Bilinear differential forms and the Löwner framework for rational interpolation

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Abstract The Löwner approach, based on the factorization of a special-structure matrix derived from data generated by a dynamical system, has been applied successfully to realization theory, generalized interpolation, and model reduction. We examine some connections between such approach and that based on bilinear- and quadratic differential forms arising in the behavioral framework.

Dedicated to Prof. Harry L. Trentelman- friend, colleague, and former supervisor- on the occasion of his "sixtieth birthday"

1 Introduction

The Löwner framework was initiated in [17, 18] in the context of tangential interpolation and partial realization problems (see also [1, 4]). Its relevance for the problem of modelling from frequency-response measurements and for model order reduction has been reported in a series of publications (see [2, 3]), resulting also in important applications in the (reduced-order) modelling of physical systems from data (see [15, 16]). Time-series modelling from a behavioral perspective has been introduced

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in [30, 31], specialized to the vector-exponential case in [32], and applied to metric interpolation problems in [13, 14, 27].

The purpose of this paper is to illustrate some connections between these two approaches. The relation between rational interpolation and partial realization problems and the behavioral framework for data modelling is well known, see [7]; we will concentrate here on the analogies and insights coming from a more recently introduced approach (see [21, 25]) that while essentially behavioral (i.e. trajectory-based) also uses Gramian-based ideas to derive models from data. An important tool in such approach is the calculus of bilinear- and quadratic differential forms (B/QDFs in the following), introduced in [33] and applied successfully in many areas of systems and control (see [22, 28]). In this paper we show that several results derived in the Löwner approach can be formulated also in terms of the two-variable polynomial matrix representations of B/QDFs derived from the system parameters. Of particular relevance is that the factorization of the Löwner matrix- an important step of the Löwner approach in obtaining state models from data- can be given a trajectory-based interpretation based on B/QDFs.

The paper is organized as follows. In section 2 we illustrate the essential concepts of the Löwner approach, of bilinear- and quadratic differential forms, and of behavioral systems theory. In section 3 we show how the Löwner matrix and some of its properties can be formulated in the polynomial language of the representations of B/QDFs. In section 4 we show how the computations of state equations based on Löwner matrix factorizations have a straightforward interpretation in terms of bilinear differential forms. Finally, section 5 contains an exposition of directions of current and future research.

Notation

The space of n dimensional real (complex) vectors is denoted by $\mathbb{R}^n$ (respectively $\mathbb{C}^n$), and that of $m \times n$ real matrices by $\mathbb{R}^{m\times n}$. $\mathbb{R}^{\bullet\times m}$ denotes the space of real matrices with m columns and an unspecified finite number of rows. Given matrices $A, B \in \mathbb{R}^{\bullet\times m}$, $\mathrm{col}(A,B)$ denotes the matrix obtained by stacking A over B.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. $\mathbb{R}^{r\times q}[\xi]$ denotes the set of all $r \times q$ matrices with entries in ξ, and $\mathbb{R}^{n\times m}[\zeta,\eta]$ that of $n \times m$ polynomial matrices in ζ and η. The set of rational $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}(\xi)$.

The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$ is denoted by $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$. $\mathfrak{D}(\mathbb{R},\mathbb{R}^q)$ is the subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ consisting of compact support functions. Given $\lambda \in \mathbb{C}$, we denote by $e^{\lambda\cdot}$ the exponential function whose value at t is $e^{\lambda t}$.

2 Background material

We restrict ourselves to the minimum amount of information necessary to understand the rest of the paper. For more details and a thorough introduction to behavioral system theory, bilinear/quadratic differential forms and the Löwner framework we refer to [19, 33, 17], respectively.

2.1 Behavioral system theory

The basic object of study in the behavioral framework is the set of trajectories, the *behavior* of a system. In this paper we consider *linear differential behaviors*, i.e. subsets of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ that consist of solutions $w:\mathbb{R}\to\mathbb{R}^q$ to systems of linear, constant-coefficient differential equations:

$$R\left(\frac{d}{dt}\right)w = 0\ . \tag{1}$$

where $R\in\mathbb{R}^{\bullet\times q}[\xi]$. A representation (1) is called a kernel representation of the *behavior*

$$\mathfrak{B} := \left\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0\right\}\ ,$$

and we associate to it in a natural way the polynomial matrix $R\in\mathbb{R}^{\bullet\times q}[\xi]$. Note that $\mathfrak{B}$ admits different kernel representations; such a representation is *minimal* if the number of rows of R is minimal among all possible representations of $\mathfrak{B}$. We denote with $\mathfrak{L}^q$ the set of all linear time-invariant differential behaviours with q variables.

If a behavior is controllable (see Ch. 5 of [19] for a definition), then it also admits an *image representation*. Let

$$w = M\left(\frac{d}{dt}\right)\ell\ , \tag{2}$$

where $M\in\mathbb{R}^{q\times l}[\xi]$ and ℓ is an auxiliary variable also called a *latent variable*; i.e.,

$$\mathfrak{B} := \{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q) \mid \exists\ \ell\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^l) \text{ such that (2) holds}\} =: \operatorname{im} M\left(\frac{d}{dt}\right)\ .$$

We call (2) an image representation of $\mathfrak{B}$.

The latent variable ℓ in (2) is called *observable* from w if $[w = M(\frac{d}{dt})\ell = 0] \Longrightarrow [\ell = 0]$. A controllable behaviour always admits an observable image representation. The set of linear differential controllable behaviours whose trajectories take their values in $\mathbb{R}^q$ is denoted by $\mathfrak{L}^q_{\text{cont}}$.

A latent variable ℓ is a *state variable* for $\mathfrak{B}$ if there exist $E, F \in \mathbb{R}^{\bullet\times\bullet}$, $G \in \mathbb{R}^{\bullet\times q}$ such that

$$\mathfrak{B} = \left\{ w \mid \exists\, \ell \text{ s.t. } E\frac{d\ell}{dt} + F\ell + Gw = 0 \right\} \;, \tag{3}$$

i.e. if $\mathfrak{B}$ has a representation of first order in ℓ and zeroth order in w. The minimal number of state variables needed to represent $\mathfrak{B}$ in this way is called the *McMillan degree* of $\mathfrak{B}$, denoted by $n(\mathfrak{B})$.

A state variable for $\mathfrak{B}$ can be computed as the image of a polynomial differential operator called a *state map* (see [9, 20, 26, 29]); such polynomial can act either on the external variable w, or on the latent variable ℓ of an image representation of $\mathfrak{B}$.

