A Comparison of High-order Finite Elements Method and Wave-based Discontinuous Galerkin Method for Helmholtz Problems

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Abstract

In this paper, we compare the performance of the wave-based discontinuous Galerkin method against the polynomial high-order finite element method (FEM) for Helmholtz problems. Previous studies demonstrate that both methods lead to a control of the dispersion error associated with low-order FEM at high frequency. Common belief is that compared to polynomial methods, physics-based methods can provide a significant improvement in performance, at the expense of a deterioration of the conditioning. However, the results presented in this paper indicate that the differences in accuracy, efficiency and conditioning between the two approaches are more nuanced than generally assumed.

Keywords: Helmholtz Problem, Wave-based Discontinuous Galerkin Method, High-order Finite Element Method.

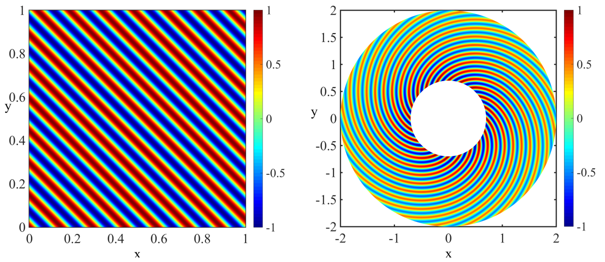
1 Introduction

We compare the performance of the wave-based discontinuous Galerkin method against the polynomial high-order finite element method (p-FEM). The methods were both devised to tackle the so-called pollution effect (accumulation of dispersion error) encountered by standard Finite Element Method when solving short wave problems. Another common characteristic between these two methods is that they easily allow local order refinement, which makes them suited for p-adaptive and hp-adaptive strategies.

The studied p-FEM replaces the low-order Lagrange polynomials with Lobatto shape functions [5], taking advantage of the improved interpolation properties of this family of functions. As the polynomial order P is increased, different types of shape functions appear: vertex, edge and bubble functions (and also face functions in 3D). Bubble functions have no connectivity with the neighbouring elements and can therefore be removed from the global system using static condensation which improves the conditioning and reduces the memory requirements.

Wave-based discontinuous Galerkin method (DGM) [1] uses plane waves to interpolate the solution in each element and the continuity between elements is weakly imposed using numerical fluxes.

Both numerical models have been identified as effective methods to address the pollution effect [2, 4] but, to the authors' knowledge, they have not been compared.



(a) Plane wave. (b) Prop. spinning waves.

Figure 1: Example of solutions.

2 Description of the test cases

To assess the performance of the methods, we use four types of solutions of the Helmholtz equation. The propagating plane wave problem in figure 1(a) involves a single direction of propagation and therefore allows a detailed study of the anisotropy of the numerical models. The spinning wave problem in figure

1(b) consists of spiral-shaped waves radiating from a cylinder. All the wave directions are equally present in the domain, which is closer to a realistic problem compared to the first test case. Propagating and evanescent waves are investigated. Finally, a singular corner solution is considered. The gradient of the solution exhibits a singularity at the origin. The objective with this last test case is to investigate how the two methods behave when confronted with such solutions. The computational domains are discretised using uniform triangular unstructured elements. To generate the solutions, an inhomogeneous Robin boundary condition is used for p-FEM and ghost cells are used for wave-based DGM [1].

3 Main results

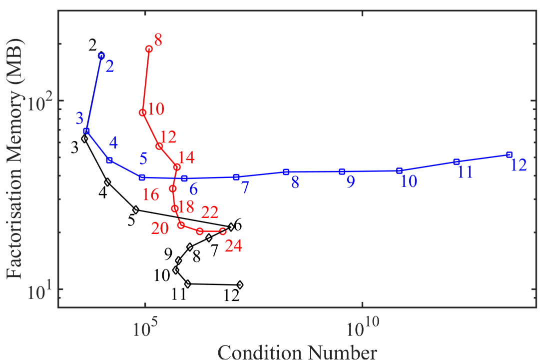


Figure 2: Propagating spinning wave problem (ka=28); factorisation memory (MB) against condition number to achieve 1% of accuracy; circles: wave-based DGM, squares: p-FEM without condensation, diamonds: p-FEM with condensation. The numbers of plane wave or polynomial order is shown next to each point.

A first part of the study is dedicated to the interpolation properties of the bases. The rest of the study focuses on the numerical models for which the following conclusions have been drawn. For the propagating wave problems, wave-based DGM and p-FEM are able to achieve the same level of accuracy and similar levels of performance. To reach the required accuracy (1% of the relative L^2-error), the wave-based systems are not ill-conditioned contrary to what is commonly assumed (figure 2). However, the studied physics-based method does not provide a step change in computational performance, even at high frequency. When dealing with evanescent waves, wave-based DGM becomes expensive compared to p-FEM which costs remained similar to that of the propagating waves. The exponential convergence of both methods for regular problems is lost when representing singular solutions. However, p-FEM is more robust and for a given mesh, the levels of accuracy reached are higher than those reached by the physics-based method.

Compared to wave-based DGM, p-FEM has a more consistent behaviour for the different types of problems. Moreover, p-FEM can directly be used on problems with non-uniform coefficients, whereas wave-based DGM would require some non-trivial developments to generalise the plane-wave basis to non-uniform media [3].

Acknowledgements

This work is performed as part of the CRANE (Community and Ramp Aircraft NoisE) project and is supported in full by European Union funding under the Framework Programme 7.

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