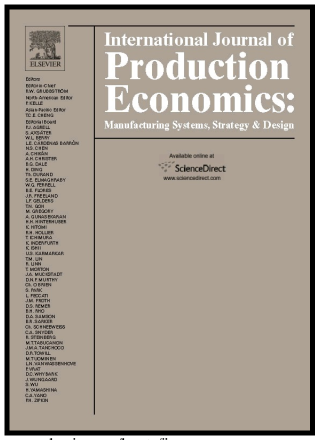

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Generalising Optimal Mean Setting for any Number and Combination of Serial and Parallel Manufacturing Operations

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Abstract

Consider a production system where products are continuously manufactured and their features inspected for conformance within specification limits. If features are produced above or below the specification limits, they are either subject to rework or the product scrapped. Optimal mean setting may be applied to adjust the manufacturing means to influence the amount of rework or scrap produced, maximising profit. Within the production system, manufacturing and then inspecting each feature in turn is termed serial production, whereas manufacturing multiple features before inspection is termed parallel production. This paper develops a generalised expression to optimise the mean values of each feature (optimal mean setting), where n number of features are produced in any combination of serial and parallel operations. Previous literature is restricted to considering two features in parallel. The production of multiple features in combinations of serial and parallel operations is not fully considered. The new generalised expression is validated by showing it is consistent with specific cases from past literature. The approach is then applied to a practical example of a gearbox shaft, considering the expected profit of eight possible manufacturing sequences, as well as the deviation of the manufactured means relative to the design intent. The generalised expression is widely applicable in component design and manufacturing planning where the process capability index (C_{pk}) of features is below one. The generalised expression also forms the basis for trade-offs between profitability and minimising deviations of manufactured means, which is the subject of further development.

Keywords: Process mean; Quality control; Markov processes; Applied probability; Production

1. Introduction

Ensuring product quality is a fundamental part of engineering design and manufacture. Aerospace components often use 100% inspection routines to ensure product quality. Implicitly, the manufacturing process responsible for producing inspectable features may produce features that are outside the specification limits, otherwise inspection would not be necessary. A normal distribution is often used to model the manufacturing process variation, as illustrated in Figure 1.

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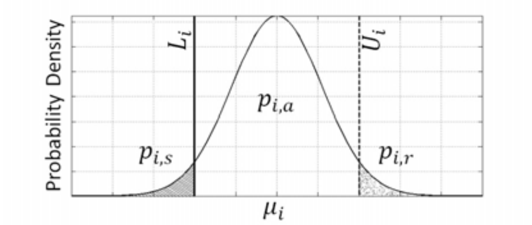


Figure 1: Illustration of the rework, conformance and scrap probabilities for the i$^{\text{th}}$ manufacturing stage

Rework is created when a manufacturing operation produces a feature outside the specification limits (non-conforming), where additional manufacturing operations can bring that feature inside the specification limits. This is indicated by the speckled region above the upper specification limit (U) (Figure 1). Scrap is created when a feature is non-conforming but no additional manufacturing operations can make that feature conform. Such a feature would lie below the lower specification limit (L), illustrated by the diagonally hatched region in Figure 1. Commonly, it is more costly to scrap a product, due to a non-conforming feature which cannot be made to conform, than it is to rework a feature. Thus, scrap cost is generally greater than rework cost. This asymmetry is exploited by optimal mean setting (process targeting) to shift the process mean to maximise the expected profit ($E(PR)$).

An item may consist of several inspectable features which may be manufactured in two principle ways relative to the inspection processes. A feature may be manufactured and then inspected before manufacturing and inspecting the next feature, known as serial production. Alternatively, several features may be manufactured then inspected in the same inspection operation, known as parallel production. Parallel manufacturing of two features is referred to as 'dual feature' production in the literature.

The main objective of this paper is to establish a general method to determine the expected profit of n-features produced by parallel operations, serial operations or a combination of both. The expression for expected profit has the general form,

$$\begin{aligned} E(PR) &= \text{Items Sold} - \text{Production Cost} \\ &- \text{Scrap Cost} - \text{Rework Cost} \end{aligned} \tag{1}$$

The principle contributions of this article are: (1) the identification of an error made in the literature with regard to the expected profit for parallel production; (2) to correctly define the expected profit for parallel operations; (3) to generalise the expected profit expression to allow expected profit to be calculated for parallel operations, serial operations and a combination of both; (4) to demonstrate how this new expression may be used to determine the optimum means and optimum combination of series and parallel operations for the production of a gearbox shaft..

2. Literature Review

Springer (1951) was the first to formally address an optimal mean setting (process targeting) problem in the form of a 'can filling' problem, where the optimal filling level of a can was sought by adjusting the specification limits to minimise cost. Hunter & Kartha (1977) and Nelson (1978) took a slightly different approach to Springer and maximised profit rather than minimised cost. A lower specification limit was used to define whether products conformed, where non-conforming product (below the specification limit) was sold in a secondary market. Carlsson (1989) took a similar approach applied to the production of steel beams. A premium was added for high quality beams (above the specification limit), while a discount was applied for lower quality beams (below the specification limit). Along similar lines, Bisgaard et al. (1984) applied a discount proportional to the degree to which the product was under the specification limit. Golhar (1987) applied the principle of optimal mean setting to pharmaceutical type products where no secondary or discount market existed. Instead under-filled containers were re-filled at additional cost and sold in the regular market. In Golhar's model the same price was charged for overfilled, as for correctly filled, containers. To limit the amount of product being given away Golhar & Pollock (1988) and Schmidt & Pfeifer (1991) introduced an upper specification limit to restrict the extent to which a container could be overfilled. Liu & Raghavachari (1997) showed that optimising the location of the upper specification limit, as well as the mean, always improved profit to a greater extent than just optimising one of these parameters. Wen & Mergen (1999) were the first to specifically apply optimal mean setting to a manufacturing feature problem; the mean for a grinding operation was optimised for the production of an inner ring of a bearing race.

The concept of quality loss functions, from the field of robust design (Taguchi (1986)), was introduced to optimal mean setting by Elsayed & Chen (1993) and Arcelus (1996). Quality loss is represented by a monetary penalty which increases as the dimension of the manufactured feature moves further from the design nominal. Thus, in optimal mean setting it limits the extent to which the mean is biased towards rework. Roan et al. (2000) utilised quality loss to not only set the optimal process mean but also the optimum production run size and material order quantity. Chen et al. (2002) introduced asymmetric loss functions to the optimal mean setting model developed by Wen & Mergen (1999) to account for smaller-the-better and larger-the-better type characteristics from robust design literature (Phadke (1989)). Chen & Lai (2007) also applied asymmetric loss functions to an optimal mean setting problem but used a sampling plan to determine conformance rather than 100% inspection.

Products are often judged on the conformance of several features, which may be produced by serial or parallel operations. Elsayed & Chen (1993), Kapur & Cho (1996) and Drain & Gough (1996) were among the first to consider a product with two inspectable features produced by a parallel manufacturing operation. The Taguchi quality loss function was also employed in these models. Teeravaraprug & Cho (2002) and Chen & Chou (2003) both developed two-feature parallel processing optimal mean setting models that encompassed different non-conformance costs depending on the feature and whether it was above the upper specification limit (rework) of below the lower specification limit (scrap). Khasawneh et al. (2008) also presented a two-feature parallel production model with differing non-conformance costs, but improved the calculation of rework cost by implementing a Markov chain model. The introduction of Markovian modelling is an important contribution as it accounts for the fact that a feature may be reworked several times before it is deemed conforming or is scrapped. The Markovian model allowed the average time components spent being reworked and the eventual probabilities of scrap and conformance to be determined.

Al-Sultan & Pulak (2000) were the first to study optimal mean setting for serial manufacture, the model was restricted to two-features. Bowling et al. (2004) developed an optimal mean setting model for n-features in serial production, which also made use of Markovian modelling to predict the rework costs. An error with the Markovian model was corrected by Selim & Al-Zubi (2011) (in Section 3.1), before they went on to discuss an efficient optimisation methodology for an n-stage serial production system. Peng & Khasawneh (2014) used the Markovian models first developed by Bowling et al. (2004) and Khasawneh et al. (2008) and applied a sample inspection routine. They also considered the manufacture of four features in two serial stages, with two features produced in parallel at each stage. This is the only example of a hybrid serial/parallel optimal mean setting model in the literature. Dodd et al. (2015) developed a novel optimisation methodology for parallel production by optimising the means that applied to single feature rework separately from the means that applied to dual feature rework. This approach outperformed the maximum profit attained by the methods used by Khasawneh et al. (2008) and Peng & Khasawneh (2014).

Goethals & Cho (2011) applied a differing approach to optimal mean setting from the literature discussed hitherto. Instead of assuming the manufacturing variation was known and the mean could be shifted, Goethals & Cho (2011) derived the most cost-effective process mean and variation through observation and design of experiment. A response surface for the process mean and variance was modelled in response to several process variables (X), which affected the conformance of a feature (Y). The mean and variance were then optimised to minimise total cost, where cost penalties were invoked if the quality characteristic was greater or lower than the specification limits (rework and scrap). A Taguchi style quality loss function was also employed. Goethals & Cho (2012) and Boylan & Cho (2014) extended the problem for multiple quality characteristics ($\boldsymbol{Y}$). They also employed the skew normal distribution (Azzalini (1985)) to represent quality characteristics classified as smaller-the-best, nominal-the-best or larger-the-better (Phadke (1989)). Another alternative approach to the optimal mean setting problem was investigated Lee et al. (2007), where the optimum mean was sought for multiple (different) products produced through a common manufacturing; as opposed to one product with several features.

In this current paper a generalised Markovian based model is developed for a single product with n-features in any combination of serial and parallel operations. The variance of the manufacturing processes are assumed to be known. The rationale for this approach is summarised below:

- Currently there is no way of computationally deriving the expected profit expression for more than two features in parallel or combinations of serial and parallel operations for n-features. The expression must be derived from first principles each time. A generalised equation allows the optimum combination of serial and parallel operations to be found along with the optimal means for each sequence. Additionally, it allows optimal mean setting to be readily applied to an existing manufacturing sequence.

- The method for determining expected profit for parallel features (developed by Khasawneh et al. (2008) and used by Peng & Khasawneh (2014)) contains errors which are corrected in this article.

- The approach developed by Goethals & Cho (2011), of deriving the most effective manufacturing variance and process mean is not taken here. Rather, the assumption that the variance of a process is known is justified by using CAPRA Tolerance Capability Expert (TCE) (CAPRA (2015)). This tool contains manufacturing capability data for a vast array of manufacturing processes. Therefore, given a manufacturing feature, it is possible to 'lookup' the industry standard variance of the process given the feature's dimensions and tolerances.

The remainder of the paper is structured as follows; Section 3.1 provides a list of modelling assumptions. Section 3.2 illustrates the calculations required to compute the rework, scrap and conformance probabilities (required to compute the expected profit) for parallel manufacturing operations for n-features. Section 3.3 examines the differences between the dual feature expected profit equation in Khasawneh et al. (2008) and the revised method, accounting for the feed-in and feed-out probabilities. Sections 3.4 and 3.5 generalise the equation for expected profit for n-features manufactured in parallel. Section 3.6 completes the fully generalised expression for any combination of serial and parallel operations. Section 4 illustrates the benefit of the new generalised expression to maximise the expected profit for the production of an intermediate shaft from a speed reducer gearbox. A conclusion is given in Section 5.

