Institute of Sound and Vibration Research

**On the Condition Number of the Matrix to be Inverted in an Acoustic Inverse Problem**

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UNIVERSITY OF SOUTHAMPTON
INSTITUTE OF SOUND AND VIBRATION RESEARCH
FLUID DYNAMICS AND ACOUSTICS GROUP

# ON THE CONDITION NUMBER OF THE MATRIX TO BE INVERTED IN AN ACOUSTIC INVERSE PROBLEM

by

**S. H. YOON and P. A. NELSON**

ISVR Technical Memorandum No. 817

April 1997

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# CONTENTS


Abstract

List of Figures

1. Outline of the theory of reconstruction of acoustic source strength ------ 1

2. Condition number ------------------------------------------------------------ 2

3. Effect of the condition number of $\mathbf{H}^H\mathbf{H}$ and perturbation in $\mathbf{S}_{pp}$ or $\mathbf{H}^H\mathbf{H}$ on $\mathbf{S}_{qq}$ ------------------------------------------------------------ 4

4. Two factors determining the condition number $CN(\mathbf{H}^H\mathbf{H})$ --------------- 6

5. Characteristics of the condition number of $\mathbf{H}^H\mathbf{H}$ depending on frequency ------------------------------------------------------------ 7

6. Effect of the geometrical arrangement of sources and microphones on the condition number of $\mathbf{H}^H\mathbf{H}$ ------------------------------------------------ 9

7. Conclusion ------------------------------------------------------------ 12

References ------------------------------------------------------------ 14

Figures ------------------------------------------------------------ 15

# ABSTRACT

In order to reconstruct acoustic sources by the inverse technique with reasonably acceptable accuracy, it is a prerequisite to have a fuller understanding of the behaviour of condition number of the matrix to be inverted. This report begins with an analytical investigation of the influence of the condition number on the accuracy of reconstruction of acoustic source strength auto- and cross-spectra. There are two parameters affecting the condition number. These are the frequency and the distances between positions of sources and microphones. Of these, an analytical discussion is presented of the oscillatory behaviour of condition number with frequency. In conjunction with the dependence of condition number on the distances determined by the geometrical arrangement of sources and microphones, an empirical investigation through a set of numerical simulations is performed in order to find a universal guide to the geometrical arrangement which produces a condition number which is as small as possible.

# LIST OF FIGURES

***Fig.1*** *A geometrical arrangement of 2 sources and 2 microphones.*

***Fig.2*** *A geometrical arrangement of 2 sources and 2 microphones (discretised source dimension a=0.2m).*

***Fig.3*** *The condition number oscillating with frequency (2 source and 3 microphone model).*

***Fig.4*** *A geometrical arrangement of 2 sources and 3 microphones (discretised source dimension a=0.2m).*

***Fig.5*** *The condition number oscillating with frequency (2 source and 3 microphone model).*

***Fig.6*** *(a) A geometrical arrangement of 2 sources and 2 microphones. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

***Fig.7*** *(a) A geometrical arrangement of 4 sources and 4 microphones. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

***Fig.8*** *(a) A geometrical arrangement of 6 sources and 6 microphones. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

***Fig.9*** *(a) A geometrical arrangement of 16 sources and 16 microphones. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

***Fig.10*** *(a) A geometrical arrangement of 35 sources and 35 microphones. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m. Variation of the condition number* $CN(\mathbf{H}^H\mathbf{H})$ *with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

***Fig.11*** *(a) A geometrical arrangement of 100 sources and 100 microphones. Variation of the condition number CN(*$\boldsymbol{H}^H\boldsymbol{H}$*) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m. Variation of the condition number CN(*$\boldsymbol{H}^H\boldsymbol{H}$*) with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

***Fig.12*** *A comparison of condition numbers of 6 models (see Fig.6 to 11). These are for the case in which the microphone array is symmetrically arranged with respect to the source array, d/a=1 and r/a=1: model number 1 denotes the 2 source and 2 microphone model, model number 2 the 4 source and 4 microphone model, model number 3 the 6 source and 6 microphone model, model number 4 the 16 source and 16 microphone model, model number 5 the 35 source and 35 microphone model, and model number 6 the 100 source and 100 microphone model.*

***Fig.13*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the horizontal line source array: thick solid line represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, plus the cross array, point the X array and thin soild line the square array.*

***Fig.14*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the vertical line source array: broken line represents the condition number obtained by the use of microphones of the horizontal line array, thick solid line the vertical line array, plus the cross array, point the X array and thin solid line the square array.*

***Fig.15*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the cross source array: plus represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, thick solid line the cross array, point the X array and thin solid line the square array.*

***Fig.16*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the x source array: point represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, plus the cross array, thick solid line the X array and thin solid line the square array.*

***Fig.17*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the square source array: thin soild line represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, plus the cross array, point the X array and thick solid line the square array.*

## 1. Outline of the theory of reconstruction of acoustic source strength

Before moving onto the discussion of the condition number, we now review briefly the theories for the reconstruction of acoustic source by the inverse technique. A detailed discussion is given by Yoon and Nelson (1995). A formula that can be used to evaluate acoustic source strength auto- and cross-spectra from measured acoustic pressure auto- and cross-spectra and acoustic transfer functions has been be obtained in the sense of least squares in the form

$$\mathbf{S}_{qqo} = \{(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\}\mathbf{S}_{pp}\{(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\}^H \tag{1}$$

where $\mathbf{H}$ is the matrix of acoustic transfer functions, $\mathbf{S}_{pp}$ is the matrix of acoustic pressure auto- and cross-spectra and $\mathbf{S}_{qqo}$ is the matrix of optimally estimated acoustic source strength auto- and cross-spectra. The key parameters dominating the accuracy of the estimate of acoustic source strength auto- and cross-spectra are the amount of measurement noise contained in $\mathbf{S}_{pp}$, error included in $\mathbf{H}$ and the condition number of the matrix to be inverted. In order to enhance the accuracy of the estimate through the reduction of the condition number, we have developed the regularised version in the form

$$\mathbf{S}_{qqr} = \{(\mathbf{H}^H\mathbf{H} + \beta\mathbf{I})^{-1}\ \mathbf{H}^H\}\mathbf{S}_{pp}\{(\mathbf{H}^H\mathbf{H} + \beta\mathbf{I})^{-1}\ \mathbf{H}^H\}^H \tag{2}$$

where subscript $_r$ represents the regularisation and $\beta$ is the regularisation parameter. Also, by singular value discarding, we produced a formula such that

$$\mathbf{S}_{qqd} = \mathbf{H}_d^{+}\ \mathbf{S}_{pp}\ (\mathbf{H}_d^{+})^H \tag{3}$$

where the superscript signifies the pseudoinverse and subscript $_d$ represents discarding some singular values of the matrix $\mathbf{H}$. Finally, by combining the regularisation and the singular value decomposition, we have procured a singular value decomposition-based-regularsation version. Thus

$$\mathbf{S}_{qqsr} = (\tilde{\mathbf{H}})^{+}\mathbf{S}_{pp}\{(\tilde{\mathbf{H}})^{+}\}^H \tag{4}$$

where $\tilde{\mathbf{H}}$ is the transfer function matrix modified from $\mathbf{H}$ such that

$$\tilde{\mathbf{H}} = \mathbf{U}\tilde{\mathbf{S}}\,\mathbf{V}^{H} \tag{5}$$

where **U** and **V** are unitary matrices of **H** and $\tilde{\mathbf{S}}$ is the matrix modified from the singular value matrix **S** of **H**, using the regularisation parameter γ and singular values $s_i$ in the form

$$\tilde{\mathbf{S}} = \mathbf{S} + \mathbf{S}^{-1}\gamma\mathbf{I} = \begin{bmatrix} s_1 + \frac{\gamma}{s_1} & & & 0 \\ & s_2 + \frac{\gamma}{s_2} & & \\ & & \ddots & \\ 0 & & & s_n + \frac{\gamma}{s_n} \end{bmatrix}. \tag{6}$$

As observed from equations (1) to (4), the matrices to be inverted are respectively $\mathbf{H}^{H}\mathbf{H}$, $\mathbf{H}^{H}\mathbf{H} + \beta\mathbf{I}$, $\mathbf{H}_{d}$, and $\tilde{\mathbf{H}}$.

## 2. Condition number

We here introduce the definition of condition number of a matrix together with ill- and well-conditioning. If a matrix **H** is square, its condition number CN(**H**) is defined as

$$\mathrm{CN}(\mathbf{H}) = \|\mathbf{H}\|\,\|\mathbf{H}^{-1}\| \ , \tag{7}$$

where $\|\bullet\|$ denotes the 2-norm or the spectral norm of matrices .

If the matrix **H** is rectangular, CN(**H**) is defined as

$$\mathrm{CN}(\mathbf{H}) = \|\mathbf{H}\|\,\|\mathbf{H}^{+}\| \ , \tag{8}$$

where superscript denotes $^{+}$ the pseudoinverse (Golub and Kahan, 1965). Meanwhile the reason why the 2-norm is used in the definition of condition number is due to the adoption of the least squares method based on the 2-norm in estimating $\mathbf{S}_{qq}$. Also the value of the condition number CN(**H**) is dependent on the particular norm used. However if it is large in one norm then it is large in all others (Golub, 1989). In addition, the condition number CN(**H**) is always equal to or larger than one, because

$$\mathbf{H}^{-1}\mathbf{H} = \mathbf{I} \quad , \tag{9}$$

and also from the property of the norm of matrices (Barnett, 1990; Strang, 1976), $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \, \|\mathbf{B}\|$ (where **A** and **B** are arbitrary matrices),

$$\|\mathbf{I}\| = \|\mathbf{H}\,\mathbf{H}^{-1}\| \leq \|\mathbf{H}\| \, \|\mathbf{H}^{-1}\| = \mathrm{CN}(\mathbf{H}) \quad , \tag{10}$$

where the 2-norm of the identity matrix is one (it should be noted that the norm of the identity matrix is equal to or larger (in the case of Frobenius norm, for example) than one). As an example, the condition numbers of orthogonal and unitary matrix are all one. In such cases, the associated equations are fully independent.

On the other hand, another way to obtain the condition number is through the use of singular value decomposition. The 2-norm of a matrix is its largest singular value (accordingly it is called the spectral norm, Stewart (1973)) and thus when **H** is a square matrix

$$\|\mathbf{H}\| = s_{max}, \qquad \|\mathbf{H}^{-1}\| = s_{min}^{-1} \quad , \tag{11}$$

where $s_{max}$ and $s_{min}$ are the maximum and minimum singular value of **H**. When **H** is a rectangular matrix

$$\|\mathbf{H}\| = s_{max}, \qquad \|\mathbf{H}^{+}\| = s_{n}^{-1} \quad , \tag{12}$$

where $s_n$ represents the smallest nonzero singular value of **H**. Accordingly the condition number of a matrix **H** is given by

$$\mathrm{CN}(\mathbf{H}) = \|\mathbf{H}\| \, \|\mathbf{H}^{-1}\| = s_{max}/s_{min} \qquad \text{(square matrix)} \; , \tag{13}$$

$$\mathrm{CN}(\mathbf{H}) = \|\mathbf{H}\| \, \|\mathbf{H}^{+}\| = s_{max}/s_{n} \qquad \text{(rectangular matrix)} \; . \tag{14}$$

When the condition number of the matrix to be inverted is small, it is said to be *well-conditioned*, otherwise it is called *ill-conditioned.* The demarcation between well- and ill-conditioning is not made by a particular value and thus it is not clear. In a linear inverse system given by $\mathbf{q}=\mathbf{H}^{-1}\mathbf{p}$, the condition number of the matrix **H** is a measure of the sensitivity to the measurement noise in **p** and error in **H**. To put it in a concrete way, if a matrix **H** is well-conditioned, small perturbations in **H** and/or **p** produce small change in **q**. On the contrary, if a matrix **H** is ill-conditioned, radical change in **q** produces for even small perturbations in **H** and/or **p**. A detailed discussion of this matter using a mathematical development will be given in the ensuing section. Meanwhile in our problem of identifying acoustic source strength auto- and cross-spectra, the matrices to be inverted are $\mathbf{H}$ and $\mathbf{H}^H\mathbf{H}$, and the condition number of $\mathbf{H}^H\mathbf{H}$ is the square of that of $\mathbf{H}$ (see Yoon and Nelson, 1995).

## 3. Effect of the condition number of $H^HH$ and perturbation in $S_{pp}$ or $H^HH$ on $S_{qq}$

If there is a small perturbation in $\mathbf{S}_{pp}$ or $\mathbf{H}^H\mathbf{H}$, how large is their effect on the estimate of $\mathbf{S}_{qq}$? This section is devoted to a mathematical discussion of this question.

Before investigating this effect, it would be useful to recall that in a linear system with multiple inputs and multiple outputs the relationship between input spectra (here the acoustic source strength cross-spectra $\mathbf{S}_{qq}$) and output spectra (here the measured acoustic pressure spectra $\mathbf{S}_{pp}$) is expressed as

$$\mathbf{H}\mathbf{S}_{qq}\mathbf{H}^H = \mathbf{S}_{pp} \quad . \tag{15}$$

where for convenience the number of inputs and outputs is assumed to be the same, say $\mathbf{H}$ is a square matrix.

