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Multichannel Simulation of Structural Response to Stochastic Pressure Fields

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by

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1. Introduction

In a number of applications the response of a structure to a complicated pressure distribution must be determined. This response can sometimes be directly measured with the actual pressure distribution acting on the structure, but such experiments can be expensive and inconvenient. In this memorandum a method is presented of simulating the statistical properties of the structural response either by simulating the pressure distribution using signals driving a number of discrete acoustic actuators or by using structural actuators, which may allow the response of the structure to be determined in the laboratory. One potential application of the technique is for the simulation of the stochastic pressure field due to a turbulent boundary layer (TBL) on the surface of a section of aircraft fuselage.

The technique is described in more detail in Section 2, together with an outline of the signal processing requirements. Sections 3 and 4 contain some preliminary observations on the application of the technique to a TBL and simulations to determine the number of independent sources per correlation length.

2. Formulation

The proposed arrangement for the simulation of a pressure field is illustrated in Figure 1. The object is to use an array of actuators to generate a pressure distribution on the panel whose statistical properties are as similar as possible to those of some desired or target pressure distribution. This can be achieved by passing a set of excitation or reference signals through an array of control filters to drive the actuators. Once the responses of this array of filters has been determined it may be more convenient to implement the system by pre-computing the waveforms of the actuator drive signals and replaying these in real time through the actuators.

Although the actuators are indicated in Figure 1 as loudspeakers, the principle of the technique is the same for an array of shakers or piezoelectric patches acting on the surface of the panel. Also the sensors are indicated in Figure 1 as microphones, but accelerometers or strain gauges could be used if the target signals are specified in terms of the structural response. There will generally be a larger number of sensors than actuators in the arrangement to ensure that the response between the sensors does not significantly diverge from the required distribution.

The equivalent block diagram for the arrangement is shown in Figure 2 in which **x** is the vector of excitation signals, which can be conveniently assumed to be uncorrelated white noise signals, **W** is the matrix of control filter responses, **G** is the matrix of responses between the actuators and sensors and **y** is the vector of response signals at the sensors. These quantities are all assumed to be in the frequency domain, so that **x** and **y** are vectors of spectra and **W** and **G** are matrices of frequency responses, and

$$\mathbf{y} = \mathbf{G}\,\mathbf{W}\,\mathbf{x}, \qquad (1)$$

where the explicit dependence on frequency has been suppressed for notational convenience.

It is assumed that the statistical properties of the stationary, zero-mean, desired or target signals at the sensors is given in terms of a spectral density matrix

$$\mathbf{S}_{dd} = E\left[\mathbf{d}\,\mathbf{d}^{\mathrm{H}}\right] \qquad (2)$$

where **d** is the vector of desired spectra and the expectation is taken across the Fourier transforms of sections of data whose length tends to infinity. The diagonal terms in this spectral density matrix contain the power spectral densities of the target signals and the off diagonal terms contain the cross spectral densities. Higher order statistical properties could be represented by higher order spectra.

The spectral density matrix generated at the sensors by the arrangement shown in Figure 1 will be equal to

$$\mathbf{S}_{yy} = E\left[\mathbf{y}\,\mathbf{y}^{\mathrm{H}}\right] = \mathbf{G}\mathbf{W}\mathbf{S}_{xx}\mathbf{W}^{\mathrm{H}}\mathbf{G}^{\mathrm{H}}, \tag{3}$$

where

$$\mathbf{S}_{xx} = E\left[\mathbf{x}\,\mathbf{x}^{\mathrm{H}}\right] \tag{4}$$

is the spectral density matrix of the excitation signals. If these excitation signals are assumed to be white, uncorrelated and of unit variance then $\mathbf{S}_{xx}$ is equal to the identity matrix and

$$\mathbf{S}_{yy} = \mathbf{G}\mathbf{W}\mathbf{W}^{\mathrm{H}}\mathbf{G}^{\mathrm{H}}. \tag{5}$$

The design of the matrix of control filters could be formulated as choosing the matrix of frequency responses, $\mathbf{W}$, such that a cost function, J, equal to a norm of the difference between $\mathbf{S}_{dd}$ and $\mathbf{S}_{yy}$ is minimised, so that

$$J = \left\|\mathbf{S}_{yy} - \mathbf{S}_{dd}\right\|. \tag{6}$$

If, for example, the square of the Frobenius norm is taken, then

$$J = \left\|\mathbf{S}_{yy} - \mathbf{S}_{dd}\right\|_F^2 = \sum_i \sum_j \left|S_{yy}(i,j) - S_{dd}(i,j)\right|^2 \tag{7}$$

where $S_{yy}(i,j)$ and $S_{dd}(i,j)$ are the i,j-th elements of $\mathbf{S}_{yy}$ and $\mathbf{S}_{dd}$.

