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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

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Authorised for issue by
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Group Chairman

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IDENTIFICATION OF NONLINEAR NORMAL MODES AND COUPLED NONLINEAR MODES

1 Introduction

The characterization of a mechanical system by its modal parameters is of great interest in many fields of vibration engineering (experimental identification, vibration control, vibration analysis, etc.,). In the case of a linear structure, the modal theory is well known, and it is normally used in order to express free or forced oscillations as a linear combination of the normal modes, which form an orthogonal closed set. Over the years, many investigators have focused on the nonlinear extension of the definition of the normal mode defined in the classical theory of vibration.

After giving the definition of Nonlinear Normal Modes (NNM) and explaining some of their proprieties, this report shall present an approximate approach, based on harmonic balance, defining the Coupled Nonlinear Modes (CNM) and its uses.

The studies have concentrated on the free vibration of mechanical systems in terms of an analytical solution and, in parallel investigations, comparing the results by modal identification using time-frequency transforms (Gabor Transform) for lightly damped systems.

This report focuses on nonlinear mechanical system endowed with linear and cubic stiffness. For some of them a procedure for the identification of modal frequency modulation laws is proposed. The next phase will be an experimental validation. The report closes by a short introduction of a proposed experimental representation, which is to be used to replicate the simulations of a coupled nonlinear system.

2 Definition and proprieties of NNM

The first original work on NNM was by Rosemberg [22, 25, 26, 30], who defined normal modes for an autonomous mechanical system as a particular kind of motion that satisfied the follow proprieties (where u_i is the motion of the i^{th} d.o.f) [1]:

1. There exists a τ =*constant*, such that

   $u_i(t)=u_i(t+\tau)$

   i.e. the motion of all masses is equiperiodic

2. If t_r is any instant of time, there exists a single t_0 in $t_r<t<t_r+\tau/2$ such that

   $u_i(t_0)=0$

   In words, during any interval of half period, the system passes precisely once through its equilibrium configuration

3. There exists a single $t_1 \neq t_0$ in $t_r<t<t_r+\tau/2$ such that

   $u_i(t_1)=0$

   i.e. during any interval of half period, the velocities of all masses vanish precisely once.

4. Let r (fixed) be any one of the $i=1,...,n$. Then every $u_i(t)$ and $u_r(t)$ may be written, for all t, in the form

   $u_i=u_i(u_r)$

   In other words, the displacement of any one mass at any instant of time determines uniquely that of every other mass at the same instant of time.

Rosemberg defined also the similar modes when it happens that:

$u_i/u_r=c_{ir}$

i.e. the modes are similar when there is a linear relationship between the motions of every d.o.f.

In the configuration space the equation $u_i=u_i(u_r)$ represent the trajectory of the mode, thus if the modes are similar, in the configuration space, they are represented by straight lines, otherwise they are represented by curved lines. In the same space one can represent the

---

[1] In these Memorandum the subscript convention applied throughout is:

$i=1,..,l$	index for d.o.f.
$j=1,...,n$	index for mode
$k=1,...,m$	index for time segment

potential energy U of the system as a closed, smooth surface surrounding the origin of the configuration space. It was demonstrated that this surface and the trajectory of the mode $u_i=u_i(u_r)$ are always symmetric with respect the origin. Moreover all the trajectories (similar – non similar modes) terminate on the surface U orthogonally.

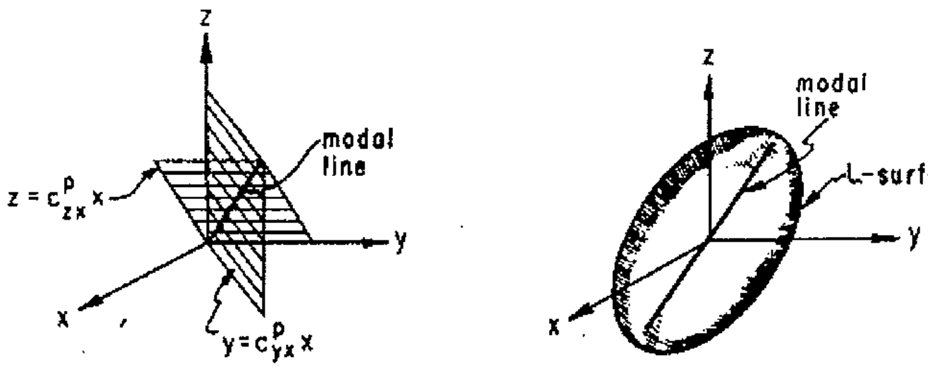


Figure 2.1: modal line and U surface in the configuration space (Ref.[25])

Rosemberg was able to demonstrate that the follow classes of mechanical system always possess similar modes:

- Linear systems
- Uniform systems, (systems with every mass identical and stiffness equal)
- Homogeneous systems (a chain of masses with restoring forces expressed by polynomial of the same degree)
- Symmetric systems

At this point it is appropriate to remember that:

1) In a linear system, the normal-mode vibrations are always similar; the normal mode coordinates decouple the equations of motion for arbitrary motions; mode and period of the normal-mode vibration are independent of the energy level of the motion. Linear combinations of solutions are also solutions.
2) If the system is nonlinear, the normal mode coordinates decouple the equations of the motions for that mode only. If the normal-mode vibration is similar, the mode is independent of the energy level, but not the period. Linear combinations of solutions are no longer solutions.

3) If the system is nonlinear and the normal mode vibration is non similar, mode as well as period depend on the energy level.

One of the most important methods to calculate the NNM is the method of multiple scale [11], by this one is able to study the problem of the bifurcation and the stability of the modes. It means that the number of the NNM could be bigger then the d.o.f. of the system and some of them could be unstable.

2.1 A 2d.o.f. symmetric system

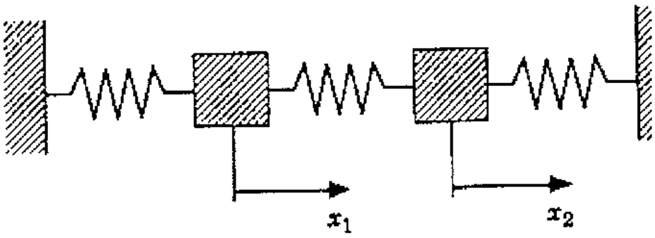


Figure 2.2: 2dof system

Vakakis [7, 13, 20] considered the symmetric system governed by the follow equations of the motions:

$$\begin{aligned} &\ddot{x}_1 + x_1 + x_1^3 + k(x_1 - x_2)^3 = 0 \\ &\ddot{x}_2 + x_2 + x_2^3 - k(x_1 - x_2)^3 = 0 \end{aligned} \tag{2.1}$$

Because the system is symmetric then the modes are similar:

$$x_2 = cx_1 \tag{2.2}$$

So substituting (2.2) into the equations of the motion (2.1)one obtains

$$\begin{aligned} &\ddot{x}_1 + x_1 + [1 + k(1-c)^3]x_1^3 = 0 \\ &\ddot{x}_1 + x_1 - \frac{1}{c}[k(1-c)^3 + c^3]x_1^3 = 0 \end{aligned} \tag{2.3}$$

and putting equal the coefficients of the cubic terms the following equation results:

$$k(1+c)(c-1)^3 = c(1-c^2)$$

from which one can have the following solutions:

$$c_{1,2} = \pm 1$$

$$c_{3,4} = \frac{1}{2k}\left(2k - 1 \pm \sqrt{(-4k+1)}\right)$$

The solutions $c_{1,2}$ always exists and do not depend upon the coupling stiffness k, whilst the solutions $c_{3,4}$ exist only when $\Delta=(1 - 4k)>0$, that is $k<0.25$.

In that situation there are 4 solutions, one of which (for $c=-1$) is unstable. This means that the solution c=-1 for $k<0.25$ bifurcate into two *localized* modes. The situation is represented in Figure 2.3.

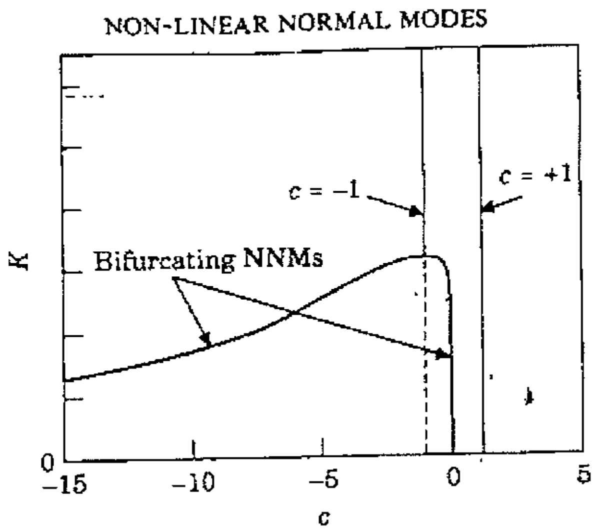


Figure 2.3: NNM of the system with cubic nonlinearities as a function of the stiffness parameter K: ——— stable; -----unstable (ref. [13])

The stability of the modes can be investigated by various methods (Poincarè maps, Multiple scales, etc..)

Since a single mode is able to decouple the equations of motions, the frequency of the mode as the frequency of a Duffing oscillator can be calculated[2].

For a Duffing oscillator

$$\ddot{x} + \omega_0^2 x + \mu x^3 = 0 \qquad (2.4)$$

The main frequency can be approximated by

$$\omega = \sqrt{\omega_0^2 + \frac{3}{4}\mu a_0^2} \qquad (2.5)$$

where a_0 is the amplitude of oscillation.

[2] In practice one can substitute one of the modal relationships found ($x_2=cx_1$) into one of the equations of motion, (2.1) in order to have an equation of a nonlinear SDOF.

Some numerical simulations of the unforced system have be produced for various values of k and initial condition in order to force the system towards a modal motion.

	Mode 1 (c_1) (1,1) $\ddot{x}+x+x^3=0$		Mode 2 (c_3) (1,-0.382) $\ddot{x}+x+1.528x^3=0$		Mode 3 (c_4) (1,-2.618) $\ddot{x}+x+10.472x^3=0$		Mode 4 (c_1) (1,-1)*Unstable* $\ddot{x}+x+2.6x^3=0$	
Amplitude	***a.f.***	***e.f.***	***a.f.***	***e.f.***	***a.f.*1**	***e.f***	***a.f.***	***e.f***
1	0.21	0.21	0.23	0.23	0.47	0.47	0.27	0.27
2	0.32	0.32	0.38	0.37	0.91	0.96	0.47	0.47
3	0.44	0.44	0.54	0.54	1.35	1.30	0.69	0.7
4	0.57	0.58	0.70	0.71	1.79	1.80	0.90	0.95
5	0.71	0.72	0.87	0.91	2.24	2.20	1.12	1.25
6	0.84	0.88	1.04	1.13	2.68	2.60	1.34	1.3
7	0.98	1.05	1.20	1.43	3.13	3.10	1.56	1.5

Table 2.1: Comparison between analytical frequencies (a.f.) and the experimental (e.f.) ones for k=0.2 (<0.25 bifurcated modes), values in Hz

A good agreement between the analytical frequency (*a.f.*) and the experimental (*e.f.*) is observed, especially at low amplitudes.

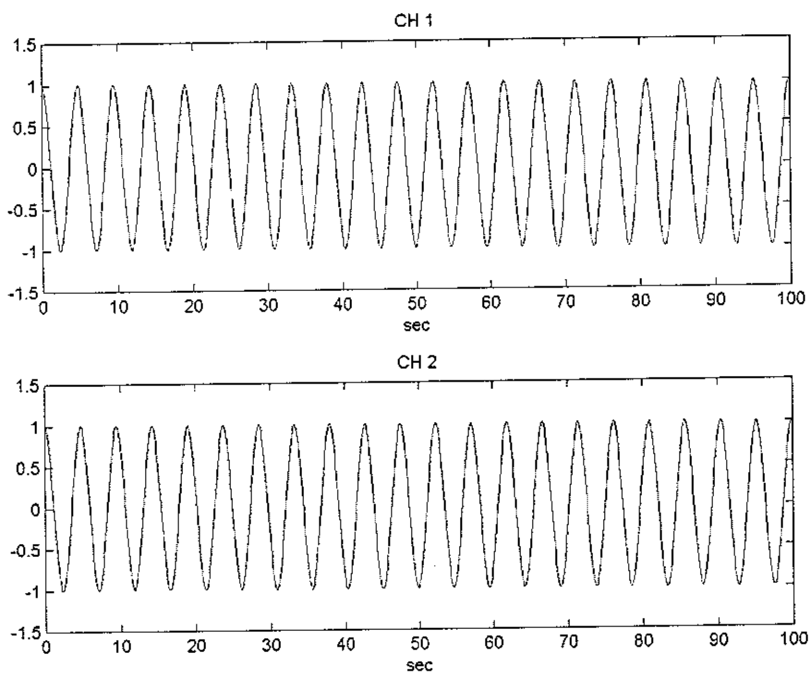


Figure 2.4: Time histories of the simulation with the following initial condition: displacement (1,1), velocity (0,0), i.e. Mode 1 with amplitude 1 in the table 2.1

