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A PERFORMANCE STUDY OF HIGH-ORDER FINITE ELEMENTS AND WAVE-BASED DISCONTINUOUS GALERKIN METHODS FOR A CONVECTED HELMHOLTZ PROBLEM

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The finite element method (FEM) remains one of the most established computational method used in industry to predict acoustic wave propagation. However, the use of standard FEM is in practice limited to low frequencies because it suffers from large dispersion errors when solving short wave problems (also called pollution effect). Various methods have been developed to circumvent this issue and we compare two numerical methods for convected Helmholtz problems. The methods chosen for this study are the polynomial high-order FEM and the wave-based Discontinuous Galerkin Method (DGM). The polynomial method takes advantage of the superior approximation properties of the Lobatto shape functions compared to the conventional Lagrange basis. The wave-based DGM is part of the physics-based methods which include *a priori* knowledge about the local behaviour of the solution into the numerical model. Previous studies have shown that both methods can control of the pollution effect. Common belief is that compared to polynomial methods, physics-based methods can provide a significant improvement in performance, at the expense of a deterioration of the conditioning. However, the results presented in this paper indicate that the differences in accuracy, efficiency and conditioning between the two approaches are more nuanced than generally assumed.

1. Introduction

The finite element method (FEM) is one of the most established computational methods used in industry for the design of acoustic systems, because of its robustness and the ease with which it deals with complex geometries and non-uniform media. However, conventional FEM suffers from large dispersion errors when solving short wave problems. As the frequency is increased, the mesh required to obtain accurate solutions becomes prohibitively fine. To circumvent this issue, a range of numerical models has been developed over the past twenty years. In this paper, we compare the polynomial high-order finite element method (p-FEM) [1] against the wave-based discontinuous Galerkin method [2] for convected Helmholtz problems. Previous studies demonstrate that both methods lead to a control of the pollution effect. But, to the authors' knowledge, p-FEM and wave-based DGM have not been compared and this paper provide quantitative elements of comparison between the two methods.

The polynomial method takes advantage of the superior interpolation properties of some families of functions (here hierarchic Lobatto shape functions) compared to the conventional Lagrange basis. The use of such functions allows a significant reduction in the total number of degrees of freedom required to solve a particular wave problem with a given accuracy [3]. Furthermore, the hierarchic nature of the Lobatto basis leads to effective p-adaptivity schemes.

The wave-based DGM is a physics-based method, such as the partition of unity method [4], the discontinuous enrichment method [5] or the Ultra Weak Variational Formulation (UWVF) [6]. The idea is to include *a priori* knowledge about the local behaviour of the solution into the numerical model (for instance the wavenumber). The wave-based DGM relies on a set of plane waves to interpolate the solution in each element. Matching conditions at the elements interfaces are weakly imposed using numerical fluxes. This method is found to be a generalisation of the UWVF [2] and the Least-squares method [7] and has been identified as a promising way to tackle the pollution effect [8]. Moreover, it easily allows local 'order' refinement which makes it well-suited for p-adaptive strategies [9].

The aim of the present article is to assess and compare the performance of the numerical methods on a simple two-dimensional convected Helmholtz problem. The structure of the paper is as follows. Section 2 describes the convected Helmholtz equation and gives the principal characteristics of the numerical methods. In Section 3, the numerical test case is described followed by the presentation of the measures of accuracy and costs. The anisotropy, convergence and performances to achieve a given accuracy are compared in Section 4.

2. Problem description

In absence of volume source, the time-harmonic propagation of acoustic perturbations p in a uniform base flow with velocity $\mathbf{u}_0$ and a $\mathrm{e}^{\mathrm{i}\omega t}$ time dependence is considered. The problem is described by the convected Helmholtz equation:

$$\frac{1}{c_0^2}\frac{D_0^2 p}{Dt^2} - \nabla^2 p = 0, \tag{1}$$

where $D(.)/Dt = \mathrm{i}\omega(.) + \mathbf{u}_0 \cdot \nabla(.)$ is the material derivative, c_0 is the speed of sound and ω is the angular frequency.

With the p-FEM, the pressure is discretised using hierarchic Lobatto shape functions [1]. As the polynomial order P of the basis is increased, different types of shape functions appear: vertex, edge and bubble functions (and also face functions in 3D). Bubble functions have no connectivity with the neighbouring elements and can therefore be removed from the global system using static condensation. This procedure improves the conditioning and reduces the memory requirements. In this work, we will consider the performance of p-FEM without condensation (referred to as p-FEM A) and p-FEM with condensation (referred to as p-FEM B).

