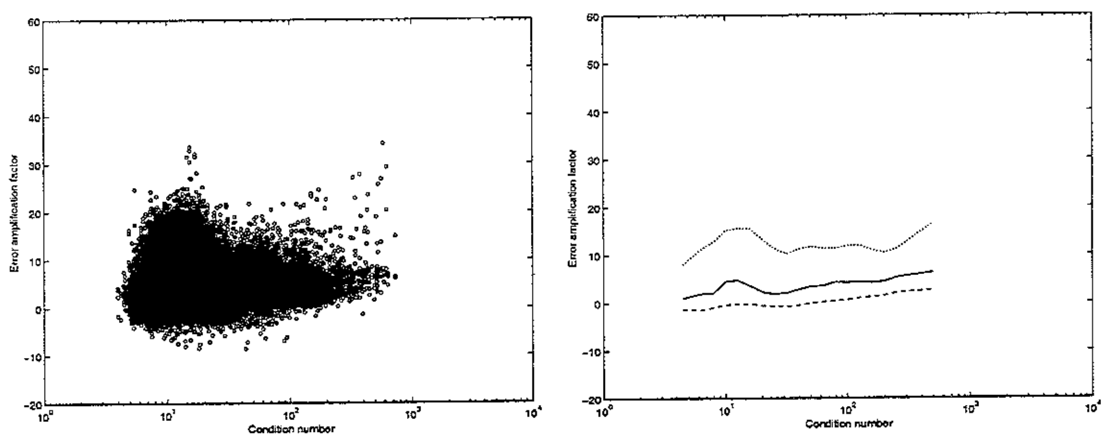
Figure 15. Correspondence between the error amplification factor and the condition number for 10 responses and 6 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

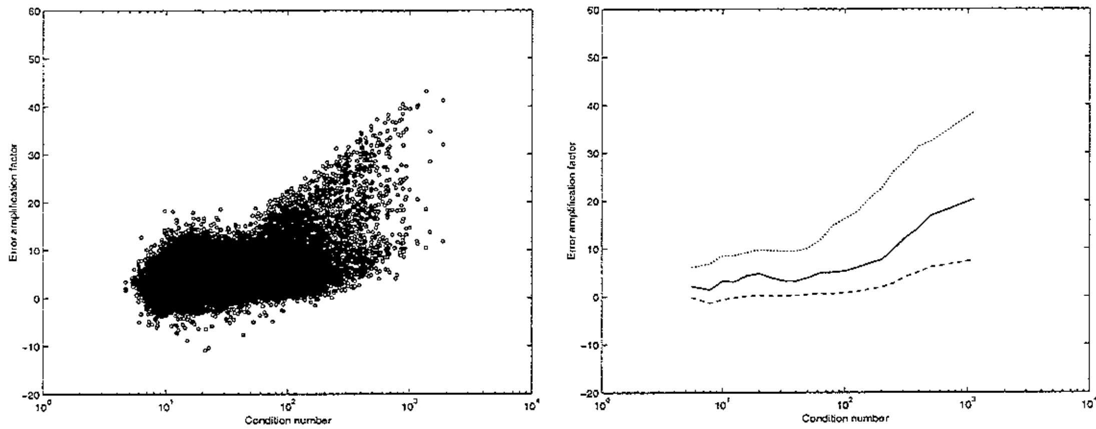


(b) Ordinary cross validation

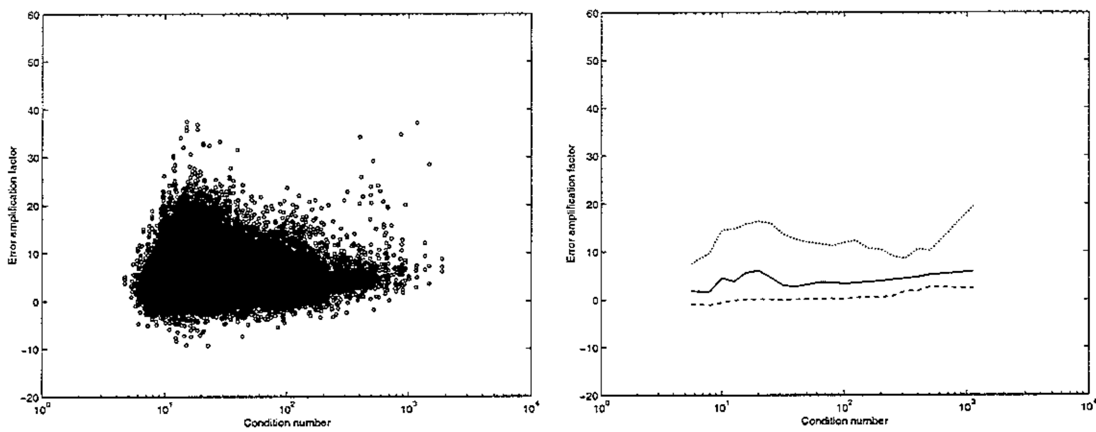


(c) Generalized cross validation

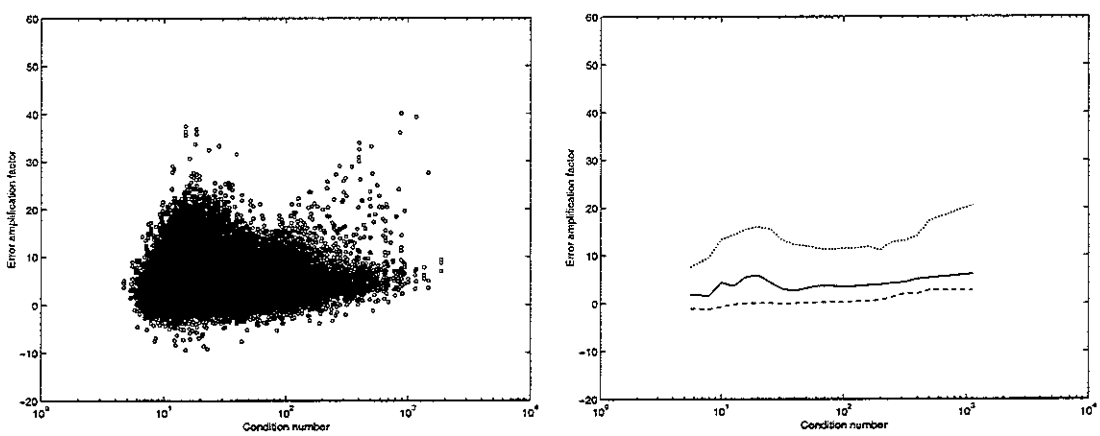
Figure 16. Correspondence between the error amplification factor and the condition number for 10 responses and 7 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

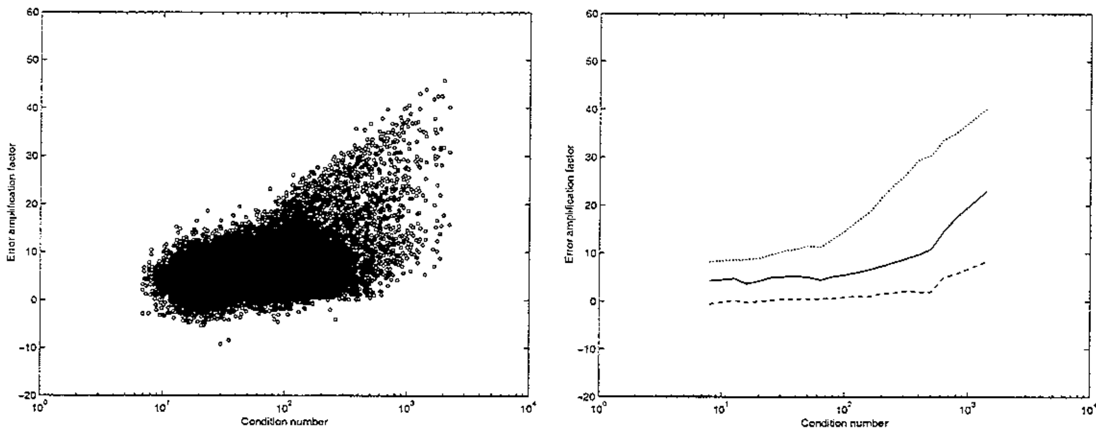


(b) Ordinary cross validation

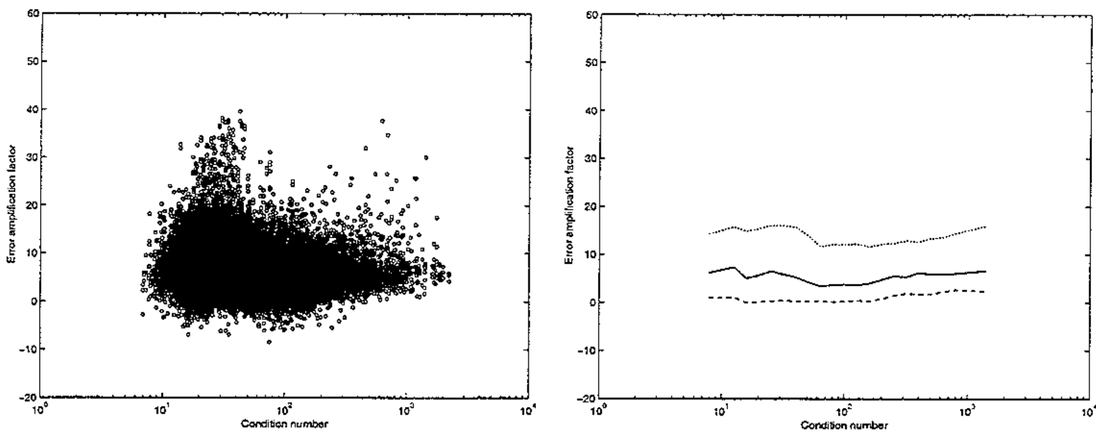


(c) Generalized cross validation

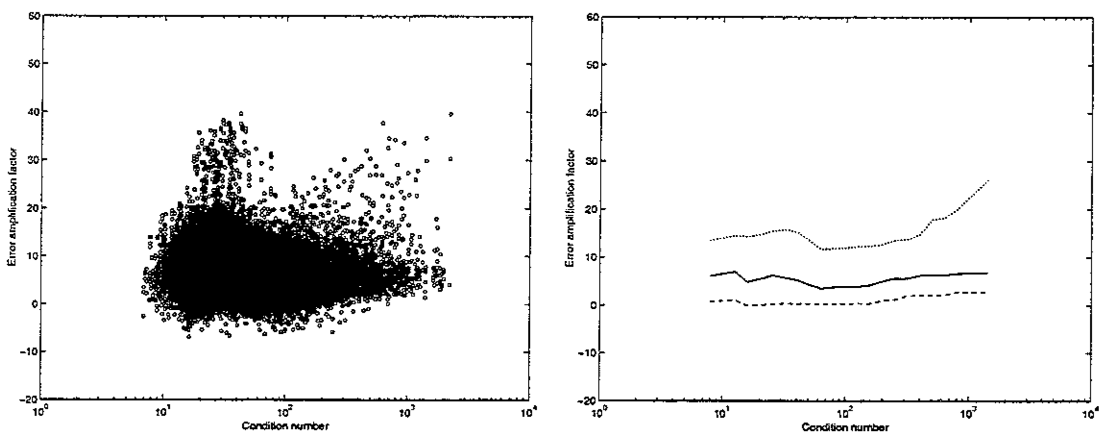
Figure 17. Correspondence between the error amplification factor and the condition number for 10 responses and 8 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

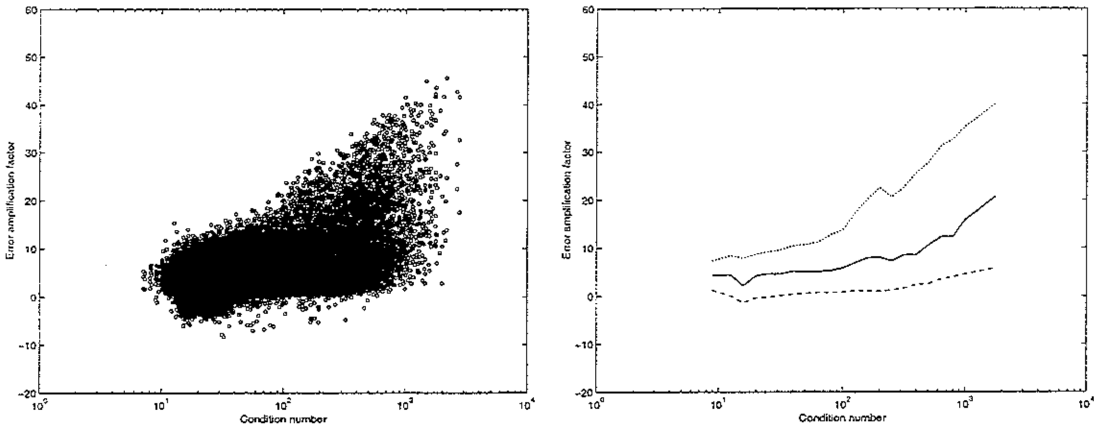


(b) Ordinary cross validation

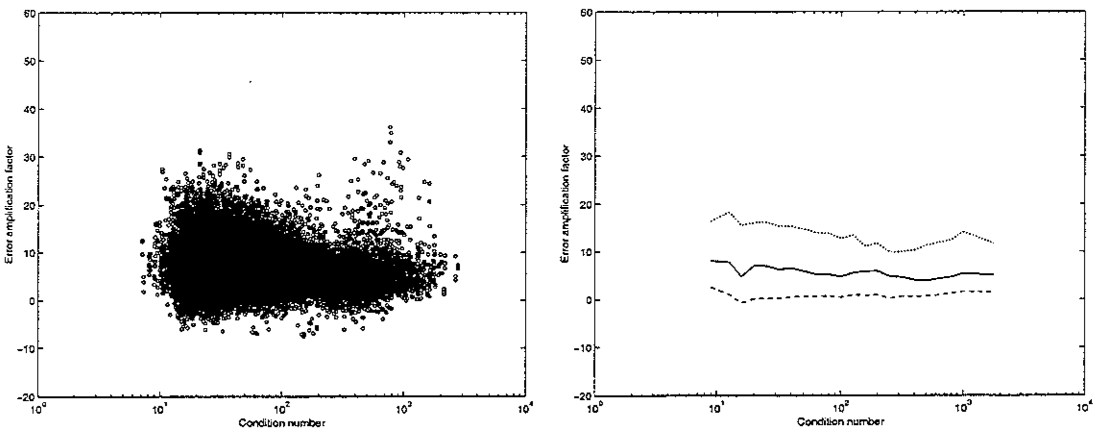


(c) Generalized cross validation

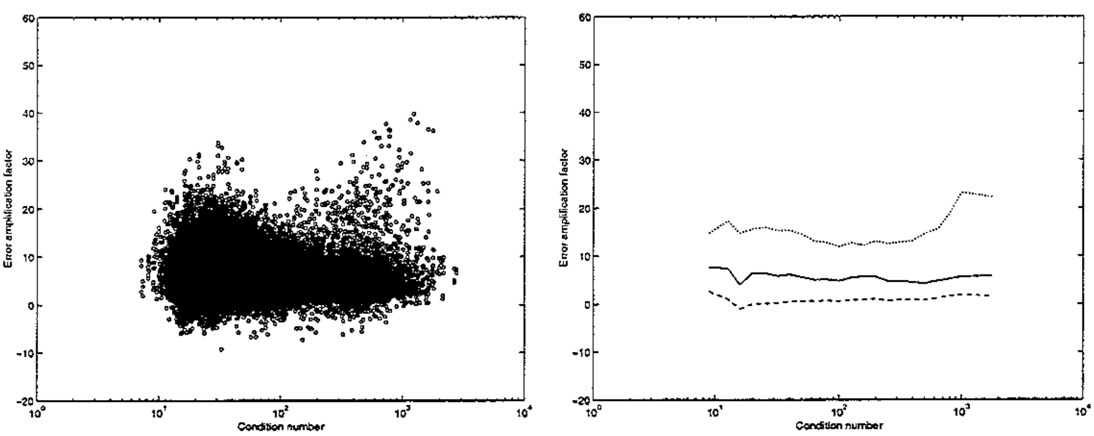
Figure 18. Correspondence between the error amplification factor and the condition number for 10 responses and 9 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

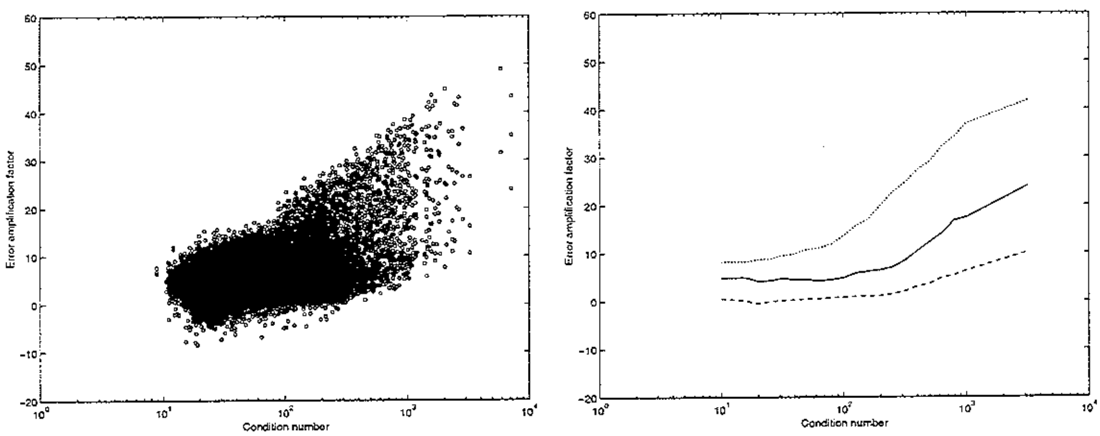


(b) Ordinary cross validation

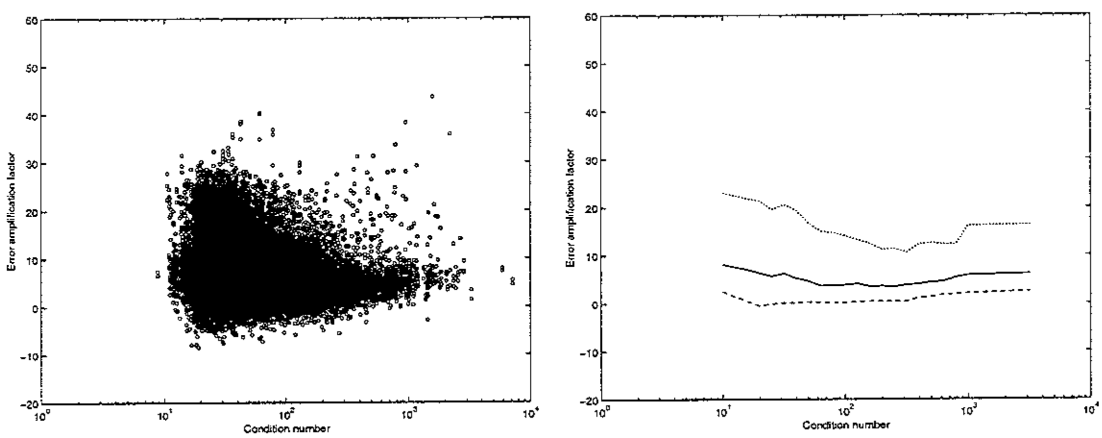


(c) Generalized cross validation

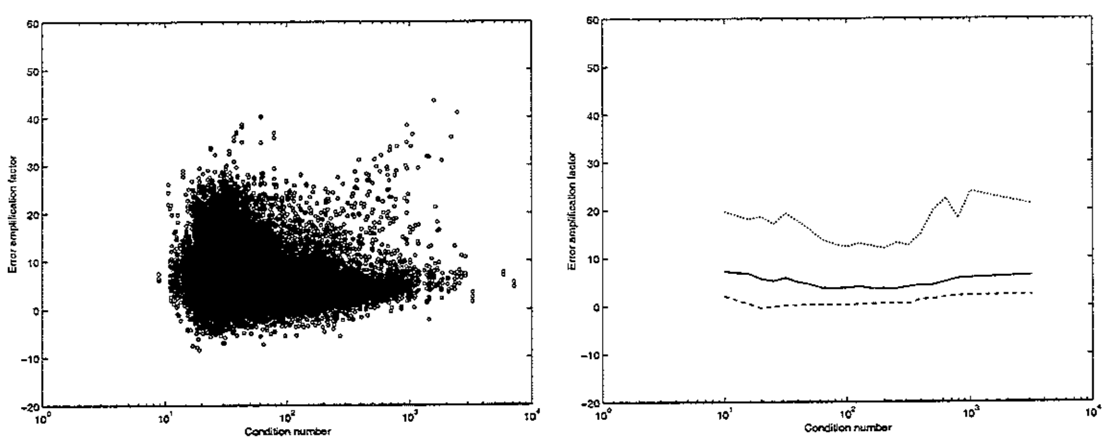
Figure 19. Correspondence between the error amplification factor and the condition number for 11 responses and 10 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