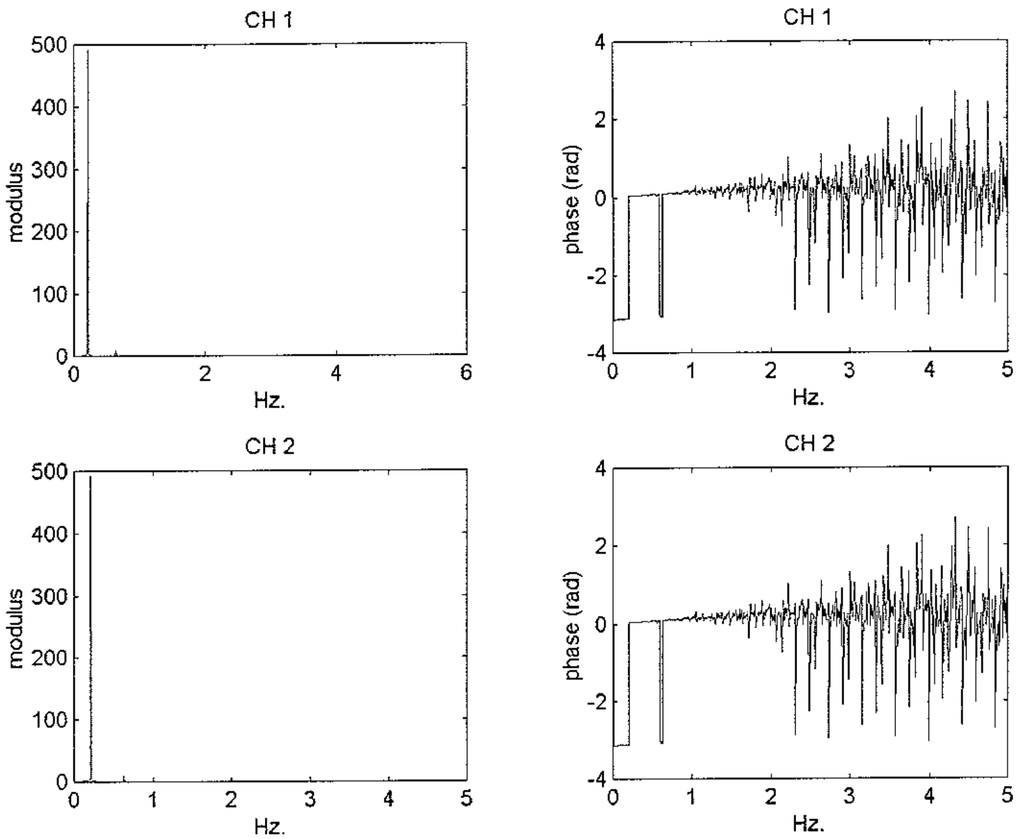


Figure 2.5: FFT of the simulation pictured in figure 2.4

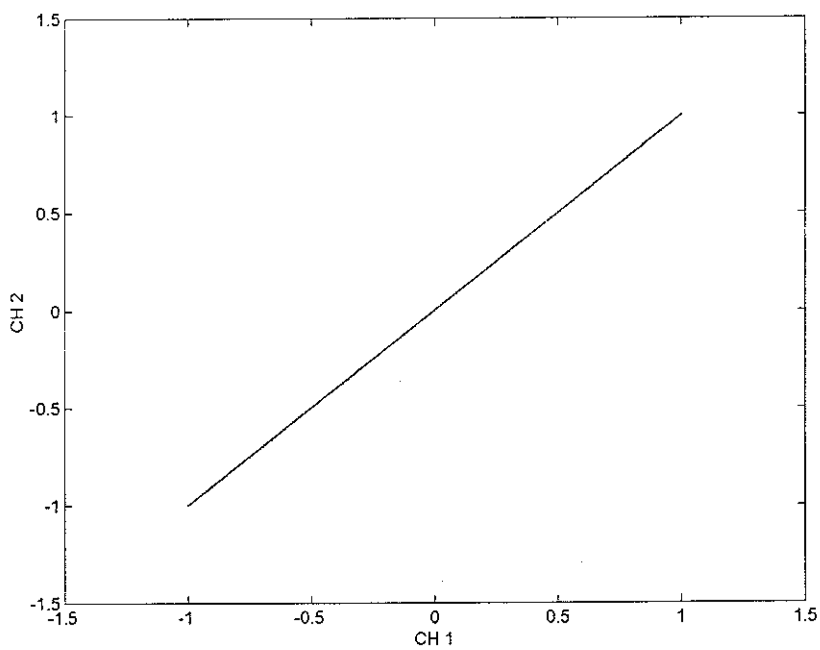


Figure 2.6: Plot of the response displacements, x_1 vs x_2, of the simulation pictured in figure 2.4

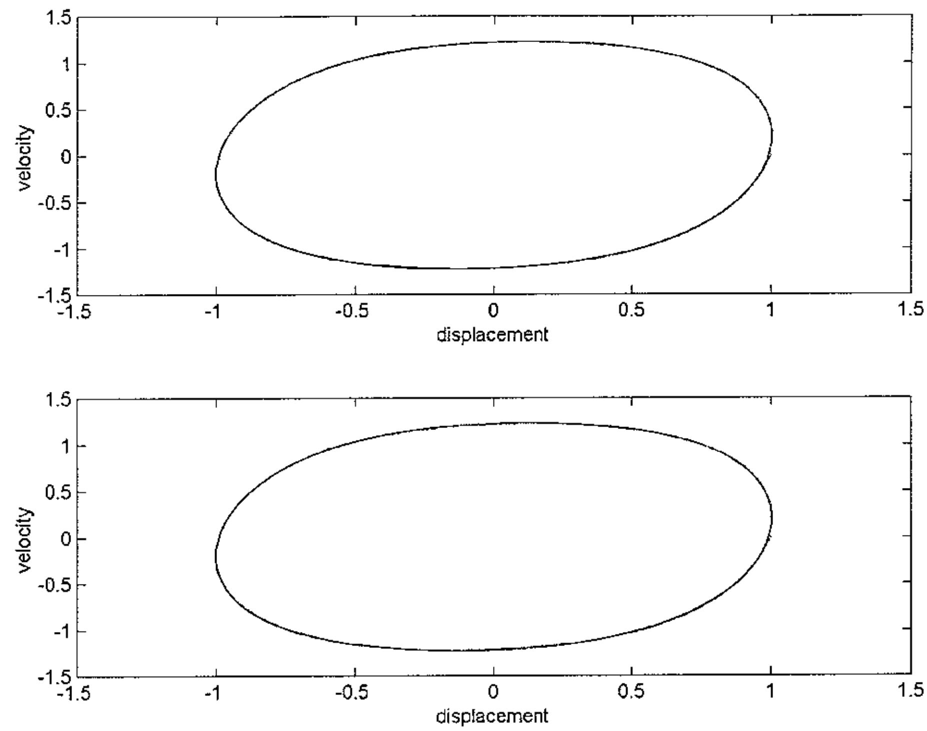


Figure 2.7 Plots of the velocity vs displacement of the simulation pictured in figure 2.4

The stability could be investigated by Poincarè maps [10], checking that the eigenvalues of the Hessian of the Poincarè maps of the simulation are between –1 and 1. This means that every couple ($\mathcal{D}$, $\mathcal{T}$) of the Hessian must be inside the triangle pictured in Figure 2.8

Where $\mathcal{D}$ and $\mathcal{T}$ are, respectively, the determinant and the trace of the Hessian.

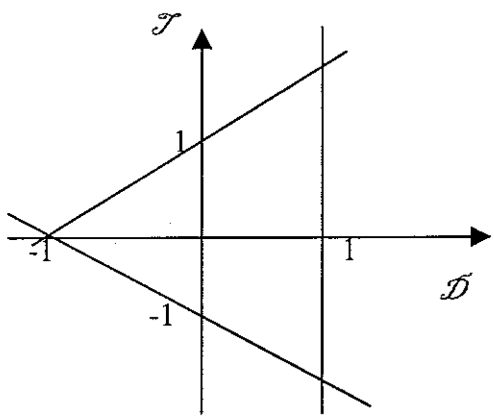


Figure 2.8: stability area in $\mathcal{D}$, $\mathcal{T}$ plane

In this way it was possible to check the stability of the motion of the mode and to find that the mode (1,-1) is unstable.

Other simulation in free vibration has been performed for a system without bifurcation. It was considered for k=0.5.

	Mode 1 (c_1) (1,1) $\ddot{x}+x+x^3=0$		Mode 2 (c_2) (1,-1) $\ddot{x}+x+5x^3=0$	
Amplitude	***a.f.***	***e.f.***	***a.f.***	***e.f.***
1	0.21	0.21	0.35	0.34
2	0.32	0.32	0.64	0.64
3	0.44	0.44	0.94	1
4	0.57	0.58	1.24	1.2
5	0.71	0.72	1.55	1.5
6	0.84	0.88	1.86	1.8
7	0.98	1.05	2.16	2.1

Table 2.2: : comparison between analytical frequencies (a.f.) and the experimental (e.f.) ones for k=0.5 (>0.25), values in Hz

At this point the equation of the system in a matrix form can be written as

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{S}_1\mathbf{x}+\mathbf{S}_2\mathbf{x}^2+\mathbf{S}_3\mathbf{x}^3=\mathbf{0} \qquad (2.6)$$

where

$$\mathbf{M}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \mathbf{S}_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbf{S}_2=3k\begin{pmatrix}-x_2 & x_1\\ x_2 & -x_1\end{pmatrix}, \quad \mathbf{S}_3=\begin{pmatrix}1+k & -k\\ -k & 1+k\end{pmatrix}$$

$$\mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}, \quad \mathbf{x}^2=\begin{pmatrix}x_1^2\\ x_2^2\end{pmatrix} \quad \mathbf{x}^3=\begin{pmatrix}x_1^3\\ x_2^3\end{pmatrix}$$

If **B** is the modal tensor between the physical and modal coordinates, that is $\mathbf{x}=\mathbf{B}\mathbf{q}$, where $\mathbf{q}=\begin{pmatrix}q_1\\ q_2\end{pmatrix}$ are the modal coordinates, the system in the modal space can be described by:

$$\mathbf{B}^{-1}\mathbf{M}\mathbf{B}\ddot{\mathbf{x}}+\mathbf{B}^{-1}\mathbf{S}_1\mathbf{B}\mathbf{x}+\mathbf{B}^{-1}\mathbf{S}_2\mathbf{B}\mathbf{x}^2+\mathbf{B}^{-1}\mathbf{S}_3\mathbf{B}\mathbf{x}^3=\mathbf{0} \qquad (2.7)$$

In this case $\mathbf{B}=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, and $\mathbf{B}^{-1}=\begin{pmatrix}0.5 & 0.5\\ 0.5 & -0.5\end{pmatrix}$

The equations of the system become:

$$\ddot{q}_1 + q_1 + q_1^3 + 3q_1q_2^2 = 0$$
$$\ddot{q}_2 + q_2 + 2q_2^3(1.5+2k) + 3(1+5k)q_2q_1^2 = 0 \tag{2.8}$$

and for k=0.5 are:

$$\ddot{q}_1 + q_1 + q_1^3 + 3q_1q_2^2 = 0$$
$$\ddot{q}_2 + q_2 + 5q_2^3 + 10.5q_2q_1^2 = 0 \tag{2.9}$$

For the initial condition $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the modal coordinates become:

$$\mathbf{q}_1 = \mathbf{B}^{-1}\mathbf{x}_1 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.10}$$

It means that only the first modal coordinate is excited and the system could be reduced to:

$\ddot{q}_1 + q_1 + q_1^3 = 0$, that is the equation of a Duffing oscillator.

The solution of the system in physical coordinates will be:

$$\mathbf{x}_1 = \mathbf{B}\mathbf{q}_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_{11} \\ 0 \end{pmatrix} = \begin{pmatrix} q_{11} \\ q_{11} \end{pmatrix} \tag{2.11}$$

In the same way one can operate for the second mode with initial condition $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,:

$$\mathbf{q}_2 = \mathbf{B}^{-1}\mathbf{x}_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.12}$$

that leads to the equation $\ddot{q}_2 + q_2 + 5q_2^3 = 0$, and then

$$\mathbf{x}_2 = \mathbf{B}\mathbf{q}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ q_{22} \end{pmatrix} = \begin{pmatrix} q_{22} \\ -q_{22} \end{pmatrix} \tag{2.13}$$

Observation: during the modal motions the coupling terms $q_1q_2^2$ and $q_2q_1^2$ disappear, so, in this way, the NNM are uncoupled into separate equations of motion.

Shaw and Pierre [21]show that, in same cases, it is possible to neglect the coupled terms in order to reconstruct the answer of the system to whatever initial conditions.

Unfortunately this is not always true, in fact, for the system above, it is not possible to reconstruct the answer from general i.e. where even if the shapes of the signal of the displacements of the physical simulations and the ones reconstructed by modal simulations are similar, they have different spectrograms. Hence the response solution of a nonlinear system is not a combination of NNM.

3 The Invariant Manifold Approaches

Shaw and Pierre reformulated the concept of NNM for a general class of nonlinear discrete oscillators. Their analysis was carried out in the real domain and was based on the computation of invariant manifolds of motion on which the NNM take place. The parameterization of the invariant manifolds of the NNM was performed by employing two independent reference variables; a reference positional displacement and a reference positional velocity.

In this formulation NNM are defined as the invariant subspaces in phase space of the nonlinear equations of motion governing the dynamical system.

That is for a nonlinear system the i^{th} equation could be written:

$$\ddot{x}_i + f_i(\mathbf{x}, \dot{\mathbf{x}}) = 0 \tag{3.1}$$

or

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -f_i(\mathbf{x}, \mathbf{y}) \end{aligned} \tag{3.2}$$

Suppose that there exists a motion for which all displacement and velocities are functionally related to a single reference displacement-velocity pair, (x_1, y_1):

$$x_i = X(x_1, y_1) \qquad y_i = Y(x_1, y_1) \tag{3.3}$$

An NNM is *defined as a motion that takes place on the two-dimensional invariant manifold defined by equation (3.3).*

The great advantage of this approach is that this definition is valid also for non-conservative systems as well. The solution is expressed in term of a power series, and substituting the expansions into the equation of the system a nonlinear algebraic system is obtained. A drawback of the invariant manifold approach was that the necessary computations became cumbersome, even for simple, lower dimensional nonlinear systems.

A further development of the manifold approach was proposed by Nayfeh and Nayfeh [14, 15], who reformulated the methodology on a complex framework. The following coordinate transformation is introduced:

$$x_i = \zeta_i + \bar{\zeta}_i \qquad \dot{x}_i = j\omega_i(\zeta_i + \bar{\zeta}_i) \tag{3.4}$$

where indicates the complex conjugate

The modes could be expressed by:

$$\zeta_i = h_i(\zeta_1, \bar{\zeta}_1) \tag{3.5}$$

4 Coupled Nonlinear Modes (CNM)

As shown previously, the solution of a nonlinear system is not a linear combination of the NNM even in an approximate way. Bellizzi and Boch [31, 32, 33] developed the concept of CNM to analyze a lightly damped mechanical system with strongly nonlinear restoring forces. The free oscillations (with arbitrary initial condition) of a conservative non-linear system are approximated by a linear combination of harmonic terms. Each harmonic term depends on a mode shape vector and its corresponding frequency. Each pair consisting of a frequency and a mode shape vector defines a CNM. One important aspect of the CNM is that the frequencies and mode shapes depend on the amplitude of all of the modes.

