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The Penrose singularity theorem in regularity $C^{1,1}$

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Abstract

We extend the validity of the Penrose singularity theorem to spacetime metrics of regularity $C^{1,1}$. The proof is based on regularization techniques, combined with recent results in low regularity causality theory.

Keywords: singularity theorems, low regularity, regularization, causality theory

1. Introduction

In 1965 Roger Penrose published his seminal paper [20] which established the first of the modern singularity theorems. In this paper Penrose introduced the notion of a trapped surface $\mathcal{T}$, which he defined as 'a closed spacelike, two-surface with the property that the two systems of null geodesics which meet $\mathcal{T}$ orthogonally converge locally in future directions at $\mathcal{T}$'. He then showed that if the spacetime M possesses both a closed trapped surface and a non-compact Cauchy surface then provided the local energy density is always positive (so that via Einstein's equations the Ricci tensor satisfies the null convergence condition) the spacetime cannot be future null complete. The Penrose paper established for the first time that the gravitational singularity found in the Schwarzschild solution was not a result of the high degree of symmetry but that provided the gravitational collapse qualitatively resembles the spherically symmetric case then (subject to the above conditions) deviations from spherical symmetry cannot prevent the formation of a gravitational singularity.

Penrose's paper was not only the first to define the notion of a trapped surface but it also introduced the idea of using geodesic incompleteness to give a mathematical characterization of a singular spacetime. The 1965 paper had immediate impact and inspired a series of papers by Hawking, Penrose, Ellis, Geroch and others which led to the development of modern singularity theorems (see the recent review paper [24] for details). Despite the great power of these theorems they follow Penrose in defining singularities in terms of geodesic incompleteness and as a result say little about the nature of the singularity. In particular there is nothing in the original theorems to say that the gravitational forces become unbounded at the singularity[3]. Furthermore the statement and proofs of the various singularity theorems assume that the metric is at least C^2 and Senovilla in [23, section 6.1] highlights the places where this assumption is explicitly used. Thus the singularities predicted by the singularity theorems could in principle be physically innocuous and simply be a result of the differentiability of the metric dropping below C^2. As emphasised by a number of authors (see e.g. [4, 15, 23]) the requirement of C^2-differentiability is significantly stronger than one would want since it fails to hold in a number of physically reasonable situations. In particular it fails across an interface (such as the surface of a star) where there is a jump in the energy density which, via the field equations, corresponds to the metric being of regularity $C^{1,1}$ (also denoted by C^{2-}, the first derivatives of the metric being locally Lipschitz continuous). For more details see e.g. [23, section 6.1]. Furthermore from the point of view of the singularity theorems themselves the natural differentiability class for the metric again is $C^{1,1}$ as this is the minimal condition which ensures existence and uniqueness of geodesics. Since the connection of a $C^{1,1}$-metric is locally Lipschitz, Rademacher's theorem implies that it is differentiable almost everywhere so that the (Ricci) curvature exists almost everywhere and is locally bounded. Any further lowering of the differentiability would result in a loss of uniqueness of causal geodesics[4] (and hence of the worldlines of observers) and generically in unbounded curvature[5], both of which correspond more closely to our physical expectations of a gravitational singularity than in the C^2-case.

The singularity theorems involve an interplay between results in differential geometry and causality theory and it is only recently that the key elements of $C^{1,1}$-causality have been established. In particular it was only in [17, theorem 1.11] and in [12, theorem 2.1] that the exponential map was shown to be a bi-Lipschitz homeomorphism, a key result needed to derive many standard results in causality theory. Building on the regularization results of [6, 13] and combining them with recent advances in causality theory [5, 6, 13, 17] the present authors in [14] gave a detailed proof of the Hawking singularity theorem for $C^{1,1}$-metrics by following the basic strategy outlined in [11, section 8.4]. In the present paper we establish the Penrose singularity theorem for a $C^{1,1}$-metric. To be precise we prove the following result:

Theorem 1.1. *Let (M, g) be an n-dimensional $C^{1,1}$-spacetime. Assume*

(i) For any Lipschitz-continuous local null vector field X, $\mathrm{Ric}(X, X) \geqslant 0$.
(ii) M possesses a non-compact Cauchy-hypersurface S.
(iii) There exists a compact achronal spacelike submanifold $\mathcal{T}$ in M of codimension 2 with past-pointing timelike mean curvature vector field H.

Then M is not future null complete.

[3] See however results on the extendability of incomplete spacetimes under suitable curvature conditions, e.g. [3, 4, 21, 25], which indicate that such spacetimes cannot be maximal unless the curvature blows up.
[4] In fact, uniqueness is lost for metrics of local Hölder regularity class $C^{1,\alpha}$ ($\alpha < 1$), see [10].
[5] While the curvature can be stably defined as a distribution even for metrics of local Sobolev regularity $W^{1,2} \cap L^\infty$ [9] the curvature will in general not be in L^∞ unless the metric is $C^{1,1} = W^{2,\infty}$.

For the definition of a $C^{1,1}$-spacetime, see below.

Remark 1.2

(a) As explained above the Ricci-tensor, Ric, of a $C^{1,1}$-metric is an (almost everywhere defined) L^∞_{loc}-tensor field. Condition (i) in theorem 1.1 is adapted to this situation and reduces to the usual pointwise condition for metrics of regularity C^2. In fact, any null vector can be extended (by parallel transport) to a local null vector field that is C^1 if the metric is C^2 and locally Lipschitz if g is $C^{1,1}$ (cf. also the proof of lemma 2.4 below). The assumption in (i) then means that the L^∞_{loc}-function $\text{Ric}(X, X)$ is non-negative almost everywhere. Since being a null vector field is not an 'open' condition (unlike the case of timelike vector fields as in Hawking's singularity theorem, see [14, remark 1.2]), it will in general not be possible to extend a given null vector to a *smooth* local null vector field.