Let us first investigate the influence of disturbance in $\mathbf{S}_{pp}$ on $\mathbf{S}_{qq}$. When $\mathbf{S}_{pp}$ changes to $\mathbf{S}_{pp}$+$\mathbf{S}_{nn}$ (where the noise cross-spectra $\mathbf{S}_{nn}$ are assumed to be uncorrelated with $\mathbf{S}_{pp}$, and it can be considered as the perturbation in $\mathbf{S}_{pp}$), $\mathbf{S}_{qq}$ becomes $\mathbf{S}_{qq} + \delta\mathbf{S}_{qq}$. Thus equation (15) becomes

$$\mathbf{H}(\mathbf{S}_{qq} + \delta\mathbf{S}_{qq})\mathbf{H}^H = \mathbf{S}_{pp} + \mathbf{S}_{nn} = \mathbf{S}_{pp} + \delta\mathbf{S}_{pp} \quad . \tag{16}$$

Since $\mathbf{H}\mathbf{S}_{qq}\mathbf{H}^H = \mathbf{S}_{pp}$, equation (16) leads to

$$\mathbf{H}(\delta\mathbf{S}_{qq})\mathbf{H}^H = \delta\mathbf{S}_{pp} \quad . \tag{17}$$

The ratio of the norms of the relative change is

$$\frac{\| \delta\mathbf{S}_{qq} \| \;/\; \| \mathbf{S}_{qq} \|}{\| \delta\mathbf{S}_{pp} \| \;/\; \|\mathbf{S}_{pp} \|} = \frac{\| \delta\mathbf{S}_{qq} \| \, \| \mathbf{S}_{pp} \|}{\| \delta\mathbf{S}_{pp} \| \, \| \mathbf{S}_{qq} \|} \quad . \tag{18}$$

From the property of matrix norms, it follows that

$$\| \mathbf{S}_{pp} \| = \| \mathbf{H}\mathbf{S}_{qq}\mathbf{H}^{H} \| \leq \| \mathbf{H} \| \, \| \mathbf{S}_{qq} \| \, \| \mathbf{H}^{H} \| \quad , \tag{19a}$$

$$\| \delta\mathbf{S}_{qq} \| = \| \mathbf{H}^{-1} (\delta\mathbf{S}_{pp}) (\mathbf{H}^{H})^{-1} \| \leq \| \mathbf{H}^{-1} \| \, \| \delta\mathbf{S}_{pp} \| \, \| (\mathbf{H}^{H})^{-1} \| \quad . \tag{19b}$$

Substituting these relationships into equation (18) gives

$$\frac{\|\delta\mathbf{S}_{qq}\| \; \|\mathbf{S}_{pp}\|}{\|\delta\mathbf{S}_{pp}\| \; \|\mathbf{S}_{qq}\|} \leq \frac{\|\mathbf{H}^{-1}\| \; \|\delta\mathbf{S}_{pp}\| \; \|(\mathbf{H}^{H})^{-1}\| \; \|\mathbf{H}\| \; \|\mathbf{S}_{qq}\| \; \|\mathbf{H}^{H}\|}{\|\delta\mathbf{S}_{pp}\| \; \|\mathbf{S}_{qq}\|}$$
$$= \|\mathbf{H}\| \; \|\mathbf{H}^{-1}\| \; \|\mathbf{H}^{H}\| \; \|(\mathbf{H}^{H})^{-1}\| \quad . \tag{20}$$

From the definition of condition number given by equation (7), equation (20) yields

$$\frac{\|\delta\mathbf{S}_{qq}\| \; \|\mathbf{S}_{pp}\|}{\|\delta\mathbf{S}_{pp}\| \; \|\mathbf{S}_{qq}\|} \leq \mathrm{CN}(\mathbf{H}^{H}) \, \mathrm{CN}(\mathbf{H}) = \mathrm{CN}^{2}(\mathbf{H}) = \mathrm{CN}(\mathbf{H}^{H}\mathbf{H}) \quad . \tag{21}$$

where $\mathrm{CN}(\mathbf{H}^{H})$ is the same as $\mathrm{CN}(\mathbf{H})$ because the singular values are not affected by the transposition and conjugation of the matrix.

Substituting this into equation (18) leads to

$$\frac{\|\delta\mathbf{S}_{qq}\|}{\|\mathbf{S}_{qq}\|} \leq \mathrm{CN}(\mathbf{H}^{H}\mathbf{H}) \frac{\|\delta\mathbf{S}_{pp}\|}{\|\mathbf{S}_{pp}\|} \quad . \tag{22}$$

Next the effect of the perturbation in $\mathbf{H}^{H}\mathbf{H}$ on $\mathbf{S}_{qq}$ is investigated. From equation (15), the 2-norm of $\mathbf{S}_{qq}$ leads to

$$\|\mathbf{S}_{qq}\| = \|\mathbf{H}^{-1}\mathbf{S}_{pp}(\mathbf{H}^{H})^{-1}\| \leq \| \mathbf{H}^{-1}\| \, \| \mathbf{S}_{pp} \| \, \|(\mathbf{H}^{H})^{-1}\| \quad . \tag{23}$$

This can be developed as

$$\|\mathbf{S}_{qq}\| \leq \|\mathbf{H}^{H}\| \, \|(\mathbf{H}^{H})^{-1}\| \, \|\mathbf{H}\| \, \|\mathbf{H}^{-1}\| \, \frac{\|\mathbf{S}_{pp}\|}{\|\mathbf{H}^{H}\| \, \|\mathbf{H}\|}$$
$$= \mathrm{CN}(\mathbf{H}^{H}) \, \mathrm{CN}(\mathbf{H}) \, \frac{\|\mathbf{S}_{pp}\|}{\|\mathbf{H}^{H}\| \, \|\mathbf{H}\|} \quad . \tag{24}$$

Since $\|\mathbf{H}^{H}\| \, \|\mathbf{H}\| \geq \|\mathbf{H}^{H}\mathbf{H}\|$ and $\mathrm{CN}(\mathbf{H}^{H}) \, \mathrm{CN}(\mathbf{H}) = \mathrm{CN}(\mathbf{H}^{H}\mathbf{H})$, equation (24) becomes

$$\|\mathbf{S}_{qq}\| \leq \mathrm{CN}(\mathbf{H}^{H}\mathbf{H}) \, \frac{\|\mathbf{S}_{pp}\|}{\|\mathbf{H}^{H}\mathbf{H}\|} \quad . \tag{25}$$

It therefore is obvious from equations (22) and (25) that the relative change of the spectra $S_{qq}$ depends on both the degree of the perturbation in $S_{pp}$ (i.e., the quantity of $S_{nn}$) and $\mathbf{H}^H\mathbf{H}$ and the condition number of the matrix $\mathbf{H}^H\mathbf{H}$ to be inverted. Thus if the matrix $\mathbf{H}^H\mathbf{H}$ is ill-conditioned, even a small variation in $S_{pp}$ and/or $\mathbf{H}^H\mathbf{H}$ results in violent changes in $S_{qq}$. Accordingly, under the situation in which the measurement noise in $S_{pp}$ and/or error in $\mathbf{H}^H\mathbf{H}$ are unavoidable, it is natural to make an attempt to improve the poor conditioning of the matrix to be inverted in order to increase stability in identifying $S_{qq}$.

## 4. Two factors determining the condition number CN($H^H H$)

As pointed out in the preceding section, in the inverse problem employing the matrix formulation, the condition number of the matrix to be inverted plays a significant role in restoring the acoustic sources with acceptably reasonable accuracy. So, it is a prerequisite to have a fuller understanding of the characteristics of the condition number. This section is concerned with the investigation into the factors determining the condition number of the matrix $\mathbf{H}^H\mathbf{H}$ which is constructed from the geometric arrangement of a number of point monopole sources and microphones. To do this, let us observe the structure of the matrix $\mathbf{H}^H\mathbf{H}$ for a simple model consisting of 2-sources and 3-microphones. In this case

$$\mathbf{H}^H\mathbf{H} = \frac{\omega^2\rho_o^2}{4\pi^2}\begin{bmatrix} \frac{1}{r_{11}^2}+\frac{1}{r_{21}^2}+\frac{1}{r_{31}^2} & \frac{e^{j\omega(r_{11}-r_{12})/c_o}}{r_{11}r_{12}}+\frac{e^{j\omega(r_{21}-r_{22})/c_o}}{r_{21}r_{22}}+\frac{e^{j\omega(r_{31}-r_{32})/c_o}}{r_{31}r_{32}} \\ \frac{e^{-j\omega(r_{11}-r_{12})/c_o}}{r_{11}r_{12}}+\frac{e^{-j\omega(r_{21}-r_{22})/c_o}}{r_{21}r_{22}}+\frac{e^{-j\omega(r_{31}-r_{32})/c_o}}{r_{31}r_{32}} & \frac{1}{r_{12}^2}+\frac{1}{r_{22}^2}+\frac{1}{r_{32}^2} \end{bmatrix}. \quad (26)$$

From equation (26), it is evident that there are two factors determining the condition number, of $\mathbf{H}^H\mathbf{H}$: frequency $\omega$ and distance $r_{ij}$ between positions of sources and microphones. The latter is in connection with the geometrical arrangement of sources and microphones. Roughly speaking, as the values of $r_{ij}$ get similar each other (i.e., $r_{ij} \approx r$), all the elements are approximated to the some value (in this case, $3\omega^2\rho_o^2/(4\pi^2 r^2)$). Also, as frequency $\omega$ approaches zero, the elements of this matrix become close to each other. Under these situations, the matrix $\mathbf{H}^H\mathbf{H}$ departs greatly from orthogonality and its condition number rapidly increases. In addition, it is implied from the phase terms given by $\exp[j\omega(r_{ij}-r_{mn})/c_o]$ that the condition number will oscillate with frequency.

## 5. Characteristics of the condition number of $\mathbf{H}^H\mathbf{H}$ depending on frequency

In two subsequent sections, we study in more detail the characteristics of the condition number of the matrix $\mathbf{H}^H\mathbf{H}$ depending on frequency and distance between sources and microphones. We first investigate analytically the behaviour of the condition number oscillating with frequency.

As mentioned already, the condition number of a square matrix $\mathbf{H}$ can be obtained from the ratio of its maximum to minimum singular value (see equation (13)). Meanwhile the singular values of $\mathbf{H}$ are the positive square roots of the eigenvalues of $\mathbf{H}^H\mathbf{H}$ or $\mathbf{H}\mathbf{H}^H$. Accordingly, the condition number of $\mathbf{H}$ can be written as

$$CN(\mathbf{H}) = \frac{s_{max}}{s_{min}} = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} \quad , \tag{27}$$

where $\lambda_{max}$ and $\lambda_{min}$ are respectively the maximum and minimum eigenvalues of $\mathbf{H}^H\mathbf{H}$ or $\mathbf{H}\mathbf{H}^H$. Since the condition number of $\mathbf{H}^H\mathbf{H}$ is the square of that of $\mathbf{H}$,

$$CN(\mathbf{H}^H\mathbf{H}) = \{CN(\mathbf{H})\}^2 = \frac{\lambda_{max}}{\lambda_{min}} \quad . \tag{28}$$

To show analytically the oscillation characteristics of $CN(\mathbf{H}^H\mathbf{H})$ with frequency, we choose a 2 source and 2 microphone model as an example, as shown in Fig.1, for simplicity of calculation of the eigenvalues. For this model, the matrix $\mathbf{H}^H\mathbf{H}$ is written as

$$\mathbf{H}^H\mathbf{H} = \frac{\omega^2\rho^2}{4\pi^2}\begin{bmatrix} \frac{1}{r_{11}^2}+\frac{1}{r_{21}^2} & \frac{e^{j\omega(r_{11}-r_{12})/c_o}}{r_{11}r_{12}}+\frac{e^{j\omega(r_{21}-r_{22})/c_o}}{r_{21}r_{22}} \\ \frac{e^{-j\omega(r_{11}-r_{12})/c_o}}{r_{11}r_{12}}+\frac{e^{-j\omega(r_{21}-r_{22})/c_o}}{r_{21}r_{22}} & \frac{1}{r_{12}^2}+\frac{1}{r_{22}^2} \end{bmatrix} \quad . \tag{29}$$

If we, for notational convenience, let $r_{11}$=a, $r_{12}$=b, $r_{21}$=c, and $r_{22}$=d, equation (29) becomes

$$\mathbf{H}^H\mathbf{H} = \frac{\omega^2\rho^2}{4\pi^2}\begin{bmatrix} \frac{1}{a^2}+\frac{1}{c^2} & \frac{e^{j\omega(a-b)/c_o}}{ab}+\frac{e^{j\omega(c-d)/c_o}}{cd} \\ \frac{e^{-j\omega(a-b)/c_o}}{ab}+\frac{e^{-j\omega(c-d)/c_o}}{cd} & \frac{1}{b^2}+\frac{1}{d^2} \end{bmatrix} \quad . \tag{30}$$

The eigenvalues of the matrix $\mathbf{H}^H\mathbf{H}$ are determined by

$$\det[\lambda\mathbf{I}-\mathbf{H}^H\mathbf{H}] = 0 \quad . \tag{31}$$

Thus

$$\begin{vmatrix} \lambda-\left(\frac{1}{a^2}+\frac{1}{c^2}\right) & \frac{e^{j\omega(a-b)/c_o}}{ab}+\frac{e^{j\omega(c-d)/c_o}}{cd} \\ \frac{e^{-j\omega(a-b)/c_o}}{ab}+\frac{e^{-j\omega(c-d)/c_o}}{cd} & \lambda-\left(\frac{1}{b^2}+\frac{1}{d^2}\right) \end{vmatrix} = 0 \quad . \tag{32}$$

Developing this yields

$$\lambda^2 - P\lambda + [Q-R\cos(S\omega)] = 0 \quad , \tag{33}$$

where

$$P = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2} \quad ,$$

$$Q = \frac{1}{a^2d^2}+\frac{1}{b^2c^2} \quad ,$$

$$R = \frac{2}{abcd} \quad ,$$

$$S = (-a+b+c-d)/c_o \quad .$$

From equation (33), the eigenvalues are calculated as

$$\lambda = \frac{P \pm \sqrt{P^2 - 4[Q - R\cos(S\omega)]}}{2} \quad . \tag{34}$$

Since all the eigenvalues of an Hermitian matrix are always real, the values of the square root in equation (34) are real and therefore the condition number of $\mathbf{H}^H\mathbf{H}$ results in

$$CN(\mathbf{H}^H\mathbf{H}) = \frac{P + \sqrt{P^2 - 4[Q - R\cos(S\omega)]}}{P - \sqrt{P^2 - 4[Q - R\cos(S\omega)]}} \quad . \tag{35}$$

Therefore, it is obvious from equation (35) that the condition number of $\mathbf{H}^H\mathbf{H}$ oscillates with frequency $\omega$. To illustrate this behaviour graphically, we arrange the acoustic sources and microphones as shown in Fig.2. Using this model, we obtain the condition number as shown in Fig.3 and it is clear that the condition number oscillates with frequency. However it should be noted that since we adopt the point source model (namely, $ka \ll 1$, a is the dimension of a discretised source and k wavenumber) this oscillation phenomenon does not appear within the frequency of interest (see Fig.3). The condition number below ka=1 decreases with frequency. In addition to this 2 source and 2 microphone model, another model consisting of 2 sources and 3 microphones as illustrated in Fig.4 is investigated. The condition number (Fig.5) of this model also reveals oscillation, but the trend is different from that (Fig.3) of the 2 source and 2 microphone model.