Finally, we introduce the notion of *dual* (or *adjoint*, see [29]) behavior. Let $\mathfrak{B} \in \mathfrak{L}^q$ and let $J = J^\top \in \mathbb{R}^{q\times q}$ be an involution, i.e. $J^2 = I_q$. We call

$$\mathfrak{B}^{\perp_J} := \left\{ w' \in \mathfrak{D}(\mathbb{R},\mathbb{R}^q) \mid \int_{-\infty}^{+\infty} w'^\top Jw\; dt = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^q) \right\} \tag{4}$$

the *J-dual behavior* of $\mathfrak{B}$; if $J = I_l$, we denote it simply by $\mathfrak{B}^\perp$. It can be shown that if $\mathfrak{B} = \operatorname{im} M\left(\frac{d}{dt}\right) = \ker R\left(\frac{d}{dt}\right)$, then $\mathfrak{B}^{\perp_J} = \operatorname{im} JR^\top\left(-\frac{d}{dt}\right) = \ker M^\top\left(-\frac{d}{dt}\right)J$. Note that if R induces a minimal kernel representation and M an observable image representation of $\mathfrak{B}$, then $M^\top(-\xi)J$ induces a minimal kernel representation and $JR^\top(-\xi)$ an observable image representation of $\mathfrak{B}^{\perp_J}$.

2.2 *Bilinear- and quadratic differential forms*

Let $\Phi \in \mathbb{R}^{q_1\times q_2}[\zeta,\eta]$; then $\Phi(\zeta,\eta) = \sum_{h,k} \Phi_{h,k}\zeta^h\eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{q_1\times q_2}$ and the sum extends over a finite set of nonnegative indices. $\Phi(\zeta,\eta)$ induces the *bilinear differential form* (abbreviated with BDF in the following) L_Φ acting on $\mathfrak{C}^\infty$-trajectories defined by

$$\begin{aligned} L_\Phi &: \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{q_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{q_2}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \\ L_\Phi(w_1,w_2) &:= \sum_{h,k} \left(\frac{d^h w_1}{dt^h}\right)^\top \Phi_{h,k} \frac{d^k w_2}{dt^k} \;. \end{aligned}$$

If $q_1 = q_2 = q$, then $\Phi(\zeta,\eta)$ also induces the *quadratic differential form* (abbreviated QDF in the following) Q_Φ acting on $\mathfrak{C}^\infty$-trajectories defined by

$$\begin{aligned} Q_\Phi : &\quad \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \\ Q_\Phi(w) := &\sum_{h,k} \left(\frac{d^h w}{dt^h}\right)^\top \Phi_{h,k} \frac{d^k w}{dt^k} \;. \end{aligned}$$

Without loss of generality we can assume that a QDF is induced by a *symmetric* two-variable polynomial matrix $\Phi(\zeta,\eta)$, i.e. one such that $\Phi(\zeta,\eta)=\Phi(\eta,\zeta)^\top$; we denote the set of such matrices by $\mathbb{R}_s^{q\times q}[\zeta,\eta]$.

$\Phi(\zeta,\eta)\in\mathbb{R}^{q_1\times q_2}[\zeta,\eta]$ (and consequently also the BDF L_Φ) can be identified with its *coefficient matrix*

$$\widetilde{\Phi}:=\left[\Phi_{h,k}\right]_{h,k=0,\dots,\infty} ,$$

in the sense that

$$\Phi(\zeta,\eta)=\left[I_{q_1}\;\zeta I_{q_1}\;\cdots\right]\widetilde{\Phi}\begin{bmatrix}I_{q_2}\\ \eta I_{q_2}\\ \vdots\end{bmatrix} .$$

Although $\tilde{\Phi}$ is infinite, only a finite number of its entries are nonzero, since the highest power of ζ and η in $\Phi(\zeta,\eta)$ is finite. Note that $\Phi(\zeta,\eta)$ is symmetric if and only if $\tilde{\Phi}^\top=\tilde{\Phi}$.

Factorizations of the coefficient matrix of a B/QDF and factorizations of the two-variable polynomial matrix corresponding to it are related as follows.

Proposition 1. *Let $\Phi\in\mathbb{R}^{q_1\times q_2}[\zeta,\eta]$, and let $\tilde{\Phi}$ be its coefficient matrix. Then the following two statements are equivalent:*

1. *There exist real matrices $\tilde{F}$, $\tilde{G}$ with n rows such that*
$$\tilde{\Phi}=\tilde{F}^\top\tilde{G} ;$$
2. *There exist polynomial matrices $F\in\mathbb{R}^{n\times q_1}[\xi]$, $G\in\mathbb{R}^{n\times q_2}[\xi]$ with coefficient matrices $\tilde{F}$, $\tilde{G}$, i.e., $F(\xi)=\tilde{F}\begin{bmatrix}I_{q_1}\\ \xi I_{q_1}\\ \vdots\end{bmatrix}$ and $G(\xi)=\tilde{G}\begin{bmatrix}I_{q_2}\\ \xi I_{q_2}\\ \vdots\end{bmatrix}$, such that*
$$\Phi(\zeta,\eta)=F(\zeta)^\top G(\eta) .$$

Proof. This follows from the discussion on p. 1709 of [33]. □

Factorizations as those of Proposition 1, which moreover correspond to the minimal value $n=\operatorname{rank}(\tilde{\Phi})$, are called *minimal* (or *canonical* as in [33]). Note that the matrices $\tilde{F}$ and $\tilde{G}$ involved in a minimal factorization of $\tilde{\Phi}$ are of *full row rank*. Minimal factorizations are not unique; using standard linear algebra arguments the following proposition can be proved in a straightforward way.

Proposition 2. *Given a minimal factorization $\tilde{\Phi}=\tilde{F}^\top\tilde{G}$, every other minimal factorization $\tilde{\Phi}=\tilde{F}'^\top\tilde{G}'$ can be obtained from it by premultiplication of $\tilde{F}$ and $\tilde{G}$ by a nonsingular $n\times n$ matrix S, respectively $S^{-\top}$. In view of Proposition 1 this implies that $\Phi(\zeta,\eta)=F(\zeta)^\top G(\eta)=F'(\zeta)^\top G'(\eta)$ with $F'(\xi):=SF(\xi)$, $G'(\xi):=S^{-\top}G(\xi)$.*

Given L_Ψ, its *derivative* is the BDF L_Φ defined by

$$L_\Phi(w_1,w_2) := \frac{d}{dt}(L_\Psi(w_1,w_2))$$

for all $w_i \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{q_i})$, $i=1,2$; this holds if and only if

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) \tag{5}$$

(see [33], p. 1710). An analogous result holds for QDFs. From this two-variable characterization it follows that if $L_\Phi = \frac{d}{dt}L_\Psi$, then $\Phi(-\xi,\xi) = 0_{q_1\times q_2}$; it can be shown (see Th. 3.1 p. 1711 of [33]) that also the converse implication holds true.