3. Model Development

3.1. Modelling Assumptions

1. Products are produced continuously.
2. The manufacturing process is under statistical control.
3. Products are scraped or reworked if a feature is under the lower or above the upper specification limit, respectively.
4. All features undergo 100% inspection.
5. Correlation between features is previsioned for through the covariance matrix $\mathbf{\Sigma}$ (Section 3.2).
6. Inspection is assumed to be 100% accurate. In practical situations this translates to requiring an accuracy far greater than the manufacturing variation.
7. All manufacturing variation is assumed to be normally distributed with specified means (μ) and variance (σ^2). This is not a limiting assumption as the method is applicable for other distributions provided a multivariate distribution model can be generated in order to compute the probabilities conformance, rework and scrap for parallel processes.
8. The process variation for rework processes is the same as the initial process variation. This implies the same, or a similar machine is used to process rework.
9. The positioning and machining accuracy of the manufacturing process is assumed to always produce the same probabilities of conformance, scrap and rework irrespective of the amount of material to be removed.

It would be possible to relax Assumptions six to nine to more exactly apply to specific practical situations without fundamentally changing the methodology presented in this paper. However, this paper concentrates on establishing the framework for the ideal case, introducing such complexities are the subjects of future work.

3.2. Parallel Production System (dual-feature)

The problem of calculating the final conformance, rework and scrap probabilities for a dual system was considered by Khasawneh et al. (2008). These probabilities are evaluated by calculating the probability of an item being in a particular rectangular region (rework, conformance or scrap state), depicted in Figure 2. The axes have been reversed and the calculations modified accordingly (from the layout given by Khasawneh et al. (2008)), to run from infinity to zero such that rework occurs towards the origin. This halves the number of cumulative distribution function (CDF) evaluations required to compute the rework, conformance and scrap probabilities compared to Khasawneh et al. (2008). Furthermore, the multivariate distribution function was used which

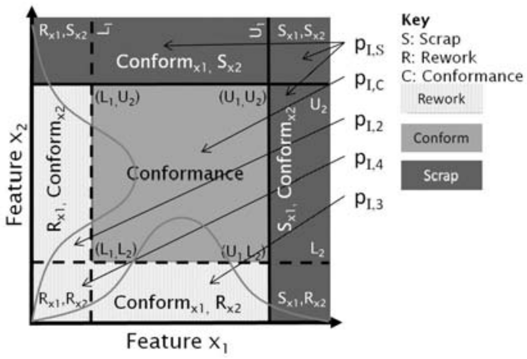


Figure 2: Bivariate rework, conformance and scrap

permits dependencies to exist between the manufactured features. Such dependencies may exist particularly if two features were produced by the same machine. The general form of a multivariate normal distribution is,

$$f_X(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left\{ -\frac{1}{2} \left(\boldsymbol{x} - \boldsymbol{\mu} \right)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right\}. \tag{2}$$

The vector $\boldsymbol{x}$ is a k-dimensional random vector $\boldsymbol{x} = [X_1, \ldots, X_k]$, $\boldsymbol{\mu}$ is a k-dimensional mean vector $\boldsymbol{\mu} = [E[X_1], \ldots, E[X_k]]$ and $\boldsymbol{\Sigma}$ is a $k \times k$ covariance matrix, $\boldsymbol{\Sigma} = [\text{Cov}[X_i, X_j]]$, $i = 1, \ldots, k$; $j = 1, \ldots, k$. The cumulative distribution of the bivariate case is,

$$F(x_1, x_2) = \int_{-\infty}^{X_1} \int_{-\infty}^{X_2} f(x_1, x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1. \tag{3}$$

The probabilities $p_{\mathrm{I},2}$ to $p_{\mathrm{I},4}$, $p_{\mathrm{I},C}$ and $p_{\mathrm{I},S}$ (Figure 2) are calculated using Equation 3 where X_1 and X_2 define the specific rectangular regions. The flow of items through the various states is depicted in Figure 3. The subscript $_\mathrm{I}$ indicates the initial operation while the proceeding subscript indicates the state where the item transfers into after the processing operation. The numerical subscripts represent the transient rework states while the $_C$ and $_S$ subscripts indicate the absorbing conformance and scrap states. A general rectangular region can be defined by; $\mathbf{L} = (L_1, \ldots, L_n)$ and $\mathbf{U} = (U_1, \ldots, U_n)$ where $L_i \leq U_i \; \forall \; i = 1, 2, \ldots, n$ and $(\boldsymbol{L}, \boldsymbol{U})$ is an n-dimensional rectangle, n is the number of features at a given stage. The vectors $\mathbf{L}$ and $\mathbf{U}$ represent the lower and upper specification limits for each feature as indicated on Figure 2. Taking the Cartesian product of n intervals, $p_{\mathrm{I},C} = (L_1, U_1) \times (L_2, U_2) \times, \ldots, \times (L_n, U_n)$. The two-dimensional rectangular conformance region is shown on Figure 2 for $p_{\mathrm{I},C}$ where $\mathbf{L} = (L_1, L_2)$ and $\mathbf{U} = (U_1, U_2)$. For a cumulative distribution function (CDF) F: $\mathbb{R}^n \rightarrow [0, 1]$ the probability enclosed in the rectangle $p_{\mathrm{I},C} = (\mathbf{L}, \mathbf{U})$ is given by:

$$p_{\mathrm{I},C} = P(L_1 < X_1 \leq U_1, \ldots, L_n < X_n \leq U_n)$$

which can be expressed as,

$$p_{\mathrm{I},C} = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{1} (-1)^{i_1+i_2+\cdots+i_n} F\left(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \ldots, \boldsymbol{x}_{i_n}\right). \tag{4}$$

where,

$$\begin{cases} x_{i_j} = L_j & \text{if } i_j = 0, \\ x_{i_j} = U_j & \text{if } i_j = 1 \end{cases} \qquad \forall \quad j = 1, 2, \ldots, n.$$

Thus, for the bivariate case in Figure 2, the conformance rectangle is given by,

$$\begin{aligned} p_{1,C} &= (-1)^2 F(U_1, U_2) \\ &+ (-1)^1 F(L_1, U_2) \\ &+ (-1)^1 F(U_1, L_2) \\ &+ (-1)^0 F(L_1, L_2). \end{aligned} \tag{5}$$

The first line of Equation 5 determines the probability within the conformance and rework rectangles in Figure 2. The second and third lines in Equation 5 account for the rework regions, while the fourth line accounts for the overlapping section of the two rework regions. It follows that $p_{\mathrm{I},2} = F(L_1, U_2) - F(L_1, L_2)$, $p_{\mathrm{I},3} = F(U_1, L_2) - F(L_1, L_2)$, $p_{\mathrm{I},4} = F(L_1, L_2)$ and $p_{\mathrm{I},S} = 1 - F(U_1, U_2)$. If a component falls into the $p_{\mathrm{I},2}$ or $p_{\mathrm{I},3}$ regions, rework is only required on a single feature. Thus, the probability of that component requiring further rework, conforming or becoming scrap is given by the univariate CDF (lines one to six of Equation 6). If the item falls into $p_{\mathrm{I},4}$, the probabilities of rework, conformance and scrap are the same as the initial probabilities (line 7 of Equation 6), where $\boldsymbol{\Phi}(\bullet)$ is the standard cumulative normal density function.

$$\begin{aligned} p_{2,2} &= 1 - \boldsymbol{\Phi}(\mathrm{U}_1), \\ p_{2,C} &= \boldsymbol{\Phi}(\mathrm{U}_1) - \boldsymbol{\Phi}(\mathrm{L}_1), \\ p_{2,S} &= \boldsymbol{\Phi}(\mathrm{L}_1), \\ p_{3,3} &= 1 - \boldsymbol{\Phi}(\mathrm{U}_2), \\ p_{3,C} &= \boldsymbol{\Phi}(\mathrm{U}_2) - \boldsymbol{\Phi}(\mathrm{L}_2), \\ p_{3,S} &= \boldsymbol{\Phi}(\mathrm{L}_2), \\ p_{4,i} &= p_{\mathrm{I},i} \quad \text{for} \quad i = [2, 3, \ldots, C, S]. \end{aligned} \tag{6}$$

Where the two subscripts are the same, items feed back into the smae state. Following the analysis presented in Khasawneh et al. (2008), the $\boldsymbol{M}$-matrix and $\boldsymbol{F}$-matrix for a two feature parallel manufacturing problem are given by Equations 7 to 9,

$$
\boldsymbol{M} = \begin{array}{c} \\ \text{I} \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccc} \text{I} & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
1 & \frac{p_{\text{I},2}}{(1-p_{2,2})(1-p_{4,4})} & \frac{p_{\text{I},3}}{(1-p_{3,3})(1-p_{4,4})} & \frac{p_{\text{I},4}}{1-p_{4,4}} \\
0 & \frac{1}{1-p_{2,2}} & 0 & 0 \\
0 & 0 & \frac{1}{1-p_{3,3}} & 0 \\
0 & \frac{p_{4,2}}{(1-p_{2,2})(1-p_{4,4})} & \frac{p_{4,3}}{(1-p_{3,3})(1-p_{4,4})} & \frac{1}{1-p_{4,4}}
\end{bmatrix}
\end{array} \tag{7}
$$

$$
\boldsymbol{F} = \begin{array}{c} \\ \text{I} \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cc} C & S \end{array} \\
\begin{bmatrix}
f_{\text{I},C} & f_{\text{I},S} \\
\frac{p_{2,C}}{1-p_{2,2}} & \frac{p_{2,S}}{1-p_{2,2}} \\
\frac{p_{3,C}}{1-p_{3,3}} & \frac{p_{3,S}}{1-p_{3,3}} \\
f_{4,C} & f_{4,S}
\end{bmatrix}
\end{array} \tag{8}
$$

where,

$$
\begin{aligned}
f_{\text{I},C} &= p_{\text{I},C} + \frac{p_{\text{I},2}p_{2,C}}{(1-p_{2,2})(1-p_{4,4})} + \frac{p_{\text{I},3}p_{3,C}}{(1-p_{3,3})(1-p_{4,4})} + \frac{p_{\text{I},4}p_{4,C}}{1-p_{4,4}}, \\
f_{\text{I},S} &= p_{\text{I},S} + \frac{p_{\text{I},2}p_{2,S}}{(1-p_{2,2})(1-p_{4,4})} + \frac{p_{\text{I},3}p_{3,S}}{(1-p_{3,3})(1-p_{4,4})} + \frac{p_{\text{I},4}p_{4,S}}{1-p_{4,4}}, \\
f_{4,C} &= \frac{p_{4,2}p_{2,C}}{(1-p_{2,2})(1-p_{4,4})} + \frac{p_{4,3}p_{3,5}}{(1-p_{3,3})(1-p_{4,4})} + \frac{p_{4,C}}{1-p_{4,4}}, \\
f_{4,S} &= \frac{p_{4,2}p_{2,S}}{(1-p_{2,2})(1-p_{4,4})} + \frac{p_{4,3}p_{3,S}}{(1-p_{3,3})(1-p_{4,4})} + \frac{p_{4,6}}{1-p_{4,4}}.
\end{aligned} \tag{9}
$$

The elements of the $\boldsymbol{M}$ and $\boldsymbol{F}$ matrices are examined in the next section.