(b) Concerning condition (iii), our conventions are as follows (see [19]): we define the mean curvature field as $H_p = \frac{1}{n-2}\sum_{i=1}^{n-2} \text{II}(e_i, e_i)$ where $\{e_i\}$ is any orthonormal basis of $T_p\mathcal{T}$ and the second fundamental form is given by $\text{II}(V, W) = \text{nor}\nabla_V W$ where nor denotes the projection orthogonal to $T_p\mathcal{T}$. Also the condition on H in (iii) is equivalent to the convergence $\mathbf{k}(v) := g(H, v)$ being strictly positive for all future pointing null vectors normal to $\mathcal{T}$ and with our conventions is therefore equivalent to the Penrose trapped surface condition.

The key idea behind Penrose's proof of the C^2-theorem is to look at the properties of the boundary of the future of the trapped surface $\mathcal{T}$. The boundary $\partial J^+(\mathcal{T})$ is generated by null geodesics but Raychaudhury's equation and the initial trapped surface condition together with the null convergence condition result in there being a focal point along every geodesic. This fact together with the assumption of null geodesic completeness may be used to show that $\partial J^+(\mathcal{T})$ is compact. On the other hand one may use the existence of the Cauchy surface S together with some basic causality theory to construct a homeomorphism between $\partial J^+(\mathcal{T})$ and S. This is not possible if S is not compact so that there must be a contradiction between the four assumptions.

In our proof of the theorem for the $C^{1,1}$-case we need to further extend the methods of [6, 12–14] and approximate g by a smooth family of Lorentzian metrics $\hat{g}_\varepsilon$ which have strictly wider lightcones than g and which are themselves globally hyperbolic. We then show that by choosing ε sufficiently small the associated Ricci tensor, Ric_ε, violates the null convergence condition by an arbitrarily small amount, which allows us to establish the compactness of $\partial J^+_\varepsilon(\mathcal{T}) = E^+_\varepsilon(\mathcal{T})$ under the assumption of null geodesic completeness. We then use the global hyperbolicity of the $\hat{g}_\varepsilon$ together with the fact that S is a Cauchy surface for g to show that $E^+_\varepsilon(\mathcal{T})$ is homeomorphic to S, which leads to a contradiction with the non-compactness of S. Finally, in theorem 3.3 we show that if M is future null complete and the assumption that S be non-compact is dropped in (ii) then $E^+(\mathcal{T})$ is a compact Cauchy-hypersurface in M. A main difficulty in these proofs, as compared to the case of Hawking's singularity theorem in [14], lies in the fact that curvature conditions on null vectors are less suitable for approximation arguments (cf. lemma 2.4 below) than conditions on timelike vectors ('timelike' being an 'open' condition, as opposed to 'null').

In the remainder of this section we fix key notions to be used throughout this paper, see also [14]. We assume all manifolds to be of class C^∞ and connected (as well as Hausdorff and second countable), and only lower the regularity of the metric. By a $C^{1,1}$-(resp C^k-, $k \in \mathbb{N}_0$) spacetime (M, g), we mean a smooth manifold M of dimension n endowed with a Lorentzian metric g of signature $(-+ \ldots +)$ possessing locally Lipschitz continuous first derivatives (resp of class C^k) and with a time orientation given by a continuous timelike vector field.

If K is a compact set in M we write $K \Subset M$. Following [19], we define the curvature tensor by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ and the Ricci tensor by $R_{ab} = R^c{}_{abc}$. Since both of these conventions differ by a sign from those of [11], the respective definitions of Ricci curvature agree. Note also that our definition of the convergence $\mathbf{k}$ follows [19] and differs by a sign from that used by some other authors.

Our notation for causal structures will basically follow [19], although as in [5, 13] we base all causality notions on locally Lipschitz curves. Any locally Lipschitz curve c is differentiable almost everywhere with locally bounded velocity. We call c timelike, causal, spacelike or null, if $c'(t)$ has the corresponding property almost everywhere. Based on these notions we define the relative chronological future $I^+(A, U)$ and causal future $J^+(A, U)$ of a set $A \subseteq M$ relative to $U \subseteq M$ literally as in the smooth case (see [13, definition 3.1], [5, 2.4]). The future horismos of A is defined as $E^+(A, U) = J^+(A, U)\setminus I^+(A, U)$. As was shown in [17, theorem 7], [13, corollary 3.1], our definitions coincide with the ones based on smooth curves.

A Cauchy hypersurface is a subset S of M which every inextendible timelike curve intersects exactly once, see [19, definition 14.28]. In the smooth case, for spacelike hypersurfaces this definition of a Cauchy hypersurface is equivalent to the one in [11], and this remains true in the $C^{1,1}$-case [14, proposition A.31]. A $C^{1,1}$-spacetime (M, g) is called globally hyperbolic if it is strongly causal and any causal diamond $J(p, q) = J^+(p) \cap J^-(q)$ is compact. It follows from [14, lemma A.20, theorem A.22] that M is globally hyperbolic if it possesses a Cauchy-hypersurface.

We will write $\exp_p$ for the exponential map of the metric g at p, and $\exp_p^{g_\varepsilon}$ for the one corresponding to the metric g_ε. For a semi-Riemannian submanifold S of M we denote by $(N(S), \pi)$ its normal bundle. By [17, theorem 13], $N(S)$ is a Lipschitz bundle.

2. Approximation results

In this section we extend the approximation results of [14] to deal with the fact that we need to be able to approximate a globally hyperbolic $C^{1,1}$-metric by a smooth family of globally hyperbolic metrics. In addition we require a more delicate estimate for the Ricci curvature than that given in [14, lemma 3.2] due to the fact that the Penrose singularity theorem makes use of the null convergence condition for the Ricci tensor rather than the timelike convergence condition used in the Hawking theorem.

We start by recalling from [18, section 3.8.2], [6, section 1.2] that for two Lorentzian metrics g_1, g_2, we say that g_2 has *strictly wider light cones* than g_1, denoted by

$$g_1 \prec g_2, \text{ if for any tangent vector } X \neq 0,\ g_1(X, X) \leqslant 0 \text{ implies that } g_2(X, X) < 0. \tag{1}$$

Thus any g_1-causal vector is g_2-timelike. The key result now is [6, proposition 1.2], which we give here in the slightly refined version of [13, proposition 2.5]. Note that the smoothness of the approximating net with respect to ε and p is vital in proposition 2.3 below.