A CNM is a harmonic approximation of the NNM when every modal amplitude is 0, except the j^{th} corresponding to the j^{th} mode.

It is assumed that:

$$x_i = \sum_j Y_{ji} a_j \cos \Phi_j(t) \qquad (4.1)^3$$

where a_j , Y_{ji} and Φ_j are, respectively, the modal amplitude, the modal component and the modal angular frequency.

---

[3] Where $\Phi_j(t) = \int_0^t \Omega_j(s)ds + \varphi_j(t)$

Substituting this into equations of motion and applying the harmonic balance procedure yields the nonlinear algebraic system:

$$\frac{1}{a_j}2\int F_i \cos\Phi_j d\Phi = [M]Y_j\Omega_j^2 \tag{4.2}$$

where $[M]$ is the mass tensor, F_i is the restoring force on the i^{th} degree of freedom and

$$\int(.)d\Phi = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}(.)d\Phi_1 d\Phi_2 \ldots d\Phi_n \tag{4.3}$$

Adding a normalizing relationship to these it is possible to solve the nonlinear algebraic system for fixed modal amplitudes, in order to obtain the coupled nonlinear modes as a function of the modal amplitudes:

$$Y_j(a_1,a_2,\ldots,a_n) \qquad \Omega_j(a_1,a_2,\ldots,a_n) \tag{4.4}$$

When this procedure is applied to a linear system the mode shapes and the frequencies do not depend on modal amplitudes and, corresponding to the eigenvalue problem for a linear system, the harmonic balance procedure yields an exact solution.

By application of the harmonic balance method it is possible to analytically calculate the Coupled Nonlinear Modes (CNM) for many types of mechanical systems, i.e. one can calculate the relationships between the modal parameters (modal components and modal frequencies) and the modal amplitudes. Likewise one can calculate Nonlinear Normal Modes (NNM), putting all modal amplitudes equal to 0 except the j^{th} corresponding to the j^{th} NNM.

Alternatively modal parameters have been determined experimentally (or from numerical simulations) both for conservative systems (for example by FFT) and for non-conservative systems (for example by Gabor Transform). A higher level identification procedure could be to try to identify the relationships between modal parameters and modal amplitudes in order to compare them with those calculated analytically by the method of harmonic balance.

5 NNM calculated by the method of harmonic balance.

The aim is to know, in order to specify, the initial conditions to give to a nonlinear conservative system, initially at rest, so that the subsequent free vibration consists of motion in a corresponding particular NNM.

Consider a MDOF nonlinear system described by the following equations of motion:

$$m_i\ddot{x}_i + F_i(x_1,...,x_l) = 0 \tag{5.1}$$

where $F_i(x_1,...,x_l)$ is the restoring force, generally nonlinear, and in the following examples assumed to be a polynomial in the degrees of freedom.

By the method of harmonic balance one can write:

$$x_i(t) = \sum_j Y_{ji}(t)a_j(t)\cos\Phi_j(t) \tag{5.2}$$

where a_j ,Y_{ji} and Φ_j are, respectively, the modal amplitude, the modal component and the modal angular frequency. If the system is conservative, these modal parameters do not depend on time, so one can write:

$$x_i(t) = \sum_j Y_{ji}a_j\cos(\omega_j t + \varphi_j) \tag{5.3}$$

and the velocity is:

$$\dot{x}_i(t) = -\sum_j Y_{ji}a_j\omega_j\sin(\omega_j t + \varphi_j) \tag{5.4}$$

Thus the initial conditions at t=0 are:

$$x_i(0) = \sum_j Y_{ji}a_j\cos(\varphi_j) \tag{5.5}$$

$$\dot{x}_i(0) = -\sum_j Y_{ji}a_j\omega_j\sin(\varphi_j) \tag{5.6}$$

Assume that the initial velocities equal zero also the phase will be equal to zero, without loss of generality.

$$\dot{x}_i(0) = 0 \quad \forall i \in 1...n \rightarrow \varphi_j = 0 \quad \forall j \in 1...n \tag{5.7}$$

Thus one can re-write the initial displacements

$$x_i(0) = \sum_j Y_{ji}a_j \tag{5.8}$$

or in a matrix form it is:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{21} & \cdots & Y_{n1} \\ Y_{12} & Y_{22} & \cdots & Y_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{5.9}$$

In order to know the initial displacement to give the system in order to obtain a NNM motion every modal amplitude must be 0 except the j^{th} corresponding to the j^{th} mode. Thus the equation of motion can be written:

$$x_i(t) = Y_{ji} a_j \cos(\omega_j t) \tag{5.10}$$

Hence substituting these solutions into the expressions for the restoring force $F_i(x_1, ..., x_l)$ one can solve the system:

$$\frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} F_i \cos\omega_j d\omega_1 ... d\omega_n = a_j m_{ji} Y_{ji} \omega_j^2 \tag{5.11}$$

Normalizing the modal component Y_{ji} with respect to division by Y_{j1}, assuming this latter is non-zero, results in l equations and l unknowns. ($Y_{ji}(a_j)$, $\omega_j(a_j)$).

To find the initial displacement in order to have a motion corresponding to the j^{th} NNM, then:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ Y_{12} & Y_{22} & \cdots & Y_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_j \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_j \\ Y_{j2} a_j \\ \vdots \\ Y_{ji} a_j \\ \vdots \\ Y_{jn} a_j \end{pmatrix} = \begin{pmatrix} 1 \\ Y_{j2} \\ \vdots \\ Y_{ji} \\ \vdots \\ Y_{jn} \end{pmatrix} a_j \tag{5.12}$$

i.e. the j^{th} column and the NNM frequency $\omega_j(a_j)$ can be determined.

5.1 A numerical example

Consider a 2d.o.f nonlinear system whose equations of motion are:

$$\begin{aligned} \ddot{x}_1 + 150x_1 + 150x_1^3 + 200(x_1 - x_2) + 100(x_1 - x_2)^3 &= 0 \\ \ddot{x}_2 + 175x_2 + 200x_2^3 - 200(x_1 - x_2) - 100(x_1 - x_2)^3 &= 0 \end{aligned} \tag{5.13}$$

Firstly the NNM is calculated.

The displacement can be written:

$$\begin{aligned} x_1(t) &= a_1 \cos\omega_1(t) \\ x_2(t) &= Y_{12} a_1 \cos\omega_1(t) \end{aligned} \tag{5.14}$$

Substitution into the expression for the restoring forces produces the following relationships:

$$\begin{cases} \dfrac{2}{4\pi^2}\displaystyle\int_0^{2\pi}\int_0^{2\pi} F_1 \cos\omega_1 d\omega_1 d\omega_2 = a_1\omega_1^2 \\ \dfrac{2}{4\pi^2}\displaystyle\int_0^{2\pi}\int_0^{2\pi} F_2 \cos\omega_1 d\omega_1 d\omega_2 = Y_{12}a_1\omega_1^2 \end{cases} \tag{5.15}$$

if a_1=1, one solution of the system is:

Y_{12}=.91164

ω_1^2=280.22 (rads/s)^2, that is equivalent to a frequency 2.66 Hz.

The corresponding initial displacements in the degrees of freedom are then:

$$x_1(0) = 1$$
$$x_2(0) = 0.91164$$

Numerical simulation and integration of the original equations of motion of this system with these initial displacements is shown below.

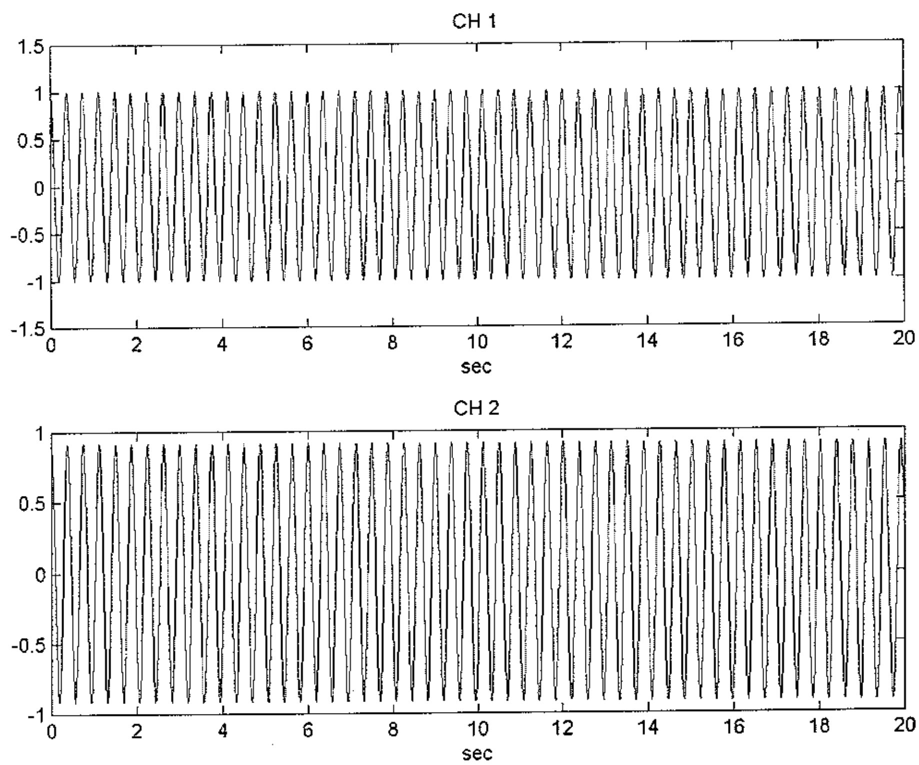


Figure 5.1: time histories of the simulation putting x_{10}=1 and x_{20}=0.91164 as initial conditions

The FFTs of the simulated response signals, given below, show a single main peak at 2.65 Hz.:

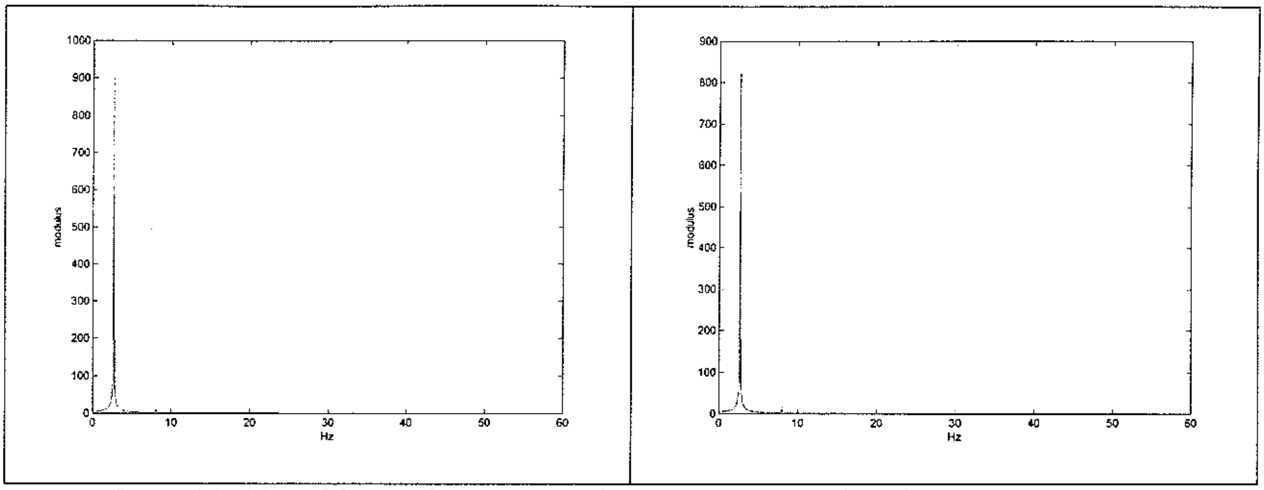


Figure 5.2: FFTs of the simulation $x_{10}=1$ and $x_{20}=0.91164$ as initial conditions

The system has 1 more real solution:

Y_{12}=-1.0895

ω_1^2=1364.6 (rads/s)^2, that is a frequency 5.88 Hz.

The corresponding initial displacements in the degrees of freedom are then:

$$x_1(0)=1$$
$$x_2(0)=-1.0895$$

This solution is the second NNM. Numerical simulation for this case is shown below.