3.3. Analysis of Dual Feature Production

The meaning of the entries in the $\boldsymbol{F}$-matrix require careful consideration as they were incorrectly interpreted by Khasawneh et al. (2008) leading to an incorrect definition of expected profit. Figure 3 depicts the flow of items with two features through the initial manufacturing stage (I) to the conforming or scrap states, C and S. Items can pass directly into these absorbing states from state I or via one of three types of rework (transient states, 2, 3 and 4). The three rework states correspond to rework for the X_1 feature, the X_2 feature or both X_1 and X_2 features together. The columns in the $\boldsymbol{F}$-matrix correspond to the absorbing states, conforming (C) and scrap (S). The rows in the $\boldsymbol{F}$-matrix correspond to the transients states; the initial manufacturing state or one of three rework states (illustrated in Figure 3). The first entry in the $\boldsymbol{F}$-matrix ($f_{\text{I},C}$) is the probability items eventually conform given the number of items in the initial manufacturing state (which is all the items). It gives the final probability of conformance and the adjacent entry $f_{\mathrm{I},S}$ gives the final probability of scrap. Entry $f_{2,C}$ is the probability items eventually conform assuming state 2 was the starting state. This does not account for the items that will have conformed or will have been scrapped directly from the initial process, or items that require X_2-feature rework or dual feature rework. Additionally it does not account for items that may feed into state 2 from state 4 if only the X_2-feature was made to conform during dual feature rework process. The same principles apply to state 3 and the entry in the $\boldsymbol{F}$-matrix, $f_{3,C}$. The entry $f_{4,C}$ defines the probability of components conforming given the number of components that passed through state 4 on their path to conformance. Therefore, $f_{4,C}$ is the combined probability of items going directly from state 4 to conformance and also items that go from state 4 to state 2 or state 3 before finally conforming.

For the purpose of finding the conforming, rework and scrap probabilities for optimal mean setting, we wish to know the number of items feeding into each transient state and the probability of them eventually transferring directly to an absorbing state. Therefore, let a value D correspond to the total probability of items being in a particular state and feeding out directly to an absorbing state. Thus the expected profit for a dual feature case is defined as,

$$\begin{aligned} E(PR) &= SP\, D_{\mathrm{I,C}} - PC - (SC_1\, D_{\mathrm{I,S}} + \\ &+ SC_2\, D_{\mathrm{2,S}} + SC_3\, D_{\mathrm{3,S}} + SC_4\, D_{\mathrm{4,S}}) \\ &- (RC_1\, m_{\mathrm{I},2} + RC_1\, m_{\mathrm{I},3} + RC_1\, m_{\mathrm{I},4}) \end{aligned} \tag{10}$$

where SP is the selling price, SC and RC are the scrap and rework costs respectively. The subscripts on each cost refers to the costs at a given state. The D-values subscripts give the probabilities of items going directly from one state (first subscript) to the absorbing states (second subscript). The specific D-values are described below.

Explanation of the $D_{I,S}$ value

The feed-in and feed-out probabilities to and from the initial state and three rework states are shown in Figure 3 . There is a direct feed-out from the initial state (I) to the scrap state (S) given by $p_{\mathrm{I},S}$ therefore,

$$D_{\mathrm{I},S} = p_{\mathrm{I},S}.$$

Explanation of the $D_{2,S}$ and $D_{3,S}$ values

The probabilities of items entering the rework states (2, 3 and 4) are $p_{\mathrm{I},2}$, $p_{\mathrm{I},3}$ and $p_{\mathrm{I},4}$ (Figure 3). These inputs into states 2 and 3 only apply during the first iteration however, there are additional inputs from state 4 during the rework iterations (Figure 3). State 4 can feed states 2 and 3 as there is the possibility that after reworking the dual features, one feature conforms but the other may require additional rework. The total probability of components from state 4 feeding state 2, over all iterations, is given by,

$$p_{\mathrm{I},4}\, p_{\mathrm{I},2} + p_{\mathrm{I},2}\, p_{\mathrm{I},4}^2 + p_{\mathrm{I},2}\, p_{\mathrm{I},4}^3 + \cdots = \sum_{r=1}^{\infty} p_{\mathrm{I},2}\, p_{\mathrm{I},4}^r,$$

where r is the iteration number[1]. This a geometric series which can be written,

$$\sum_{r=1}^{\infty} p_{\mathrm{I},2}\, p_{\mathrm{I},4}^{r} = p_{\mathrm{I},4}\, \frac{p_{\mathrm{I},2}}{1-p_{\mathrm{I},4}} \Leftrightarrow |p_{\mathrm{I},4}| < 1. \tag{11}$$

The output from state 2 into the absorbing scrap state must account for the two feed-ins, one from the first iteration and another from state 4, therefore,

$$D_{2,S} = f_{2,S}\left(\frac{p_{\mathrm{I},4}p_{\mathrm{I},2}}{(1-p_{\mathrm{I},4})} + p_{\mathrm{I},2}\right). \tag{12}$$

Similar terms apply for the output from state 3 into the absorbing scrap state. Replacing the $_S$ subscripts with $_C$ subscripts on the D and f terms (Equation 12) gives the outputs from states 2 and 3 into the absorbing conforming state (shown in Figure 3).

Explanation of the $D_{4,S}$ value

The dual feature rework state (4) initially receives items from the first iteration given by $p_{I,4}$. There are no other feed-ins to this state. The final output from state 4 into the absorbing scrap state must not include the feed-outs into the two transient single feature rework states (2 and 3). Recall the $\boldsymbol{F}$-matrix value, $f_{4,S}$, includes these feed-out probabilities to states 2 and 3. The output from state four into the scrap state follows a similar principle defined in Equation 11, given by,

$$D_{4,S} = p_{\mathrm{I},4}\left(\frac{p_{\mathrm{I},2}}{1-p_{\mathrm{I},4}}\right). \tag{13}$$

A similar equation exists to define the probability of components going to the conformance states from state 4 as displayed in Figure 3.

The total rework cost is calculated by multiplying the average number of rework iterations (first row of the $\boldsymbol{M}-$matrix) by the rework cost for each state. For example, $m_{\mathrm{I},2}$ (from the $\boldsymbol{M}-$matrix) is the number of times the second state is occupied given the first state as the starting point (hence $m_{\mathrm{I},2} < 1$). The same argument follows for $m_{\mathrm{I},3}$ and $m_{\mathrm{I},4}$. The same probabilities can also be found from $m_{4,2}$ and $m_{4,2}$ as the dual feature rework process (state 4) is identical to the first state (I), albeit with fewer components. Note, $m_{4,4}$ is the probability the fourth state is occupied, given the starting point is the fourth state, hence $m_{4,4} \geq 1$, due to rework.

[1]In a continuous production system there is always a diminishing probability of feed-outs after r-iterations.

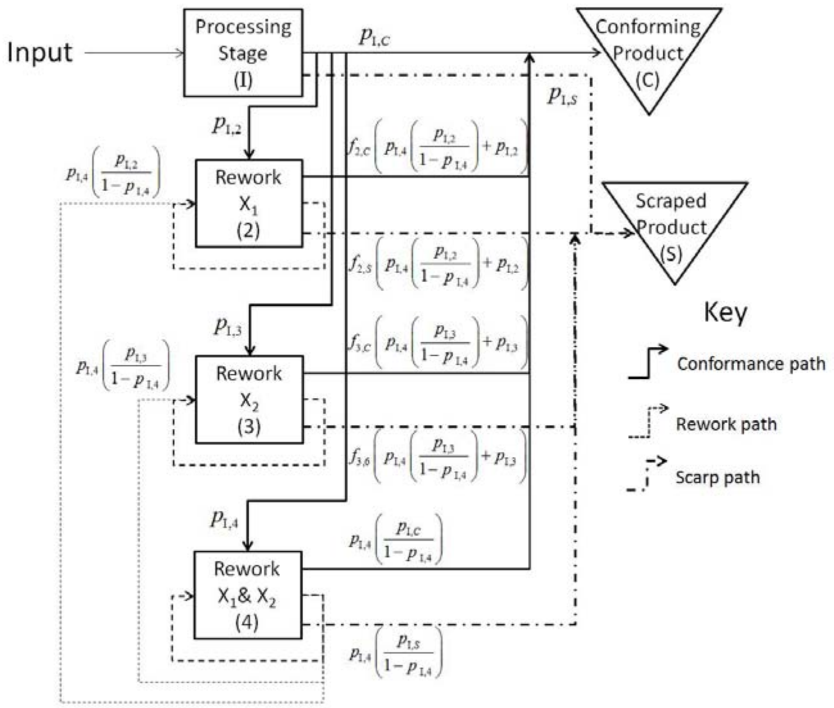


Figure 3: A dual feature production process

The final form of the expected profit for this dual feature process is given by,

$$
\begin{aligned}
E(PR) &= SP\, f_{\mathrm{I},C} - PC - SC_1\, p_{\mathrm{I},S} \\
&- SC_2 \left(p_{\mathrm{I},4} \left(\frac{p_{\mathrm{I},2}}{1 - p_{\mathrm{I},4}} \right) + p_{\mathrm{I},2} \right) f_{2,S} \\
&- SC_3 \left(p_{\mathrm{I},4} \left(\frac{p_{\mathrm{I},3}}{1 - p_{\mathrm{I},4}} \right) + p_{\mathrm{I},3} \right) f_{3,S} \\
&- SC_4 \left(\frac{p_{\mathrm{I},S}}{1 - p_{\mathrm{I},4}} \right) p_{\mathrm{I},4} - RC_2\, m_{\mathrm{I},2} \\
&- RC_3\, m_{\mathrm{I},3} - RC_4\, m_{\mathrm{I},4}.
\end{aligned} \tag{14}
$$

The probability terms multiplying the four scrap costs (SC_I to SC_4) are the difference between this correct expression for expected profit and Equation 5 from Khasawneh et al. (2008).

The correct method of modelling the flow of components through a two feature parallel manufacturing system has been shown by accounting for the feed-in and feed-out probabilities. As the number of features increases, these feed-in and feed-out terms increase exponentially as $3^n - 2^n$, where n is the number of features. It quickly becomes impractical to establish which feed-ins and feed-outs apply to which state and non-trivial to determine the expected profit equation. For just four features produced in parallel, one must calculate 65 feed-in and feed-out terms and apply them to the correct rework states. The next section develops a method for defining these terms for n-features processed in parallel and applying them to the relevant rework state. This method allows the expected profit equation to be written directly for n-features processed in parallel.

3.4. General Solution for Parallel Production

As the number of features produced by parallel manufacturing increases, the number of rework options also increase, (i.e. the number of states on Figure 3 increases). As with the fourth state in Figure 3, any multiple feature rework state will feed components into other rework states. A general solution must account for all the feed-ins to each rework state as well as the outputs from each rework state. This was achieved by setting up two types of matrix. The $\boldsymbol{S}$-matrix is binary matrix and indicates the existence of rework states. The $\boldsymbol{D}$-matrix has a similar form to the $\boldsymbol{S}$-matrix but determines the number of components transferring into each rework state. The combination of $\boldsymbol{S}$ and $\boldsymbol{D}$ matrices correctly determines the feed-ins and feed-outs to each rework state, dependent on the number of features. This Section shows how generalised $\boldsymbol{S}$ and $\boldsymbol{D}$ matrices were derived.