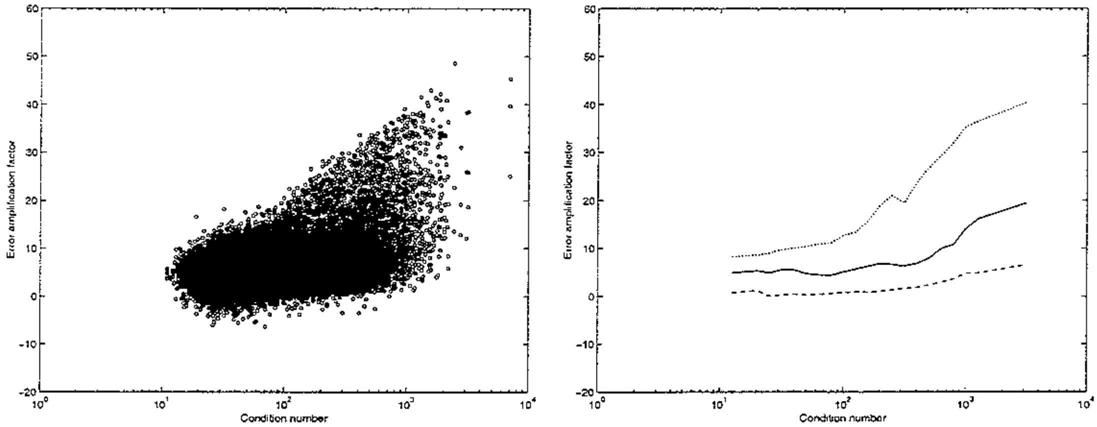


(b) Ordinary cross validation

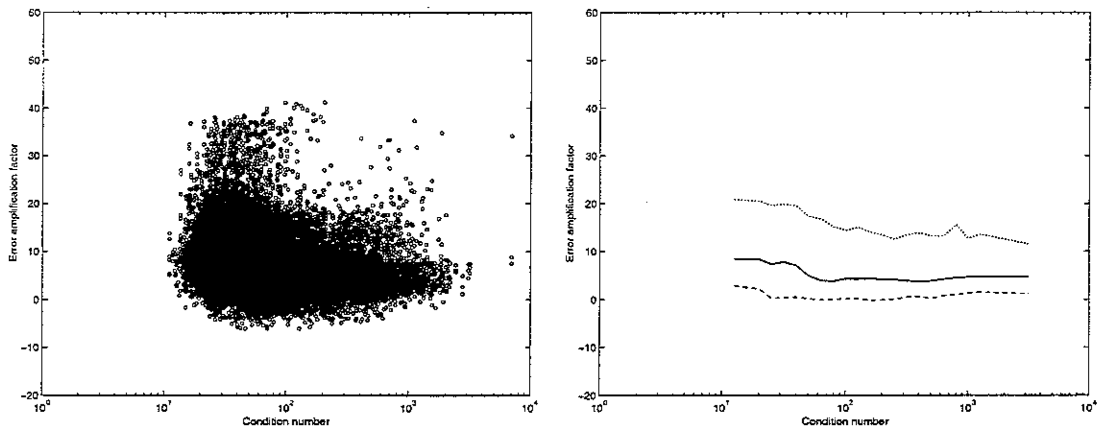


(c) Generalized cross validation

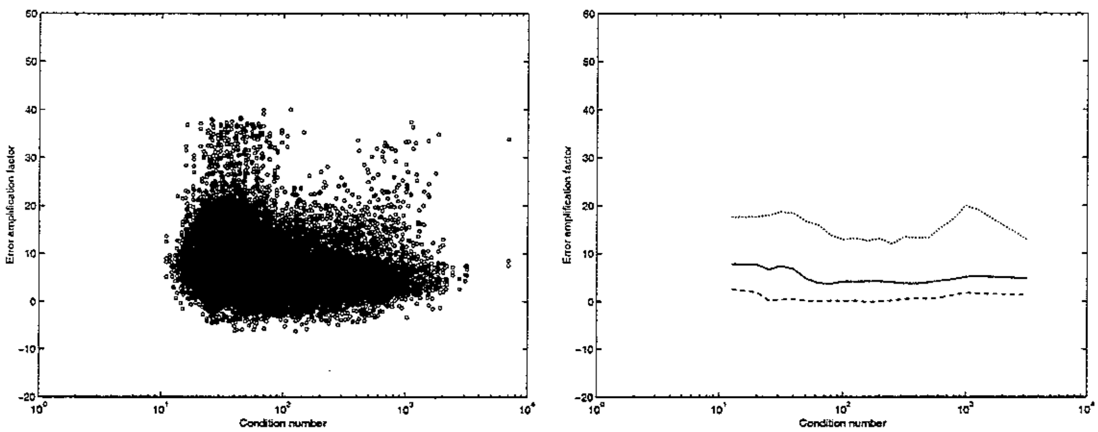
Figure 20. Correspondence between the error amplification factor and the condition number for 12 responses and 11 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

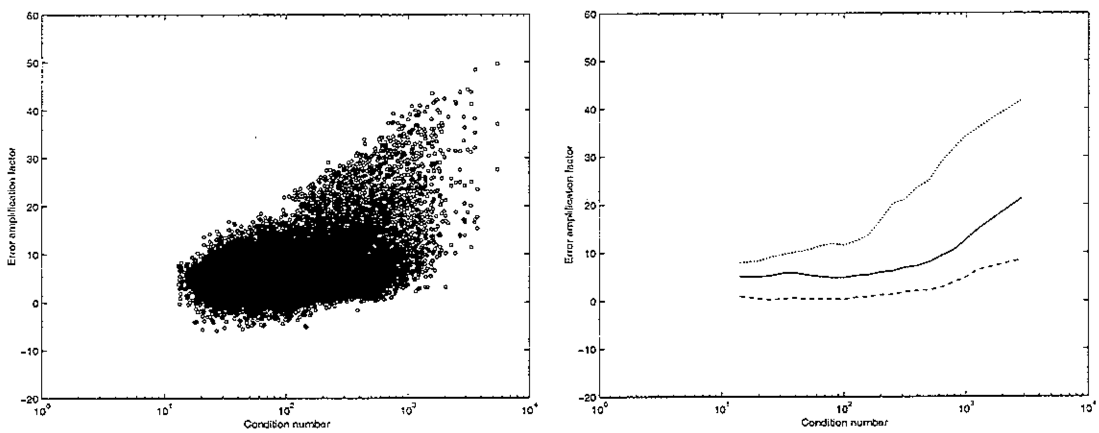


(b) Ordinary cross validation

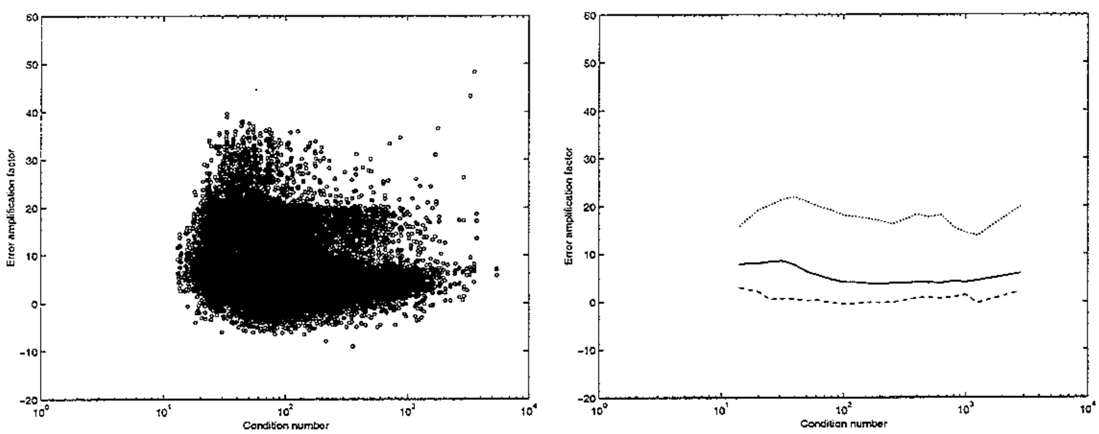


(c) Generalized cross validation

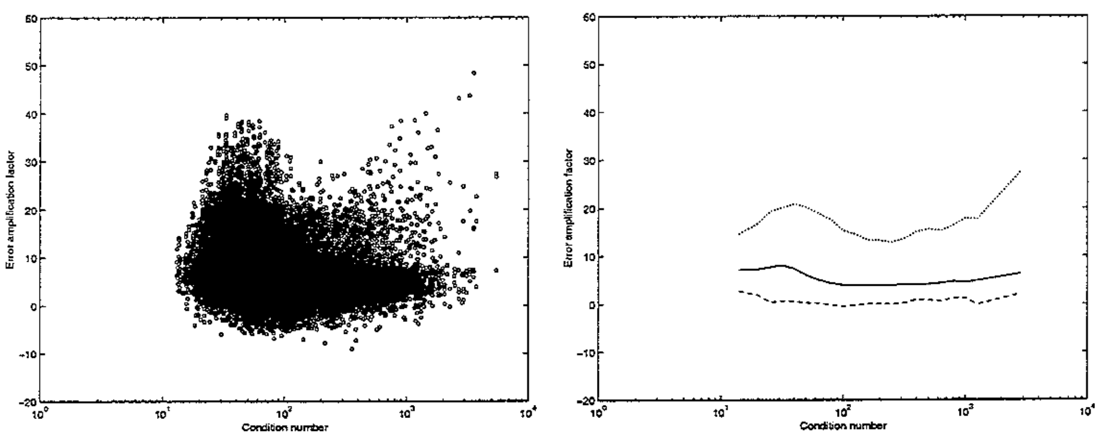
Figure 21. Correspondence between the error amplification factor and the condition number for 13 responses and 12 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method

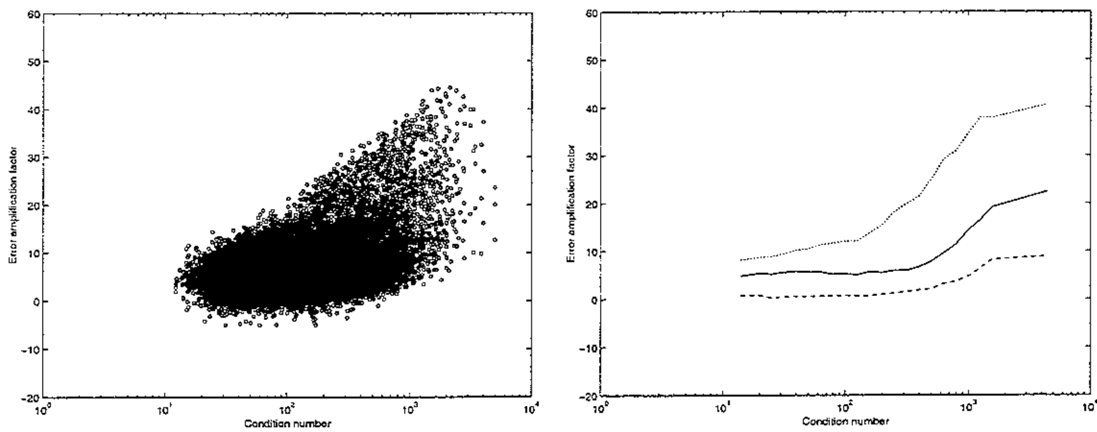


(b) Ordinary cross validation

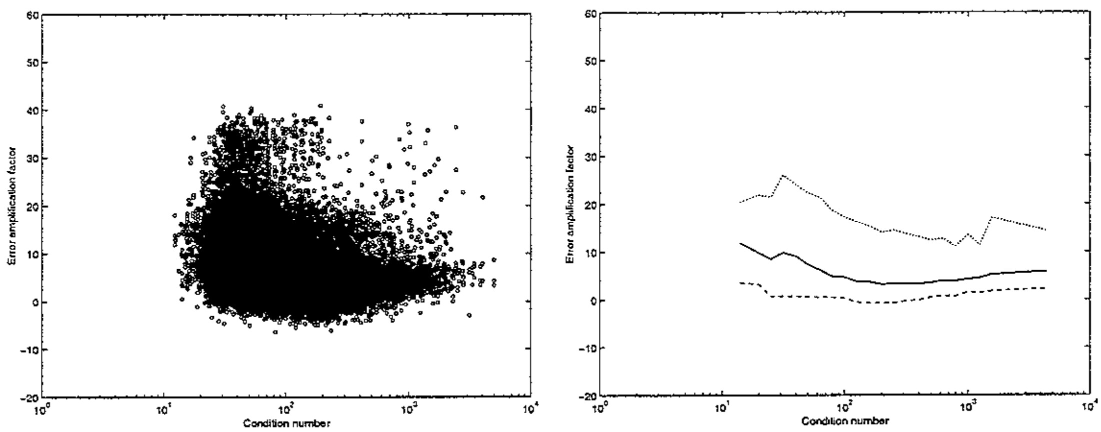


(c) Generalized cross validation

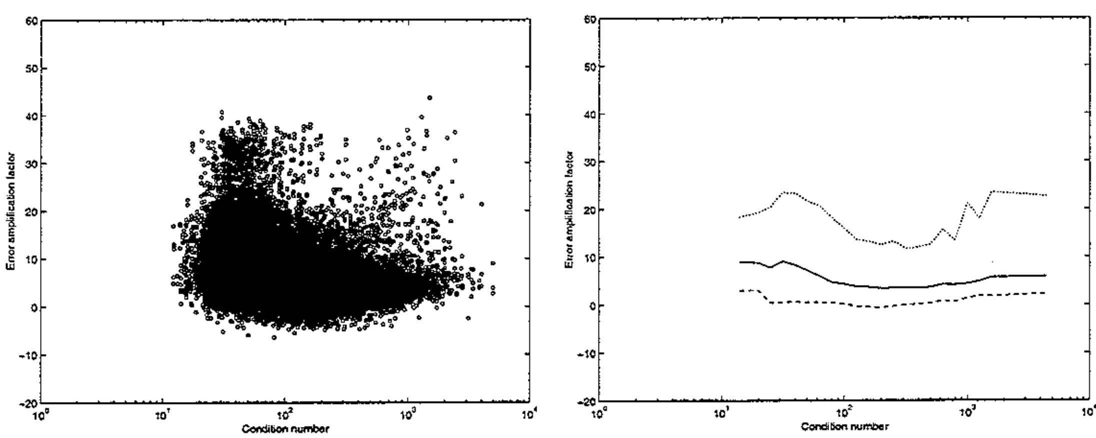
Figure 22. Correspondence between the error amplification factor and the condition number for 14 responses and 13 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.



(a) Pseudo-inverse method



(b) Ordinary cross validation



(c) Generalized cross validation

Figure 23. Correspondence between the error amplification factor and the condition number for 15 responses and 14 forces with three noise levels. – – – 5 % confidence level, —— 50 %, -------- 95 %.

To obtain the cross-over condition number, the fitted curves of the error amplification factor with respect to the condition number obtained by the pseudo-inverse method are compared with those from the Tikhonov regularization based on the ordinary and generalized cross validation. These results are shown in Figure 24 for the series of response and force positions used previously. These figures are the assemblies of the figures obtained by each method shown in Figures 3 to 23. Out of the three (5, 50 and 95 %) fitted curves, the 95 % fitted curve is of most importance as it is the larger error amplifications that must be avoided. Besides, as seen in Figure 24, the cross-over for the 50 % confidence level is at a lower value of condition number in each case than for the 95 % confidence level. This means that the forces reconstructed by Tikhonov regularization around the cross-over condition number for the 50 % confidence level are prone to involve larger errors than those above the cross-over condition number for the 95 % level. Therefore, the cross-over condition numbers are obtained by comparing the 95 % fitted curves obtained by the pseudo-inverse method and those obtained by each cross validation method of Tikhonov regularization.

Table 2 shows the values of the cross-over condition numbers for all systems. Figure 25 (a) shows a plot of the cross-over condition numbers for $10\times n$ systems, and Figure 25 (b) for $n\times(n-1)$ systems. It can be seen that the selection of Tikhonov regularization is related not only with the condition number but also with the system matrix size, because the cross-over condition number varies between systems. For this rectangular plate, Tikhonov regularization should be selected if the condition number of an FRF matrix at a particular frequency is greater than the corresponding cross-over condition number. This threshold varies from 10 for a 10×2 matrix to around 200 for a 15×14 matrix. It increases as the matrix size increases and as the level of over-determination reduces. The thresholds for ordinary and generalized cross validation are very similar in each case.

In conclusion, it is important to consider the condition number and the system matrix size together when selecting the Tikhonov regularization to reconstruct the operational forces.

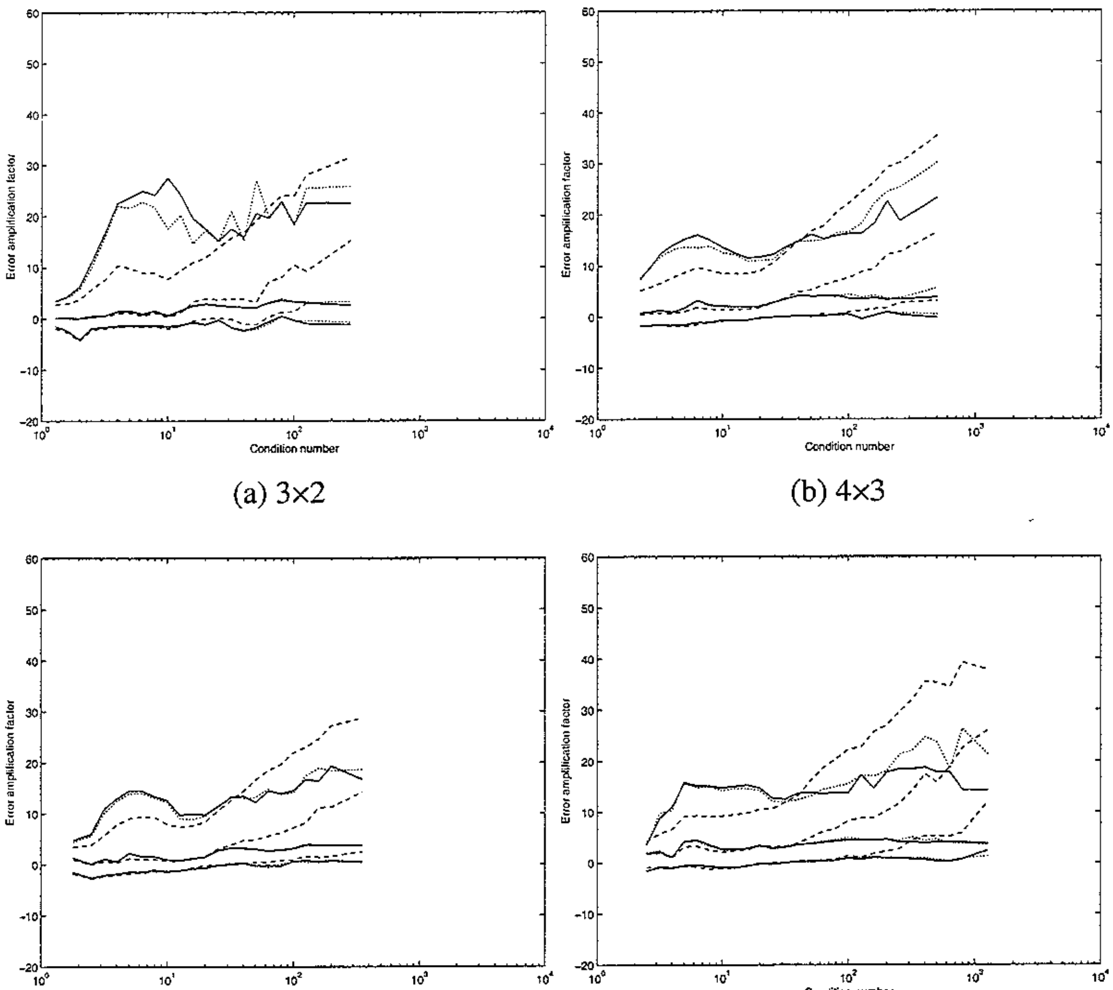
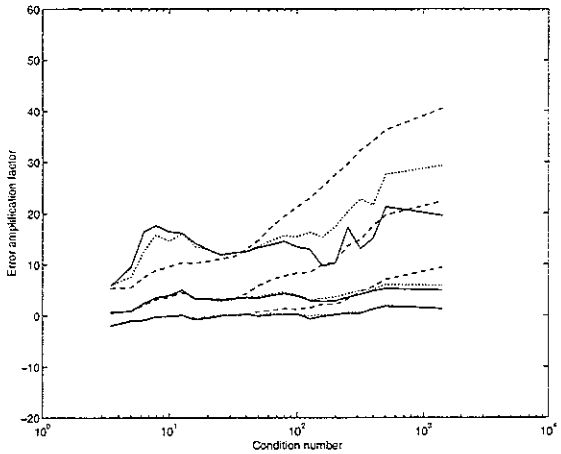
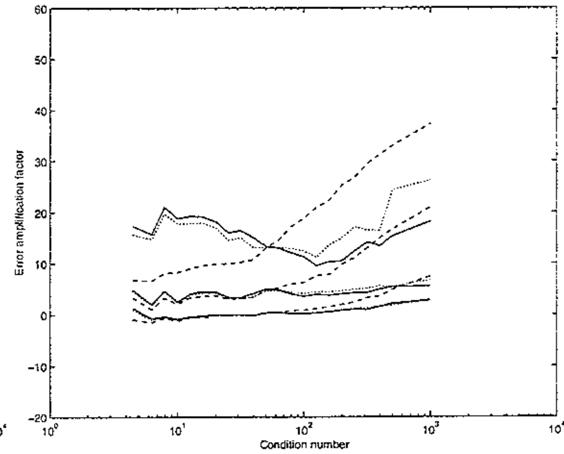


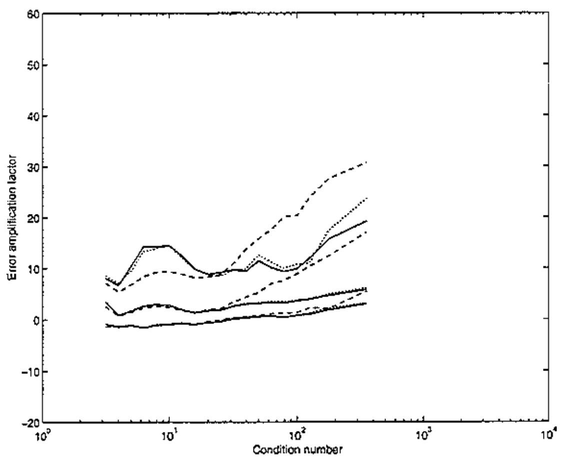
Figure 24. Comparison of the fitted lines obtained by the pseudo-inverse (– – –), ordinary (——) and generalized cross validation (--------) for a different number of response and force positions.



