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UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

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by

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MOVING GREEN'S FUNCTIONS FOR A LAYERED CIRCULAR CYLINDER OF INFINITE LENGTH

X. Sheng, C. J. C. Jones and D. J. Thompson

ABSTRACT

This paper derives the Fourier transformed steady state responses (Fourier transformed Green's functions) of an infinitely long cylinder subject to harmonic loads which may be stationary or moving in the cylinder-axis direction. The cylinder may have a layered structure in the radial direction. By letting the inner radius of a hollow cylinder approach zero and its outer radius approach infinity, the Fourier transformed moving Green's functions of a homogeneous and visco-elastic whole-space are found. They are expressed in terms of Bessel functions of the second kind. The Green's functions, either for a layered cylinder or for a homogeneous whole-space, are useful in a number of engineering applications such as ground vibration induced by underground trains and leakage detection of water, gas and oil pipes buried underground. The formulae and the computer program that implements them are validated first by means of the plane-strain Green's functions which correspond to the case of zero axial wavenumber. They are further tested by comparison with the exact solution for a point stationary harmonic load and the exact solution for a moving constant load for a homogeneous elastic whole-space.

CONTENTS

1. INTRODUCTION

With regard to the environmental impact of rail traffic, an important issue is the problem of ground vibrations produced by trains running in cut-and-cover or bored tunnels. The dominant frequencies associated with underground train-induced ground vibration are from about 15 Hz to 200 Hz [Grootenhuis, 1977]. Vibration in this range of frequency gives rise to structure borne or 'ground-borne' noise. To explore the mechanism of this problem, several modelling approaches have been proposed.

A two-dimensional (in the plane normal to the tunnel axis) FE model was reported [Balendra *et al* 1989, 1991, 1992] to investigate the steady-state vibration of tunnel-soil-building systems. In this model, an artificial boundary was introduced for the application of the finite element technology.

At ISVR, Southampton University, a two-dimensional (in the plane normal to the tunnel) BE/FE coupled model has been developed for a ground including lined or unlined tunnels [Jones, Thompson and Petyt 1999]. This model is capable of accounting for arbitrary shapes of the cross-section of the tunnel and lateral geometry of the ground, and automatically satisfies the condition that there is no wave reflected from infinity. However this model cannot account for the movement of trains or wave propagation in the tunnel direction.

At Cambridge University, a cylinder theory-based model has been produced [Forrest 1999]. In this work, the rails and track-slab are modelled as beams of infinite length, with elastic layers to represent the railpads and slab bearings. The ground and the tunnel are regarded as a viscoelastic whole-space including a circular thin shell of infinite length. Excitation is provided by roughness displacement inputs between the rails and a train of axle masses spaced regularly. The motion of the vehicles has not been taken into account, though the phase differences of the inputs at different wheel/rail contact points have been implemented in the formulation. The advantage of this model lies in that it is an analytical model so that it is of very high computational efficiency. However, when the tunnel does not possess a circular cross section, when the tunnel is not very far below the ground surface, or when the ground has a layered structure, the usefulness of this model will be limited.

At Delft University of Technology, Netherlands, a two-dimensional (*in the vertical plane along the tunnel axis*) analytical model has been suggested [Metrikine et al 2000]. This model is composed of two two-dimensional visco-elastic layers (representing the soil above and below the tunnel) and two identical Euler-Bernoulli beams (modelling the tunnel). The beams are connected by continuously distributed springs. This model accounts for the movement of wheel/rail forces; however, it omits the wave propagation in the lateral direction.

A model is required that can account not only for the wave propagation in the vertical plane normal to the tunnel axis, but also for the movement of trains in the tunnel and wave propagation in the tunnel direction. The latter is important since the inclusion of a tunnel in the ground provides a waveguide in that direction. A three-dimensional model is therefore needed and for some studies is essential. A three-dimensional FE/BE model has been implemented [Andersen and Jones 2001 a, b and 2002] but is very complicated to use and, because of the extremely large computing resources that are required, cannot be run many times in order to carry out an investigative study. The three-dimensional FE/BE model is useful for many foundation problems where the structure is of limited extent and in which the response of the ground is not required at great distances. For a tunnel, however, and for vibration response at distance on the ground surface, the model cannot be used for the whole frequency range of interest for structure-borne noise from tunnels. Even on advanced computing equipment the computation time is too long and the memory requirement is too large.

Having considered that, in general situations, the geometry of the ground-tunnel structure is invariant with respect to any translation along the tunnel axis, an approach was recommended by, for example, Aubry [Aubry et al 1994] in which the problem is transformed into a sequence of two dimensional models depending on wavenumbers in the tunnel direction. For each longitudinal (in the tunnel direction) wavenumber, the finite cross-section of the tunnel is modelled using the FE method and the wave propagation in the surrounding soil is modelled using the BE method. The simplicity of this approach comes from the fact that the discretization is only made over the cross-section of the ground-tunnel structure. It requires Green's functions, which are responses due to moving harmonic loads, either for a half-space or for a whole-space, expressed in terms of coordinates in the plane normal to the tunnel axis and the wavenumber in the tunnel direction. Such Green's functions for a layered half-space are available [Sheng et al 1999]; however they are expressed in terms of an infinite integral which can only be evaluated numerically. The Green's functions for a whole-space due to stationary harmonic loads, expressed in terms of the three Cartesian coordinates, were obtained a long time ago [see e.g. Miklowitz 1978]. Recently, Tadeu and Kausel [Tadeu and Kausel 2000] derived the Fourier transforms of these Green's functions with respect to one of the three coordinates, expressing them in terms of Hankel functions.

As a theoretical preparation for a thorough investigation into the tunnel-ground vibration problem, this paper deals with the steady-state responses of a solid or hollow circular cylinder to moving harmonic loads, or in other words, it aims at obtaining the moving Green's functions of such cylindrical structures. The cylinder is assumed to have a layered structure in the radial direction, i.e. it consists of a number, n, of concentric sub-cylinders. Starting from the most inner one, the sub-cylinders are numbered as sub-cylinder 1, sub-cylinder 2 etc. The geometry of the jth sub-cylinder is described by its inner radius, R_{j0}, and its outer radius, R_{j1}, where, $R_{j1} = R_{j+1,0}$. The density, Young's modulus, Possion ratio and loss factor of the material of the jth sub-cylinder are denoted by ρ_j, E_j, ν_j and η_j.

When $R_{10} = 0$, the first sub-cylinder is a solid one. The outer radius of the last sub-cylinder may be infinite. For a deeply buried circular tunnel, the tunnel is the first sub-cylinder and the surrounding soil may be regarded as the second sub-cylinder which has an infinite outer radius. A homogeneous elastic whole-space is a special cylinder the inner radius of which is zero and the outer radius of which is infinite.

Investigations have been performed for solid or hollow circular cylinders by many researchers. Flügge & Kelkar [1968] proposed a general method to solve the statics of a solid or hollow circular cylinder of any length, allowing arbitrary conditions prescribed on the curved and flat surfaces. Mirsky [1965] carried out a study on the wave propagation of free harmonic waves in hollow and solid circular cylinders of a transversely anisotropic material, following the approach that Gazis [1959] used for an isotropic material.

As shown in Figure 1, the x-axis forms the axis of the cylinder. Two coordinate systems are used; one is the Cartesian coordinate (x, y, z) system and the other is the cylindrical coordinate (x, r, θ) system. The relation between them is

$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases} \tag{1}$$

Three harmonic loads of angular frequency Ω are applied at the interface of the ($l_P - 1$)th and l_Pth sub-cylinders in the axial, radial and circumferential directions. These loads move at speed c in the axial, i.e., the x-direction. These harmonic loads may be distributed in the axial direction but concentrated in the other two directions, thus they are denoted by $p_x(x)e^{i\Omega t}$, $p_r(x)e^{i\Omega t}$ and $p_\theta(x)e^{i\Omega t}$. Their action point at each cross-section of the cylinder is determined by its circumferential coordinate, $\theta = \theta_0$, and its radial coordinate, $r = R_{l_P 0}$. The displacement vector of the cylinder is decomposed into three components, either as $w_x(x,y,z,t), w_y(x,y,z,t)$ and $w_z(x,y,z,t)$ in the Cartesian co-ordinate system, or as $w_x(x,r,\theta,t), w_r(x,r,\theta,t)$ and $w_\theta(x,r,\theta,t)$ in the cylindrical co-ordinate system (see Figure 1).

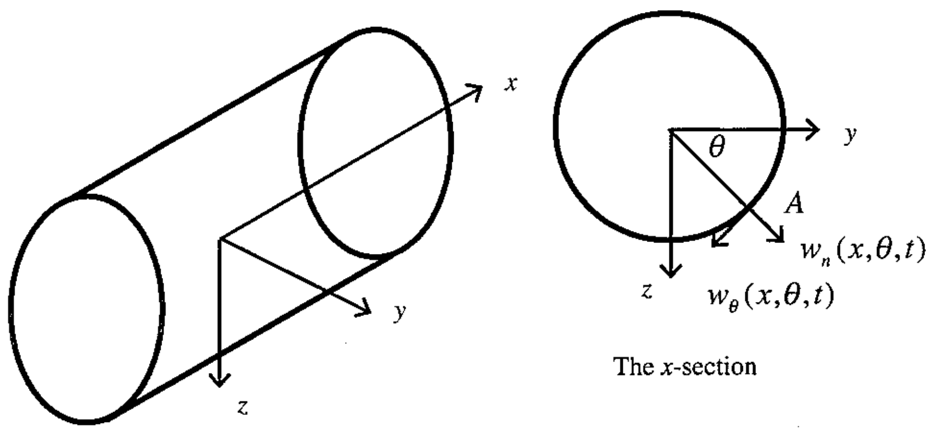

Figure 1. Coordinate systems.

2. SOLUTIONS FOR A SINGLE HOMOGENEOUS CYLINDER

2.1 FORMULAE FOR THE DILATATIONAL AND EQUIVOLUMINAL POTENTIALS

The motion of each sub-cylinder is governed by the free elastodynamics equation (the subscript indicating the sub-cylinder has been omitted)

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{2}$$

where, $\mathbf{u}$ is the displacement vector of the sub-cylinder, μ and λ are Lamé constants determined by

$$\lambda = \frac{\nu E(1+i\eta \,\mathrm{sgn}(\omega))}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E(1+i\eta \,\mathrm{sgn}(\omega))}{2(1+\nu)} \tag{3}$$

To solve equation (2), the displacement vector $\mathbf{u}$ is decomposed in terms of a dilatational scalar potential ϕ and an equivoluminal vector potential $\mathbf{H}$, i.e.

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{H} \tag{4}$$

Equation (2) is satisfied if the potentials ϕ and $\mathbf{H}$ satisfy the wave equations

$$c_1^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} \tag{5}$$

$$c_2^2 \nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2} \tag{6}$$

where,

$$c_1 = \sqrt{(\lambda + 2\mu)/\rho}, \; c_2 = \sqrt{\mu/\rho} \tag{7}$$

are the dilatational wave speed and the shear wave speed of the material.