Alternatively a simple formulation for the matrix of optimal control filter responses can be obtained if the cost function is equal to the sum of the modulus squared errors between the elements of **d** and **y**, so that

$$J = E\left[\mathbf{e}^{\mathrm{H}}\mathbf{e}\right] = \operatorname{trace} E\left[\mathbf{e}\,\mathbf{e}^{\mathrm{H}}\right], \tag{8}$$

where the superscript H denotes the Hermitian, complex conjugate, transpose and

$$\mathbf{e} = \mathbf{d} - \mathbf{G}\mathbf{W}\mathbf{x}. \tag{9}$$

In this case the derivative of the cost function with respect to the real and imaginary parts of **W** is equal to (Elliott, 1999)

$$\frac{\partial J}{\partial \mathbf{W}_R} + j\frac{\partial J}{\partial \mathbf{W}_I} = 2\,\mathbf{G}^H\mathbf{G}\,\mathbf{W}\,\mathbf{S}_{xx} - 2\,\mathbf{G}^{\mathrm{H}}\mathbf{S}_{xd}. \tag{10}$$

If **d** is assumed to be generated from the excitation signals **x** by a matrix of responses **P**, so that

$$\mathbf{d} = \mathbf{P}\,\mathbf{x}, \tag{11}$$

then $\mathbf{S}_{xd} = \mathbf{P}\mathbf{S}_{xx}$. Assuming $\mathbf{S}_{xx}$ is full rank, the optimum matrix of control filter responses can then be obtained by setting equation (10) to zero to give

$$\mathbf{W} = [\mathbf{G}^H\mathbf{G}]^{-1}\,\mathbf{G}^{\mathrm{H}}\mathbf{P}. \tag{12}$$

The matrix **P** can be obtained from the cross spectral matrix of the desired or target signals by noting that if $\mathbf{S}_{xx} = \mathbf{I}$ then

$$\mathbf{S}_{dd} = \mathbf{P}\,\mathbf{P}^{\mathrm{H}}, \qquad (13)$$

and so $\mathbf{P}$ can be obtained from a factorisation of $\mathbf{S}_{dd}$, using its eigenvalue, eigenvector decomposition for example. The eigenvalue spectrum of $\mathbf{S}_{dd}$ could also be used to calculate the number of independent excitation signals required to generate the target signals, as described in Section 4.

3. TBL Excitation

In this section we consider the particular example of turbulent boundary layer (TBL) excitation. A popular model of the pressure distribution due to a TBL is that due to Corcos (1967), which has been used for example by Thomas and Nelson (1995) and Graham (1996). Corcos suggested that the cross spectral density between the pressures measured at two points A and B at a frequency ω has the form

$$S_{AB}(\omega) = S_{pp}(\omega)\, e^{-|r_1|/L_1}\, e^{-|r_2|/L_2}\, e^{-j\omega r_1/U_c}, \qquad (14)$$

where

$S_{pp}(\omega)$ is the power spectral density of the pressure fluctuations

r_1 is the distance between points A and B in the streamwise direction

r_2 is the distance between points A and B in the spanwise direction

$$L_1 = \alpha_1\, U_c/\omega \qquad (15)$$

is the correlation length in the streamwise direction

$$L_2 = \alpha_2\, U_c/\omega \qquad (16)$$

is the correlation length in the spanwise direction

U_c is the eddy convection velocity (generally taken as $0.6\,U_\infty$, where U_∞ is the velocity at the edge of the boundary layer)

α_1 is an empirically determined constant ≈ 10

α_2 is another empirically determined constant ≈ 2.

Equation (14) indicates that the cross spectral density between two points falls off more quickly as the points are separated in the spanwise direction than in the streamwise direction, and that there is a phase shift between the pressures only in the streamwise direction.