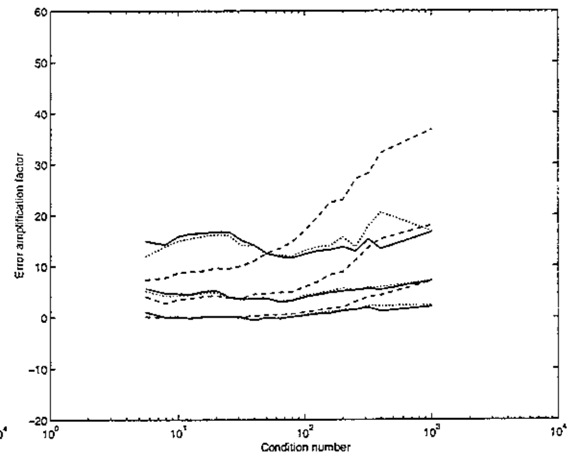
(e) 6×5



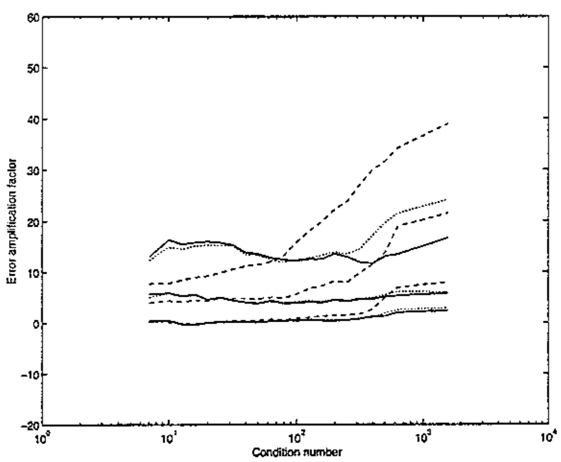
(f) 7×6



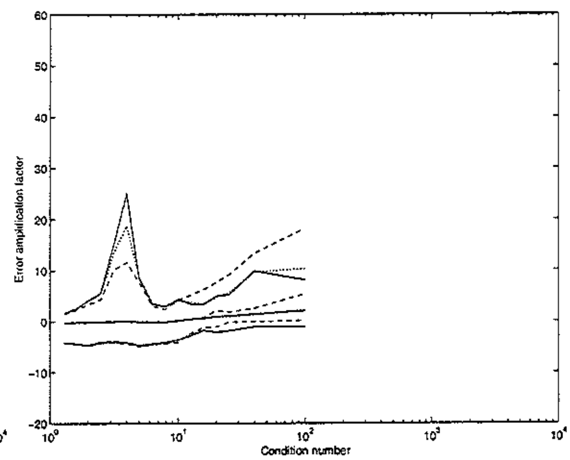
(g) 8×5



(h) 8×7

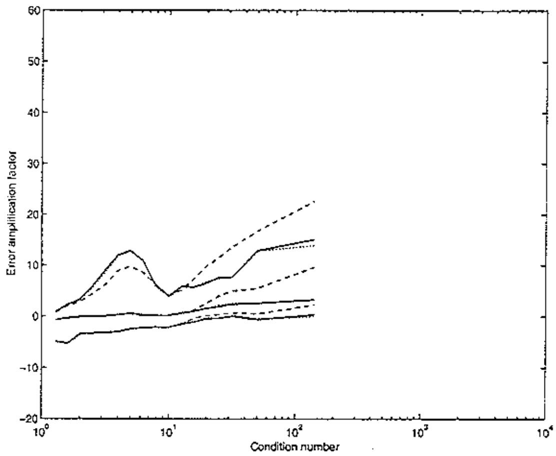


(i) 9×8

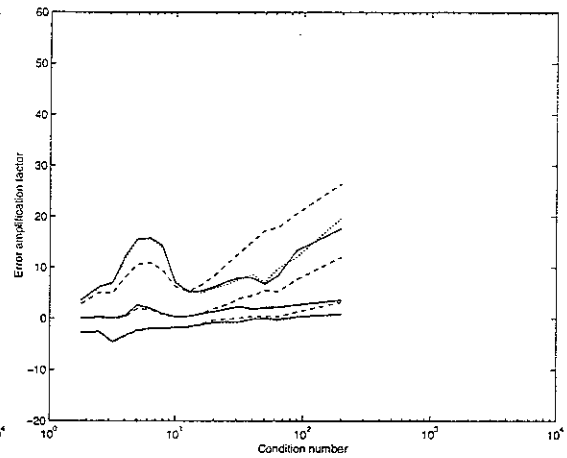


(j) 10×2

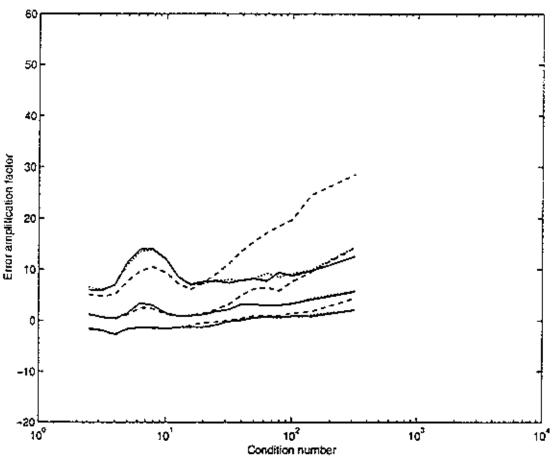
Figure 24 (contd). Comparison of the fitted lines obtained by the pseudo-inverse (– – –), ordinary (——) and generalized cross validation (--------) for a different number of response and force positions.



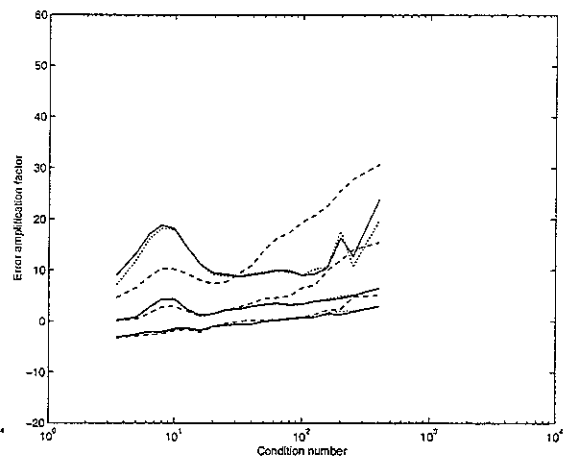
(k) 10×3



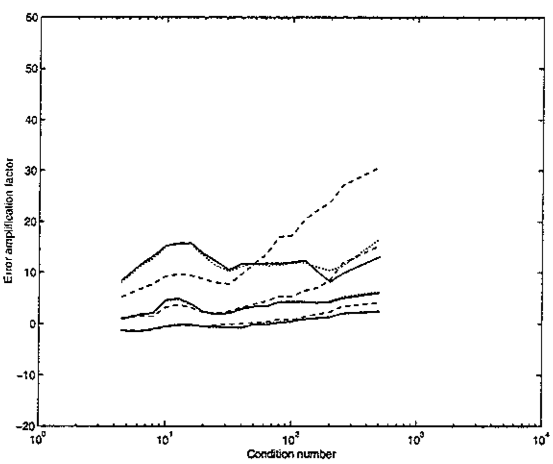
(l) 10×4



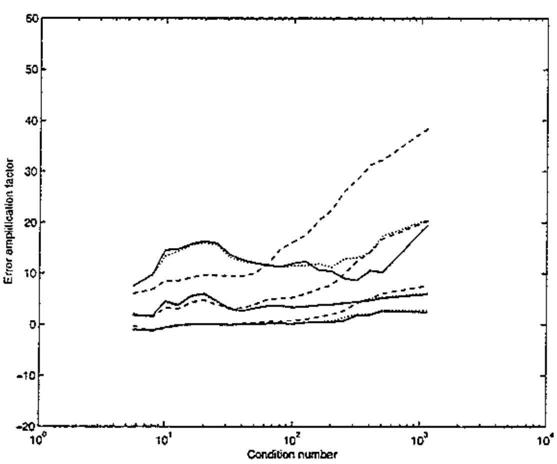
(m) 10×5



(n) 10×6

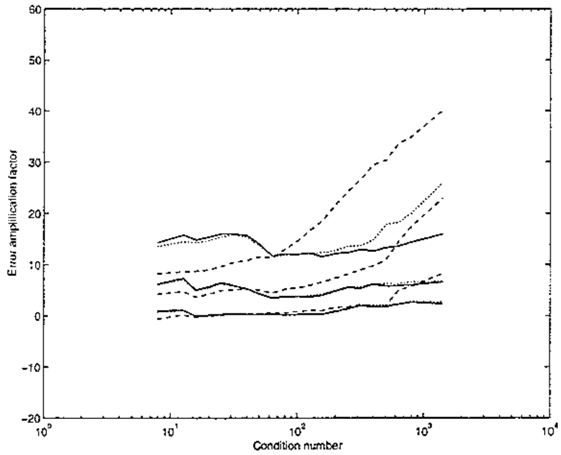


(o) 10×7

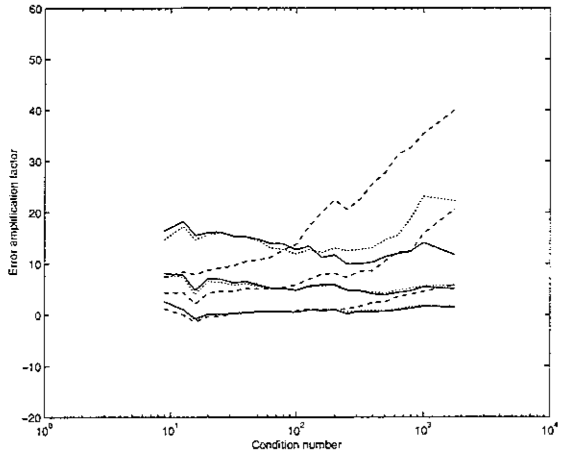


(p) 10×8

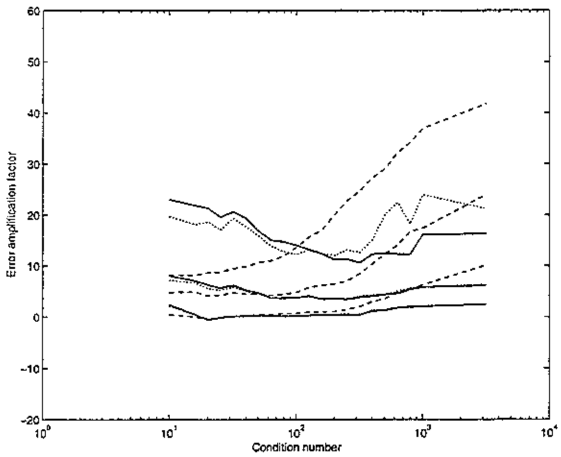
Figure 24 (contd). Comparison of the fitted lines obtained by the pseudo-inverse (– – –), ordinary (——) and generalized cross validation (--------) for a different number of response and force positions.



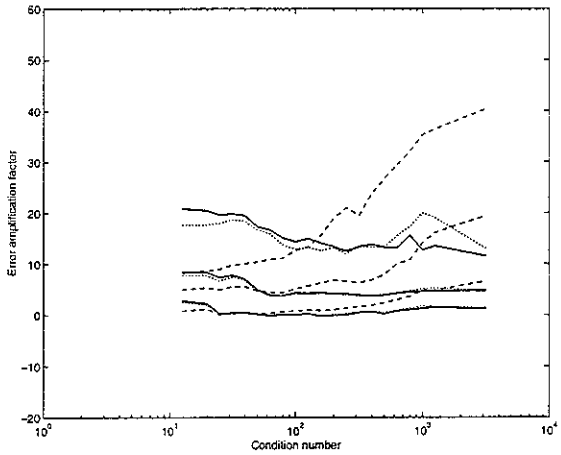
(q) 10×9



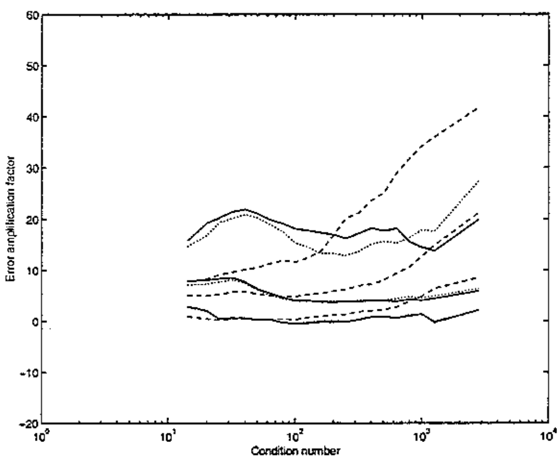
(r) 11×10



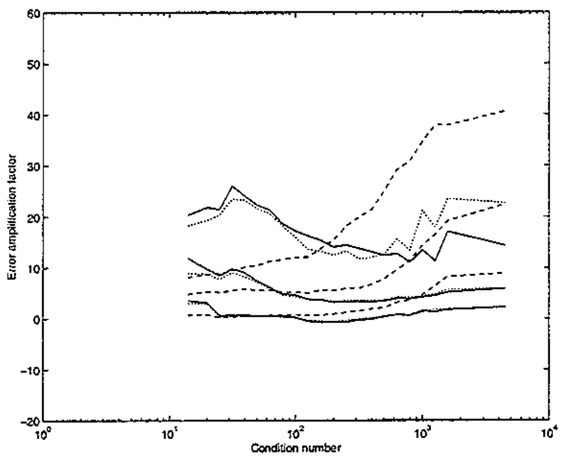
(s) 12×11



(t) 13×12



(u) 14×13

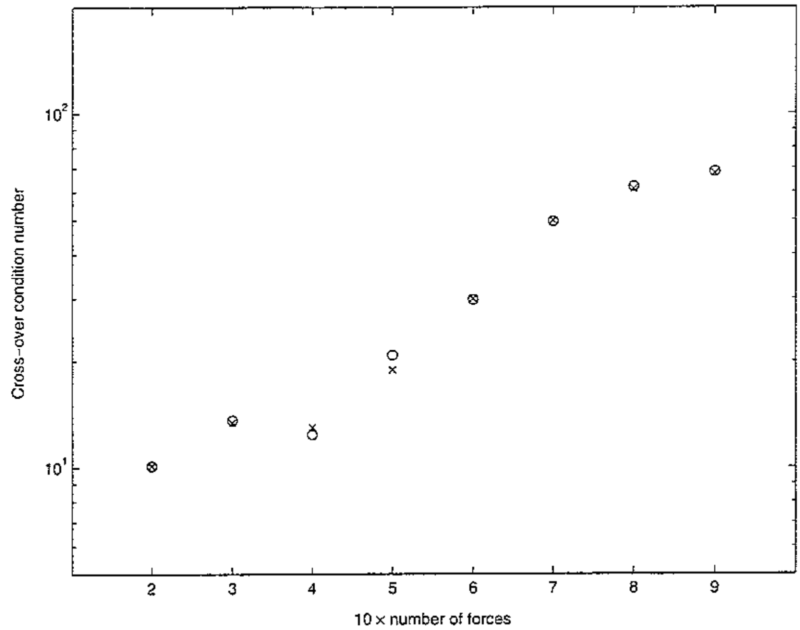


(v) 15×14

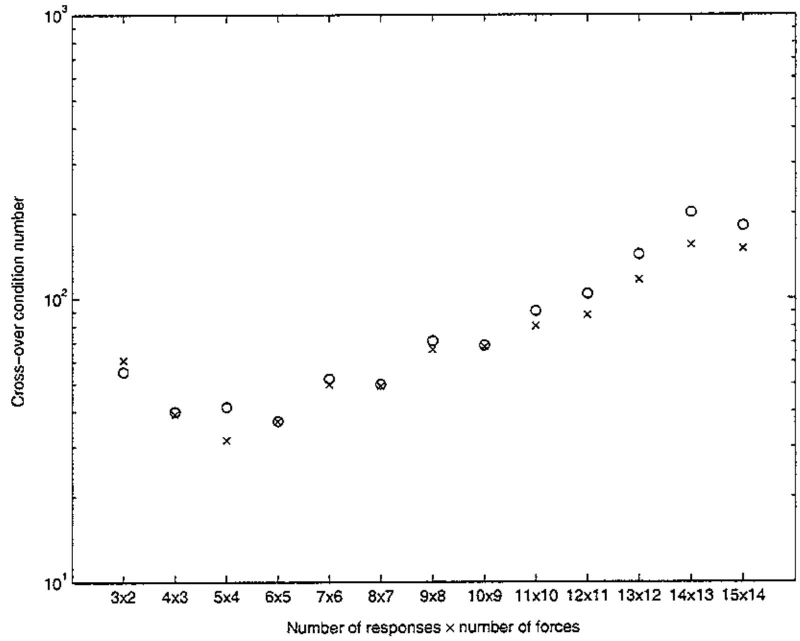
Figure 24 (contd). Comparison of the fitted lines obtained by the pseudo-inverse (– – –), ordinary (——) and generalized cross validation (--------) for a different number of response and force positions.

Table 2. Condition numbers corresponding to the cross-over between the 95 % confidence levels of the error amplification factor with respect to the condition number, obtained by the pseudo-inverse method and Tikhonov regularization. When a condition number at a frequency of a system is greater than or equal to the corresponding cross-over condition number, Tikhonov regularization generally gives better results than the pseudo-inverse method.

Numbers of responses and forces	Cross-over condition number between the pseudo-inversion and	
	the ordinary cross validation	the generalized cross validation
3×2	55.2	60.6
4×3	39.9	39.0
5×3	34.6	34.7
5×4	41.5	31.7
6×5	37.0	36.9
7×6	52.2	49.8
8×5	26.7	22.6
8×7	49.9	49.2
9×8	70.9	66.1
10×2	10.1	10.1
10×3	13.6	13.4
10×4	12.4	13.0
10×5	20.8	18.9
10×6	29.8	30.0
10×7	49.5	49.8
10×8	62.1	61.1
10×9	68.4	67.6
11×10	90.4	80.0
12×11	103.9	87.6
13×12	142.9	116.5
14×13	201.0	154.4
15×14	180.5	149.6



(a) 10 response and n force locations, where n is from 2 to 9



(b) n response and n–1 force locations, where n is from 3 to 15

Figure 25. Condition number corresponding to the cross-over between the 95 % confidence levels of the error amplification factor with respect to the condition numbers obtained by the pseudo-inverse method and Tikhonov regularization based on the ordinary (○) and generalized (×) cross validation method.

2.3. Summary

In this section, the error amplification has been used to distinguish the use of Tikhonov regularization from the pseudo-inverse method. The error amplification factor is defined as the difference between the force error and the operational response error in dB. In other words, the error amplification factor is a value indicating how much larger the force error is than the operational response error. When the error amplification factor is large, this means that the performance of the method used to reconstruct the operational forces is not so good.

The error amplification factors are estimated for various sets of response and force positions, i.e., $n \times (n-1)$ systems and $10 \times n$ systems. To investigate the relationship between the error amplification factor and the condition number, the 5 %, 50 % and 95 % confidence levels have been determined. As can be seen in the results, the error amplification factor is related not only with the condition number but also with the system matrix size. The cross-over condition numbers are determined from the 95 % confidence levels obtained by the pseudo-inverse method and the ordinary cross validation. These cross-over condition numbers are very useful to select the method to reconstruct the operational forces. Tikhonov regularization should only be used where the condition number is greater than the threshold (cross-over condition number). This threshold varies from 10 for a 10×2 matrix to around 200 for a 15×14 matrix. Consequently the fact that the selection of the Tikhonov regularization is related not only to the condition number but also to the system matrix size is also seen to be of importance.

3. SENSOR LOCATION SELECTION BASED ON COMPOSITE CONDITION NUMBER

3.1. Introduction

As seen in the previous section, the condition number can be used to estimate the sensitivity of the solution to the matrix inversion. The sensor locations used can be selected based on the conditioning of the measured frequency response function matrix from force locations to selected sensor locations [1]. Thus to use this method it is important to estimate the condition numbers at each frequency, and their average across the frequency range of interest, in order to select a suitable set of sensor locations. However, the estimation of these average condition numbers involves many calculations at each frequency to cover the various combinations of locations. When five locations are to be selected from 20 locations, for example, it is required to estimate the condition numbers of $5 \times n$ matrices (where n is the number of force locations) of FRF's corresponding to 15504 combinations of sensor locations at each frequency. To cope with this, an approximate method based on the composite condition number (or so-called amplification factor) [1] is used in this study and the relation between the condition number and the composite condition number is investigated. Also, one-third octave band averaging is introduced when calculating the average of the condition numbers and the results are compared with each other.