To solve equations (5) and (6), the Fourier transforms

$$\bar{\phi}(\beta, y, z, t) = \int_{-\infty}^{\infty} \phi(x, y, z, t) e^{-i\beta x} dx, \; \overline{\mathbf{H}}(\beta, y, z, t) = \int_{-\infty}^{\infty} \mathbf{H}(x, y, z, t) e^{-i\beta x} dx \tag{8}$$

are performed, where β is the wavenumber in the x-direction. By letting

$$\bar{\phi}(\beta, y, z, t) = \tilde{\phi}(\beta, y, z) e^{i(\Omega - \beta c)t}, \; \overline{\mathbf{H}}(\beta, y, z, t) = \tilde{\mathbf{H}}(\beta, y, z) e^{i(\Omega - \beta c)t} \tag{9}$$

equations (5) and (6) yield

$$\frac{\partial^2 \tilde{\phi}}{\partial y^2} + \frac{\partial^2 \tilde{\phi}}{\partial z^2} + \left(\frac{\omega^2}{c_1^2} - \beta^2\right)\tilde{\phi} = 0 \tag{10}$$

$$\frac{\partial^2 \tilde{\mathbf{H}}}{\partial y^2} + \frac{\partial^2 \tilde{\mathbf{H}}}{\partial z^2} + \left(\frac{\omega^2}{c_2^2} - \beta^2\right)\tilde{\mathbf{H}} = 0 \tag{11}$$

where,

$$\omega = \Omega - \beta c \tag{12}$$

Equations (10) and (11) are expressed in terms of the Cartesian coordinates. To express them in terms of the cylindrical coordinates, recall equation (1), then

$$\frac{\partial^2 \tilde{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\phi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{\phi}}{\partial \theta^2} + (\frac{\omega^2}{c_1^2} - \beta^2)\tilde{\phi} = 0 \tag{13}$$

$$\frac{\partial^2 \tilde{H}_x}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_x}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2)\tilde{H}_x = 0 \tag{14}$$

$$\frac{\partial^2}{\partial r^2}\begin{Bmatrix}\tilde{H}_y \\ \tilde{H}_z\end{Bmatrix} + \frac{1}{r}\frac{\partial}{\partial r}\begin{Bmatrix}\tilde{H}_y \\ \tilde{H}_z\end{Bmatrix} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\begin{Bmatrix}\tilde{H}_y \\ \tilde{H}_z\end{Bmatrix} + (\frac{\omega^2}{c_2^2} - \beta^2)\begin{Bmatrix}\tilde{H}_y \\ \tilde{H}_z\end{Bmatrix} = 0 \tag{15}$$

where, $\tilde{H}_x, \tilde{H}_y, \tilde{H}_z$ are the components of $\tilde{\mathbf{H}}$ in the *x*-, *y*- and *z*-directions. If $\tilde{\mathbf{H}}$ is decomposed into $\tilde{H}_x, \tilde{H}_r, \tilde{H}_\theta$, which are related to $\tilde{H}_x, \tilde{H}_y, \tilde{H}_z$ by

$$\tilde{H}_y = \tilde{H}_r \cos\theta - \tilde{H}_\theta \sin\theta,\ \tilde{H}_z = \tilde{H}_r \sin\theta + \tilde{H}_\theta \cos\theta \tag{16}$$

then equation (15) becomes

$$\begin{aligned}&[\frac{\partial^2 \tilde{H}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_r}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{1}{r^2})\tilde{H}_r - \frac{2}{r^2}\frac{\partial \tilde{H}_\theta}{\partial \theta}]\cos\theta - \\ &[\frac{\partial^2 \tilde{H}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_\theta}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{1}{r^2})\tilde{H}_\theta + \frac{2}{r^2}\frac{\partial \tilde{H}_r}{\partial \theta}]\sin\theta = 0\end{aligned} \tag{17}$$

$$\begin{aligned}&[\frac{\partial^2 \tilde{H}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_r}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{1}{r^2})\tilde{H}_r - \frac{2}{r^2}\frac{\partial \tilde{H}_\theta}{\partial \theta}]\sin\theta + \\ &[\frac{\partial^2 \tilde{H}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_\theta}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{1}{r^2})\tilde{H}_\theta + \frac{2}{r^2}\frac{\partial \tilde{H}_r}{\partial \theta}]\cos\theta = 0\end{aligned} \tag{18}$$

each of which implies that

$$\frac{\partial^2 \tilde{H}_r}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_r}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{1}{r^2})\tilde{H}_r - \frac{2}{r^2}\frac{\partial \tilde{H}_\theta}{\partial \theta} = 0 \tag{19}$$

and that

$$\frac{\partial^2 \tilde{H}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{H}_\theta}{\partial \theta^2} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{1}{r^2})\tilde{H}_\theta + \frac{2}{r^2}\frac{\partial \tilde{H}_r}{\partial \theta} = 0 \tag{20}$$

Since $\tilde{\phi}$ and $\tilde{\mathbf{H}}$ are periodic functions of θ with a period of 2π, they may be expressed in terms of the Fourier series

$$\tilde{\phi} = \sum_{k=-\infty}^{\infty} \tilde{\phi}_k(\beta, r)e^{ik\theta} \tag{21a}$$

$$\tilde{H}_x = \sum_{k=-\infty}^{\infty} \tilde{H}_{xk}(\beta, r)e^{ik\theta} \tag{21b}$$

$$\tilde{H}_r = \sum_{k=-\infty}^{\infty} \tilde{H}_{rk}(\beta, r)e^{ik\theta} \tag{21c}$$

$$\tilde{H}_\theta = \sum_{k=-\infty}^{\infty} \tilde{H}_{\theta k}(\beta, r)e^{ik\theta} \tag{21d}$$

Inserting equation (21) into equations (13), (14), (19) and (20), then multiplying the results by $e^{-im\theta}$, and finally integrating with respect to θ between 0 and 2π, yields

$$\frac{\partial^2 \tilde{\phi}_m}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\phi}_m}{\partial r} + (\frac{\omega^2}{c_1^2} - \beta^2 - \frac{m^2}{r^2})\tilde{\phi}_m = 0 \tag{22}$$

$$\frac{\partial^2 \tilde{H}_{xm}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_{xm}}{\partial r} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{m^2}{r^2})\tilde{H}_{xm} = 0 \tag{23}$$

$$\frac{\partial^2 \tilde{H}_{rm}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_{rm}}{\partial r} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{m^2+1}{r^2})\tilde{H}_{rm} - \frac{2im}{r^2}\tilde{H}_{\theta m} = 0 \tag{24}$$

$$\frac{\partial^2 \tilde{H}_{\theta m}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{H}_{\theta m}}{\partial r} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{m^2+1}{r^2})\tilde{H}_{\theta m} + \frac{2im}{r^2}\tilde{H}_{rm} = 0 \tag{25}$$

where, $m = -\infty, \cdots, -1, 0, 1, \cdots, \infty$. Equations (24) and (25) can be rearranged to give

$$\frac{\partial^2(\tilde{H}_{rm} + i\tilde{H}_{\theta m})}{\partial r^2} + \frac{1}{r}\frac{\partial(\tilde{H}_{rm} + i\tilde{H}_{\theta m})}{\partial r} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{m^2+2m+1}{r^2})(\tilde{H}_{rm} + i\tilde{H}_{\theta m}) = 0 \tag{26}$$

$$\frac{\partial^2(\tilde{H}_{rm} - i\tilde{H}_{\theta m})}{\partial r^2} + \frac{1}{r}\frac{\partial(\tilde{H}_{rm} - i\tilde{H}_{\theta m})}{\partial r} + (\frac{\omega^2}{c_2^2} - \beta^2 - \frac{m^2-2m+1}{r^2})(\tilde{H}_{rm} - i\tilde{H}_{\theta m}) = 0 \tag{27}$$

Equations (22), (23), (26) and (27) are modified Bessel differential equations of order m, m, $m+1$ and $m-1$, respectively, if $p_1 \neq 0$ and $p_2 \neq 0$, where,

$$p_1 = \sqrt{\beta^2 - \frac{\omega^2}{c_1^2}}, \quad p_2 = \sqrt{\beta^2 - \frac{\omega^2}{c_2^2}} \tag{28}$$

Thus, the general solutions to those four differential equations may be written as

$$\tilde{\phi}_m(r) = C_1 W_m(p_1 r) + C_2 Z_m(p_1 r) \tag{29}$$

$$\tilde{H}_{xm}(r) = C_3 W_m(p_2 r) + C_4 Z_m(p_2 r) \tag{30}$$

$$\tilde{H}_{rm}(r) + i\tilde{H}_{\theta m} = 2C_5 W_{m+1}(p_2 r) + 2C_6 Z_{m+1}(p_2 r) \tag{31}$$

$$\tilde{H}_{rm}(r) - i\tilde{H}_{\theta m} = 2C_7 W_{m-1}(p_2 r) + 2C_8 Z_{m-1}(p_2 r) \tag{32}$$

where, $C_1, \cdots C_8$ are constants of integration, and

$$W_m(pr) = \begin{cases} I_m(pr), \text{ if } p \neq 0; \\ r^{|m|}, \text{ if } p = 0, m \neq 0; \\ 1, \text{ if } p = 0, m = 0. \end{cases} \tag{33}$$

$$Z_m(pr) = \begin{cases} K_m(pr), \text{ if } p \neq 0; \\ r^{-|m|}, \text{ if } p = 0, m \neq 0; \\ \ln r, \text{ if } p = 0, m = 0. \end{cases} \tag{34}$$

I_m, K_m denote the modified Bessel functions of order m of the first kind and the second kind with complex argument.

Equations (31) and (32) give

$$\tilde{H}_{rm}(r) = C_5 W_{m+1}(p_2 r) + C_6 Z_{m+1}(p_2 r) + C_7 W_{m-1}(p_2 r) + C_8 Z_{m-1}(p_2 r) \tag{35}$$

$$\tilde{H}_{\theta m}(r) = -iC_5 W_{m+1}(p_2 r) - iC_6 Z_{m+1}(p_2 r) + iC_7 W_{m-1}(p_2 r) + iC_8 Z_{m-1}(p_2 r) \tag{36}$$

Now there are eight integration constants. Since the elastodynamics equation of motion is of order 6, only six integration constants can be determined by boundary conditions. This indicates that ϕ and $\mathbf{H}$ should be subjected to additional constraints. This condition is normally taken as [Miklowitz 1978]

$$\nabla \cdot \mathbf{H} = 0 \tag{37}$$

which is consistent with the Helmholtz decomposition of a vector. From equation (37),

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_r}{\partial r} + \frac{H_r}{r} + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} = 0 \tag{38}$$

Now inserting equations (8), (9), (21) into equation (38) yields

$$i\beta \tilde{H}_{xm} + \frac{\partial \tilde{H}_{rm}}{\partial r} + \frac{\tilde{H}_{rm}}{r} + \frac{im}{r}\tilde{H}_{\theta m} = 0 \tag{39}$$

which, combined with equations (30), (31) and (32), becomes

$$\begin{aligned} & i\beta[C_3 W_m(p_2 r) + C_4 Z_m(p_2 r)] + C_5[\tfrac{m+1}{r} W_{m+1}(p_2 r) + \frac{\partial W_{m+1}(p_2 r)}{\partial r}] + \\ & C_6[\tfrac{m+1}{r} Z_{m+1}(p_2 r) + \frac{\partial Z_{m+1}(p_2 r)}{\partial r}] + C_7[-\tfrac{m-1}{r} W_{m-1}(p_2 r) + \frac{\partial W_{m-1}(p_2 r)}{\partial r}] + \\ & C_8[-\tfrac{m-1}{r} Z_{m-1}(p_2 r) + \frac{\partial Z_{m-1}(p_2 r)}{\partial r}] = 0 \end{aligned} \tag{40}$$