Let $\boldsymbol{X} = [x_1, x_2, \ldots, x_N]$ be a vector of inspectable features, where N is the total number of inspectable features. The total number of manufacturing states, including the initial manufacturing state is given by,

$$\eta = 1 + \sum_{k=1}^{N} \frac{N!}{k!(N-k)!}, \tag{15}$$

where k is the number of features requiring rework at each state. Thus, a process with three inspectable features requires an initial manufacturing state, three single feature reworks, three dual feature reworks and one triple feature rework. Let ${}^k\boldsymbol{C} = \{{}^k c_{\beta+1}, {}^k c_{\beta+2}, \ldots, {}^k c_{\beta+m}\}$ be the set containing the k-type combinations where m determines the cardinality of the set,

$$m = \frac{N!}{k!(N-k)!}, \tag{16}$$

corresponding to the number of k-type combinations. The value, β, determines the starting element for each k-type set where,

$$\beta = \sum_{k=1}^{k-1} \frac{N!}{k!(N-k)!}. \tag{17}$$

For $k-1=0$, the β term is zero. Examining the combinations when $k=1$ a three feature process, $\boldsymbol{X} = [x_1, x_2, x_3]$, gives ${}^1\boldsymbol{C} = \{{}^1c_1, {}^1c_2, {}^1c_3\} = \{[x_1], [x_2], [x_3]\}$. For $k=2$ the combinations are ${}^2\boldsymbol{C} = \{{}^2c_4, {}^2c_5, {}^2c_6\} = \{[x_1, x_2], [x_1, x_3], [x_2, x_3]\}$ and ${}^3\boldsymbol{C} = \{{}^3c_7\} = \{[x_1, x_2, x_3]\}$ for $k=3$. All the possible combinations are given by $\boldsymbol{\Upsilon}$ where $\left\{{}^k\boldsymbol{C} \subset \boldsymbol{\Upsilon}\right\}_{\forall\, k \in \{1,2,\ldots,k\}}$. The set $\boldsymbol{\Upsilon}$ is monotonic in k such that the first subset of combinations, ${}^1\boldsymbol{C}$ always represents single feature rework(s). The second subset of combinations, ${}^2\boldsymbol{C}$, always contains dual feature rework and so on. In general form $\boldsymbol{\Upsilon} = \{{}^1\boldsymbol{C}, {}^2\boldsymbol{C}, \ldots, {}^k\boldsymbol{C}\}$. Thus, for three inspectable features,

$$\begin{aligned} \boldsymbol{\Upsilon} &= \{\{{}^1\boldsymbol{C}\}, \{{}^2\boldsymbol{C}\}, \{{}^3\boldsymbol{C}\}\} \\ &= \{\{{}^1c_1, {}^1c_2, {}^1c_3\}, \{{}^2c_4, {}^2c_5, {}^2c_6\}, \{{}^3c_7\}\} \\ &= \{[x_1], [x_2], [x_3], [x_1, x_2], [x_1, x_3], \ldots \\ &\quad [x_2, x_3], [x_1, x_2, x_3]\}. \end{aligned} \tag{18}$$

It is possible to construct a matrix whose elements determine the inputs into the various rework states. The generalised form of such a matrix $\boldsymbol{S}$ is given by Equation 19. To condense the subscript notation let $\delta = \beta + m$. Each column in Equation 19 represents a k-type combination while the rows represent each rework state.

$$
{}^{k}\boldsymbol{S} = \begin{array}{c} \\ \alpha \\ \alpha+1 \\ \vdots \\ \eta \end{array}
\begin{array}{c}
\begin{array}{cccc} {}^{k}c_{\beta+1} & {}^{k}c_{\beta+2} & \cdots & {}^{k}c_{\delta} \end{array} \\
\begin{bmatrix}
\lambda_{\alpha-1,\beta+1} & \lambda_{\alpha-1,\beta+2} & \cdots & \lambda_{\alpha-1,\delta} \\
\lambda_{\alpha,\beta+1} & \lambda_{\alpha,\beta+2} & \cdots & \lambda_{\alpha,\delta} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{\eta-1,\beta+1} & \lambda_{\eta-1,\beta+2} & \cdots & \lambda_{\eta-1,\delta}
\end{bmatrix}
\end{array}
\begin{array}{l} \\ (\boldsymbol{\Upsilon}_{\alpha-1}) \\ (\boldsymbol{\Upsilon}_{\alpha}) \\ \vdots \\ \boldsymbol{\Upsilon}_{\eta-1} \end{array}
\tag{19}
$$

Note the matrix rows start from α, given by,

$$
\alpha = 2 + \sum_{k=1}^{N} \frac{N!}{k!(N-k)!} - \left[\sum_{k=N}^{k} \frac{N!}{k!(N-k)!} \right], \tag{20}
$$

where the minimum value of α is two, corresponding to the first possible rework state as shown in Figure 3. The matrix elements are given by,

$$
\lambda_{i,j} = \begin{cases} 0 & \iff {}^{k}c_j \nexists \{\boldsymbol{\Upsilon}_i\}; \\ 1 & \iff {}^{k}c_j \exists \{\boldsymbol{\Upsilon}_i\}, \end{cases} \tag{21}
$$

where i and j refer to row and column number respectively. The $\boldsymbol{\Upsilon}$ elements on the right side of Equation 19 refer to the subsets of $\boldsymbol{\Upsilon}$. For a three feature production system the values $\lambda_{i,j}$ are populated as follows: For $k = 1$, Equation 16 indicates there are three $k = 1$ type combinations. The first $k = 1$ combination gives ${}^{1}c_1 = [x_1]$ and $\boldsymbol{\Upsilon}_1 = [x_1]$ thus Equation 21 indicates $\lambda_{1,1} = 1$. The second element $\lambda_{1,2} = 0$, as ${}^{1}c_2 = [x_2]$ is not part of the set $\boldsymbol{\Upsilon}_1 = [x_1]$ and similarly for ${}^{1}c_3$. All the λ values for the ${}^{1}\boldsymbol{S}$ are displayed in Equation 22 where the c-values along the top row and $\boldsymbol{\Upsilon}$-combinations on the right column are written out.

$$
{}^{1}\boldsymbol{S} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\end{array}
\begin{array}{l} \\ (x_1) \\ (x_2) \\ (x_3) \\ (x_1, x_2) \\ (x_1, x_3) \\ (x_2, x_3) \\ (x_1, x_2, x_3). \end{array}
\tag{22}
$$

An $\boldsymbol{S}$-matrix exists for all k, where the size of the matrix is $(\eta - 1) \times m$ where m is k-dependent.

The $\boldsymbol{S}^k$-matrix for $k = 2$ is given by,

$$ {}^2\boldsymbol{S} = \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} x_1,x_2 & x_1,x_3 & x_2,x_3 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ (x_1, x_2) \\ (x_1, x_3) \\ (x_2, x_3) \\ (x_1, x_2, x_3). \end{array} \tag{23} $$

For $k = 3$ the $\boldsymbol{S}$-matrix is,

$$ {}^3\boldsymbol{S} = 8 \begin{array}{c} x_1,x_2,x_3 \\ \left[\begin{array}{c} 1 \end{array}\right] \end{array} (x_1, x_2, x_3). \tag{24} $$

The $\boldsymbol{S}$-matrix determines where the rework feed-ins come from for each rework state. The probabilities for each of these feed-ins is given by the $\boldsymbol{D}$-matrix which has the general form,

$$ {}^k\boldsymbol{D} = \begin{array}{c} \\ \alpha \\ \alpha{+}1 \\ \vdots \\ \delta{+}1 \\ \delta{+}2 \\ \vdots \\ \eta \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \alpha{+}1 & \cdots & \delta{+}1 \end{array} \\ \left[\begin{array}{cccc} p_{\mathrm{I},\alpha} & 0 & \cdots & 0 \\ 0 & p_{\mathrm{I},\alpha+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{\mathrm{I},\delta+1} \\ \frac{p_{\mathrm{I},\delta+2}p_{\mathrm{I},\alpha}}{1-p_{\mathrm{I},\delta+2}} & \frac{p_{\mathrm{I},\delta+2}p_{\mathrm{I},\alpha+1}}{1-p_{\mathrm{I},\delta+2}} & \cdots & \frac{p_{\mathrm{I},\delta+2}p_{\mathrm{I},\delta+1}}{1-p_{\mathrm{I},\delta+2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_{\mathrm{I},\eta}p_{\mathrm{I},\alpha}}{1-p_{\mathrm{I},\eta}} & \frac{p_{\mathrm{I},\eta}p_{\mathrm{I},\alpha+1}}{1-p_{\mathrm{I},\eta}} & \cdots & \frac{p_{\mathrm{I},\eta}p_{\mathrm{I},\delta+1}}{1-p_{\mathrm{I},\eta}} \end{array}\right] \end{array}. \tag{25} $$

The $\boldsymbol{D}$-matrix is the same size as the $\boldsymbol{S}$-matrix where the top layer of the matrix is a $m \times m$ matrix with only leading diagonal elements. These probabilities relate to the probability of feed-ins to a rework state from the initial manufacturing state. The lower layer of the matrix has size $(\eta - \delta - 1) \times m$, where the probabilities in each element refer to rework feed-ins from other rework states. The $\boldsymbol{D}$-matrices for $k = [1, 2, 3]$, for a three feature system are given below:

$$ {}^1\boldsymbol{D} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccc} p_{\mathrm{I},2} & 0 & 0 \\ 0 & p_{\mathrm{I},3} & 0 \\ 0 & 0 & p_{\mathrm{I},4} \\ \frac{p_{\mathrm{I},5}p_{\mathrm{I},2}}{1-p_{\mathrm{I},5}} & \frac{p_{\mathrm{I},5}p_{\mathrm{I},3}}{1-p_{\mathrm{I},5}} & \frac{p_{\mathrm{I},5}p_{\mathrm{I},4}}{1-p_{\mathrm{I},5}} \\ \frac{p_{\mathrm{I},6}p_{\mathrm{I},2}}{1-p_{\mathrm{I},6}} & \frac{p_{\mathrm{I},6}p_{\mathrm{I},3}}{1-p_{\mathrm{I},6}} & \frac{p_{\mathrm{I},6}p_{\mathrm{I},4}}{1-p_{\mathrm{I},6}} \\ \frac{p_{\mathrm{I},7}p_{\mathrm{I},2}}{1-p_{\mathrm{I},7}} & \frac{p_{\mathrm{I},7}p_{\mathrm{I},3}}{1-p_{\mathrm{I},7}} & \frac{p_{\mathrm{I},7}p_{\mathrm{I},4}}{1-p_{\mathrm{I},7}} \\ \frac{p_{\mathrm{I},8}p_{\mathrm{I},2}}{1-p_{\mathrm{I},8}} & \frac{p_{\mathrm{I},8}p_{\mathrm{I},3}}{1-p_{\mathrm{I},8}} & \frac{p_{\mathrm{I},8}p_{\mathrm{I},4}}{1-p_{\mathrm{I},8}} \end{array}\right] \end{array}, $$

$$
{}^{2}\boldsymbol{D} = \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc} p_{\mathrm{I},5} & 0 & 0 \\ 0 & p_{\mathrm{I},6} & 0 \\ 0 & 0 & p_{\mathrm{I},7} \\ \frac{p_{\mathrm{I},8}p_{\mathrm{I},5}}{1-p_{\mathrm{I},8}} & \frac{p_{\mathrm{I},8}p_{\mathrm{I},6}}{1-p_{\mathrm{I},8}} & \frac{p_{\mathrm{I},8}p_{\mathrm{I},7}}{1-p_{\mathrm{I},8}} \end{array} \right] \end{array},
$$

$$
{}^{3}\boldsymbol{D} = 8 \begin{array}{c} 8 \\ \left[p_{\mathrm{I},8} \right] \end{array}.
$$

The product of $\boldsymbol{D}$-matrix columns and $\boldsymbol{S}$-matrix columns determines the probability of components entering the transient rework states from other rework states and the initial manufacturing state. The expected profit for multiple features from a single-stage can be written as,

$$
E(PR) = SP\, f_{\mathrm{I},C} - PC - Sr - Rw, \tag{26}
$$

where,

$$
Sr = SC_1\, p_{\mathrm{I},S} + \sum_{k=1}^{N} \sum_{j=\alpha}^{\delta+1} \left(SC_j \sum \left[{}^{k}\boldsymbol{S}_{\forall\,[\alpha,\eta],j} \; {}^{k}\boldsymbol{D}_{\forall\,[\alpha,\eta],j} \, \frac{p_{j,S}}{1-p_{j,j}} \right] \right), \tag{27}
$$

$$
Rw = \sum_{j=1}^{\eta} RC_j\, m_{1,j}.
$$

For the scrap (Sr) term in Equation 27, the column numbering of the $\boldsymbol{S}$-matrices is the same as the $\boldsymbol{D}$-matrices. The fraction term (last term) of the scrap equation in Equation 27 determines the final probability of items in the transient states reaching the scrap state. Thus, it follows the scrap cost from the transient states is the total number of items that can feed into the transient states (given by the combination of the $\boldsymbol{S}$ and $\boldsymbol{D}$ matrices), multiplied by the final probability items transfer to the scrap state (given by the fraction term) and the cost of scrap at each transient state (SC_j). As there are no feed-ins to the initial state (I), the scrap cost from the initial state is simply the cost of scrap multiplied by the probability of scrap from the initial state ($p_{\mathrm{I},S}$) .