Finally, we introduce a standard result in B/QDF theory of great importance for the rest of this paper. The first part of the result is a straightforward consequence of the relation (5) between the two-variable representation of a B/QDF and its derivative; the second part follows from Prop. 10.1 p. 1730 of [33].

Proposition 3. *Let $R \in \mathbb{R}^{g\times q}[\xi]$ and $M \in \mathbb{R}^{q\times l}[\xi]$ induce a minimal kernel, respectively observable image representation of $\mathfrak{B} \in \mathfrak{L}^q$. There exists $\Psi \in \mathbb{R}^{g\times l}[\zeta,\eta]$ such that*

$$R(-\zeta)M(\eta) = (\zeta+\eta)\Psi(\zeta,\eta)\ . \tag{6}$$

Moreover, there exist polynomial matrices $Z \in \mathbb{R}^{\bullet\times g}[\xi]$ and $X \in \mathbb{R}^{\bullet\times l}[\xi]$ such that

$$\Psi(\zeta,\eta) = Z(\zeta)^\top X(\eta)\ , \tag{7}$$

and $Z\left(\frac{d}{dt}\right)$ is a minimal state map for $\mathfrak{B}^\perp$ and $X\left(\frac{d}{dt}\right)$ is a minimal state map for $\mathfrak{B}$.

State maps such as Z and X in (7) are called *matched*. Factorizations such as (7) can be computed factorizing canonically the coefficient matrix $\widetilde{\Psi}$ as illustrated in Prop. 1, see also Prop. 2.

2.3 Rational interpolation and modelling of vector exponential time-series

Define the *left* and *right interpolation data* as the triples in $\mathbb{C}\times\mathbb{C}^p\times\mathbb{C}^m$ and $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^p$, respectively:

$$\begin{aligned} &\{(\mu_i,\ell_i^*,v_i^*)\}_{i=1,\dots,k_1}\ , && \mu_i\in\mathbb{C}, \ell_i^*\in\mathbb{C}^{1\times p}, v_i^*\in\mathbb{C}^{1\times m} \\ &\{(\lambda_i,r_i,w_i)\}_{i=1,\dots,k_2}\ , && \lambda_i\in\mathbb{C}, r_i\in\mathbb{C}^{m\times 1}, w_i\in\mathbb{C}^{p\times 1}\ . \end{aligned} \tag{8}$$

In the rest of this paper we will assume for simplicity of exposition that the *μ_is and λ_is are distinct*; the general case follows with straightforward modifications of the statements and the arguments. We will also assume that $\{\mu_i\}_{i=1,\dots,k_1}\cap\{\lambda_j\}_{j=1,\dots,k_2} = \emptyset$.

Let $H \in \mathbb{R}^{p\times m}(\xi)$ be a proper rational matrix. *H satisfies the interpolation constraints* if

$$\begin{aligned} \ell_i^* H(\mu_i) &= v_i^* \; , \; i=1,\ldots,k_1 \\ H(\lambda_i) r_i &= w_i \; , \; i=1,\ldots,k_2 \; . \end{aligned} \tag{9}$$

Rational interpolation can be stated as behavioral modelling of vector-exponential functions (see [7]). Assume that $H \in \mathbb{R}^{p\times m}(\xi)$ satisfies the interpolation constraints, and let $H(\xi) = N(\xi)D(\xi)^{-1} = P(\xi)^{-1}Q(\xi)$ be right-, respectively left coprime factorisations of $H(\xi)$, with $N \in \mathbb{R}^{p\times m}[\xi]$, $D \in \mathbb{R}^{m\times m}[\xi]$, $P \in \mathbb{R}^{p\times p}[\xi]$, $Q \in \mathbb{R}^{p\times m}[\xi]$. We associate to the right coprime factorisation of $H(\xi)$ the observable image representation

$$M(\xi) := \begin{bmatrix} D(\xi) \\ N(\xi) \end{bmatrix} \tag{10}$$

and to the left-coprime factorisation the minimal kernel representation

$$R(\xi) := \begin{bmatrix} Q(\xi) & -P(\xi) \end{bmatrix} \; . \tag{11}$$

It follows from standard results in behavioral system theory (see Ch. 5 of [19]) that

$$\ker \; \begin{bmatrix} Q\left(\frac{d}{dt}\right) & -P\left(\frac{d}{dt}\right) \end{bmatrix} = \operatorname{im} \begin{bmatrix} D\left(\frac{d}{dt}\right) \\ N\left(\frac{d}{dt}\right) \end{bmatrix} =: \mathfrak{B} \; . \tag{12}$$

Under the standing assumption that $D(\mu_i)$ *and* $P(\lambda_i)$ *are nonsingular at* μ_i*, respectively* λ_i, we rewrite (9) equivalently as

$$\begin{aligned} \begin{bmatrix} v_i^* & -\ell_i^* \end{bmatrix} \begin{bmatrix} D(\mu_i) \\ N(\mu_i) \end{bmatrix} &= 0 \; , \; i=1,\ldots,k_1 \\ \begin{bmatrix} Q(\lambda_i) & -P(\lambda_i) \end{bmatrix} \begin{bmatrix} r_i \\ w_i \end{bmatrix} &= 0 \; , \; i=1,\ldots,k_2 \; . \end{aligned} \tag{13}$$

From the equalities (13) it follows that

$$\begin{aligned} \begin{bmatrix} v_j^* & -\ell_j^* \end{bmatrix} &\in \text{row span} \begin{bmatrix} Q(\mu_j) & -P(\mu_j) \end{bmatrix} \\ \begin{bmatrix} r_i \\ w_i \end{bmatrix} &\in \operatorname{im} \begin{bmatrix} D(\lambda_i) \\ N(\lambda_i) \end{bmatrix} \; , \end{aligned}$$

$j = 1,\ldots,k_1$, $i = 1,\ldots,k_2$. We conclude that the interpolation constraints (9) (and the equations (13)) are equivalent with

$$\begin{aligned} w_i(\cdot) &:= \begin{bmatrix} r_i \\ w_i \end{bmatrix} e^{\lambda_i \cdot} \in \mathfrak{B} \; , \; i=1,\ldots,k_2 \\ w_j'(\cdot) &:= \begin{bmatrix} v_j \\ -\ell_j \end{bmatrix} e^{-\mu_j \cdot} \in \mathfrak{B}^\perp \; , \; j=1,\ldots,k_1 \; , \end{aligned} \tag{14}$$

where $\mathfrak{B}^{\perp}$ is the dual behavior $\mathfrak{B}^{\perp} = \operatorname{im} \begin{bmatrix} Q^{\top}\left(-\frac{d}{dt}\right) \\ -P^{\top}\left(-\frac{d}{dt}\right) \end{bmatrix} = \ker \left[D^{\top}\left(-\frac{d}{dt}\right)\ N^{\top}\left(-\frac{d}{dt}\right)\right]$. In the language of [31], $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$, respectively, are *unfalsified models* for the trajectories (14). Thus every solution of the interpolation problem yields an unfalsified model for the exponential trajectories associated with the data; and conversely, every minimal kernel or observable image representation of such an unfalsified model for such trajectories yields a solution of the interpolation problem.

