Online Appendix
On the Benefits of a Monetary Union: Does it Pay to Be Bigger?

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1 The zero-inflation deterministic steady state

In this appendix we argue that, given appropriate initial conditions, zero inflation is a Nash equilibrium policy at the deterministic steady state under both regimes A and B. This result is the first step to find the optimal policies under regimes A, B and C using the linear quadratic approach pioneered by Benigno and Woodford (JET 2012).

Under regime A, the *timelessly* optimal policy problem of a monetary authority of country i in area H can be formulated as the maximization of the following Lagrangian:

$$\begin{aligned} L_s = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \\ & + \zeta_{1,t}^{s,i}\left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta}\left(\alpha_s C_t^i + (\alpha_b-\alpha_s)C_t^{i\,\sigma\eta}\mathcal{C}_{H,t} + (1-\alpha_b)C_t^{i\,\sigma\eta}\mathcal{C}_{F,t}\right)\right] \\ & + \zeta_{2,t}^{s,i}\left[K_t^i - \left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1} Z_t^{i\,\varphi}(1+\mu_t^i)(1-\tau^i)\frac{\varepsilon}{\varepsilon-1}\right] - \zeta_{2,t-1}^{s,i}\theta\Pi_{i,t}^{\varepsilon}K_t^i \\ & + \zeta_{3,t}^{s,i}\left[F_t^i - Y_t^i C_t^{i\,-\sigma}\frac{P_{i,t}}{P_{C^i,t}}\right] - \zeta_{3,t-1}^{s,i}\theta\Pi_{i,t}^{(\varepsilon-1)}F_t^i \\ & + \zeta_{4,t}^{s,i}\left[F_t^i - K_t^i\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}}\right] \\ & + \zeta_{5,t}^{s,i}\left[Z_t^i - \theta Z_{t-1}^i\Pi_{i,t}^{\varepsilon} - (1-\theta)\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]\Bigg\} \end{aligned} \tag{1}$$

where $P_{i,t}/P_{C^i,t}$ is determined as:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\gamma_s + (\gamma_b-\gamma_s)\left(\frac{C_{H,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)} + (1-\gamma_b)\left(\frac{C_{F,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)}\right]^{\frac{1}{1-\eta}} \tag{2}$$

while $C_{H,t}^*$, $C_{F,t}^*$, $\mathcal{C}_{H,t}$ and $\mathcal{C}_{F,t}$ are taken as given.[1] Assume that $\mu_t^j = \mu$, $A_t^j = 1$, $\tau^j = \tau$, $Z_t^j = \Pi_{j,t} = 1$ and $Z_{-1}^j = 1$ for all t and $j \neq i$ with $j \in [0,1]$. Then, since $\tilde{\tau} = 1-(1-\tau)(1+\mu)\frac{\varepsilon}{\varepsilon-1}$ it can be shown that zero inflation is an optimal policy at the deterministic steady state. In other words, zero inflation is a solution to the first-order conditions of the Lagrangian (1), and therefore the best response to the zero-inflation policies of the other policy makers in areas F and H. Indeed, if $Z_t^i = \Pi_{i,t} = 1$ at all t, from the first-order conditions of (1) with respect to C_t^i, Y_t^i, Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$

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[1]Notice that this Lagrangian incorporates the additional constraints at time 0 that render the policy timelessly optimal.

evaluated at the symmetric deterministic steady state it follows that:

$$
\begin{aligned}
&C^{-\sigma} = \zeta_1^s \delta_s - \zeta_3^s \sigma \gamma_s Y C^{-\sigma-1} \\
&Y^{\varphi} = \zeta_1^s - \zeta_2^s(\varphi+1) Y^{\varphi}(1-\tilde{\tau}) - \zeta_3^s C^{-\sigma} \\
&Y^{\varphi+1} = -\zeta_2^s \varphi Y^{\varphi+1}(1-\tilde{\tau}) + \zeta_5^s(1-\beta\theta) \\
&\zeta_2^s(1-\theta) = \zeta_4^s \\
&\zeta_3^s(1-\theta) = -\zeta_4^s \\
&\zeta_2^s \theta \varepsilon K = -\zeta_3^s \theta(\varepsilon-1) F + \zeta_4^s \frac{\theta}{1-\theta} K
\end{aligned} \tag{3}
$$

with $\gamma_s = \frac{1}{\alpha_s}$ and $\delta_s = \alpha_s(1-\sigma\eta) + \gamma_s \eta \sigma$. Hence:

$$
\begin{aligned}
&Y = (1-\tilde{\tau})^{-\frac{1}{\sigma+\varphi}} \\
&C = Y \\
&F = K = \frac{Y C^{-\sigma}}{1-\beta\theta} = \frac{Y^{\varphi+1}(1-\tilde{\tau})}{1-\beta\theta} \\
&\Pi = \Pi_H = \Pi_F = Z = 1 \\
&\zeta_1^s = Y^{\varphi}(1-\varphi\zeta_s) \qquad \zeta_2^s = -\frac{\zeta_s}{(1-\tilde{\tau})} \qquad \zeta_3^s = \frac{\zeta_s}{(1-\tilde{\tau})} \\
&\zeta_4^s = -\frac{(1-\theta)\zeta_s}{(1-\tilde{\tau})} \qquad \zeta_5^s = \frac{Y^{\varphi+1}(1-\varphi\zeta_s)}{(1-\beta\theta)}
\end{aligned}
$$

where $\zeta_s \equiv \frac{\delta_s-(1-\tilde{\tau})}{\gamma_s\sigma+\delta_s\varphi}$ is a steady-state symmetric solution to the optimal policy problem just stated.

Consider now the monetary union in area F.[2] Suppose that for all $i \in [0, \frac{1}{2})$, $\Pi_t^i = Z_t^i = 1$ at all times. Hence, $F_t^i = F$, $K_t^i = K$ and $F_t^i/K_t^i = 1$ for all i and t. We want to argue that also in this case, $\Pi_t^i = 1$ for all t and $i \in [\frac{1}{2}, 1]$ is the optimal best response of the central bank in area F to the other policy makers' optimal strategies. If for all $i \in [\frac{1}{2}, 1]$, $\Pi_t^i = 1$ at all times, the optimal policy problem of the monetary authority in area F can be written as maximizing:

$$
\begin{aligned}
L_b = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \int_{\frac{1}{2}}^{1} & \left[\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \right] \\
&+ \zeta_{1,t}^{b,i}\left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta}\left(\alpha_s C_t^i + (\alpha_b-\alpha_s) C_t^{i\,\sigma\eta}\mathcal{C}_{F,t} + (1-\alpha_b) C_t^{i\,\sigma\eta}\mathcal{C}_{H,t}\right)\right] \\
&+ \zeta_{2,t}^{b,i}\left[K_t^i - \left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1} Z_t^{i\,\varphi}(1+\mu_t^i)(1-\tau^i)\frac{\varepsilon}{\varepsilon-1}\right] - \zeta_{2,t-1}^{b,i}\theta \Pi_{i,t}^{\varepsilon} K_t^i \\
&+ \zeta_{3,t}^{b,i}\left[F_t^i - Y_t^i C_t^{i\,-\sigma}\frac{P_{i,t}}{P_{C^i,t}}\right] - \zeta_{3,t-1}^{b,i}\theta \Pi_{i,t}^{(\varepsilon-1)} F_t^i \\
&+ \zeta_{4,t}^{b,i}\left[F_t^i - K_t^i\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{\varepsilon-1}}\right]
\end{aligned}
$$

[2] We follow closely Benigno and Benigno (JME 2006).