In wave-based DGM, a set of plane waves is used to approximate the solution on each element and continuity between elements is weakly imposed using an upwind flux-vector splitting method. The method used is described in detail in [2]. The acoustic waves can propagate in any direction, a simple way to build the basis is to evenly space the N_w wave directions in the interval $[0, 2\pi]$ [2, 8]. However, in the presence of a mean flow, a uniform distribution of the waves directions in the basis is not optimum. The convective effect leads to a shortening of the wavelength λ when the wave is propagating against the flow direction (upstream direction) and it leads to a stretching of the wavelength when the wave is propagating in the flow direction (downstream direction). When λ is small, waves can propagate over many wavelength within the computational domain Ω which allows large accumulations of the error on the phase speed. A way to bypass this convective effect is to adjust the plane waves to obtain a better discretisation of the upstream direction. The plane waves in the basis have a direction θ and a ray direction α that takes into account the mean flow. If the flow is

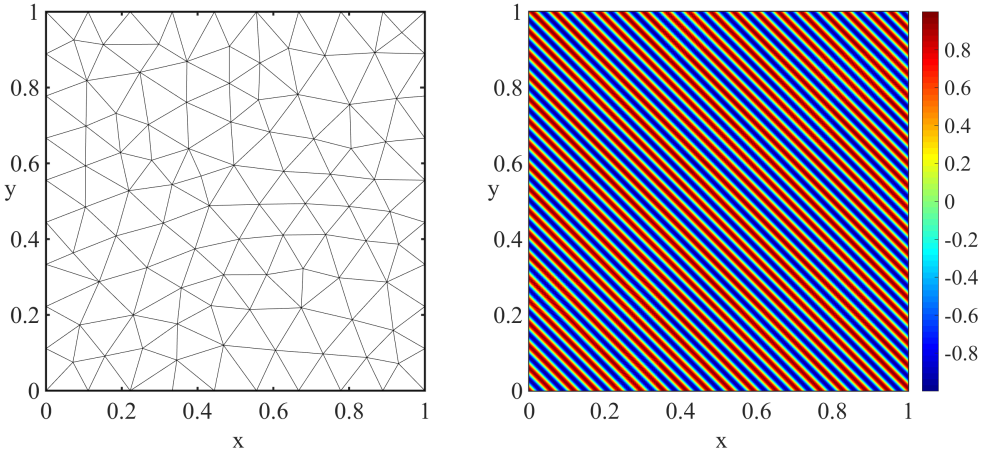

Figure 1: Example of mesh and solution.

propagating in the x direction then from trigonometry, the following expression is derived:

$$\theta = \alpha + \text{asin}(M\text{sin}(\alpha)), \tag{2}$$

where $M = \|\mathbf{u}_0\|/c_0$ is the flow Mach number. Our approach is to evenly distribute the ray directions and infer the wave directions using Eq. 2. The consequence is a clustering of the plane waves in the upstream direction. The method with a uniform distribution of plane wave in the basis is referred to as DGM A whereas the second method with a clustering of the plane waves in the upstream direction is referred to as DGM B.

For this study, the polynomial orders P and numbers of plane waves N_w are taken uniform over the entire mesh.

3. Numerical test case

As a test case, let us consider the propagation of a plane wave in free field. This problem is very similar to a dispersion analysis, and allows a detailed study of the properties of the numerical models such as the anisotropy. This test case has been used in [2, 4, 10] to analyse the properties of wave-based DGM, Partition of Unity method and Discontinuous Enrichment method. The computational domain is a unit square, discretised using an unstructured triangular mesh (Fig. 1) with a typical element size h. As the solution is known outside the numerical domain, a Robin boundary condition for p-FEM [11] and ghost cells for wave-based DGM [2] are used to generate a single acoustic plane wave with a direction θ_0. Outside the domain, the prescribed pressure field p reads:

$$p(\mathbf{x}) = \text{e}^{-\text{i}k\boldsymbol{\theta}_0\cdot\mathbf{x}}, \tag{3}$$

where $\boldsymbol{\theta}_0 = [\cos\theta_0, \sin\theta_0]^\text{T}$ and $k = \omega/(c_0 + u_0\cos(\theta_0))$ is the acoustic wavenumber. For simplicity, the speed of sound, c_0 and the density ρ_0 are set to unity. The mean flow is uniform in the x direction with $M = 0.5$.