From (13) it follows that there exist vectors $s_j \in \mathbb{C}^{1\times p}$, $j = 1,\dots,k_1$ and p_i, $i = 1,\dots,k_2$, uniquely defined because of observability and minimality and controllability, such that

$$\begin{aligned} \begin{bmatrix} v_j^* & -\ell_j^* \end{bmatrix} &= s_j^* \begin{bmatrix} Q(\mu_j) & -P(\mu_j) \end{bmatrix} \\ \begin{bmatrix} r_i \\ w_i \end{bmatrix} &= \begin{bmatrix} D(\lambda_i) \\ N(\lambda_i) \end{bmatrix} p_i \,. \end{aligned} \tag{15}$$

It is straightforward to check that such vectors define (unique) latent variable trajectories $p_i e^{\lambda_i \cdot}$ and $s_j e^{-\mu_j \cdot}$ for the image representations $\mathfrak{B} = \operatorname{im} M\left(\frac{d}{dt}\right)$, $\mathfrak{B}^{\perp} = \operatorname{im} R^{\top}\left(-\frac{d}{dt}\right)$, respectively.

3 The Löwner matrix and its properties

The *Löwner matrix* associated with the interpolation data (8) is defined by

$$\mathbb{L} := \left[\frac{v_i^* r_j - \ell_i^* w_j}{\mu_i - \lambda_j}\right]_{i=1,\dots,k_1;j=1,\dots,k_2} \,. \tag{16}$$

The *shifted Löwner matrix* is defined by

$$\sigma\mathbb{L} := \left[\frac{\mu_i v_i^* r_j - \lambda_j \ell_i^* w_j}{\mu_i - \lambda_j}\right]_{i=1,\dots,k_1;j=1,\dots,k_2} \,. \tag{17}$$

The first result of this paper connects the Löwner matrix and the two-variable polynomial matrix $\Psi(\zeta,\eta)$ in (6), and is the fundamental connection between the two approaches.

Proposition 4. *Let $\Psi(\zeta,\eta) \in \mathbb{R}^{p\times m}[\zeta,\eta]$ be defined by (6), with M and R defined by (10) and (11), and s_i and p_j defined as in (15). Then*

$$\mathbb{L} = -\left[s_i^* \Psi(-\mu_i,\lambda_j) p_j\right]_{i=1,\dots,k_1;j=1,\dots,k_2} \,. \tag{18}$$

Proof. It follows from the equations (15) that if $H \in \mathbb{R}^{p\times m}(\xi)$ satisfies the interpolation constraints, then the Löwner matrix (16) can also be written as

$$\mathbb{L} = \left[\frac{s_i^* \begin{bmatrix} Q(\mu_i) & -P(\mu_i) \end{bmatrix} \begin{bmatrix} D(\lambda_j) \\ N(\lambda_j) \end{bmatrix} p_j}{\mu_i - \lambda_j} \right]_{i=1,\dots,k_1, j=1,\dots,k_2} , \tag{19}$$

where s_i and p_j are defined by (15). The claim follows easily from this equation and the definition of $\Psi(\zeta,\eta)$. □

If all $-\mu_i$ and λ_i are all on one and the same side of the imaginary axis (e.g. the left-hand side) then the two-variable polynomial (22) is associated with a BDF, and the Löwer matrix has the interpretation of a Gramian, as illustrated in the following result.

Proposition 5. *Partition the variables in* $\mathfrak{B}$*, respectively* $\mathfrak{B}^\perp$ *by* $w' := \begin{bmatrix} y' \\ u' \end{bmatrix} \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^{m+p})$*, respectively* $w := \begin{bmatrix} u \\ y \end{bmatrix} \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^{m+p})$*. Assume that* $\lambda_i, -\mu_j \in \mathbb{C}_-$*,* $i = 1,\dots,k_1,\ j = 1,\dots,k_2$*.*

Define the bilinear form $\langle , \rangle$ *on* $\mathfrak{B}' \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^q) \times \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^q)$ *by*

$$\langle w', w \rangle := \int_0^{+\infty} y'^* u + u'^* y \; dt \; .$$

Then

$$\mathbb{L}_{i,j} = \langle w_i', w_j \rangle \; ,$$

where w_i'*,* w_j *are defined by (14).*

Proof. The claim follows integrating $w_i'^\top w_j$ on the half-line. □

The equality (18) is instrumental in obtaining the following result, analogous to Lemma 2.1 in [17].

Proposition 6. *Denote by* n *the McMillan degree of* $\mathfrak{B}$*. If* $k_1, k_2 \geq n$*, then* $\operatorname{rank} \mathbb{L} = n$*.*

Proof. Using the factorization (7) of $\Psi(\zeta,\eta)$, conclude that $\mathbb{L} = -S^* P$, where S and P are defined by

$$\begin{aligned} S &:= \begin{bmatrix} Z(-\mu_1^*) s_1 \; \dots \; Z(-\mu_{k_1}^*) s_{k_1} \end{bmatrix} \in \mathbb{C}^{n \times k_1} \\ P &:= \begin{bmatrix} X(\lambda_1) p_1 \; \dots \; X(\lambda_{k_2}) p_{k_2} \end{bmatrix} \in \mathbb{C}^{n \times k_2} \; . \end{aligned}$$

We now prove that under the assumption that the λ_is are distinct, the matrix P has full row rank n; a similar argument yields the same property for S.