$$
\begin{aligned}
&+\zeta_{5,t}^{b,i}\left[Z_t^i-\theta Z_{t-1}^i\Pi_{i,t}^{\varepsilon}-(1-\theta)\left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]\\
&+\zeta_{6,t}^{b,i}\left[\left(\frac{C_{F,t}^*}{C_{F,t-1}^*}\right)^{-\sigma}\frac{P_{F,t}}{P_{F,t}^*}\frac{P_{F,t-1}^*}{P_{F,t-1}}-\left(\frac{C_t^i}{C_{t-1}^i}\right)^{-\sigma}\frac{P_{i,t}}{P_{C^i,t}}\frac{P_{C^i,t-1}}{P_{i,t-1}}\Pi_{i,t}^{-1}\right]di\\
&+\int_0^{\frac{1}{2}}\zeta_{7,t}^{b,i}\left[Y_t^i-\left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta}\left(\alpha_s C_t^i+(\alpha_b-\alpha_s)C_t^{i\sigma\eta}\mathcal{C}_{H,t}+(1-\alpha_b)C_t^{i\sigma\eta}\mathcal{C}_{F,t}\right)\right]\\
&+\zeta_{8,t}^{b,i}\left[(1+\mu_t^i)(1-\tau)\frac{\varepsilon}{\varepsilon-1}\left(\frac{Y_t^i}{A_t^i}\right)^{\varphi+1}-\frac{P_{i,t}}{P_{C^i,t}}Y_t^iC_t^{i-\sigma}\right]di\Bigg\}
\end{aligned}
\tag{4}
$$

where $\mathcal{C}_{H,t}\equiv 2\int_0^{\frac{1}{2}}C_t^{i1-\sigma\eta}di$ and $\mathcal{C}_{F,t}\equiv 2\int_{\frac{1}{2}}^1 C_t^{i1-\sigma\eta}di$ and $P_{i,t}/P_{C^i,t}$ and $P_{F,t}^*/P_{F,t}$ are determined consistently with (2), the foreign counterpart of (2) and:

$$
\frac{P_{F,t}^*}{P_{F,t}}=\left[\alpha_b+(1-\alpha_b)\left(\frac{P_{H,t}}{P_{F,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{5}
$$

$$
\left(\frac{C_{F,t}^*}{C_{H,t}^*}\right)=\left[\frac{\alpha_b\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta}+(1-\alpha_b)}{(1-\alpha_b)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta}+\alpha_b}\right]^{-\frac{1}{\sigma(1-\eta)}} \tag{6}
$$

Assume that $\mu_t^i=\mu$, $A_t^i=1$, $\tau^i=\tau$ and $Z_{-1}^i=1$ for all $i\in[0,1]$ and t. Then, it can be shown that zero inflation is an optimal policy, because such a policy is consistent with the first-order conditions of (4). Indeed, if $Z_t^i=\Pi_{i,t}=1$ for all t and $i\in[\frac{1}{2},1]$, the first-order conditions with respect to C_t^i, Y_t^i for all i and Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$ all $i\in[\frac{1}{2},1]$ at the symmetric deterministic steady state can written as:

$$
\begin{aligned}
&C^{-\sigma}=\zeta_1^b\delta_b+\zeta_7^b(1-\delta_b)-\zeta_3^b\sigma\gamma_bYC^{-\sigma-1}-\zeta_8^b\sigma(1-\gamma_b)YC^{-\sigma-1}\\
&Y^{\varphi}=\zeta_1^b-\zeta_2^b(\varphi+1)Y^{\varphi}(1-\tilde{\tau})-\zeta_3^bC^{-\sigma}\\
&0=\zeta_1^b(1-\delta_b)+\zeta_7^b\delta_b-\zeta_3^b\sigma(1-\gamma_b)YC^{-\sigma-1}-\zeta_8^b\sigma\gamma_bYC^{-\sigma-1}\\
&0=\zeta_7^b+\zeta_8^b\left[(\varphi+1)Y^{\varphi}(1-\tilde{\tau})-C^{-\sigma}\right]\\
&Y^{\varphi+1}=-\zeta_2^b\varphi Y^{\varphi+1}(1-\tilde{\tau})+\zeta_5^b(1-\beta\theta)\\
&\zeta_2^b(1-\theta)=\zeta_4^b\\
&\zeta_3^b(1-\theta)=-\zeta_4^b\\
&\zeta_2^b\theta\varepsilon K=-\zeta_3^b\theta(\varepsilon-1)F+\zeta_4^b\frac{\theta}{1-\theta}K
\end{aligned}
\tag{7}
$$

where $\gamma_b=\frac{\alpha_b}{2\alpha_b-1}$ and $\delta_b\equiv(1-\sigma\eta)\alpha_b+\eta\sigma\gamma_b$. As a consequence:

$$
\begin{aligned}
&Y=(1-\tilde{\tau})^{-\frac{1}{\sigma+\varphi}}\\
&C=Y\\
&F=K=\frac{YC^{-\sigma}}{1-\beta\theta}=\frac{Y^{\varphi+1}(1-\tilde{\tau})}{1-\beta\theta}
\end{aligned}
$$

$$\Pi = \Pi_H = \Pi_F = Z = 1$$

$$\zeta_1^b = Y^{\varphi}\left(1 - \varphi\zeta_b\right) \qquad \zeta_2^b = -\frac{\zeta_b}{(1-\tilde{\tau})} \qquad \zeta_3^b = \frac{\zeta_b}{(1-\tilde{\tau})} \qquad \zeta_4^b = -\frac{(1-\theta)\zeta_b}{(1-\tilde{\tau})}$$

$$\zeta_5^b = \frac{Y^{\varphi+1}\left(1-\varphi\zeta_b\right)}{1-\beta\theta} \qquad \zeta_7^b = -Y^{\varphi}\varphi\left(\zeta_w - \zeta_b\right) \qquad \zeta_8^b = \frac{(\zeta_w - \zeta_b)}{(1-\tilde{\tau})} \tag{8}$$

where $\zeta_b \equiv \frac{1}{2}\frac{\tilde{\tau}}{\sigma+\varphi} - \frac{\delta_b - 1 + (1/2)\tilde{\tau}}{(1-2\gamma_b)\sigma + (1-2\delta_b)\varphi}$ and $\zeta_w \equiv \frac{\tilde{\tau}}{\sigma+\varphi}$ Hence, being the best response for both monetary union and the small open economy policy makers, zero inflation is a Nash equilibrium solution under regime A.

Consider now the case of regime B and suppose that the central bank of area H set $\Pi_{H,t} = 1$ for all t. In this case, given the symmetry of area H's small open economies, it has to be that in the absence of shocks $\Pi_{i,t} = 1$ for all t and $i \in [0, \frac{1}{2})$. Hence, under regime B the solution to the optimal policy problem in area F is identical to the one in (7) for regime A and zero inflation is the best response of the policy maker in area F to a zero-inflation policy of the policy maker in area H. However under regime B, the optimal policy problem of the policy maker in area H is symmetric to the one in area F. Thus, we can conclude that zero inflation is a Nash equilibrium at the deterministic steady state also under regime B.

2 Welfare criteria, welfare-relevant targets and optimal policies under regimes A, B and C

In this section, we recover the purely quadratic approximations to the welfare criteria, the welfare-relevant targets and the optimal policies under regimes A, B and C. We will consider the case of the small open economy, the monetary union in the areas H and F and the world economy separately.

2.1 The case of the small open economy

We start by looking at the case of a small region located in area H.

2.1.1 The approximation of the welfare criterion

As a first step we need to approximate the small open economy representative agent lifetime utility up to the second order. Recall that the period t utility is:

$$U_t \equiv \frac{{C_t^i}^{1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1}. \tag{9}$$

Then we can approximate the utility derived from private consumption for generic region i as:

$$\frac{{C_t^i}^{1-\sigma}}{1-\sigma} \simeq \frac{C^{1-\sigma}}{1-\sigma} + C^{1-\sigma}\left(\hat{c}_t^i + \frac{1}{2}(\hat{c}_t^i)^2\right) - \frac{\sigma}{2}C^{1-\sigma}(\hat{c}_t^i)^2 + t.i.p. \tag{10}$$

where $\hat{c}_t^i$ stands for the log deviations of private consumption from the deterministic steady state and $t.i.p.$ for "terms independent of policy". By the same token we can approximate labor disutility. Since $Z_t^i = \int_0^1 \left(\frac{p_t(h^i)}{P_{i,t}}\right)^{-\varepsilon} dh^i$, it follows that, as showed by Galí and Monacelli (REStud 2005):

$$\hat{z}_t^i \simeq \frac{\varepsilon}{2} Var_{h^i}(p_t(h^i)). \tag{11}$$

In words, the approximation of Z_t^i around the symmetric steady state is purely quadratic. Moreover, following Woodford (NBER WP 8071 2001), it is possible to show that $\sum_{t=0}^{\infty} \beta^t Var_{h^i}(p_t(h^i)) = \frac{1}{\lambda}\sum_{t=0}^{\infty}\beta^t \pi_{i,t}^2$ with $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. Therefore, labor disutility can be approximated up to the second order as:

$$\frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \simeq \frac{1}{\varphi+1} Y^{\varphi+1} + Y^{\varphi+1}\left(\hat{y}_t^i + \frac{1}{2}(\hat{y}_t^i)^2\right) + Y^{\varphi+1}\frac{\varepsilon}{2\lambda}(\pi_{i,t})^2 + \frac{\varphi}{2}Y^{\varphi+1}(\hat{y}_t^i)^2$$
$$-(\varphi+1)Y^{\varphi+1}\hat{y}_t^i a_t^i + t.i.p. \tag{12}$$