3.5. Transition probabilities

The transition matrix ($\boldsymbol{P}$) is required to compute $f_{1,C}$ and the $m_{1,j}$ values in Equations 26 and 27. A generalised $\boldsymbol{P}$-matrix is presented which can be determined from the $\boldsymbol{S}$-matrix. Computationally it is easy to generate the k-type combinations to find $\boldsymbol{\Upsilon}$ using a combination package such as '*combinator*' for Matlab. Thus, the $\boldsymbol{S}$-matrix is readily computable computationally and the $\boldsymbol{P}$-matrix can be found without further logical operations. The general form of the $\boldsymbol{P}$-matrix is given by Equation 28. The $\boldsymbol{\Gamma}$ terms are k dependent such that,

$$
{}^{k}\boldsymbol{\Gamma} = \sum {}^{k}\boldsymbol{S}_{\forall\,[\alpha,\eta],j} \; {}^{k}\boldsymbol{J}_{\forall\,[\alpha,\eta],j} \tag{29}
$$

$$\boldsymbol{P} = \begin{array}{c|cccccccccc|} & \mathrm{I} & 2 & 3 & \cdots & \cdots & \cdots & \cdots & \eta & C & S \\ \hline \mathrm{I} & 0 & p_{\mathrm{I},2} & p_{\mathrm{I},3} & \cdots & \cdots & \cdots & \cdots & p_{\mathrm{I},\eta} & p_{\mathrm{I},C} & p_{\mathrm{I},S} \\ 2 & 0 & \multicolumn{3}{c}{} & \cdots & \cdots & \cdots & 0 & p_{2,C} & p_{2,S} \\ \vdots & \vdots & \multicolumn{3}{c}{} & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \multicolumn{3}{c}{{}^{1}\boldsymbol{\Gamma}} & & & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \multicolumn{3}{c}{} & \multicolumn{3}{c}{{}^{2}\boldsymbol{\Gamma}} & \vdots & \vdots & \vdots \\ \eta & 0 & \multicolumn{3}{c}{} & \multicolumn{3}{c}{} & {}^{k}\boldsymbol{\Gamma} & p_{\eta,C} & p_{\eta,S} \\ C & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ S & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ \hline \end{array}. \tag{28}$$

where j is the column number. The ${}^{k}\boldsymbol{J}$ terms are given by,

$${}^{k}\boldsymbol{J} = \begin{array}{c} \\ \alpha \\ \alpha{+}1 \\ \vdots \\ \eta \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \alpha{+}1 & \cdots & \delta{+}1 \end{array} \\ \begin{bmatrix} p_{\alpha,\alpha} & p_{\alpha,\alpha+1} & \cdots & p_{\alpha,\delta+1} \\ p_{\alpha+1,\alpha} & p_{\alpha+1,\alpha+1} & \cdots & p_{\alpha+1,\delta+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\eta,\alpha} & p_{\eta,\alpha+1} & \cdots & p_{\eta,\delta+1} \end{bmatrix} \end{array}, \tag{30}$$

where both the subscripts α and δ are k-dependent. For a three feature example, the following $\boldsymbol{J}$-matrices are obtained:

$${}^{1,2}\boldsymbol{J} = \begin{array}{c} \\ 2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,2} & p_{3,3} & p_{3,4} \\ p_{4,2} & p_{4,3} & p_{4,4} \\ p_{5,2} & p_{5,3} & p_{5,4} \\ p_{6,2} & p_{6,3} & p_{6,4} \\ p_{7,2} & p_{7,3} & p_{7,4} \\ p_{8,2} & p_{8,3} & p_{8,4} \end{bmatrix} \end{array}, \quad \begin{array}{c} \\ 5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} p_{5,5} & p_{5,6} & p_{5,7} \\ p_{6,5} & p_{6,6} & p_{6,7} \\ p_{7,5} & p_{7,6} & p_{7,7} \\ p_{8,5} & p_{8,6} & p_{8,7} \end{bmatrix} \end{array},$$

and

$${}^{3}\boldsymbol{J} = \begin{array}{c} \\ 8 \end{array} \begin{array}{c} 8 \\ \begin{bmatrix} p_{8,8} \end{bmatrix} \end{array}.$$

Multiplying each column of ${}^{k}\boldsymbol{J}$ by each column of ${}^{k}\boldsymbol{S}$ gives the ${}^{k}\boldsymbol{\Gamma}$ entries of the transition matrix (Equation 28). For a three feature example the transition matrix is given in Equation 31. Khasawneh et al. (2008) detail the mechanism to translate the $\boldsymbol{P}$-matrix into the $\boldsymbol{M}$ and $\boldsymbol{F}$ matrices necessary for the $f_{1,C}$ and $m_{1,j}$ values (Equations 26 and 27).

3.6. General solution for n-stage serial and parallel production

A multi-stage production system may involve several parallel processes following serially from one another, or parallel processes mixed with serial processes. Multi-stage serial production systems

$$\boldsymbol{P} = \begin{array}{c} \\ \text{I} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ C \\ S \end{array} \begin{array}{c} \begin{array}{cccccccccc} \text{I} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & C & S \end{array} \\ \left[\begin{array}{cccccccccc} 0 & p_{\text{I},2} & p_{\text{I},3} & p_{\text{I},4} & p_{\text{I},5} & p_{\text{I},6} & p_{\text{I},7} & p_{\text{I},8} & p_{\text{I},C} & p_{\text{I},S} \\ 0 & p_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & p_{2,C} & p_{2,S} \\ 0 & 0 & p_{3,3} & 0 & 0 & 0 & 0 & 0 & p_{3,C} & p_{3,S} \\ 0 & 0 & 0 & p_{4,4} & 0 & 0 & 0 & 0 & p_{4,C} & p_{4,S} \\ 0 & p_{5,2} & p_{5,3} & 0 & p_{5,5} & 0 & 0 & 0 & p_{5,C} & p_{5,S} \\ 0 & p_{6,2} & 0 & p_{6,4} & 0 & p_{6,6} & 0 & 0 & p_{6,C} & p_{6,S} \\ 0 & 0 & p_{7,3} & p_{7,4} & 0 & 0 & p_{7,7} & 0 & p_{7,C} & p_{7,S} \\ 0 & p_{8,2} & p_{8,3} & p_{8,4} & p_{8,5} & p_{8,6} & p_{8,7} & p_{8,8} & p_{8,C} & p_{8,S} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad (31)$$

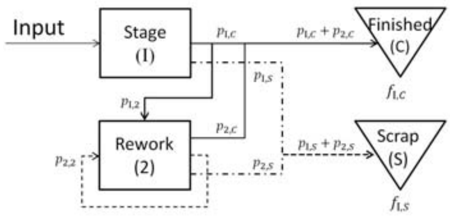


Figure 4: Single-stage process

were discussed by Bowling et al. (2004) and Selim & Al-Zubi (2011) but could not deal with parallel production. A small but significant change is made in this section, such that the formulation of the serial transition matrix and subsequent formulations of the $\boldsymbol{M}$ and $\boldsymbol{F}$ matrices are consistent with the methodology for parallel systems discussed in the previous section. Figure 4 shows an initial manufacturing stage (I), from which scrap, rework and conforming items are generated. A hypothetical distinction is made between the rework state (2) and initial manufacturing stage (I) such that rework is a separate operation to the initial manufacturing state. This is simply a mathematical convenience and does not imply a practical manufacturing operation must follow this segregation. It is conceptually different from past literature where rework simply fed back into the initial manufacturing stage. Figure 4 indicates the feed-ins and feed-outs of each state. The transition matrix for this process is,

$$\boldsymbol{P} = \begin{array}{c} \\ \text{I} \\ 2 \\ C \\ S \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & 2 & C & S \end{array} \\ \left[\begin{array}{cccc} 0 & p_{\text{I},2_\text{I}} & p_{\text{I},C} & p_{\text{I},S} \\ 0 & p_{2_\text{I},2_\text{I}} & p_{2_\text{I},C} & p_{2_\text{I},S} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}, \qquad (32)$$

where 2_I indicates stage one rework (state 2). For a single rework state process, the probabilities of conformance, rework and scrap are identical to the initial probabilities of conformance, rework and scrap. Therefore, the following simplifying conditions are met,

$$p_{\text{I},2_\text{I}} = p_{2_\text{I},2_\text{I}}, \quad \text{and} \quad p_{\text{I},\text{C}} = p_{2_\text{I},\text{C}}. \qquad (33)$$

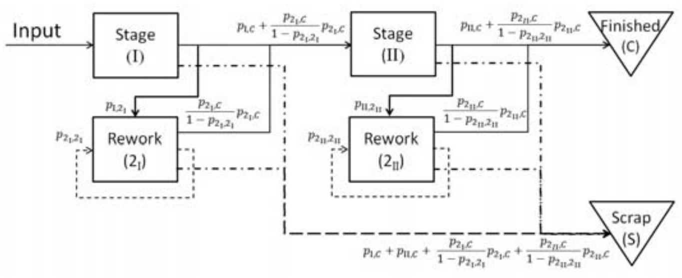


Figure 5: A two-stage serial production process

Despite the re-formulation of the transition matrix, the profit equation can be written,

$$\begin{aligned} E(PR) =& SP\,\frac{p_{\mathrm{I},C}}{1-p_{\mathrm{I},2_\mathrm{I}}} - PC_1 - SC_1\frac{p_{\mathrm{I},S}}{1-p_{\mathrm{I},2_\mathrm{I}}} \\ & - RC_1\,\frac{p_{\mathrm{I},2_\mathrm{I}}}{1-p_{\mathrm{I},2_\mathrm{I}}}, \end{aligned} \tag{34}$$

where greater detail regarding this derivation is given in Appendix A. This is the same form as Equation 2 from Bowling et al. (2004) or generated by Equation 2 from Selim & Al-Zubi (2011).