Assume by contradiction that $\operatorname{rank}(P) = r < n$; then there exist $\alpha_i \in \mathbb{C}$, $i = 1,\dots,k_2$, not all zero, such that $P\operatorname{col}(\alpha_i)_{i=1,\dots,k_2} = 0$. Let $F \in \mathbb{R}^{m \times m}[\xi]$ be such that $\ker \left(F\left(\frac{d}{dt}\right)\right)$ equals the subspace of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^m)$ spanned by $v_i e^{\lambda_i \cdot}$, $i = 1,\dots,k_2$; such F always exists (see section XV of [32]). Now consider the following equations:

$$\begin{aligned} w &= M\left(\frac{d}{dt}\right)\ell \\ x &= X\left(\frac{d}{dt}\right)\ell \\ 0 &= F\left(\frac{d}{dt}\right)\ell \,. \end{aligned} \tag{20}$$

The external behavior $\mathfrak{B}' \subset \mathfrak{B}$ described by these equations is autonomous (see [19]), of dimension k_2. Moreover $X\left(\frac{d}{dt}\right)$ is a state map for $\mathfrak{B}'$, since it is a state map for $\mathfrak{B}$. Consider the trajectory $\hat{\ell}$ defined by $\hat{\ell}(t) := \sum_{i=1}^{N} \alpha_i p_i e^{\lambda_i t}$, and let $\ell = \hat{\ell}$ in (20); then the value of $\hat{x} := X\left(\frac{d}{dt}\right)\hat{\ell}$ at $t = 0$ is zero. Since $\mathfrak{B}'$ is autonomous, it follows that $\hat{w} := M\left(\frac{d}{dt}\right)\hat{\ell}$ is also zero. From the observability of M it follows then that $\hat{\ell} = 0$, which is in contradiction with the assumption that not all α_i's are equal to zero. Consequently P has rank n. $\square$

Another result well-known in the Löwner framework (see the first formula in (12) p. 640 of [17]) follows in a straightforward way from (18) and Prop. 3.

Proposition 7. *Define the matrices*

$$\begin{aligned} M &:= \mathrm{diag}(-\mu_i)_{i=1,\dots,k_1} \\ \Lambda &:= \mathrm{diag}\,(\lambda_j)_{j=1,\dots,k_2} \\ S &:= \left[s_i^* \left[Q(\mu_i)\ -P(\mu_i)\right]\right]_{i=1,\dots,k_1} \in \mathbb{C}^{k_1 \times q} \\ W &:= \left[\begin{bmatrix} D(\lambda_j) \\ N(\lambda_j) \end{bmatrix} p_j\right]_{j=1,\dots,k_2} \in \mathbb{C}^{q \times k_2}. \end{aligned}$$

$\mathbb{L}$ *satisfies the Sylvester equation*

$$M\mathbb{L} + \mathbb{L}\Lambda = -S^* W \,. \tag{21}$$

Proof. Observe that

$$\begin{aligned} Q(-\zeta)^\top D(\eta) - P(-\zeta)^\top N(\eta) = \quad & \zeta\, \frac{Q(-\zeta)^\top D(\eta) - P(-\zeta)^\top N(\eta)}{\zeta + \eta} \\ & + \eta\, \frac{Q(-\zeta)^\top D(\eta) - P(-\zeta)^\top N(\eta)}{\zeta + \eta} \,. \end{aligned}$$

The claim follows in a straightforward way substituting ζ with $-\mu_i^*$, η with λ_j, and multiplying on the left by s_i^* and on the right by p_j. $\square$

Remark 1. In the special case of lossless- and self-adjoint port-Hamiltonian systems, the results of Prop.s 6 and 7 coincide with results obtained in the B/QDF approach in [25]. Note that Prop. 4, on which the Löwner approach is fundamentally based, is valid for any linear differential system, while the results illustrated in Rem. 1 and **??** are valid only under the assumption of conservativeness or self-adjointness.

The transfer function $H(s) \in \mathbb{R}^{m\times m}[s]$ of a *lossless port-Hamiltonian system* (see [25, 22] for the definition) satisfies the equality $-H(-s)^\top = H(s)$. From such property, using the right- and left-coprime factorisations already introduced we conclude that given the image representation M, a kernel representation is

$$R(s) = M(-s)^\top \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} N(-s)^\top & D(-s)^\top \end{bmatrix} .$$

Thus for this class of systems the two-variable polynomial matrix $\Psi(\zeta,\eta)$ defined in Prop. 3 is

$$\Psi(\zeta,\eta) = \frac{\begin{bmatrix} N(\zeta)^\top & D(\zeta)^\top \end{bmatrix} \begin{bmatrix} D(\eta) \\ N(\eta) \end{bmatrix}}{\zeta+\eta} .$$

If we consider *symmetric* data, i.e. $k_1 = k_2$, $\mu_i = \lambda_i$ and $s_i = p_i$, $i = 1,\dots,k_1$, then it is a matter of straightforward verification to check that the Löwner matrix (16) coincides with the *Pick* matrix defined in formula (1) in [25]. Moreover, if the frequencies μ_i and λ_j lie all on one and the same side of the complex plane, the Pick (i.e. Löwner) matrix has a straightforward interpretation as a Gramian for the trajectories in the indefinite inner product on the half real line induced by

$$J := \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} ,$$

see formulas (8) and (11) of [25].

Under the assumptions mentioned above, the rank result of Prop. 6 of this paper coincides with the result of Prop. 1 of [25], and the Sylvester equation result of Prop. 7 coincides with that of Prop. 2 of [25].

The transfer function $H(s) \in \mathbb{R}^{m\times m}[s]$ of a *self-adjoint port-Hamiltonian system* (see [25] for the definition) satisfies the equality $H(s)^\top = H(s)$, from which using the right- and left-coprime factorisations already introduced we conclude that given an image representation M, a kernel representation is

$$R(s) = M(s)^\top \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} = \begin{bmatrix} N(s)^\top & -D(s)^\top \end{bmatrix} .$$

Thus for this class of systems the two-variable polynomial matrix $\Psi(\zeta,\eta)$ defined in Prop. 3 is

$$\Psi(\zeta,\eta) = \frac{\begin{bmatrix} N(-\zeta)^\top & -D(-\zeta)^\top \end{bmatrix} \begin{bmatrix} D(\eta) \\ N(\eta) \end{bmatrix}}{\zeta+\eta} .$$

If we consider *symmetric* data, i.e. $k_1 = k_2$, $\mu_i = \lambda_i$ and $s_i = p_i$, $i = 1,\dots,k_1$, and if the frequencies λ_i lie all on the right- or left-half plane, then the Löwner matrix (16) coincides with the *Pick* matrix of formula (34) in [25]. In this case, the Löwner matrix has an interpretation as Gramian for the indefinite inner product on the half real line induced by

$$J' := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} .$$

Results analogous to Prop. 6 and Prop. 7 of this paper appear as Prop. 6 and Prop. 7, respectively, in [25]. □

Remark 2. In this paper we restrict ourselves to the problem of modelling continuous-time trajectories. Gramian-based ideas for the identification of state-space systems in the discrete-time case under the assumption of losslessness have been illustrated in [23]. □

The shifted Loewner matrix (17) can be associated with a two-variable polynomial matrix in the following way. From the right- and left-coprime factorizations of H define

$$\Psi'(\zeta,\eta) := \frac{\zeta Q(-\zeta)^\top D(\eta) + P(-\zeta)^\top N(\eta)\eta}{\zeta+\eta} ; \tag{22}$$

note that $\Psi'(\zeta,\eta)$ is a polynomial matrix, since substituting $-\xi$ in place of ζ and ξ in place of η in $\zeta Q(-\zeta)^\top D(\eta) + P(-\zeta)^\top N(\eta)\eta$ yields the zero matrix. The following result follows in a straightforward way from (22).