By combining (10) and (12) and recalling that at the steady state $C^{-\sigma} = (1-\tilde{\tau})Y^{\varphi}$, we can express the second-order approximation of U_t as a fraction of steady state consumption in the following way:

$$\frac{U_t - U}{U_C C} \simeq \frac{1}{1-\tilde{\tau}}\left[(1-\tilde{\tau})\hat{c}_t^i - (1-\tilde{\tau})\frac{(\sigma-1)}{2}(\hat{c}_t^i)^2 - \hat{y}_t^i - \frac{\varphi+1}{2}(\hat{y}_t^i)^2 - \frac{\varepsilon}{2\lambda}(\pi_{i,t})^2\right] \tag{13}$$

Therefore, the second-order approximation to the welfare of region i representative agent is given by:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[(1-\tilde{\tau})\hat{c}_t^i - (1-\tilde{\tau})\frac{(\sigma-1)}{2}(\hat{c}_t^i)^2 - \hat{y}_t^i - \frac{\varphi+1}{2}(\hat{y}_t^i)^2 - \frac{\varepsilon}{2\lambda}(\pi_{i,t})^2\right] + t.i.p. \tag{14}$$

When $i \in \left[0, \frac{1}{2}\right)$, the expression in (14) can be rewritten in matrix form as:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\hat{s}_t^{i\prime} w_s - \frac{1}{2}\hat{s}_t^{i\prime} W_{s,s}\hat{s}_t^i + \hat{s}_t^{i\prime} W_{s,e}\hat{e}_t^i\right] + t.i.p. \tag{15}$$

where:

$$\hat{s}_t^{i\prime} \equiv \left[\hat{y}_t^i,\ \hat{c}_t^i,\ \pi_{i,t}\right] \qquad w_s' \equiv [-1,\ (1-\tilde{\tau}),\ 0] \qquad \hat{e}_t^{i\prime} \equiv \left[\hat{c}_{H,t},\ \hat{c}_{F,t},\ \hat{a}_t^i,\ \hat{\mu}_t^i\right]$$

$$W_{s,s} \equiv \begin{bmatrix} (\varphi+1) & 0 & 0 \\ 0 & (1-\tilde{\tau})(\sigma-1) & 0 \\ 0 & 0 & \frac{\varepsilon}{\lambda} \end{bmatrix} \qquad W_{s,e} \equiv \begin{bmatrix} 0 & 0 & (\varphi+1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\hat{c}_{H,t} \equiv 2\int_0^{\frac{1}{2}} \hat{c}_t^j dj$ and $\hat{c}_{F,t} \equiv 2\int_{\frac{1}{2}}^1 \hat{c}_t^j dj$.

In order to rewrite the approximation in (15) in a purely quadratic way, we need to use the second-order approximations to both the aggregate demand and the Phillips curve. The second-order approximation to the demand curve can be written as:

$$0 \simeq \left[\hat{s}_t^{i\prime} g_s - \hat{e}_t^{i\prime} g_e + \frac{1}{2}\hat{s}_t^{i\prime} G_{s,s}\hat{s}_t^i - \hat{s}_t^{i\prime} G_{s,e} e_t^i\right] + t.o.c. \tag{16}$$

where:

$$g_s' \equiv [-1,\ \delta_s,\ 0] \qquad g_e' \equiv [-(\delta_b - \delta_s),\ -(1-\delta_b),\ 0,\ 0]$$

$$G_{s,s} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & \delta_s+\omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s,e} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_1+\omega_2 & -\omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\delta_s \equiv \alpha_s(1-\eta\sigma)+\eta\sigma\gamma_s$, $\delta_b \equiv \alpha_b(1-\eta\sigma)+\gamma_b\eta\sigma$, $\gamma_s \equiv \frac{1}{\alpha_s}$ and $\gamma_b \equiv \frac{\alpha_b}{2\alpha_b-1}$, while:

$$\omega_1 \equiv \frac{(1-\alpha_s)\eta\sigma(\sigma-(1-\alpha_s)\alpha_s(1-\eta\sigma))}{\alpha_s^2}$$

$$\omega_2 \equiv \frac{(1-\alpha_b)\eta\sigma\left(\sigma+\left(\alpha_s^2+(1-2\alpha_b)\right)(1-\eta\sigma)\right)}{\alpha_s(1-2\alpha_b)}.$$

Moreover, *t.o.c.* stands for terms out of control" of the policy maker which include the average area variables besides the terms independent of policy.

Consistently with Benigno Woodford (JEEA 2005), the second-order approximation to the Phillips curve can be written as:

$$V_0 = \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\left[\hat{s}_t^{i'}v_s - \hat{e}_t^{i'}v_e + \frac{1}{2}\hat{s}_t^{i'}V_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}V_{s,e}\hat{e}_t^i\right] + t.o.c. \qquad (17)$$

where

$$v_s' \equiv [\varphi,\ \sigma\gamma_s,\ 0] \qquad v_e' \equiv [\sigma(\gamma_s-\gamma_b),\ -\sigma(1-\gamma_b),\ -(\varphi+1),\ 1]$$

$$V_{s,s} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & -\eta\sigma^2(\gamma_s-1)\gamma_s-\sigma^2\gamma_s & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix}$$

$$V_{s,e} \equiv \begin{bmatrix} -\sigma(\gamma_b-\gamma_s) & -\sigma(1-\gamma_b) & (\varphi+1)^2 & -(\varphi+1) \\ \eta\sigma^2\gamma_s(\gamma_b-\gamma_s) & \eta\sigma\gamma_s^2(1-\gamma_b) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given (16) and (17), it is easy to show that:

$$w_s = (1-\varphi\zeta_s)g_s - \zeta_s v_s$$

where $\zeta_s = \frac{\delta_s-(1-\tilde{\tau})}{\delta_s\varphi+\gamma_s\sigma}$.[3] Then, we can write the second-order approximation to region i welfare as:

$$-\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_t^{i'}\Omega_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}\Omega_{s,e}\hat{e}_t^i\right] + t.o.c. \qquad (18)$$

where:

$$\Omega_{s,s} \equiv W_{s,s} + (1-\varphi\zeta_s)G_{s,s} - \zeta_s V_{s,s} \qquad \Omega_{s,e} \equiv W_{s,e} + (1-\varphi\zeta_s)G_{s,e} - \zeta_s V_{s,e}$$

and $\Omega_{s,s}$ and $\Omega_{s,e}'$ are respectively equal to:

$$\begin{bmatrix} (1-\zeta_s(\varphi+1))\varphi & -\zeta_s\gamma_s\sigma & 0 \\ -\zeta_s\gamma_s\sigma & (1-\tilde{\tau})(\sigma-1)-\zeta_s\sigma^2\eta\gamma_s(1-\gamma_s)+\zeta_s\sigma^2\gamma_s+(1-\zeta_s\varphi)(\delta_s+\omega_1) & 0 \\ 0 & 0 & \frac{(1-\zeta_s(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} \zeta_s\sigma(\gamma_b-\gamma_s) & -\zeta_s\eta\sigma^2\gamma_s(\gamma_b-\gamma_s)+(1-\zeta_s\varphi)(\omega_1+\omega_2) & 0 \\ \zeta_s\sigma(1-\gamma_b) & -\zeta_s\eta\sigma^2\gamma_s(1-\gamma_b)-(1-\zeta_s\varphi)\omega_2 & 0 \\ (1-\zeta_s(\varphi+1))(\varphi+1) & 0 & 0 \\ \zeta_s(\varphi+1) & 0 & 0 \end{bmatrix}$$

[3]Notice that ζ_s determines also the Lagrange multipliers previously recovered for the optimal policy problem of the small economy policy maker. See Benigno and Woodford (JEEA 2005).