The number of elements per wavelength can be used to describe the resolution of the problem but when dealing with high-order methods, it is more relevant to use the number of degrees of freedom per wavelength D_λ. This parameter can be understood as the density of information used to describe one wavelength. We use the same expression as in [2]:

$$D_\lambda = \frac{2\pi}{k}\frac{\sqrt{N_\text{DOF}} - 1}{\sqrt{\text{area}(\Omega)}}, \tag{4}$$

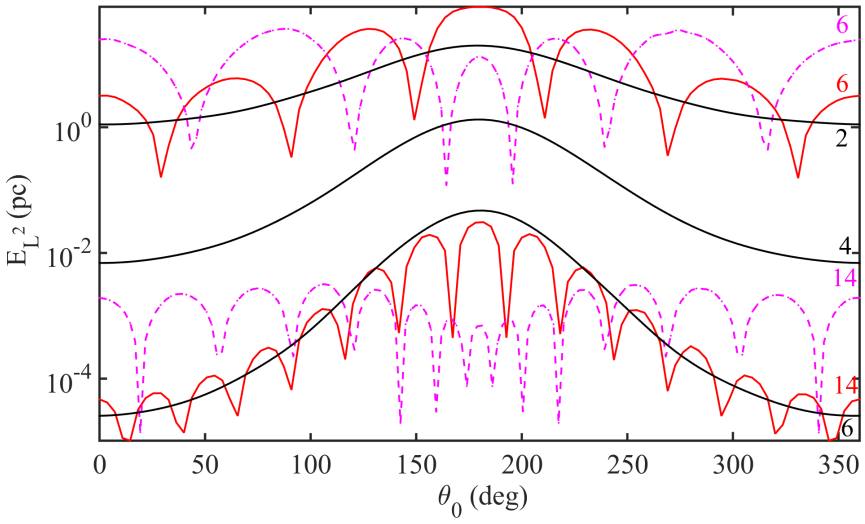

Figure 2: Relative L^2-error against the wave direction for $M = 0.5$, $h = 0.1$ and $\omega = 20$; black lines: p-FEM A/B, red lines: DGM A, magenta dashed lines: DGM B. The numbers of plane wave or polynomial order is shown next to each point.

where N_{DOF} is the total number of degrees of freedom. For p-FEM when condensation is applied, N_{DOF} does not take into account the degrees of freedom corresponding to the bubble functions.

To measure the error induced by the numerical methods, we use the relative L^2-error on the pressure. Both methods transform differential equations into sparse matrix equations $\mathbf{Ax} = \mathbf{f}$. These equations are then solved using an $\mathbf{LU}$ factorisation ($\mathbf{LDL}$ if the matrix is symmetric). The measures of interest are:

- the number of non-zero entries in the sparse $\mathbf{A}$ matrices, NNZ. This is directly proportional to the memory required to store the matrix before factorisation.
- the factorisation memory as reported by the solver MUMPS [14].
- the condition number using the 1-norm.

4. Results

4.1 Anisotropy

We first examine the anisotropy of the models which is the behaviour of the numerical error when the propagating wave direction is changed. For this investigation, we set $h = 0.1$ and $\omega = 20$. The plane wave direction θ_0 is varied from 0 to 2π, for each polynomial order and number of plane waves. The relative L^2-error is measured and plotted against θ_0 in Fig. 2.

As explained previously, the numerical error for p-FEM does not change whether condensation is applied or not. The error obtained with p-FEM varies with the propagation angle. Due to the shortening of the wavelength in the upstream direction, D_λ is smaller when $\theta_0 = 180°$ compared to the case where $\theta_0 = 0°$. Therefore, for a given order, the relative error reaches its minimum (resp. maximum) in the downstream (resp. upstream) direction. For angles close to the downstream case, the errors are decreased by more than two orders of magnitude when the polynomial degree is increased from P to $P + 2$. Close to the upstream case, the benefit of increasing P is reduced but remains significant.

As observed in Fig. 2, having a non-uniform distribution in the basis for wave-based DGM increases the error in the downstream direction but reduces the error due to the convective effect compared to the uniform distribution. Moreover, for a given N_{w}, the error obtained with the physics-based methods (A and B) varies with the angular distance between the propagating plane wave and the plane