The coherence between the pressure at two points in the TBL can be evaluated by using equation (14) in the definition of the coherence,

$$\gamma_{AB}^2(\omega) \;=\; \frac{|S_{AB}(\omega)|^2}{S_{AA}(\omega)\,S_{BB}(\omega)}, \tag{17}$$

where in this case $S_{AA}(\omega) = S_{BB}(\omega) = S_{pp}(\omega)$ so that

$$\gamma_{AB}^2(\omega) \;=\; \exp\left(-\left|\frac{2\,\omega r_1}{\alpha_1 U_c}\right|\right)\;\exp\left(-\left|\frac{2\,\omega r_2}{\alpha_2 U_c}\right|\right). \tag{18}$$

4. TBL Simulations

A spectral density matrix for the pressure in a TBL has been calculated using equations (14) and (2) for a linear array of 100 evenly-spaced microphones arranged over a distance of 3 correlation lengths in the spanwise direction. The cross spectral densities between the microphones in this array fall off exponentially with distance as shown in Figure 3, and are entirely real in this case. If the microphones had been arranged over three correlation lengths in the streamwise direction, the magnitudes of the cross spectral densities would be identical to those shown in Figure 3, and apart from the phase shift between the sensors, the analysis presented below would be identical. Also note that since a fixed number of microphones per unit correlation length have been assumed, the results scale directly with correlation length and hence with frequency. The spectral density matrix may be calculated using equation (14) for each of the elements and then expanded out using its eigenvector, eigenvalue decomposition so that it can be expressed as

$$\mathbf{S}_{dd} = \sum_{i=1}^{L} \lambda_i \, \mathbf{q}_i \, \mathbf{q}_i^{\mathrm{H}} \tag{19}$$

where L is the number of sensors (100 in this case), λ_i is the i-th eigenvalue of $\mathbf{S}_{dd}$ and $\mathbf{q}_i$ is the ith eigenvector. The eigenvalues are all real and positive, since $\mathbf{S}_{dd}$ is positive definite, and their values for this example are shown in Figure 4. If the sensors are evenly spaced then $\mathbf{S}_{dd}$ has equal elements along each of its diagonals and is thus Toeplitz. A property of Toeplitz matrices is that as they grow larger, their eigenvectors tend to those of a circulant matrix, which are equal to the coefficients of a Discrete Fourier Transform (Gray, 1972). Thus for a large array the eigenvalues of the spectral density matrix will tend to the sampled values of the spatial Fourier transform of the cross-spectral density variation about a single point.

An estimate of the spectral density matrix can be obtained by only considering N eigenvalues in the summation above to give

$$\hat{\mathbf{S}}_{dd} = \sum_{i=1}^{N} \lambda_i \, \mathbf{q}_i \, \mathbf{q}_i^{\mathrm{H}}. \tag{20}$$

The pressure distribution which gives rise to this spectral density matrix can be thought of as being generated by N independent random noise sources, of variance λ_i, driving an array of fixed filters, with frequency responses $\mathbf{q}_i$, whose outputs are summed together to give the observed pressure distribution. If the required spectral density matrix in equation (19) can be adequately approximated by equation (20), then only N independent excitation signals are required in Figure 1, and so N provides a lower bound on the number of actuators required to simulate a given pressure distribution.

The error incurred in using $\hat{\mathbf{S}}_{dd}$ as an approximation to $\mathbf{S}_{dd}$ may be quantified by defining the normalised error function

$$\varepsilon = \frac{\left\| \mathbf{S}_{dd} - \hat{\mathbf{S}}_{dd} \right\|_F}{\left\| \mathbf{S}_{dd} \right\|_F} \tag{21}$$

where $\| \ \|_F$ denotes the Frobenius norm of the matrix, which is equal to the square root of the sum of its square eigenvalues. The error function may thus be expressed in decibels as

$$20 \log_{10} \varepsilon = 10 \log_{10} \left[\frac{\sum_{i=N+1}^{L} \lambda_i^2}{\sum_{i=1}^{L} \lambda_i^2} \right]. \tag{22}$$