Alternatively, (18) can be written as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\Big[\frac{1}{2}\varpi_{1,s}(\hat{y}_t^i)^2+\varpi_{2,s}\hat{c}_t^i\hat{y}_t^i+\frac{1}{2}\varpi_{3,s}(\hat{c}_t^i)^2+\frac{1}{2}\varpi_{4,s}(\pi_{i,t})^2-\varpi_{5,s}\hat{y}_t^i\hat{c}_{H,t}$$

$$-\varpi_{6,s}\hat{y}_t^i\hat{c}_{F,t}-\varpi_{7,s}\hat{c}_t^i\hat{c}_{H,t}-\varpi_{8,s}\hat{c}_t^i\hat{c}_{F,t}-\varpi_{9,s}\hat{y}_t^i\hat{a}_t^i-\varpi_{10,s}\hat{y}_t^i\hat{\mu}_t^i\Big]+t.o.c. \tag{19}$$

where:

$$\begin{aligned}
\varpi_{1,s} &\equiv [1-\zeta_s(\varphi+1)]\,\varphi\\
\varpi_{2,s} &\equiv -\zeta_s\gamma_s\sigma\\
\varpi_{3,s} &\equiv (1-\tilde{\tau})(\sigma-1)-\zeta_s\sigma^2\eta\gamma_s\,(1-\gamma_s)+\zeta_s\sigma^2\gamma_s+(1-\zeta_s\varphi)(\delta_s+\omega_1)\\
\varpi_{4,s} &\equiv [1-\zeta_s(\varphi+1)]\frac{\varepsilon}{\lambda}\\
\varpi_{5,s} &\equiv \zeta_s\sigma(\gamma_b-\gamma_s)\\
\varpi_{6,s} &\equiv \zeta_s\sigma(1-\gamma_b)\\
\varpi_{7,s} &\equiv -\zeta_s\eta\sigma^2\gamma_s\,(\gamma_b-\gamma_s)+(1-\zeta_s\varphi)\,(\omega_1+\omega_2)\\
\varpi_{8,s} &\equiv -\zeta_s\eta\sigma^2\gamma_s(1-\gamma_b)-(1-\zeta_s\varphi)\,\omega_2\\
\varpi_{9,s} &\equiv [1-\zeta_s(\varphi+1)]\,(\varphi+1)\\
\varpi_{10,s} &\equiv \zeta_s(\varphi+1)
\end{aligned} \tag{20}$$

2.1.2 The welfare-relevant target

The next step is to express the approximations in (18) in terms of deviations from the welfare-relevant target of the small open economy policy maker. This target can be determined by maximizing (18) subject to good market clearing condition of region i, namely:

$$\hat{y}_t^i=\delta_s\hat{c}_t^i+(\delta_b-\delta_s)\hat{c}_{H,t}+(1-\delta_b)\hat{c}_{F,t} \qquad i\in[0,\frac{1}{2}) \tag{21}$$

The Lagrangian associated with this problem can be written as:

$$L^s=\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_t^{s\prime}\Omega_{s,s}\hat{s}_t^s-\hat{s}_t^{s\prime}\Omega_{s,e}\hat{e}_t^i+\phi_t^s\left(\hat{s}_t^{s\prime}g_s-\hat{e}_t^{i\prime}g_e\right)\right] \tag{22}$$

where $\hat{s}_t^s$ indicates the welfare-relevant target for the variable $\hat{s}_t^i$. The first-order conditions of L^s with respect to $\hat{s}_t^{s\prime}$ and ϕ_t^s are:

$$\begin{aligned}
\Omega_{s,s}\hat{s}_t^s-\Omega_{s,e}\hat{e}_t^i &= -\phi_t^s g_s\\
g_s\hat{s}_t^s-g_e\hat{e}_t^i &= 0
\end{aligned} \tag{23}$$

for all t and $i\in[0,\frac{1}{2})$ and where ϕ_t^s is the Lagrange multiplier of (21). Condition (23) can be also read as:

$$\begin{aligned}
&\varpi_{1,s}\hat{y}_t^{i,s}+\varpi_{2,s}\hat{c}_t^{i,s}-\varpi_{5,s}\hat{c}_{H,t}-\varpi_{6,s}\hat{c}_{F,t}-\varpi_{9,s}\hat{a}_t^i-\varpi_{10,s}\hat{\mu}_t^i=\phi_t^s\\
&\varpi_{3,s}\hat{c}_t^s+\varpi_{2,s}\hat{y}_t^s-\varpi_{7,s}\hat{c}_{H,t}-\varpi_{8,s}\hat{c}_{F,t}=-\delta_s\phi_t^s\\
&\varpi_{4,s}\pi_{i,t}=0\\
&\hat{y}_t^s=\delta_s\hat{c}_t^s+(\delta_b-\delta_s)\hat{c}_{H,t}+(1-\delta_b)\hat{c}_{F,t}
\end{aligned} \tag{24}$$

Notice that the small open monetary authority takes $\hat{c}_{H,t}$ and $\hat{c}_{F,t}$ as exogenous. Then the system of equations in (24) allows to determine the target for $\hat{c}_t^i$ $\hat{y}_t^i$ and $\pi_{i,t}$. In addition we can recover the target for $\hat{s}_{iH,t}$ and $\hat{s}_{iF,t}$ using the next equilibrium conditions:

$$\hat{s}_{iH,t} = -\sigma\gamma_s(\hat{c}_{H,t} - \hat{c}_t^i) \tag{25}$$

$$\hat{s}_{iF,t} = -\sigma\gamma_s(\hat{c}_{H,t} - \hat{c}_t^i) - \sigma(2\gamma_b - 1)(\hat{c}_{F,t} - \hat{c}_{H,t}) \tag{26}$$

Moreover, from the conditions in (24), (25) and (26) it follows that:

$$\begin{aligned}
&[1 - \zeta_s(\varphi + 1)]\widehat{mc}_{i,t}^{e,s} = \zeta_s(\varphi + 1)\hat{\mu}_t^i + \kappa_H^s \hat{s}_{iH,t}^s + \kappa_F^s \hat{s}_{iF,t}^s \\
&\hat{s}_{iF,t}^s = \hat{s}_{iH,t}^s - \sigma(2\gamma_b - 1)(\hat{c}_{F,t} - \hat{c}_{H,t}) \\
&\hat{s}_{iH,t}^s = \kappa_a^s \hat{a}_t^i + \kappa_\mu^s \hat{\mu}_t^i + \kappa_{cH}^s \hat{c}_{H,t} + \kappa_{cF}^s \hat{c}_{F,t}
\end{aligned} \tag{27}$$

where $\widehat{mc}_{i,t}^{e,s} = (\varphi+\sigma)\hat{y}_t^i + \left[\frac{\gamma_s - \delta_s}{\gamma_s} - \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\right]\hat{s}_{iH,t} + \gamma_s + \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\hat{s}_{iF,t}$, while κ_H^s, κ_F^s, κ_a^s, κ_μ^s, κ_{cH}^s and κ_{cF}^s are defined as:

$$\begin{aligned}
\kappa_H^s &\equiv -\frac{a_1^s}{\sigma\gamma_s\delta_s} + \frac{a_2^s}{\sigma(2\gamma_b - 1)\delta_s} + \left(\frac{\gamma_b}{2\gamma_b - 1} - \frac{1}{\gamma_s}\right)[1 - \zeta_s(\varphi + 1)] \\
\kappa_F^s &\equiv -\frac{a_2^s}{\sigma(2\gamma_b - 1)\delta_s} - \frac{1 - \gamma_b}{2\gamma_b - 1}[1 - \zeta_s(\varphi + 1)] \\
\kappa_a^s &\equiv \frac{1}{a_3^s}[1 - \zeta_s(\varphi + 1)](1 + \varphi) \\
\kappa_\mu^s &\equiv \frac{1}{a_3^s}\zeta_s(1 + \varphi) \\
\kappa_{cH}^s &\equiv -\frac{1}{a_3^s}\left[\frac{a_2^s}{\delta_s} + (1 - \zeta_s(1 + \varphi))(\sigma + \varphi\delta_b)\right] \\
\kappa_{cF}^s &\equiv \frac{1}{a_3^s}\left[\frac{a_2^s}{\delta_s} - (1 - \zeta_s(1 + \varphi))(1 - \delta_b)\varphi\right]
\end{aligned} \tag{28}$$

with:

$$\begin{aligned}
a_1^s &\equiv -\zeta_s\sigma^2\eta\gamma_s(1 - \gamma_s) + (1 - \zeta_s\varphi)\omega_1 + \zeta_s\sigma\delta_s(1 - \gamma_s) + \zeta_s\sigma\gamma_s(1 - \delta_s) \\
a_2^s &\equiv -\zeta_s\sigma^2\eta\gamma_s(1 - \gamma_b) - (1 - \zeta_s\varphi)\omega_2 + \zeta_s\sigma\delta_s(1 - \gamma_b) + \zeta_s\sigma\gamma_s(1 - \delta_b) \\
a_3^s &\equiv [1 - \zeta_s(1 + \varphi)]\left[(\varphi + \sigma)\frac{\delta_s}{\gamma_s\sigma} + \frac{(1 - \delta_s)}{\gamma_s}\right] + \frac{a_1^s}{\gamma_s\delta_s\sigma}.
\end{aligned} \tag{29}$$

The first condition in (27) expresses the target for the fluctuations of the efficient firms' marginal cost as a function of mark-up shocks and the two terms of trade that are relevant from the small open economy viewpoint; the second condition combines (25) with (26), while the last condition expresses $\hat{s}_{iH,t}^s$ in terms of the exogenous shocks.