For $p_2 \neq 0$ (for damped material, p_1 and p_2 are always non-zero), based on the properties of the modified Bessel functions [Abramowitz and Stegun 1964],

$$\frac{1+m}{r} W_{m+1}(p_2 r) + \frac{\partial W_{m+1}(p_2 r)}{\partial r} = p_2 W_m(p_2 r) \tag{41}$$

$$-\frac{m-1}{r} W_{m-1}(p_2 r) + \frac{\partial W_{m-1}(p_2 r)}{\partial r} = p_2 W_m(p_2 r) \tag{42}$$

$$\frac{1+m}{r} Z_{m+1}(p_2 r) + \frac{\partial Z_{m+1}(p_2 r)}{\partial r} = -p_2 Z_m(p_2 r) \tag{43}$$

$$-\frac{m-1}{r} Z_{m-1}(p_2 r) + \frac{\partial Z_{m-1}(p_2 r)}{\partial r} = -p_2 Z_m(p_2 r) \tag{44}$$

thus equation (40) becomes

$$\begin{aligned} & i\beta[C_3 W_m(p_2 r) + C_4 Z_m(p_2 r)] + C_5 p_2 W_m(p_2 r) \\ & - C_6 p_2 Z_m(p_2 r) + C_7 p_2 W_m(p_2 r) - C_8 p_2 Z_m(p_2 r) = 0 \end{aligned} \tag{45}$$

Since this must hold for any r within the sub-cylinder, and $W_m(p_2 r)$ and $Z_m(p_2 r)$ are independent of each other, equation (45) implies

$$i\beta C_3 + C_5 p_2 + C_7 p_2 = 0,\ i\beta C_4 - C_6 p_2 - C_8 p_2 = 0$$

from which C_7 and C_8 are determined as

$$C_7 = -(\frac{i\beta}{p_2} C_3 + C_5),\ C_8 = \frac{i\beta}{p_2} C_4 - C_6 \tag{46}$$

Thus

$$\tilde{\phi}_m(r) = C_1 W_m(p_1 r) + C_2 Z_m(p_1 r) \tag{47a}$$

$$\tilde{H}_{xm}(r) = C_3 W_m(p_2 r) + C_4 Z_m(p_2 r) \tag{47b}$$

$$\begin{aligned} \tilde{H}_{rm}(r) = &-\frac{i\beta}{p_2} C_3 W_{m-1}(p_2 r) + \frac{i\beta}{p_2} C_4 Z_{m-1}(p_2 r) \\ &+ C_5[W_{m+1}(p_2 r) - W_{m-1}(p_2 r)] + C_6[Z_{m+1}(p_2 r) - Z_{m-1}(p_2 r)] \end{aligned} \tag{47c}$$

$$\begin{aligned} \tilde{H}_{\theta m}(r) = &\frac{\beta}{p_2} C_3 W_{m-1}(p_2 r) - \frac{\beta}{p_2} C_4 Z_{m-1}(p_2 r) \\ &- iC_5[W_{m-1}(p_2 r) + W_{m+1}(p_2 r)] - iC_6[Z_{m-1}(p_2 r) + Z_{m+1}(p_2 r)] \end{aligned} \tag{47d}$$

2.2 FORMULAE FOR DISPLACEMENTS

Equation (4) gives

$$w_x = \frac{\partial \phi}{\partial x} + \frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r} - \frac{1}{r}\frac{\partial H_r}{\partial \theta} \tag{48a}$$

$$w_r = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial H_x}{\partial \theta} - \frac{\partial H_\theta}{\partial x} \tag{48b}$$

$$w_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial H_r}{\partial x} - \frac{\partial H_x}{\partial r} \tag{48c}$$

If the Fourier transforms of w_x, w_r, w_θ with respect to x, denoted by $\tilde{w}_x, \tilde{w}_r, \tilde{w}_\theta$, are expressed in the same way as for ϕ and $\mathbf{H}$, then equations (48) yield

$$\tilde{w}_x = i\beta\tilde{\phi} + \frac{\partial \tilde{H}_\theta}{\partial r} + \frac{\tilde{H}_\theta}{r} - \frac{1}{r}\frac{\partial \tilde{H}_r}{\partial \theta} \tag{49a}$$

$$\tilde{w}_r = \frac{\partial \tilde{\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tilde{H}_x}{\partial \theta} - i\beta \tilde{H}_\theta \tag{49b}$$

$$\tilde{w}_\theta = \frac{1}{r}\frac{\partial \tilde{\phi}}{\partial \theta} + i\beta \tilde{H}_r - \frac{\partial \tilde{H}_x}{\partial r} \tag{49c}$$

With the substitution of equations (21), equation (49) becomes

$$\tilde{w}_{xm} = i\beta \tilde{\phi}_m + \frac{\partial \tilde{H}_{\theta m}}{\partial r} + \frac{\tilde{H}_{\theta m}}{r} - \frac{im}{r}\tilde{H}_{rm} \tag{50a}$$

$$\tilde{w}_{rm} = \frac{\partial \tilde{\phi}_m}{\partial r} + \frac{im}{r}\tilde{H}_{xm} - i\beta \tilde{H}_{\theta m} \tag{50b}$$

$$\tilde{w}_{\theta m} = \frac{im}{r}\tilde{\phi}_m + i\beta \tilde{H}_{rm} - \frac{\partial \tilde{H}_{xm}}{\partial r} \tag{50c}$$

Inserting equation (47) into equation (50) yields

$$\begin{Bmatrix} \tilde{w}_{xm} \\ \tilde{w}_{rm} \\ \tilde{w}_{\theta m} \end{Bmatrix} = [C(r)]\{C\} \tag{51}$$

where, $\{C\} = (C_1, C_2, \cdots, C_6)^T$ denotes the vector of the integral constants, $[C(r)]$ is a 3×6 matrix, called *the displacement matrix of circumferential order m*. The elements $c_{kj}\ (k=1,2,3;\ j=1,2,\cdots,6)$ are as follows:

$$c_{11} = i\beta W_m(p_1 r),\quad c_{12} = i\beta Z_m(p_1 r),\quad c_{13} = \beta W_m(p_2 r),\quad c_{14} = \beta Z_m(p_2 r)$$

$$c_{15} = -2ip_2 W_m(p_2 r),\quad c_{16} = 2ip_2 Z_m(p_2 r)$$

$$c_{21} = \frac{\partial W_m(p_1 r)}{\partial r}, c_{22} = \frac{\partial Z_m(p_1 r)}{\partial r},\quad c_{23} = \frac{im}{r}W_m(p_2 r) - \frac{i\beta^2}{p_2}W_{m-1}(p_2 r)$$

$$c_{24} = \frac{im}{r}Z_m(p_2 r) + \frac{i\beta^2}{p_2}Z_{m-1}(p_2 r), c_{25} = -\beta[W_{m-1}(p_2 r) + W_{m+1}(p_2 r)]$$

$$c_{26} = -\beta[Z_{m-1}(p_2 r) + Z_{m+1}(p_2 r)]$$

$$c_{31} = \frac{im}{r}W_m(p_1 r),\quad c_{32} = \frac{im}{r}Z_m(p_1 r)$$

$$c_{33} = \frac{\beta^2}{p_2} W_{m-1}(p_2 r) - \frac{\partial W_m(p_2 r)}{\partial r}, \quad c_{34} = -\frac{\beta^2}{p_2} Z_{m-1}(p_2 r) - \frac{\partial Z_m(p_2 r)}{\partial r}$$

$$c_{35} = i\beta[W_{m+1}(p_2 r) - W_{m-1}(p_2 r)], \quad c_{36} = i\beta[Z_{m+1}(p_2 r) - Z_{m-1}(p_2 r)]$$

2.3 FORMULAE FOR STRESSES

Now the strain-displacement relations

$$\varepsilon_{xr} = \frac{1}{2}(\partial w_r / \partial x + \partial w_x / \partial r) \tag{52a}$$

$$\varepsilon_{rr} = \partial w_r / \partial r \tag{52b}$$

$$\varepsilon_{r\theta} = \frac{1}{2}[r \frac{\partial}{\partial r}(\frac{w_\theta}{r}) + \frac{1}{r}\frac{\partial w_r}{\partial \theta}] \tag{52c}$$

and the strain-stress relations

$$\sigma_{xr} = 2\mu\varepsilon_{xr} \tag{53a}$$

$$\sigma_{rr} = \lambda\Delta + 2\mu\varepsilon_{rr} \tag{53b}$$

$$\sigma_{r\theta} = 2\mu\varepsilon_{r\theta} \tag{53c}$$

are used to express the Fourier transformed stresses of circumferential order m in terms of the potential functions. In the above Δ is the dilatation given by

$$\Delta = \nabla^2 \phi \tag{54}$$

Equations (52) and (54) yield

$$\tilde{\varepsilon}_{xrm} = \frac{1}{2}(i\beta\tilde{w}_{rm} + \partial\tilde{w}_{xm} / \partial r) \tag{55a}$$

$$\tilde{\varepsilon}_{rrm} = \partial\tilde{w}_{rm} / \partial r \tag{55b}$$

$$\tilde{\varepsilon}_{r\theta m} = \frac{1}{2}[r \frac{\partial}{\partial r}(\frac{\tilde{w}_{\theta m}}{r}) + \frac{im}{r}\tilde{w}_{rm}] \tag{55c}$$

$$\tilde{\Delta} = -\beta^2\tilde{\phi} + \frac{\partial^2\tilde{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\tilde{\phi}}{\partial \theta^2} \tag{56}$$

$$\widetilde{\Delta}_m = -\beta^2 \widetilde{\phi}_m + \frac{\partial^2 \widetilde{\phi}_m}{\partial r^2} + \frac{1}{r}\frac{\partial \widetilde{\phi}_m}{\partial r} - \frac{m^2}{r^2}\widetilde{\phi}_m \tag{57}$$

Having considered equation (22), equation (57) becomes

$$\widetilde{\Delta}_m = -\frac{\omega^2}{c_1^2}\widetilde{\phi}_m \tag{58}$$

From equations (53),

$$\begin{aligned}
\widetilde{\sigma}_{xrm} &= 2\mu\widetilde{\varepsilon}_{xrm} = \mu(i\beta\widetilde{w}_{rm} + \partial\widetilde{w}_{xm}/\partial r) \\
&= i\mu\beta[\frac{\partial \widetilde{\phi}_m}{\partial r} + \frac{im}{r}\widetilde{H}_{xm} - i\beta\widetilde{H}_{\theta m}] \\
&\quad + \mu[i\beta\frac{\partial \widetilde{\phi}_m}{\partial r} + \frac{\partial^2 \widetilde{H}_{\theta m}}{\partial r^2} + \frac{\partial}{\partial r}(\frac{\widetilde{H}_{\theta m}}{r} - \frac{im}{r}\widetilde{H}_{rm})]
\end{aligned} \tag{59a}$$

$$\begin{aligned}
\widetilde{\sigma}_{rrm} &= \lambda\widetilde{\Delta}_m + 2\mu\widetilde{\varepsilon}_{rrm} = -\lambda\frac{\omega^2}{c_1^2}\widetilde{\phi}_m + 2\mu\frac{\partial}{\partial r}(\frac{\partial \widetilde{\phi}_m}{\partial r} + \frac{im}{r}\widetilde{H}_{xm} - i\beta\widetilde{H}_{\theta m}) \\
&= -\lambda\frac{\omega^2}{c_1^2}\widetilde{\phi}_m + 2\mu(\frac{\partial^2 \widetilde{\phi}_m}{\partial r^2} + \frac{im}{r}\frac{\partial \widetilde{H}_{xm}}{\partial r} - \frac{im}{r^2}\widetilde{H}_{xm} - i\beta\frac{\partial \widetilde{H}_{\theta m}}{\partial r})
\end{aligned} \tag{59b}$$

$$\begin{aligned}
\widetilde{\sigma}_{r\theta m} &= 2\mu\widetilde{\varepsilon}_{r\theta m} = \mu(\frac{\partial \widetilde{w}_{\theta m}}{\partial r} - \frac{\widetilde{w}_{\theta m}}{r} + \frac{im}{r}\widetilde{w}_{rm}) \\
&= -\frac{2i\mu m}{r^2}\widetilde{\phi}_m + \frac{2i\mu m}{r}\frac{\partial \widetilde{\phi}_m}{\partial r} - \frac{\mu m^2}{r^2}\widetilde{H}_{xm} + \frac{\mu}{r}\frac{\partial \widetilde{H}_{xm}}{\partial r} \\
&\quad - \mu\frac{\partial^2 \widetilde{H}_{xm}}{\partial r^2} - \frac{i\mu\beta}{r}\widetilde{H}_{rm} + i\mu\beta\frac{\partial \widetilde{H}_{rm}}{\partial r} + \frac{\mu m\beta}{r}\widetilde{H}_{\theta m}
\end{aligned} \tag{59c}$$