## 6. Effect of the geometrical arrangement of sources and microphones on the condition number of $H^HH$

Now the effect of distances between sources and microphones on $CN(\mathbf{H}^H\mathbf{H})$ is investigated empirically through computer simulations using a set of models. From this study, we wish to find a universal way of arranging microphones, independent of the number of sources and microphones used and frequency, in order to make the condition number as small as possible. The parameters to be considered are selected as follows:

1. The number of sources and microphones;
2. Geometrical arrangement of the microphone array with respect to the source array: symmetric or asymmetric;
3. Ratio of the source array plane-to-microphone array plane distance (r) to the source-to-source distance (a), r/a;
4. Ratio of the microphone-to-microphone distance (d) to the source-to-source distance (a), d/a;
5. Helmholtz number, ka;
6. Geometry of the microphone array.

First of all, in order to seek the effect of the parameters given by the factors numbered 1 to 5 above, we build two line source models (2 and 6 source models), one rectangular plane source model (35 source model), and three square plane models (4, 16 and 100 source models). In each model, the number of microphones is selected to be the same as that of the sources and the centre of microphone array is placed symmetrically (namely, the eccentricity e=0, see Fig.6(a) to 11(a)) or asymmetrically (e≠0 say) with respect to that of the source array. Utilising these models, the condition numbers are computed at a certain value of ka (here ka=0.549, or 300Hz, a=0.1m), varying the values of m/a and r/a respectively from 0.1 to 2 with increment of 0.1 (see (b)s, (c)s and (d)s of Fig.6 to 11). Also with the symmetric placement of microphone array, the condition numbers are computed from ka=0.0018 (1Hz, a=0.1m) to 9.16 (5000Hz), varying the ratio d/a from 0.1 to 2 with increment of 0.1 (whilst fixing the value of r/a with 1 or 0.5, see (e)s and (f)s of Fig.6 to 11) or r/a from 0.1 to 2 with increment of 0.1(whilst setting the value of m/a by 1 or 0.5, see (g)s and (h)s of Fig.6 to 11).

The results of this study are shown by the condition numbers illustrated in Fig.6 to 11 and from these we can observe the following:

1. Comparing (b)s, (c)s and (d)s of Fig.6 to 11 indicates that the symmetric arrangement of microphone array with respect to the source array produces a smaller condition number than the asymmetric arrangement;

2. The results given by (b)s of Fig.6 to 11 show that when the microphone array is placed symmetrically, the condition number decreases as the distance d between two neighbouring microphones becomes close to the distance between two adjacent sources (namely, d/a=1). Also in such a case, the condition number becomes small as the microphone array plane approaches to the source array plane (i.e., when r/a is smaller). However, it should not be overlooked that in practice, the smaller r/a becomes the less accurate are the acoustic pressures measured, because of the reflection and scattering from the microphone array including the scanning system;

3. The results of (e)s, (f)s, (g)s and (h)s of Fig.6 to 11 reveal that the statements given above (1 and 2) which are for a certain value of ka (here ka=0.549) are also valid for all values of up to ka≈1, but not valid for ka>1. This means that as long as we adopt a source modelling in which a real source is discretised into a number of point sources, the statements described above (1 and 2) for the behaviour of condition number are valid, regardless of frequency;

4. Observing all the figures of Fig.6 to 11, the statements regarding the condition number given above (1, 2 and 3) hold, regardless of the number of sources and microphones.

5. Fig.12 compares the condition numbers of six models shown in Fig.6 to 11 in which the microphone array is located symmetrically with respect to the source array and the ratios d/a and r/a are set respectively to 1. The result shows that as the number of sources and microphones increases the condition number increases. This statement holds below ka=1 but is violated for ka>1.

Next we investigate how the condition number is affected by the type of microphone array (the parameter numbered 6 above). To do this, we consider five types of microphone array, together with five types of source array. They are respectively the horizontal line, vertical line, cross, X, and square array. It is now imagined that the acoustic field generated from the acoustic sources which are arranged by a form of five types of source array is measured in turn by using the five types of microphone array. So we have 25 models as shown in Fig.13 to 17. In these models, the microphone-to-microphone distance (d) and the source plane-to-microphone plane distance (r) are set equal to the source-to-source distance (a). Also, the microphone array is placed symmetrically with respect to the source array. On the other hand, whilst the use of horizontal line, vertical line and square microphone array are commonly used in the source identification, the X and cross microphone array have been employed in identifying moving sources such as motorbike, car and train (Kim *et al*, 1996; Vos, 1996; Takano, 1996). Fig.13 to 17 show the condition numbers CN($\mathbf{H}^H\mathbf{H}$) for 25 models. According to these, it is evident that the condition number becomes smallest when the microphones are arranged with the geometry as the source array.

## 7. Conclusion

The accuracy of reconstruction of acoustic source strength auto- and cross-spectra is determined by the amount of measurement noise contained in acoustic pressure auto- and cross-spectra, the degree of error included in transfer functions linking sources and field points, and the conditioning of the matrix to be inverted. Under the condition in which measurement noise and error are unavoidable, the only way to enhance the restoration is to make the condition number as small as possible. Basically, the condition number of the matrix $\mathbf{H}^H\mathbf{H}$ which is constructed from the geometrical arrangement of a set of point monopole sources and microphones is determined by the frequency of interest and the distances between the positions of the sources and field points. The condition number shows the oscillation behaviour with frequency, with different patterns depending on the number of sources and field points. However, as long as the real source is modelled by a number of point monopole sources (requiring ka<<1), the condition number decreases with frequency, with no oscillation. Regarding the results of investigation into the effect of geometrical placement of sources and microphones on the condition number, the follows are concluded. First of all, it is observed that when the centre of microphone array is adjusted to that of source array with no eccentricity the condition number decreases. Also, the condition number becomes small as the microphone-to-microphone distance (d) approaches to the source-to-source distance (a). Under the condition in which microphone array is placed with no eccentricity and d/a≈1, the condition number decreases as the microphone array plane becomes close to the source array plane. Further simulation results reveal that these characteristics stated above are valid until ka≈1, regardless of the number of sources and microphones. In addition, if the geometrical arrangement is the same, as the number of source and microphones increases, the condition number increases. Finally, it is concluded that the microphone array geometry similar to the source array geometry produces much better conditioning than others.

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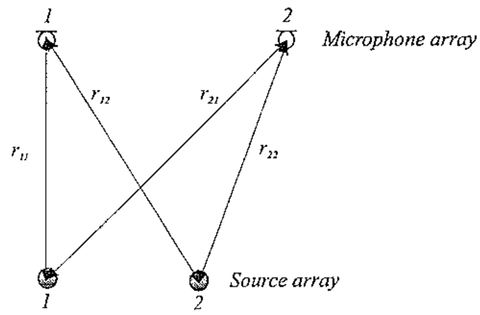