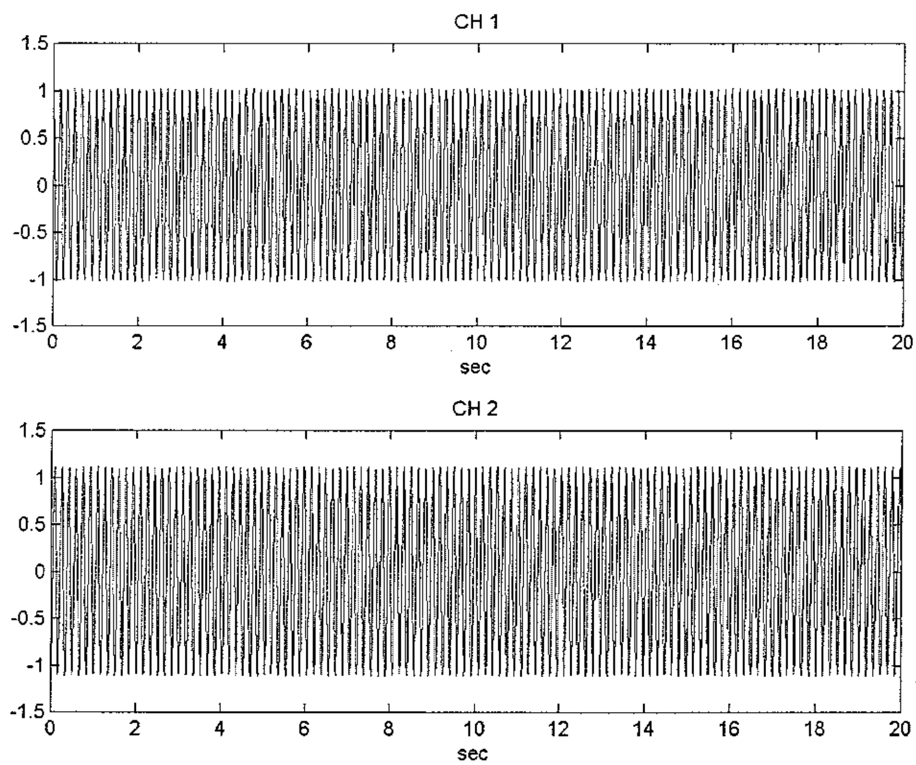
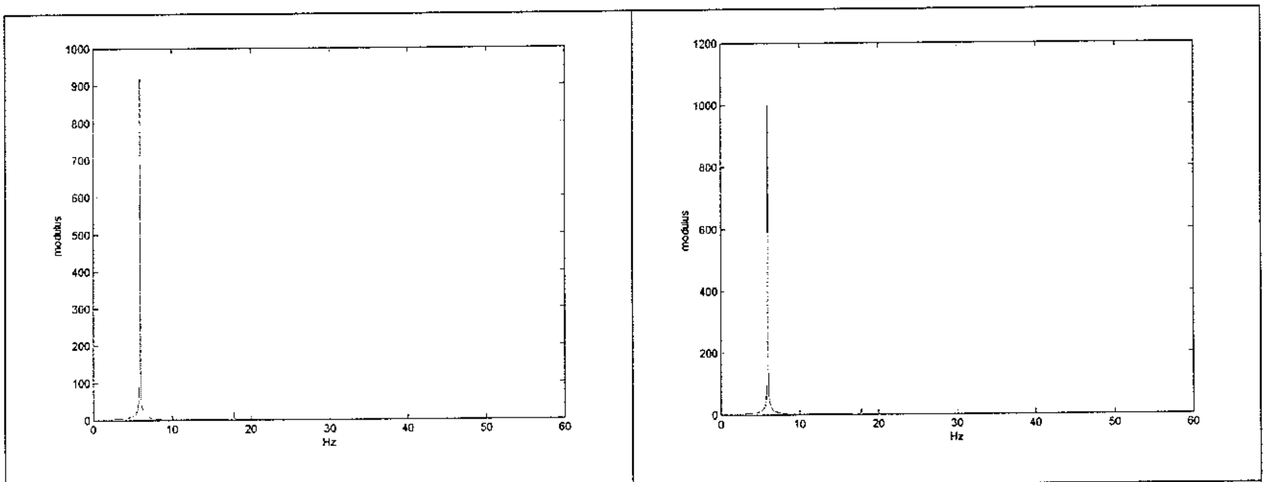


Figure 5.3: time histories of the simulation putting $x_{10}=1$ and $x_{20}=-1.0895$ as initial conditions

The FFT of the response signals displays a single peak at 5.95 Hz.:



FFT CH 1	FFT CH 2

Figure 5.4: FFTs of the simulation x_{10}=1 and x_{20}=-1.0895 as initial conditions

Alternatively, by specifying the amplitudes $a_1=0$ and $a_2=1$, the same solutions are obtained, namely

Y_{22}=-1.0895 and frequency squared ω_2^2=1364.6 (rads/s)^2

And Y_{22}=0.91164 and frequency squared ω_2^2=280.22 (rads/s)^2, as previously.

5.2 A particular case

The previous example examined in section 2.1, (equation of motion (2.1) with k=0.2) is now analysed by harmonic balance, in order to extract the NNM. Setting one of the two modal amplitudes to zero then the response is expressed by:

$$\begin{aligned} x_1(t) &= a\cos\omega(t) \\ x_2(t) &= Y_{12}a\cos\omega(t) \end{aligned} \qquad (5.16)$$

Solving the nonlinear algebraic system, the solutions obtained are:

Mode 1 $\quad Y_{12}=1 \quad \omega^2=1+0.75a^2$

Mode 2 $\quad Y_{12}=-1 \quad \omega^2=1+0.75a^2+6ka^2$

and

Mode 3 (and 4) $$\omega^2 = 0.25\frac{4k+3ka^2Y_{12}-3a^2Y_{12}}{k} \qquad (5.17)$$

It is observed that

1) the mode shapes do not depend on modal amplitudes, in fact this system exhibit similar modes because it is symmetric.
2) These solutions correspond to that ones found previously (in 2.1).
3) Mode 3 and 4 are the same mode, depending upon the normalization adopted.
4) When k>0.25 then one has just modes 1 and 2.

For an example, k=0.2 (<0.25), one obtains the following numerical solutions:

Mode 1	$Y_{12}=1$	$\omega^2=1+0.75a^2$
Mode 2	$Y_{12}=-1$	$\omega^2=1+1.95a^2$
Mode 3	$Y_{12}=-0.38197$	$\omega^2=1+1.1416a^2$
Mode 4	$Y_{12}=-2.618$	$\omega^2=1+7.854a^2$

As shown previously it is possible demonstrate that for this value of k Mode 2 is unstable, but it exists, that is, if the system is excited with some particular initial condition one can have a motion corresponding to the second mode.

The CNM corresponding to arbitrary initial conditions is

$$\begin{aligned} x_1(t) &= a_1 \cos\omega_1(t) + a_2 \cos\omega_2(t) \\ x_2(t) &= Y_{12}a_1 \cos\omega(t) + Y_{22}a_2 \cos\omega_2(t) \end{aligned} \tag{5.18}$$

Choosing some initial condition, for example an initial displacement of (2,0) and initial velocity (0,0), one obtains just one solution with physical meaning corresponding to two real CNMs, that is:

Mode 1 : $Y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\omega_1^2 = 3.25\,(\text{rad/sec})^2$ f_1=0.29 Hz

Mode 2 $Y_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\omega_2^2 = 4.45\,(\text{rad/sec})^2$ f_2=0.34 Hz

It is possible to check these results simulating the system with the same initial conditions. From these simulations it is clear that there are two main frequencies (or modal components). By analysis of the simulations in a frequency domain, one can extract these two main frequencies corresponding to the two main peaks of FFT and the two modal shapes from the FFT modulus and phase. This is illustrated in the following figures.

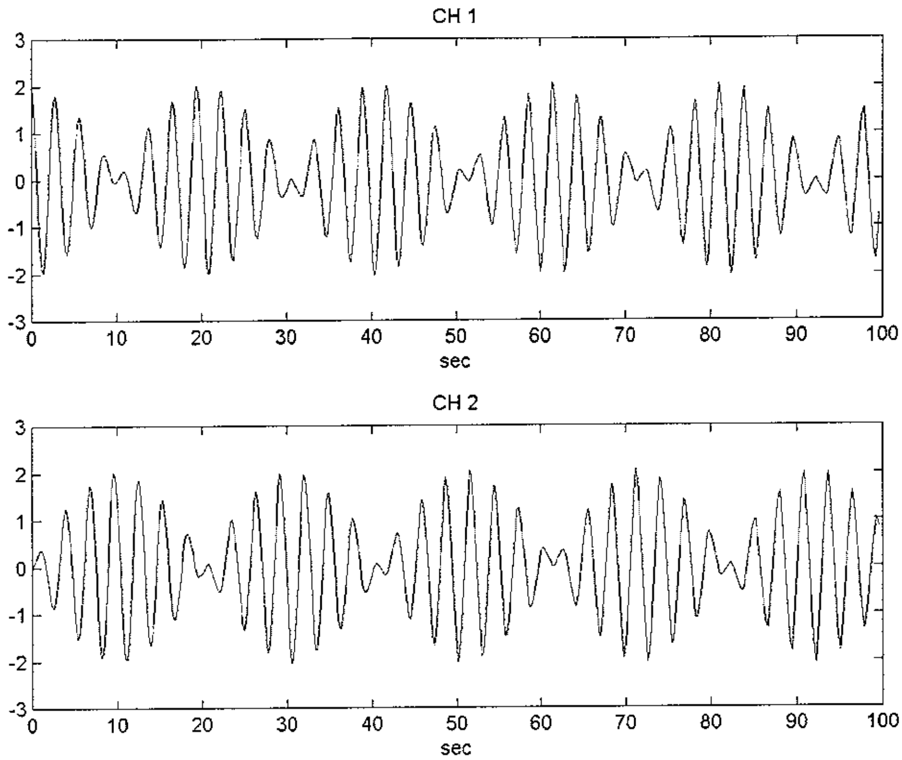


Figure 5.5: time histories of the simulation putting x_{10}=2 and x_{20}=0 as initial conditions

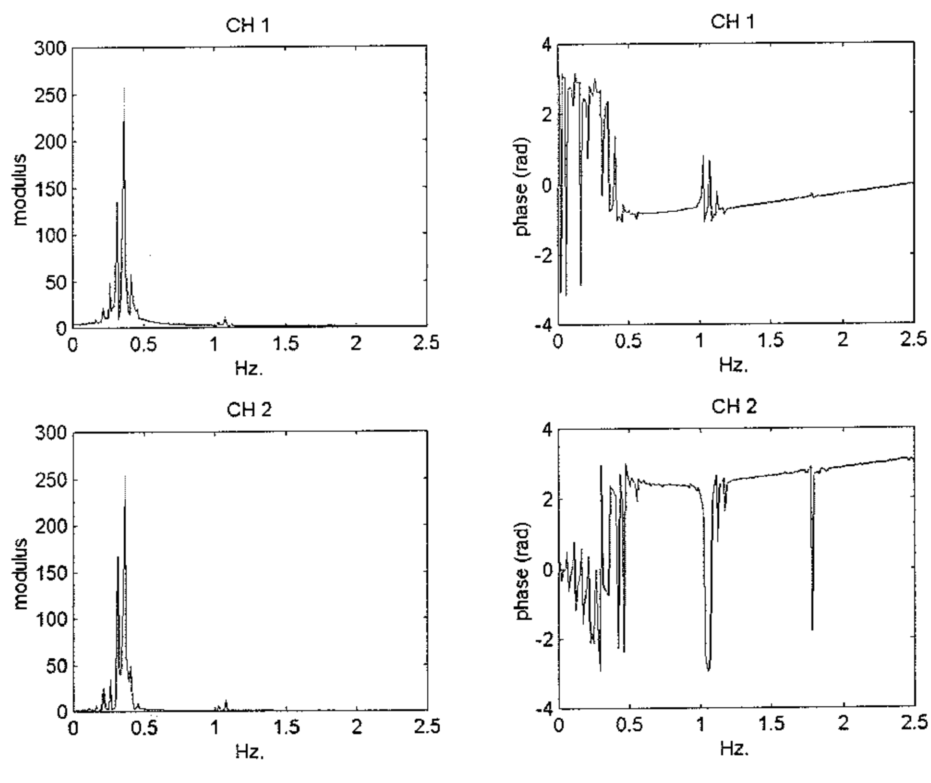


Figure 5.6: FFTs in modulus and phase of the simulation putting x_{10}=2 and x_{20}=0 as initial conditions

Observe the 2 main peaks at:

	Frequency (Hz.)	Modulus	Phase (rad.)
CH 1	0.31	134.8	-0.35
	0.36	257.5	-0.754
CH 2	0.31	166.6	-0.4
	0.36	254.6	2.364

Table 5.1: main peaks of FFTs of the simulation putting x_{10}=2 and x_{20}=0 as initial conditions

These correspond to the two CNM:

Mode 1: (1, 1.24) f_1=0.31 Hz.

Mode 2: (1, -0.99) f_2=0.36 Hz.

6 Similar modes (NNM-CNM) for nonlinear (cubic stiffness) mechanical system.

It is known that a nonlinear mechanical system could possess either similar or non-similar NNM. If the system has similar NNM it also has similar CNM, and this means that its modal components Y_{ij} do not depend on the modal amplitude a_j. Moreover if in the system there is only linear and cubic stiffness, by the method of harmonic balance it can be shown that the square of the modal angular frequency associated with the j^{th} CNM is always a linear combination of the square of the modal amplitudes, that is:

$$\omega_j^2 = \omega_{0j}^2 + \sum_{j=1}^{n} c_j a_j^2 \tag{6.1}$$

where ω_j is the modal angular frequency, ω_{0j} is the j^{th} modal angular frequency associated with the linear stiffness, n is the number of the modal component, c_j is the j^{th} constant.

The square of the j^{th} modal angular frequency associated with the j^{th} NNM is:

$$\omega_j^2 = \omega_{0j}^2 + c_j a_j^2 \tag{6.2}$$

And, in particular, for 1 d.o.f system (Duffing Oscillator) c is:

$$c = \frac{3}{4}\frac{\mu}{M} \tag{6.3}$$

where μ is the cubic stiffness.

7 Damping considerations

In the first instance one can introduce linear viscous damping ($C\dot{x}$). Suppose the system is a single d.o.f. nonlinear damped system:

$$M\ddot{x} + C\dot{x} + Kx + \mu x^3 = 0 \tag{7.1}$$

where c is the viscous damping ratio. Normalization with respect to the mass M:

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x + \frac{\mu}{M} x^3 = 0 \tag{7.2}$$

In [4] the following frequency relationship is derived:

$$\left(-\omega_D^2 + \omega_0^2 + \frac{3p^2 \mu a^2}{4M} \right)^2 + 4\nu^2 \omega_D^2 = 0 \tag{7.3}$$

where ω_D^2 is the damped angular frequency.

Letting $\left(\omega_0^2 + \frac{3p^2 \mu a^2}{4M} \right) = \omega^2$ (the square of the undamped angular frequency),

so

$$\left(-\omega_D^2 + \omega^2 \right)^2 + 4\nu^2 \omega_D^2 = 0 \tag{7.4}$$

and hence the damped natural frequency is given by: $\omega_D^2 = (\omega^2 - 2\nu^2) \pm 2\nu i \sqrt{\omega^2 - \nu^2}$

If the damping is small, neglect the term ν^2 and re-write:

$$\omega_D^2 \cong \omega^2 (1 \pm i2\nu) \tag{7.5}$$

whose modulus is

$$\left| \omega_D^2 \right| \cong \omega^2 \sqrt{1 - 4\nu^2} \tag{7.6}$$

When $\nu < 10^{-1}$, then one can assume $\omega_D = \omega$

[4] Timonshenko,Young,Weaver, "Vibration Problem in Engineering" Wiley & Sons 1974 pp.176-186

For a single d.o.f. system hysteretic damping, which is proportional to the linear stiffness, is a more realistic model and thus, as in the linear case, the NNM are able to decouple the system in the j^{th} equation of the type:

$$\ddot{q}_j + 2\nu_j\dot{q}_j + k_j q_j + f_j(q_1, q_2, ...q_n) = 0 \qquad (7.7)$$

where $f_j(q_1, q_2, ...q_n)$ is a cubic function in the variables $(q_1, q_2, ...q_n)$.

