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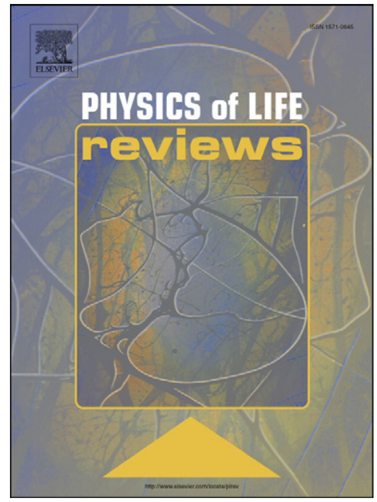

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A Tale of Two Theorems
Comment on "Universal Scaling for the Dilemma Strength in Evolutionary Games" by Z. Wang et. al.

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Wang et. al. [12] give an extensive review on evolutionary game theory, with a particular focus on the evolution of cooperation; an area that is of central importance to understanding many disparate biological topics [2, 1, 8, 9]. One of the key focuses of the paper is that there are potentially two differing ways of defining two-by-two games so that they can be parameterised with only two variables. This is a highly worthwhile endeavour in that it allows one to systematically investigate and categorise all such games with much greater ease. Perhaps the key result of the paper is that the authors argue that one such parameterisation is far superior to the other, as it retains predictive power even with the additional assumption of a number of reciprocity mechanisms, such as network reciprocity. We wish to draw attention to the fact that these two parameterisations can be understood as the application of two invariance theorems in evolutionary game theory. Furthermore, we complement the authors' extensive simulations with a mathematical argument that might help explain their result in a more transparent way. We will show that one of these invariance theorems generalises to the case of an assorted population and the other does not. It may be argued that many, if not all, of the particular mechanisms of reciprocity crucially rely on the positive assortment of cooperative behaviours [10, 3, 5, 6, 7, 4]; if one accepts this then our result acts as an argument towards why only one game parameterisation will be sufficient for models of reciprocity.

Evolutionary game theory has two "invariance" theorems, which roughly correspond to the two alternative ways of parameterising a payoff matrix. These two results are ways in which the payoff matrix can be transformed such that the dynamics of selection remain unaltered. Firstly, the *local invariance of payoffs* states that one may add a constant value to each column of the payoff matrix, that is replace M with M' where $M'_{ij} = M_{ij} + a_j$ without altering the dynamics of selection. This allows the payoff matrix to be transformed as follows:

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix} \rightarrow \begin{pmatrix} 0 & S-P \\ T-R & 0 \end{pmatrix} = \begin{pmatrix} 0 & -D_r \\ D_g & 0 \end{pmatrix} \tag{1}$$

The other major invariance result is that selection dynamics are unaltered under an *affine* transformation of the payoff matrix, that is $M'_{ij} = aM_{ij} + b$, which allows one to express the standard payoff matrix as:

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{S-P}{R-P} \\ \frac{T-P}{R-P} & 0 \end{pmatrix} = \begin{pmatrix} 1 & -D'_r \\ 1+D'_g & 0 \end{pmatrix} \tag{2}$$

see [13] for a proof of both of these results. We show that only the latter result generalises to the case of positive assortment.

Assortment can be modelled as follows: with propbaility α an individual is paired with an individual of the same strategy as itself, otherwise it is paired with a random individual [3, 11]. In which case payoff to an individual of type i is given by: $\pi_i = \alpha M_{ii} + (1-\alpha)\sum_j x_j M_{ij}$ and the mean payoff is given by $\bar{\pi} = \sum_i x_i \pi_i$ (see section 3.3.1 of original paper). Let π'_i be the payoff that player i receives under the transformed payoff matrix M', and likewise $\bar{\pi}'$ the mean payoff under this transformation. For the transformation to be invariant it must be the case that: $\pi_i' - \bar{\pi}' \propto \pi_i - \bar{\pi}$ for all i, (note that a constant of proportionality simply changes the time scale in which selection takes place).

Firstly, the affine transformation:

$$\pi_i' = \alpha\left(aM_{ii} + b\right) + (1-\alpha)\sum_j x_j\left(aM_{ij} + b\right) \quad (3)$$

$$= a\pi_i + b \quad (4)$$

and also:

$$\bar{\pi}' = \sum_i x_i(a\pi_i + b) \quad (5)$$

$$= a\bar{\pi} + b \quad (6)$$

and thus: $\pi_i' - \bar{\pi}' = a\left(\pi_i - \bar{\pi}\right)$, which demonstrates that this proof generalises to assortment.

In the case of local transformations:

$$\pi_i' = \alpha\left(M_{ii} + a_i\right) + (1-\alpha)\sum_j x_j\left(M_{ij} + a_j\right) \quad (7)$$

$$= \pi_i + \alpha a_i + (1-\alpha)\sum_j a_j x_j \quad (8)$$

and

$$\bar{\pi}' = \sum x_i\left(\pi_i + \alpha a_i + (1-\alpha)\sum_j a_j x_j\right) \quad (9)$$

$$= \bar{\pi} + \alpha\sum_i x_i a_i + (1-\alpha)\sum_j a_j x_j \quad (10)$$

thus the necessary cancelation occurs only if $\alpha = 0$ or if a is a constant vector and can thus be taken out of the summation in the second term, which is merely a special type of affine transformation.

We conclude by remarking that the incredible simplicity and generality of the parametrisation involving D_r' and D_g' suggests that all studies of two player symmetric games should be cast in these terms and, given that the result does not depend upon only two strategies being employed, it may be worthwhile to extend such conventions to higher dimensions.

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