This error function is plotted in Figure 5 for the TBL simulation described above. The level of the error function is more than 30dB down when $\mathbf{S}_{dd}$ is approximated by only 9 eigencomponents in equation (20). The spatial decay of the cross spectral density at all microphones, calculated from the estimated spectral density matrix, equation (20), with 9 eigencomponents, is shown in Figure 6. Apart from the truncation effects at the end of the array the estimated cross spectral density can be seen to have a small ripple in its peak value, and the average peak value is about 93% of that of the true Corcos model. This is shown more clearly in Figure 7 in which the cross spectral density is plotted when moving away from a single microphone in the centre of the array, as calculated from the Corcos model and when using 5, 9 and 29 eigencomponents in the estimated spectral density matrix, equation (20). The number of components required for a good TBL simulation depends on the confidence with which the Corcos model can be used, but 9 components appear to give a reasonable fit over three correlation lengths in these simulations. This simple simulation thus implies that 3 uncorrelated sources per correlation length are required to simulate the pressure fluctuations of a TBL to this degree of accuracy.

If three uncorrelated sources are required per correlation length in both the streamwise and the spanwise direction in a two dimensional system, then the total number of sources required per unit area is

$$N_{\text{Total}} = \frac{9}{L_1 L_2}. \tag{23}$$

Using the expressions for L_1 and L_2 given in Section 3, the total number of sources per unit area may be expressed as

$$N_{\text{Total}} = \frac{9\,\omega^2}{\alpha_1\,\alpha_2\,U_c^2}. \tag{24}$$

Assuming $U_\infty = 225$ ms^{-1} so that $U_c = 135$ ms^{-1}, and that $\alpha_1 = 10$ and $\alpha_2 = 2$, then the number of uncorrelated sources per square metre required to simulate a TBL up to a frequencies of 300 Hz is thus about 88. Because of the dependence of N_{Total} on ω^2, however, four times as many sources would be required to simulate the TBL up to a frequency of 600 Hz.

5. Conclusions

A method has been presented of simulating a complicated pressure distribution using a number of actuators. The signals driving these actuators are arranged so that the statistical properties of a distribution of responses, measured at a number of sensors, is as similar as possible to those of the desired set of responses.

For the particular case of a turbulent boundary layer (TBL), the cross spectral densities, required to define the statistical properties of the desired responses, can be approximated using the Corcos model. An eigenvalue, eigenvector decomposition of the spectral density matrix at an array of microphones suggests that about 3 independent sources per correlation length are sufficient to give a reasonable simulation of the TBL pressure distribution of the Corcos model.

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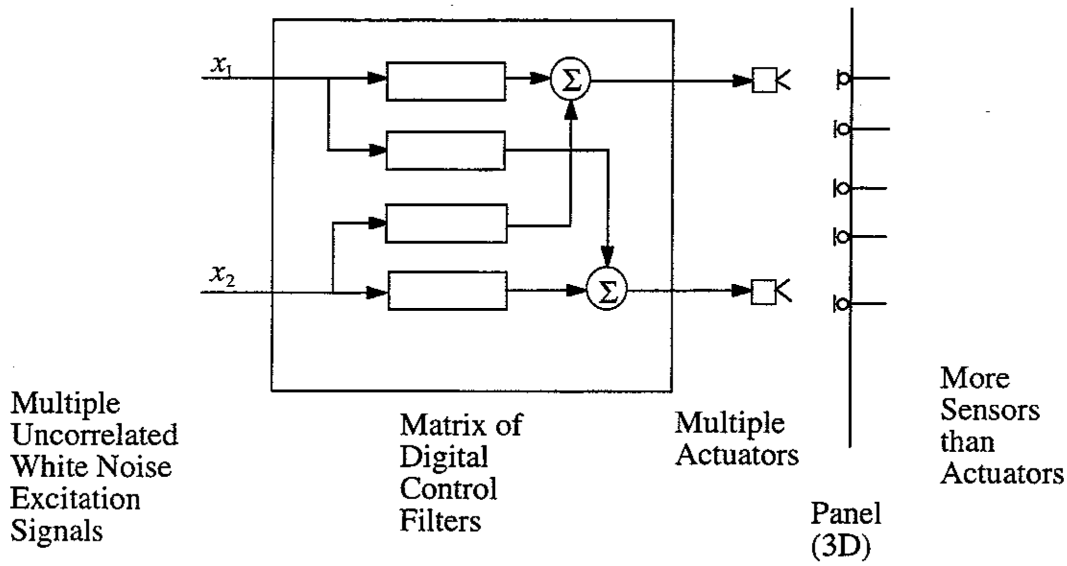