From the analysis one has the square of the j^{th} modal angular frequency associated with j^{th} NNM in terms of the modal amplitude and undamped natural frequency ω_j.

With analogy to the result for the single d.o.f. system, if we consider $\nu < 10^{-1}$, one can assume $\omega_{jD} = \omega_j$

8 A procedure for the identification of the frequency modulation law of a Nonlinear mechanical system possessing cubic stiffness and exhibiting similar modes by the Gabor Transform (GT) of the "experimental" simulation of a known system (inverse problem)

At this point it is appropriate to propose a procedure to identify the relationship between the modal frequencies and the modal amplitudes for a nonlinear system possessing cubic stiffness terms and exhibiting similar modes.

The approach taken is to analyse the "experimental" simulation of free vibration of a mechanical nonlinear system by the Gabor Transform (GT) in order to extract the mode shapes and frequency modulation laws for the CNM, from which it is possible to calculate the frequency modulation laws of the NNM.

This procedure is summarized in four principal steps:

- **Step 1**

  After calculating the l Gabor transforms (GT_i) of l experimental signals x_i , for each signal, it is possible to extract the n modal frequencies in correspondence with the n ridges of the modulus of GT_i. In this way one has:

  1. Assignment of n extracted frequencies at the instant of time $t(1)$ (by operator)

2. At the instant $t(1)$ examine the n ridges (by Boolean operations)
   In this way one has the first value of F_i (matrix of the i^{th} signal of frequencies), A_i (matrix of the i^{th} signal of Amplitudes), P_i (matrix of the i^{th} signal of Phases)
3. At the next time segment $t(k+1)$ examine the extracted ridges beginning with the values at the k^{th} instance of time.
   In this way one can complete F_i, A_i, P_i.

This can be repeated for each step of each response signal x_i in order to have $3l$ mxn matrices (F_i, A_i, P_i)

- **Step 2**

Extraction of similar mode shape. A decision is made to normalize the mode Y_j by the amplitude A_{ii}, but it is possible to normalize with respect to other components, for example towards A_{lj}; the procedure is, in general, always valid.

a) calculate the modulus (vs time): $\left|Y_{ij}(t_k)\right| = \left|\dfrac{A_{ij}(t_k)}{A_{ii}(t_k)}\right|$

b) calculate the sign (vs time):

$$s_{ij}(t_k) = sign(\cos(\left|P_j(t_k) - P_i(t_k)\right|))$$

c) calculate the mode shape: $Y_{ij}(t_k) = s_{ij}(t_k)\left|Y_{ij}(t_k)\right|$

d) extraction of similar mode: $Y_{ij} = \dfrac{\sum_{k=1}^{m} Y_{ij}(t_k)}{m}$

- **Step 3**

Extraction of modal amplitude $\mathbf{a}_i(t_k)$. As the system possesses similar modes the modal tensor **Y** is constant with respect to the modal amplitude (and time) so calculating $\mathbf{q}=\mathbf{Y}^{-1}\mathbf{x}$ and extract $\mathbf{a}_i$ (**q** is a lxm matrix) from **q** by the Hilbert transform.

- **Step 4**

Extraction of the modulation laws:

Let

$\mathbf{W}_{ij} = 4\pi^2\mathbf{F}_i^2(t_k, j)$, j^{th} square angular frequency on the j^{th} signal

$\mathbf{c}_{ij} = (c_{1j}, c_{2j}, \ldots, c_{n+1,j})_i$

Then $\mathbf{W}_{ij} = \mathbf{X}\mathbf{c}_{ij}$, where $\mathbf{X} = \begin{bmatrix} 1 & a_1^2(t_1) & \dots & a_n^2(t_1) \\ 1 & a_1^2(t_2) & \cdots & a_n^2(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_1^2(t_m) & \cdots & a_n^2(t_m) \end{bmatrix}$,

if $\mathbf{e}_{ij} = \widetilde{\mathbf{W}}_{ij} - \widetilde{\mathbf{X}}\mathbf{c}_{ij}$, is the error between the experimental "data" one can define the functional $E_{ij} = \mathbf{e}_{ij}^{\mathrm{T}}\mathbf{e}_{ij}$, equivalent to the sum of the squared errors, and by minimisation (LS method) one can calculate the matrix $\mathbf{c}$.

This step can be repeated for the l d.o.f., and then one has a statistical estimation for $\mathbf{c}_j$.

In brief :

$\mathbf{W}_{ij}$ is a m x 1 vector	(there are n x l)
$\mathbf{X}$ is a m x $(n+1)$ matrix	(there is 1)
$\mathbf{c}_{ji}$ is a $(n+1)$ x 1 vector	(there are n x l)

8.1 A numerical example

Consider a 2 d.o.f damped symmetric system whose equations of motion are:

$$\ddot{x}_1 + 200x_1 + 100x_1^3 + 200(x_1 - x_2) + 100(x_1 - x_2)^3 + 0.1\dot{x}_1 + 0.05(\dot{x}_1 - \dot{x}_2) = 0$$
$$\ddot{x}_2 + 200x_2 + 100x_2^3 - 200(x_1 - x_2) - 100(x_1 - x_2)^3 + 0.1\dot{x}_2 + 0.05(\dot{x}_2 - \dot{x}_1) = 0$$

The system possesses similar modes. If the equations are solved analytically, using harmonic balance, the following CNM are obtained:

Mode 1 $Y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \omega_1^2 = 200 + 75a_1^2 + 150a_2^2$

Mode 2 $Y_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \omega_2^2 = 600 + 150a_1^2 + 675a_2^2$

The time histories for various initial condition can also be simulated. In order to excite the two modes choose the initial condition (2,0). The picture below illustrates sections of the generated time histories:

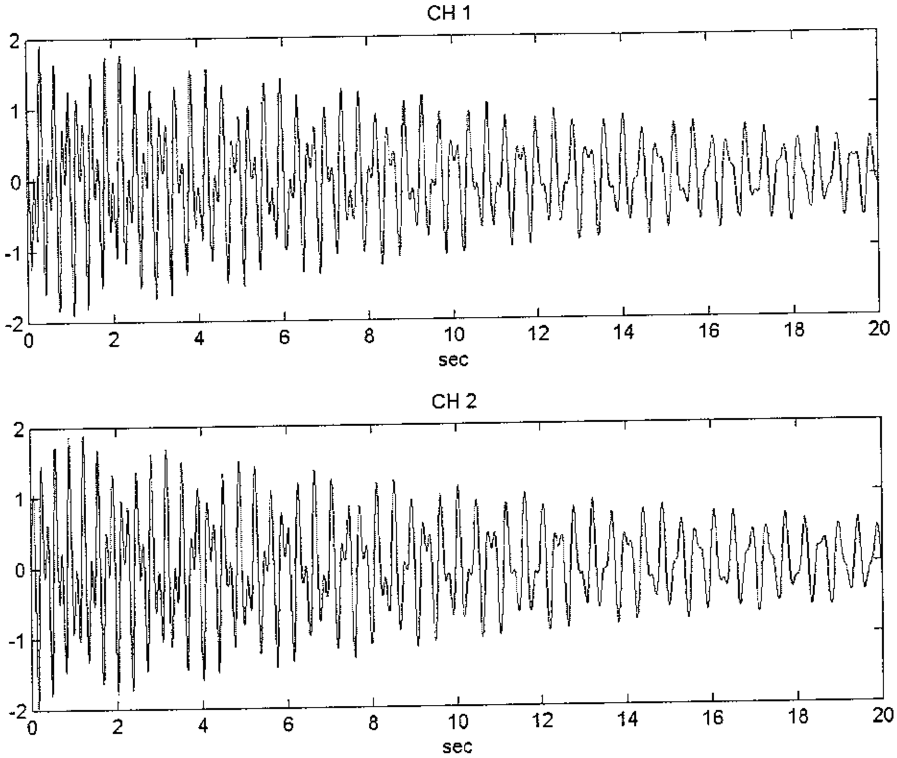
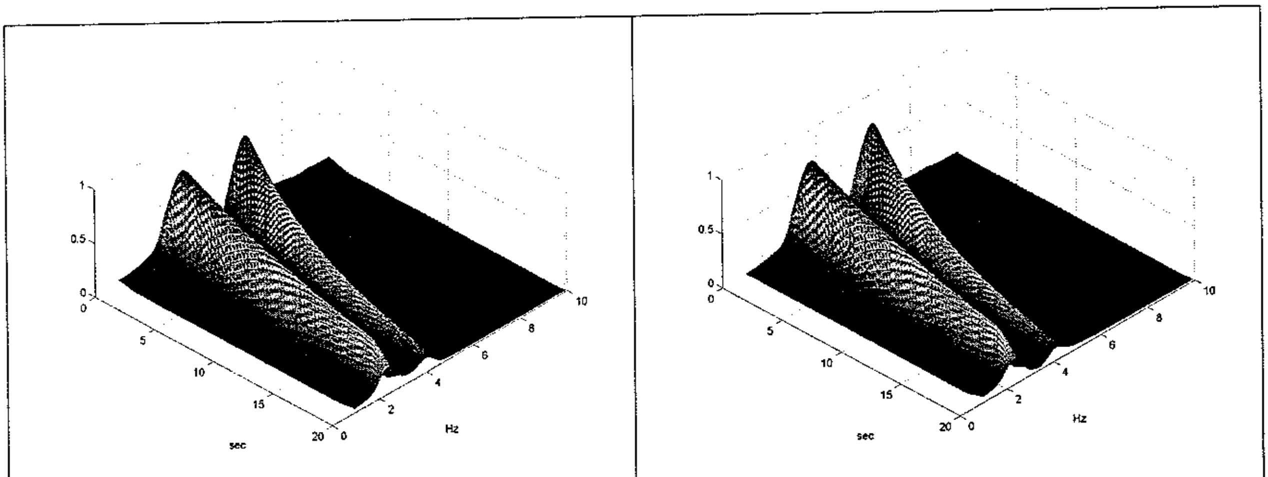


Figure 8.1: time histories of the simulation putting $x_{10}=2$ and $x_{20}=0$ as initial conditions

The modal parameters are to be identified from the Gabor transform of signals shown below:



Gabor Transform on CH 1	Gabor Transform on CH 2

Figure 8.2: Gabor transforms (GT) of the simulation putting $x_{10}=2$ and $x_{20}=0$ as initial conditions

From which it is possible to extract $\omega_1(t)$, $\omega_2(t)$, and $Y_1(t)$, $Y_2(t)$:

$$Y_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{thus } \mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The modal components are always constant, so one can calculate $\mathbf{q}=\mathbf{Y}^{-1}\,\mathbf{x}$,

where $\mathbf{x} = [x_1(t_k)\ x_2(t_k)]^T$; q_1 and q_2 are pictured below:

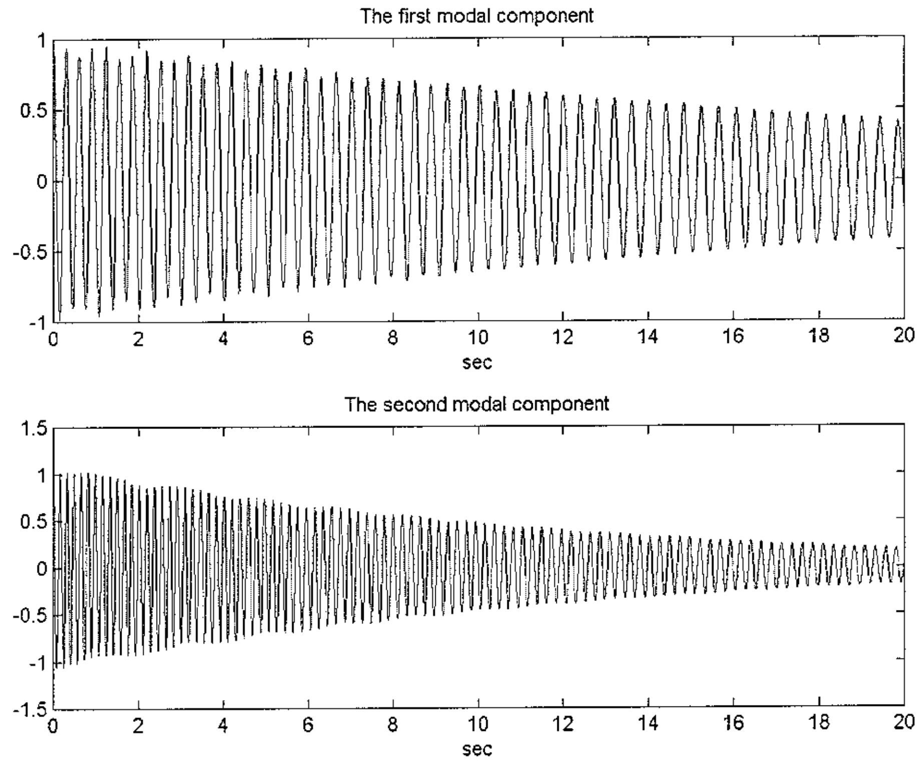


Figure 8.3: the two modal components calculated by $\mathbf{q}=\mathbf{Y}^{-1}\,\mathbf{x}$

From which it is possible to extract the modal amplitudes.

Finally the frequency modulation laws are identified as:

$$\omega_1^{\,2} = 201.72 + 75.68 a_1^{\,2} + 132.17 a_2^2$$

$$\omega_2^{\,2} = 604.55 + 165.96 a_1^{\,2} + 630.36 a_2^2$$

In a similar way one can consider various initial conditions to identify the same frequency modulation laws. The table below shows the different values of the laws identified for different initial displacements.