***Fig.1*** *A geometrical arrangement of 2 sources and 2 microphones.*

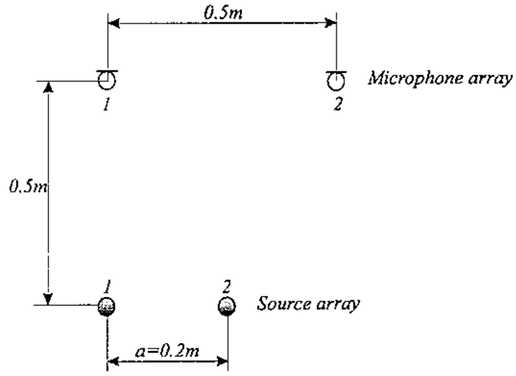



***Fig.2*** *A geometrical arrangement of 2 sources and 2 microphones (discretised source dimension a=0.2m).*

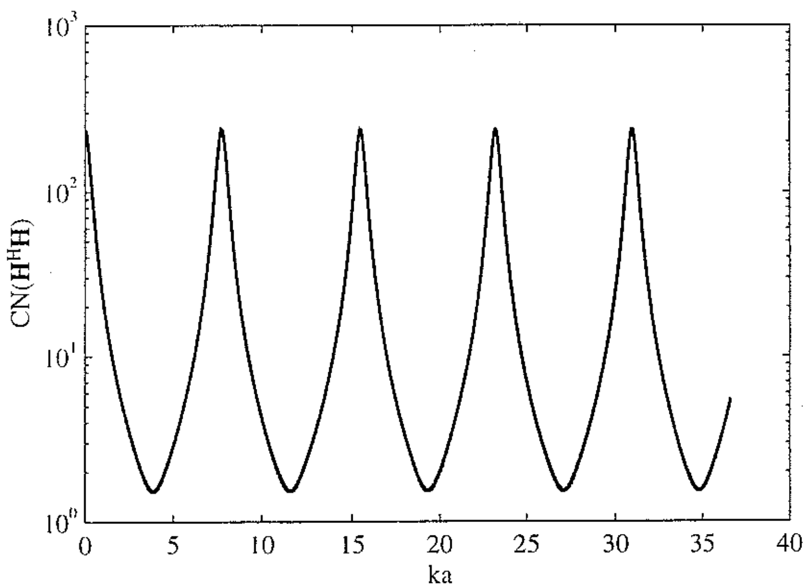



***Fig.3*** *The condition number oscillating with frequency (2 source and 3 microphone model).*

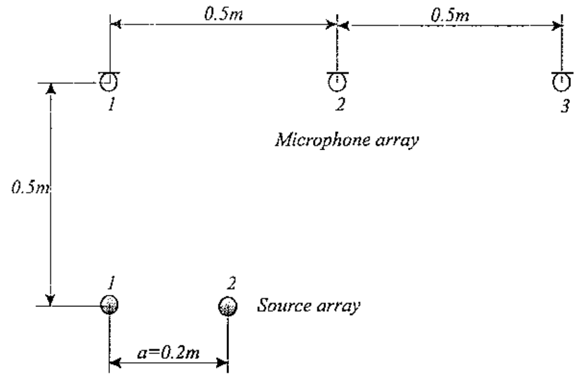



***Fig.4*** *A geometrical arrangement of 2 sources and 3 microphones (discretised source dimension a=0.2m).*

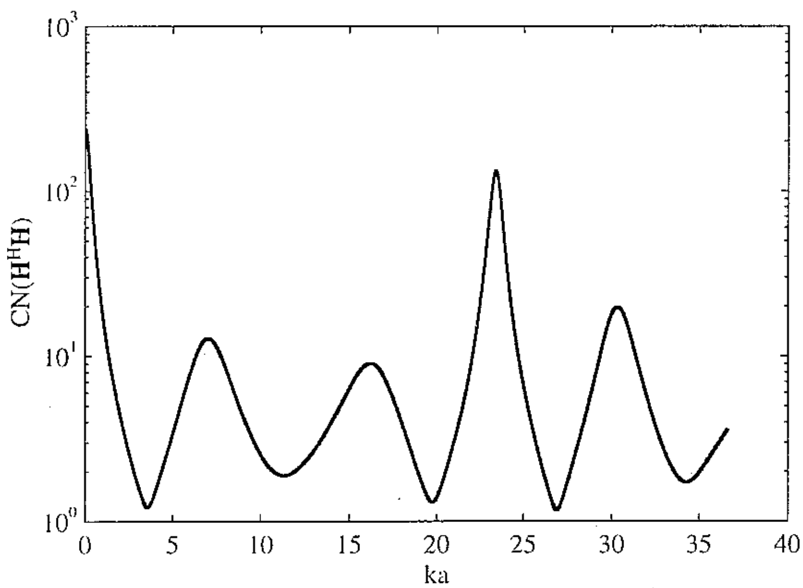



***Fig.5*** *The condition number oscillating with frequency (2 source and 3 microphone model).*

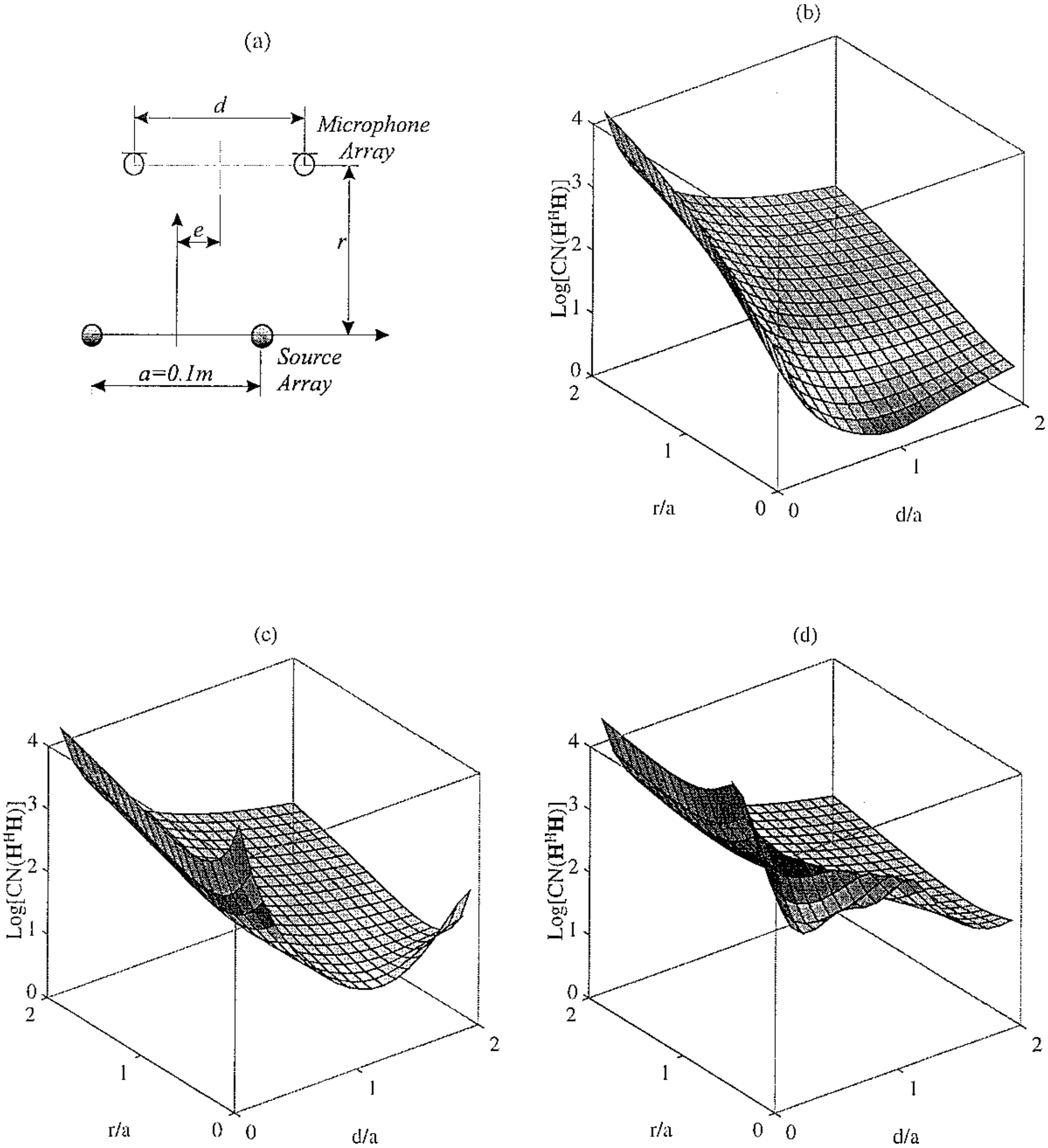



***Fig.6*** *(a) A geometrical arrangement of 2 sources and 2 microphones. Variation of the condition number CN($\boldsymbol{H}^H\boldsymbol{H}$) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m.*

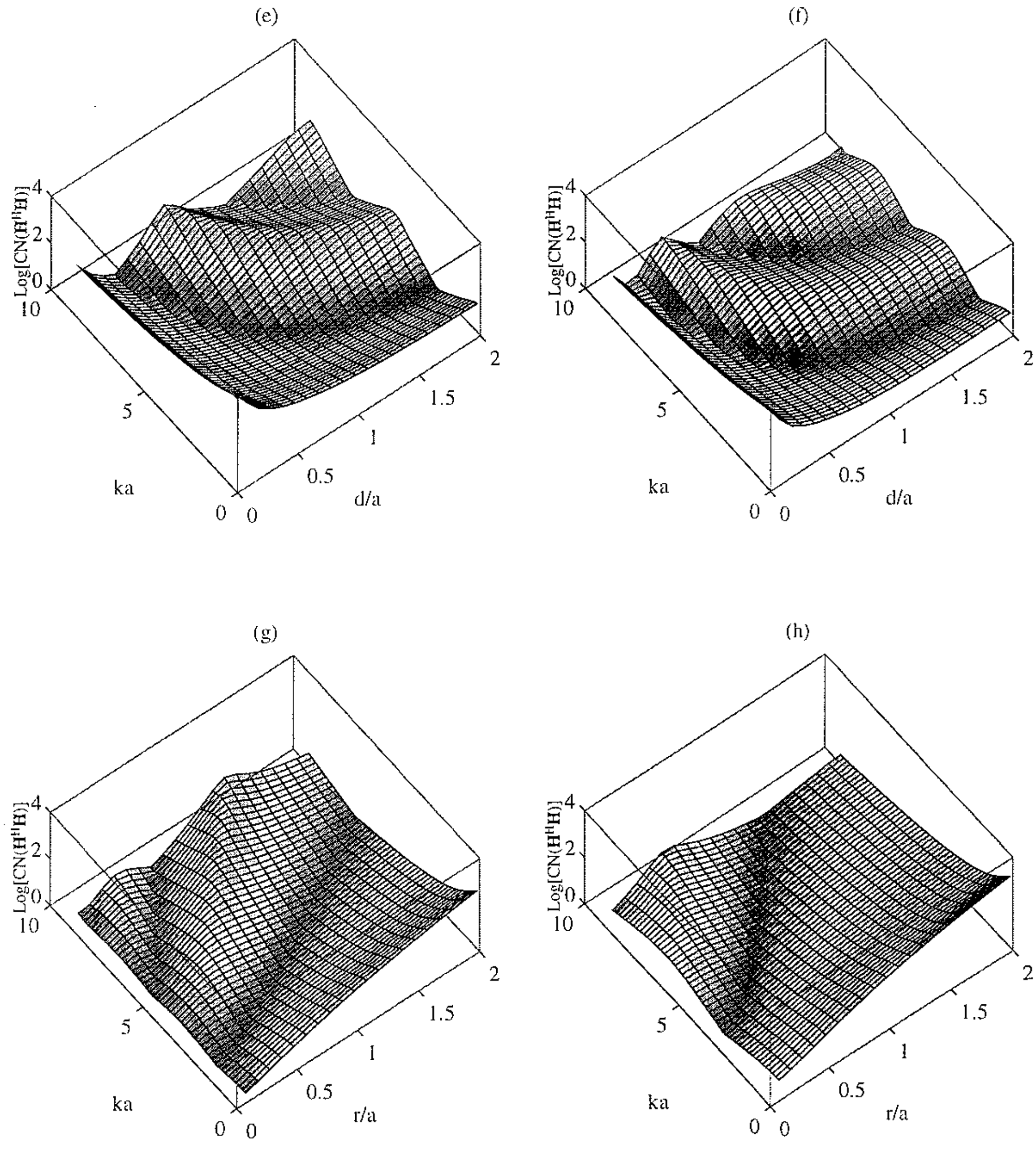



***Fig.6 contd*** *Variation of the condition number CN($\boldsymbol{H}^H\boldsymbol{H}$) with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

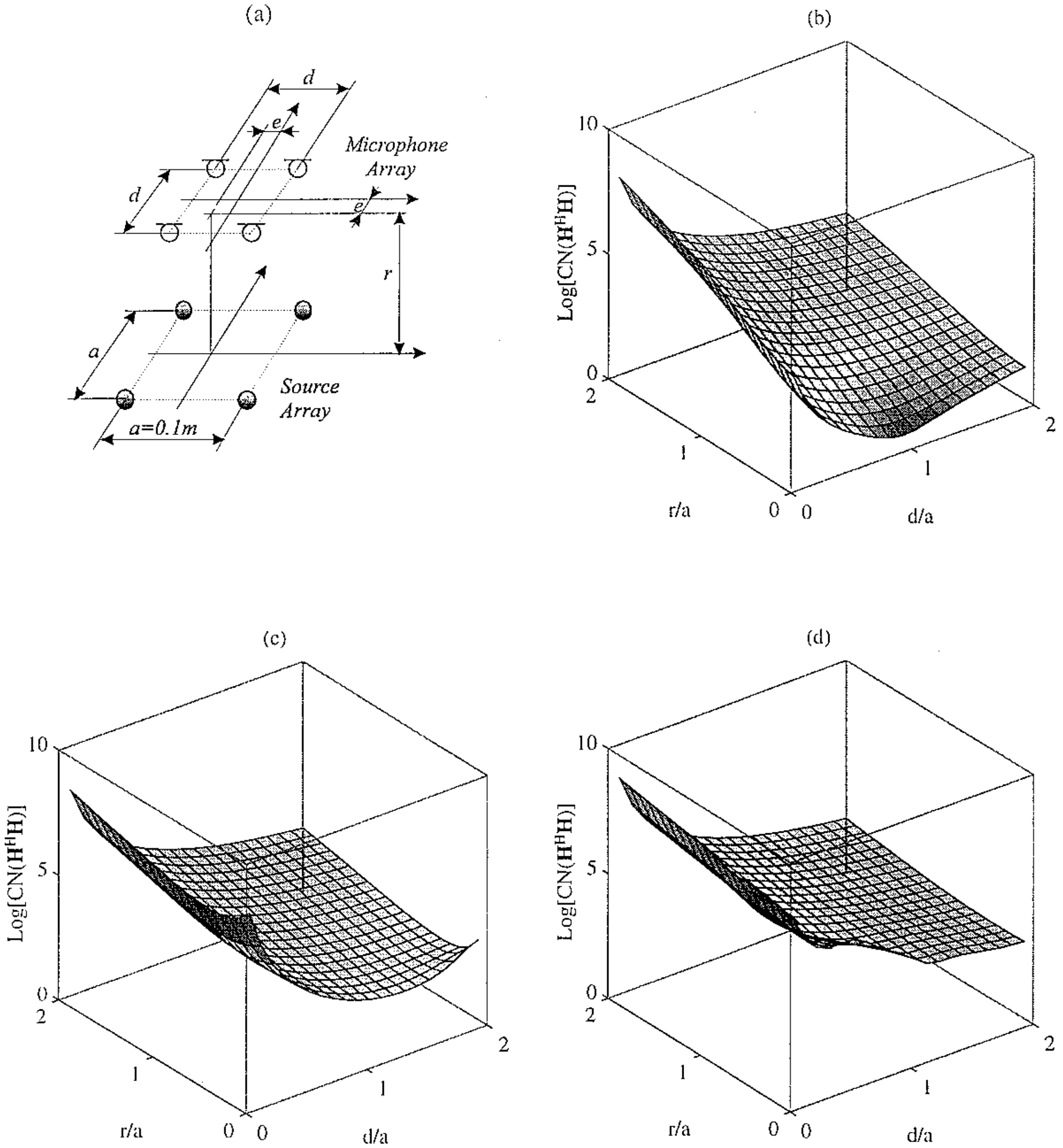



***Fig.7*** *(a) A geometrical arrangement of 4 sources and 4 microphones. Variation of the condition number CN($\mathbf{H}^H\mathbf{H}$) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m.*