3.2. Composite condition number

The condition number of the $m \times n$ matrix is approximated based on the condition numbers of $2 \times n$ FRF sub-matrices formed by using just two response locations from those available and all n force locations. In other words, in order to estimate approximately the condition number of FRF matrices for a set (e.g. $5\times n$) of the locations selected, the condition numbers for all the combinations of the pairs of the locations selected are calculated and then combined together. By comparing these quantities (called the composite condition number) for all possible sets, the set giving the smallest value is chosen as the 'best' combination of locations. The detailed procedure is as follows [1]:

1) Select m_{all} response locations that are candidates for measuring the operational response.
2) Construct $2\times n$ matrix of FRF's for one set of 2 response locations (i, j) at each frequency and calculate the condition numbers of this matrix (where n is the number of force positions).
3) Average the condition numbers over the frequency range interested and write this as κ_{ij} with i, j corresponding to the two response locations considered.
4) From these κ_{ij}, construct the $m_{all}\times m_{all}$ matrix $[\kappa]$, the diagonal elements of which being set to zero.
5) Decide the number of responses m to be used.
6) Determine all possible combinations of m responses taken from m_{all} i.e. $m_{all}!/m!(m_{all}-m)!$ combinations, where ! is a factorial.
7) Select a combination, for example, 1, 2, ..., m, and find all possible pairs of picking 2 unordered from m elements of this combination, e.g. {1,2}, {1,3}, {2,3}, etc. The sum of all κ_{ij} corresponding these all pairs is called the composite condition number X in relation of the combination selected. The composite condition number used here is slightly different from the amplification factor defined by Thite [1].

$$X = \sum_{i,j} \kappa_{ij} \tag{4}$$

8) Construct a vector of all composite condition numbers from step 7 for all combinations identified in step 6.
9) The combination giving the minimum composite condition number is taken as the optimal combination of locations.

3.3. Condition number averaged in one-third octave band

As described in the previous section (as in [1]), the average of the condition numbers over the frequency range is the arithmetic mean of the condition numbers at the frequencies of calculation, usually linearly spaced over some range. Also the condition numbers κ_{ij} used to estimate the composite condition number are the arithmetic means of the condition numbers for FRF matrix corresponding to *i*th and *j*th response locations. However, since the condition numbers at low frequencies (low modal overlap) are found to be larger than at high frequencies, consideration is given to the use of different averaging with unequal weighting over frequencies. Thus, the one-third octave band conversion is considered in this study.

First, the average of the condition numbers is calculated in each one-third octave band. And then the average over all one-third octave bands is formed.

To evaluate this average, obtained by using the one-third octave band conversion, and to compare the composite condition number with this average value, some simulations are carried out.

3.4. Simulation objects

An analytical model of a simply supported rectangular plate with dimensions of 600 *mm* × 500 *mm* × 1.5 *mm* is used in order to estimate the composite condition number and the average of the condition numbers. Its material is steel, i.e. Young's modulus of 2.07×10^{11} N/m^2, Poisson's ratio of 0.3 and density of 7850 kg/m^3.

To introduce variations in the condition numbers, their averages and the composite condition number, three distributions of force and response locations and three values of the damping loss factor are used. The numbers of force and response locations used are 4 and 20, respectively. The numbers of response locations to be selected out of 20 locations are 4, 5 and 6. Therefore the numbers of possible combinations are ${}_{20}C_4 = 4845$, ${}_{20}C_5 = 15504$ and ${}_{20}C_6 = 38760$, respectively.

The first distribution of force and response locations is selected at random in the whole area of the plate, the second in the region covering the middle 80 % of the length and width and the third in the central 30 % of the length and width, as shown in Figure 26.

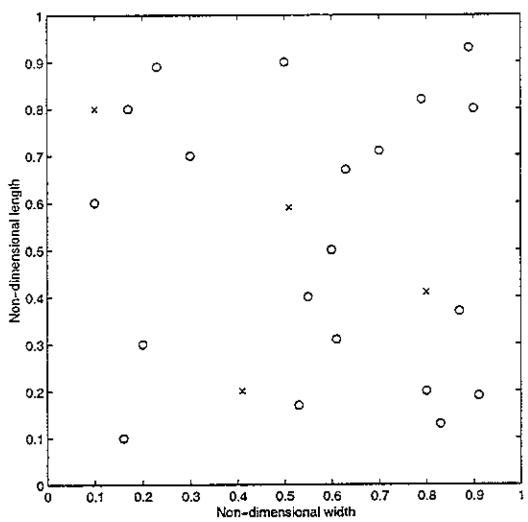
To investigate the effect of the damping loss factor on the relationship between the composite condition number and the average condition number, three values of the damping loss factor are used. The initial damping loss factor is 0.03, a smaller one is 0.01 and a larger one is 0.10.

3.5. Comparison of the composite condition number and the average of the condition numbers

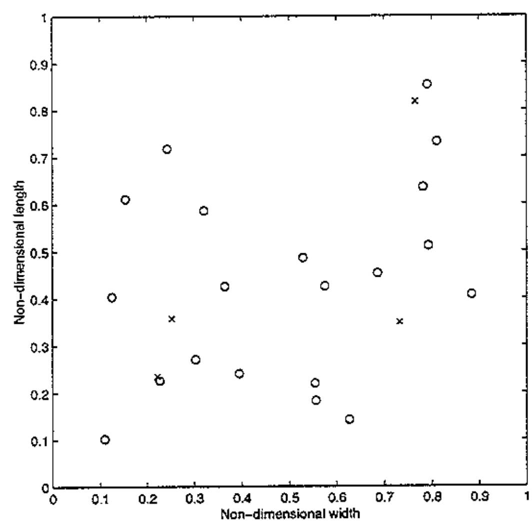
To estimate the composite condition number and the average of the condition numbers, two kinds of averaging method are used, as mentioned earlier, that is, the first one is the arithmetic mean, and the second one is the average of all mean values in each one-third octave band. Thus, two types of composite condition number and two types of average condition number are produced corresponding to the two methods.

The numbers of the force and response locations selected for analysis are

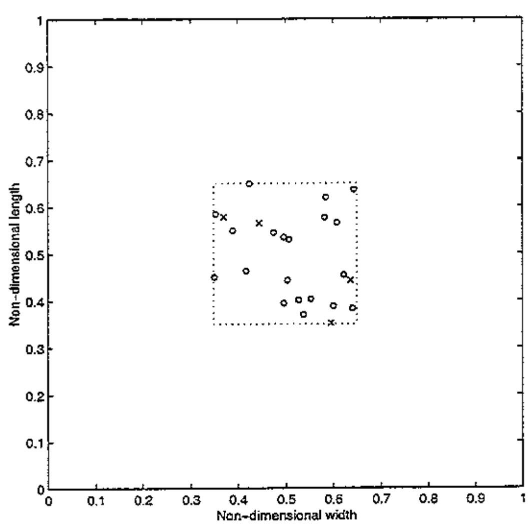
1) four response and four force locations (4×4),
2) five response and four force locations (5×4), and
3) six response and four force locations (6×4).



(a) 1st distribution



(b) 2nd distribution



(c) 3rd distribution

Figure 26. Force and response positions. ×: force location, ○: response location.

The frequency ranges of analysis are

1) 10 – 500 Hz, and
2) 10 – 3600 Hz.

The results obtained by using the condition numbers are

1) the average condition number using the arithmetic mean,
2) the average condition number using 1/3 octave band conversion,
3) the composite condition number using the arithmetic mean, and
4) the composite condition number using 1/3 octave band conversion.

And, the comparisons of these values are as follows:

1) the average condition number and the composite condition number over 10 – 3600 Hz,
2) the average condition number and the composite condition number over 10 – 500 Hz,
3) the average condition number and the composite condition number after 1/3 octave conversion over 10 – 3600 Hz, and
4) the average condition number and the composite condition number after 1/3 octave conversion over 10 – 500 Hz.

Figures 27 to 29 show the correspondences between the composite condition number and the average condition number for the first distribution of the force and response positions (see Figure 26 (a)). Figures 27, 28 and 29 are the results corresponding to 4, 5 and 6 response locations, respectively. In each case figures (a) and (b) are plotted using the arithmetic mean, and figures (c) and (d) are obtained by using the average after averaging in each 1/3 octave band. Figures (a) and (c) show the average over the frequency range of 10 – 3600 Hz, and figures (b) and (d) over 10 – 500 Hz.

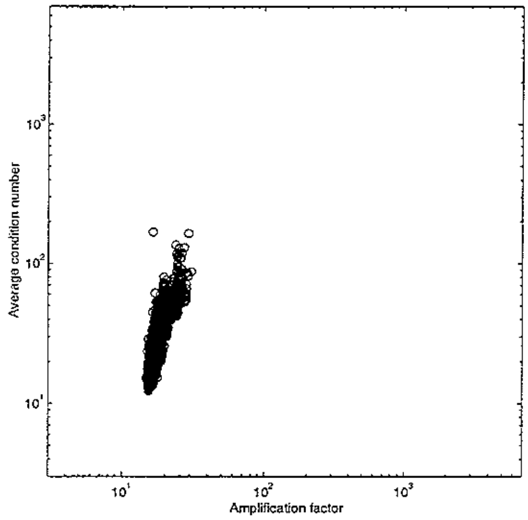
As shown in these figures, the minimum of the composite condition numbers does not always correspond to the minimum of the average condition numbers. However, it can be seen that, in each case, as the composite condition number become larger, the average condition number increases. Also, the point corresponding to the minimum of the composite condition number is close the point corresponding to the minimum of the average condition number and these two points are in the vertex close to the origin but they are not exactly the same point. Therefore it is necessary to consider how close the two minimum points are and how the average condition number varies as the composite condition number increases. These are considered in the next section.

Comparing the averaging methods, the mean using 1/3 octave bands is more appropriate than the arithmetic mean, because the mean using 1/3 octave bands can allow for the fact that the results are more sensitive to the condition numbers at low frequencies than those at high frequencies, as mentioned earlier. Moreover, in the plots obtained using the mean after averaging in 1/3 octave bands it is easier to compare the average condition number with the composite condition number because the plots have a greater slope suggesting a better correspondence between composite condition number and average condition number.

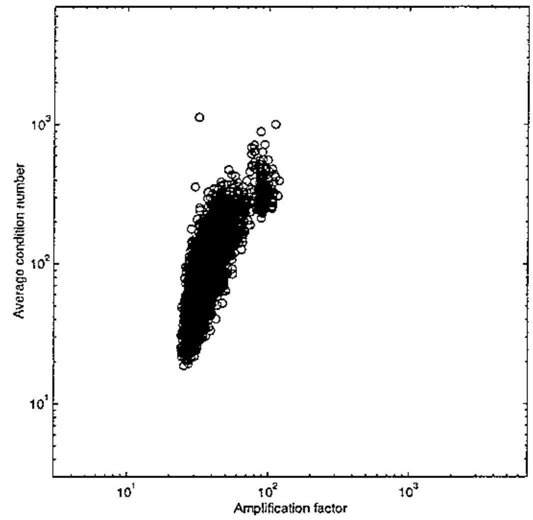
In the case of choosing the frequency range, the frequency range of 10 – 500 Hz is more appropriate than that of 10 – 3600 Hz because the condition numbers are dominant in the low frequency range. However, when using 1/3 octave band averaging the results are less sensitive to the frequency range.

Comparing Figures 27, 28 and 29, i.e., using 4, 5 or 6 responses, as the number of responses increases the areas become more rectangular and also are shifted to the right (in the direction of increasing the composite condition number) and downward (in the direction of decreasing the average condition number). These results are different from those using the amplification factor proposed by Thite [1]. As the over-determination increases, the minimum of the average condition numbers reduces a little and also the range of the average condition numbers becomes smaller. In the case of the composite condition number, the increase in the level of over-determination means an increase in the number of pairs of response locations and the composite condition number also increases.

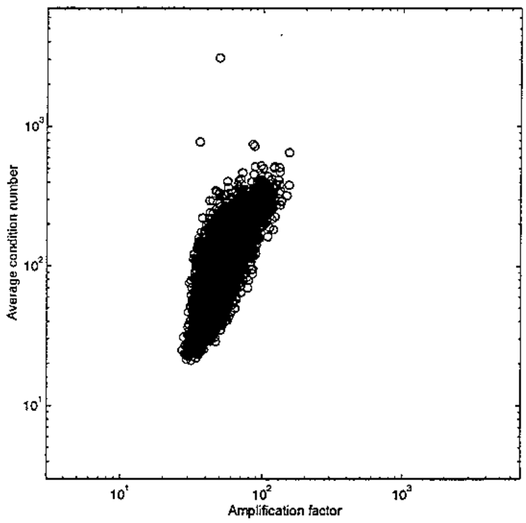
Next the correspondences between the composite condition number and the average condition number for the 2nd and the 3rd distribution and other damping loss factors are considered.



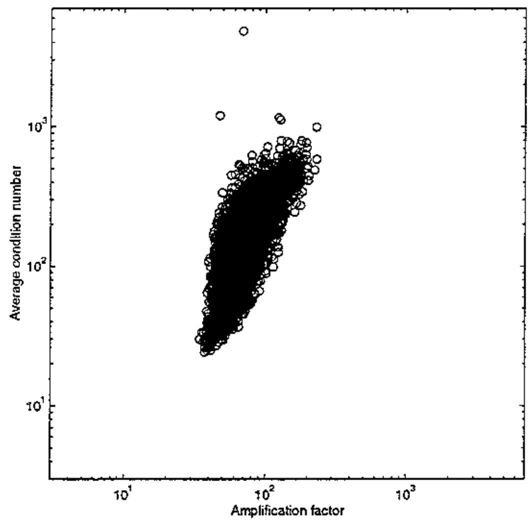
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

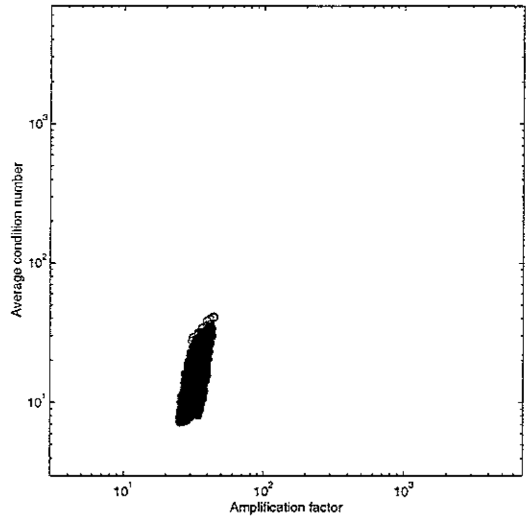


(c) 1/3 octave; 10 – 3600 Hz

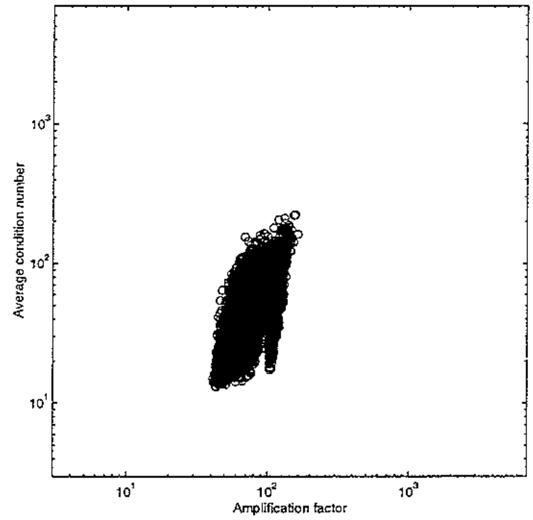


(d) 1/3 octave; 10 – 500 Hz

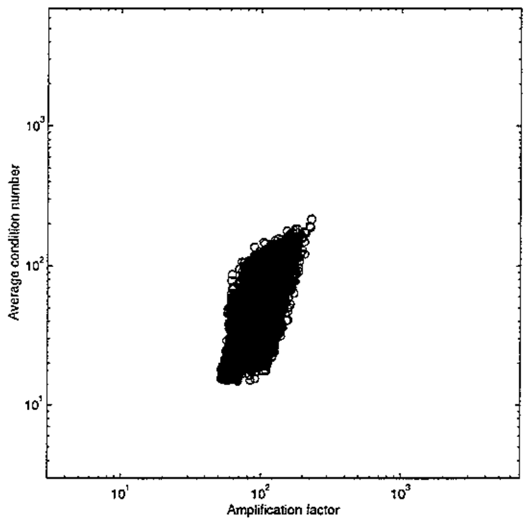
Figure 27. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 4 responses chosen from 20 responses of the first distribution and the initial damping loss factor.



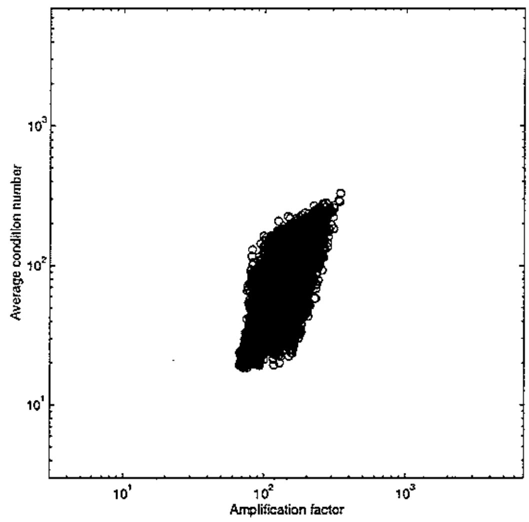
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

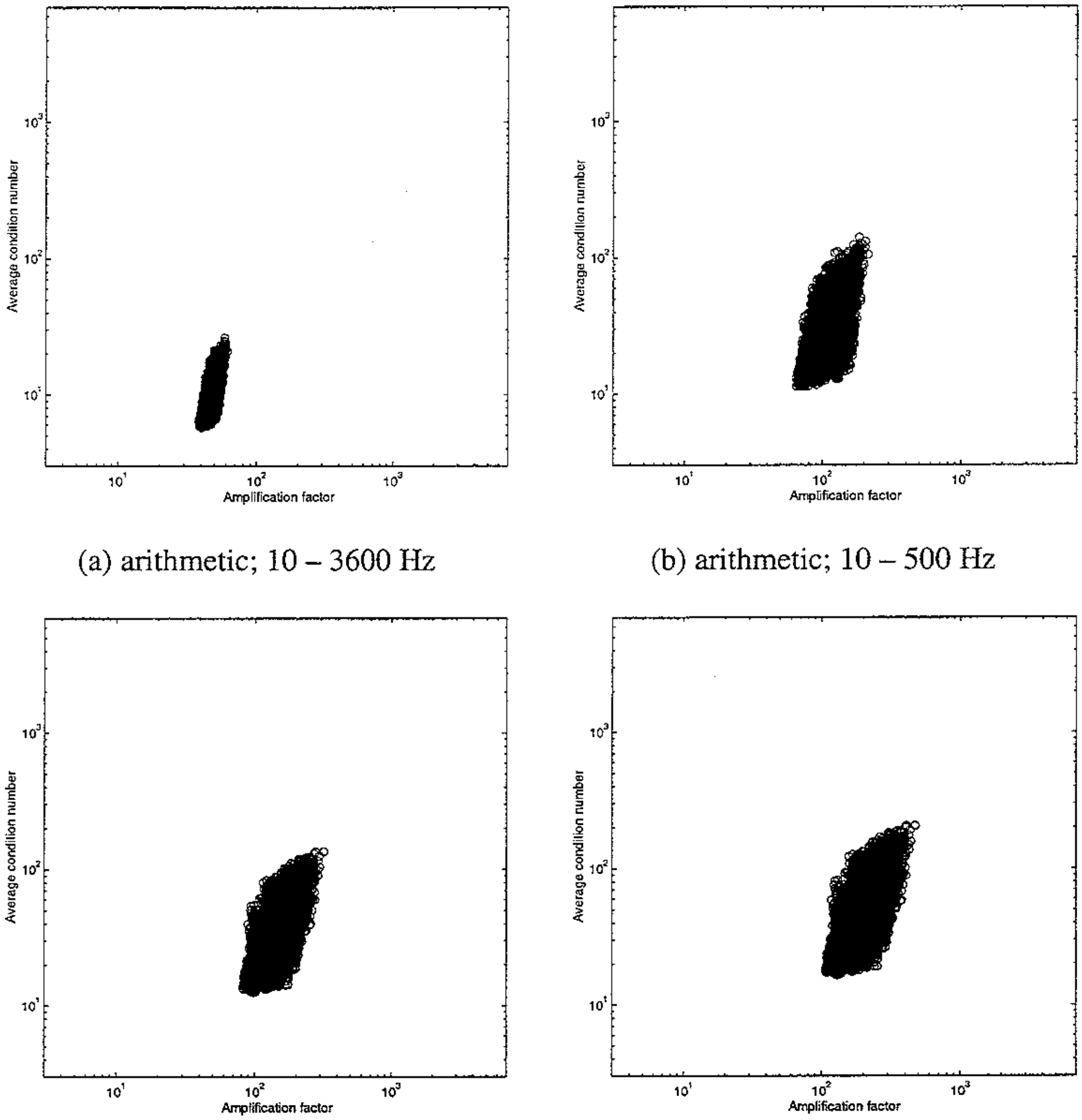


(c) 1/3 octave; 10 – 3600 Hz



(d) 1/3 octave; 10 – 500 Hz

Figure 28. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 5 responses chosen from 20 responses of the first distribution and the initial damping loss factor.