Proposition 8. *Let $\Psi' \in \mathbb{R}^{k_1\times k_2}[\zeta,\eta]$ be defined by (22). Then*

$$\sigma\mathbb{L} = -\left[s_i^* \Psi'(-\mu_i,\lambda_j) p_j\right]_{i=1,\dots,k_1; j=1,\dots,k_2} .$$

If the frequencies λ_i, $-\mu_i$ are all on one and the same side of the imaginary axis (e.g. the left-hand side) then the two-variable polynomial (22) is associated with the following BDF, and the Löwner matrix has the interpretation of a Gramian, as illustrated in the following result. .

Proposition 9. *Assume that $\lambda_i, -\mu_i \in \mathbb{C}_-$ and partition w' and w as in Prop. 5. Define the following BDF on $\mathfrak{B}' \times \mathfrak{B}$:*

$$\langle\langle w', w\rangle\rangle := \int_0^{+\infty} \left(\frac{d}{dt}y'\right)^\top u + u'^\top \left(\frac{d}{dt}y\right) \, dt ;$$

then

$$\sigma\mathbb{L}_{i,j} = \langle\langle w_i', w_j\rangle\rangle ,$$

where w_i', w_j are defined in (14).

Proof. The claim follows integrating $\left(\frac{d}{dt}v_i\right)^\top r_j + \ell_i'^\top \left(\frac{d}{dt}w_j\right)$ on the half-line. □

Another dynamical interpretation of the shifted Löwner matrix can be given as follows: associate to the behavior $\mathfrak{B}$ defined in (12) the behavior

$$\mathfrak{B}' := \left\{ \mathrm{col}(y', u') \mid \exists\, \mathrm{col}(y,u) \in \mathfrak{B} \text{ s.t. } y' := \frac{d}{dt}y, u' = u \right\} . \tag{23}$$

To each trajectory (14) in $\mathfrak{B}$, $\mathfrak{B}^\perp$ one can associate a corresponding trajectory in $\mathfrak{B}'$ by "differentiating the output variable". It is straightforward to see that the shifted

Löwner matrix is the Löwner matrix of such new set of interpolation data, or equivalently, the Löwner matrix associated with the transfer function $sH(s)$. Now following an argument analogous to that used in proving Prop. 7, one can prove that $\sigma\mathbb{L}$ satisfies the following Sylvester equation:

$$M\sigma\mathbb{L}+\sigma\mathbb{L}\Lambda=-S'P' ,$$

where M, L are as in Prop. 7 and

$$\begin{aligned} S' &:= \left[s_i^* \left[Q(\mu_i)\mu_i \ -P(\mu_i)\right]\right]_{i=1,\dots,k_1} \in \mathbb{C}^{k\times(l+g)} \\ P' &:= \left[\begin{bmatrix} D(\lambda_j) \\ \lambda_j N(\lambda_j)\end{bmatrix} p_j\right]_{j=1,\dots,k_2} \in \mathbb{C}^{(l+g)\times q} . \end{aligned}$$

This is the counterpart of the second formula in (12) p. 640 of [17].

4 Computation of interpolants

Generalized state-space formulas of interpolants based on the Löwner matrix and the shifted Löwner matrix are given in Lemma 5.1 p. 643 of [17]. The dimension of the generalized state variable equals the number of right-interpolation data, and thus in general this procedure does not produce a minimal order interpolant; on the other hand, the interpolant is constructed directly from the Löwner and shifted Löwner matrices, without need of further computations. In section 5.2 of [17] formulas for a minimal order interpolant are obtained in terms of the *short singular value decomposition* of the matrix $\nu\mathbb{L}-\sigma\mathbb{L}$, where $\nu\in\{\mu_j\}\cup\{\lambda_i\}$, under the assumption (20) on p. 645 *ibid*. In this section we show how analogous results can be derived in the B/QDF approach; we examine separately the mono-directional interpolation problem (where only the right- or left interpolation constraints need to be satisfied) and the bi-directional one.

Given a matrix $S\in\mathbb{R}^{k_1\times k_2}$, a *rank-revealing factorization* of S is any factorization $S=U_1U_2$ with $U\in\mathbb{R}^{k_1\times n}$, $U_2\in\mathbb{R}^{n\times k_2}$ of full rank $n=\text{rank } S$; such a factorization can be computed in a straightforward way from a singular value decomposition of S. The results presented in this section are based on the following fundamental result connecting rank-revealing factorizations of the Löwner matrix and state trajectories corresponding to the vector-exponential ones (14) in the external variables of the primal- and the dual system.

Proposition 10. *Let $\mathbb{L}=Z^*V$ be any rank-revealing factorization of the Löwner matrix associated with the data (8); denote by V_i, respectively Z_i, the i-th column of V, respectively Z.*

There exists a minimal state representation (3) of $\mathfrak{B}$, respectively $\mathfrak{B}^\perp$, such that $V_ie^{\lambda_i\cdot}$, respectively $Z_ie^{-\mu_i\cdot}$, are minimal state trajectories of $\mathfrak{B}$, respectively $\mathfrak{B}^\perp$.

Proof. The claim follows straightforwardly from Prop. 3 and Prop. 4. □

Different rank-revealing factorizations of $\mathbb{L}$ yield different state trajectories and thus different realizations; see [24] for an application to the computation of canonical realizations.

4.1 Mono-directional interpolants and factorizations of the Löwner matrix

We first show that under suitable assumptions on the number of interpolation data, a minimal state representation (3) of an interpolant of the *right* interpolation data can be computed from a rank-revealing factorization of $\mathbb{L}$.