With the substitution of equations (47), equation (59) yields

$$\begin{Bmatrix} \widetilde{\sigma}_{xrm} \\ \widetilde{\sigma}_{rrm} \\ \widetilde{\sigma}_{r\theta m} \end{Bmatrix} = [B(r)]\{C\} \tag{60}$$

where, $[B(r)]$, called *the stress matrix of circumferential order m*, is of order 3×6. The elements of $[B(r)], b_{kj}$ $(k = 1,2,3; j = 1,2,\cdots,6)$, are as follows:

$$b_{11} = 2i\mu\beta\frac{\partial W_m(p_1 r)}{\partial r}, \quad b_{12} = 2i\mu\beta\frac{\partial Z_m(p_1 r)}{\partial r}$$

$$b_{13} = \mu\beta p_2 W_{m+1}(p_2 r) + \frac{\mu\beta^3}{p_2}W_{m-1}(p_2 r)$$

$$b_{14} = -\mu\beta p_2 Z_{m+1}(p_2 r) - \frac{\mu\beta^3}{p_2} Z_{m-1}(p_2 r)$$

$$b_{15} = -i\mu\beta^2[W_{m-1}(p_2 r) + W_{m+1}(p_2 r)] - 2i\mu p_2 \frac{\partial W_m(p_2 r)}{\partial r}$$

$$b_{16} = -i\mu\beta^2[Z_{m-1}(p_2 r) + Z_{m+1}(p_2 r)] + 2i\mu p_2 \frac{\partial Z_m(p_2 r)}{\partial r}$$

$$b_{21} = -\lambda \frac{\omega^2}{c_1^2} W_m(p_1 r) + 2\mu \frac{\partial^2 W_m(p_1 r)}{\partial r^2}, \quad b_{22} = -\lambda \frac{\omega^2}{c_1^2} Z_m(p_1 r) + 2\mu \frac{\partial^2 Z_m(p_1 r)}{\partial r^2}$$

$$b_{23} = \frac{2i\mu m}{r}[\frac{\partial W_m(p_2 r)}{\partial r} - \frac{W_m(p_2 r)}{r}] - \frac{2i\mu\beta^2}{p_2} \frac{\partial W_{m-1}(p_2 r)}{\partial r}$$

$$b_{24} = \frac{2i\mu m}{r}[\frac{\partial Z_m(p_2 r)}{\partial r} - \frac{Z_m(p_2 r)}{r}] + \frac{2i\mu\beta^2}{p_2} \frac{\partial Z_{m-1}(p_2 r)}{\partial r}$$

$$b_{25} = -2\mu\beta[\frac{\partial W_{m-1}(p_2 r)}{\partial r} + \frac{\partial W_{m+1}(p_2 r)}{\partial r}]$$

$$b_{26} = -2\mu\beta[\frac{\partial Z_{m-1}(p_2 r)}{\partial r} + \frac{\partial Z_{m+1}(p_2 r)}{\partial r}]$$

$$b_{31} = \frac{2i\mu m}{r}[-\frac{W_m(p_1 r)}{r} + \frac{\partial W_m(p_1 r)}{\partial r}], \quad b_{32} = \frac{2i\mu m}{r}[-\frac{Z_m(p_1 r)}{r} + \frac{\partial Z_m(p_1 r)}{\partial r}]$$

$$b_{33} = -\frac{\mu m^2}{r^2} W_m(p_2 r) + \frac{\mu}{r} \frac{\partial W_m(p_2 r)}{\partial r} - \mu \frac{\partial^2 W_m(p_2 r)}{\partial r^2} + \frac{\mu\beta^2(m-1)}{p_2 r} W_{m-1}(p_2 r) + \frac{\mu\beta^2}{p_2} \frac{\partial W_{m-1}(p_2 r)}{\partial r}$$

$$b_{34} = -\frac{\mu m^2}{r^2} Z_m(p_2 r) + \frac{\mu}{r} \frac{\partial Z_m(p_2 r)}{\partial r} - \mu \frac{\partial^2 Z_m(p_2 r)}{\partial r^2} - \frac{\mu\beta^2(m-1)}{p_2 r} Z_{m-1}(p_2 r) - \frac{\mu\beta^2}{p_2} \frac{\partial Z_{m-1}(p_2 r)}{\partial r}$$

$$b_{35} = i\mu\beta[-\frac{m+1}{r} W_{m+1}(p_2 r) - \frac{m-1}{r} W_{m-1}(p_2 r) + \frac{\partial W_{m+1}(p_2 r)}{\partial r} - \frac{\partial W_{m-1}(p_2 r)}{\partial r}]$$

$$b_{36} = i\mu\beta[-\frac{m+1}{r} Z_{m+1}(p_2 r) - \frac{m-1}{r} Z_{m-1}(p_2 r) + \frac{\partial Z_{m+1}(p_2 r)}{\partial r} - \frac{\partial Z_{m-1}(p_2 r)}{\partial r}]$$

The derivatives that appear in the matrices [C(r)] and [B(r)] are given by

$$\frac{\partial W_m(p_1 r)}{\partial r} = \begin{cases} p_1 I_{m+1}(p_1 r) + \frac{m}{r} I_m(p_1 r), & p_1 \neq 0 \\ |m| r^{|m|-1}, & p_1 = 0, m \neq 0 \\ 0, & p_1 = 0, m = 0 \end{cases} \tag{61a}$$

$$\frac{\partial^2 W_m(p_1 r)}{\partial r^2} = \begin{cases} (p_1^2 + \frac{m^2 - m}{r^2}) I_m(p_1 r) - \frac{p_1}{r} I_{m+1}(p_1 r), & p_1 \neq 0 \\ |m|(|m|-1) r^{|m|-2}, & p_1 = 0, m \neq 0 \\ 0, & p_1 = 0, m = 0 \end{cases} \tag{61b}$$

$$\frac{\partial Z_m(p_1 r)}{\partial r} = \begin{cases} -p_1 K_{m+1}(p_1 r) + \frac{m}{r} K_m(p_1 r), & p_1 \neq 0 \\ -|m| r^{-|m|-1}, & p_1 = 0, m \neq 0 \\ \frac{1}{r}, & p_1 = 0, m = 0 \end{cases} \tag{61c}$$

$$\frac{\partial^2 Z_m(p_1 r)}{\partial r^2} = \begin{cases} (p_1^2 + \frac{m^2 - m}{r^2}) K_m(p_1 r) + \frac{p_1}{r} K_{m+1}(p_1 r), & p_1 \neq 0 \\ |m|(|m|+1) r^{-|m|-2}, & p_1 = 0, m \neq 0 \\ -\frac{1}{r^2}, & p_1 = 0, m = 0 \end{cases} \tag{61d}$$

3. DETERMINATION OF THE INTEGRATION CONSTANTS AND RESPONSES OF THE LAYERED CYLINDER

3.1 DETERMINATION OF THE INTEGRATION CONSTANTS

Equations (51) and (60) give the displacements and stresses of any sub-cylinder in terms of six integration constants. The six constants of integration differ from one sub-cylinder to another. All the integration constants are determined using the boundary conditions on the inner surface of the first sub-cylinder and those on the outer surface of the last sub-cylinder, as well as using the continuity of displacement and the balance of stress at each interface of the sub-cylinders. The following cases are dealt with separately.

(1) EQUILIBRIUM OF THE STRESSES AT THE LOADING INTERFACE. Balance equations of stress at the loading interface are derived from the three-dimensional stress equations of motion expressed in terms of cylindrical coordinates. These equations are [Leissa 1973]

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{r}\sigma_{xr} + \frac{\partial \sigma_{xr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{x\theta}}{\partial \theta} = \rho \frac{\partial^2 w_x}{\partial t^2} - p_x(x - ct) e^{i\Omega t} \delta(r - R_{l_p 0}) \delta(\theta - \theta_0) / R_{l_p 0} \tag{62a}$$

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})+\frac{\partial\sigma_{xr}}{\partial x}+\frac{1}{r}\frac{\partial\sigma_{x\theta}}{\partial\theta}=\rho\frac{\partial^2 w_r}{\partial t^2}-p_r(x-ct)e^{i\Omega t}\delta(r-R_{l_p0})\delta(\theta-\theta_0)/R_{l_p0} \quad (62b)$$

$$\frac{\partial\sigma_{x\theta}}{\partial x}+\frac{2}{r}\sigma_{r\theta}+\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}=\rho\frac{\partial^2 w_\theta}{\partial t^2}-p_\theta(x-ct)e^{i\Omega t}\delta(r-R_{l_p0})\delta(\theta-\theta_0)/R_{l_p0} \quad (62c)$$

Using equations (8), (9) and (21), equation (62a) can be simplified as

$$i\beta\tilde{\sigma}_{xxm}+\frac{1}{r}\tilde{\sigma}_{xrm}+\frac{\partial\tilde{\sigma}_{xrm}}{\partial r}+\frac{im}{r}\tilde{\sigma}_{x\theta m}=-\rho\omega^2\tilde{w}_{xm}-\frac{1}{2\pi R_{l_p0}}\tilde{p}_x(\beta)\delta(r-R_{l_p0})e^{-im\theta_0} \quad (63)$$

where, $\omega=\Omega-\beta c$. Integration of equation (62) with respect to r from $R^-_{l_p0}$ to $R^+_{l_p0}$ gives

$$\tilde{\sigma}_{xrm}\Big|_{R^+_{l_p0}}-\tilde{\sigma}_{xrm}\Big|_{R^-_{l_p0}}=-\frac{1}{2\pi R_{l_p0}}\tilde{p}_x(\beta)e^{-im\theta_0} \quad (64a)$$

Similarly,

$$\tilde{\sigma}_{rrm}\Big|_{R^+_{l_p0}}-\tilde{\sigma}_{rrm}\Big|_{R^-_{l_p0}}=-\frac{1}{2\pi R_{l_p0}}\tilde{p}_r(\beta)e^{-im\theta_0} \quad (64b)$$

$$\tilde{\sigma}_{r\theta m}\Big|_{R^+_{l_p0}}-\tilde{\sigma}_{r\theta m}\Big|_{R^-_{l_p0}}=-\frac{1}{2\pi R_{l_p0}}\tilde{p}_\theta(\beta)e^{-im\theta_0} \quad (64c)$$

Equation (64) gives the condition of balance of stress at the loading interface.