(a) arithmetic; 10 – 3600 Hz

(b) arithmetic; 10 – 500 Hz

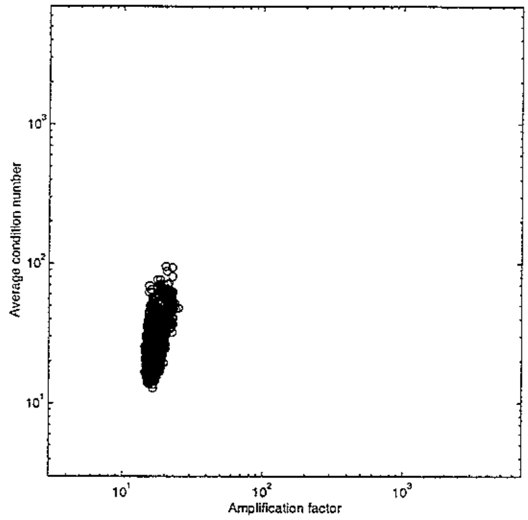
(c) 1/3 octave; 10 – 3600 Hz

(d) 1/3 octave; 10 – 500 Hz

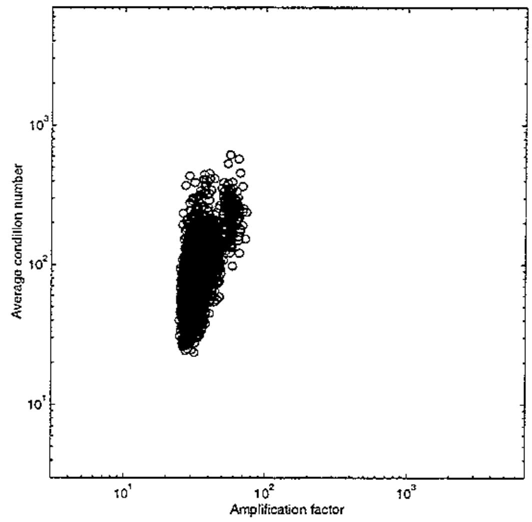
Figure 29. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 6 responses chosen from 20 responses of the first distribution and the initial damping loss factor.

For the second distribution of force and response positions (see Figure 26 (b)), the correspondence between the composite condition number and the average condition number is shown in Figures 30 to 32. The pattern of the areas plotted in these figures is similar to that in Figures 27 to 29. This is because the distribution of force and response positions is also similar to the first distribution. These positions are selected at random, except for the difference in the area within which the positions can be generated. As mentioned earlier, the first distribution has a possibility for points to lie anywhere on the whole area of the plate, but the second distribution is limited to a range of 80 % of the length and 80 % of the width. Consequently, the effect of the possible area on the condition number is small.

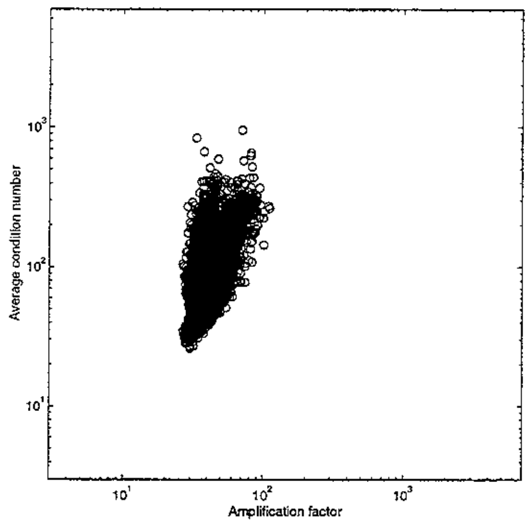
However, the other aspects of the results, the averaging method and the frequency range of analysis, give similar results to the case of the first distribution.



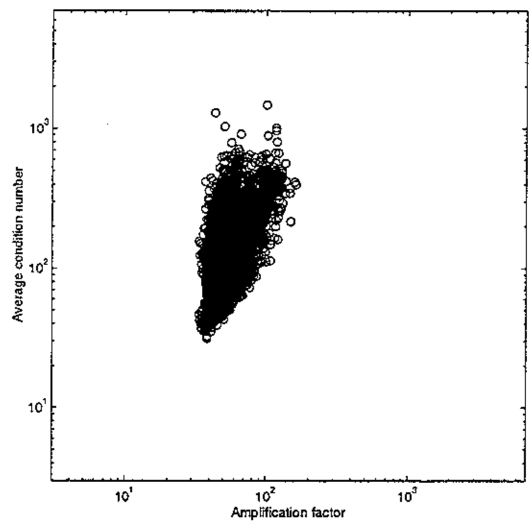
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

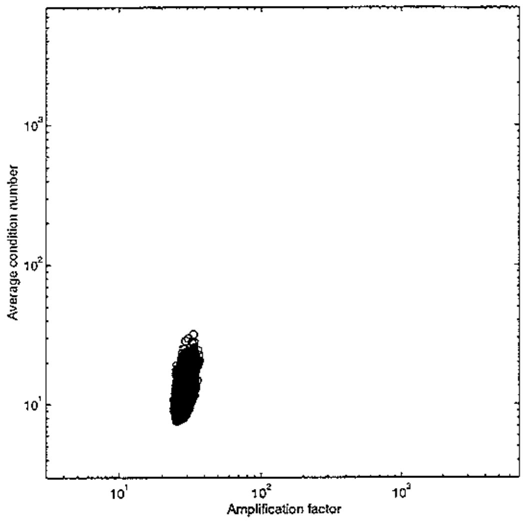


(c) 1/3 octave; 10 – 3600 Hz

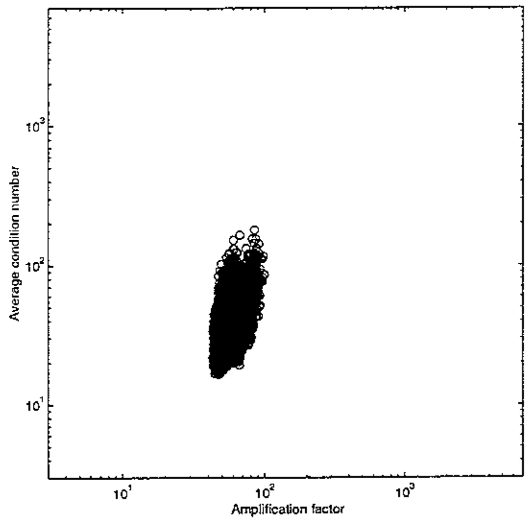


(d) 1/3 octave; 10 – 500 Hz

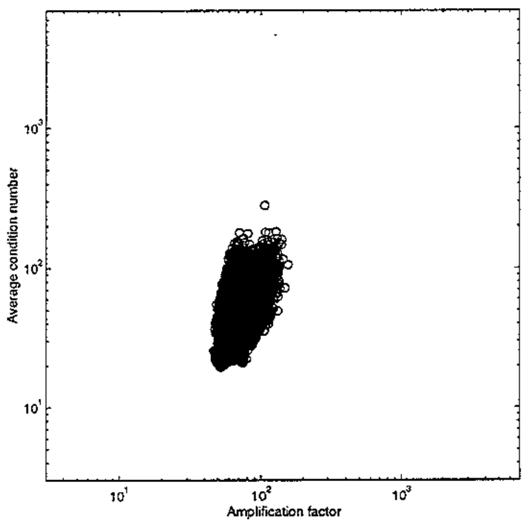
Figure 30. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 4 responses chosen from 20 responses of the second distribution and the initial damping loss factor.



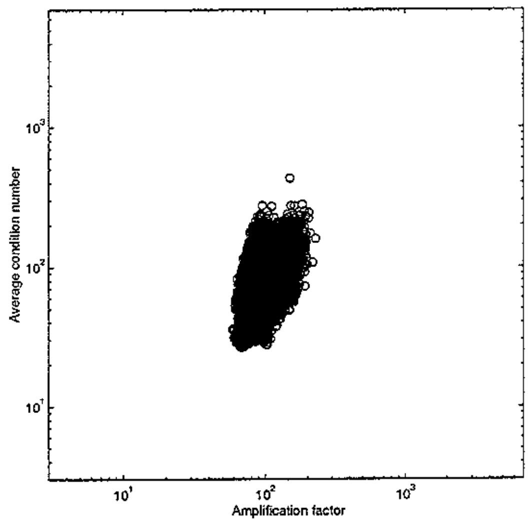
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

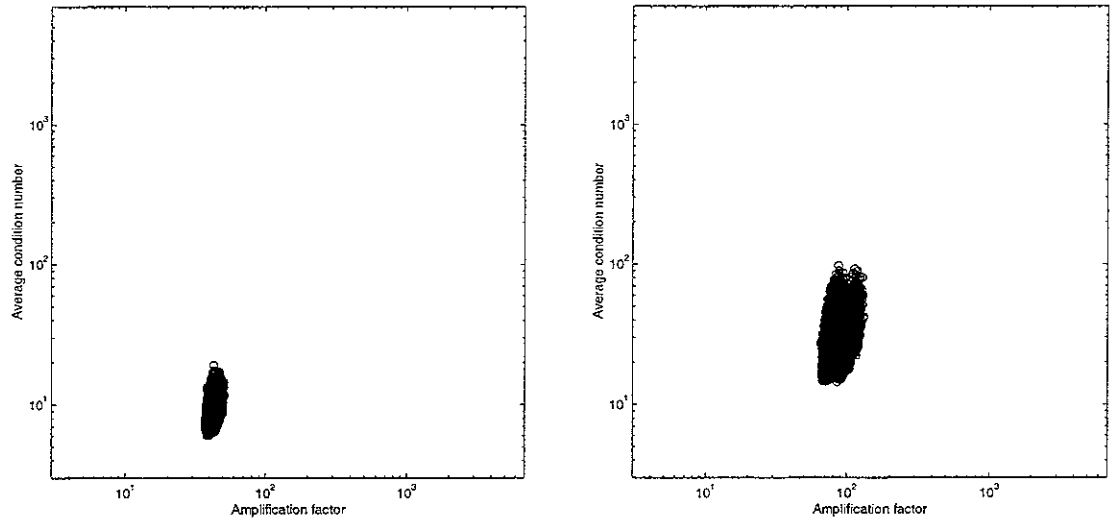


(c) 1/3 octave; 10 – 3600 Hz

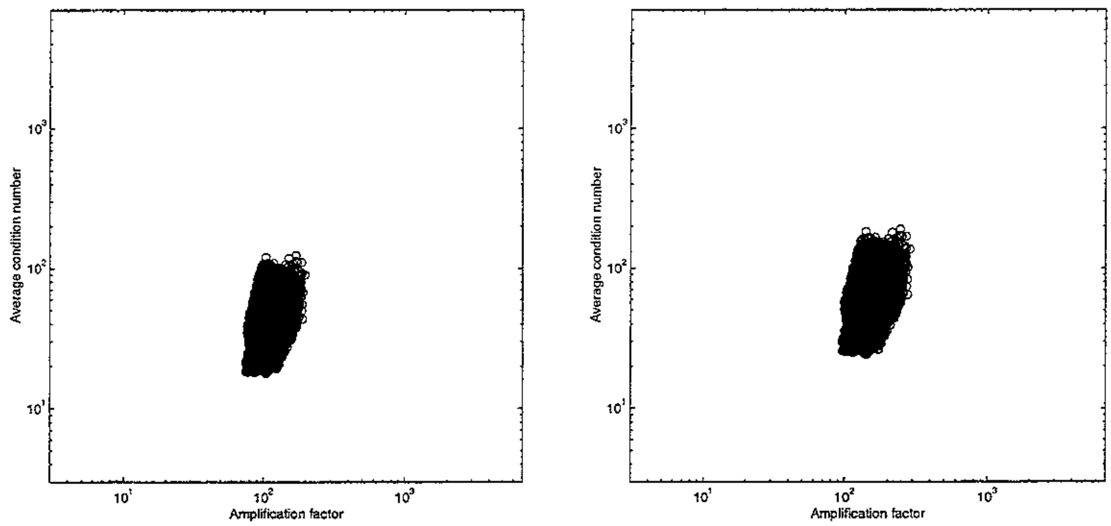


(d) 1/3 octave; 10 – 500 Hz

Figure 31. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 5 responses chosen from 20 responses of the second distribution and the initial damping loss factor.



(a) arithmetic; 10 – 3600 Hz (b) arithmetic; 10 – 500 Hz



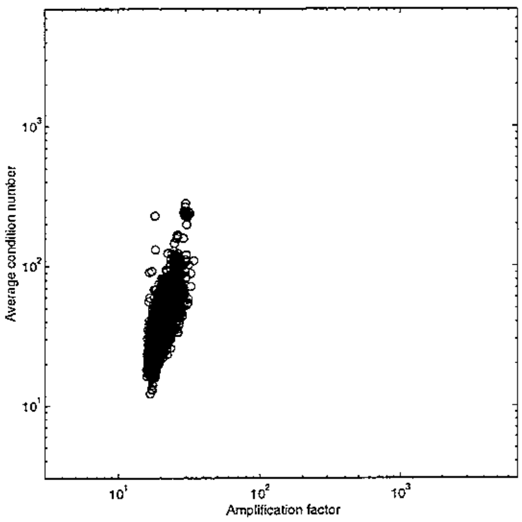
(c) 1/3 octave; 10 – 3600 Hz (d) 1/3 octave; 10 – 500 Hz

Figure 32. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 6 responses chosen from 20 responses of the second distribution and the initial damping loss factor.

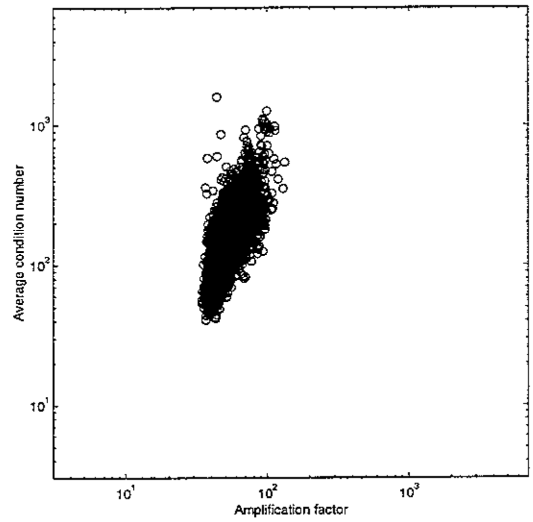
Figures 33 to 35 gives the results of the correspondences between the composite condition number and the average condition number for the case of using the third distribution of the force and response locations, as shown in Figure 26 (c). This distribution is concentrated in the centre of the plate.

Comparing the results obtained from using this distribution with those using the first distribution, the minimum of the average condition number increases except the one obtained by using the arithmetic mean over the frequency range of 10 – 3600 Hz. However, the difference between the minimum and the maximum of the average condition numbers decreases so that the area also becomes smaller. These changes are more conspicuous when using the frequency range of 10 – 500 Hz and using 1/3 octave band calculation. Since the force and response locations are close, there is less difference between them.

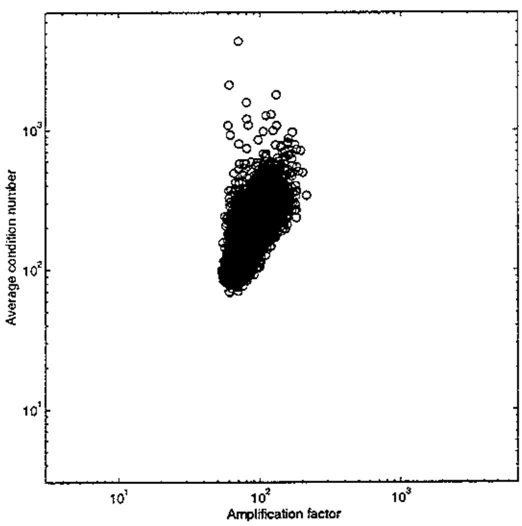
Next, the effect of the damping loss factor is considered on the relationship between the composite condition number and the average condition number.



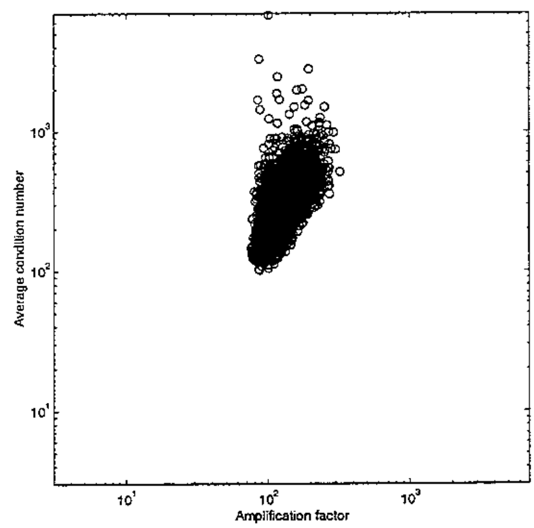
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

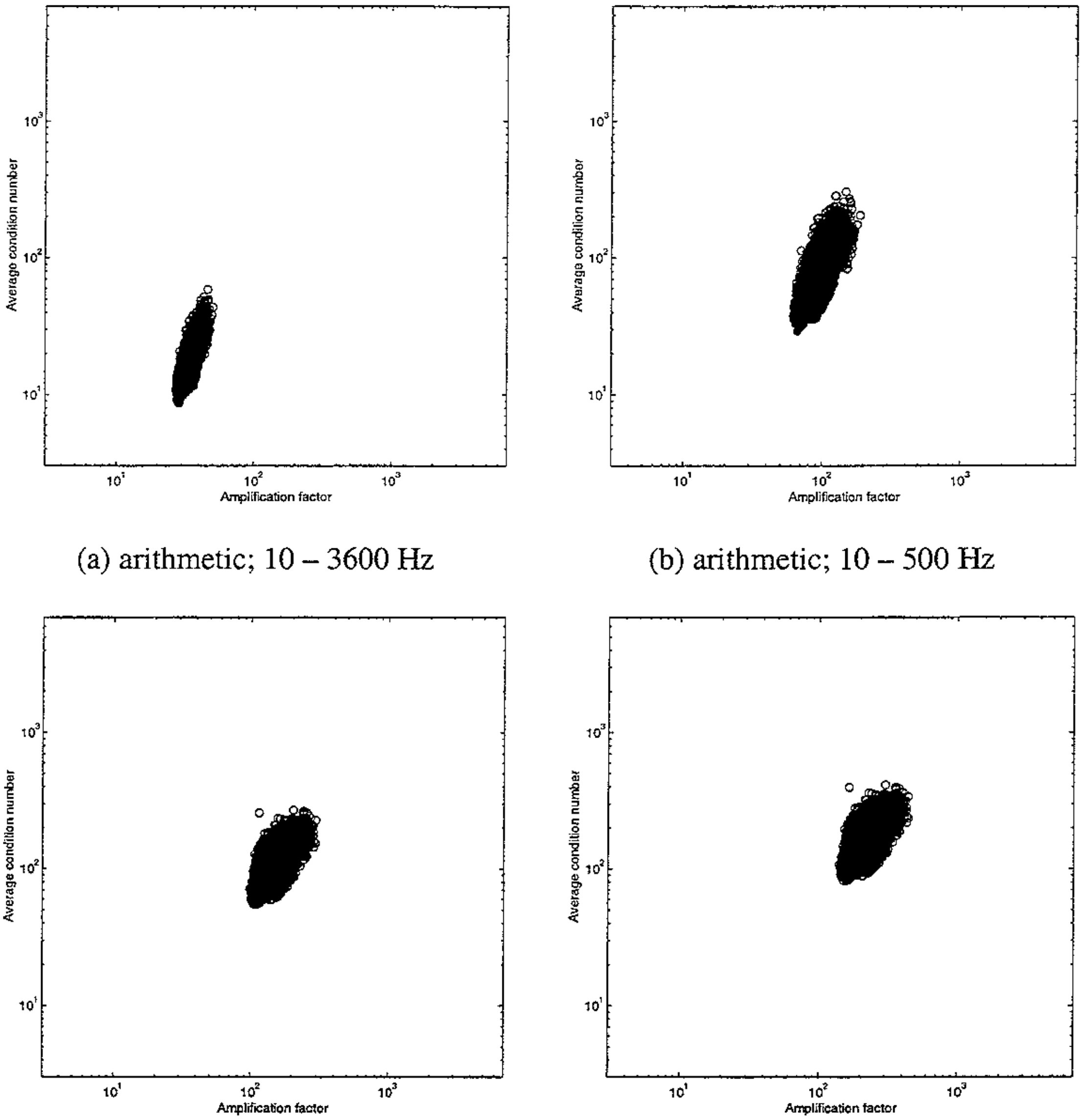


(c) 1/3 octave; 10 – 3600 Hz



(d) 1/3 octave; 10 – 500 Hz

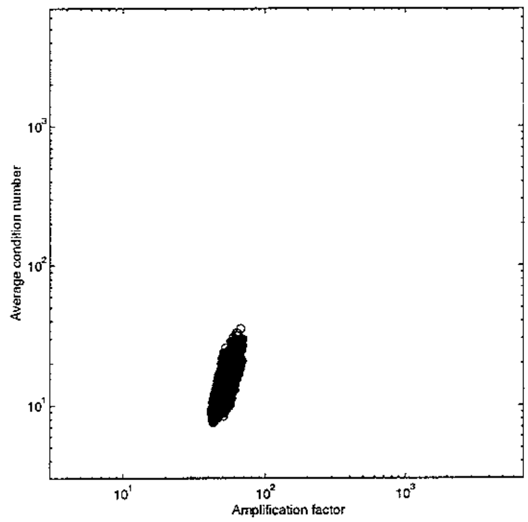
Figure 33. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 4 responses chosen from 20 responses of the third distribution and the initial damping loss factor.