2.1.3 The optimal policy

By using (24), (25) and (26), we can rewrite (19) in terms of gaps, namely:[4]

$$-\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t\frac{1}{2}E_0\left[\varpi_{1,s}(\tilde{y}_t^{i,s})^2+\varpi_{11,s}(\tilde{s}_{iH,t}^s)^2+\varpi_{12,s}(\tilde{s}_{iF,t}^s)^2+\varpi_{4,s}(\pi_{i,t})^2\right]+t.o.c. \tag{30}$$

where

$$\begin{aligned}\varpi_{11,s} &\equiv \frac{1}{\sigma\gamma_s}\left[2\varpi_{2,s}\left(\frac{\delta_s}{\sigma\gamma_s}+\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\right)+\frac{1}{\sigma^2\gamma_s^2}\varpi_{3,s}\right]\\ \varpi_{12,s} &\equiv -2\frac{\varpi_{2,s}}{\sigma\gamma_s}\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\end{aligned} \tag{31}$$

and where $\tilde{x}_t^s$ represents the gap of the variable $\hat{x}_t$ from its target $\hat{x}_t^s$. The *timelessly* optimal monetary policy can be retrieved by maximizing (30) with respect to $\tilde{y}_t^s$, $\tilde{s}_{iH,t}^s$, $\tilde{s}_{iF,t}^s$ and $\pi_{i,t}$ subject to the following sequence of constraints:

$$\begin{aligned}\pi_{i,t} &= \lambda\left[(\varphi+\sigma)\tilde{y}_t^{i,s}+\left(\frac{\gamma_s-\delta_s}{\gamma_s}-\frac{\gamma_b-\delta_b}{2\gamma_b-1}\right)\tilde{s}_{iH,t}^s+\frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}_{iF,t}^s\right]+\beta E_t\{\pi_{i,t+1}\}+\upsilon_{i,t}^s\\ \tilde{y}_t^{i,s} &= \left[\frac{\delta_s}{\sigma\gamma_s}+\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\right]\tilde{s}_{iH,t}^s-\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\tilde{s}_{iF,t}^s\\ \tilde{s}_{iH,t}^s &= \tilde{s}_{iF,t}^s\end{aligned} \tag{32}$$

where:

$$\begin{aligned}\upsilon_{i,t}^s = &\frac{\lambda}{1-\zeta_s(\varphi+1)}\left[\left(1+\kappa_\mu^s(\kappa_H^s+\kappa_F^s)\right)\hat{\mu}_t^i+\left(\kappa_a^s(\kappa_H^s+\kappa_F^s)\right)\hat{a}_t^i\right.\\ &\left.+\left(\kappa_{cH}^s(\kappa_H^s+\kappa_F^s)+\kappa_F^s\sigma(2\gamma_b-1)\right)\hat{c}_{H,t}+\left(\kappa_{cF}^s(\kappa_H^s+\kappa_F^s)-\kappa_F^s\sigma(2\gamma_b-1)\right)\hat{c}_{F,t}\right]\end{aligned} \tag{33}$$

and taking into account the constraint on $\pi_{i,0}$ implied by the timeless perspective. The constraints in (32) are recovered from the Phillips curve and conditions (21), (25) and (26). The Lagrangian of the optimal monetary policy problem of the small open economy can then be written as:

$$\begin{aligned}\mathcal{L}^s = \sum_{t=0}^{\infty}\beta^t E_0\Bigg\{&\frac{1}{2}\left[\varpi_{1,s}(\tilde{y}_t^{i,s})^2+\varpi_{11,s}(\tilde{s}_{iH,t}^s)^2+\varpi_{12,s}(\tilde{s}_{iF,t}^s)^2+\varpi_{4,s}(\pi_{i,t})^2\right]\\ &+\psi_{1,t}^s\left[\pi_{i,t}-\lambda\left((\varphi+\sigma)\tilde{y}_t^{i,s}+\left(\frac{\gamma_s-\delta_s}{\gamma_s}-\frac{\gamma_b-\delta_b}{2\gamma_b-1}\right)\tilde{s}_{iH,t}^s+\frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}_{iF,t}^s\right)-\upsilon_{i,t}^s\right]-\psi_{1,t-1}^s\pi_{i,t}\\ &+\psi_{2,t}^s\left[\tilde{y}_t^{i,s}-\left(\frac{\delta_s}{\sigma\gamma_s}+\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\right)\tilde{s}_{iH,t}^s+\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\tilde{s}_{iF,t}^s\right]\\ &+\psi_{3,t}^s\left(\tilde{s}_{iH,t}^s-\tilde{s}_{iF,t}^s\right)\Bigg\}\end{aligned} \tag{34}$$

[4]To recover condition (30) first we rewrite the approximation in (19) in deviations from the target using (24). Then, we use conditions (25) and (26) in deviations from the target to express the welfare approximation as a function solely of $\tilde{y}_t^{i,s}$, $\tilde{s}_{iH,t}^s$, $\tilde{s}_{iF,t}^s$ and $\pi_{i,t}$.

Minimizing $\mathcal{L}^s$ with respect to $\tilde{y}_t^{is}$, $\tilde{s}_{iH,t}^s$, $\tilde{s}_{iF,t}^s$ and $\pi_{i,t}$ leads to the following first-order conditions:

$$\varpi_{1,s}\tilde{y}_t^{i,s} = \psi_{1,t}^s\lambda(\varphi+\sigma) - \psi_{2,t}^s$$

$$\varpi_{11,s}\tilde{s}_{iH,t}^s = \psi_{1,t}^s\lambda\left(\frac{\gamma_s-\delta_s}{\gamma_s} - \frac{\gamma_b-\delta_b}{2\gamma_b-1}\right) + \psi_{2,t}^s\left(\frac{\delta_s}{\sigma\gamma_s} + \frac{1-\delta_b}{\sigma(2\gamma_b-1)}\right) - \psi_{3,t}^s$$

$$\varpi_{12,s}\tilde{s}_{iF,t}^s = \psi_{1,t}^s\lambda\frac{\gamma_b-\delta_b}{2\gamma_b-1} - \psi_{2,t}^s\frac{1-\delta_b}{\sigma(2\gamma_b-1)} + \psi_{3,t}^s$$

$$\varpi_{4,s}\pi_{i,t} = -\left(\psi_{1,t}^s - \psi_{1,t-1}^s\right). \tag{35}$$

The solution to this problem allows us to determine the best response of a small open region i in area H under regime A, given the state-contingent paths of $\pi_{F,t}$ and all $\pi_{j,t}$ with $j \in [0, \frac{1}{2})$.

2.2 The case of the monetary union

We now move to the case of the monetary union in area H.

2.2.1 The approximation of the welfare criterion

By (14), the second-order approximation to the average welfare of the households living in area H can be read as:

$$\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t 2\int_0^{\frac{1}{2}} E_0\left[\hat{s}_t^{i\prime} w_s - \frac{1}{2}\hat{s}_t^{i\prime} W_{s,s}\hat{s}_t^i + \hat{s}_t^{i\prime} W_{s,u}\hat{u}_t^i\right] di + t.i.p. \tag{36}$$

$$\hat{s}_t^{i\prime} \equiv \left[\hat{y}_t^i,\ \hat{c}_t^i,\ \pi_{i,t}\right] \qquad w_s' \equiv [-1,\ (1-\tilde{\tau}),\ \ 0] \qquad \hat{u}_t^{i\prime} \equiv \left[a_t^i,\ \mu_t^i\right]$$

$$W_{s,s} \equiv \begin{bmatrix} (\varphi+1) & 0 & 0 \\ 0 & (1-\tilde{\tau})(\sigma-1) & 0 \\ 0 & 0 & \frac{\varepsilon}{\lambda} \end{bmatrix} \qquad W_{s,u} \equiv \begin{bmatrix} (\varphi+1) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

As for the case of the small open economy, we retrieve a purely quadratic approximation to the welfare of the households living in area H using the second-order approximations to the demand and supply curves.