Proposition 2.1. *Let (M, g) be a C^0-spacetime and let h be some smooth background Riemannian metric on M. Then for any $\varepsilon > 0$, there exist smooth Lorentzian metrics $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ on M such that $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ and $d_h(\check{g}_\varepsilon, g) + d_h(\hat{g}_\varepsilon, g) < \varepsilon$, where*

$$d_h(g_1, g_2) := \sup_{p\in M, 0 \neq X, Y\in T_pM} \frac{\left|g_1(X, Y) - g_2(X, Y)\right|}{\|X\|_h \|Y\|_h}. \tag{2}$$

Moreover, $\hat{g}_\varepsilon(p)$ and $\check{g}_\varepsilon(p)$ depend smoothly on $(\varepsilon, p) \in \mathbb{R}^+ \times M$, and if $g \in C^{1,1}$ then letting g_ε be either $\check{g}_\varepsilon$ or $\hat{g}_\varepsilon$, we additionally have

(i) g_ε converges to g in the C^1-topology as $\varepsilon \to 0$, and
(ii) the second derivatives of g_ε are bounded, uniformly in ε, on compact sets.

Remark 2.2. In several places below we will need approximations as in proposition 2.1, but with additional properties. In particular, we will require that for globally hyperbolic metrics there exist approximations with strictly wider lightcones that are themselves globally hyperbolic. Extending methods of [8], it was shown in [1] that global hyperbolicity is stable in the interval topology. Consequently, if g is a smooth, globally hyperbolic Lorentzian metric then there exists some smooth globally hyperbolic metric $g' \succ g$. In [7, theorem 1.2], the stability of global hyperbolicity was established for continuous cone structures. It has to be noted, however, that the definition of global hyperbolicity in [7] requires stable causality (in addition to the compactness of the causal diamonds), which is stronger than the usual assumption of strong causality, so this result is not directly applicable in our setting. In [22] it is proved directly that if g is a continuous metric that is non-totally imprisoning and has the property that all causal diamonds are compact (as is the case for any globally hyperbolic $C^{1,1}$-metric by the proof of [19, lemma 14.13]) then there exists a smooth metric $g' \succ g$ that has the same properties, hence in particular is causal with compact causal diamonds and thereby globally hyperbolic by [2].

Proposition 2.3. *Let (M, g) be a C^0-spacetime with a smooth background Riemannian metric h.*

(i) Let $\check{g}_\varepsilon$, $\hat{g}_\varepsilon$ as in proposition 2.1. Then for any compact subset $K \Subset M$ there exists a sequence $\varepsilon_j \searrow 0$ such that $\hat{g}_{\varepsilon_{j+1}} \prec \hat{g}_{\varepsilon_j}$ on K (resp. $\check{g}_{\varepsilon_j} \prec \check{g}_{\varepsilon_{j+1}}$ on K) for all $j \in \mathbb{N}_0$.
(ii) If g′ is a continuous Lorentzian metric with $g \prec g'$ (resp. $g' \prec g$) then $\hat{g}_\varepsilon$ (resp. $\check{g}_\varepsilon$) as in proposition 2.1 can be chosen such that $g \prec \hat{g}_\varepsilon \prec g'$ (resp. $g' \prec \check{g}_\varepsilon \prec g$) for all ε.
(iii) There exist sequences of smooth Lorentzian metrics $\check{g}_j \prec g \prec \hat{g}_j$ ($j \in \mathbb{N}$) such that $d_h(\check{g}_j, g) + d_h(\hat{g}_j, g) < 1/j$ and $\check{g}_j \prec \check{g}_{j+1}$ as well as $\hat{g}_{j+1} \prec \hat{g}_j$ for all $j \in \mathbb{N}$.
(iv) If g is $C^{1,1}$ and globally hyperbolic then the $\hat{g}_\varepsilon$ from proposition 2.1, as well as the $\hat{g}_j$ from (iii) can be chosen globally hyperbolic as well.
(v) If g is $C^{1,1}$ then the regularizations constructed in (i)–(iv) can in addition be chosen such that they converge to g in the C^1-topology and such that their second derivatives are bounded, uniformly in ε (resp. j) on compact sets.

Proof

(i) We follow the argument of [22, lemma 1.5]: pick any $\varepsilon_0 > 0$. Since $g \prec \hat{g}_{\varepsilon_0}$, there exists some $\delta > 0$ such that $\{X \in TM|_K \,|\, \|X\|_h = 1, g(X, X) < \delta\}$ is contained in $\{X \in TM \,|\, \hat{g}_{\varepsilon_0}(X, X) < 0\}$. In fact, otherwise there would exist a convergent sequence $X_k \to X$ in $TM|_K$ with $\|X_k\|_h = 1$, $g(X_k, X_k) < 1/k$, and $\hat{g}_{\varepsilon_0}(X_k, X_k) \geqslant 0$. But then $g(X, X) \leqslant 0$ and $\hat{g}_{\varepsilon_0}(X, X) \geqslant 0$, contradicting $g \prec \hat{g}_{\varepsilon_0}$. Next, we choose $\varepsilon_1 < \min(\varepsilon_0, \delta)$, so $d_h(g, \hat{g}_{\varepsilon_1}) < \delta$. Then if $X \in TM|_K$, $\|X\|_h = 1$ and $\hat{g}_{\varepsilon_1}(X, X) \leqslant 0$, we obtain $g(X, X) < \hat{g}_{\varepsilon_1}(X, X) + \delta \leqslant \delta$, so $\hat{g}_{\varepsilon_0}(X, X) < 0$, i.e., $\hat{g}_{\varepsilon_1} \prec \hat{g}_{\varepsilon_0}$ on K. The claim therefore follows by induction. Analogously one can construct the sequence $\check{g}_{\varepsilon_j}$.