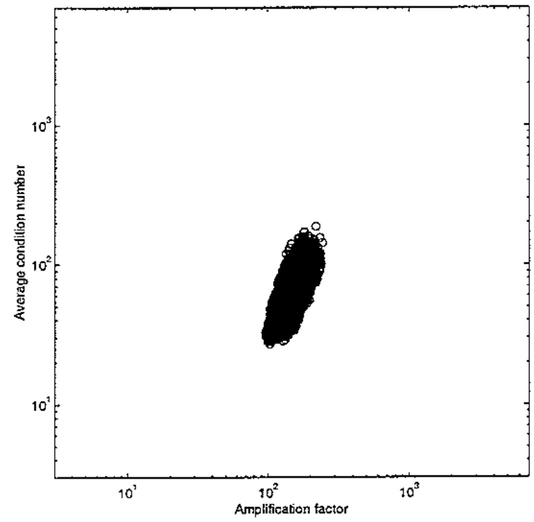
(a) arithmetic; 10 – 3600 Hz (b) arithmetic; 10 – 500 Hz

(c) 1/3 octave; 10 – 3600 Hz (d) 1/3 octave; 10 – 500 Hz

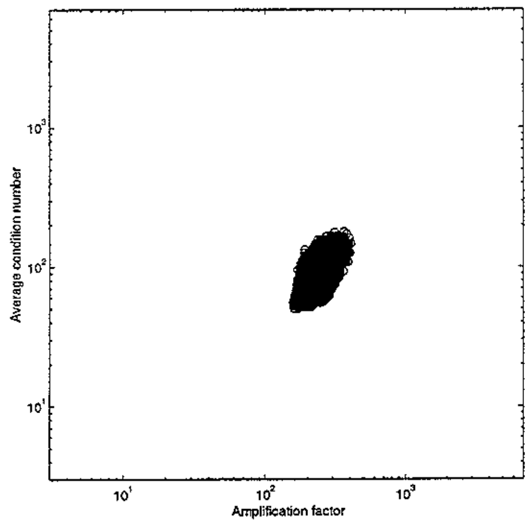
Figure 34. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 5 responses chosen from 20 responses of the third distribution and the initial damping loss factor.



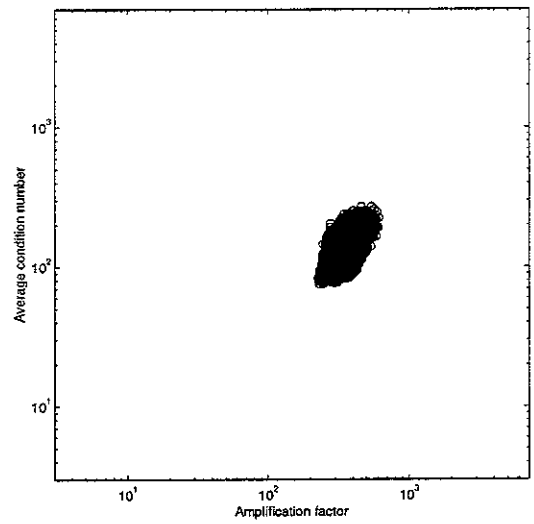
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz



(c) 1/3 octave; 10 – 3600 Hz

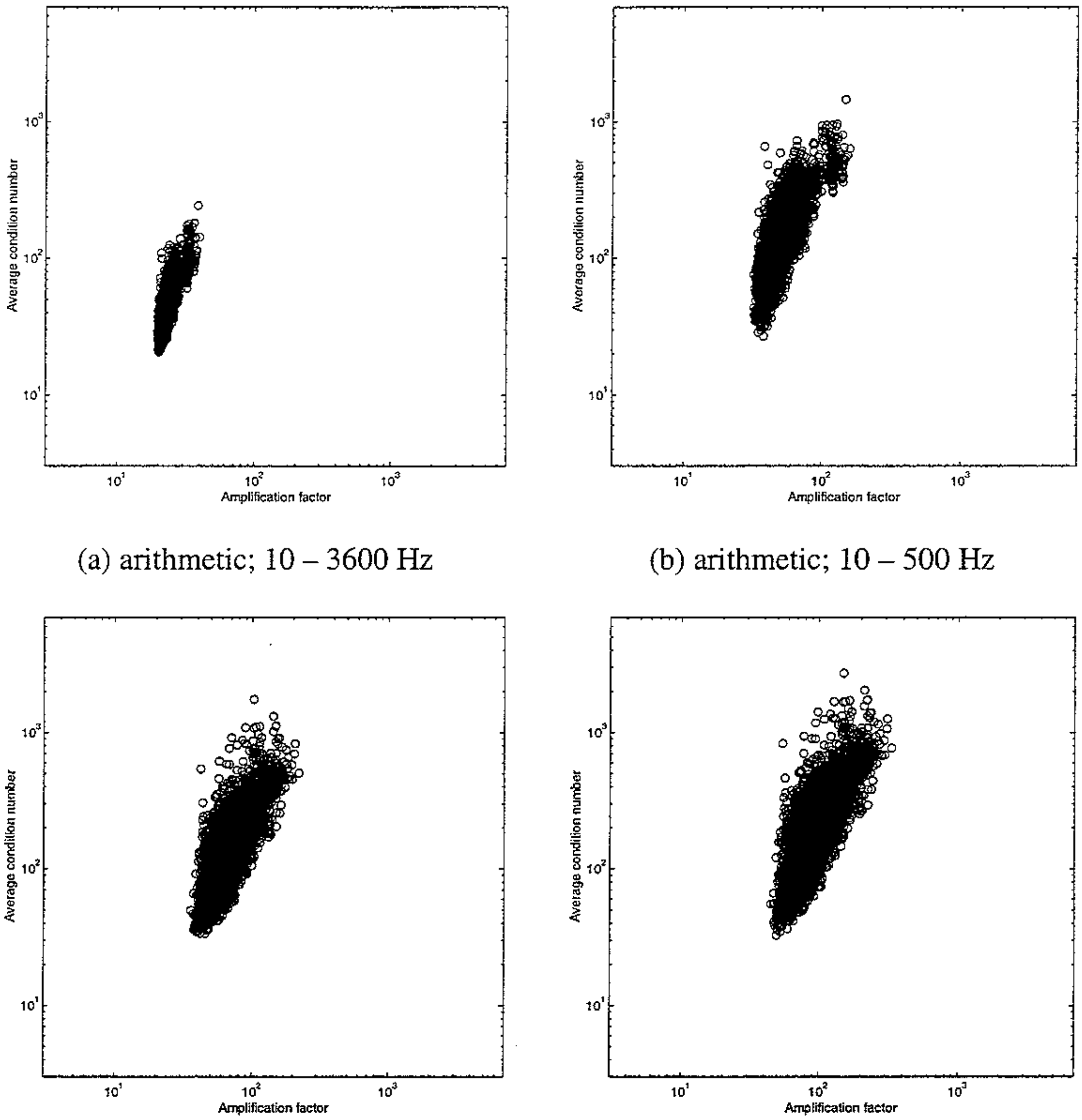


(d) 1/3 octave; 10 – 500 Hz

Figure 35. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 6 responses chosen from 20 responses of the third distribution and the initial damping loss factor.

Figures 36 to 38 show the results for the case of a smaller damping loss factor of 0.01 and Figures 39 to 41 gives the results for a larger damping loss factor of 0.1. The initial damping loss factor is 0.03 and the corresponding results using this value are plotted in Figures 27 to 29. In each case, the effect of over-determination, the averaging method, and the frequency range of analysis are also investigated.

Comparing the results for the damping loss factors of 0.01, 0.03 and 0.1, the areas made by the points, indicating the relationship between the composite condition number and the average condition number, have very similar shapes. The difference between the results is only that the areas are shifted relative to the origin. In other words, as the damping loss factor increases the average condition number and the composite condition number both decrease, but the shapes are unchanged in logarithmic scale. Therefore, it can be said that the damping loss factor is not important for choosing the optimal sensor locations in this dynamic problem. However, if the test object is more complex, it would be necessary to investigate the effect of the damping loss factor on the relationship between the composite condition number and the average condition number.



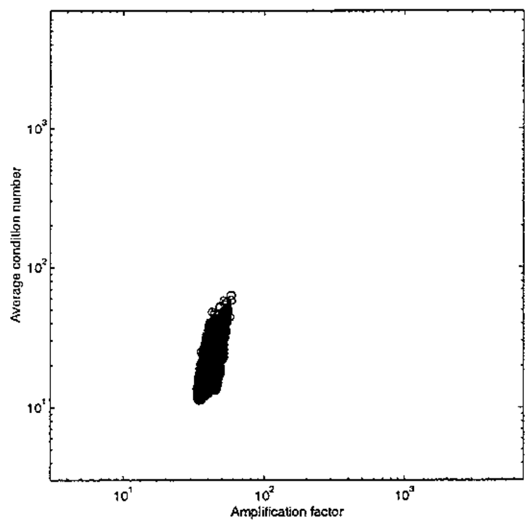
(a) arithmetic; 10 – 3600 Hz

(b) arithmetic; 10 – 500 Hz

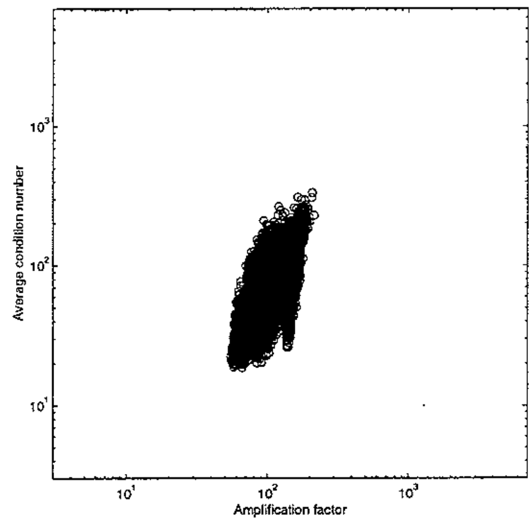
(c) 1/3 octave; 10 – 3600 Hz

(d) 1/3 octave; 10 – 500 Hz

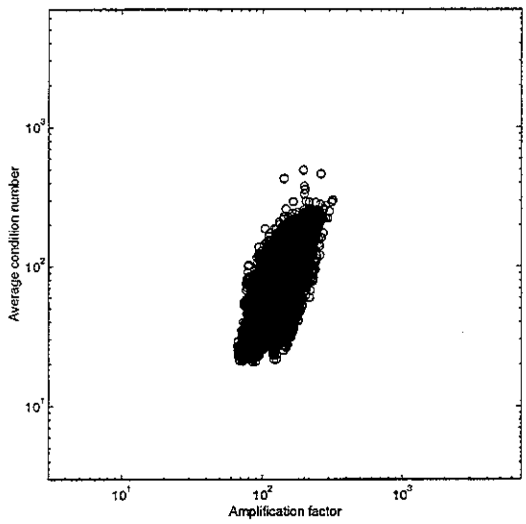
Figure 36. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 4 responses chosen from 20 responses of the first distribution and a smaller damping loss factor.



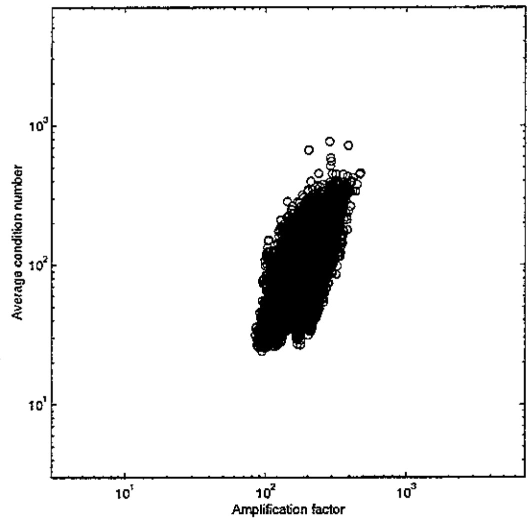
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

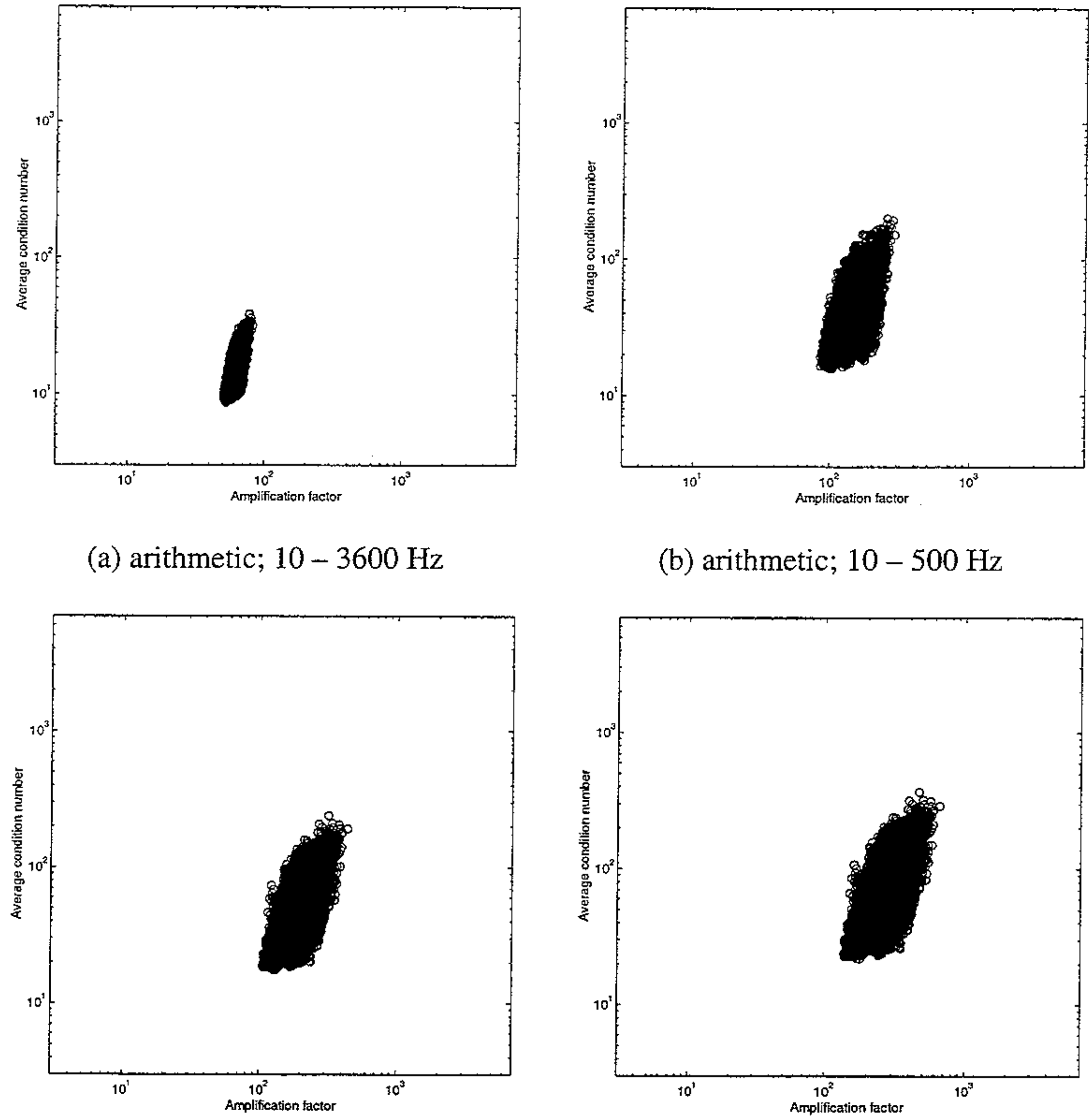


(c) 1/3 octave; 10 – 3600 Hz



(d) 1/3 octave; 10 – 500 Hz

Figure 37. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 5 responses chosen from 20 responses of the first distribution and a smaller damping loss factor.



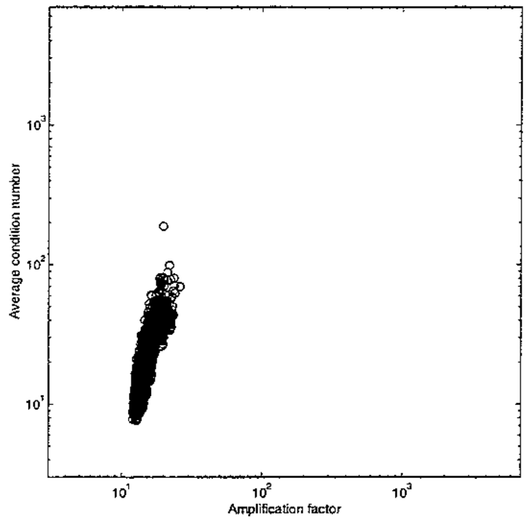
(a) arithmetic; 10 – 3600 Hz

(b) arithmetic; 10 – 500 Hz

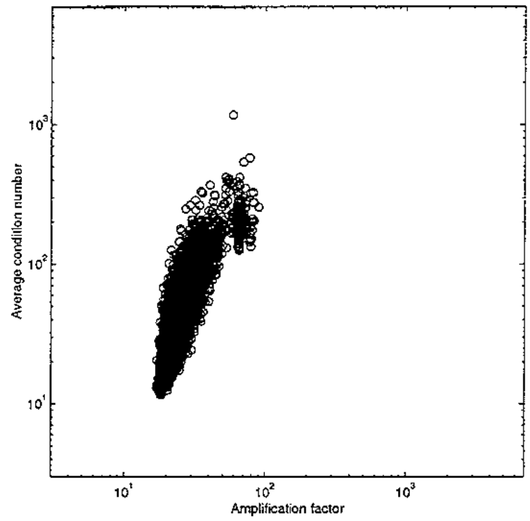
(c) 1/3 octave; 10 – 3600 Hz

(d) 1/3 octave; 10 – 500 Hz

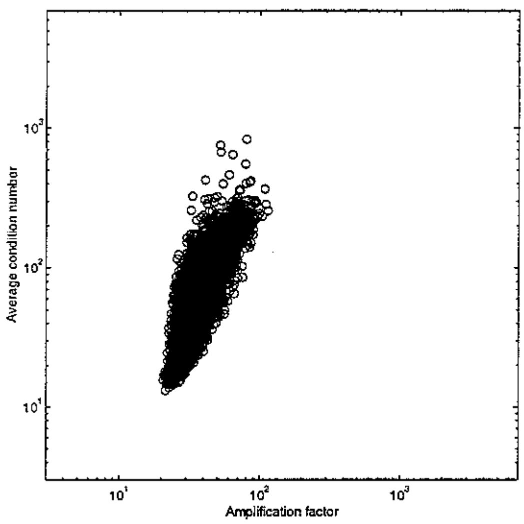
Figure 38. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 6 responses chosen from 20 responses of the first distribution and a smaller damping loss factor.



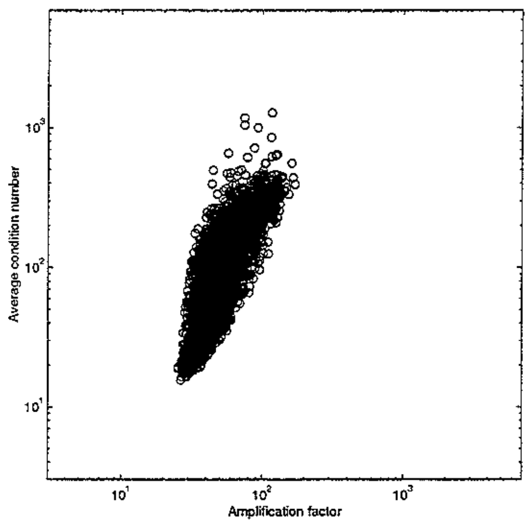
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz

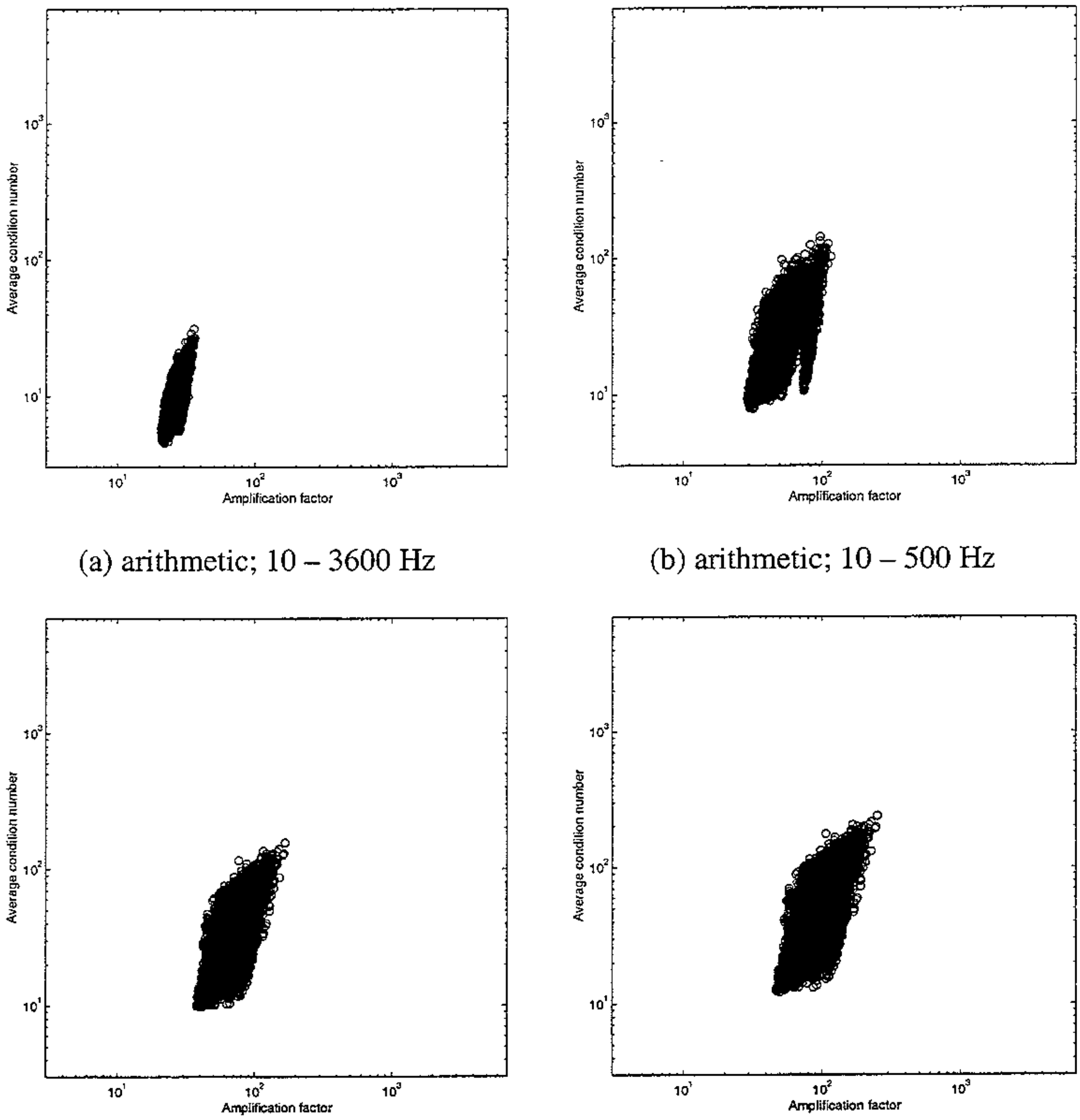


(c) 1/3 octave; 10 – 3600 Hz



(d) 1/3 octave; 10 – 500 Hz

Figure 39. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 4 responses chosen from 20 responses of the first distribution and a larger damping loss factor.