(2) FOR THE CASE OF A SOLID CYLINDER. If the first sub-cylinder is a solid cylinder, then as the functions $Z_m(pr), Z_{m+1}(pr)$ and $Z_{m-1}(pr)$ tend to infinity when $r\to 0$, it is required that $C_2=C_4=C_6=0$. From equation (60)

$$\begin{Bmatrix}\tilde{\sigma}_{xrm}\\ \tilde{\sigma}_{rrm}\\ \tilde{\sigma}_{r\theta m}\end{Bmatrix}_{r=R_{11}}=[B(R_{11})]\{C\} \quad (65)$$

which combined with equation (51) gives

$$\{\tilde{\sigma}_m\}_{11}=[U]\{\tilde{w}_m\}_{11} \quad (66)$$

where, $\{\tilde{w}_m\}_{11} = (\tilde{w}_{xm}, \tilde{w}_{rm}, \tilde{w}_{\theta m})^T\big|_{r=R_{11}}$, $\{\tilde{\sigma}_m\}_{11} = (\tilde{\sigma}_{xrm}, \tilde{\sigma}_{rrm}, \tilde{\sigma}_{r\theta m})^T\big|_{r=R_{11}}$, and [*U*] is a 3×3 matrix. Further, the following vectors are defined

$$\{\tilde{S}_m\}_{j0} = \begin{Bmatrix} \{\tilde{w}_m\} \\ \{\tilde{\sigma}_m\} \end{Bmatrix}_{r=R_{j0}}, \quad \{\tilde{S}_m\}_{j1} = \begin{Bmatrix} \{\tilde{w}_m\} \\ \{\tilde{\sigma}_m\} \end{Bmatrix}_{r=R_{j1}} \tag{67}$$

to describe the 'states' of the inner and the outer surfaces of the *j*th sub-cylinder.

(3) FOR THE CASE OF A HOLLOW CYLINDER WITH AN INFINITE OUTER RADIUS. If the last sub-cylinder has an infinite outer radius, i.e., if $R_{n1} = \infty$, then it is required that $C_1 = C_3 = C_5 = 0$. This is because, for β big enough, $W_m(pr)$, $W_{m+1}(pr)$ and $W_{m-1}(pr)$ tend to infinity when $r \to \infty$. Similar to equation (66)

$$\{\tilde{w}_m\}_{n0} = [V]\{\tilde{\sigma}_m\}_{n0} \tag{68}$$

where, [*V*] is a 3×3 matrix.

(4) FOR THE GENERAL CASE. For any, e.g. the *j*th, sub-cylinder, from equations (51) and (60),

$$\begin{Bmatrix} \{\tilde{w}_m\} \\ \{\tilde{\sigma}_m\} \end{Bmatrix} = \begin{bmatrix} [C(r)] \\ [B(r)] \end{bmatrix} \{C\} = [A'(r)]\{C\} \tag{69}$$

where, $[A'(r)]$ is a 6×6 matrix. This may be rewritten in the form of

$$[A'(R_{j1})] = \begin{bmatrix} \xi_{j1}[E] & 0 \\ 0 & \zeta_{j1}[E] \end{bmatrix} [A]_{j1}, \quad [A'(R_{j0})] = \begin{bmatrix} \xi_{j0}[E] & 0 \\ 0 & \zeta_{j0}[E] \end{bmatrix} [A]_{j0} \tag{70}$$

where [*E*] is a 3×3 unit matrix, and ξ and ζ are real scaling factors chosen so that the magnitudes of the elements in matrix [*A*] are less than or equal to unity. By this means, possible numerical difficulties can be avoided in the manipulations of those matrices. Note that the unit of ξ is m^{-1} while that of ζ is N/m^4. From equations (69) and (70)

$$\{\tilde{S}_m\}_{j1} = \begin{bmatrix} \xi_{j1}[E] & 0 \\ 0 & \zeta_{j1}[E] \end{bmatrix} [A]_{j1}[A]_{j0}^{-1} \begin{bmatrix} \xi_{j0}^{-1}[E] & 0 \\ 0 & \zeta_{j0}^{-1}[E] \end{bmatrix} \{\tilde{S}_m\}_{j0} \tag{71}$$

which links the state of the outer surface and that of the inner surface of the jth sub-cylinder.

Now, the continuity of displacement and the balance of stress at each interface of the sub-cylinders require that

$$\{\tilde{S}_m\}_{j1} = \{\tilde{S}_m\}_{j+1,0}\,,\quad (j=1,2,\cdots,n,\, j \neq l_P - 1)\,,\quad \{\tilde{S}_m\}_{l_P-1,1} = \{\tilde{S}_m\}_{l_P 0} + \{\tilde{S}_m\}^P \tag{72}$$

where, according to equation (64),

$$\{\tilde{S}_m\}^P = \frac{1}{2\pi R_{l_P 0}} e^{-im\theta_0} (0,0,0,\tilde{p}_x(\beta), \tilde{p}_r(\beta), \tilde{p}_\theta(\beta))^T \tag{73}$$

From equations (71) and (72)

$$\begin{aligned}\{\tilde{S}_m\}_{n1} = &\begin{bmatrix} \xi_{n1}[E] & 0 \\ 0 & \zeta_{n1}[E] \end{bmatrix} \begin{bmatrix} [T]_{11} & [T]_{12} \\ [T]_{21} & [T]_{22} \end{bmatrix} \begin{bmatrix} \xi_{10}^{-1}[E] & 0 \\ 0 & \zeta_{10}^{-1}[E] \end{bmatrix} \{\tilde{S}_m\}_{10} \\ &- \begin{bmatrix} \xi_{n1}[E] & 0 \\ 0 & \zeta_{n1}[E] \end{bmatrix} \begin{bmatrix} [F]_{11} & [F]_{12} \\ [F]_{21} & [F]_{22} \end{bmatrix} \begin{bmatrix} \xi_{l_P 0}^{-1}[E] & 0 \\ 0 & \zeta_{l_P 0}^{-1}[E] \end{bmatrix} \{\tilde{S}_m\}^P \end{aligned} \tag{74}$$

and (suppose $l_R \leq l_P$)

$$\{\tilde{S}_m\}_{l_R 0} = \begin{bmatrix} \xi_{l_R-1,1}[E] & 0 \\ 0 & \zeta_{l_R-1,1}[E] \end{bmatrix} \begin{bmatrix} [G]_{11} & [G]_{12} \\ [G]_{21} & [G]_{22} \end{bmatrix} \begin{bmatrix} \xi_{10}^{-1}[E] & 0 \\ 0 & \zeta_{10}^{-1}[E] \end{bmatrix} \{\tilde{S}_m\}_{10} \tag{75}$$

where $[T]_{11}$, $[F]_{11}$ and $[G]_{11}$ etc. are 3×3 matrices and

$$\begin{aligned}[T] = \begin{bmatrix} [T]_{11} & [T]_{12} \\ [T]_{21} & [T]_{22} \end{bmatrix} = [A]_{n1}[A]_{n0}^{-1} \begin{bmatrix} \xi_{n0}^{-1}[E] & 0 \\ 0 & \zeta_{n0}^{-1}[E] \end{bmatrix} \begin{bmatrix} \xi_{n-1,1}[E] & 0 \\ 0 & \zeta_{n-1,1}[E] \end{bmatrix} \times \\ [A]_{n-1,1}[A]_{n-1,0}^{-1} \begin{bmatrix} \xi_{n-1,0}^{-1}[E] & 0 \\ 0 & \zeta_{-1,n0}^{-1}[E] \end{bmatrix} \cdots \begin{bmatrix} \xi_{1,1}[E] & 0 \\ 0 & \zeta_{1,1}[E] \end{bmatrix} [A]_{11}[A]_{10}^{-1}\end{aligned}$$

That is,

$$[T] = [A]_{n1}[A]_{n0}^{-1} \begin{bmatrix} \frac{\xi_{n-1,1}}{\xi_{n0}}[E] & 0 \\ 0 & \frac{\zeta_{n-1,1}}{\zeta_{n0}}[E] \end{bmatrix} \cdot [A]_{n-1,1}[A]_{n-1,0}^{-1} \cdots \begin{bmatrix} \frac{\xi_{11}}{\xi_{20}}[E] & 0 \\ 0 & \frac{\zeta_{11}}{\zeta_{20}}[E] \end{bmatrix} [A]_{11}[A]_{10}^{-1}$$

(76)

$$[F]=[A]_{n1}[A]_{n0}^{-1}\begin{bmatrix}\frac{\xi_{n-1,1}}{\xi_{n0}}[E] & 0\\ 0 & \frac{\zeta_{n-1,1}}{\zeta_{n0}}[E]\end{bmatrix}\cdot[A]_{n-1,1}[A]_{n-1,0}^{-1}\cdots\begin{bmatrix}\frac{\xi_{l_p1}}{\xi_{l_p+1,0}}[E] & 0\\ 0 & \frac{\zeta_{l_p1}}{\zeta_{l_p+1,0}}[E]\end{bmatrix}[A]_{l_p1}[A]_{l_p0}^{-1}$$

(77)

$$[G]=[A]_{l_R-1,1}[A]_{l_R-1,0}^{-1}\begin{bmatrix}\frac{\xi_{l_R-2,1}}{\xi_{l_R-1,0}}[E] & 0\\ 0 & \frac{\zeta_{l_R-2,1}}{\zeta_{l_R-1,0}}[E]\end{bmatrix}\cdot[A]_{l_R-2,1}[A]_{l_R-2,0}^{-1}\cdots\begin{bmatrix}\frac{\xi_{11}}{\xi_{20}}[E] & 0\\ 0 & \frac{\zeta_{11}}{\zeta_{20}}[E]\end{bmatrix}[A]_{11}[A]_{10}^{-1}$$

(78)

Thus

$$\begin{Bmatrix}\{\tilde{w}_m\}_{n1}\\ \{\tilde{\sigma}_m\}_{n1}\end{Bmatrix}=\begin{bmatrix}\frac{\xi_{n1}}{\xi_{10}}[T]_{11} & \frac{\xi_{n1}}{\zeta_{10}}[T]_{12}\\ \frac{\zeta_{n1}}{\xi_{10}}[T]_{21} & \frac{\zeta_{n1}}{\zeta_{10}}[T]_{22}\end{bmatrix}\begin{Bmatrix}\{\tilde{w}_m\}_{10}\\ \{\tilde{\sigma}_m\}_{10}\end{Bmatrix}-\frac{1}{2\pi R_{l_p0}}e^{-im\theta_0}\begin{bmatrix}\frac{\xi_{n1}}{\xi_{l_p0}}[F]_{11} & \frac{\xi_{n1}}{\zeta_{l_p0}}[F]_{12}\\ \frac{\zeta_{n1}}{\xi_{l_p0}}[F]_{21} & \frac{\zeta_{n1}}{\zeta_{l_p0}}[F]_{22}\end{bmatrix}\begin{Bmatrix}0\\ \{\tilde{p}\}\end{Bmatrix}$$

(79)

$$\begin{Bmatrix}\{\tilde{w}_m\}_{l_R0}\\ \{\tilde{\sigma}_m\}_{l_R0}\end{Bmatrix}=\begin{bmatrix}\frac{\xi_{l_R-1,1}}{\xi_{10}}[G]_{11} & \frac{\xi_{l_R-1,1}}{\zeta_{10}}[G]_{12}\\ \frac{\zeta_{l_R-1,1}}{\xi_{10}}[G]_{21} & \frac{\zeta_{l_R-1,1}}{\zeta_{10}}[G]_{22}\end{bmatrix}\begin{Bmatrix}\{\tilde{w}_m\}_{10}\\ \{\tilde{\sigma}_m\}_{10}\end{Bmatrix} \quad (80)$$

where, $\{\tilde{p}\}=(\tilde{p}_x,\tilde{p}_r,\tilde{p}_\theta)^T$.

3.2 DISPLACEMENTS AND STRESSES OF THE FIRST INTERFACE

Now the state of the first interface is evaluated for four different situations. It should be noted that matrices [*T*], [*F*], [*G*] etc. are different for different situations.