(ii) The proof of (i) shows that for any $K \Subset M$ there exists some ε_K such that for all $\varepsilon < \varepsilon_K$ we have $g \prec \hat{g}_\varepsilon \prec g'$ on K, and $d_h(g|_K, \hat{g}_\varepsilon|_K) < \varepsilon$. Clearly all these properties are stable under shrinking K or ε_K. Therefore, [13, lemma 2.4] shows that there exists a smooth map $(\varepsilon, p) \mapsto \tilde{g}_\varepsilon(p)$ such that for each fixed ε, $\tilde{g}_\varepsilon$ is a Lorentzian metric on M with $g \prec \tilde{g}_\varepsilon \prec g'$ and such that $d_h(g, \tilde{g}_\varepsilon) < \varepsilon$ on M. Again the proof for $\check{g}_\varepsilon$ is analogous.

(iii) This follows from (ii) by induction.

(iv) By remark 2.2 there exists a smooth globally hyperbolic metric $g' \succ g$. Constructing $\hat{g}_\varepsilon$ resp. $\hat{g}_j$ as in (ii) resp. (iii) then automatically gives globally hyperbolic metrics (see [1, section II]).

(v) By [13, lemma 2.4], in the construction given in (ii) above, for any $K \Subset M$, $\tilde{g}_\varepsilon$ coincides with the original $\hat{g}_\varepsilon$ on K for ε sufficiently small. Thus by (i) and (ii) from proposition 2.1 the $\tilde{g}_\varepsilon$ (i.e., the new $\hat{g}_\varepsilon$) have the desired properties, and analogously for the new $\check{g}_\varepsilon$. Concerning (iii), fix any atlas $\mathcal{A}$ of M and an exhaustive sequence K_n of compact sets in M with $K_n \subseteq K_{n+1}^\circ$ for all n. Then in the inductive construction of the $\hat{g}_j$ we may additionally require that the C^1-distance of g and $\hat{g}_j$ on K_j (as measured with respect to the C^1-seminorms induced by the charts in $\mathcal{A}$) be less than $1/j$. Moreover, for any K_j there is some constant C_j bounding the second derivatives of the $\hat{g}_\varepsilon$ from (ii) (again w.r.t. the charts in $\mathcal{A}$) for ε smaller than some ε_j. It is therefore also possible to have the second derivatives of $\hat{g}_k$ bounded by C_j on K_j for all $k \geqslant j$. Altogether, this gives the claimed properties for the sequence $(\hat{g}_j)$, and analogously for $(\check{g}_j)$.

Lemma 2.4. *Let (M, g) be a $C^{1,1}$-spacetime and let h, $\tilde{h}$ be Riemannian metrics on M and TM, respectively. Suppose that $\mathrm{Ric}(Y, Y) \geqslant 0$ for every Lipschitz-continuous g-null local vector field Y. Let $K \Subset M$ and let C, $\delta > 0$. Then there exist $\eta > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we have: If $p \in K$ and $X \in T_pM$ is such that $\|X\|_h \leqslant C$ and there exists a g-null vector $Y_0 \in TM|_K$ with $d_{\tilde{h}}(X, Y_0) \leqslant \eta$ and $\|Y_0\|_h \leqslant C$ then $\mathrm{Ric}_\varepsilon(X, X) > -\delta$. Here Ric_ε is the Ricci-tensor corresponding to a metric $\hat{g}_\varepsilon$ as in proposition 2.1.*

Proof. We first note that as in the proof of [14, lemma 3.2] it follows that we may assume that $M = \mathbb{R}^n$, $\|.\|_h = \|.\|$ is the Euclidean norm and we may replace $\hat{g}_\varepsilon$ by $g_\varepsilon := g * \rho_\varepsilon$ (component-wise convolution), and prove the claim for Ric_ε calculated from g_ε. For the distance on $TM \cong \mathbb{R}^{2n}$ we may then simply use $d(X_p, Y_q) := \|p - q\| + \|X - Y\|$ (which is equivalent to the distance function induced by the natural product metric on $T\mathbb{R}^n$).

Denote by E the map $v \mapsto (\pi(v), \exp(v))$, defined on an open neighbourhood of the zero section in $T\mathbb{R}^n$. Let L be a compact neighbourhood of K. Then E is a homeomorphism from some open neighbourhood $\mathcal{U}$ of $L \times \{0\}$ in $T\mathbb{R}^n$ onto an open neighbourhood $\mathcal{V}$ of $\{(q, q)|q \in L\}$ in $\mathbb{R}^n \times \mathbb{R}^n$ and there exists some $r > 0$ such that for any $q \in L$ the set $U_r(q) := \exp_q(B_r(0))$ is a totally normal neighbourhood of q and $\bigcup_{q \in L}(U_r(q) \times U_r(q)) \subseteq \mathcal{V}$ (see the proof of [12, theorem 4.1]). We may assume that $\mathcal{U}$ is of the form $\{(q, v)|q \in L', \|v\| < a\}$ for some open $L' \supseteq L$ and some $a > 0$ and that $\overline{\mathcal{U}}$ is contained in the domain of E. It follows from standard ODE theory (see [12, section 2]) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q^{g_\varepsilon}(tv)\right) \to \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q(tv)\right) \quad (\varepsilon \to 0), \tag{3}$$

uniformly in $v \in \mathbb{R}^n$ with $\|v\| \leqslant 1$, $t \in [0, a]$, and $q \in L$. Hence for ε small and such v, t and q and we have

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q(tv) \right) \right\| \leqslant \left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q^{g_\varepsilon}(tv) \right) \right\| + 1. \tag{4}$$

Furthermore, for ε small the operator norms of $T_v \exp_q^{g_\varepsilon}$ are bounded, uniformly in ε, $v \in \mathbb{R}^n$ with $\| v \| \leqslant a$ and $q \in L$ by some constant $\tilde{C}_1$: this follows from (7) in [12], noting that we may assume that a as above is so small that this estimate is satisfied uniformly in ε, $\| v \| \leqslant a$, and $q \in L$. Consequently, for ε small, $q \in L$, $t \in [0, a]$ and $\| v \| \leqslant 1$ we have