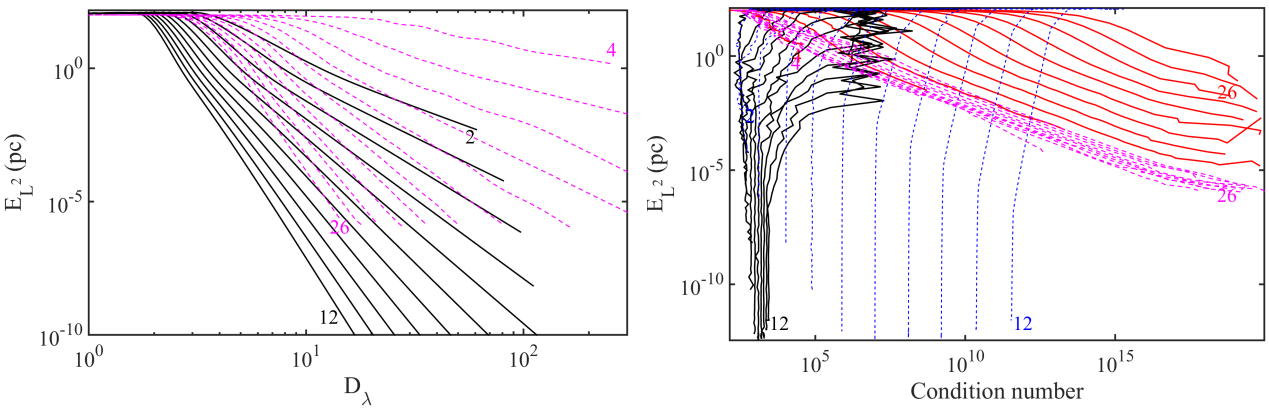

Figure 3: L^2-error against D_λ (left) and the condition number (right) for $h = 0.1$; blue lines: p-FEM A, black lines: p-FEM B, red lines: DGM A, magenta lines: DGM B. The numbers of plane wave or polynomial order is shown next to each point.

waves constituting the basis. When the propagating wave is aligned with a wave of the basis, the error drops to zero as the solution belongs to the basis. Locally, the maximum error corresponds to a plane wave propagating half way in between two waves of the basis.

In the remainder of the paper, for each method, we consider the propagating wave direction leading to the largest error. In realistic cases, the waves are propagating in several directions and the global error is driven by the wave direction resulting in the largest error. For the wave-based methods, this means that the wave propagates half way between two plane waves of the basis in the upstream (resp. downstream) direction for DGM A (resp. B). For p-FEM, the upstream case is considered.

4.2 Convergence and conditioning

In this section, the mesh is fixed and we study the behaviour of the numerical model when the resolution of the problem is changed (Fig. 3). For clarity, the convergence plot (L^2-error against D_λ) only shows the behaviour of p-FEM B and DGM B.

For a given D_λ, when increasing the order or the number of plane waves, the error is significantly reduced: when $D_\lambda = 10$, between order P and $P + 1$ or N_w and $N_\mathrm{w} + 2$, the numerical error is decreased by nearly an order of magnitude. As expected for high-order methods, rather than refining the mesh, it is more efficient to increase the order of the method to improve the error [4, 11]. In the asymptotic region, it is possible to infer that the error for the polynomial basis is decaying like $D_\lambda^{-(P+1)}$ [11] whereas for wave-based basis the error behaves like $D_\lambda^{-[(N_w-1)/2]}$, where $[a]$ stands for the integer part of a. These behaviours are recovered with the A versions of each model.

The condition numbers of p-FEM B linear systems are low compared to the wave-based method. This is due to the application of condensation. We find that for a polynomial order 12, applying condensation leads to a 7 order of magnitude decrease of the condition number which is consistent with the results shown in [12].

As mentioned previously, a common criticism concerning physics-based methods is their poor conditioning. The condition numbers increase rapidly with D_λ and the growth rate becomes larger when the number of plane waves in the basis is increased [2]. However, in Fig. 3 showing the L^2-error as a function of the condition number, we can see that for reasonable levels of accuracy, the conditioning of the wave-based DGM B is not an issue. For instance, to achieve 1% accuracy, the condition numbers of the two methods are comparable. In contrast with DGM A, for a given accuracy, the condition number of DGM B systems depends only slightly on the number of plane waves in the basis.