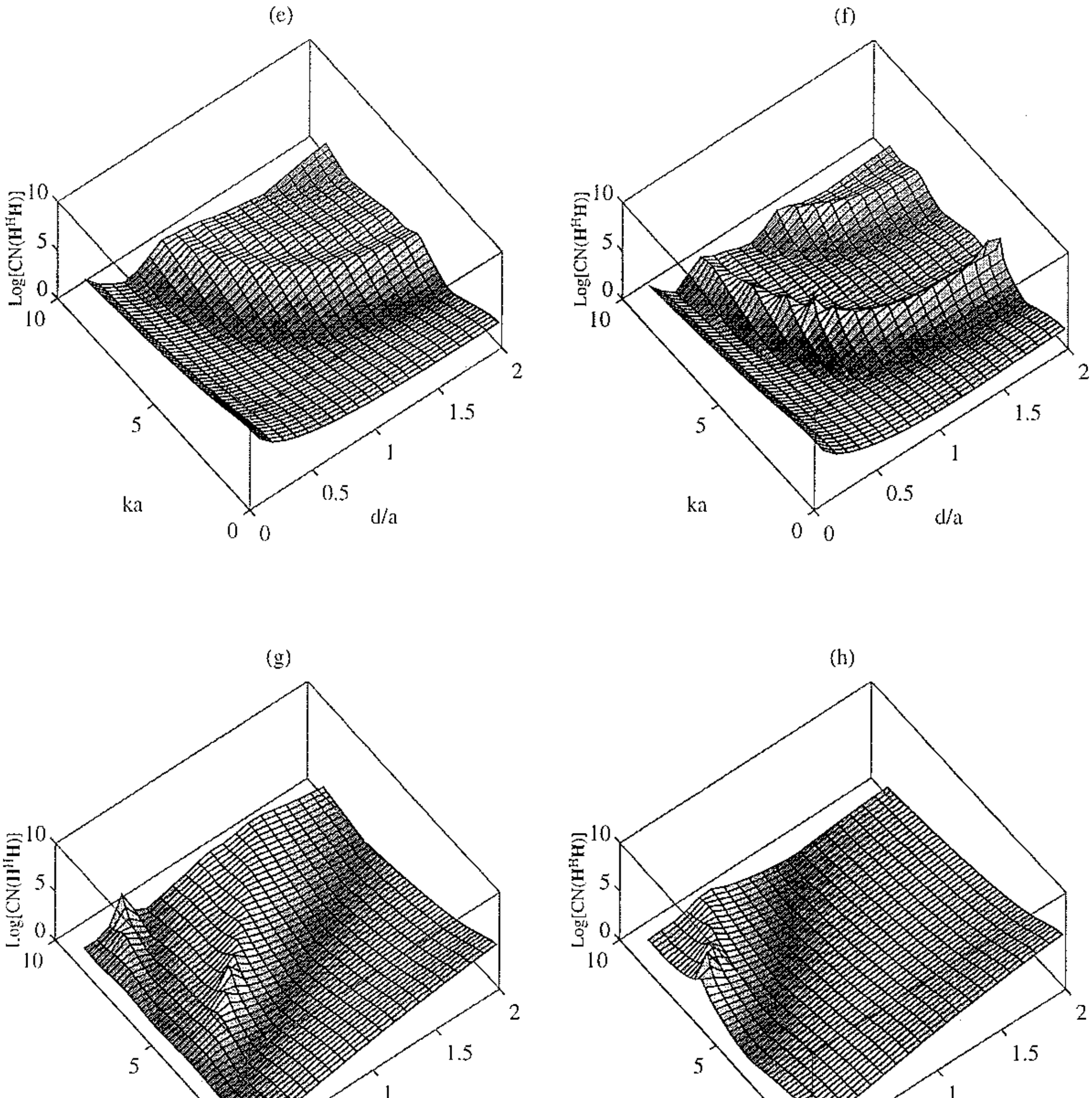



***Fig.7 contd*** *Variation of the condition number CN($\boldsymbol{H}^H\boldsymbol{H}$) with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

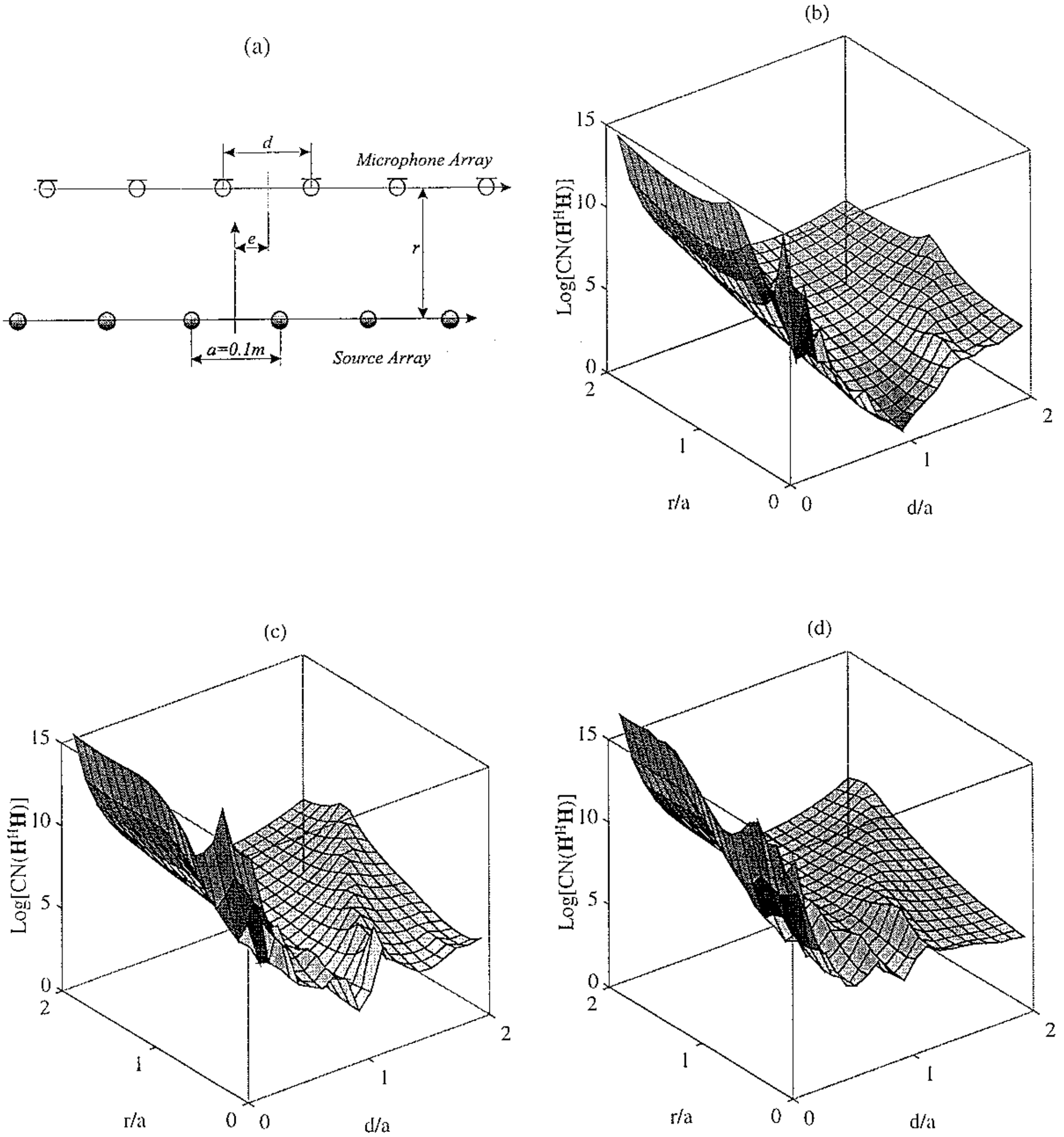



***Fig.8*** *(a) A geometrical arrangement of 6 sources and 6 microphones. Variation of the condition number CN($\boldsymbol{H}^H\boldsymbol{H}$) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m.*

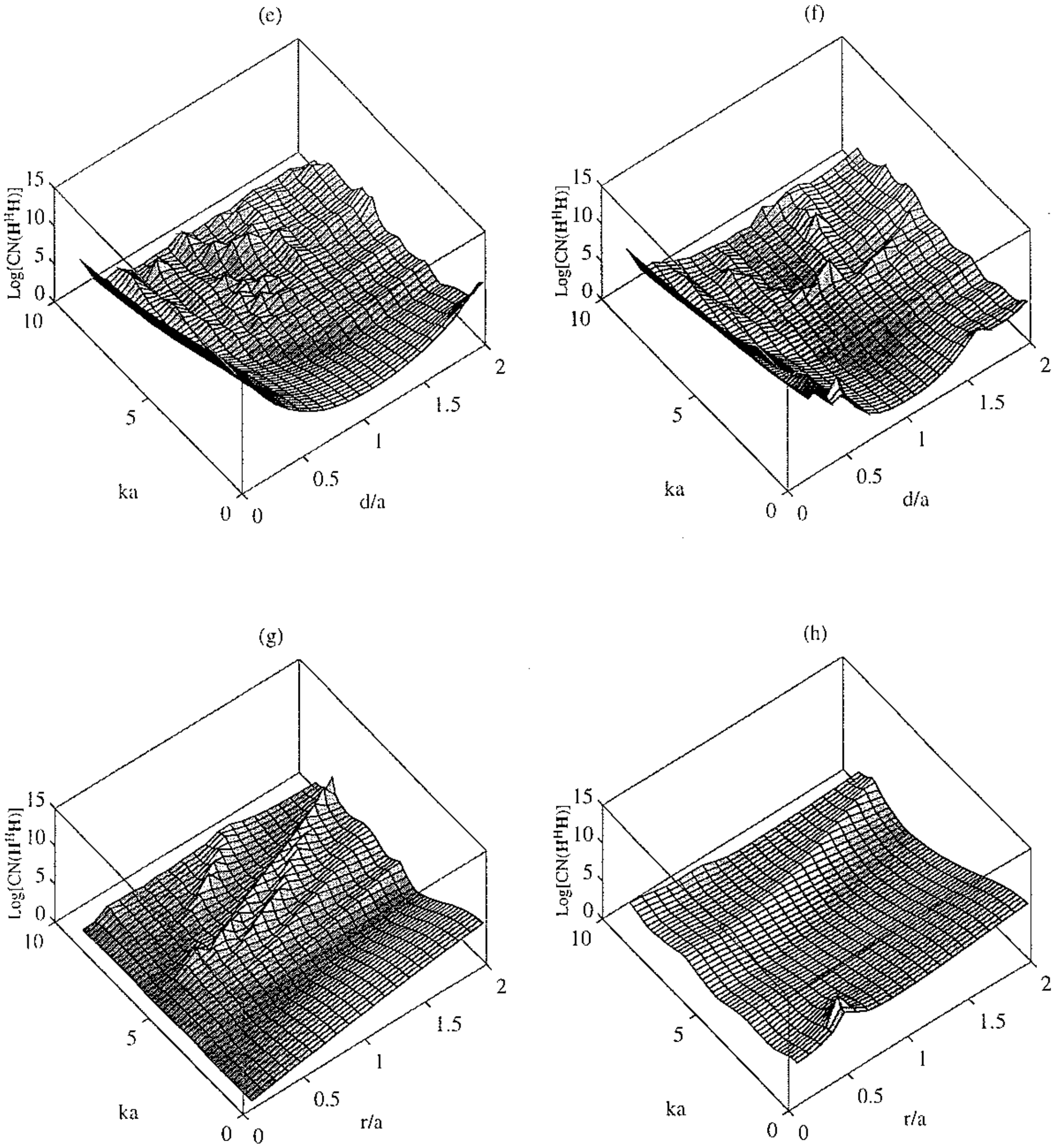



***Fig.8 contd*** *Variation of the condition number CN($\boldsymbol{H}^H\boldsymbol{H}$) with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

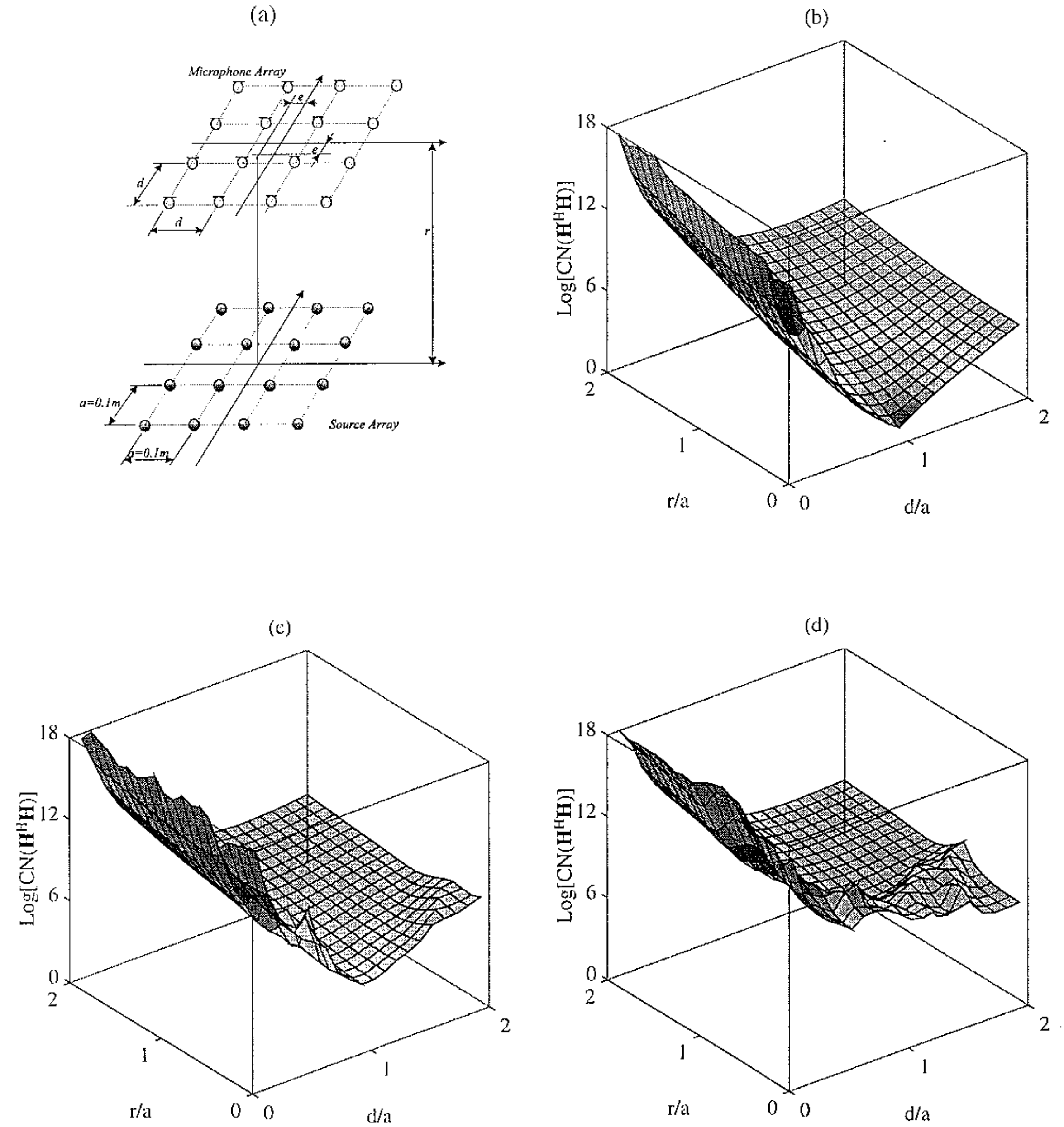



***Fig.9*** *(a) A geometrical arrangement of 16 sources and 16 microphones. Variation of the condition number CN($\boldsymbol{H}^H\boldsymbol{H}$) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m.*