The second-order approximation to the demand curve of a generic region i located in area H can be read as:

$$0 \simeq \hat{s}_t^{i\prime} g_s + \int_0^{\frac{1}{2}} \hat{s}_t^i di' g_{S_H} + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' g_{S_F} + \frac{1}{2}\hat{s}_t^{i\prime} G_{s,s}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^{i\prime} G_{s_H,s_H}\hat{s}_t^i di$$

$$+ \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i\prime} G_{s_F,s_F}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' G_{S_H,S_H}\int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^i di' G_{S_F,S_F}\int_{\frac{1}{2}}^1 \hat{s}_t^i di$$

$$+ \hat{s}_t^{i\prime} G_{s,S_H}\int_0^{\frac{1}{2}} \hat{s}_t^i di + \hat{s}_t^{i\prime} G_{s,S_F}\int_{\frac{1}{2}}^1 \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' G_{S_H,S_F}\int_{\frac{1}{2}}^1 \hat{s}_t^i di + s.o.t.i.p. \tag{37}$$

where:

$$g_s' \equiv [-1,\ \delta_s,\ 0] \qquad g_{S_H}' \equiv [0,\ 2(\delta_b-\delta_s),\ 0] \qquad g_{S_F}' \equiv [0,\ 2(1-\delta_b),\ 0,]$$

$$G_{s,s} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & \delta_s + \omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(1-\delta_b) + 2\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_s^2) + 2(\delta_b - \delta_s) - 2(\omega_1 + \omega_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(\eta\sigma^2(1-\gamma_s^2) - \eta\sigma^2\gamma_b(1-\gamma_b) + 2\omega_1 + 2\omega_2 + \omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\left(\eta\sigma^2\gamma_b(1-\gamma_b) + \omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\left(\omega_1 + \omega_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{s,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{S_H,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(\eta\sigma^2\gamma_b(1-\gamma_b) - \omega_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and with:

$$\omega_3 \equiv \frac{(1-\alpha_b)\eta\sigma(\sigma + 2(1-\alpha_b)(1-\eta\sigma))}{1-2\alpha_b}$$

By integrating (37) over $i \in [0, \frac{1}{2})$, we obtain:

$$0 \simeq \int_0^{\frac{1}{2}} \hat{s}_t^i di' h_{S_H} + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' h_{S_F} + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^{i'} H_{s_H,s_H} \hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i'} H_{s_F,s_F} \hat{s}_t^i di$$

$$+ \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' H_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^i di' H_{S_F,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' H_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di$$

$$+ s.o.t.i.p.$$

with

$$h'_{S_H} \equiv [-1,\ \delta_b,\ 0] \qquad h'_{S_F} \equiv [0,\ (1-\delta_b),\ 0,]$$

$$H_{s_H,s_H} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\eta\sigma^2(1-\gamma_s^2) + \delta_b - \omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-\delta_b) + \omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta\sigma^2(1-\gamma_s^2) - 2\eta\sigma^2\gamma_b(1-\gamma_b) + 2\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\left(\eta\sigma^2(1-\gamma_b)\gamma_b + 2\omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_{S_H,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta\sigma^2\gamma_b(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A symmetric approximation can be retrieved from the good market clearing conditions of the regions in area F, namely:

$$0 \simeq \int_{\frac{1}{2}}^1 \hat{s}_t^i di' f_{S_F} + \int_0^{\frac{1}{2}} \hat{s}_t^i di' f_{S_H} + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i'} F_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^{i'} F_{s_H,s_H} \hat{s}_t^i di$$

$$+ \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^i di' F_{S_F,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' F_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' F_{S_F,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di$$

$$+ s.o.t.i.p.$$

where $f_{S_F} = h_{S_H}$, $f_{S_H} = h_{S_F}$, $F_{s_F,s_F} = H_{s_H,s_H}$, $F_{s_H,s_H} = H_{s_F,s_F}$, $F_{S_F,S_F} = H_{S_H,S_H}$, $F_{S_H,S_H} = H_{S_F,S_F}$ and $F_{S_F,S_H} = H_{S_H,S_F}$. Conversely, the second-order approximation to the Phillips curve of area F can be recovered from:

$$\begin{aligned} V_0 \;&=\; \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\hat{s}_t^{i\prime} v_s + \int_{\frac{1}{2}}^{1}\hat{s}_t^i di' v_{S_F} + \int_0^{\frac{1}{2}}\hat{s}_t^i di' v_{S_H} - \hat{u}_t^{i\prime} v_u + \frac{1}{2}\hat{s}_t^{i\prime} V_{s,s}\hat{s}_t^i \\ &+\hat{s}_t^{i\prime} V_{s,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \hat{s}_t^{i\prime} V_{s,s_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_{t'}^i V_{s_F,s_F}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^{i\prime} V_{s_H,s_H}\hat{s}_t^i di \\ &+\frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' V_{S_F,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^i di' V_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di + \int_0^{\frac{1}{2}}\hat{s}_t^i di' V_{S_H,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di \\ &-\hat{s}_t^{i\prime} V_{s,u}\hat{u}_t^i\Big] + s.o.t.i.p. \end{aligned} \tag{38}$$

where:

$$v_s' \equiv [\varphi,\ \sigma\gamma_s,\ 0] \quad v_{S_F}' \equiv [0,\ 2\sigma(\gamma_b-\gamma_s),\ 0] \quad v_{S_H}' \equiv [0,\ 2\sigma(1-\gamma_b),\ 0] \quad v_u' \equiv [\ (\varphi+1),\ -1]$$

$$V_{s,s} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & \eta\sigma^2(1-\gamma_s)\gamma_s - \sigma^2\gamma_s & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix} \quad V_{s,S_F} \equiv \begin{bmatrix} 0 & 2\sigma(1-\gamma_b) & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)\gamma_s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s,S_H} \equiv \begin{bmatrix} 0 & 2\sigma(\gamma_b-\gamma_s) & 0 \\ 0 & 2\eta\sigma^2\gamma_s(\gamma_s-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(\eta-1)\sigma^2(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(\eta-1)\sigma^2(\gamma_b-\gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(1-\gamma_b)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(\gamma_b-\gamma_s)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V_{S_F,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(1-\gamma_b)(\gamma_b-\gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s,u} \equiv \begin{bmatrix} (\varphi+1)^2 & -(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By integrating (38) over $i \in [\frac{1}{2}, 1]$, we find that:

$$\begin{aligned} \frac{1}{2}V_0 \;&=\; \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' r_{S_F} + \int_0^{\frac{1}{2}}\hat{s}_t^i di' r_{S_H} - \int_{\frac{1}{2}}^{1}\hat{u}_t^i di' r_u + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^{i\prime} R_{s_F,s_F}\hat{s}_t^i di \\ &+\frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^{i\prime} R_{s_H,s_H}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' R_{S_F,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^i di' R_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di \\ &+\int_0^{\frac{1}{2}}\hat{s}_t^i di' R_{S_F,S_H}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di - \int_{\frac{1}{2}}^{1}\hat{s}_t^{i\prime} R_{s_F,u}\hat{u}_t^i di\Big] + s.o.t.i.p. \end{aligned} \tag{39}$$

where:

$$r'_{S_F} \equiv [\varphi,\ \sigma\gamma_b,\ 0] \qquad r'_{S_H} \equiv [0,\ \sigma(1-\gamma_b),\ 0] \qquad r'_u \equiv [(\varphi+1),\ -1]$$

$$R_{s_F,s_F} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & -\eta\gamma_s^2\sigma^2+\eta\gamma_b\sigma^2-\gamma_b\sigma^2 & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix} R_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\eta-1)\sigma^2(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{S_F,S_F} \equiv \begin{bmatrix} 0 & 2\sigma(\gamma_b-\gamma_s) & 0 \\ 2\sigma(\gamma_b-\gamma_s) & 2\eta\sigma^2\left(\gamma_s^2-\gamma_b^2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} R_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{S_F,S_H} \equiv \begin{bmatrix} 0 & 2\sigma(1-\gamma_b) & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)\gamma_b & 0 \\ 0 & 0 & 0 \end{bmatrix} R_{s_F,u} \equiv \begin{bmatrix} (\varphi+1)^2 & -(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Again, by using symmetry we can find a symmetric condition for the regions of area H, namely:

$$\begin{aligned} \frac{1}{2}V_0 &= \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' k_{S_H} + \int_0^{\frac{1}{2}}\hat{s}_t^i di' k_{S_F} - \int_{\frac{1}{2}}^{1}\hat{u}_t^i di' k_u + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'} K_{s_H,s_H}\hat{s}_t^i di \\ &+\frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' K_{S_H,S_H}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^i di' K_{S_F,S_F}\int_0^{\frac{1}{2}}\hat{s}_t^i di + \int_0^{\frac{1}{2}}\hat{s}_t^i di' K_{S_H,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di \\ &-\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'} K_{s_H,u}\hat{u}_t^i di\Big] + s.o.t.i.p. \end{aligned}$$

with $k_{S_H} = r_{S_F}$, $k_{S_F} = r_{S_H}$, $k_u = r_u$, $K_{s_H,s_H} = R_{s_F,s_F}$, $K_{S_H,S_H} = R_{S_F,S_F}$, $K_{S_F,S_F} = R_{S_H,S_H}$, $K_{S_H,S_F} = R_{S_F,S_H}$ and $K_{s_H,u} = R_{s_F,u}$. Then, it can be shown that:

$$w_s = (1-\varphi\zeta_b)h_{S_H} - (\zeta_w-\zeta_b)\varphi f_{S_H} - \zeta_b k_{S_H} - (\zeta_w-\zeta_b)r_{S_H}$$

$$0 = (1-\varphi\zeta_b)h_{S_F} - (\zeta_w-\zeta_b)\varphi f_{S_F} - \zeta_b k_{S_F} - (\zeta_w-\zeta_b)r_{S_F}$$

with $\zeta_b = \frac{1}{2}\frac{\tilde{\tau}}{\sigma+\varphi} - \frac{\delta_b-1+(1/2)\tilde{\tau}}{(1-2\gamma_b)\sigma+(1-2\delta_b)\varphi}$ and $\zeta_w = \frac{\tilde{\tau}}{\sigma+\varphi}$. Hence, we can write the second-order approximation to the average welfare of area H as:

$$\begin{aligned} &-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,s_H}\hat{s}_t^i di + \int_{\frac{1}{2}}^{1}\hat{s}_t^{i'}\Omega_{s_F,s_F}\hat{s}_t^i di + 2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di\right. \\ &+2\int_{\frac{1}{2}}^{1}\hat{s}_t^i di'\Omega_{S_F,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + 4\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di - 2\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,u}\hat{u}_t^i di \\ &\left.-2\int_{\frac{1}{2}}^{1}\hat{s}_t^{i'}\Omega_{s_F,u}\hat{u}_t^i di\right] + t.o.c. \end{aligned} \tag{40}$$

where:

$$\begin{aligned} \Omega_{s_H,s_H} &\equiv W_{s,s} + (1-\varphi\zeta_b)H_{s_H,s_H} - (\zeta_w-\zeta_b)\varphi F_{s_H,s_H} - \zeta_b K_{s_H,s_H} - (\zeta_w-\zeta_b)R_{s_H,s_H} \\ \Omega_{s_F,s_F} &\equiv (1-\zeta_b\varphi)H_{s_F,s_F} - (\zeta_w-\zeta_b)\varphi F_{s_F,s_F} - \zeta_b K_{s_F,s_F} - (\zeta_w-\zeta_b)R_{s_F,s_F} \\ \Omega_{S_H,S_H} &\equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_H,S_H} - \tfrac{1}{2}(\zeta_w-\zeta_b)\varphi F_{S_H,S_H} - \tfrac{1}{2}\zeta_b K_{S_H,S_H} - \tfrac{1}{2}(\zeta_w-\zeta_b)R_{S_H,S_H} \\ \Omega_{S_F,S_F} &\equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_F,S_F} - \tfrac{1}{2}(\zeta_w-\zeta_b)\varphi F_{S_F,S_F} - \tfrac{1}{2}\zeta_b K_{S_F,S_F} - \tfrac{1}{2}(\zeta_w-\zeta_b)R_{S_F,S_F} \\ \Omega_{S_H,S_F} &\equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_H,S_F} - \tfrac{1}{2}(\zeta_w-\zeta_b)\varphi F'_{S_F,S_H} - \tfrac{1}{2}\zeta_b K_{S_H,S_F} - \tfrac{1}{2}(\zeta_w-\zeta_b)R'_{S_F,S_H} \\ \Omega_{s_H,u} &\equiv W_{s,u} - \zeta_b K_{s_H,u} \qquad \Omega_{s_F,u} \equiv -(\zeta_w-\zeta_b)R_{s_F,u} \end{aligned} \tag{41}$$

and Ω_{s_H,s_H}, Ω_{s_F,s_F}, Ω_{S_H,S_H}, Ω_{S_F,S_F}, Ω_{S_H,S_F}, $\Omega_{s_H,u}$ and $\Omega_{s_F,u}$ are respectively equal to:

$$\begin{bmatrix} (1-\zeta_b(\varphi+1))\varphi & -\zeta_b\sigma\gamma_s & 0 \\ -\zeta_b\sigma\gamma_s & \omega_{sHsH} & 0 \\ 0 & 0 & \frac{(1-\zeta_b(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} -(\zeta_w-\zeta_b)(\varphi+1)\varphi & -(\zeta_w-\zeta_b)\sigma\gamma_s & 0 \\ -(\zeta_w-\zeta_b)\sigma\gamma_s & \omega_{sFsF} & 0 \\ 0 & 0 & -\frac{((\zeta_w-\zeta_b)(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\zeta_b\sigma\left(\gamma_b-\gamma_s\right) & 0 \\ -\zeta_b\sigma\left(\gamma_b-\gamma_s\right) & \omega_{SHSH} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -(\zeta_w-\zeta_b)\sigma\left(\gamma_b-\gamma_s\right) & 0 \\ -(\zeta_w-\zeta_b)\sigma\left(\gamma_b-\gamma_s\right) & \omega_{SFSF} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\zeta_b\sigma\left(1-\gamma_b\right) & 0 \\ -(\zeta_w-\zeta_b)\sigma\left(1-\gamma_b\right) & \omega_{SHSF} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} (1-\zeta_b(\varphi+1))(\varphi+1) & \zeta_b(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -(\zeta_w-\zeta_b)(\varphi+1)^2 & (\zeta_w-\zeta_b)(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with:

$$\begin{aligned}
\omega_{sHsH} \equiv\ & (\sigma-1)\left(1-\tilde{\tau}\right) \\
& + (1-\zeta_b\varphi)\left(-\eta\sigma^2\left(1-\gamma_s^2\right)+\delta_b-\omega_3\right) \\
& - (\zeta_w-\zeta_b)\varphi(1-\delta_b+\omega_3) \\
& - \zeta_b\left(-\eta\sigma^2\gamma_s^2+\eta\sigma^2\gamma_b-\sigma^2\gamma_b\right) \\
& - (\zeta_w-\zeta_b)\left(\eta\sigma^2\left(1-\gamma_b\right)-\sigma^2\left(1-\gamma_b\right)\right) \\
\omega_{sFsF} \equiv\ & (1-\zeta_b\varphi)(1-\delta_b+\omega_3) \\
& - (\zeta_w-\zeta_b)\varphi(-\eta\sigma^2\left(1-\gamma_s^2\right)+\delta_b-\omega_3) \\
& - \zeta_b\left(\eta\sigma^2\left(1-\gamma_b\right)-\sigma^2\left(1-\gamma_b\right)\right) \\
& + (\zeta_w-\zeta_b)\left(\eta\sigma^2\gamma_s^2-\eta\sigma^2\gamma_b+\sigma^2\gamma_b\right) \\
\omega_{SHSH} \equiv\ & (1-\zeta_b\varphi)(\eta\sigma^2\left(1-\gamma_s^2-\gamma_b\left(1-\gamma_b\right)\right)+\omega_3) \\
& + (\zeta_w-\zeta_b)\varphi(\eta\sigma^2\gamma_b\left(1-\gamma_b\right)+\omega_3) \\
& + \zeta_b\eta\sigma^2\left(\gamma_b^2-\gamma_s^2\right) \\
& + (\zeta_w-\zeta_b)\eta\sigma^2\left(1-\gamma_b\right)^2 \\
\omega_{SFSF} \equiv\ & -(1-\zeta_b\varphi)(\eta\sigma^2\gamma_b\left(1-\gamma_b\right)+\omega_3) \\
& - (\zeta_w-\zeta_b)\varphi(\eta\sigma^2\left(1-\gamma_s^2-\gamma_b\left(1-\gamma_b\right)\right)+\omega_3) \\
& + \zeta_b\eta\sigma^2\left(1-\gamma_b\right)^2 \\
& + (\zeta_w-\zeta_b)\eta\sigma^2\left(\gamma_b^2-\gamma_s^2\right) \\
\omega_{SHSF} \equiv\ & (1-\zeta_b\varphi)\eta\sigma^2\gamma_b\left(1-\gamma_b\right) \\
& - (\zeta_w-\zeta_b)\varphi\eta\sigma^2\gamma_b\left(1-\gamma_b\right) \\
& + \zeta_b\eta\sigma^2\left(1-\gamma_b\right)\gamma_b \\
& + (\zeta_w-\zeta_b)\eta\sigma^2\left(1-\gamma_b\right)\gamma_b
\end{aligned}$$