		Initial Conditions (Displacement)						
	Analytical Values	(0.5, 0)	(1, 0)	(1.5, 0)	(2, 0)	(2.5 0)	(3, 0)	
Mode 1	200	200.23	200.33	201.18	201.72	208.17	208.12	constant
	75	100.98	82.06	74.67	75.68	57.55	62.55	a1^2
	150	125.7	140.4	140.55	132.17	141.22	134.2	a2^2
Mode 2	600	600.74	602.84	598.79	604.55	606.38	625.24	constant
	150	340.5	186.6	206.96	165.96	154.59	111.63	a1^2
	675	551.15	646.92	616.5	630.36	628.4	656.29	a2^2

Table 8.1: Comparison between the values of the frequency modulation laws found analytically and identified with various initial conditions

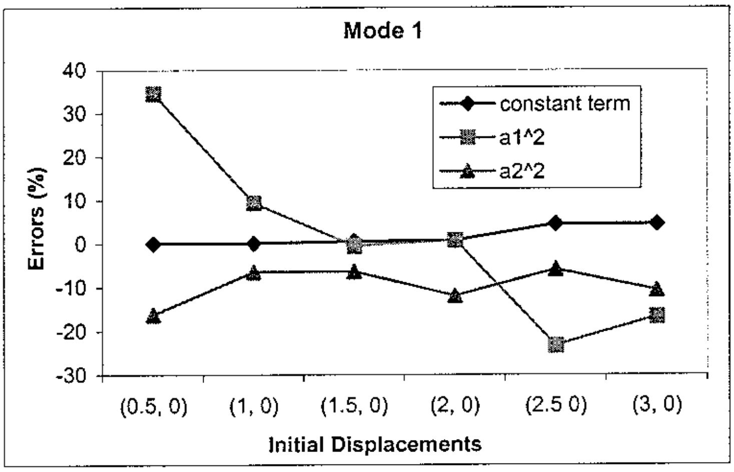


Figure 8.4: Error in percentage between the values of the first frequency modulation laws found analytically and identified with various initial conditions.

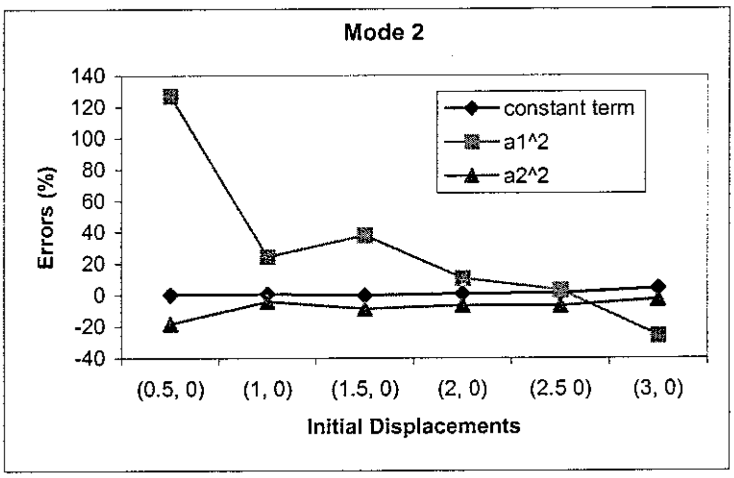


Figure 8.5: Error in percentage between the values of the first frequency modulation laws found analytically and identified with various initial conditions

Examination of figures 8.4 and 8.5 show that the linear term, that is the constant term in the frequency relationship, is only slightly influenced by the modal amplitudes and it is always well identified. The small increase in the error of this first term, at the higher initial conditions, is due to a sort of compensation of the increasing error and reduction in the nonlinear terms. In fact, errors of the nonlinear terms, i.e. the coefficients of the square of modal amplitudes, at the higher initial displacements are due to a major approximation of the analytical results. It is to be remembered that the harmonic balance method works better under small amplitude responses. A large error in the nonlinear terms is noticed at lower initial displacement. This is due to the fact that the system is lightly damped and analysing the time histories in a too short time segment the amplitudes do not vary significantly. Hence the procedure is not able to accurately identify the variation of frequency due to the nonlinear term.

9 Non similar modes

In this section two examples are presented: a conservative and a non-conservative system respectively, involving the calculation of CNM by the harmonic balance method. The two systems are consist of linear and cubic stiffness and they are completely generic, that is their CNM are nonsimilar.

In the first example, after finding analytical solutions the results of the numerical simulation are presented and then compared. In this example a procedure is presented so as to enable the selection of the physical solution from amongst all of those given by the nonlinear algebraic system imposed by the harmonic balance method.

In the second example the analytical solution is presented in an implicit form. Then the modal parameters of the mechanical system are evaluated according to the step 1-4, presented in section 8. In the end it is verified that the analytical relationships, between the modal frequency and other modal parameters, is a good approximation of the physical one.

9.1 Conservative System

Consider the following 2 d.o.f nonlinear system, described by the equations of motion

$$\begin{aligned}
&\ddot{x}_1 + 150x_1 + 150x_1^3 + 200(x_1 - x_2) + 100(x_1 - x_2)^3 = 0 \\
&\ddot{x}_2 + 175x_2 + 200x_2^3 - 200(x_1 - x_2) - 100(x_1 - x_2)^3 = 0
\end{aligned} \tag{9.1}$$

excited by the initial displacement $(x_1, x_2)=(2, 0)$.

The analytical solution for the system, expressing the values for the amplitudes in terms of the modal component and the initial condition:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Y_{12} & Y_{22} \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

that is $a_1 = -2\dfrac{Y_{22}}{Y_{12} - Y_{22}}, a_2 = 2\dfrac{Y_{12}}{Y_{12} - Y_{22}}$

Solving the system there are several solutions; most of them are complex (and conjugate), the real ones are:

$$\begin{aligned}
&Y_{12} = -1.1204, Y_{22} = 0.89951, \omega_1^2 = 1511.4, \omega_2^2 = 540.88 \\
&Y_{12} = 0.89951, Y_{22} = -1.1204, \omega_1^2 = 540.88, \omega_2^2 = 1511.4 \\
&Y_{12} = -1.7835, Y_{22} = 0.6303, \omega_1^2 = 2083.7, \omega_2^2 = 1646.1 \\
&Y_{12} = 0.6303, Y_{22} = -1.7835, \omega_1^2 = 1646.1, \omega_2^2 = 2083.7
\end{aligned}$$

The solutions appear in pairs and correspond to the following two solutions:

1)
$$\begin{aligned}
x_1 &= 1.1094\cos 23.27t + 0.8904\cos 38.88t \\
x_2 &= 0.9979\cos 23.27t + 0.9979\cos 38.88t
\end{aligned} \tag{9.2a}$$

2)
$$\begin{aligned}
x_1 &= 3.0931\cos 40.57t - 1.0931\cos 45.65t \\
x_2 &= 1.9496\cos 40.57t - 1.9495\cos 45.65t
\end{aligned} \tag{9.2b}$$

Only one pair has a physical meaning, as it is evident plotting them in figures 9.1 and 9.2.

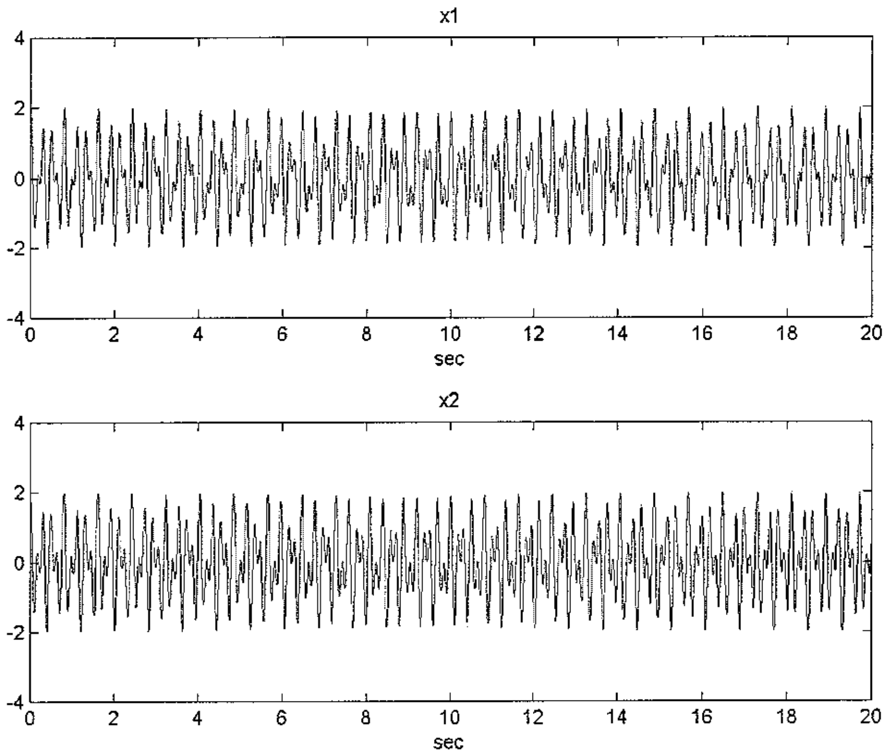


Figure 9.1: Analytical time histories (9.2a)

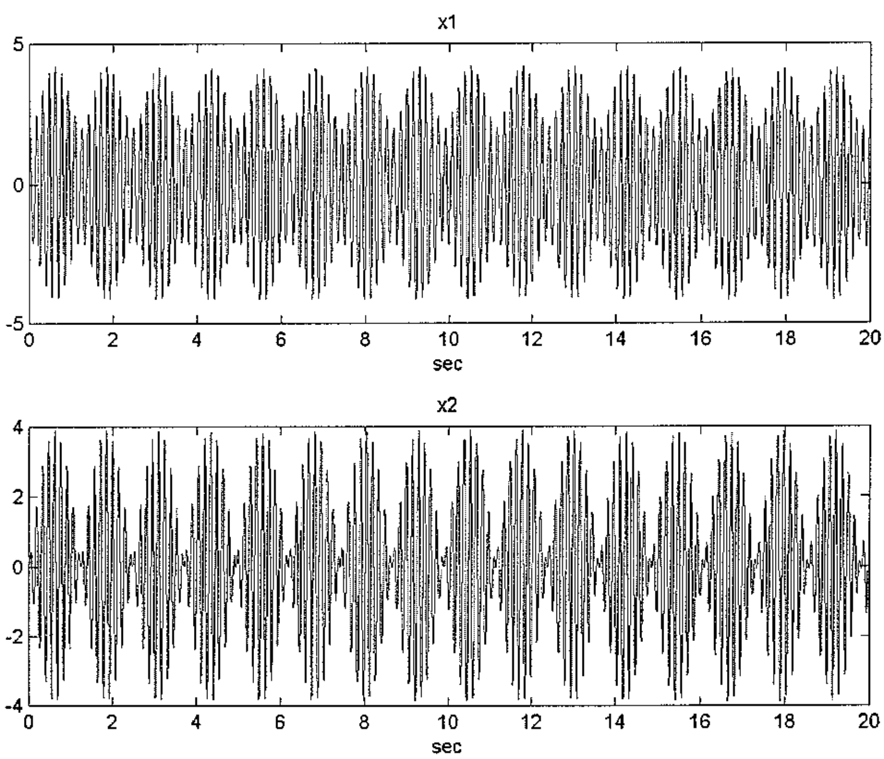


Figure 9.2: Analytical time histories (9.2b)

It is possible to choose the correct solution by energy considerations. Because the system considered is a conservative one, the total mechanical energy must be the same at every instant of time. Thus if T is the kinetic energy, U is the potential energy and E is the mechanical energy, for this system it is:

$$E = T + U = E_0 \tag{9.3}$$

where E_0 is the initial energy.

Unfortunately the solution calculated using harmonic balance is approximate, it means that energy contributions of higher frequencies are lost, and the previous relationship could not be used. However, since the system is conservative, the root mean square of the total mechanical energy must be less than or equal to the initial energy, in other words it must be:

$$\frac{\overline{E}}{E_0} = \frac{\sqrt{\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau}(T+U)^2\,dt}}{E_0} \le 1 \tag{9.4}$$

The solutions that do not satisfy relationship (9.4) do not have a physical meaning.

In the example considered:

$$U = 75x_1^2 + 37.5x_1^4 + 100(x_1 - x_2)^2 + 25(x_1 - x_2)^4 + 87.5x_2^2 + 50x_1^4 \tag{9.5}$$

$$T = 0.5(\dot{x}_1^2 + \dot{x}_2^2) \tag{9.6}$$

Substituting equation (9.2a) into (9.5) and (9.6) and then into (9.4), one has $\overline{E}/E_0 = 0.965$ (<1), whilst substituting (9.2b) similarly $\overline{E}/E_0 = 8.85$ (>1). This means that the physical solution is the first one, it has, for that excitation, the following CNM parameters:

Mode 1: (1, 0.90) f_1= 3.70 Hz.

Mode 1: (1, −1.12) f_2= 6.19 Hz.

Simulating the system and exciting it with the same initial conditions, that is (2,0), the following time histories are obtained.