***Figure 1** Technique for simulating pressure distributions*

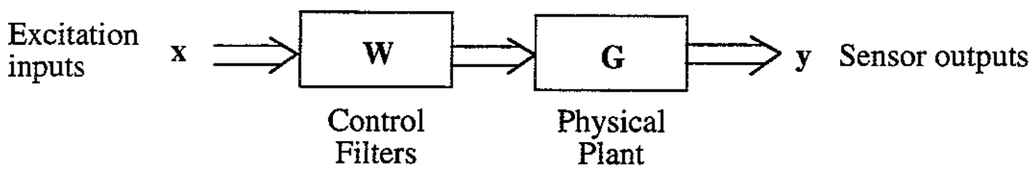

***Figure 2** Equivalent block diagram*

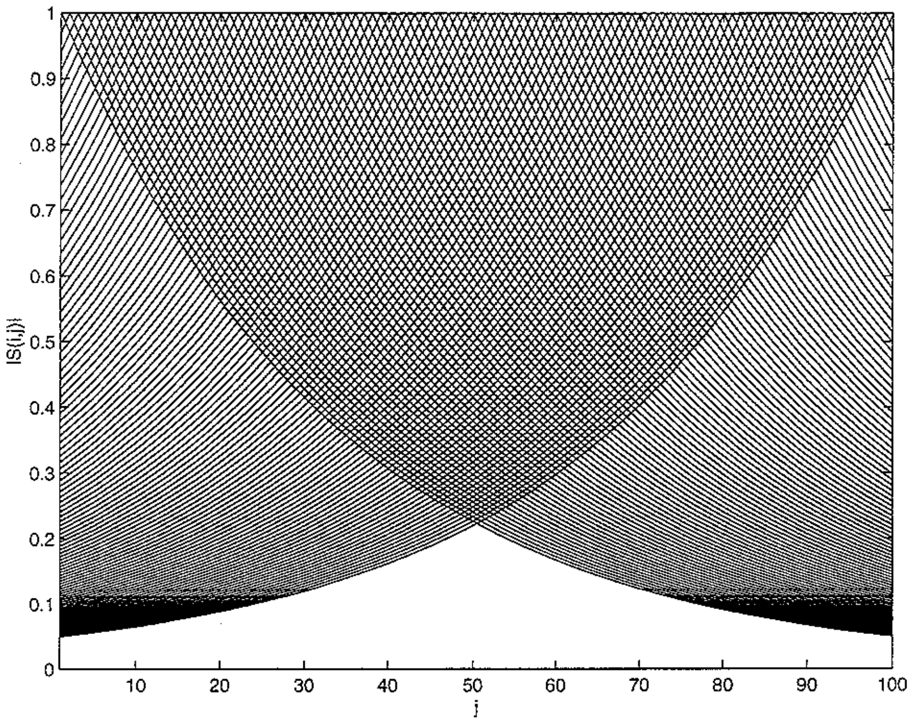

Figure 3 *The calculated cross spectral densities between the 100 microphones in the TBL model.*