(1) *The first sub-cylinder is a hollow cylinder, and the outer radius of the last sub-cylinder is finite.* Since $\{\tilde{\sigma}_m\}_{10}=\{\tilde{\sigma}_m\}_{n1}=0$, from equation (79),

$$\{\tilde{w}_m\}_{10}=\frac{1}{2\pi R_{l_p0}}e^{-im\theta_0}\frac{\xi_{10}}{\zeta_{l_p0}}[T]_{21}^{-1}[F]_{22}\{\tilde{p}\} \quad (81)$$

(2) *The first sub-cylinder is a hollow cylinder, and the outer radius of the last one is infinite*. Since $\{\tilde{\sigma}_m\}_{10}=0$ and $\{\tilde{w}_m\}_{n0}=[V]\{\tilde{\sigma}_m\}_{n0}$ (see equation (68)), from equation (79)

$$\{\tilde{w}_m\}_{10}=\frac{1}{2\pi R_{l_p0}}e^{-im\theta_0}\left(\frac{\xi_{n-1,1}}{\xi_{10}}[T]_{11}-\frac{\zeta_{n-1,1}}{\xi_{10}}[V][T]_{21}\right)^{-1}\left(\frac{\xi_{n-1,1}}{\zeta_{l_p0}}[F]_{12}-\frac{\zeta_{n-1,1}}{\zeta_{l_p0}}[V][F]_{22}\right)\{\tilde{p}\} \tag{82}$$

(3) *The first sub-cylinder is a solid cylinder, and the outer radius of the last one is finite*. Since $\{\tilde{\sigma}_m\}_{11}=[U]\{\tilde{w}_m\}_{11}$ (equation (66)) and $\{\tilde{\sigma}_m\}_{n1}=0$, from equation (79)

$$\{\tilde{w}_m\}_{11}=\frac{1}{2\pi R_{l_p0}}e^{-im\theta_0}\left(\frac{\zeta_{n1}}{\xi_{20}}[T]_{21}+\frac{\zeta_{n1}}{\zeta_{20}}[T]_{22}[U]\right)^{-1}\frac{\zeta_{n1}}{\zeta_{l_p0}}[F]_{22}\{\tilde{p}\} \tag{83}$$

(4) *The first sub-cylinder is a solid cylinder, and the outer radius of the last one is infinite*. Since $\{\tilde{\sigma}_m\}_{11}=[U]\{\tilde{w}_m\}_{11}$ and $\{\tilde{w}_m\}_{n0}=[V]\{\tilde{\sigma}_m\}_{n0}$, from equation (79)

$$\begin{aligned}\{\tilde{w}_m\}_{11}=&\frac{e^{-im\theta_0}}{2\pi R_{l_p0}}\left(\frac{\xi_{n-1,1}}{\xi_{20}}[T]_{11}+\frac{\xi_{n-1,1}}{\zeta_{20}}[T]_{12}[U]-\frac{\zeta_{n-1,1}}{\xi_{20}}[V][T]_{21}-\frac{\zeta_{n-1,1}}{\zeta_{20}}[V][T]_{22}[U]\right)^{-1}\\&\cdot\left(\frac{\xi_{n-1,1}}{\zeta_{l_p0}}[F]_{12}-\frac{\zeta_{n-1,1}}{\zeta_{l_p0}}[V][F]_{22}\right)\{\tilde{p}\}\end{aligned} \tag{84}$$

3.3 DISPLACEMENTS AND STRESSES OF THE OBSERVER INTERFACE

Inserting equations (81) to (84) into equation (80) yields the responses (displacements and stresses) of the interface of the (l_R-1) th and the l_R th sub-cylinders, if $l_R \le l_P$. The responses are dealt with for the four situations stated above.

(1) *When the first sub-cylinder is a hollow cylinder and the outer radius of the last sub-cylinder is finite,*

$$\{\tilde{w}_m\}_{l_R0}=\frac{1}{2\pi R_{l_p0}}e^{-im\theta_0}\frac{\xi_{l_R-1,1}}{\zeta_{l_p0}}[G]_{11}[T]_{21}^{-1}[F]_{22}\{\tilde{p}\} \tag{85a}$$

$$\{\tilde{\sigma}_m\}_{l_R0}=\frac{1}{2\pi R_{l_p0}}e^{-im\theta_0}\frac{\zeta_{l_R-1,1}}{\zeta_{l_p0}}[G]_{21}[T]_{21}^{-1}[F]_{22}\{\tilde{p}\} \tag{85b}$$

(2) *The first sub-cylinder is a hollow cylinder, and the outer radius of the last one is infinite,*

$$\{\tilde{w}_m\}_{l_R 0} = \frac{1}{2\pi R_{l_p 0}} e^{-im\theta_0} \frac{\xi_{l_R-1,1}}{\zeta_{l_p 0}} [G]_{11} \left([T]_{11} - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][T]_{21} \right)^{-1} \left([F]_{12} - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][F]_{22} \right) \{\tilde{p}\} \tag{86a}$$

$$\{\tilde{w}_m\}_{l_R 0} = \frac{1}{2\pi R_{l_p 0}} e^{-im\theta_0} \frac{\zeta_{l_R-1,1}}{\zeta_{l_p 0}} [G]_{21} \left([T]_{11} - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][T]_{21} \right)^{-1} \left([F]_{12} - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][F]_{22} \right) \{\tilde{p}\} \tag{86b}$$

(3) *The first sub-cylinder is a solid cylinder, and the outer radius of the last one is finite,*

$$\{\tilde{w}_m\}_{l_R 0} = \frac{1}{2\pi R_{l_p 0}} e^{-im\theta_0} \frac{\xi_{l_R-1,1}}{\zeta_{l_p 0}} \left([G]_{11} + \frac{\xi_{20}}{\zeta_{20}} [G]_{12} [U] \right) \left([T]_{21} + \frac{\xi_{20}}{\zeta_{20}} [T]_{22} [U] \right)^{-1} [F]_{22} \{\tilde{p}\} \tag{87a}$$

$$\{\tilde{\sigma}_m\}_{l_R 0} = \frac{1}{2\pi R_{l_p 0}} e^{-im\theta_0} \frac{\zeta_{l_R-1,1}}{\zeta_{l_p 0}} \left([G]_{21} + \frac{\xi_{20}}{\zeta_{20}} [G]_{22} [U] \right) \left([T]_{21} + \frac{\xi_{20}}{\zeta_{20}} [T]_{22} [U] \right)^{-1} [F]_{22} \{\tilde{p}\} \tag{87b}$$

(4) *The first sub-cylinder is a solid cylinder, and the outer radius of the last one is infinite,*

$$\begin{aligned}\{\tilde{w}_m\}_{l_R 0} = \frac{e^{-im\theta_0}}{2\pi R_{l_p 0}} \frac{\xi_{l_R-1,1}}{\zeta_{l_p 0}} \left([G]_{11} + \frac{\xi_{20}}{\zeta_{20}} [G]_{12} [U] \right) \Bigg([T]_{11} + \frac{\xi_{20}}{\zeta_{20}} [T]_{12} [U] - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][T]_{21} \\ - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} \frac{\xi_{20}}{\zeta_{20}} [V][T]_{22}[U] \Bigg)^{-1} \cdot \left([F]_{12} - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][F]_{22} \right) \{\tilde{p}\}\end{aligned} \tag{88a}$$

$$\begin{aligned}\{\tilde{\sigma}_m\}_{l_R 0} = \frac{e^{-im\theta_0}}{2\pi R_{l_p 0}} \frac{\xi_{l_R-1,1}}{\zeta_{l_p 0}} \left([G]_{21} + \frac{\xi_{20}}{\zeta_{20}} [G]_{22} [U] \right) \Bigg([T]_{11} + \frac{\xi_{20}}{\zeta_{20}} [T]_{12} [U] - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][T]_{21} \\ - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} \frac{\xi_{20}}{\zeta_{20}} [V][T]_{22}[U] \Bigg)^{-1} \cdot \left([F]_{12} - \frac{\zeta_{n-1,1}}{\xi_{n-1,1}} [V][F]_{22} \right) \{\tilde{p}\}\end{aligned} \tag{88b}$$

Equations (85) to (88) may be rewritten as

$$\{\tilde{w}_m\}_{l_R 0} = [\tilde{Q}(\beta,m)]\{\tilde{p}\} = \begin{bmatrix} \tilde{Q}_{11}(\beta,m) & \tilde{Q}_{12}(\beta,m) & \tilde{Q}_{13}(\beta,m) \\ \tilde{Q}_{21}(\beta,m) & \tilde{Q}_{22}(\beta,m) & \tilde{Q}_{23}(\beta,m) \\ \tilde{Q}_{31}(\beta,m) & \tilde{Q}_{32}(\beta,m) & \tilde{Q}_{33}(\beta,m) \end{bmatrix} \{\tilde{p}\} \tag{89}$$

where, matrix $[\tilde{Q}(\beta,m)]$ may be termed *the Fourier transformed moving dynamic flexibility matrix of circumferential order m.*

According to equation (21), the Fourier transformed displacements of the interface of the (l_R-1)th and l_Rth sub-cylinders are given by

$$\begin{Bmatrix} \tilde{w}_x(\beta,r,\theta) \\ \tilde{w}_r(\beta,r,\theta) \\ \tilde{w}_\theta(\beta,r,\theta) \end{Bmatrix} = \left(\sum_{m=-\infty}^{\infty} [\tilde{Q}(\beta,m)] e^{im(\theta-\theta_0)} \right) \begin{Bmatrix} \tilde{p}_x(\beta) \\ \tilde{p}_r(\beta) \\ \tilde{p}_\theta(\beta) \end{Bmatrix} \tag{90}$$

where, $r = R_{l_R 0}$.

4. PROPERTIES OF MATRIX $[\tilde{Q}(\beta,m)]$

Two properties of the matrix $[\tilde{Q}(\beta,m)]$ are observed which will increase the efficiency of computation.

(1) In accordance with the symmetry or asymmetry of the displacements, it can be shown that

$$\begin{bmatrix} \tilde{Q}_{11}(\beta,m) & \tilde{Q}_{12}(\beta,m) & \tilde{Q}_{13}(\beta,m) \\ \tilde{Q}_{21}(\beta,m) & \tilde{Q}_{22}(\beta,m) & \tilde{Q}_{23}(\beta,m) \\ \tilde{Q}_{31}(\beta,m) & \tilde{Q}_{32}(\beta,m) & \tilde{Q}_{33}(\beta,m) \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{11}(\beta,-m) & \tilde{Q}_{12}(\beta,-m) & -\tilde{Q}_{13}(\beta,-m) \\ \tilde{Q}_{21}(\beta,-m) & \tilde{Q}_{22}(\beta,-m) & -\tilde{Q}_{23}(\beta,-m) \\ -\tilde{Q}_{31}(\beta,-m) & -\tilde{Q}_{32}(\beta,-m) & \tilde{Q}_{33}(\beta,-m) \end{bmatrix}$$

(91)

(2) When the loads are applied at the interface of the (l_R-1)th and l_Rth sub-cylinders, the displacements at the interface of the (l_P-1)th and l_Pth sub-cylinders, similar to equation (89), are expressed as

$$\{\tilde{w}_m\}_{l_P 0} = [\tilde{\delta}(\beta,m)]\{\tilde{p}\} = \begin{bmatrix} \tilde{\delta}_{11} & \tilde{\delta}_{12} & \tilde{\delta}_{13} \\ \tilde{\delta}_{21} & \tilde{\delta}_{22} & \tilde{\delta}_{23} \\ \tilde{\delta}_{31} & \tilde{\delta}_{32} & \tilde{\delta}_{33} \end{bmatrix} \{\tilde{p}\} \tag{92}$$

Then it can be shown that

$$\begin{bmatrix} \tilde{\delta}_{11} & \tilde{\delta}_{12} & \tilde{\delta}_{13} \\ \tilde{\delta}_{21} & \tilde{\delta}_{22} & \tilde{\delta}_{23} \\ \tilde{\delta}_{31} & \tilde{\delta}_{32} & \tilde{\delta}_{33} \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{11} & -\tilde{Q}_{21} & -\tilde{Q}_{31} \\ -\tilde{Q}_{12} & \tilde{Q}_{22} & \tilde{Q}_{32} \\ -\tilde{Q}_{13} & \tilde{Q}_{23} & \tilde{Q}_{33} \end{bmatrix} \tag{93}$$

Equation (93) is an extension of the Betti reciprocity relations in elastodynamics.