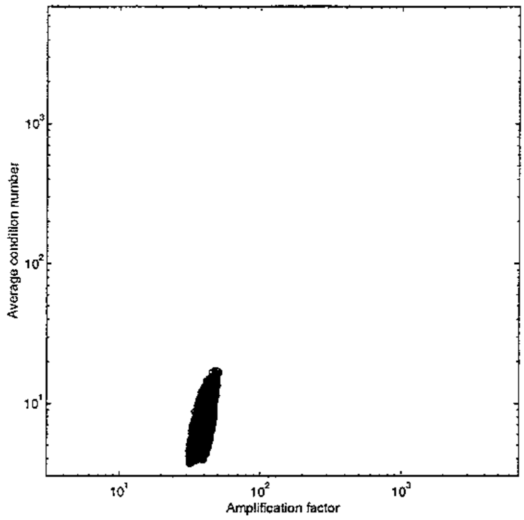
(a) arithmetic; 10 – 3600 Hz

(b) arithmetic; 10 – 500 Hz

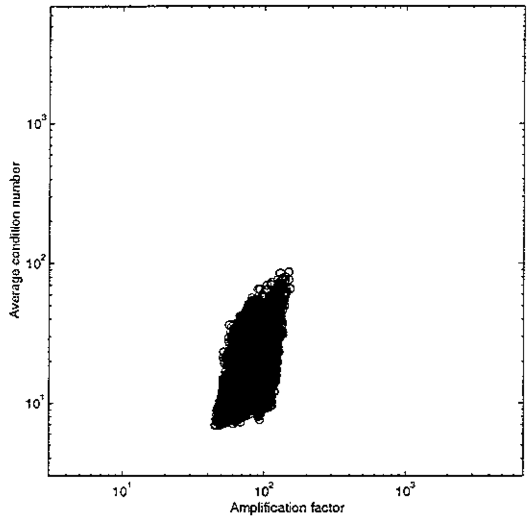
(c) 1/3 octave; 10 – 3600 Hz

(d) 1/3 octave; 10 – 500 Hz

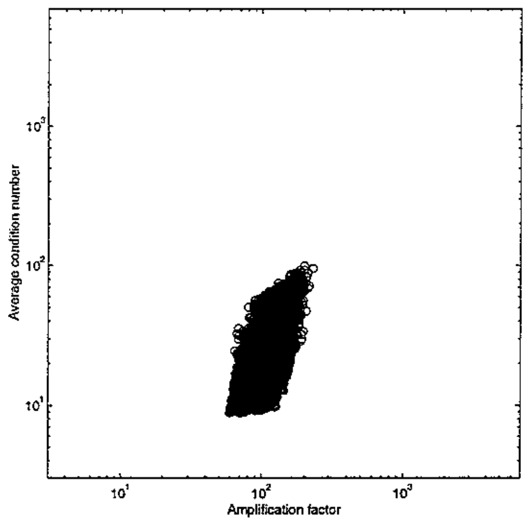
Figure 40. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 5 responses chosen from 20 responses of the first distribution and a larger damping loss factor.



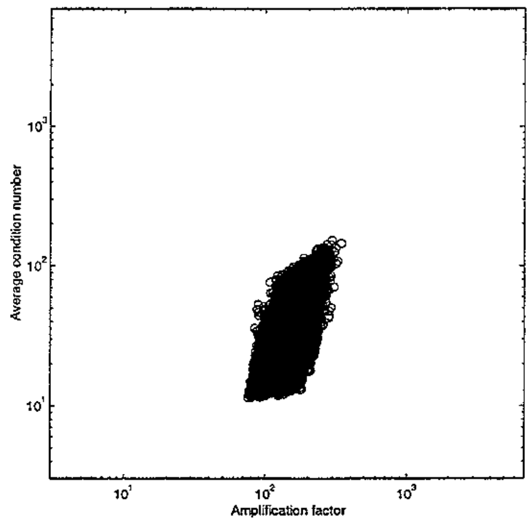
(a) arithmetic; 10 – 3600 Hz



(b) arithmetic; 10 – 500 Hz



(c) 1/3 octave; 10 – 3600 Hz



(d) 1/3 octave; 10 – 500 Hz

Figure 41. Correspondence between the composite condition number (amplification factor) and the condition number of the FRF matrix of 4 forces and 6 responses chosen from 20 responses of the first distribution and a larger damping loss factor.

3.6. Sensor location selection based on the composite condition number

As mentioned earlier, the composite condition number is proposed as the basis for selecting the optimal set of sensor locations instead of the average condition numbers in order to avoid an excessive amount of calculations. A set having the minimum composite condition numbers of all possible sets of the sensor locations is selected as the 'optimal' set of the sensor locations. However, this 'optimal' set is not always the best set corresponding to the minimum of the average of the condition numbers. A refinement of the method would involve selecting a small range of possible combinations on the basis of the composite condition number and then calculating the average condition number for all of these sets of locations. Therefore, it is necessary to investigate how large the sets of combinations should be to ensure that the location with the minimum average condition number would be correctly found. The sets of locations are arranged in order according to their composite condition numbers.

Figures 42 to 46 show the minimum of the average condition number for a set containing the first N % of points according to the composite condition number. The horizontal axis is the percentage of combinations in ascending order of the composite condition number over the total number of the composite condition number, not the actual value of the composite condition number. The vertical axis is the ratio of the 'local' minimum of the average condition number corresponding to a particular value of composite condition number to the global minimum of the average condition number. Therefore, it can be seen from the figures that as the range of composite condition number considered increases, the minimum average condition number converges to the overall minimum.

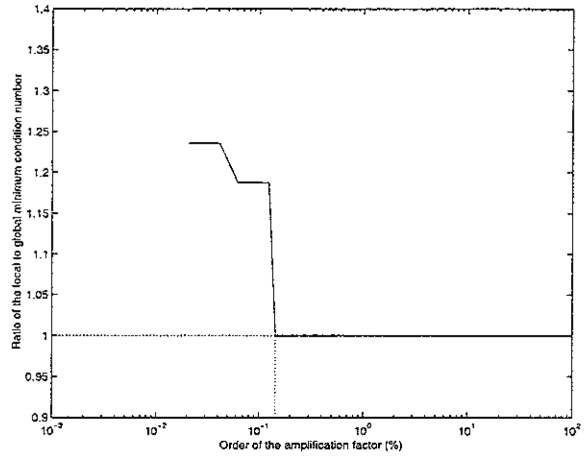
Figure 42 shows the results for the case of the first distribution of the force and response locations, Figure 43 for the second distribution and Figure 44 for the third distribution. The results for a smaller and a larger damping loss factor are plotted in Figures 45 and 46, respectively. The frequency range used is 10 – 500 Hz, and averaging method used is the one-third octave band averaging.

However, it is not easy to find a general trend from Figures 42 to 46. Therefore it would be better to make a plot to summarize these results. To avoid confusion, the results are selected for various specific percentages of the total number of the possible combinations in the ascending order of the composite condition numbers, i.e. 0.1, 0.3, 1, 3 and 10 %. Figure 47 shows these results.

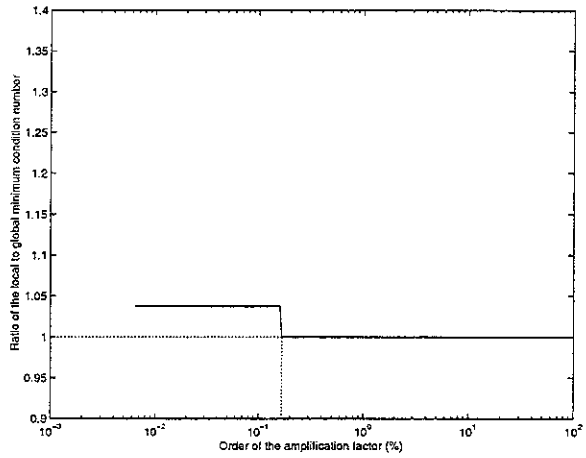
From Figure 47, it can be seen that

1) when using 10 % of combinations of the response locations corresponding to the lower 10 % of the composite condition numbers, the best set that can be obtained corresponds to between 0 and 3 % larger than the global minimum of the average of the condition number,
2) when using 3 % of combinations, the best set corresponds to 0 – 5 % larger,
3) when using 1 % of combinations, the best set corresponds to 0 – 16 % larger,
4) when using 0.3 % of combinations, the best set corresponds to 0 – 22 % larger, and
5) when using 0.1 % of combinations, the best set corresponds to 0 – 38 % larger.

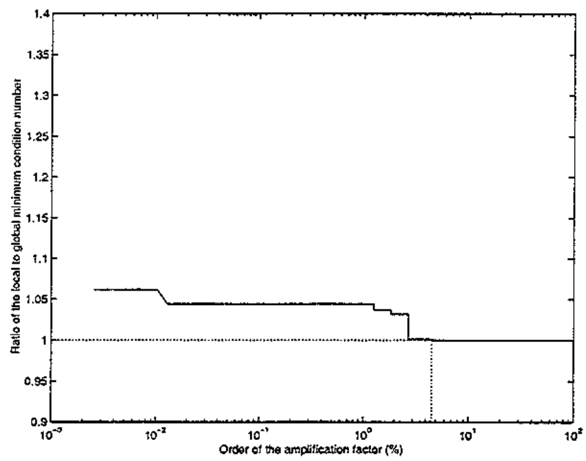
These may be compared with average condition numbers which may be more than a factor of 10 greater in the worst case than the best case. Therefore, by using these results, the number of the combinations to be investigated in detail can be decided in order to obtain an acceptable level of the minimum average condition numbers. However, it must not be forgotten that these results are specific to the dynamic problem considered of a flat rectangular plate, although it may be expected that similar results should be obtained for other systems.



(a) 4 responses

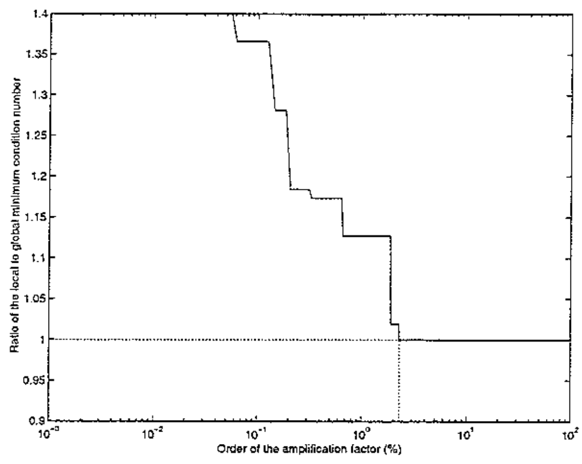


(b) 5 responses

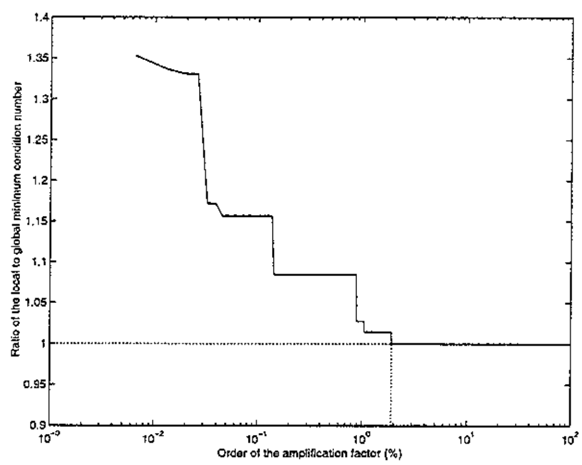


(c) 6 responses

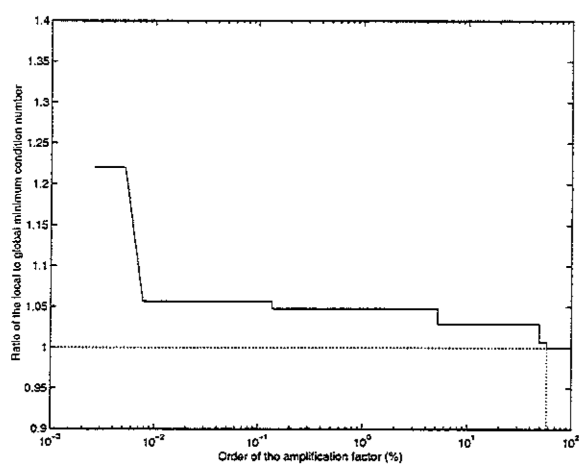
Figure 42. Variation of the local minimum of the average condition number with respect to the ascending order of the composite condition number for the first distribution and the initial damping loss factor.



(a) 4 responses

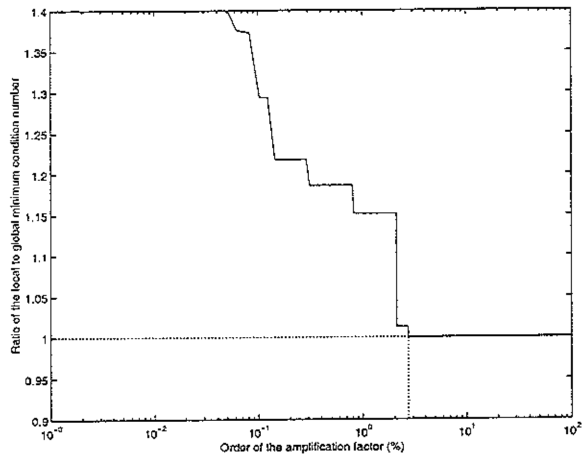


(b) 5 responses

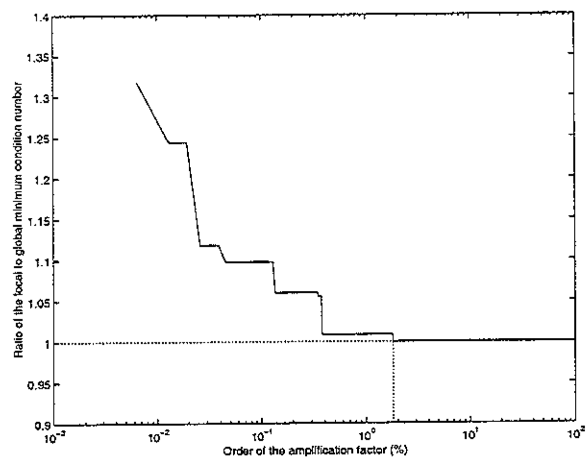


(c) 6 responses

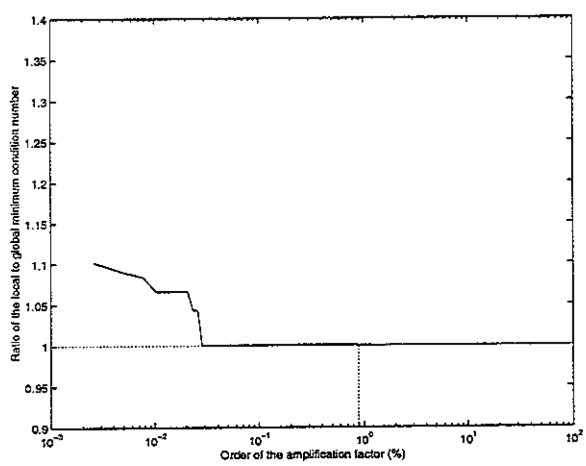
Figure 43. Variation of the local minimum of the average condition number with respect to the ascending order of the composite condition number for the second distribution and the initial damping loss factor.



(a) 4 responses

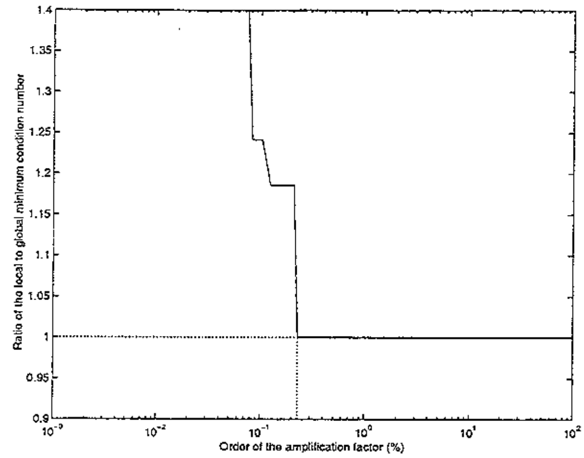


(b) 5 responses

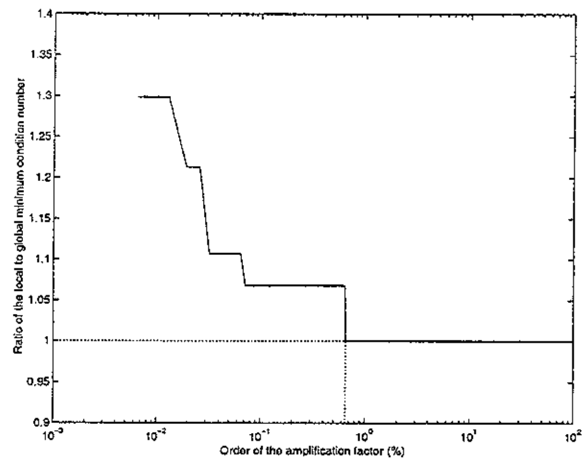


(c) 6 responses

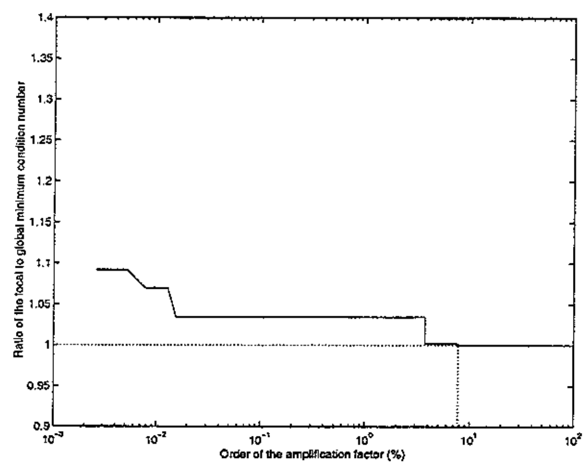
Figure 44. Variation of the local minimum of the average condition number with respect to the ascending order of the composite condition number for the third distribution and the initial damping loss factor.



(a) 4 responses

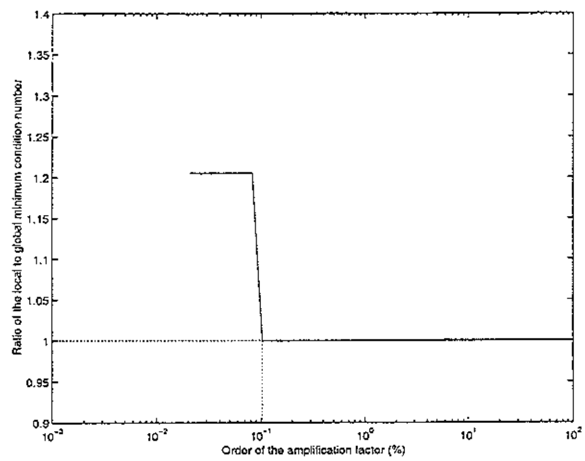


(b) 5 responses

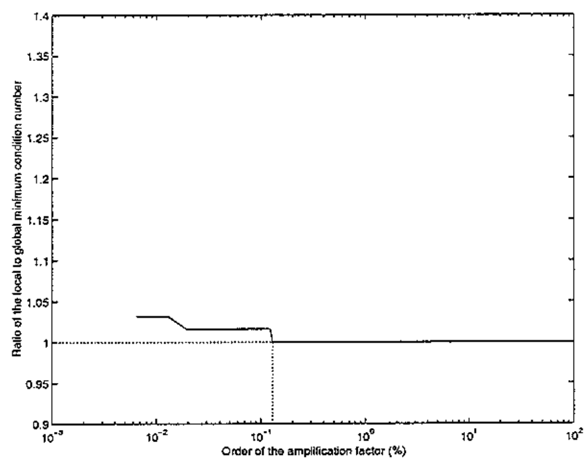


(c) 6 responses

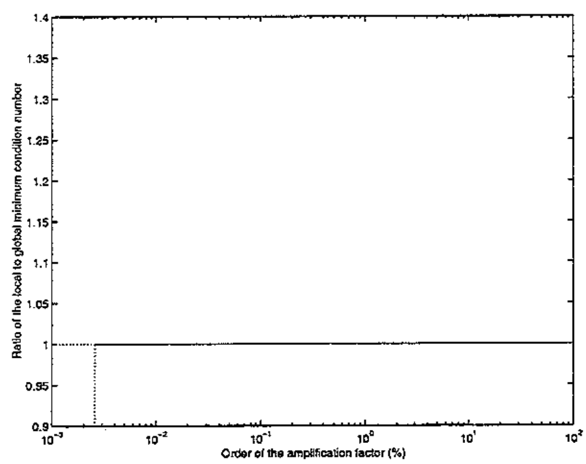
Figure 45. Variation of the local minimum of the average condition number with respect to the ascending order of the composite condition number for the first distribution and the smaller damping loss factor.