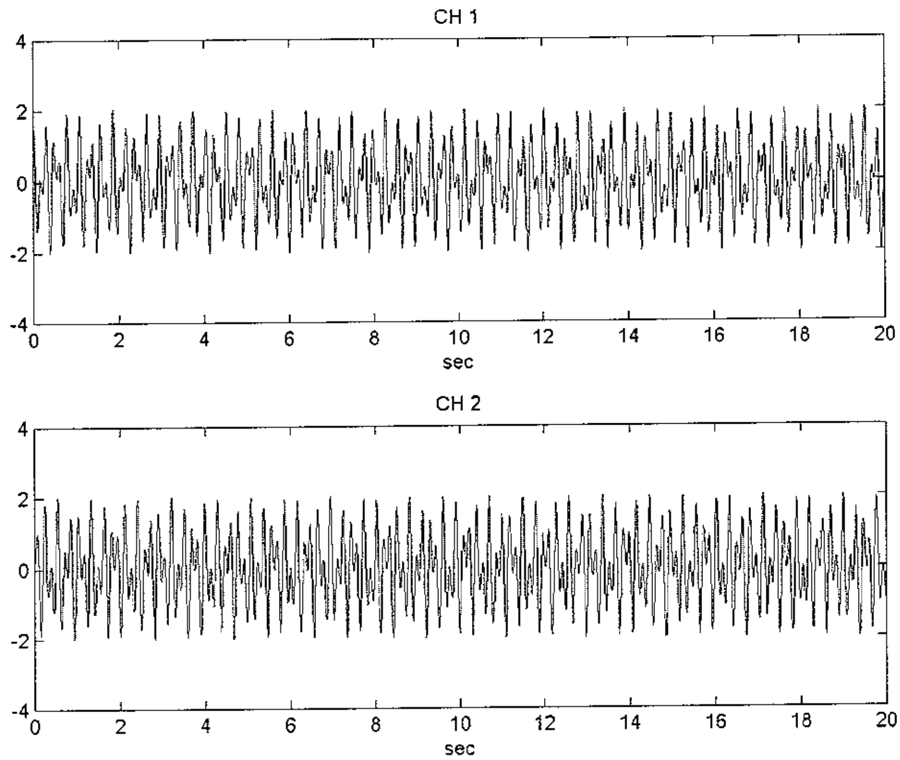


Figure 9.3: Time histories obtained by simulating the system with initial condition (2,0)

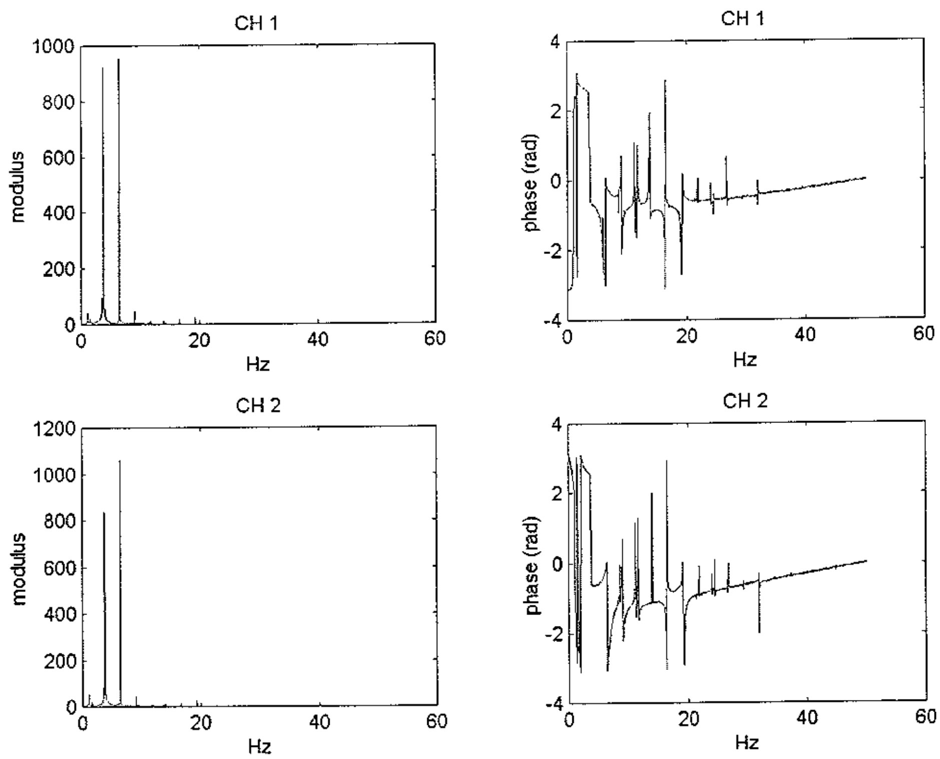


Figure 9.4: FFT in modulus and phase of the time histories simulated

The FFT of these response signals, are used to identify the modal parameters; the 2 main peaks are observed at:

	Frequency (Hz.)	Modulus	Phase (rad.)
CH 1	3.75	923.722	-0.6
	6.40	955.8316	0.0592
CH 2	3.75	836.0151	-0.6
	6.40	1061.3456	-3.0762

Table 9.1:modal parameter extracted by FFT

These correspond to the two CNM:

Mode 1: (1, 0.905) f_1=3.75 Hz.

Mode 2: (1, −1.11) f_2=6.40 Hz.

It is observed that the identified CNM are in good agreement with the analytical ones.

9.2 Non-Conservative System

Viscous damping is introduced into the previous 2 d.o.f symmetric system, resulting in the equations of motion:

$$\begin{aligned} &\ddot{x}_1 + 150x_1 + 150x_1^3 + 200(x_1 - x_2) + 100(x_1 - x_2)^3 + 0.1\dot{x}_1 + 0.05(\dot{x}_1 - \dot{x}_2) = 0 \\ &\ddot{x}_2 + 175x_2 + 200x_2^3 - 200(x_1 - x_2) - 100(x_1 - x_2)^3 + 0.1\dot{x}_2 + 0.05(\dot{x}_2 - \dot{x}_1) = 0 \end{aligned} \quad (9.7)$$

Using the method of harmonic balance one can calculate $Y_1(a_1,a_2)$, $Y_2(a_1,a_2)$, $\omega_1(a_1,a_2), \omega_2(a_1,a_2)$.

The solution in an implicit form is:

$$\omega_1^2=(225Y_{12}^2-75Y_{12}^3-225Y_{12}+187.5)a_1^2+$$
$$+(375-300Y_{22}-150Y_{12}+150Y_{22}^2+300Y_{12}Y_{22}-150Y_{12}Y_{22}^2)a_2^2+350-200Y_{12} \quad (9.8a)$$
$$\omega_2^2=(225Y_{22}^2-75Y_{22}^3-225Y_{22}+187.5)a_2^2+$$
$$+(375-300Y_{12}-150Y_{22}+150Y_{12}^2+300Y_{12}Y_{22}-150Y_{22}Y_{12}^2)a_1^2+350-200Y_{22} \quad (9.8b)$$

Exciting the system with initial condition (2,0) results in the simulations:

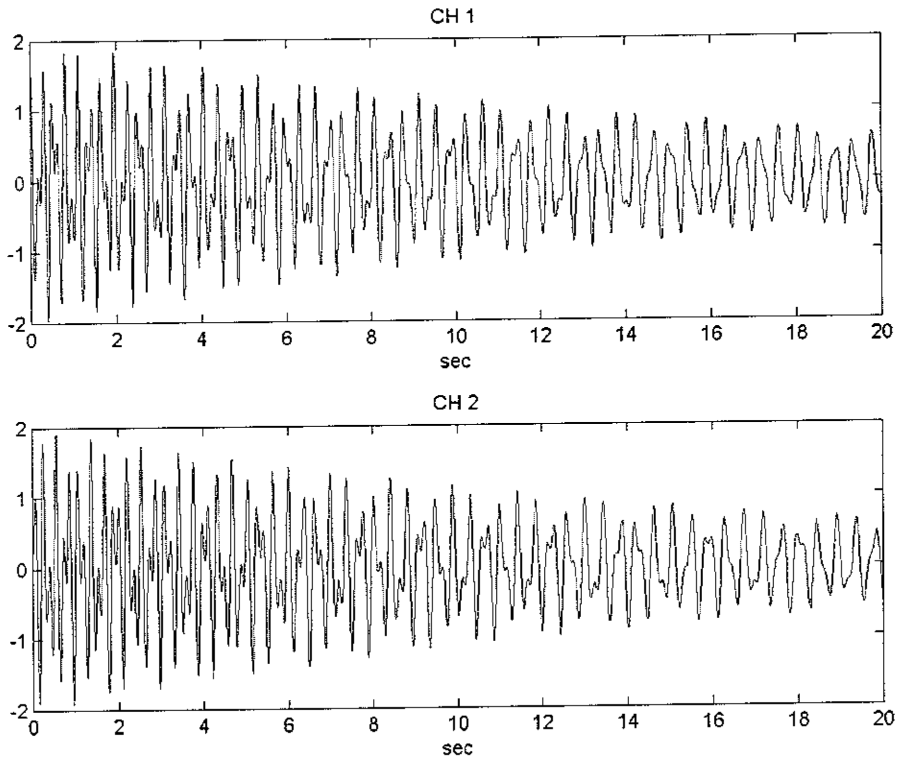


Figure 9.5: Time histories obtained by simulating then non-conservative system with initial condition (2,0)

Analysis of the corresponding Gabor Transform identifies the modal parameters:

1. Frequencies:

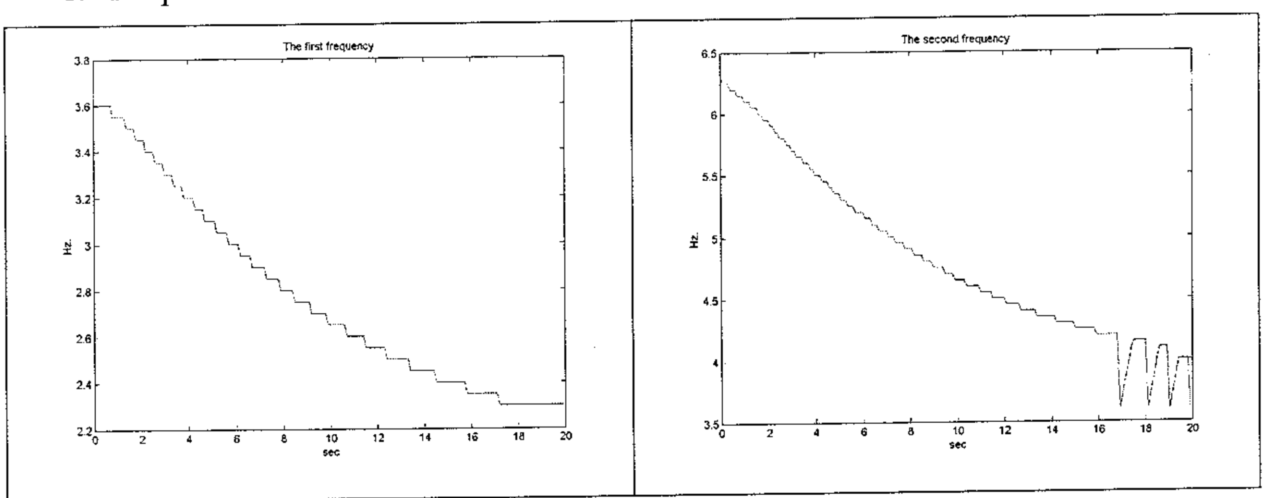
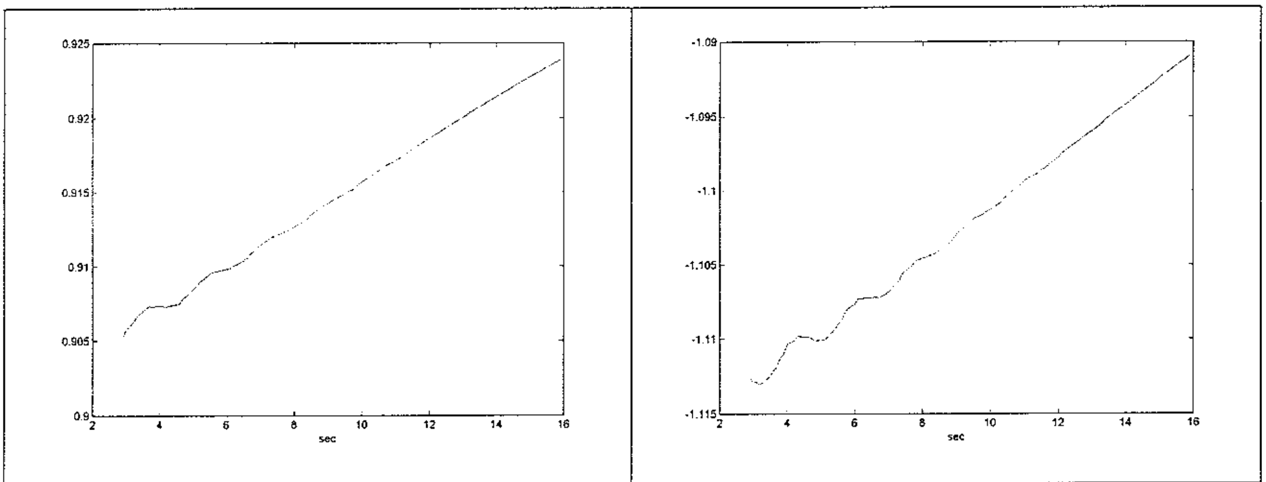


Figure 9.6: Identified modal frequencies vs time

2. Modal components (normalization towards CH 1):

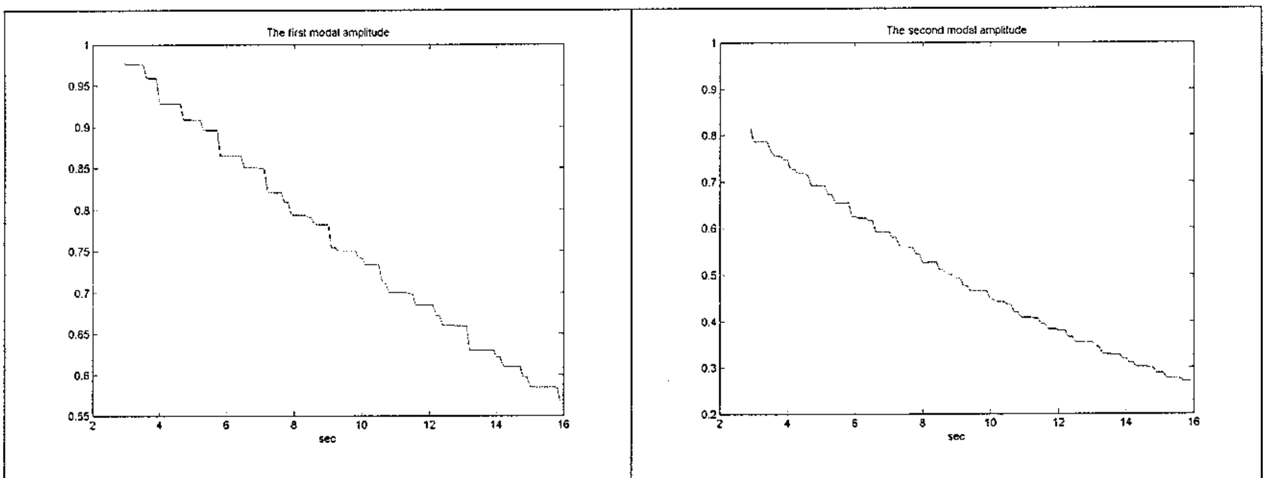


Y_{12}	Y_{22}

Figure 9.7: Identified modal components vs time

It is assumed that the modal tensor **Y** is constant for every short time-window, in this way one can calculate the modal amplitudes as step 3 of section 5.