5. FOURIER TRANSFORMED MOVING GREEN'S FUNCTIONS FOR A HOMOGENEOUS WHOLE-SPACE

The harmonic responses (Green's functions) of a homogeneous whole-space without any inclusion due to stationary harmonic loads were first derived analytically by Stokes and can be found in many books on elastodynamics [e.g. Miklowitz 1978, Dominguez, 1993]. A Fourier transform of these Green's functions with respect to x gives the Fourier transformed counterparts, i.e. the Fourier transformed stationary Green's functions. Such Green's functions, also referred as the Green's function for the 'two-and-half-dimensional problem', have been derived by Tadeu et al [Tadeu and Kausel 2000].

When the moving load is a constant load, i.e., its frequency is zero, the steady state responses of an **undamped** whole-space were obtained by Eason *et al* [Eason, Fulton and Sneddon 1955/1956, Frýba 1982]. The Fourier transforms of these responses have been derived by Apostolos et al [Apostolos, Papageorgiou and Duoli 1998].

The Fourier transformed moving (displacement) Green's functions for a whole-space with material damping due to moving harmonic loads, may be derived using the cylindrical formulations developed in this paper. To do so, first let the loads be applied on the inner surface of a cylinder with an infinite outer radius and then let the inner radius be allowed to approach zero. With the procedure of derivation omitted, the final results are listed below.

When $\tilde{p}_x(\beta) \neq 0, \tilde{p}_y(\beta) = \tilde{p}_r(\beta) = 0, \tilde{p}_z(\beta) = \tilde{p}_\theta(\beta) = 0$, the response field of the whole-space consists only of that of circumferential order zero and the integral constants are determined as

$$C_1 = C_3 = C_4 = C_5 = 0, \quad C_2 = -\frac{i\beta}{2\pi\rho\omega^2}\tilde{p}_x(\beta), \quad C_6 = \frac{ip_2}{4\pi\rho\omega^2}\tilde{p}_x(\beta) \tag{94}$$

Thus, the transformed displacements in the x-, y- and z-directions are (see equation (51))

$$\tilde{u}(\beta) = \frac{1}{2\pi\rho\omega^2}[\beta^2 K_0(p_1 r) - p_2^2 K_0(p_2 r)]\tilde{p}_x(\beta) \tag{95a}$$

$$\tilde{\upsilon}(\beta) = \frac{i\beta}{2\pi\rho\omega^2}\frac{y}{r}[p_1 K_1(p_1 r) - p_2 K_1(p_2 r)]\tilde{p}_x(\beta) \tag{95b}$$

$$\tilde{w}(\beta) = \frac{i\beta}{2\pi\rho\omega^2}\frac{z}{r}[p_1 K_1(p_1 r) - p_2 K_1(p_2 r)]\tilde{p}_x(\beta) \tag{95c}$$

It should be noted that, since $\omega = \Omega - \beta c$ (equation (12)), ω vanishes at $\beta = \Omega / c$. However, equation (94) has a finite limit as $\omega \to 0$.

When $\tilde{p}_x(\beta) = 0, \tilde{p}_y(\beta) = \tilde{p}_r(\beta) \neq 0, \tilde{p}_z(\beta) = \tilde{p}_\theta = 0$, the response field of the whole-space consists only of that of circumferential order one, and the integral constants are determined as

$$C_1 = C_3 = C_5 = C_6 = 0, \quad C_2 = \frac{p_1}{2\pi\rho\omega^2}\tilde{p}_y(\beta), \quad C_4 = -\frac{ip_2}{2\pi\rho\omega^2}\tilde{p}_y(\beta) \tag{96}$$

Thus

$$\tilde{u}(\beta) = \frac{i\beta}{2\pi\rho\omega^2}\frac{y}{r}[p_1 K_1(p_1 r) - p_2 K_1(p_2 r)]\tilde{p}_y(\beta) \tag{97a}$$

$$\begin{aligned}\tilde{\upsilon}(\beta) = \frac{1}{2\pi\rho\omega^2}\{&\frac{y^2}{r^2}[p_1 \frac{\partial K_1(p_1 r)}{\partial r} + \frac{p_2}{r}K_1(p_2 r) + \beta^2 K_0(p_2 r)] \\ &+ \frac{z^2}{r^2}[\frac{p_1}{r}K_1(p_1 r) + \beta^2 K_0(p_2 r) + p_2 \frac{\partial K_1(p_2 r)}{\partial r}]\}\tilde{p}_y(\beta)\end{aligned}$$

i.e.

$$\begin{aligned}\tilde{\upsilon}(\beta) = \frac{1}{2\pi\rho\omega^2}\{&\frac{y^2}{r^2}[p_2^2 K_2(p_2 r) - p_1^2 K_2(p_1 r)] + \omega^2 K_0(p_2 r)/c_2^2 \\ &+ \frac{p_1}{r}K_1(p_1 r) - \frac{p_2}{r}K_1(p_2 r)\}\tilde{p}_y(\beta)\end{aligned} \tag{97b}$$

$$\tilde{w}(\beta)=\frac{1}{2\pi\rho\omega^2}\{\frac{yz}{r^2}[p_1\frac{\partial K_1(p_1r)}{\partial r}+\frac{p_2}{r}K_1(p_2r)+\beta^2K_0(p_2r)]$$
$$-\frac{yz}{r^2}[\frac{p_1}{r}K_1(p_1r)+\beta^2K_0(p_2r)+p_2\frac{\partial K_1(p_2r)}{\partial r}]\}\tilde{p}_y(\beta)$$

$$\tilde{w}(\beta)=\frac{1}{2\pi\rho\omega^2}\frac{yz}{r^2}[p_2^2K_2(p_2r)-p_1^2K_2(p_1r)]\}\tilde{p}_y(\beta) \tag{97c}$$

It is noticed that for $\beta=0$, a plane-strain problem results, and equation (97) gives the Green's functions for a whole-plane (*yz* plane), which are well known and normally expressed as [Dominguez 1993]

$$w_{lk}=\frac{1}{2\pi\rho c_2^2}[\psi\delta_{lk}-\chi\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_k}] \tag{98}$$

where, $l, k = 2$ or 3, $x_2=y, x_3=z$, $r=\sqrt{y^2+z^2}$, and

$$\psi=K_0(p_2r)+\frac{1}{p_2r}[K_1(p_2r)-\frac{c_2}{c_1}K_1(p_1r)] \tag{99}$$

$$\chi=K_2(p_2r)-\frac{c_2^2}{c_1^2}K_2(p_1r) \tag{100}$$

$$p_1=i\omega/c_1,\ p_2=i\omega/c_2 \tag{101}$$

The Fourier transformed moving stress Green's functions can be obtained following the Hooke's law which, after being Fourier transformed, reads

$$\tilde{\sigma}_x=i\beta(\lambda+2\mu)\tilde{u}+\lambda(\frac{\partial\tilde{\upsilon}}{\partial y}+\frac{\partial\tilde{w}}{\partial z}) \tag{102}$$

$$\tilde{\sigma}_y=i\beta\lambda\tilde{u}+(\lambda+2\mu)\frac{\partial\tilde{\upsilon}}{\partial y}+\lambda\frac{\partial\tilde{w}}{\partial z} \tag{103}$$

$$\tilde{\sigma}_z=i\beta\lambda\tilde{u}+\lambda\frac{\partial\tilde{\upsilon}}{\partial y}+(\lambda+2\mu)\frac{\partial\tilde{w}}{\partial z} \tag{104}$$

$$\tilde{\tau}_{yz}=\mu(\frac{\partial\tilde{w}}{\partial y}+\frac{\partial\tilde{\upsilon}}{\partial z}) \tag{105}$$

$$\tilde{\tau}_{zx}=\mu\frac{\partial\tilde{u}}{\partial z}+i\beta\mu\tilde{w} \tag{106}$$

$$\tilde{\tau}_{xy}=\mu\frac{\partial\tilde{u}}{\partial y}+i\beta\mu\tilde{\upsilon} \tag{107}$$

6. VALIDATION OF SOLUTION

The solution derived in the previous sections and the corresponding computer program can be validated by three means. Firstly, as has been shown in Section 5, when a zero axial wavenumber is used in the moving Green's functions, the plane-strain Green's function are indeed recovered (Equations (98) to (101)).

Secondly, the Fourier transformed moving Green's functions for a layered cylinder can be validated numerically by means of the exact solution for a point stationary harmonic load in a homogeneous whole-space. Suppose the point load is applied at point A which has a distance R_1 from the x-axis, and the displacements along a straight line, which is parallel to the x-axis and at a distance R_2 (suppose $R_2 > R_1$), is evaluated. For this source-observer configuration, the homogeneous whole-space can be regarded as a layered cylinder consisting of three sub-cylinders. The first one is a solid cylinder with an outer radius of R_1. The second one is a hollow cylinder, the inner and the outer radii being R_1 and R_2 respectively. The third one is a hollow cylinder with an infinite outer radius and an inner radius of R_2.

TABLE 1

The parameters for the whole-space

Young's modulus ($\times 10^6$ Nm^{-2})	Possion's ratio	Density (kg/m^3)	Loss factor	P-wave speed (m/s)	S-wave speed (m/s)
1770	0.4	1700	0.15	1500	610

Figure 2 presents a comparison between these two methods by showing the calculated vertical displacements along the straight line (y = 20 m, z = 0 m) due to a unit vertical harmonic load of 250 Hz applied at the origin. The material parameters are listed in Table 1. In the application of the Fourier transformed moving Green's functions for a layered cylinder, the FFT technique is used to transform the displacement from the wavenumber domain into the spatial domain. In the performance of the FFT, 2048 samples are used with a spacing of β equal to $0.0025\times 2\pi$. This figure indicates a high computational accuracy can be achieved by the FFT. This is true even for higher frequency and for near observer, as shown in Figures 3 and 4.

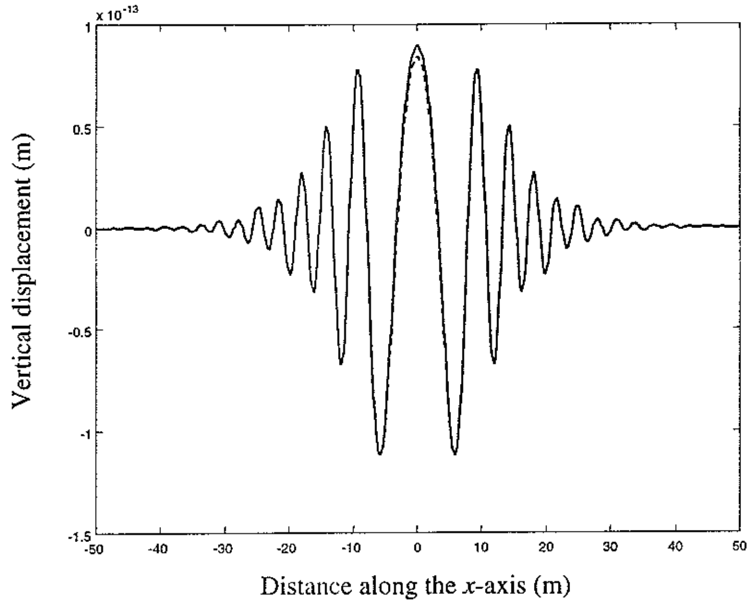

Figure 2. Vertical displacements along the straight line ($y = 20$ m, $z = 0$ m) due to a unit vertical harmonic load of 250 Hz applied at the origin. ——, by the layered cylinder theory; – – –, by the exact solution for a point stationary harmonic load.