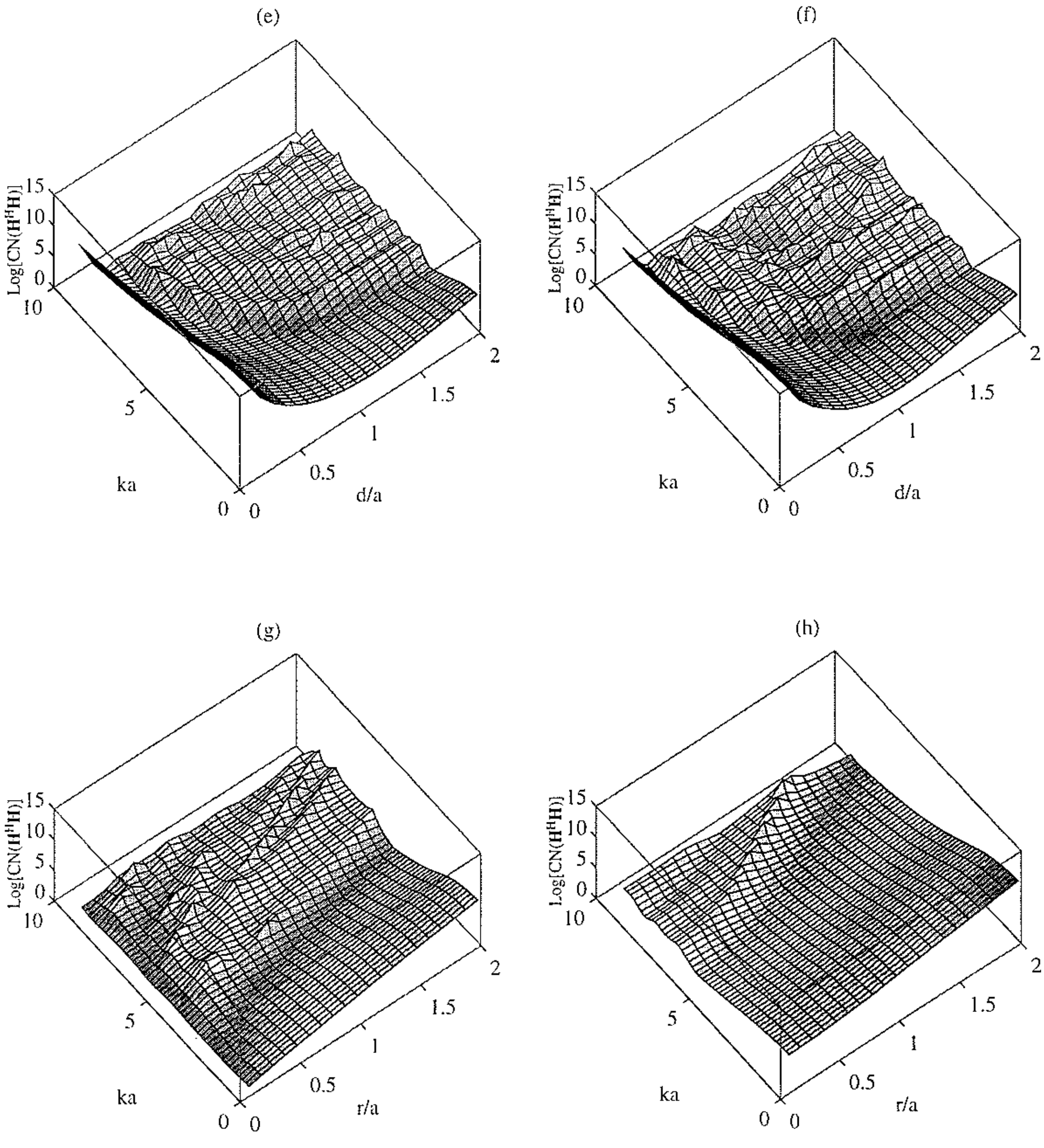



***Fig.9 contd*** *Variation of the condition number CN($\mathbf{H}^H\mathbf{H}$) with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

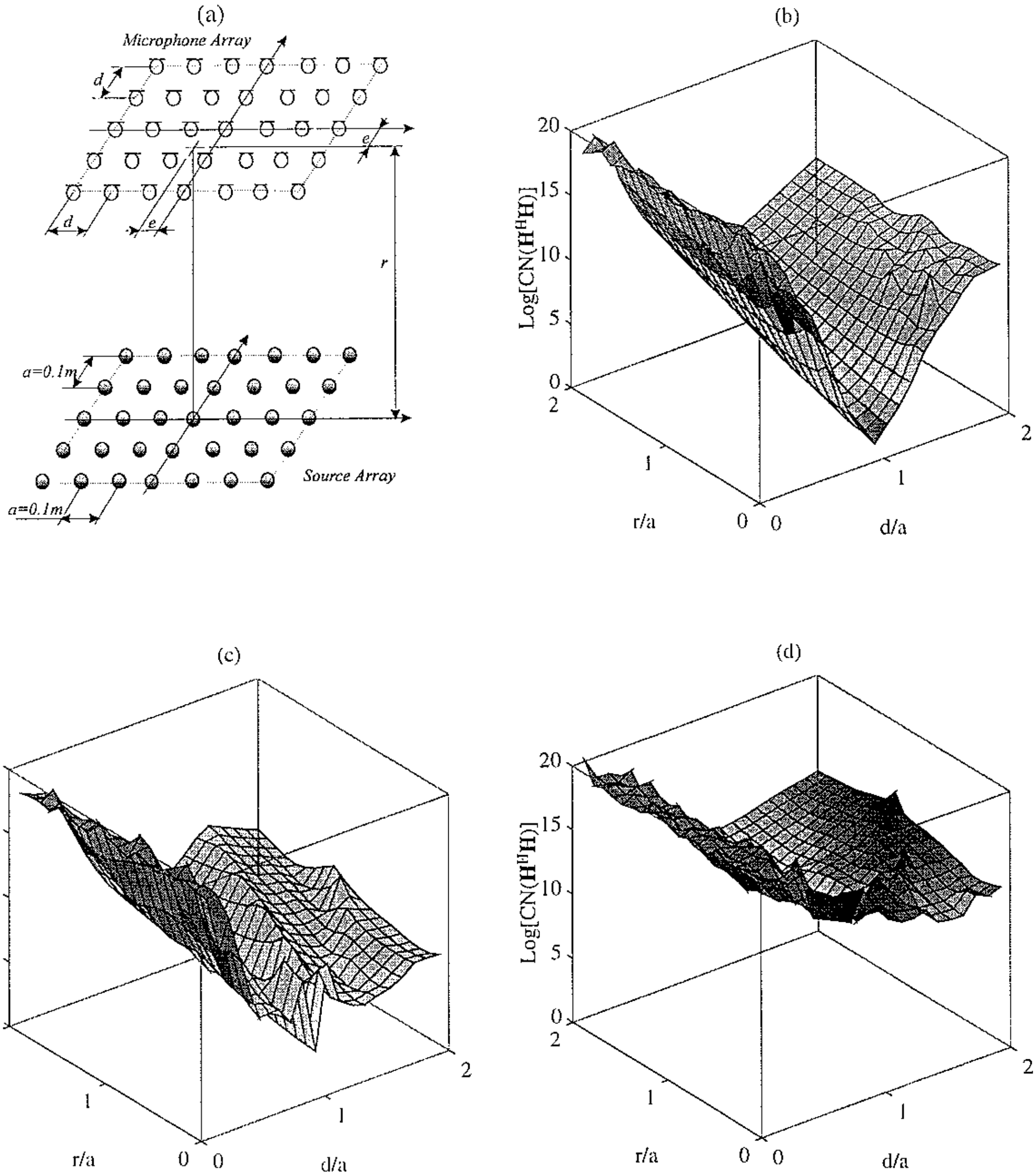



***Fig.10*** *(a) A geometrical arrangement of 35 sources and 35 microphones. Variation of the condition number CN(**H**$^H$**H**) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m.*

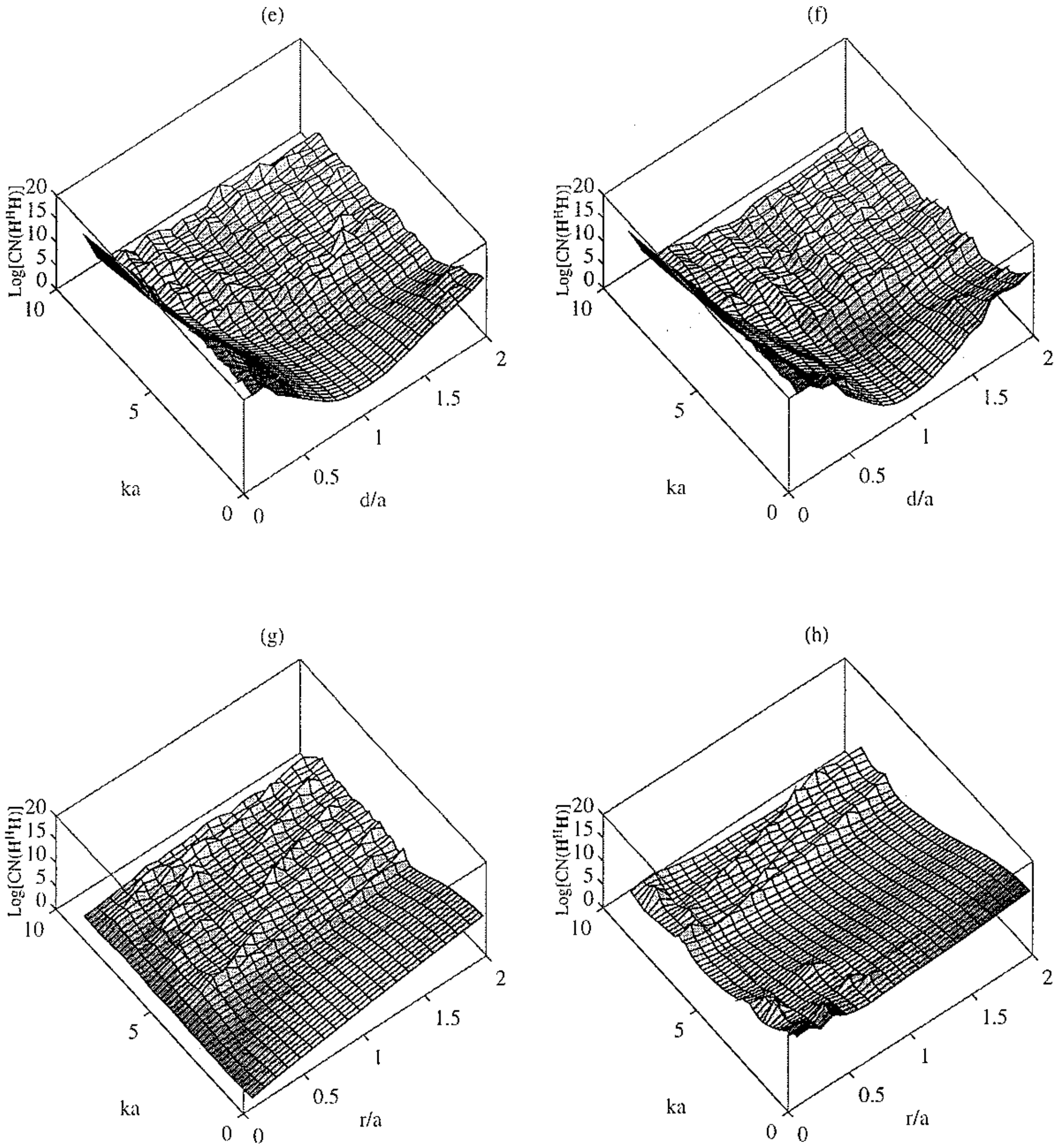



***Fig.10 contd*** *Variation of the condition number* $CN(\boldsymbol{H}^H\boldsymbol{H})$ *with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