Proposition 11. *Assume $k_1, k_2 \geq n = \mathrm{rank}(\mathbb{L})$, and let $\mathbb{L} = Z^* V$ be a rank-revealing factorization with $Z \in \mathbb{C}^{n\times k_1}$ and $V \in \mathbb{C}^{n\times k_2}$. Define*

$$\begin{aligned} M &:= \mathrm{diag}(-\mu_i)_{i=1,\dots,k_1} \in \mathbb{C}^{k_1\times k_1} \\ S &:= \left[s_i^* \left[Q(\mu_i) \ -P(\mu_i) \right] \right]_{i=1,\dots,k_1} \in \mathbb{C}^{k_1\times q} \end{aligned}$$

Then a minimal state-representation (3) of a right-interpolant for the data $\left(\lambda_i, \begin{bmatrix} r_i \\ w_i \end{bmatrix}\right)$, $i = 1,\dots,k_2$ is

$$Z^* \frac{d}{dt} x + (MZ^*) x + Sw = 0 \; . \tag{24}$$

Proof. We prove that the external behavior of (24) contains the trajectories $\begin{bmatrix} r_i \\ w_i \end{bmatrix} e^{\lambda_i \cdot}$, $i = 1,\dots,k_2$, i.e. that there exist trajectories x_i, $i = 1,\dots,k_2$ such that (24) is satisfied. Denote by v_i the i-th column of the matrix V of the rank-revealing factorization of $\mathbb{L}$, and define $x_i(\cdot) := v_i e^{\lambda_i \cdot}$, $i = 1,\dots,k_2$. It follows from Prop. 10 and the Sylvester equation (21) that with such positions (24) is satisfied. □

Remark 3. Formula (24) is similar to formula (15) p. 642 of [17], which gives a input-state-output representation of an interpolant of McMillan degree k_1. Note however that the McMillan degree of (24) equals $\mathrm{rank}(\mathbb{L})$.

Remark 4. Prop. 11 implies that the rational matrix $-(sZ^* + MZ^*)^{-1} S$ satisfies the equations

$$(\lambda_i Z^* + MZ^*)^{-1} S \begin{bmatrix} r_i \\ w_i \end{bmatrix} = v_i \; , \; i = 1,\dots,k_2 \; ,$$

where v_i is the i-th column of the matrix V associated with the rank-revealing factorization of $\mathbb{L}$. Thus the matrix V plays a role analogous to that of the generalized tangential controllability matrix of p. 639 of [17]. □

Remark 5. When minimal, respectively observable, kernel and image representations of $\mathfrak{B}$ are known, a state representation (3) of $\mathfrak{B}$ can be obtained directly from the coefficient matrices of $Z(\xi)$ and $X(\xi)$ in (7), see sect. 2.5 of [29]. □

In order to find an input-state-output (iso) representation

$$\begin{aligned} E\frac{d}{dt}x &= Ax+Bu \\ y &= Cx+Du \end{aligned} \tag{25}$$

of an interpolant, assume $k_1, k_2 \geq n = \mathrm{rank}(\mathbb{L})$, and compute a rank-revealing factorization $\mathbb{L} = Z^* V$. Define

$$\begin{aligned} U &:= \begin{bmatrix} r_1 & \dots & r_{k_1} \end{bmatrix} \in \mathbb{C}^{m\times k_1} \\ Y &:= \begin{bmatrix} w_1 & \dots & w_{k_1} \end{bmatrix} \in \mathbb{C}^{p\times k_1} \ . \end{aligned}$$

The following result, whose proof is straightforward and hence omitted, characterizes iso representations of right interpolants.

Proposition 12. *A quintuple* $(E,A,B,C,D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{m\times m}$ *defines an iso representation of a right interpolant if and only if*

$$\begin{bmatrix} E & -A & -B & 0_{n\times p} \\ 0 & C & D & -I_p \end{bmatrix} \begin{bmatrix} V\Lambda \\ V \\ U \\ Y \end{bmatrix} = 0\ . \tag{26}$$

It follows from Prop. 25 that in order to find an iso representation of a right interpolant it suffices to find a matrix whose rows form a basis for the space orthogonal to im $\begin{bmatrix} V\Lambda \\ V \\ U \\ Y \end{bmatrix}$, and with the special structure

$$\begin{bmatrix} E & -A & -B & 0_{n\times p} \\ 0 & C & D & -I_p \end{bmatrix}\ .$$

This can be achieved with standard linear algebra computations; we will not deal with such details here.

Remark 6. In Prop. 4 and section VI of [25] explicit formulas in terms of the matrices arising from a rank-revealing factorization of $\mathbb{L}$ are given for computing A, B, C, D of an input-state-output representation

$$\begin{aligned} \frac{d}{dt}x &= Ax+Bu \\ y &= Cx+Du \end{aligned}$$

of a right interpolant for data generated by conservative- and adjoint port-Hamiltonian systems (see Remark 1 of this paper). Moreover, a parametrization for all such interpolants is also given. $\square$

Remark 7. Following an argument analogous to that used in proving Prop. 11 it can be shown that a state representation (3) of an interpolant for the *left* interpolation data can be computed defining $E := V^*$, $F := V^*\mathrm{diag}(\lambda_i)$, $G := W^*$. Moreover, a result analogous to that of Prop. 12 holds true also for left-interpolants; we will not state it explicitly. □

4.2 Bi-directional interpolation and BDFs

In Th. 5.1 of [17] formulas are given for the matrices E, A, B and C of an iso representation (25) of a left- and right-interpolant. In the following we show that these can be given an interpretation in terms of BDFs, and in case the interpolation points are all on the same side of the imaginary axis, in terms of factorization of the Löwner and shifted Löwner matrix.

In the following, besides the iso representation (25) we consider its *dual* (note that the terminology "dual" is not uniform in the literature; on this issue see also [8, 10, 11]), defined by

$$\begin{aligned} E^\top \frac{d}{dt} z &= -A^\top z - C^\top u' \\ y' &= -B^\top z\,, \end{aligned} \tag{27}$$

where $z \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^n)$, $u' \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^p)$, $y' \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^m)$.

The following two results are crucial for computing E and A from factorizations of the Löwner matrices.

Proposition 13. *Let* $\mathrm{col}(x,u,y)$ *and* $\mathrm{col}(z,u',y')$ *be full trajectories of the behaviors described by (25) and (27), respectively. Then*

$$\frac{d}{dt}\left(z^\top E x\right) = -u'^\top y - y'^\top u = -\begin{bmatrix} u'^\top & y'^\top \end{bmatrix} \begin{bmatrix} 0 & I_p \\ I_m & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} . \tag{28}$$

Proof. The claim follows from the following chain of equalities:

$$\begin{aligned} \frac{d}{dt}\left(z^\top E x\right) &= \left(\frac{d}{dt} z^\top\right) E x + z^\top E \left(\frac{d}{dt} x\right) = \left(-z^\top A - u'^\top C\right) x + z^\top \left(Ax + Bu\right) \\ &= -u'^\top y - y'^\top u\,. \end{aligned}$$

We now state another important result.