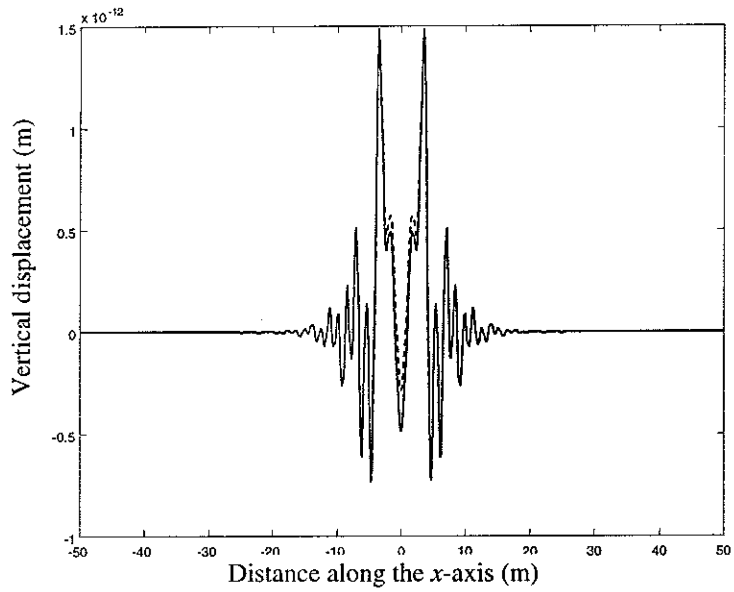

Figure 3. Vertical displacements along the straight line ($y = 0$ m, $z = 5$ m) due to a unit vertical harmonic load of 500 Hz applied at the origin. ——, by the layered cylinder theory; – – –, by the exact solution for a point stationary harmonic load.

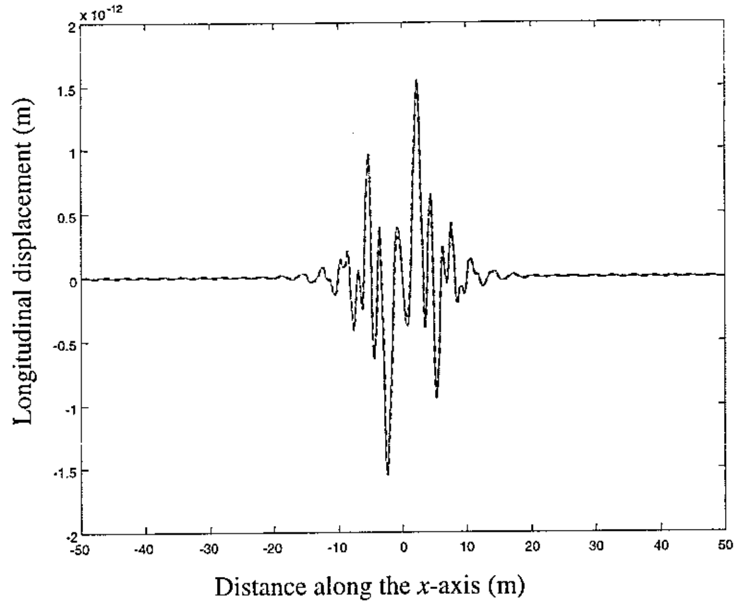

Figure 4. Longitudinal displacements along the straight line (y = 0 m, z = 5 m) due to a unit vertical harmonic load of 500 Hz applied at the origin. ——, by the layered cylinder theory; – – –, by the exact solution for a point stationary harmonic load.

Finally, the solution is validated numerically by means of the exact solution for an **undamped** homogeneous whole-space subject to a constant moving point load [Eason, Fulton and Sneddon 1955/1956]. Suppose the whole-space includes an infinitely long cylindrical hole. The axis of the hole is parallel to the x-axis. The presence of the hole would not make a significant difference if its radius is small (e.g. 0.05 m). Now a constant point load is applied on the wall of the hole and moves in the x-direction. The response of the whole-space with the hole to this moving load can be evaluated using the cylinder theory and may be compared with that calculated from the exact solution for the whole-space without any inclusion subject to the same load. The results of such a calculation are shown in Figures 5 and 6 for the 'subsonic' case in which the load speed, 500 m/s, is lower than the shear wave speed in the whole-space, 610 m/s (see Table 1). In the application of the Fourier transformed moving Green's functions for a layered cylinder, the FFT technique is also used. Again 2048 samples are used with a spacing of β equal to $0.0025\times2\pi$. Figures 5 and 6 also show that a high computational accuracy can be achieved by the FFT.

Figure 7 shows the vertical and longitudinal displacements for the 'transonic' case in which the load speed, 800 m/s, is between the shear wave speed and the P-wave speed of the whole-space. This figure indicates a pulse at

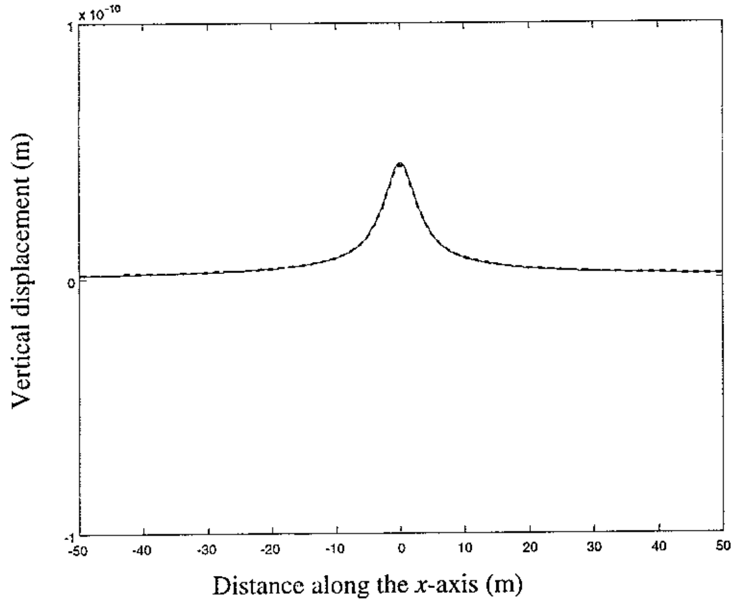

Figure 5. Vertical displacements along the straight line ($y = 0$ m, $z = 5$ m) due to a unit vertical constant load moving along the x-axis at 500 m/s. ——, by the layered cylinder theory; – – –, by the exact solution for a point moving load.

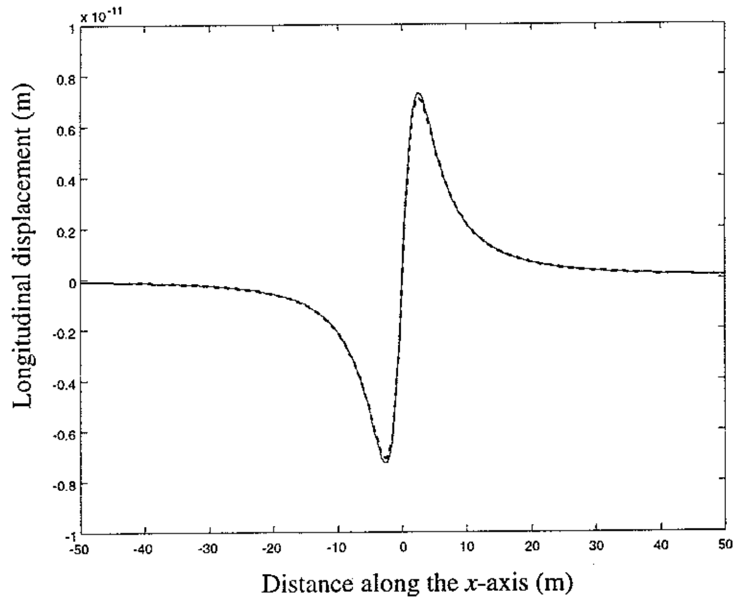

Figure 6. Longitudinal displacements along the straight line ($y = 0$ m, $z = 5$ m) due to a unit vertical constant load moving along the x-axis at 500 m/s. ——, by the layered cylinder theory; – – –, by the exact solution for a point moving load.

$$x = \pm\sqrt{[(c/c_2)^2 - 1](y^2 + z^2)} = \pm\sqrt{[(800/610)^2 - 1](0^2 + 5^2)} = 4.24\,\text{m}$$

This feature is confirmed by the exact solution which states that, at these positions, a singularity occurs. Comparison between the layered cylinder theory and the exact solution for a constant moving load is not applicable due to this singularity.

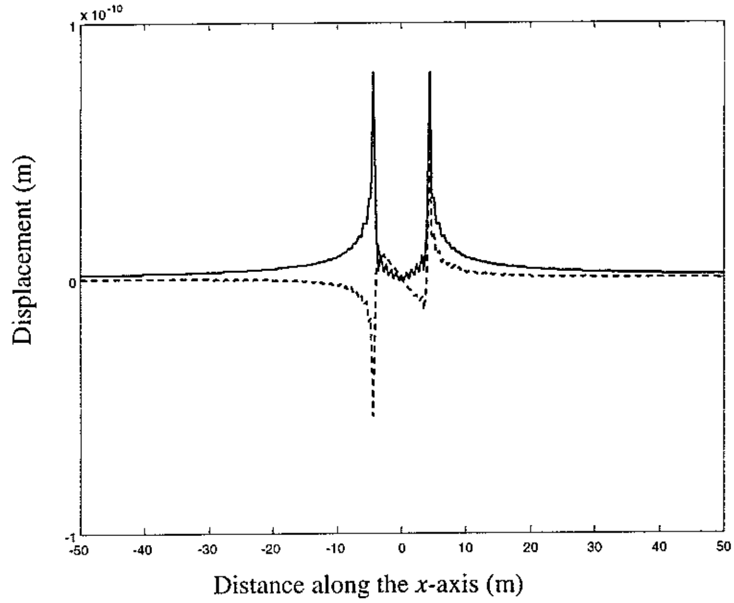

Figure 7. Vertical (——) and longitudinal (– – –) displacements along the straight line ($y = 0$ m, $z = 5$ m) due to a unit vertical constant load moving along the x-axis at 800 m/s, calculated using the layered cylinder theory.

7. CONCLUSIONS

In this paper, the Fourier transformed steady state responses (Fourier transformed Green's functions) are derived for a radially layered circular cylinder of infinite length subject to harmonic loads moving uniformly in the direction of the cylinder axis. These have not been found in previous publications. By letting the inner radius of a hollow cylinder approach zero and its outer radius increase to infinity, the Fourier transformed moving Green's functions of a homogeneous visco-elastic whole-space are formulated in terms of Bessel functions of the second kind. This provides the Green's functions that can be used in the 'two-and half-dimensional boundary element method'. These moving Green's functions are comparable with the stationary Green's functions recently published by Tadeu and Kausel. The formulae and the computer program that implements the moving Green's functions have been validated by means of the plane-strain Green's functions which correspond to the case of zero axial wavenumber, and by comparison with the exact solution for a point stationary harmonic load and the exact solution for a moving constant load.

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