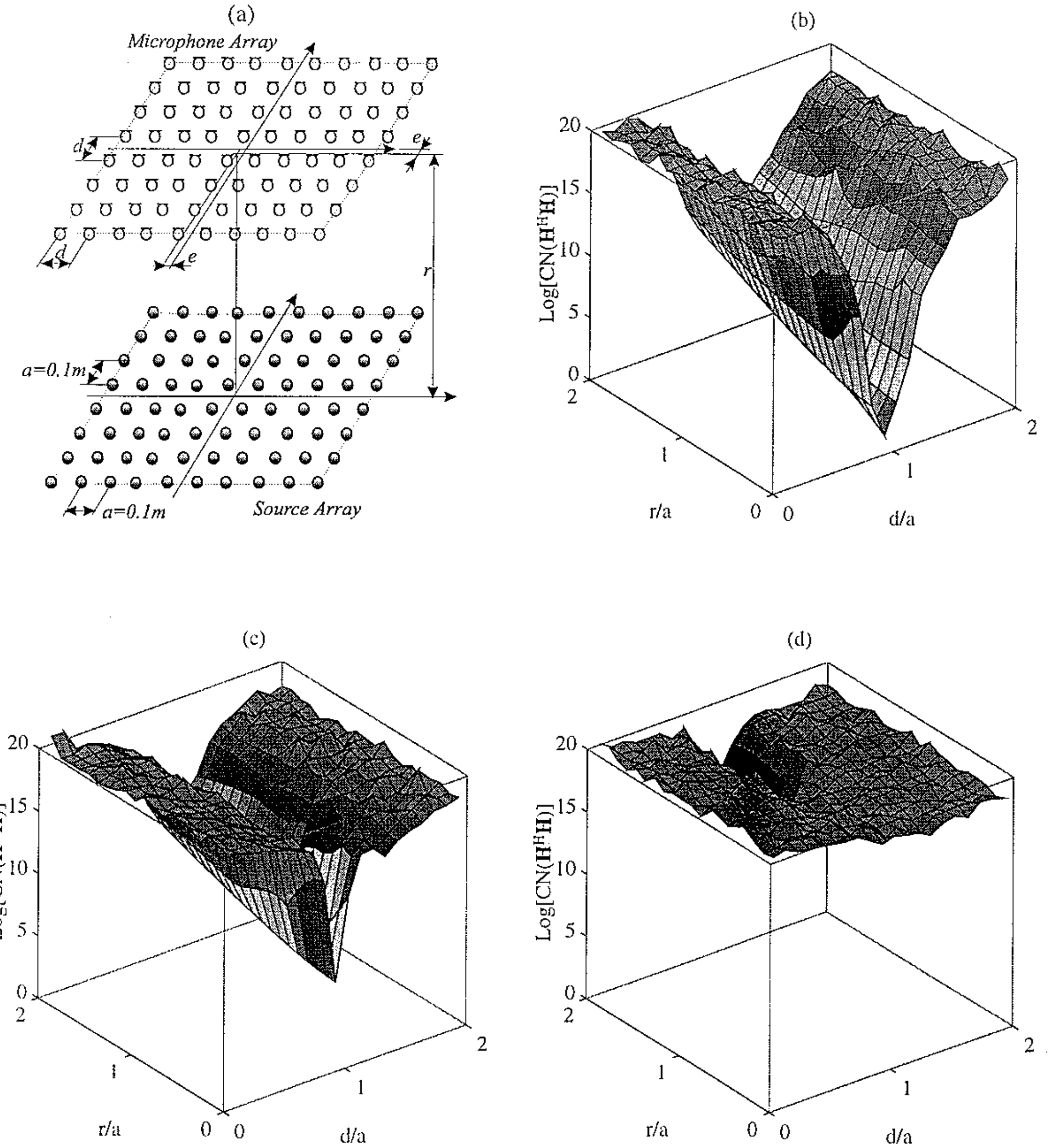



***Fig.11*** *(a) A geometrical arrangement of 100 sources and 100 microphones. Variation of the condition number CN(*$\boldsymbol{H}^H\boldsymbol{H}$*) with r/a and d/a: (b) ka=0.549 (=300Hz), e=0, (c) ka=0.549 (=300Hz), e=-0.05m, (d) ka=0.549 (=300Hz), e=0.08m.*

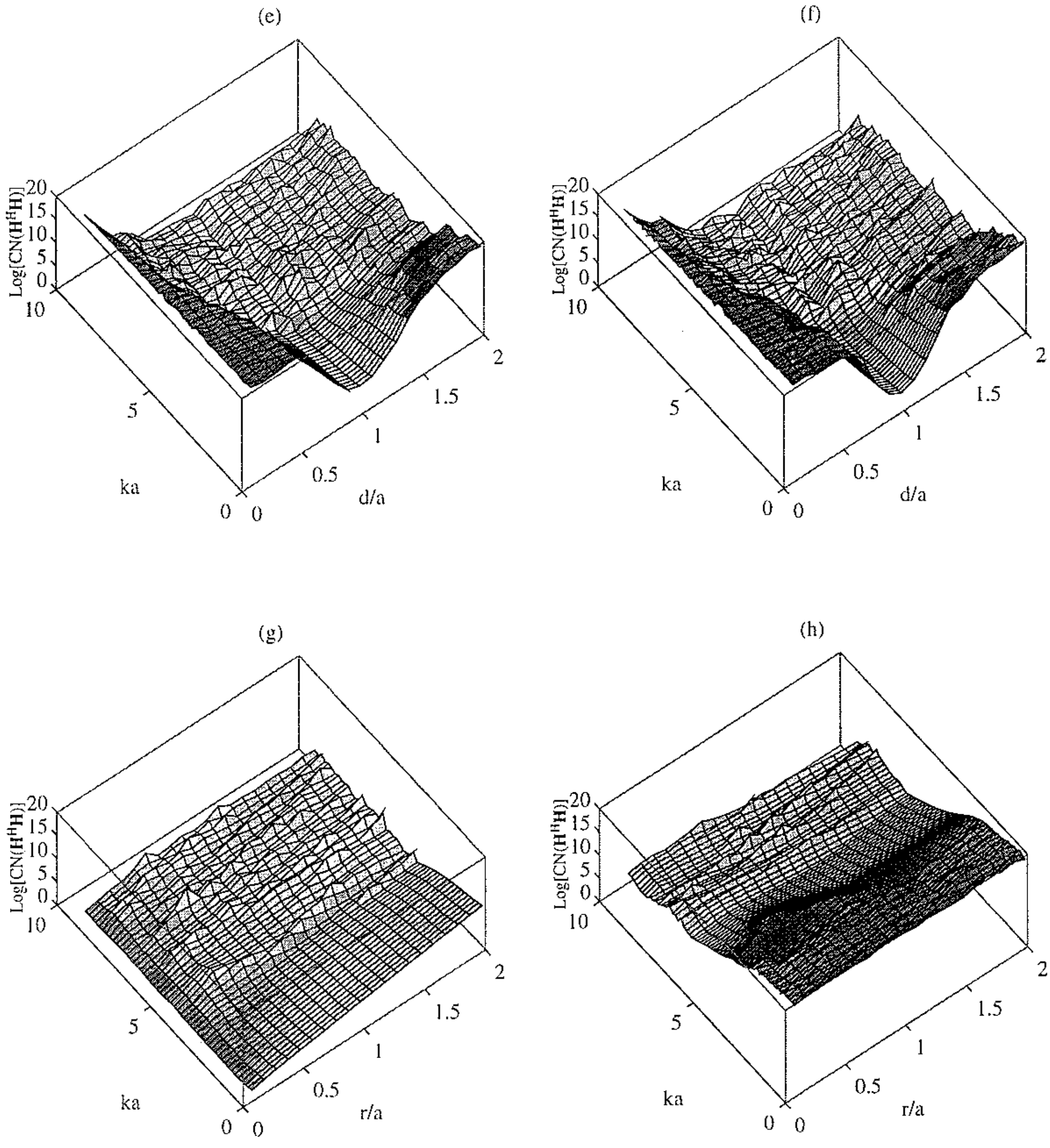



***Fig.11 contd*** *Variation of the condition number CN($\mathbf{H}^H\mathbf{H}$) with d/a and ka: (e) r/a=1, e=0, (f) r/a=0.5, e=0, (g) d/a=1, e=0, (h) d/a=0.5, e=0.*

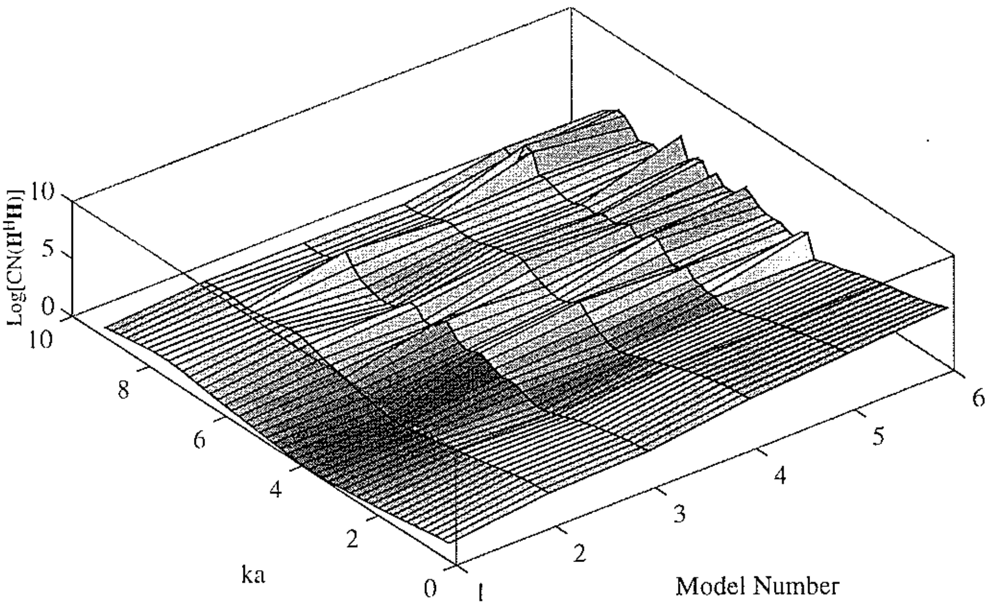
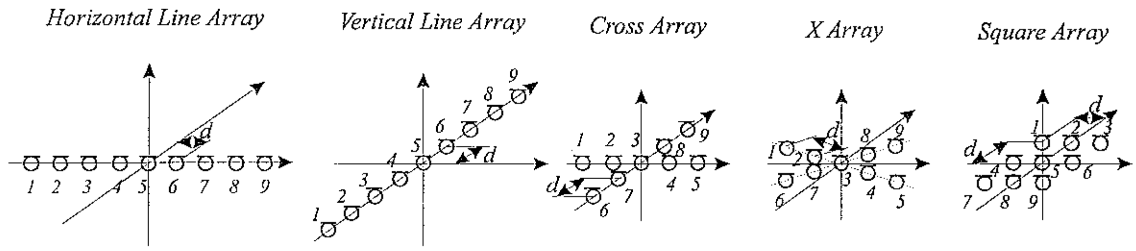
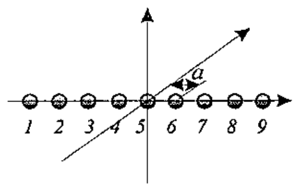



***Fig.12** A comparison of condition numbers of 6 models (see Fig.6 to 11). These are for the case in which the microphone array is symmetrically arranged with respect to the source array, d/a=1 and r/a=1: model number 1 denotes the 2 source and 2 microphone model, model number 2 the 4 source and 4 microphone model, model number 3 the 6 source and 6 microphone model, model number 4 the 16 source and 16 microphone model, model number 5 the 35 source and 35 microphone model, and model number 6 the 100 source and 100 microphone model.*




Horizontal Line Array

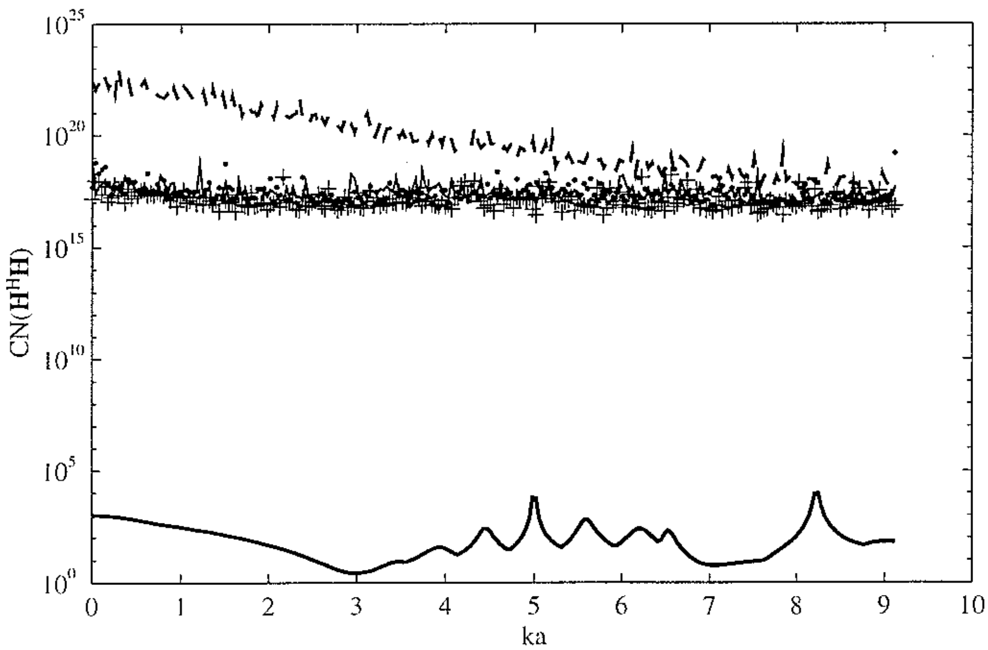



***Fig.13*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the horizontal line source array: thick solid line represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, plus the cross array, point the X array and thin solid line the square array.*