Proposition 14. *Let* $\mathrm{col}(x,u,y)$ *and* $\mathrm{col}(z,u',y')$ *be full trajectories of the behaviors described by (25) and (27), respectively. Then*

$$\frac{d}{dt}\left(z^\top A x\right) = -u'^\top \left(\frac{d}{dt} y\right) + \left(\frac{d}{dt} y'^\top\right) u\,. \tag{29}$$

Proof. The claim follows from the following chain of equalities:

$$\begin{aligned}
u'^{\top}\left(\frac{d}{dt}y\right)-\left(\frac{d}{dt}y'^{\top}\right)u &= u'^{\top}\left(C\frac{d}{dt}x\right)-\left(-\frac{d}{dt}z^{\top}B\right)u\\
&= \left(u'^{\top}C\right)\frac{d}{dt}x+\frac{d}{dt}z^{\top}\left(Bu\right)\\
&= -\left(\frac{d}{dt}z^{\top}E+z^{\top}A\right)\frac{d}{dt}x+\frac{d}{dt}z^{\top}\left(E\frac{d}{dt}x-Ax\right)\\
&= -\frac{d}{dt}\left(z^{\top}Ax\right)
\end{aligned}$$

The next result follows in a straightforward way from Prop. s 13 and 14 and reformulates (28) and (29) in two-variable polynomial terms.

Proposition 15. *Let $R \in \mathbb{R}^{p\times(p+m)}[\xi]$, respectively $M \in \mathbb{R}^{(p+m)\times m}[\xi]$ be a minimal kernel, respectively observable image representation of the external behavior $\mathfrak{B}$ of (25). Define*

$$\begin{aligned}
\Psi(\zeta,\eta) &:= R(-\zeta)M(\eta)\\
\Psi'(\zeta,\eta) &:= R(-\zeta)\begin{bmatrix}0 & -I_p\eta\\ \zeta I_m & 0\end{bmatrix}M(\eta)\ .
\end{aligned}$$

There exist state maps $X,Z \in \mathbb{R}^{\bullet\times m}[\xi]$ for $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$, respectively, such that

$$\begin{aligned}
\Psi(\zeta,\eta) &= (\zeta+\eta)Z(\zeta)^{\top}EX(\eta)\\
\Psi'(\zeta,\eta) &= (\zeta+\eta)Z(\zeta)^{\top}AX(\eta)\ .
\end{aligned} \tag{30}$$

The following is an important consequence of Prop.s 13, 14 and 15.

Proposition 16. *Let (25) be an iso representation of a bi-directional interpolant. There exist $X',X \in \mathbb{C}^{n\times k}$ such that*

$$\begin{aligned}
\mathbb{L} &= X'^{*}EX\\
\mathbb{L}_s &= X'^{*}AX\ .
\end{aligned} \tag{31}$$

Moreover, the columns of X', respectively X correspond to the directions of (exponential) state trajectories of the dual, respectively primal system, corresponding to the external trajectories (14).

Proof. The claim follows by substituting μ_i in place of ζ and λ_i in place of η in (30), and multiplying on the left by s_i^* and on the right by p_j. □

Remark 8. If $-\mu_i$ and λ_j lie on the same half-plane, the result of Prop. 16 can be proved integrating by parts (28) and (29) along the trajectories (14). □

To compute E and A from $\mathbb{L}$ and $\mathbb{L}_s$, respectively, observe that from (31) it follows that

$$\begin{aligned} \begin{bmatrix} \mathbb{L} & \mathbb{L}_s \end{bmatrix} &= X'^* \begin{bmatrix} EX & AX \end{bmatrix} \\ \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} &= \begin{bmatrix} X'^*E \\ X'^*A \end{bmatrix} X \ . \end{aligned} \tag{32}$$

These factorisations are the counterpart of those in formula (2.25) of [5], with $Y = X'^*$, $\Sigma_\ell \widetilde{X}^* = \begin{bmatrix} EX & AX \end{bmatrix}$ and $\widetilde{Y}\Sigma_r = \begin{bmatrix} X'^*E \\ X'^*A \end{bmatrix}$. A "short" SVD of the two matrices on the left-hand side of (32) yields matrices X'^* and X with orthonormal rows; under such assumption we recover E and A by projection of $\mathbb{L}$ and $\mathbb{L}_s$ as

$$\begin{aligned} E &= X'\mathbb{L}X^* \\ A &= X'\mathbb{L}_s X^* \ , \end{aligned}$$

respectively, see the first two formulas (22) p. 646 of [17].

The matrices B, C of a representation (25) can be obtained as follows. From the output equation $y' = -B^\top z$ of the dual system (27) it follows that $V = -B^\top X'$, where

$$V := \begin{bmatrix} \ell_1 & \dots & \ell_{k_1} \end{bmatrix} \in \mathbb{C}^{m \times k_1} \ .$$

Assuming that X' has been obtained via a short SVD, it follows that

$$B = -X'V^* \ .$$

This is the third equation in (2.28) p. 17 of [6]. Analogously, from the output equation $y = Cx$ of the primal system (25) it follows that $W = CX$, where

$$W := \begin{bmatrix} w_1 & \dots & w_{k_2} \end{bmatrix} \in \mathbb{C}^{m \times k_2} \ .$$

Consequently

$$C = WX^* \ ,$$

the fourth equation in (2.28) p. 17 of [6].

Remark 9. The BDFs used to compute E and A in Prop.s 13 and 14 are not the same; such difference goes against the interpretation of the shifted Löwner matrix as the Löwner matrix associated with the transfer function $sH(s)$. It is currently investigated whether such asymmetry depends on our possibly non-standard definition of the dual system (27), or whether there is an intrinsic motivation to it. $\square$

5 Conclusions

We have shown that several results in the Löwner framework for interpolation can be given a direct interpretation in the language of bilinear differential forms and their two-variable polynomial matrix representations. We have shed new light on known results in the Löwner framework (e.g. the rank result of Prop. 6, the Sylvester equation in Prop. 7), and we have also given insights of a more fundamental nature (e.g. the correspondence between state trajectories and factorizations in Prop. 10, the interpretation of the Löwner matrices as Gramians, see Prop.s 5 and 9).

For reasons of space we have refrained from illustrating the correspondences between the Löwner approach to model order reduction and that based on BDFs (see section 3 of [6], section V of [25]); this will be pursued elsewhere. Current research questions include the formulation of recursive interpolation in the BDF framework, and the extension to parametric interpolation and parametric model order reduction (see [12]).

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