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q^{g_\varepsilon}(tv) \right) \right\| = \left\| T_{tv} \exp_q^{g_\varepsilon}(v) \right\| \leqslant \tilde{C}_1. \tag{5}$$

It follows from (4), (5) that there exists some $\varepsilon' > 0$ such that for any $\varepsilon \in (0, \varepsilon')$, any $q \in L$, any $v \in \mathbb{R}^n$ with $\| v \| \leqslant a$ and any $t \in [0, 1]$ we have

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q(tv) \right) \right\| = \left\| \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t\|v\|} \left(\exp_q\left(s\frac{v}{\| v \|} \right) \right) \right\| \| v \| \leqslant \left(\tilde{C}_1 + 1 \right) \| v \|. \tag{6}$$

Set

$$C_1 := \left(\tilde{C}_1 + 1 \right) \sup_{p \in L} \| \Gamma(p) \|, \qquad C_2 := \sup_{p \in L} \| \mathrm{Ric}(p) \|. \tag{7}$$

Given any $C > 0$ and $\delta > 0$, pick $\eta_1 \in (0, 1)$ so small that $6C_2 C\eta_1 < \delta/2$ and let

$$\tilde{r} := \sup\left\{ \| E^{-1}(p, p') \| | p, p' \in U_r(q), q \in L \right\}. \tag{8}$$

Then $\tilde{r} < a$ and by compactness we may suppose that r from above is so small that $\mathrm{e}^{C_1 \tilde{r}} < 2$, $2C_1 C\tilde{r} < \eta_1$, and $U_r(q) \subseteq L$ for all $q \in K$.

We may then cover K by finitely many such sets $U_r(q_1), \ldots, U_r(q_N)$. Then $K = \bigcup_{j=1}^N K_j$ with $K_j \Subset U_j := U_r(q_j)$ for each j. Set $s := \min_{1 \leqslant j \leqslant N} \mathrm{dist}(K_j, \partial U_j)$ and let $0 < \eta < \min(\eta_1, s/2)$.

Next, let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a standard mollifier, i.e., $\rho \geqslant 0$, $\mathrm{supp}(\rho) \subseteq B_1(0)$ and $\int \rho(x)\, \mathrm{d}x = 1$. From (3) in [14] we know that

$$R_{\varepsilon ik} - R_{ik} * \rho_\varepsilon \to 0 \text{ uniformly on compact sets.} \tag{9}$$

Hence there exists some $\varepsilon'' \in (0, \varepsilon')$ such that for all $0 < \varepsilon < \varepsilon''$ we have

$$\sup_{x \in K} \left| R_{\varepsilon ik}(x) - R_{ik} * \rho_\varepsilon(x) \right| < \frac{\delta}{2C^2}. \tag{10}$$

To conclude our preparations, we set $\varepsilon_0 := \min(\varepsilon'', s/2)$ and consider any $\varepsilon < \varepsilon_0$.

Now let $p \in K$ and $X \in \mathbb{R}^n$ such that $\| X \| \leqslant C$ and suppose there exists some $g(q)$-null vector $Y_0 \in \mathbb{R}^n$ with $q \in K$,

$$d\left(X_p, \left(Y_0 \right)_q \right) = \| p - q \| + \| X - Y_0 \| \leqslant \eta, \tag{11}$$

and $\| Y_0 \| \leqslant C$. Then for some $j \in \{1, \ldots, N\}$ we have $p \in K_j$, and since $\eta < s/2$ we also have $q \in U_j$.

Since $g(q)(Y_0, Y_0) = 0$, we may extend Y_0 to a Lipschitz-continuous null vector field, denoted by Y, on all of U_j by parallelly transporting it radially outward from q. Let $p' \in U_j$ be any point different from q and let $v := \overrightarrow{qp'} = E^{-1}(q, p')$. Then $Y(p') = Z(1)$, where $Z(t) = Y(\exp_q(tv))$ for all $t \in [0, 1]$ and Z satisfies the linear ODE

$$\frac{\mathrm{d}Z^k}{\mathrm{d}t} = -\Gamma^k_{ij}\Big(\exp_q(tv)\Big)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp^i_q(tv)\Big)Z^j(t) \tag{12}$$

with initial condition $Z(0) = Y(q) = Y_0$. By Gronwall's inequality it follows that

$$\|Z(t)\| \leqslant \|Y_0\|\, \mathrm{e}^{t\|\Gamma\|_{L^\infty(U_j)}\sup_{t\in[0,1]}\|\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp_q(tv)\right)\|} \quad (t \in [0, 1]). \tag{13}$$

Therefore, (6)–(8) give

$$\|Y(p')\| \leqslant \|Y_0\|\, \mathrm{e}^{C_1\tilde{r}} < 2\,\|Y_0\| \tag{14}$$

for all $p' \in U_j$. Moreover, for all $t \in [0, 1]$ we have

$$\|Z(t) - Y_0\| \leqslant t \cdot \sup_{t\in[0,1]}\left\|\frac{\mathrm{d}Z^k}{\mathrm{d}t}\right\|, \tag{15}$$

which, due to $\|Y_0\| \leqslant C$, by (12)–(14) leads to

$$\|Y(p') - Y_0\| \leqslant \sup_{t\in[0,1]}\left\|\frac{\mathrm{d}Z^k}{\mathrm{d}t}\right\| \leqslant C_1C\tilde{r}\mathrm{e}^{C_1\tilde{r}} < 2C_1C\tilde{r} < \eta_1. \tag{16}$$

We also extend X to a constant vector field on U_j, again denoted by X. Then $\|Y\| < 2C$ by (14), and

$$\|X - Y\| \leqslant \|X - Y_0\| + \|Y_0 - Y\| < 2\eta_1 \tag{17}$$

on U_j. It follows that, on U_j, we have the following inequality

$$\begin{aligned}|\mathrm{Ric}(X, X) - \mathrm{Ric}(Y, Y)| &= |\mathrm{Ric}(X - Y, X) + \mathrm{Ric}(X - Y, Y)| \\ &\leqslant C_2\,\|X - Y\|\|X\| + C_2\,\|X - Y\|\|Y\| \leqslant 6C_2C\eta_1 < \delta/2.\end{aligned} \tag{18}$$

Since $\mathrm{Ric}(Y, Y) \geqslant 0$, we conclude that $\mathrm{Ric}(X, X) > -\delta/2$ on U_j.