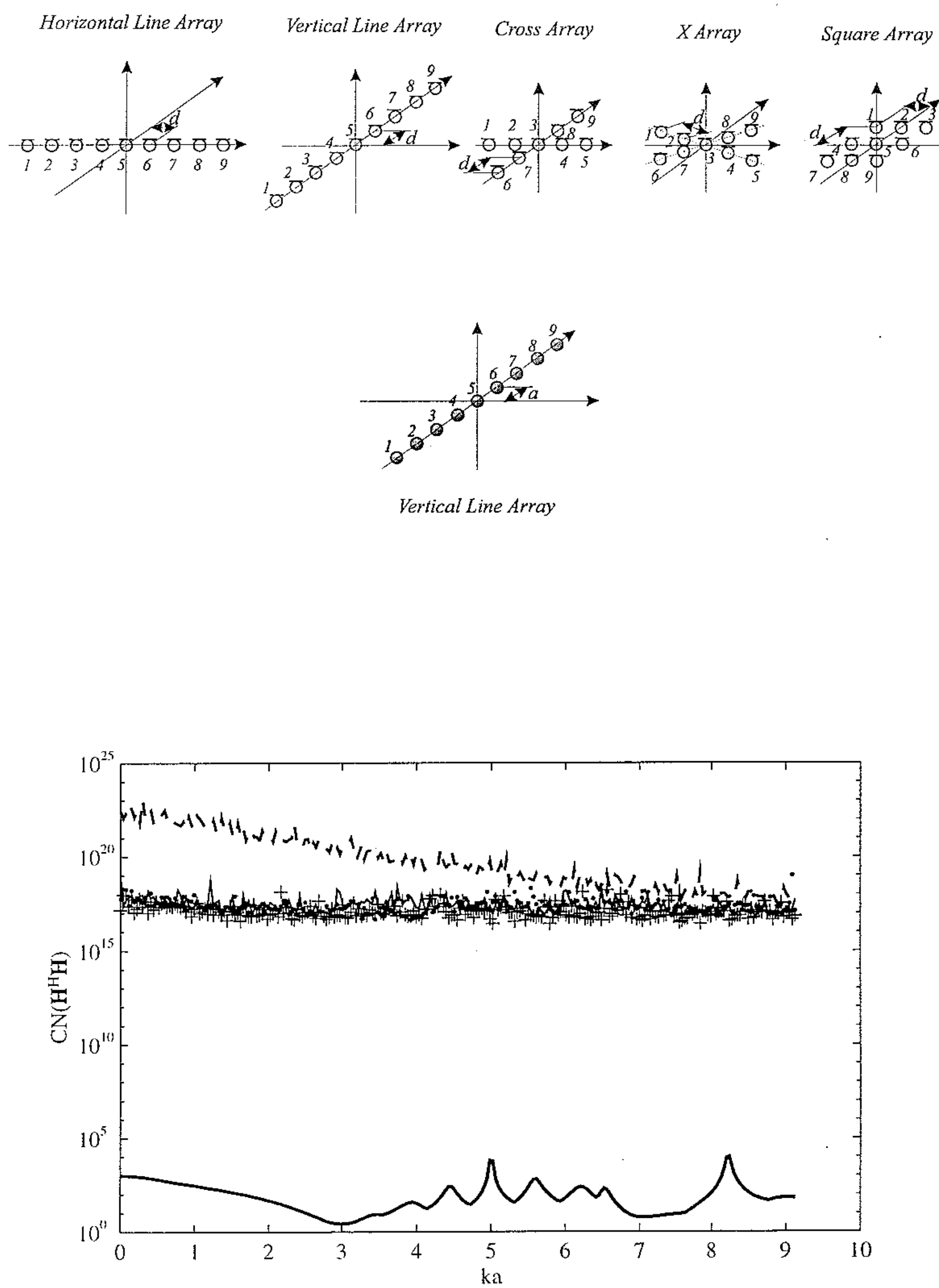
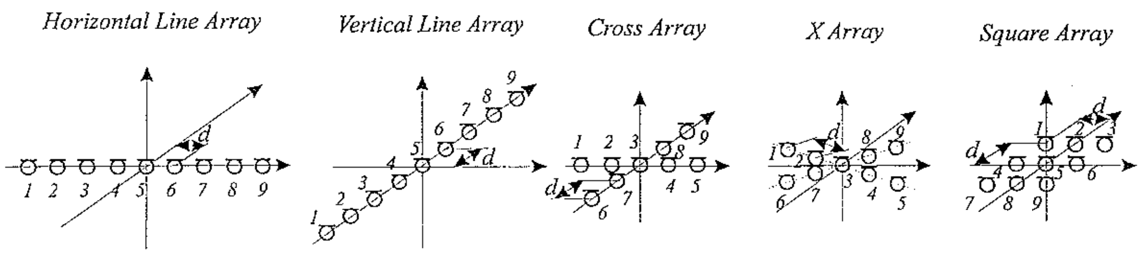
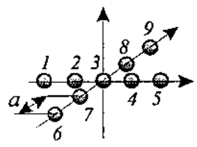



*Fig.14 A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the vertical line source array: broken line represents the condition number obtained by the use of microphones of the horizontal line array, thick solid line the vertical line array, plus the cross array, point the X array and thin solid line the square array.*



Cross Array

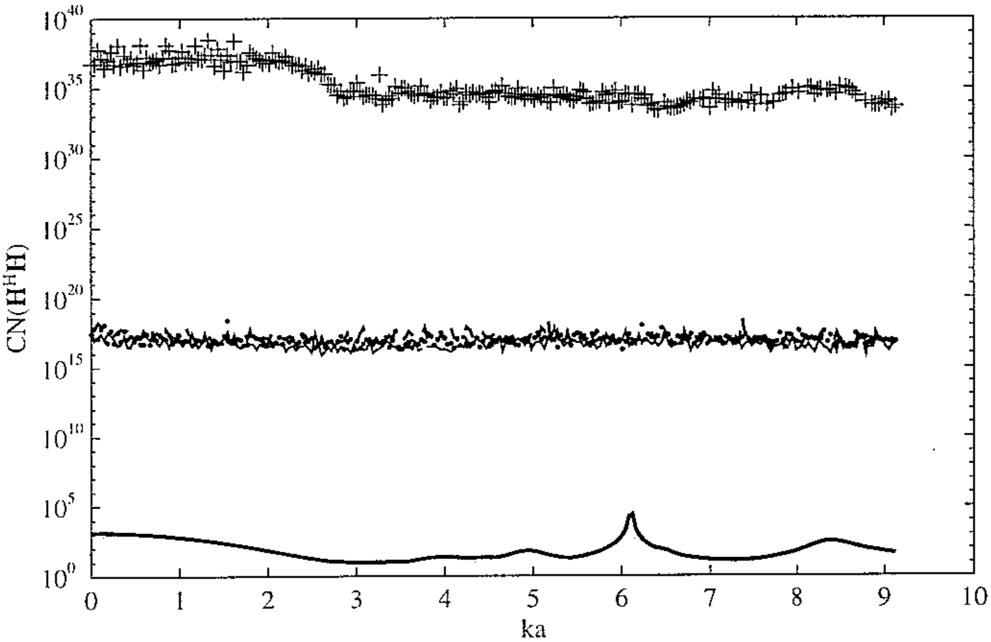
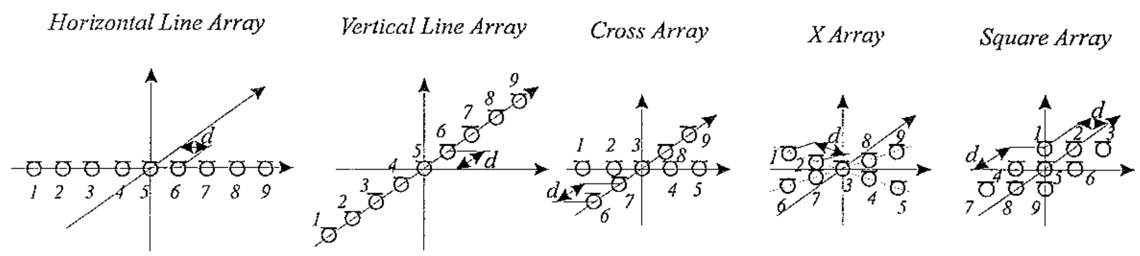
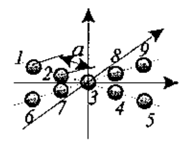



***Fig.15*** *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the cross source array: plus represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, thick solid line the cross array, point the X array and thin solid line the square array.*



X Array

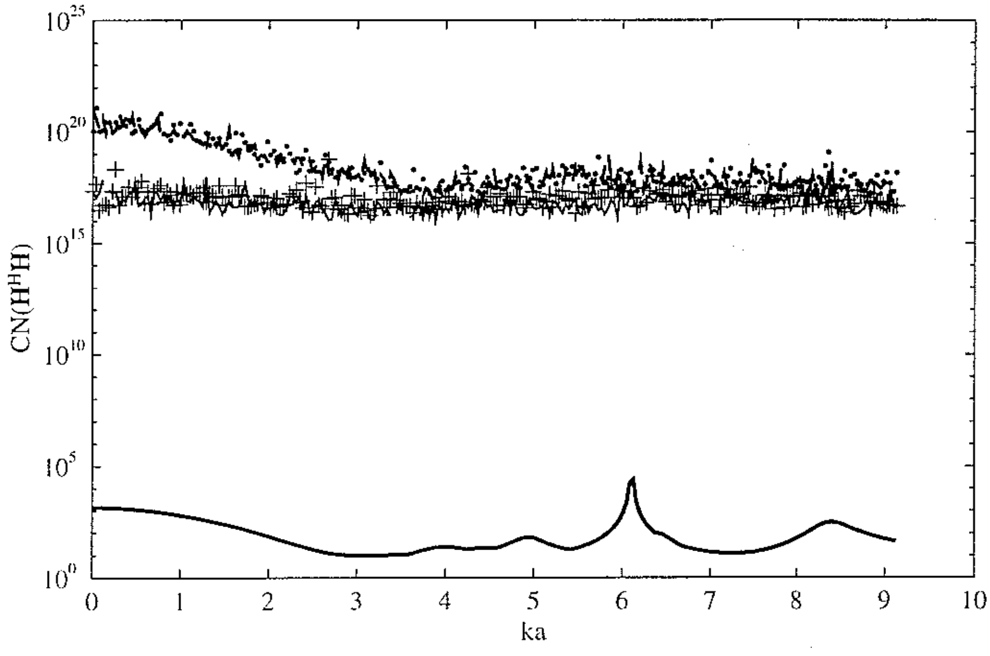
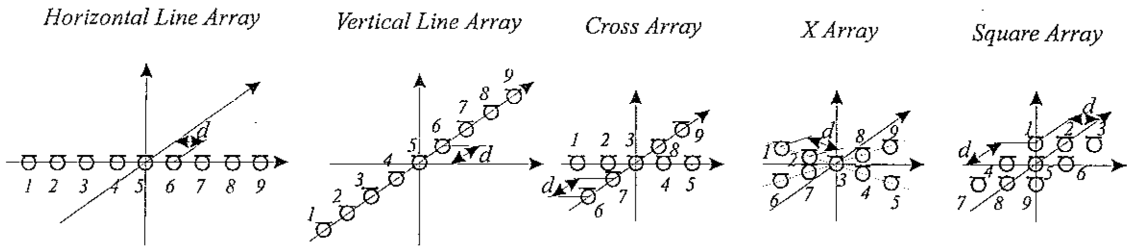
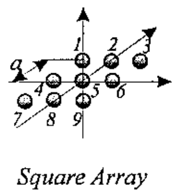



*Fig.16* *A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the x source array: point represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, plus the cross array, thick solid line the X array and thin solid line the square array.*



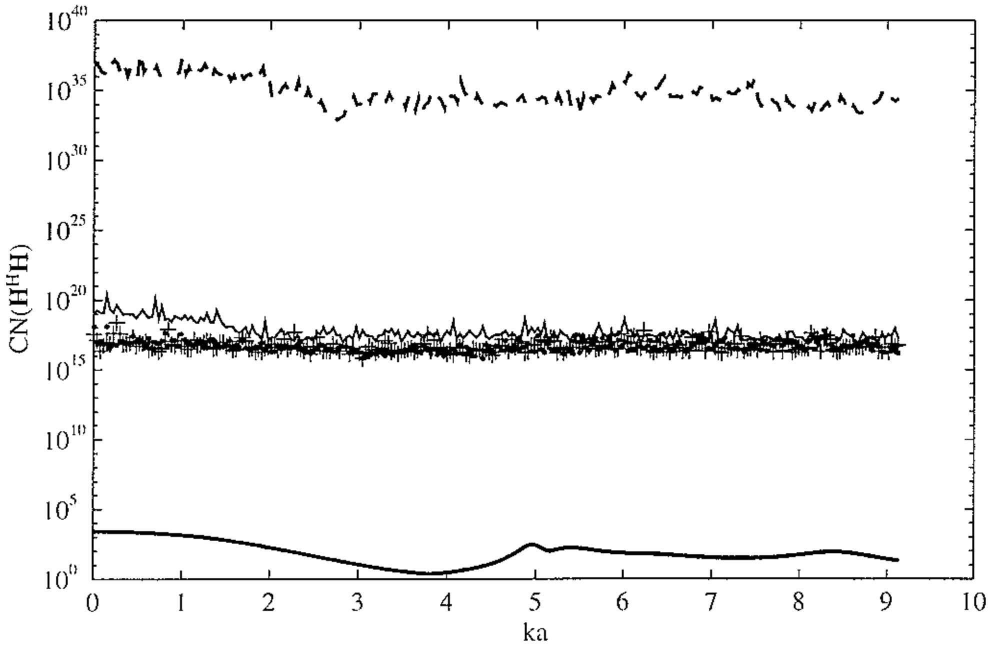




*Fig.17 A comparison of condition numbers obtained from the use of five different types of microphone array in measuring the acoustic field generated from the square source array: thin solid line represents the condition number obtained by the use of microphones of the horizontal line array, broken line the vertical line array, plus the cross array, point the X array and thick solid line the square array.*