The format shown in Figure 4 can easily be extended to a multi-stage process. A two-stage system is illustrated by Figure 5 indicating the feed-in and feed-out probabilities. The transition matrices for stages I and II are given by Equation 35,

$$\boldsymbol{P}_\mathrm{I} = \begin{array}{c} \\ \mathrm{I} \\ 2 \\ C \\ S \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{I} & 2 & C & S \end{array} \\ \begin{bmatrix} 0 & p_{\mathrm{I},2_\mathrm{I}} & p_{\mathrm{I},C} & p_{\mathrm{I},S} \\ 0 & p_{2_\mathrm{I},2_\mathrm{I}} & p_{2_\mathrm{I},C} & p_{2_\mathrm{I},S} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad \boldsymbol{P}_\mathrm{II} = \begin{array}{c} \\ \mathrm{I} \\ 2 \\ C \\ S \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{I} & 2 & C & S \end{array} \\ \begin{bmatrix} 0 & p_{\mathrm{II},2_\mathrm{II}} & p_{\mathrm{II},C} & p_{\mathrm{II},S} \\ 0 & p_{2_\mathrm{II},2_\mathrm{II}} & p_{2_\mathrm{II},C} & p_{2_\mathrm{II},S} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{35}$$

The value 2_I indicates the rework state from the first stage and 2_II is the rework state from the second stage. Additionally to Equation 33, the probabilities of conformance, scrap and rework in the rework state (2) for the second stage are identical to the initial stage probabilities such that,

$$p_{\mathrm{II},2_\mathrm{II}} = p_{2_\mathrm{I},2_\mathrm{II}} \quad \text{and} \quad p_{\mathrm{II},\mathrm{C}} = p_{2_\mathrm{II},\mathrm{C}}. \tag{36}$$

The expected profit can be written,

$$\begin{aligned} E(PR) =& SP\, f_{\mathrm{I},C} f_{\mathrm{II},C} - (PC_\mathrm{I} + PC_\mathrm{II}\, f_{\mathrm{I},C}) \\ & - \left[SC_\mathrm{I}\left(\frac{p_{\mathrm{I},S}}{1-p_{2_\mathrm{I},2_\mathrm{I}}}\right) + SC_\mathrm{II}\left(\frac{p_{\mathrm{II},S}}{1-p_{\mathrm{II},2_\mathrm{II}}}\right) f_{\mathrm{I},C}\right] \\ & - RC_{2_\mathrm{I}}\, m_{\mathrm{I},2_\mathrm{I}} - RC_{2_\mathrm{II}}\, m_{\mathrm{II},2_\mathrm{II}}\, f_{\mathrm{I},C}, \end{aligned} \tag{37}$$

where extra detail regarding this derivation is given in Appendix B.

Notice that the second stage scrap and rework terms are multiplied by the final conformance probability from the previous stage, $f_{\mathrm{I},C} = p_{\mathrm{I},C}/(1-p_{\mathrm{I},2_\mathrm{I}})$. This allows each stage to be modelled as a single-stage process using the same form as the single stage equations (Equation 32 and the $\boldsymbol{F}$ and $\boldsymbol{M}$ matrices in Equation A.1). In essence, the scrap, rework and conformance probabilities from each stage are multiplied by the final conformance from the previous stage. Thus, the expected profit for any number of features in any combinations of serial and/or parallel operations is,

$$\begin{aligned} E(PR) = & SP \prod_{i=\mathrm{I}}^{W} f_{i,C} \\ & - \sum_{i=\mathrm{I}}^{W} \{(PC_i + Sr_i + Rw_i)\} \prod_{l=\mathrm{I}}^{l-\mathrm{I}} f_{l,C}. \end{aligned} \tag{38}$$

The value W is the total number of stages and the Sr_i and Rw_i terms are given from Equation 38. The $f_{\mathrm{I},C}$ and $f_{l,C}$ terms account for the conformance from the previous stage (for multi-stage manufacturing sequences).

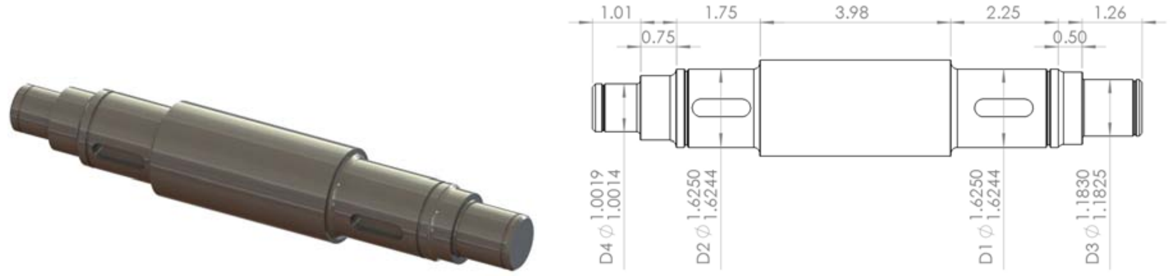
4. Numerical Example

The practical benefit of Equation 38 is it's ability to be matched to a given manufacturing sequence with any number of serial and parallel combinations. This has applications beyond simply representing a single method of manufacture. For example, the number of inspection operations in a manufacturing sequence may be optimised by considering all possible permutations of serial and parallel operations to produce several features. Parallel operations combine the production of at least two features, thus removing at least one inspection stage. This may have a financial benefit but also comes with a potentially higher risk of scrap. Such a principle is analogous to a study by Mittal & McNally (1994) (with a numerical example in Marsh et al. (2010)), where the number of inspection stages were optimised to maximise profit for semiconductor manufacturing.

A case study is given here, where the objective was to maximise the expected profit for the production of an intermediate shaft from a speed reducer gearbox (Figure 6(a)). The shaft design was taken from Section 18 in Shigley et al. (2011); the geometry and tolerances of the shaft are critical for the shaft performance and are justified within Shigley's text. The shaft was considered to be manufactured from two inch AISI 1050 medium carbon steel bar. Optimal mean setting only has a benefit where the variation of the manufacturing process is greater than the tolerances. To establish which dimensions were likely to fall into this category, CAPRA's TCE tool (Tolerance Capability Expert) was used to investigate the capability of the manufacturing processes (turning) against each dimension for the shaft given by Figure 18-3 in Shigley et al. (2011) (reproduced in Figure 6). In TCE, a manufacturing process can be selected as well as the associated dimension and tolerance for the feature. There are additional parameters relating to the machinability of the selected material and geometry as well as other factors that may increase manufacturing variation (Figure 7). In this study the 'Material Effect' automatically increased to 1.4 due to the hardness of the steel. The predicted C_{pk}[2] and number of defects per million parts were forecast from the capability map (top right of Figure 7). The C_{pk} values for four of the shaft's diameters, D1, D2, D3 and D4 (Figure 6(b)) were 0.33, 0.33, 0.27 and 0.32 respectively. Therefore, these features

[2] C_{pk} is a measure of process capability defined as $C_{\mathrm{pk}} = \min[U - \mu, \mu - L]/3\sigma$. Where $C_{\mathrm{pk}} < 1$ scrap and/or rework are produced.

would systematically create scrap and rework as the predicted standard deviation of the turning process was larger than the tolerances.



(a) Intermediate speed reducer shaft

(b) Intermediate speed reducer shaft drafting drawing (all dimensions in inches)

Figure 6: Shigley's speed reducer shaft from Figure 18-3 in Shigley et al. (2011)

The possible number of manufacturing sequences (combinations of serial and parallel operations) and orders in which the diameters (D1 to D4) could be produced were limited by the following assumptions.

1. The larger diameters were turned before the smaller diameters. Therefore, D1 was turned before D3 and D4, D2 was turned before D3 and D4, lastly D3 was turned before D4.
2. Each manufacturing operation added value to the component. The more time consuming the operation, the greater the added value. Therefore, the longer axial diameters were cut before the shorter axial diameters to minimise the scrap cost if a feature failed to conform later in the manufacturing sequence. This implied D1 (2.25 inches) was cut before D2 (1.75 inches).

The combination of assumptions 1 and 2 limited the method of manufacture to eight possible sequences starting with D1 and progressing in numerical order[3]. The eight sequences are conceptualised in Figure 8, an inspection process occurs after each stage (after each block or column of blocks) and applies just to the feature or features in that block or column. Therefore, Sequence 1 has four inspection processes, Sequence 2 has three inspection processes and Sequence 8 has just one inspection process, inspecting all four features. Table 1 gives the costs, normalised process variations (σ') and normalised upper and lower specification limits (U' and L') used by the numerical example[4]. The selling price was 200 currency units for each item with an initial material cost of 50 units. The processing costs were arbitrarily chosen and rework costs were half of the processing costs. The scrap costs were defined from the sum of the material cost (50 units) and the cumulative total of processing costs. The three bullet points below further describe how the costs were applied to the sequences shown in Figure 8.

- In a parallel process, if any one feature was scrap, the sum of all the feature scrap costs in that parallel process were taken. For example, if feature D2 was scrap from Sequence 3

[3]Only turning is considered for the production of the four diameters, which TCE predicts has a variation greater than the tolerance limits. A grinding operation could also be used for finishing but was not considered here. An additional operation would also increase cost and TCE predicts the grinding to have a $C_{pk} > 1$ only under the most favourable conditions.

[4]The values were normalised by the standard deviations predicted from TCE, with the form $U' = \frac{U-\mu}{\sigma}$ and similarly for the lower specification limit (L).

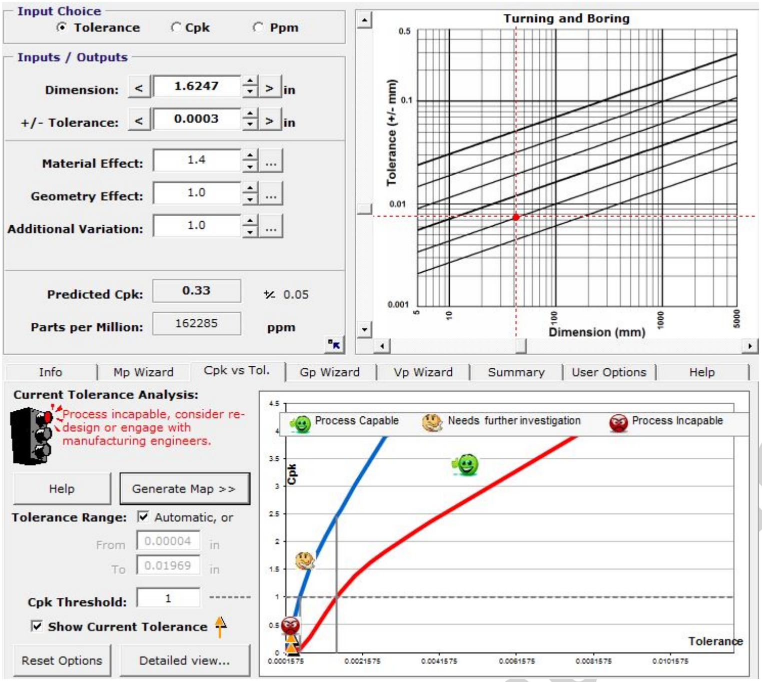


Figure 7: Screen shot of TCE showing the capability of the turning operation for diameter D1 (Figure 6(b))

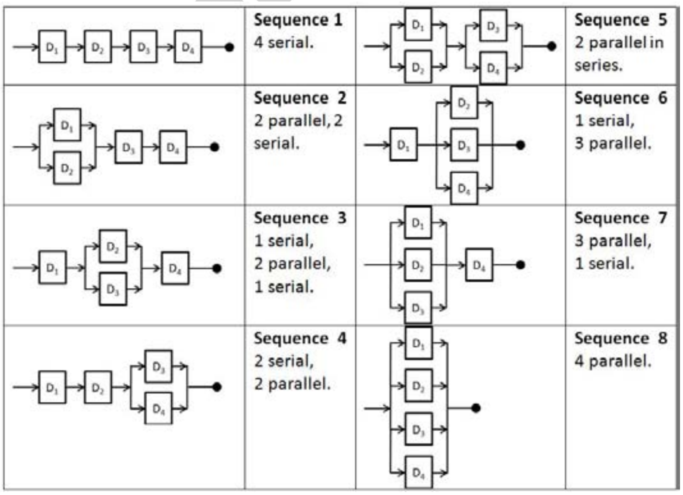


Figure 8: Possible manufacturing sequences for four features

	Feature			
Parameter	**D1**	**D2**	**D3**	**D4**
Processing cost (PC)	22.5	17.5	12.5	10
Rework cost (RC)	11.25	8.75	6.25	5
Scrap cost (SC)	72.5	90	102.5	112.5
L'	-0.99	-0.99	-0.81	-0.96
U'	0.99	0.99	0.81	0.96
σ'	1	1	1	1

Table 1: Cost and process data for the four features

(Figure 8), the scrap cost was calculated as $72.5 + 17.5 + 12.5 = 102.5$. Even though D1 and D3 may have conformed, the item was still scrap (due to D2). Therefore, it incurred the D3 scrap cost (as D3 was still manufactured).