Set

$$\tilde{R}_{ik}(x) := \begin{cases} R_{ik}(x) & \text{for } x \in B_{s/2}(p), \\ 0 & \text{otherwise.}\end{cases} \tag{19}$$

By our assumption and the fact that $\rho \geqslant 0$ we then have $(\tilde{R}_{ik}X^iX^k) * \rho_\varepsilon \geqslant -\delta/2$ on $\mathbb{R}^n$. Furthermore, since $\varepsilon < s/2$ it follows that $(R_{ik} * \rho_\varepsilon)(p) = (\tilde{R}_{ik} * \rho_\varepsilon)(p)$, so (10) gives:

$$\begin{aligned}\left|R_{\varepsilon ik}(p)X^iX^k - \left(\left(\tilde{R}_{ik}X^iX^k\right) * \rho_\varepsilon\right)(p)\right| &= \left|\left(R_{\varepsilon ik}(p) - \left(R_{ik} * \rho_\varepsilon\right)(p)\right)X^iX^k\right| \\ &\leqslant C^2\sup_{x\in K}\left|R_{\varepsilon ik}(x) - R_{ik} * \rho_\varepsilon(x)\right| < \delta/2.\end{aligned} \tag{20}$$

It follows that $R_{\varepsilon ik}(p)X^iX^k > -\delta$, as claimed. □

3. Proof of the main result

Based on the approximation results of the previous section we are now ready to prove theorem 1.1. As a final preliminary result we need:

Proposition 3.1. *Let (M, g) be a $C^{1,1}$-spacetime that is future null complete and suppose that assumptions (i) and (iii) of theorem 1.1 are satisfied. Moreover, suppose that $\hat{g}_\varepsilon$ $(\varepsilon > 0)$*

is a net of smooth Lorentzian metrics on M as in proposition 2.1. Then there exists some $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *the future horismos* $E^+_\varepsilon(\mathcal{T})$ *of* $\mathcal{T}$ *with respect to the metric* $\hat{g}_\varepsilon$ *is relatively compact.*

Proof. Let h be a smooth background Riemannian metric and define

$$\tilde{T} := \{v \in N(\mathcal{T}) | v \text{ future-directed } g\text{-null and } h(v, v) = 1\},$$

where $N(\mathcal{T})$ is the g-normal bundle of $\mathcal{T}$, and analogously

$$\tilde{T}_\varepsilon := \left\{ v \in N_\varepsilon(\mathcal{T}) | v \text{ future-directed } \hat{g}_\varepsilon\text{-null and } h(v, v) = 1 \right\},$$

where $N_\varepsilon(\mathcal{T})$ is the $\hat{g}_\varepsilon$-normal bundle of $\mathcal{T}$. Moreover, we set (cf. remark 1.2(b))

$$m := (n-2)\min_{v\in\tilde{T}} \mathbf{k}(v) = (n-2)\min_{v\in\tilde{T}} g(\pi(v))(H, v) > 0$$

and pick $b > 0$ such that $(n-2)/b < m$. Denote by H_ε the mean curvature vector field of $\mathcal{T}$ with respect to $\hat{g}_\varepsilon$, and similarly for $\mathbf{k}_\varepsilon$. Then $H_\varepsilon \to H$ uniformly on $\mathcal{T}$ and we claim that for ε sufficiently small and all $v \in \tilde{T}_\varepsilon$ we have $\mathbf{k}_\varepsilon(v) > 1/b$. To see this, suppose to the contrary that there exist a sequence $\varepsilon_k \searrow 0$ and vectors $v_k \in \tilde{T}_{\varepsilon_k}$ such that $\hat{g}_{\varepsilon_k}(\pi(v_k))(H_{\varepsilon_k}, v_k) \leqslant 1/b$ for all k. By compactness we may suppose without loss of generality that $v_k \to v$ as $k \to \infty$. Then $v \in \tilde{T}$ but $\mathbf{k}(v) \leqslant 1/b$, a contradiction.

Now we show that there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we have

$$E^+_\varepsilon(\mathcal{T}) \subseteq \exp^{\hat{g}_\varepsilon}\left(\left\{ sv | s \in [0, b], v \in \tilde{T}_\varepsilon \right\} \right) \Subset M. \tag{21}$$

Again arguing by contradiction, suppose that there exists a sequence $\varepsilon_j \searrow 0$ and points $q_j \in E^+_{\varepsilon_j}(\mathcal{T}) \setminus \exp^{\hat{g}_{\varepsilon_j}}(\{sv | s \in [0, b], v \in \tilde{T}_{\varepsilon_j}\})$. By [19, theorem 10.51, corollary 14.5], for each $j \in \mathbb{N}$ there exists a $\hat{g}_{\varepsilon_j}$-null-geodesic γ_j from $\mathcal{T}$ to q_j which is $\hat{g}_{\varepsilon_j}$-normal to $\mathcal{T}$ and has no focal point before q_j. Let $\gamma_j(t) = \exp^{\hat{g}_{\varepsilon_j}}(t\tilde{v}_j)$ with $\tilde{v}_j \in \tilde{T}_{\varepsilon_j}$. Let t_j be such that $\gamma_j(t_j) = q_j$. Then by our indirect assumption, $t_j > b$ for all j. In particular, each γ_j is defined at least on $[0, b]$.