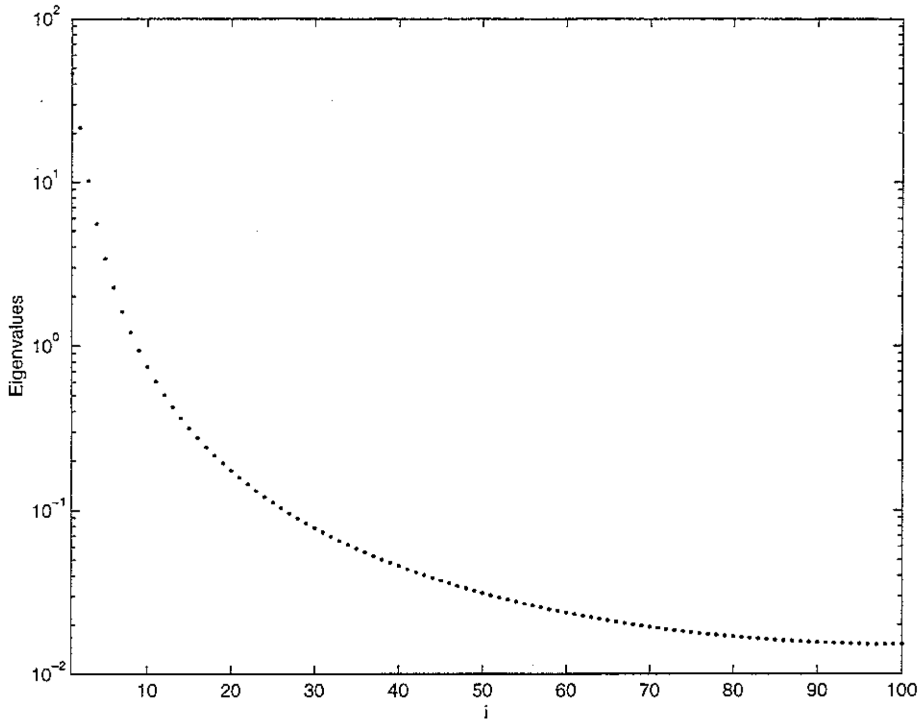

Figure 4 *The eigenvalues of the spectral density matrix for the TBL model*

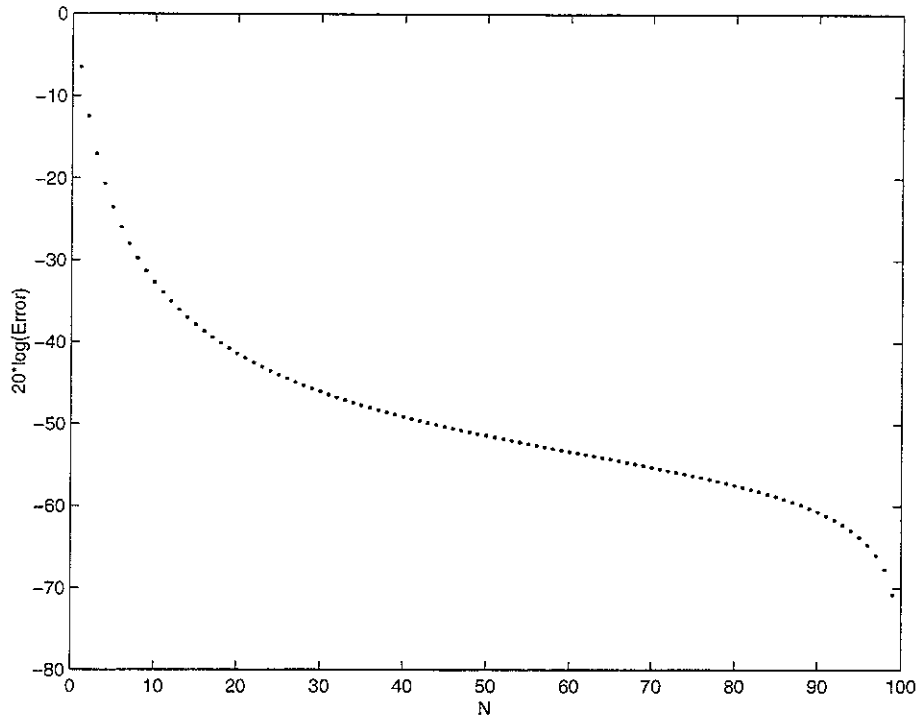

Figure 5 *The level of the error function involved in approximating the spectral density matrix by its first N eigencomponents for the TBL model.*