- The processing cost for a parallel operation was the sum of all the feature processing costs in that parallel operation. Therefore, the processing cost for the second stage of Sequence 3 was, $17.5 + 12.5 = 30$ (the D2 and D3 processing costs).
- Rework created from parallel operations could apply to single or multiple features. For the parallel operation (stage 2) in Sequence 3, the rework operations were either single feature rework on D2 or D3, or dual feature rework on both D2 and D3. The rework costs were 8.75, 6.25 and 15, respectively.

4.1. Results and Discussion

Equation 38 was used to determine the expression for expected profit for each of the eight sequences shown in Figure 8. Three sets of simulations were run for each sequence, corresponding to negative, neutral and positive correlation. Correlation was defined through the correlation coefficient (ρ, where $-1 \geq \rho \geq 1$) to multiply the off-diagonal elements of the covariance matrix (Σ), given in Section 3.2. The mean values for each feature were then optimised to maximise profit using Matlab's *fmincon* function (Mathworks (2012)). Results were obtained for the expected profit, $E(PR)$, and the expected profit minus the inspection cost, $E(PR) - I$ (Figure 9). The $E(PR)$ value is the direct result from Equation 38, while $E(PR) - I$ is the profit when accounting for the cost of an inspection station, I. The presence of an inspection operation incurred a cost of 2 units with a further 0.5 units for each additional feature inspected. The inspection costs for each sequence are tabulated in Table 2. The variation of the inspection costs was designed to represent the likelihood that it would cost less to inspect several features together, than to have a dedicated inspection stop for each feature before manufacturing the next feature.

Figure 9 shows Sequence 1 was the most profitable method of manufacture and Sequence 8 the least profitable, where $\rho = 0$. Sequence 1 was more profitable as the total scrap cost was minimised since the failure of any feature to conform only resulted in the scrap cost up to that point. The failure of any feature to conform in Sequence 8 resulted in the scrap cost for all four features, irrespective of which one failed. Accounting for inspection cost ($E(PR) - I$), Sequence 2 was the least profitable and Sequence 6 was the most profitable. Reducing the number of inspection operations was clearly beneficial, however, not worth the extra scrap cost risk by producing all four features in parallel (Sequence 8).

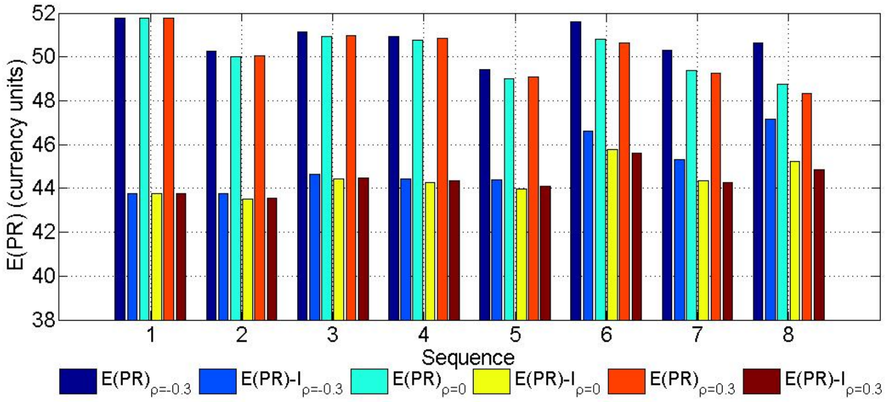


Figure 9: Expected profit for the eight manufacturing sequences with $\rho = -0.3$, $\rho = 0$ and $\rho = 0.3$

Correlation did not affect the expected profits for Sequence 1 as each feature was manufactured independently. Sequence 1 was also the most profitable for $E(PR)$ for both correlated cases. When inspection cost $(E(PR) - I)$ was accounted for, Sequence 6 was the most profitable for positive correlation ($\rho = 0.3$), as it was for $\rho = 0$. However, for negative correlation ($\rho = -0.3$), Sequence 8 yielded the greatest profit. It is also apparent negative correlation generally produced higher profits than for $\rho = 0.3$ or $\rho = 0$. This is explained by considering how the distribution of features lie on the conformance, rework and scrap regions shown by Figure 10(a). The black points indicate positive correlation between two arbitrary features and grey points show negative correlation. The grey points may be moved out of the expensive scrap regions to produce single feature rework by increasing the mean value of the two features (X_1 and X_2). However, doing the same with the black points would increase dual feature rework (R_{X_1,X_2}). As dual feature rework is more costly than single feature rework and can also generate additional single feature rework, increasing the mean values for the positive correlated points increases rework cost faster than increasing the mean values of the grey points. Thus, the mean values of the negatively correlated features could be increased to a greater extent (to reduce the probability of scrap) than the positively correlated points, confirmed by Table 2. This led to a lower overall rework and scrap cost and hence a higher profit for the negatively correlated features[5].

A consequence of using optimal mean setting to maximise profit is the modification of the product's feature means and thus the mean dimensions of the product. Figure 10(b) illustrates the normalised optimal mean values necessary to maximise profitability for each manufacturing sequence; all are greater than the nominal (zero). This information may have a design impact, for instance although sequence 6 was the most profitable (for $E(PR)-I$), sequence 1 may be preferred from a design perspective as it minimises the two largest diameters, reducing the mass of the shaft (although it was the second least profitable sequence). Sequence 8 may be a suitable trade-off between weight and profitability[6]. Such trade-offs between the most economical manufacturing route, shape of the manufactured geometry and the performance impact this may have, are the subject of future research.

[5]Although only two features are shown in Figure 10(a), the principle is applicable to three and four features.

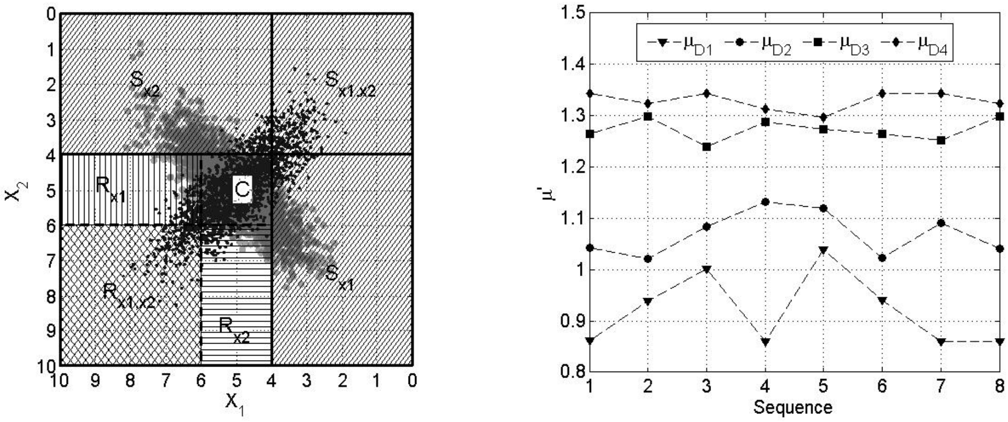
[6]The absolute shift in the means is small, the largest shift for optimal mean for the D4 feature is only 0.00035 inches greater than the original nominal.

Seq.	ρ	**Optimal Means**	$E(PR)$	I	$E(PR)-I$	**Rank** $E(PR)$	**Rank** $E(PR)-I$
1	0	[0.8620, 1.0420, 1.2648, 1.3427]	51.78	8	43.78	1	7
-	-0.3	[0.9388, 1.0218, 1.2984, 1.3244]	51.78	8	43.78	1_n	7_n
-	0.3	[0.8620, 1.0420, 1.2648, 1.3427]	51.78	8	43.78	1_p	7_p
2	0	[0.9388, 1.0218, 1.2984, 1.3244]	50.00	8	43.50	5	8
-	-0.3	[0.9366, 1.0184, 1.2967, 1.3224]	50.28	6.5	43.78	7_n	8_n
-	0.3	[0.9297, 1.0111, 1.2933, 1.3159]	50.04	6.5	43.54	5_p	8_p
3	0	[1.0019, 1.0836, 1.2389, 1.3427]	50.93	6.5	44.43	2	3
-	-0.3	[1.0058, 1.0868, 1.2363, 1.3427]	51.14	6.5	44.64	3_n	4_n
-	0.3	[0.9815, 1.0615, 1.2234, 1.3427]	50.97	6.5	44.47	2_p	3_p
4	0	[0.8599, 1.1315, 1.2887, 1.3137]	50.78	6.5	44.28	4	5
-	-0.3	[0.8617, 1.1352, 1.2864, 1.3153]	50.92	6.5	44.42	4_n	5_n
-	0.3	[0.8595, 1.1138, 1.2772, 1.2949]	50.83	6.5	44.33	3_p	4_p
5	0	[1.0381, 1.1187, 1.2744, 1.2961]	48.99	5	43.99	7	6
-	-0.3	[1.0334, 1.1160, 1.2705, 1.2853]	49.41	5	44.41	8_n	6_n
-	0.3	[0.9985, 1.1111, 1.2608, 1.2681]	49.08	5	44.08	7_p	6_p
6	0	[0.9406, 1.0235, 1.2648, 1.3427]	50.08	5	45.80	3	1
-	-0.3	[0.9381, 1.0199, 1.2648, 1.3427]	51.61	5	46.61	2_n	2_n
-	0.3	[0.9314, 1.0128, 1.2648, 1.3427]	50.62	5	45.62	4_p	1_p
7	0	[0.8602, 1.0916, 1.2517, 1.3427]	49.37	5	44.37	6	4
-	-0.3	[0.8606, 1.0901, 1.2486, 1.3427]	50.30	5	45.30	6_n	3_n
-	0.3	[0.8603, 1.0827, 1.2447, 1.3427]	49.26	5	44.26	6_p	5_p
8	0	[0.8598, 1.0403, 1.2983, 1.3244]	48.74	3.5	45.24	8	2
-	-0.3	[0.8601, 1.0405, 1.2967, 1.3224]	50.64	3.5	47.14	5_n	1_n
-	0.3	[0.8599, 1.0404, 1.2933, 1.3159]	48.36	3.5	44.86	8_p	2_p

Table 2: Expected profit and means for the eight manufacturing sequences. The Rank columns indicate the most profitable sequences for the $E(PR)$ and $E(PR) - I$ results respectively. The n and p subscripts indicate negative and positive correlation, respectively.

5. Conclusion

A review of the literature examining ways of optimising expected profit in serial and parallel manufacturing processes led to the discovery of an error in previous literature (Khasawneh et al. (2008)). The method described in Khasawneh et al. (2008) neglected to account for the flow of features in and out of the rework states. An explanation of these feed-in and feed-out terms and a correct expression for expected profit was given in Section 3.3. Previous literature had only considered two features manufactured in parallel, and as the number of features produced in parallel increased, the number of feed-in and feed-out terms grew exponentially. Determining these terms and which rework state they applied to became impractical as the number of features increased. Therefore, a generalised expression was derived to directly determine the expected profit for n-features manufactured in parallel without having to derive the expression from first principles. A further development was made to allow the expected profit to be determined for any



(a) Illustration of positive and negative correlation. R_{X_1}, R_{X_2}, R_{X_1,X_2}, S_{X_1}, S_{X_2} and S_{X_1,X_2} are the single feature and dual feature rework probabilities and scrap probabilities, respectively.