By compactness, we may assume that $\tilde{v}_j \to \tilde{v}$ as $j \to \infty$. Then $\tilde{v} \in \tilde{T}$, and we set $\gamma(t) := \exp^g(t\tilde{v})$. As (M, g) is future-null complete, γ is defined on $[0, \infty)$. It now follows from standard ODE-results (see [12, section 2]) that $\gamma_j \to \gamma$ in the C^1-topology on $[0, b]$. In particular, $\gamma_j'(t) \to \gamma'(t)$ uniformly on $[0, b]$. Pick $C > 0$ and a compact set $K \Subset M$ such that $\|\gamma_j'(t)\|_h \leqslant C$ and $\gamma_j(t) \in K$ for all $t \in [0, b]$ and all $j \in \mathbb{N}$. Then by lemma 2.4, for any $\delta > 0$ there exists some $j_0 \in \mathbb{N}$ such that $\mathrm{Ric}_{\varepsilon_j}(\gamma_j'(t), \gamma_j'(t)) > -\delta$ for all $j \geqslant j_0$ and all $t \in [0, b]$.

Denoting by θ_j the expansion of γ_j we have by the Raychaudhuri equation

$$\frac{\mathrm{d}\left(\theta_j^{-1}\right)}{\mathrm{d}t} \geqslant \frac{1}{n-2} + \frac{1}{\theta_j^2}\mathrm{Ric}_{\hat{g}_{\varepsilon_j}}\left(\gamma_j', \gamma_j'\right) > \frac{1}{n-2} - \frac{\delta}{\theta_j^2}. \tag{22}$$

At this point we fix $\delta > 0$ so small that

$$a := \frac{n-2}{m} < \frac{n-2}{\alpha m} < b, \tag{23}$$

where $\alpha := 1 - (n-2)m^{-2}\delta$ and choose j_0 as above for this δ. For $j \geqslant j_0$ let $m_j := (n-2)\min_{v\in\tilde{T}_{\varepsilon_j}} \mathbf{k}_{\varepsilon_j}(v)$, then $m_j \to m$ $(j \to \infty)$ and $\alpha_j := 1 - (n-2)m_j^{-2}\delta \to \alpha$ $(j \to \infty)$, so for j large, (23) implies

$$a < \frac{n-2}{\alpha_j m_j} < b. \tag{24}$$

Consequently, choosing j so large that $\alpha_j > 0$, the right-hand side of (22) is strictly positive at $t = 0$. Thus θ_j^{-1} is initially strictly increasing and $\theta_j(0) = -(n-2)\mathbf{k}_j(\gamma_j'(0)) < -m_j < 0$, so from (22) we conclude that $\theta_j^{-1}(t) \in [-m_j^{-1}, 0)$ on its entire domain of definition. Hence θ_j has no zero on $[0, b]$, whereby θ_j^{-1} exists on all of $[0, b]$. From this, using (22), it follows that $\theta_j^{-1}(t) \geqslant f_j(t) := -m_j^{-1} + t\frac{\alpha_j}{n-2}$ on $[0, b]$. In particular this means that θ_j^{-1} must go to zero at or before the zero of f_j, i.e., there exists some $\tau \in (0, \frac{n-2}{\alpha_j m_j})$ such that $\theta_j^{-1}(t) \to 0$ as $t \to \tau$.

But for j sufficiently large (24) implies that $\theta_j^{-1} \to 0$ within $[0, b]$. However, since γ_j does not incur a focal point between $t = 0$ and $t = t_j > b$, θ_j is smooth, hence bounded, on $[0, b]$, a contradiction.

Remark 3.2. As an inspection of the proofs of lemma 2.4 and proposition 3.1 shows, both results remain valid for any approximating net g_ε (or sequence g_j) of metrics that satisfy properties (i) and (ii) from proposition 2.1. In particular, this applies to the approximations $\check{g}_\varepsilon$ from the inside. For the proof of the main result, however, it will be essential to use approximations from the outside that themselves are globally hyperbolic.

Proof of theorem 1.1. Suppose, to the contrary, that M is future null complete. Proposition 3.1 applies, in particular, to a net $\hat{g}_\varepsilon$ as in proposition 2.3 (iv), approximating g from the outside and such that each $\hat{g}_\varepsilon$ is itself globally hyperbolic.

Fix any $\varepsilon < \varepsilon_0$, such that by proposition 3.1 $E_\varepsilon^+(\mathcal{T})$ is relatively compact. Then since $\hat{g}_\varepsilon$ is globally hyperbolic, smooth causality theory (see the proof of [19, theorem 14.61]) implies that $E_\varepsilon^+(\mathcal{T}) = \partial J_{\hat{g}_\varepsilon}^+(\mathcal{T})$ is a topological hypersurface that is $\hat{g}_\varepsilon$-achronal. We obtain that $E_\varepsilon^+(\mathcal{T})$ is compact and since $g \prec \hat{g}_\varepsilon$, it is also g-achronal.

As in the proof of [19, theorem 14.61] let now X be a smooth g-timelike vector field on M and denote by ρ: $E_\varepsilon^+(\mathcal{T}) \to S$ the map that assigns to each $p \in E_\varepsilon^+(\mathcal{T})$ the intersection of the maximal integral curve of X through p with S. Then due to the achronality of $E_\varepsilon^+(\mathcal{T})$, ρ is injective, so by invariance of domain it is a homeomorphism of $E_\varepsilon^+(\mathcal{T})$ onto an open subset of S. By compactness this set is also closed in S. But also in the $C^{1,1}$-case, any Cauchy hypersurface is connected (the proof of [19, proposition 14.31] also works in this regularity). Thus $\rho(E_\varepsilon^+(\mathcal{T})) = S$, contradicting the fact that S is non-compact. This concludes the proof of theorem 1.1. □

We also have the following analogue of [19, theorem 14.61]:

Theorem 3.3. *Let (M, g) be an n-dimensional $C^{1,1}$-spacetime. Assume that*

(i) For any Lipschitz-continuous local null vector field X, $\mathrm{Ric}(X, X) \geqslant 0$.
(ii) M possesses a Cauchy-hypersurface S.
(iii) There exists a compact spacelike achronal submanifold $\mathcal{T}$ in M of codimension 2 with past-pointing timelike mean curvature vector field H.
(iv) M is future null complete.