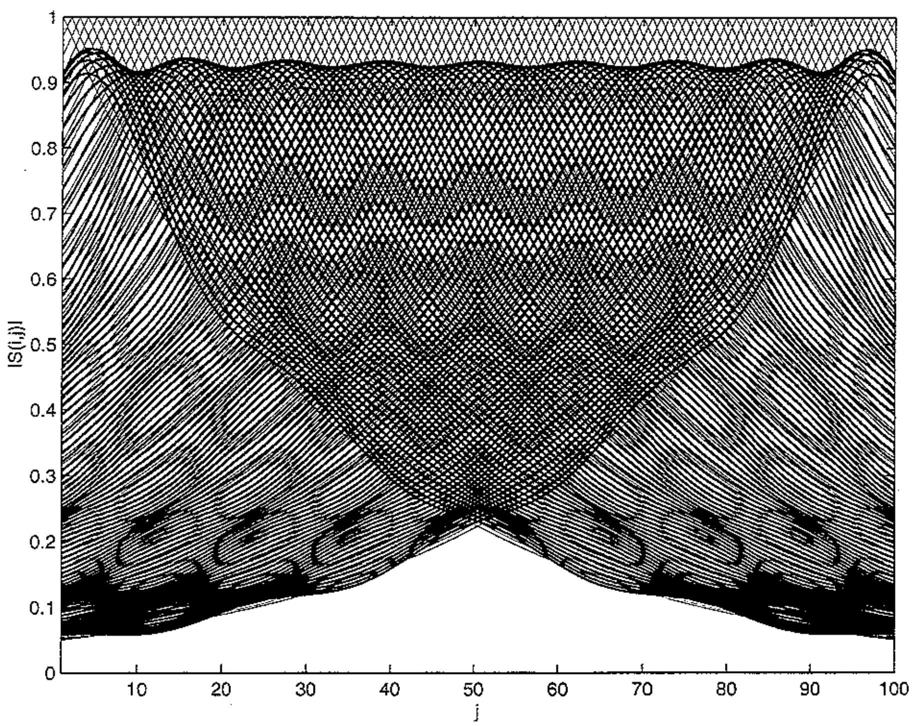

Figure 6 *Cross spectral densities calculated between the 100 microphones over three correlation lengths for the full TBL model (faint lines) and the estimated cross spectral densities using only the first 9 eigen components of the spectral density matrix.*

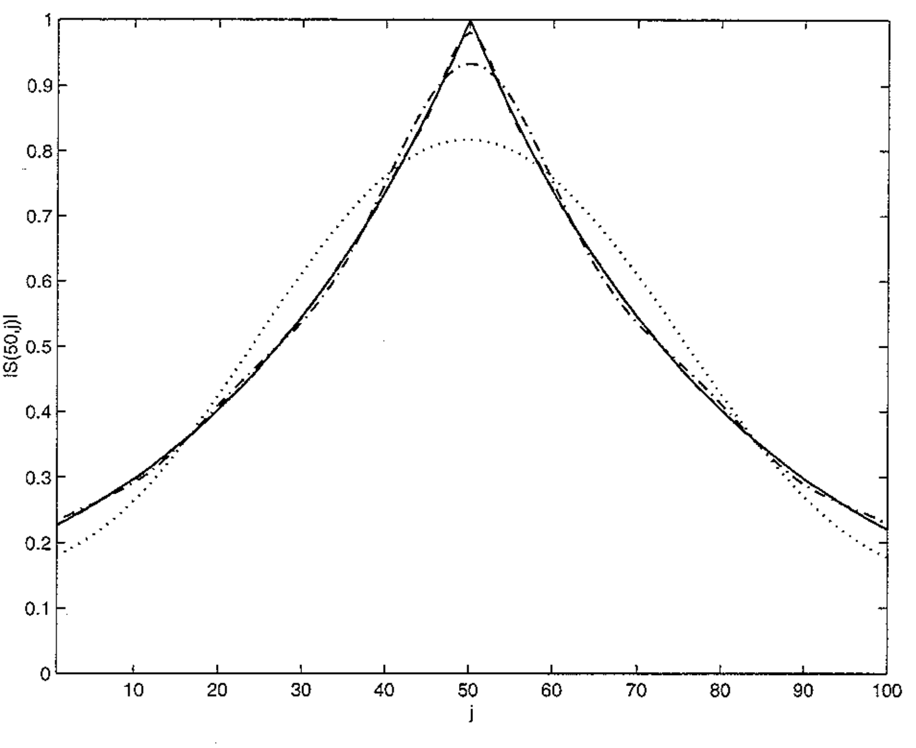

Figure 7 *The spatial decay of the cross spectral density for the Corcos model of the TBL (solid) and when the spectral density matrix in the TBL simulation is approximated by 4 (dotted), 9 (dot dashed) and 29 (dashed) components in the eigen expansion*