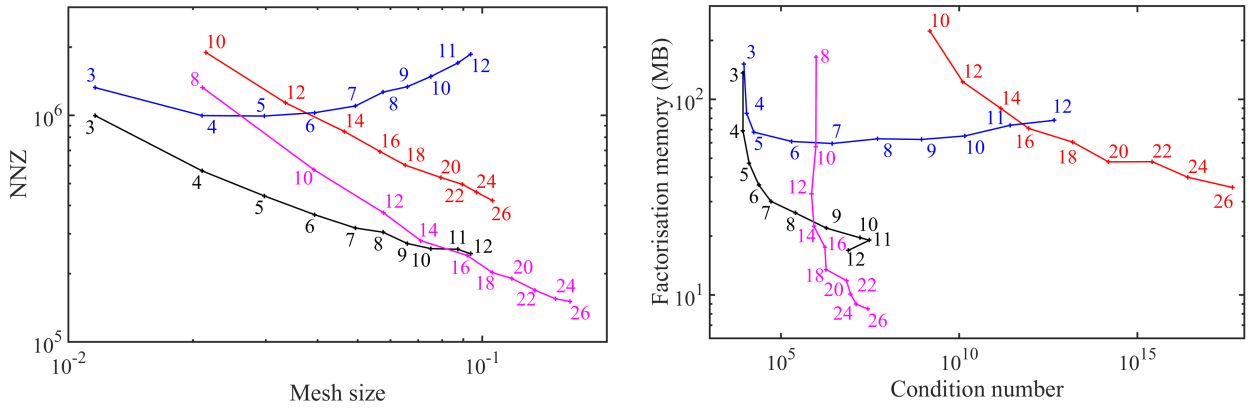

Figure 4: NNZ against mesh size (left) and factorisation memory against condition number (right) to achieve 1% of accuracy at $\omega = 75$; blue lines: p-FEM A, black lines: p-FEM B, red lines: DGM A, magenta dashed lines: DGM B. The numbers of plane wave or polynomial order are shown next to each point.

4.3 Fixed accuracy

In this section, we investigate the cost of each numerical model to achieve a defined accuracy. The following frequencies $\omega = 25, 50, 75, 100, 150$ were investigated. The trends obtained are very similar and therefore, for the sake of conciseness, only the results obtained for $\omega = 75$ are reported here (Fig. 4).

To reach 1% error, as P or N_{w} is increased, the required meshes become coarser. For p-FEM without condensation, $P = 5$ constitutes an optimum situation: it corresponds to the lowest NNZ and factorisation memory needed. On the contrary, using p-FEM with condensation or the wave-based method leads to a systematic decrease of the NNZ when increasing P or N_{w}. Static condensation for p-FEM reduces drastically not only the conditioning but also the storage and factorisation memory needed. The same effect is observed for wave-based DGM when the basis is adjusted. As noted previously, the conditioning of the systems for DGM B is not limiting: the condition numbers are similar to that of p-FEM B and the conditioning of the systems mainly depends on the accuracy, rather than the number of plane waves in the basis. The conditioning of DGM A and p-FEM A systems to reach 1% of accuracy significantly increase with P and N_{w}, which is an issue if iterative solvers are to be used.

With high-order methods, increasing the number of plane waves or the polynomial order allows to achieve a fixed accuracy with less degrees of freedom, time and memory. However, to achieve a given accuracy, using higher orders means adopting coarse meshes that might not describe complex geometries accurately. Furthermore, with increasing orders, the costs improvement rate slows down. Therefore, going up to a high polynomial orders or numbers of plane waves in the basis is unnecessary because the savings are reduced.

For the range of orders and number of plane waves, DGM B reaches the lowest demands in terms of memory. Contrary to the polynomial method, the basis can be adapted to the problem to be solved and as it is noted in this study, this feature makes the wave-based method an efficient method.

5. Conclusions

The conventional finite element method is not well-suited for short wave problems, due to the presence of cumulative dispersion errors (pollution effect). In this paper, we compare two advanced methods: polynomial higher-order FEM and wave-based discontinuous Galerkin method. We considered two variations of the wave-based method: the first basis is built with evenly spaced plane waves

whereas the second basis is improved by clustering the plane waves in the upstream direction.

For the propagation of a plane wave in free field problem, wave-based DGM with the improved basis and p-FEM with condensation have been able to achieve the same high levels of accuracy. The levels of performance were slightly better for wave-based DGM B thanks to the malleability of the basis. To reach the accuracy required for typical engineering problem, contrary to what is usually suggested, the wave-based systems are not necessarily ill-conditioned. In fact, they are when the plane wave in the basis are uniformly distributed but if the flow is taken into account into the discretisation, the condition number is not a limiting factor.

Additional test cases closer to realistic problems have to be studied in order to complete the present study and give a clear idea of the advantages of using wave-based DGM instead of p-FEM in particular ranges of frequencies. In particular, singular solutions and evanescent waves will be investigated. Some practical aspects have also to be taken into account that is p-FEM can directly be used on problems with non-uniform coefficients, whereas wave-based DGM would require some non-trivial developments to generalise the plane-wave basis to non-uniform media [13]. For frequency sweeps i.e. when varying the frequency of interest solely, p-FEM does not require a complete calculation of the element matrices for each frequency, unlike for a wave-based method which basis is frequency dependent.

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