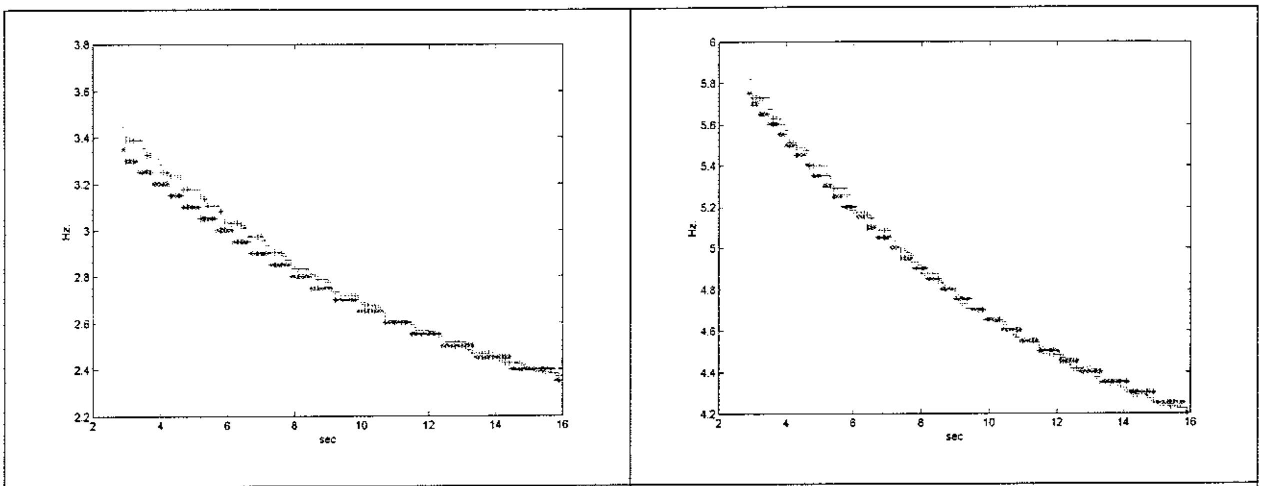
3. Modal Amplitudes:



a_1	a_2

Figure 9.8: Identified modal amplitudes vs time

Comparison of the identified frequencies with those calculated by identified modal amplitudes and modal components using harmonic balance, equation (9.8), is displayed in the following figure (9.9).



f_1	f_2

Figure 9.9: comparison between identified modal frequencies (***) and calculated by modal amplitudes and modal components using harmonic balance and equation (9.8) (+++)

10 Physical Model

The next stage of the study will be to produce a physical system that replicates both the linear and nonlinear characteristics. The simplest to investigate is a tension controlled system with attached discrete masses. The nonlinear behaviour is given by the tension produced in a cable by the motion of the masses.

In order to better explain the phenomenon one can consider the single d.o.f system pictured below:

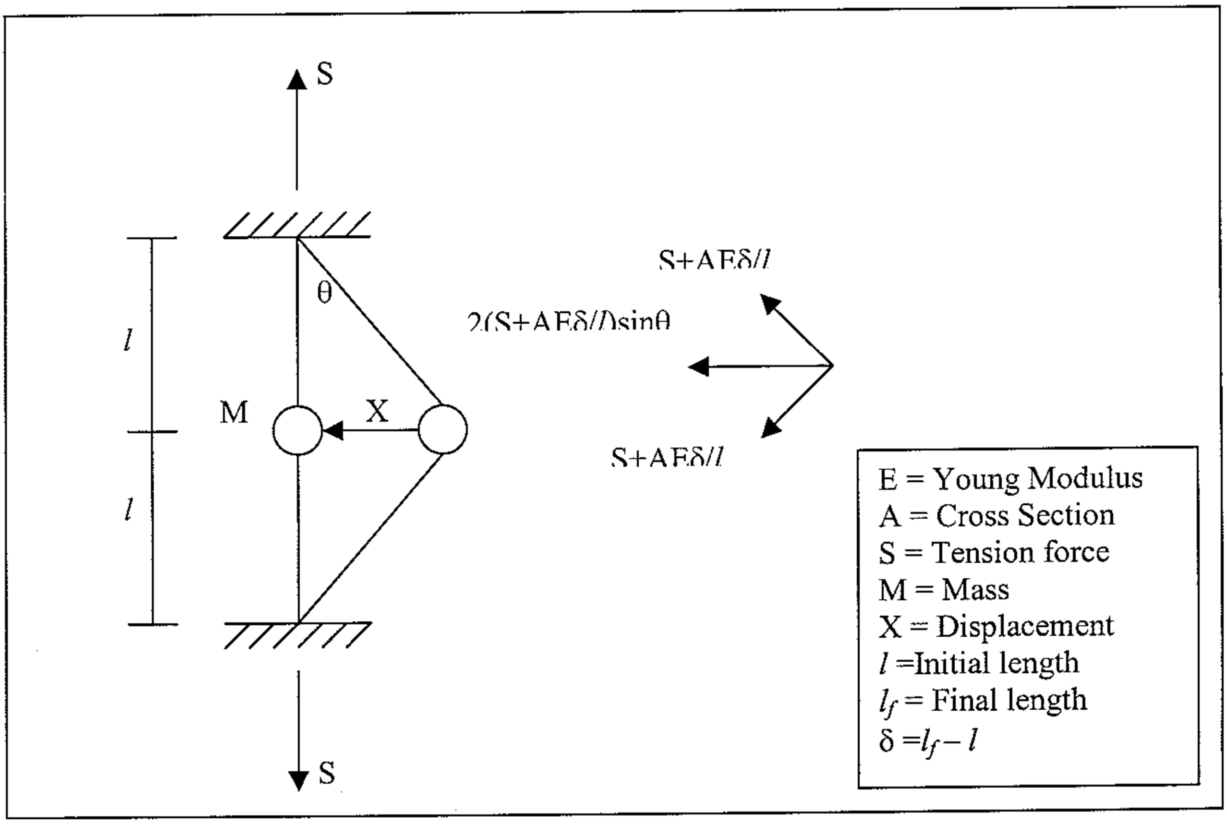


Figure 10.1: s.d.o.f. physical scheme

The equilibrium equation of this system is:

$$M\ddot{x} + 2(S + \frac{AE}{l}\delta)\sin\theta = 0$$

where: $\delta = \sqrt{l^2 + x^2} - l$ and $\sin\theta = \frac{x}{\sqrt{l^2 + x^2}}$

one can re-write

$$\delta = \frac{x^2}{\sqrt{l^2 + x^2} + l}$$

if $l >> x \Rightarrow \sqrt{l^2 + x^2} \cong l$

so it is simply

$$\delta = \frac{x^2}{2l} \qquad \text{and} \qquad \sin\theta = \frac{x}{l}$$

In this way the equation of motion becomes

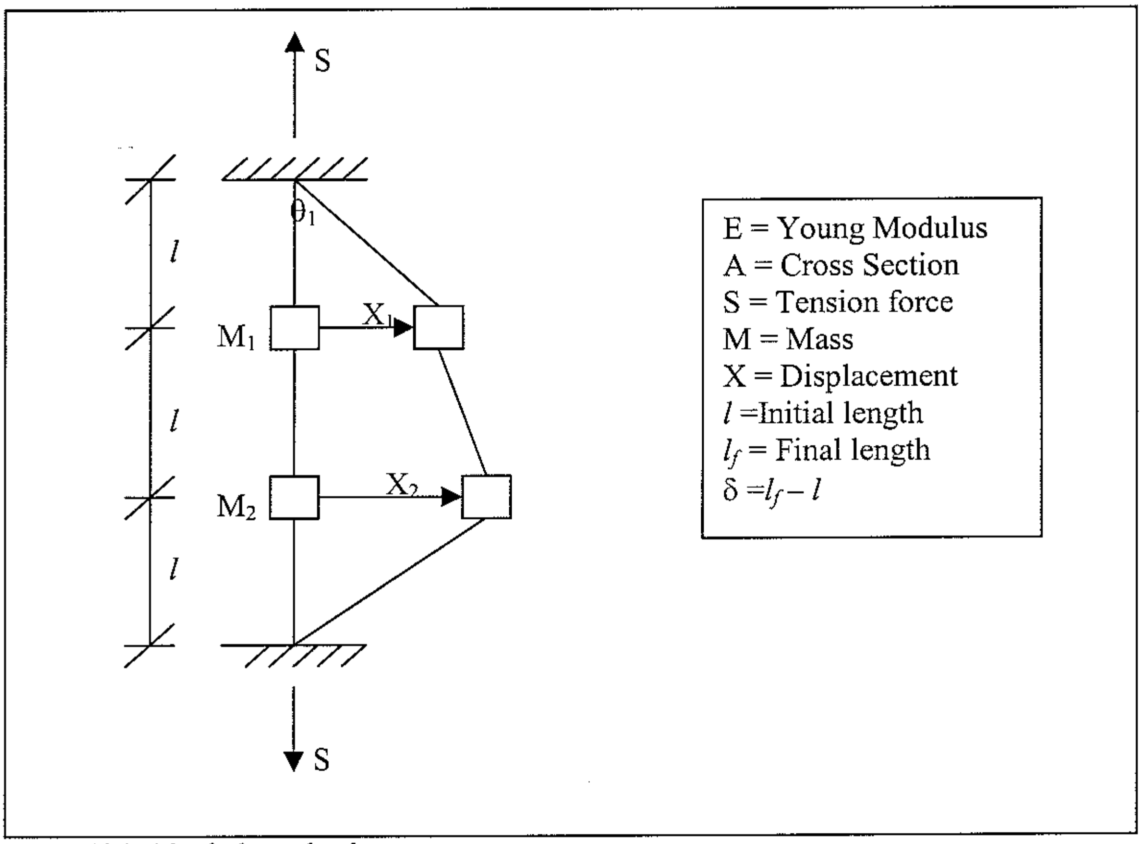


Figure 10.2: 2d.o.f. physical scheme

$$M\ddot{x} + \frac{2S}{l}x + \frac{AE}{l^3}x^3 = 0$$

This sytem can now be extended to a two degree of freedom realisation letting:

$k_l = \dfrac{S}{l}$ linear stiffness

$k_{nl} = \dfrac{AE}{2l^3}$ nonlinear stiffness

Similar to the first system the 2 d.o.f. system equations of motion are:

$$\begin{cases} M_1\ddot{x}_1 + \frac{S}{l}(2x_1 - x_2) + \frac{AE}{2l^3}x_1^3 + \frac{AE}{2l^3}(x_1 - x_2)^3 = 0 \\ M_2\ddot{x}_2 + \frac{S}{l}(2x_2 - x_1) + \frac{AE}{2l^3}x_2^3 + \frac{AE}{2l^3}(x_2 - x_1)^3 = 0 \end{cases}$$

The solution of this system, by the method of harmonic balance, is:

$$\psi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega_1^2 = \frac{1}{M_1}\left(\frac{S}{l} + 0.375\frac{AE}{l^3}a_1^3 + 0.75\frac{AE}{l^3}a_2^2\right)$$

$$\psi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega_2^2 = \frac{1}{M_2}\left(\frac{3S}{l} + 0.75\frac{AE}{l^3}a_1^3 + 3.375\frac{AE}{l^3}a_2^2\right)$$

10.1 Design

l=40 cm (l_{tot}=120cm)

Φ=0.5 mm

E=2.1 10^6 Kg/cm^2

A_l=1.96 10^{-3} cm^2 assuming using 2 wires so A_{tot}=3.927 10^{-3}cm^2

EA=82467 N

Consider a displacement on the first mass d=2 cm then the additional strain is ε=1.25 10^{-3}

So the maximum tension due to the displacement is T=103.08 N

The wires have an initial static tension S=20 N to add to that due to the initial displacement.

Thus T+S=123.08 N

The static restraining force, to keep the first mass at the position d=2 cm is:

F=2(T+S)sinθ=12.3N

In this situation one should have for the first mode a starting frequency of 7.02 Hz and a final frequency of 3.56 Hz., while for the second mode one should have a starting frequency of 13.14 Hz and a final frequency of 6.16 Hz. It appears, for this system, that the frequencies are always well separated.

10.2 A Prelimanary Test Program

- **Step 1 Check of the Mechanical Proprieties of the wire**

  It consists simply of a static test on the wire: without dynamic masses various loads can be applied on the cable in order to measure the strain and to confirm information on the mechanical proprieties of the wire.

- **Step 2 Tests on 1 DOF**

  With the information collected in step 1 it is possible to realize an analytical model of a 1 DOF (that is with just 1 mass). Then the result should be checked with the experimental tests that could be performed varying:

  - -the load S
  - -the initial condition

- **Step 3 Tests on 2 DOF symmetrical system**

  With the information collected in 1 & 2 it is possible to make

  - analytical model
  - numerical simulations solving numerically motion equations
  - numerical simulation by finite element methods.

  Then to check the results with experimental tests that could be performed varying:

  - The load S
  - The initial conditions
  - The relative position of the two masses (maintaining always a symmetric system)
  - The damping

- **Step 4 Tests on a 2 DOF unsymmetrical system**

  Similar to step 4 but putting the masses in an unsymmetrical position in order to have non similar modes.

11 Conclusion and future developments

In this report it is shown how to calculate NNM and CNM for a NDOF nonlinear system by the method of harmonic balance. It is shown that there is a good agreement between the analytical approximate solutions and results identified by numerical simulations.

For nonlinear mechanical systems possessing cubic stiffness and exhibiting similar modes a procedure for the identification of the frequency modulation law is proposed. In conditions of small amplitudes and examination of time histories in which the frequency variation is appreciable it gives good results. At this moment an extension of the approach to systems exhibiting non similar modes does not seem to be trivial.

The following table summarises what is now possible to do for a NDOF mechanical system possessing cubic stiffness, by harmonic balance, in free vibrations.

	Similar Modes	Non Similar Modes
To calculate NNM for a conservative system	Yes	Yes
To calculate NNM for a non conservative system	Yes	No
To calculate analytically CNM for conservative systems	Yes	Yes
To calculate analytically CNM for non conservative system	Yes	Yes
To calculate an easy relationship between frequency and amplitudes	Yes	Yes, but in an implicit way.
Identification of modal parameters of conservative system	Yes	Yes
Identification of modal parameters of non conservative system	Yes	Yes
Identification of relationship between frequencies and amplitudes	Yes	No

Table 11.1: Summary on what has been achieved so far.

The realization of the physical system described in chapter 10 is now in progress, and the intention of authors is to realize the experimental program presented in 10.2.

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