Then the future horismos of $\mathcal{T}$, $E^+(\mathcal{T})$, is a compact Cauchy-hypersurface in M.

Proof. Since (M, g) is globally hyperbolic, [14, proposition A.28] implies that the causality relation $\leqslant$ on M is closed. Thus since $\mathcal{T}$ is compact it follows that $J^+(\mathcal{T})$ is closed. Also, by [13, corollary 3.16], $J^+(\mathcal{T})^\circ = I^+(\mathcal{T})$, so $E^+(\mathcal{T}) = \partial J^+(\mathcal{T})$. It is thereby the topological boundary of a future set and the proof of [19, corollary 14.27] carries over to the $C^{1,1}$-setting (using [14, theorem A.1, proposition A.18]) to show that $E^+(\mathcal{T})$ is a closed achronal topological hypersurface. It remains to show that any inextendible timelike curve intersects it.

Suppose to the contrary that there exists some inextendible timelike (locally Lipschitz) curve $\tilde{\alpha}$ that is disjoint from $E^+(\mathcal{T})$. Then as in (the proof of) [14, lemma A.10] we may also construct an inextendible timelike C^2-curve α that does not meet $E^+(\mathcal{T})$ (round off the breakpoints of the piecewise geodesic obtained in [14, lemma A.10] in a timelike way). By [19, example 14.11], since (M, g) is strongly causal, α is an integral curve of a timelike C^1-vector field X on M.

Next, let $\hat{g}_j$ be an approximating net as in proposition 2.3 (iv), (v) (to which thereby all arguments from the proof of theorem 1.1 apply, cf. remark 3.2). Denote by $I_j^+(\mathcal{T})$, $J_j^+(\mathcal{T})$, $E_j^+(\mathcal{T})$ the chronological and causal future, and the future horismos, respectively, of $\mathcal{T}$ with respect to $\hat{g}_j$. Set $K := \{sv \,|\, s \in [0, b],\, v \in TM|_{\mathcal{T}},\, \|v\|_h = 1\} \Subset TM$, where h is some complete smooth Riemannian background metric on M. It then follows from the locally uniform convergence of $\exp^{\hat{g}_j}$ to $\exp^g$, together with (21) that there exists some $j_0 \in \mathbb{N}$ such that for $j \geqslant j_0$ we have

$$\partial J_j^+(\mathcal{T}) = E_j^+(\mathcal{T}) \subseteq \exp^{\hat{g}_j}(K) \subseteq \overline{\left\{p \in M \,|\, \mathrm{dist}_h\big(p, \exp^g(K)\big) \leqslant 1\right\}} =: L \Subset M. \tag{25}$$

Let the map ρ from the proof of theorem 1.1 be constructed from the vector field X from above. Then by the proof of theorem 1.1 we may additionally suppose that j_0 is such that, for each $j \geqslant j_0$, $E_j^+(\mathcal{T})$ is a compact achronal topological hypersurface in (M, g) that is homeomorphic via ρ to S. Therefore α (which is timelike for all $\hat{g}_j$) intersects every $E_j^+(\mathcal{T})$ $(j \geqslant j_0)$ precisely once. Let q_j be the intersection point of α with $\partial J_j^+(\mathcal{T}) = E_j^+(\mathcal{T})$. We now pick t_j such that $q_j = \alpha(t_j)$ for all $j \in \mathbb{N}$. Each q_j is contained in L, so since (M, g) is globally hyperbolic, hence non-partially-imprisoning (as already noted in remark 2.2, the proof of [19, lemma 14.13] carries over verbatim to the $C^{1,1}$-case), it follows that (t_j) is a bounded sequence in $\mathbb{R}$ and without loss of generality we may suppose that in fact $t_j \to \tau$ for some $\tau \in \mathbb{R}$. Then also $q_j = \alpha(t_j) \to q = \alpha(\tau) \in L$.

As $q_j \in \partial J_j^+(\mathcal{T})$ there exist $p_j \in \mathcal{T}$ and $\hat{g}_j$-causal curves β_j from p_j to q_j (in fact, the β_j are $\hat{g}_j$-normal $\hat{g}_j$-null geodesics). Again without loss of generality we may assume that $p_j \to p \in \mathcal{T}$. By [16, theorem 3.1] (or [5, proposition 2.8.1]) there exists an accumulation curve β of the sequence β_j such that β goes from p to q. Moreover, since $\hat{g}_{j+1} \prec \hat{g}_j$ for all j, each β_k is $\hat{g}_j$-causal for all $k \geqslant j$. Therefore, β is $\hat{g}_j$-causal for each j. Thus by (the proof of) [6, proposition 1.5], β is g-causal and we conclude that $q = \alpha(\tau) \in J^+(\mathcal{T})$. If we had $q \in I^+(\mathcal{T})$ then for some j_1 we would also have $q_j \in I^+(\mathcal{T}) \subseteq I_j^+(\mathcal{T})$ for all $j \geqslant j_1$ (using [13, corollary 3.12]). But this is impossible since $q_j \in \partial J_j^+(\mathcal{T}) = E_j^+(\mathcal{T})$. Thus

$$q = \alpha(\tau) \in E^+(\mathcal{T}), \tag{26}$$

a contradiction to our initial assumption. We conclude that $E^+(\mathcal{T})$ is indeed a Cauchy-hypersurface in M.

Finally, as in the proof of theorem 1.1, the map ρ is a homeomorphism from $E_j^+(\mathcal{T})$ onto $E^+(\mathcal{T})$ (for $j \geqslant j_0$), so $E^+(\mathcal{T})$ is compact. □

In particular, as in [19, corollary B of theorem 14.61] it follows that if (i)–(iii) from theorem 3.3 hold and there exists some inextendible causal curve that does not meet $E^+(\mathcal{T})$ then (M, g) is future null incomplete. Indeed by [14, lemma A.20] the existence of such a curve shows that $E^+(\mathcal{T})$ cannot be a Cauchy-hypersurface.

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