(b) Variation of each optimised normalised mean value with manufacturing sequence, where $\rho = 0$.

Figure 10: Illustration of correlation and the variation of the normalised mean values with manufacturing sequence

combination of serial and parallel operations, making the method generally applicable. A numerical demonstration on the practicality of this new generalised expression (Equation 38) was given in Section 4, where the optimal means and optimal sequence of serial and parallel operations were found for the manufacture of a gearbox shaft. The ability to numerically compute the optimal manufacturing sequence (using Equation 38) also opens up new ways to explore optimal value design solutions where the most profitable manufacturing sequences can be determined and the effect on the product's geometry evaluated.

Appendix A. Reformulation of a single stage production system

Following the procedure detailed in Bowling et al. (2004), the $\boldsymbol{F}$ and $\boldsymbol{M}$ matrices generated from the transition matrix given by Equation 32 are,

$$\boldsymbol{M} = \begin{matrix} & \mathrm{I} & 2 \\ \mathrm{I} & 1 & \frac{p_{\mathrm{I},2_\mathrm{I}}}{1-p_{\mathrm{I},2_\mathrm{I}}} \\ 2 & 0 & \frac{1}{1-p_{\mathrm{I},2_\mathrm{I}}} \end{matrix}, \quad \text{and} \quad \boldsymbol{F} = \begin{matrix} & C & S \\ \mathrm{I} & \frac{p_{\mathrm{I},C}}{1-p_{\mathrm{I},2_\mathrm{I}}} & \frac{p_{\mathrm{I},S}}{1-p_{\mathrm{I},2_\mathrm{I}}} \\ 2 & \frac{p_{\mathrm{I},C}}{1-p_{\mathrm{I},2_\mathrm{I}}} & \frac{p_{\mathrm{I},S}}{1-p_{\mathrm{I},2_\mathrm{I}}} \end{matrix}. \tag{A.1}$$

Thus the expected profit for this reformulated single stage, single feature example is,

$$\begin{aligned} E(PR) = &\overbrace{SP\,\frac{p_{\mathrm{I},C}}{1-p_{2_\mathrm{I},2_\mathrm{I}}}}^{\text{Conformance}} - PC_1 - \overbrace{SC_\mathrm{I}p_{\mathrm{I},S}}^{\text{Scrap 1}^{\text{st}}\text{ state}} \\ &- \underbrace{SC_{2_\mathrm{I}}\frac{p_{\mathrm{I},2_\mathrm{I}}\,p_{2_\mathrm{I},S}}{1-p_{2_\mathrm{I},2_\mathrm{I}}}}_{\text{Scrap 2}^{\text{nd}}\text{ state}} - \underbrace{RC_{\mathrm{II}}\,\frac{1}{1-p_{2_\mathrm{I},2_\mathrm{I}}}}_{\text{Rework 2}^{\text{nd}}\text{ state}}. \end{aligned} \tag{A.2}$$

Equation A.2 reduces to Equation 34 by acknowledging the scrap terms can be reduced by writing,

$$C = SC_{\mathrm{I}}\, p_{\mathrm{I},S} + SC_{2_{\mathrm{I}}} \frac{p_{\mathrm{I},2_{\mathrm{I}}} p_{2_{\mathrm{I}},S}}{1 - p_{2_{\mathrm{I}},2_{\mathrm{I}}}}.$$

The cost of scrap is the same from state I and from state 2_{I} hence, $SC_{\mathrm{I}} = SC_{2_{\mathrm{I}}}$. The probability of scrap from state I is the same as the probability of scrap from state 2_{I} hence, $p_{\mathrm{I},S} = p_{2_{\mathrm{I}},S}$. The probability of rework components feeding into state 2_{I} from state I is the same as the probability of rework from state 2_{I} back to state 2_{I} hence, $p_{\mathrm{I},2_{\mathrm{I}}} = p_{2_{\mathrm{I}},2_{\mathrm{I}}}$, therefore,

$$\begin{aligned} C(1 - p_{\mathrm{I},2_{\mathrm{I}}}) =& (1 - p_{\mathrm{I},2_{\mathrm{I}}}) SC_{\mathrm{I}}\, p_{\mathrm{I},S} - SC_{\mathrm{I}}\, p_{\mathrm{I},2_{\mathrm{I}}}\, p_{\mathrm{I},S} \\ C(1 - p_{\mathrm{I},2_{\mathrm{I}}}) =& SC_{\mathrm{I}}\, p_{\mathrm{I},S}(1 - p_{2_{\mathrm{I}},2_{\mathrm{I}}} + p_{\mathrm{I},2_{\mathrm{I}}}) \\ C =& SC_{\mathrm{I}} \frac{p_{\mathrm{I},S}}{1 - p_{\mathrm{I},2_{\mathrm{I}}}}. \end{aligned} \tag{A.3}$$

Substituting C into Equation A.2, gives Equation 34.

Appendix B. Reformulation of a two stage production system

As in Appendix A, the procedure detailed in Bowling et al. (2004) was followed to transform the transition matrices (Equation 35) into the $\boldsymbol{F}$ and $\boldsymbol{M}$ matrices for both stages,

$$\boldsymbol{F}_{\mathrm{I}} = \begin{array}{c} \\ \mathrm{I} \\ 2_{\mathrm{I}} \end{array} \begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} \frac{p_{\mathrm{I},C}}{1-p_{\mathrm{I},2_{\mathrm{I}}}} & \frac{p_{\mathrm{I},S}}{1-p_{\mathrm{I},2_{\mathrm{I}}}} \\ \frac{p_{\mathrm{I},C}}{1-p_{\mathrm{I},2_{\mathrm{I}}}} & \frac{p_{\mathrm{I},S}}{1-p_{\mathrm{I},2_{\mathrm{I}}}} \end{bmatrix} \end{array} \text{for stage I} \quad \boldsymbol{F}_{\mathrm{II}} = \begin{array}{c} \\ \mathrm{II} \\ 2_{\mathrm{II}} \end{array} \begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} \frac{p_{\mathrm{II},C}}{1-p_{\mathrm{II},2_{\mathrm{II}}}} & \frac{p_{\mathrm{II},S}}{1-p_{\mathrm{II},2_{\mathrm{II}}}} \\ \frac{p_{\mathrm{II},C}}{1-p_{\mathrm{II},2_{\mathrm{II}}}} & \frac{p_{\mathrm{II},S}}{1-p_{\mathrm{II},2_{\mathrm{II}}}} \end{bmatrix} \end{array} \text{for stage II} \tag{B.1}$$

and

$$\boldsymbol{M}_{\mathrm{I}} = \begin{array}{c} \\ \mathrm{I} \\ 2_{\mathrm{I}} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{I} & 2_{\mathrm{I}} \end{array} \\ \begin{bmatrix} 1 & \frac{p_{\mathrm{I},2_{\mathrm{I}}}}{1-p_{\mathrm{I},2_{\mathrm{I}}}} \\ 0 & \frac{1}{1-p_{\mathrm{I},2_{\mathrm{I}}}} \end{bmatrix} \end{array} \text{for stage I} \qquad \boldsymbol{M}_{\mathrm{II}} = \begin{array}{c} \\ \mathrm{II} \\ 2_{\mathrm{II}} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{I} & 2_{\mathrm{II}} \end{array} \\ \begin{bmatrix} 1 & \frac{p_{\mathrm{II},2_{\mathrm{II}}}}{1-p_{\mathrm{II},2_{\mathrm{II}}}} \\ 0 & \frac{1}{1-p_{\mathrm{II},2_{\mathrm{II}}}} \end{bmatrix} \end{array} \text{for stage II.} \tag{B.2}$$

Thus expected profit is given by,

$$
\begin{aligned}
E(PR) = \; & \overbrace{SP\, f_{\mathrm{I},C}\, f_{\mathrm{II},C}}^{\text{Final conformance}} - PC_{\mathrm{I}} - \overbrace{SC_{\mathrm{I}}\, p_{\mathrm{I},S}}^{\text{Scrap I 1}^{\text{st}}\text{ state}} - \overbrace{SC_{2_{\mathrm{I}}} \frac{p_{\mathrm{I},2_{\mathrm{I}}}\, p_{2_{\mathrm{I}},S}}{1 - p_{\mathrm{I},2_{\mathrm{I}}}}}^{\text{Scrap I 2}^{\text{nd}}\text{ state}} \\
& - \overbrace{RC_{2_{\mathrm{I}}} \frac{1}{1 - p_{\mathrm{I},2_{\mathrm{I}}}}}^{\text{Rework I 2}^{\text{nd}}\text{ state}} - \Bigg[PC_{\mathrm{II}} - \underbrace{SC_{\mathrm{II}}\, p_{\mathrm{II},S_{\mathrm{I}}}}_{\text{Scrap II 1}^{\text{st}}\text{ state}} \\
& - \underbrace{SC_{2_{\mathrm{II}}} \frac{p_{\mathrm{II},2_{\mathrm{II}}}\, p_{2_{\mathrm{II}},S}}{1 - p_{\mathrm{II},2_{\mathrm{II}}}}}_{\text{Scrap II 2}^{\text{nd}}\text{ state}} - \underbrace{RC_{2_{\mathrm{II}}} \frac{1}{1 - p_{\mathrm{II},2_{\mathrm{II}}}}}_{\text{Rework II 2}^{\text{nd}}\text{ state}} \Bigg] f_{\mathrm{I},C}
\end{aligned} \tag{B.3}
$$

In the same manner to Equation A.3, the scrap terms related to the scrap from stage I are reduced to,

$$SC_{\mathrm{I}} \frac{p_{\mathrm{I},S}}{1 - p_{\mathrm{I},2_{\mathrm{I}}}}.$$

The scrap terms related to scrap from stage II can be written,

$$C = SC_{\mathrm{II}}\, p_{\mathrm{II},S} + SC_{2_{\mathrm{II}}} \frac{p_{\mathrm{II},2_{\mathrm{II}}}\, p_{2_{\mathrm{II}},S}}{1 - p_{2_{\mathrm{II}},2_{\mathrm{II}}}},$$

where the cost of scrap is the same irrespective whether the scrap originates from state II or 2_{II}, $SC_{\mathrm{II}} = SC_{2,\mathrm{II}}$. The probability of scrap from state II is the same as the probability of scrap from state 2_{II} hence, $p_{\mathrm{II},S} = p_{2_{\mathrm{II}},S}$. The probability of rework components feeding into state 2_{II} from state II is the same as the probability of rework from state 2_{II} back to state 2_{II} hence, $p_{\mathrm{I},2_{\mathrm{II}}} = p_{2_{\mathrm{I}},2_{\mathrm{I}}}$, therefore,

$$
\begin{aligned}
C(1 - p_{\mathrm{II},2_{\mathrm{II}}}) =& (1 - p_{2_{\mathrm{II}},2_{\mathrm{II}}})\, SC_{\mathrm{II}}\, p_{\mathrm{II},S} - SC_{\mathrm{II}}\, p_{\mathrm{II},2_{\mathrm{II}}}\, p_{\mathrm{II},S} \\
C(1 - p_{\mathrm{II},2_{\mathrm{I}}}) =& SC_{\mathrm{II}}\, p_{\mathrm{II},S}(1 - p_{2_{\mathrm{II}},2_{\mathrm{II}}} + p_{\mathrm{II},2_{\mathrm{II}}}) \\
C =& SC_{\mathrm{II}} \frac{p_{\mathrm{II},S}}{1 - p_{\mathrm{II},2_{\mathrm{II}}}}.
\end{aligned} \tag{B.4}
$$

Substituting C into Equation B.3 gives Equation 37.

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