(a) 4 responses



(b) 5 responses



(c) 6 responses

Figure 46. Variation of the local minimum of the average condition number with respect to the ascending order of the composite condition number for the first distribution and the larger damping loss factor.

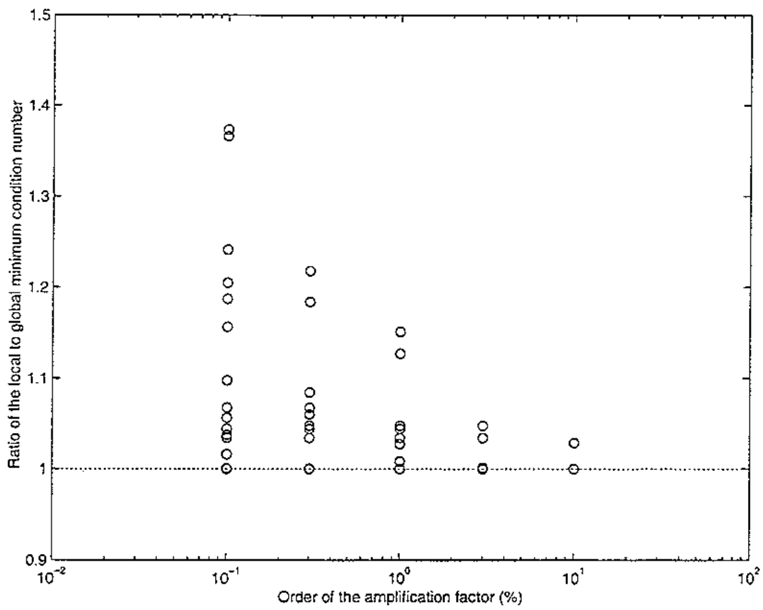


Figure 47. Variation of the local minimum of the average condition number with respect to the size of subset based on the composite condition number for all cases.

3.7. Summary

The composite condition number and the average of the condition number are investigated. The composite condition number is used to reduce the amount of calculation involved in finding the condition number at each frequency and for all possible combinations. Two methods of averaging the condition numbers are also discussed, that is, the method of averaging first in each 1/3 octave band and the more straightforward method of using the arithmetic mean. From the correspondence between the composite condition number and the average of the condition number, the result obtained using the method of averaging the intermediate average in each 1/3 octave band shows the better correspondence and is less sensitive to the frequency range considered.

However the optimal set determined from the minimum of the composite condition number is not always the best set in terms of the minimum average condition number. It is proposed to select a small number of combinations using the composite condition number and then to find the actual condition number for these cases.

The size of set considered determines how close to the overall minimum condition number it is possible to achieve using this method. For example using only 0.1 % of combinations, the best set found was at worst 38 % larger than the true minimum condition number, whereas the range of condition numbers in the whole set includes values more than 10 times the minimum value.

4. SENSOR LOCATION SELECTION USING GENETIC ALGORITHMS

4.1. Introduction to genetic algorithms

As shown in [1], the selection of the sensor locations can have a large influence on the estimates of the operational forces, but the work of finding the optimal sensor locations is a difficult and time-consuming process. In the previous section, the method based on the composite condition number (or amplification factor) [1] is used to search for the optimal locations. However, this method does not always give the best result because the set corresponding to the minimum composite condition number does not coincide with the set corresponding to the minimum of the average condition number. Moreover, since this problem of sensor location selection does not have a continuous objective function but a discrete one, which also does not have any derivatives, the conventional optimisation techniques cannot be applied. Therefore, genetic algorithms (GAs) are very suitable for these problems of the sensor location selection.

Genetic algorithms have the facility to search for a global optimum solution without requiring the objective function to be differentiable and so can be applied to discrete as well as continuous functions as mentioned above [6].

In general, genetic algorithms are stochastic global search methods based on an analogy with natural evolution. In natural evolution, members of a population compete with each other to survive and reproduce successfully. If a member of a population has superior genes to an other member, this member has more possibility to breed successfully. Over many generations, natural populations are believed to evolve according to the principles of natural selection and "survival of the fittest", as first clearly stated by Charles Darwin in *The Origin of Species*. Like the pattern of these evolution processes, genetic algorithms make the solutions grow the optimal solutions having the "fittest" objective function.

4.2. Fundamentals of genetic algorithms

To use the genetic algorithms, the parameters called *chromosomes* have to be defined first. Each chromosome includes a series of input variables and its *fitness* value given by the objective function. The commonly used representations of chromosome in the genetic algorithms are binary, integer, real-valued, etc. [8]. Next, using some chromosomes a number of potential solutions, called a *population*, are made.

To estimate the fitness of members of a population, a new function, called the *fitness function*, is normally used and this function is obtained by transforming the objective function value into a measure of relative fitness that is a non-negative value. A commonly used transformation from the objective function to the fitness function is that of proportional fitness assignment. It is explained in detail in the following section.

There are three genetic operators, which are used to generate a new population of chromosomes from an old population: *selection* (or *reproduction*), *crossover* and *mutation*.

The selection operator assigns each chromosome a relative probability for reproduction according to the fitness of the chromosomes. Based on each relative probability, the number of times for each chromosome to be reproduced is determined. Many employed selection techniques are based on the so-called roulette wheel selection methods: the stochastic sampling with replacement, the stochastic sampling with partial replacement, the remainder stochastic sampling with replacement, the remainder stochastic sampling without replacement, etc. [9].

The crossover operator is the basic operator for producing new chromosomes. Like its counterpart in nature, the chromosomes are randomly paired together and a part of each chromosome is exchanged with each other so that new chromosomes are produced. The commonly used crossover operators are single-point crossover, multi-point crossover, uniform crossover, shuffle, reduced surrogate, etc. The crossover operator is applied to each pair of chromosomes with a probability typically in the range 0.6 and 1.0 [9].

The mutation operator is a random process where some genes in a chromosome are replaced by another gene to produce a new chromosome. In other words, the mutation operator has the potential to re-introduce genetic information that has been lost from the population. Contrary to the crossover operator, the mutation operator is randomly applied with low probability, typically in the range 0.001 and 0.01, and modifies genes in the chromosomes.

To terminate the genetic algorithms, it is necessary to specify convergence criteria, but it is difficult to define formally the termination criteria because the genetic algorithms are stochastic search methods. Commonly used termination methods are a method of terminating the genetic algorithms after a pre-specified number of generations or a method of checking the convergence of the best individual of the population.

4.3. Application of GAs to the sensor location selection problem

The analysis object is the same as the one used in the previous section, i.e. an analytical model of a rectangular plate with 4 force locations. The total number of sensor locations is 20 and the number of selected sensor locations is 5. Therefore the number of all possible combinations is ${}_{20}C_5 = 15504$. The chromosome used in GAs is defined as 5 integers using an integer-string representation, like (3,8,1,18,13), but the order is not important. The 5 integers must all be different.

The objective function used here is the average of the condition number over the range of frequencies. The fitness value of ith combination x_i is defined as the reciprocal of the objective function,

$$f(x_i) = \kappa_i^{-1}, \tag{5}$$

and the corresponding fitness function is defined as the proportion of each fitness value to the total sum of all fitness values,

$$F(x_i) = f(x_i) \Big/ \sum_{i=1}^{N_{ind}} f(x_i), \tag{6}$$

where N_{ind} is the number of individuals in the population in a given generation. Based on the fitness function, in this study, the roulette wheel selection method (the stochastic sampling with replacement) is used to determine the number of times each individual is used for reproduction. First, a real-valued interval, *Sum*, is determined as the sum of the raw fitness values over all the individuals in the current population. Then individuals are mapped one-to-one into contiguous intervals in the range [0, *Sum*]. The size of each individual interval corresponds to the fitness value of the associated individual. To select an individual, a random number is generated in the interval [0, *Sum*] and the individual whose segment spans the random number is selected. This process is repeated until the desired number of individuals have been selected.

The crossover operator used in this study is somewhat different from other crossover operators. Because the genes of each chromosome in this study may not be duplicated, the subset crossover operator proposed by Lucasius and Kateman [10] is used here. Offspring subsets are first produced from parents, preserving their intersection. The remaining elements are then shuffled and exchanged to produce two offspring (see Figure 48). This is because in this problem no offspring may have two identical genes (here, integers). For example, if the parents in Figure 48 with two identical genes (i.e., 9 and 15) produce their offspring by the general single-point operator, setting a crossover point between second and third genes, then new offspring would be (9,2,4,3,9) and (10,15,7,15,12). In this case, the first offspring has a duplicate gene of 9 and second has a duplicate gene of 15 so that these offspring are invalid. Therefore, as mentioned earlier, the common elements of the parent strings are isolated and then the remaining elements of each parent are shuffled and exchanged to produce two new strings which are used to be added to the isolated common elements and to produce two offspring.

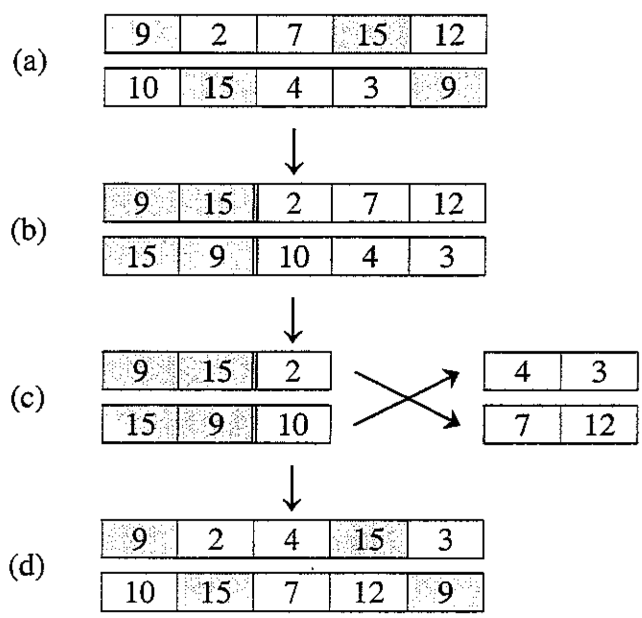


Figure 48. Subset crossover. (a) Parent. (b) Retain common parts.

(c) Cross. (d) Re-arrange in original order.

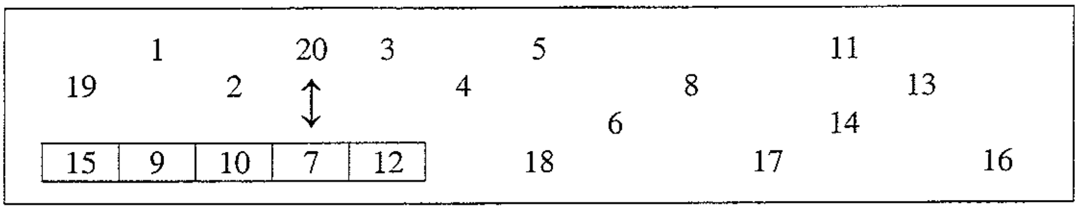


Figure 49. Subset mutation.

Also the mutation operator used here simply replaces terms in individuals with another term from the complementary subset, as shown in Figure 49, in order to avoid duplicate terms.

To improve the performance of the GAs, the "elitist model" is used [7]. In this, the best individual in the current generation automatically survives by replacing the worst one in the next generation. If the elitist model is not used, it is possible that, since the selection is a stochastic sampling, the best individual in the current generation is not selected at all so that the best individual in the next generation is worse than the current best individual.

A further method to improve the performance of GAs is applied to this sensor location selection problem: the variable probability of mutation [11]. According to the repetition of the best individual in every generation, the population becomes dominated by the best individual and its relatives. Consequently it becomes difficult to search for new individuals that are different from the current individuals. To avoid this problem, the probability of mutation is increased in proportion to how often the same best individual has been retained. The increase of the probability of mutation forces new individuals to be sought in the complementary solution area.

4.4. Simulations

Using the GAs, the optimal sensor locations are sought. The numbers of forces and responses are 4 and 5 respectively, and the total number of response positions available is 20 as in the previous section. The frequency range is 10 to 500 Hz. The objective function is taken as the average (simple mean) of the condition number over frequencies. The best combination corresponds to an average condition number of 13.0497.

The probability of crossover p_c used in this study is 0.8, which corresponds to the center of the typical range 0.6 to 1.0 mentioned earlier. In the case of the mutation probability, its typical range is from 0.001 to 0.01 and when using larger mutation probability than 0.01, the search area becomes more complicated and diverse and the searching is prone to converge to local optima. Conversely when using smaller mutation probability, the searching is prone to converge to local optima near to the start points. Therefore the initial probability of mutation p_m used is 0.01, and the increase of the probability of mutation Δp_m used each time the same best individual is retained is also 0.01.

The size of the population N_{ind} is important in determining the speed of searching. The number of generations N_{gen} is related with the size of the population.

Since the GAs are stochastic search methods, the best sensor locations will not always be found. Therefore, a number of simulations have been carried out repeatedly to assess the effectiveness of the method.

First, the effect of the population size on the performance of GAs is investigated by considering various cases of the population size (10, 20, 30, 40, 50) and corresponding numbers of generations (100, 50, 35, 25, 20). At this stage no elitist model and no increase of the probability of mutation are used. The number of generations is determined to maintain a roughly constant maximum number of evaluations of the average condition number. This is set to 1000, that is about 7 % of the total combinations. Twenty different calculations of each case are made and the average of the results is calculated. The average of the condition number of the selected optimal set is shown in Table 3. These may be compared with the overall best value of 13.0497. The last column in the table indicates the average and standard deviation of rankings of the selected optimal sets out of all possible sets. These are expressed as percentages of the total set of 15504 combinations, where the best combination corresponds to 0.007 %. From Table 3, it can be seen that, as the population size increases, the average of the best objective functions decreases. In other words, as the number of individuals in a generation increases, a better set is selected. However the results in Table 3 are not yet acceptable because of the large solution domain.

Since the elitist model is not used, the best set in each generation cannot be guaranteed to survive in the next generation. For the same reason, the last best set may not be the best of all sets considered during the GA process. To overcome this, the elitist model is used next.

Table 4 shows the results for the case of using the elitist model with constant probability of mutation.

Comparing the results of Tables 3 and 4, it can be seen that the elitist model is very effective in improving the search for the best set of sensor locations. If the population consists of 30 or more individuals, then it is certain that the GAs choose an individual within the 0.5 % best ones with 84 % confidence.

Table 3. The average of the objective functions of the optimal sets selected by using GAs according to varying the population size and the generation number with no elitist model and no increase of the mutation probability ($N_{ind} \times N_{gen} \approx 1000$).

Population size	Number of generations	Average of best objective functions	% of total combinations
10	100	17.28	5.60 ± 5.86
20	50	15.85	2.23 ± 3.06
30	35	15.60	1.34 ± 1.22
40	25	15.07	0.97 ± 1.61
50	20	14.78	0.62 ± 0.89

Table 4. The average of the objective functions of the optimal sets selected by using GAs according to varying the population size and the generation number with the elitist model and no increase of the mutation probability ($N_{ind} \times N_{gen} \approx 1000$).

Population size	Number of generations	Average of best objective functions	% of total combinations
10	100	15.61	1.77 ± 2.93
20	50	14.76	0.52 ± 0.67
30	35	14.18	0.18 ± 0.20
40	25	14.13	0.16 ± 0.18
50	20	13.71	0.07 ± 0.11

To improve the performance of the GAs, the method of increasing the probability of mutation is next used. The results obtained using this method are shown in Table 5.

The optimal sets selected by the GA are within 0.6 % of the best set for all population sizes with 84 % confidence. Comparing the results of Tables 3 to 5, the elitist model and the increase of the probability of mutation according to the re-generation of the best sets give very effective improvement to the sensor location selection.

Table 5. The average of the objective functions of the optimal sets selected by using GAs according to varying the population size and the generation number with the elitist model and an increase of the mutation probability ($N_{ind} \times N_{gen} \approx 1000$).

Population size	Number of generations	Average of best objective functions	% of total combinations
10	100	14.34	0.23 ± 0.32
20	50	14.00	0.11 ± 0.14
30	35	14.26	0.20 ± 0.27
40	25	14.05	0.17 ± 0.30
50	20	13.72	0.07 ± 0.10

The maximum numbers of individuals considered during the GA procedure in the above is about 1000, which is about 7 % of the total number of possible combinations. To investigate the effect of reducing the maximum numbers of individuals considered on the performance of GAs, the sensor location selection is evaluated using the GAs for both half and a quarter of the number of generations previously used. The results are shown in Tables 6 and 7. Even though half of the numbers of generations are used, the selected sets are in the range of 0.5 % or smaller of the best set except for the case of using only 10 individuals. Therefore, using about 500 combinations of response locations shows good results and is useful. However, when using a quarter of the numbers of generations, the selected sets are in the range of 0.6 % or smaller of the best set for the case of using 30 or more individuals. Consequently, as the number of all combinations used in the evolutionary process decreases, it would be better that the number of individuals in each generation should increase.

Table 6. The average of the objective functions of the optimal sets selected by using GAs according to varying the population size and the generation number with the elitist model and an increase of the mutation probability ($N_{ind} \times N_{gen} \approx 500$).

Population size	Number of generations	Average of best objective functions	% of total combinations
10	50	14.65	0.51 ± 0.74
20	25	14.21	0.15 ± 0.11
30	17	14.39	0.22 ± 0.22
40	13	14.06	0.15 ± 0.22
50	10	13.99	0.12 ± 0.13

Table 7. The average of the objective functions of the optimal sets selected by using GAs according to varying the population size and the generation number with the elitist model and an increase of the mutation probability ($N_{ind} \times N_{gen} \approx 250$).

Population size	Number of generations	Average of best objective functions	% of total combinations
10	25	15.46	1.35 ± 1.74
20	13	14.96	0.89 ± 1.45
30	9	14.46	0.27 ± 0.30
40	7	14.58	0.31 ± 0.30
50	5	14.55	0.27 ± 0.18

4.5. Evaluation of the performance of genetic algorithms

To evaluate the performance of the genetic algorithms, the results obtained by the genetic algorithms are compared with those obtained by random selection. The method of random selection selects 1000 (and 500, 250) sets of possible response location combinations at random and then the minimum of the objective functions is obtained. Repeating this process 20 times, the average and the standard deviation are calculated from the 20 minima of the objective functions. The average of the 20 minima and the average and standard deviation of the % orders are as follows:

from 1000 random sets: average = 14.09, 0.11 ± 0.07 %,

from 500 random sets: average = 14.36, 0.19 ± 0.18 %,

from 250 random sets: average = 14.54, 0.31 ± 0.27 %.

The GA method gives better results than random selection when the population is at least 40 and the elitist model and increase of mutation probability are included. Otherwise GAs actually give worse results than random selection.

4.6. Summary

In this section, the genetic algorithms have been used to search for the best set of response locations. From some simulations, it can be concluded that the genetic algorithms perform better to select the optimal sensor locations. The GAs with consideration of the elitist model and the variable probability of mutation also give better results than the simple GAs. Selection by using 20 or more individuals in each generation in this rectangular flat plate problem gives the optimal sets within at least 0.5 % of the best sets with 84 % confidence.

However, random selection of an equal number of individuals can provide location selection that is just as good unless at least 40 individuals are considered per generation.

5. CONCLUSION

In this study, the use of Tikhonov regularization was investigated in terms of the amplification of the errors in reconstructed forces due to the response errors. The error amplification factors were estimated for various sets of response and force positions for Tikhonov regularization and the pseudo-inverse method. To investigate the relationship between the error amplification factor and the condition number, the 5, 50 and 95 % confidence levels have been determined. From the 95 % confidence levels obtained by the pseudo-inversion and Tikhonov regularization the threshold, called the cross-over condition numbers, are determined. Tikhonov regularization should only be used where the condition number is greater than this threshold; below the threshold pseudo-inversion gives better results. This threshold varies from 10 for a 10×2 matrix to around 200 for a 15×14 matrix. Consequently it is found that the criterion for the use of Tikhonov regularization is related not only to the condition number but also to the system matrix size.

In this study the optimal selection of the sensor locations has also been considered. Methods based on the 'amplification factor' (composite condition number) and using the genetic algorithms are used to select the best set from all possible combinations of response locations. In order to obtain the composite condition number and the average condition number, the method of averaging first in each 1/3 octave band has been introduced. Different frequency ranges have also been considered. From some simulations, however, the optimal set for the minimum of the composite condition numbers is not always the best set in terms of the minimum average condition number. Therefore it is proposed to select a small number of combinations using the composite condition number and then to find the actual condition number for these cases. It has been seen that the size of set considered determines how close to the overall minimum condition number it is possible to achieve using this method. Nevertheless using only 0.1 % of combinations, the best set found had an average condition number at worst 38 % larger than the true minimum, whereas the range of average condition numbers in the whole set includes values more than 10 times the minimum value.

The use of genetic algorithms to select the optimal set of sensor locationshas also been considered. The subset crossover operator is used to avoid the duplicated elements in an individual. The genetic algorithm using the elitist model and a variable mutation possibility gives better results than a simple genetic algorithm. When twenty or more individuals are used in each generation, and enough generations are used to ensure about 7 % of all combinations are considered the selection of the optimal set was found to lie within at least 0.5 % of the best sets with 84 % confidence. However, comparing the performance of genetic algorithms with a simple random selection, it is found that the random selection of an equal number of individuals can provide location selection that is just as good unless at least 40 individuals are considered per generation.

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