Now, we rewrite the welfare approximation in (40) as:

$$
\begin{aligned}
-\frac{2}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0 \Bigg[& \frac{1}{2}\int_0^{\frac{1}{2}} \left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)' \Omega_{s_H,s_H} \left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right) di \\
& + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)' \Omega_{s_F,s_F} \left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right) di \\
& - \int_0^{\frac{1}{2}} \left(\hat{s}_t^i - 2\int_0^{\frac{1}{2}} \hat{s}_t^i di\right)' \Omega_{s_H,u} \left(\hat{u}_t^i - 2\int_0^{\frac{1}{2}} \hat{u}_t^i di\right) di \\
& - \int_{\frac{1}{2}}^1 \left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)' \Omega_{s_F,u} \left(\hat{u}_t^i - 2\int_{\frac{1}{2}}^1 \hat{u}_t^i di\right) di \\
& + \int_0^{\frac{1}{2}} \hat{s}_t^i di' (\Omega_{s_H,s_H} + \Omega_{S_H,S_H}) \int_0^{\frac{1}{2}} \hat{s}_t^i di \\
& + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' (\Omega_{s_F,s_F} + \Omega_{S_F,S_F}) \int_{\frac{1}{2}}^1 \hat{s}_t^i di \\
& + 2\int_0^{\frac{1}{2}} \hat{s}_t^i di' \Omega_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di \\
& - 2\int_0^{\frac{1}{2}} \hat{s}_t^i di' \Omega_{s_H,u} \int_0^{\frac{1}{2}} \hat{u}_t^i di \\
& - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di' \Omega_{s_F,u} \int_{\frac{1}{2}}^1 \hat{u}_t^i di \Bigg] + t.o.c. \qquad (42)
\end{aligned}
$$

Notice that the components expressed as the difference between specific-country and average-union variables can be considered terms out of control of the policy maker (even if they should be taken into account in the welfare evaluation). Indeed, movements in the common nominal interest rate can just influence the average union economic performance. Thus, (42) can be read as:

$$
\begin{aligned}
-\frac{2}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0 \Bigg[& \int_0^{\frac{1}{2}} \hat{s}_t^i di' (\Omega_{s_H,s_H} + \Omega_{S_H,S_H}) \int_0^{\frac{1}{2}} \hat{s}_t^i di + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' (\Omega_{s_F,s_F} + \Omega_{S_F,S_F}) \int_{\frac{1}{2}}^1 \hat{s}_t^i di \\
& + 2\int_0^{\frac{1}{2}} \hat{s}_t^i di' \Omega_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di - 2\int_0^{\frac{1}{2}} \hat{s}_t^i di' \Omega_{s_H,u} \int_0^{\frac{1}{2}} \hat{u}_t^i di - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di' \Omega_{s_F,u} \int_{\frac{1}{2}}^1 \hat{u}_t^i di \Bigg] \\
& + t.o.c. \qquad (43)
\end{aligned}
$$

which approximates the welfare criterion in (36) in a purely quadratic way.

2.2.2 The welfare-relevant target

The next step consists in expressing (43) in deviations from the welfare-relevant target of the monetary union. In order to do so, we need first to determine this target. Consider that (43) can be rewritten as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}'_{H,t}(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\hat{s}_{H,t}+\frac{1}{2}\hat{s}'_{F,t}(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\hat{s}_{F,t}\right.$$
$$\left.+\hat{s}'_{H,t}\Omega_{S_H,S_F}\hat{s}_{F,t}-\hat{s}'_{H,t}\Omega_{s_H,u}\hat{u}_{H,t}-\hat{s}'_{F,t}\Omega_{s_F,u}\hat{u}_{F,t}\right]+t.o.c. \quad (44)$$

which corresponds to:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\varpi_{1,b}\hat{y}^2_{H,t}+\varpi_{2,b}\hat{c}_{H,t}\hat{y}_{H,t}+\frac{1}{2}\varpi_{3,b}\hat{c}^2_{H,t}+\frac{1}{2}\varpi_{4,b}\pi^2_{H,t}+\frac{1}{2}\varpi_{5,b}\hat{y}^2_{F,t}\right.$$
$$+\varpi_{6,b}\hat{c}_{F,t}\hat{y}_{F,t}+\frac{1}{2}\varpi_{7,b}\hat{c}^2_{F,t}+\varpi_{8,b}\hat{y}_{H,t}\hat{c}_{F,t}+\varpi_{9,b}\hat{y}_{F,t}\hat{c}_{H,t}+\varpi_{10,b}\hat{c}_{H,t}\hat{c}_{F,t}$$
$$\left.-\varpi_{11,b}\hat{y}_{H,t}\hat{a}_{H,t}-\varpi_{12,b}\hat{y}_{H,t}\hat{\mu}_{H,t}-\varpi_{13,b}\hat{y}_{F,t}\hat{a}_{F,t}-\varpi_{14,b}\hat{y}_{F,t}\hat{\mu}_{F,t}\right]+t.o.c. \quad (45)$$

where:

$$\begin{aligned}
\varpi_{1,b} &\equiv [1-\zeta_b(\varphi+1)]\varphi\\
\varpi_{2,b} &\equiv -\zeta_b\gamma_b\sigma\\
\varpi_{3,b} &\equiv (\sigma-1)(1-\tilde{\tau})+(1-\zeta_b\varphi)\delta_b-(\zeta_w-\zeta_b)\varphi(1-\delta_b)-\eta\sigma^2\gamma_b(1-\gamma_b)(1-\zeta_w\varphi+\zeta_w)\\
&\quad+\sigma^2[\gamma_b\zeta_b+(\zeta_w-\zeta_b)(1-\gamma_b)]\\
\varpi_{4,b} &\equiv [1-\zeta_b(\varphi+1)]\frac{\varepsilon}{\lambda}\\
\varpi_{5,b} &\equiv -(\zeta_w-\zeta_b)(\varphi+1)\varphi\\
\varpi_{6,b} &\equiv -(\zeta_w-\zeta_b)\gamma_b\sigma\\
\varpi_{7,b} &\equiv (1-\zeta_b\varphi)(1-\delta_b)-(\zeta_w-\zeta_b)\varphi\delta_b-\eta\sigma^2\gamma_b(1-\gamma_b)(1-\zeta_w\varphi+\zeta_w)\\
&\quad+\sigma^2[\zeta_b(1-\gamma_b)+(\zeta_w-\zeta_b)\gamma_b]\\
\varpi_{8,b} &\equiv -\zeta_b(1-\gamma_b)\sigma\\
\varpi_{9,b} &\equiv -(\zeta_w-\zeta_b)(1-\gamma_b)\sigma\\
\varpi_{10,b} &\equiv \eta\sigma^2\gamma_b\,(1-\gamma_b)\,(1-\zeta_w\varphi+\zeta_w)\\
\varpi_{11,b} &\equiv [1-\zeta_b(\varphi+1)](\varphi+1)\\
\varpi_{12,b} &\equiv \zeta_b(\varphi+1)\\
\varpi_{13,b} &\equiv -(\zeta_w-\zeta_b)(\varphi+1)^2\\
\varpi_{14,b} &\equiv (\zeta_w-\zeta_b)(\varphi+1)
\end{aligned} \quad (46)$$

The welfare-relevant target of the monetary authority in area H can be found by maximizing (44) subject to the aggregate market clearing conditions for areas H and F, namely:

$$\begin{aligned}
\hat{y}_{H,t} &= \delta_b\hat{c}_{H,t}+(1-\delta_b)\hat{c}_{F,t}\\
\hat{y}_{F,t} &= \delta_b\hat{c}_{F,t}+(1-\delta_b)\hat{c}_{H,t}
\end{aligned} \quad (47)$$

The Lagrangian associated with this problem can be written:

$$L^b=\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}^{b'}_{H,t}(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\hat{s}^b_{H,t}+\frac{1}{2}\hat{s}^{b'}_{F,t}(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\hat{s}^b_{F,t}\right.$$
$$+\hat{s}^{b'}_{H,t}\Omega_{S_H,S_F}\hat{s}^b_{F,t}-\hat{s}^{b'}_{H,t}\Omega_{s_H,u}\hat{u}_{H,t}-\hat{s}^{b'}_{F,t}\Omega_{s_F,u}\hat{u}_{F,t}$$
$$+\phi^b_{H,t}\left(\hat{s}^{b'}_{H,t}h_{S_H}+\hat{s}^{b'}_{F,t}h_{S_F}\right)$$
$$\left.+\phi^b_{F,t}\left(\hat{s}^{b'}_{F,t}f_{S_F}+\hat{s}^{b'}_{H,t}f_{S_H}\right)\right] \quad (48)$$

where the superscript b indicates the target of the big economy monetary policy maker. Using the first-order conditions of L^b with respect to $\hat{s}^{b'}_{H,t}$, $\hat{s}^{b'}_{F,t}$, $\phi^b_{H,t}$ and $\phi^b_{F,t}$ we obtain:

$$
\begin{aligned}
&(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\hat{s}^b_{H,t}+\Omega_{S_H,S_F}\hat{s}^b_{F,t}-\Omega_{s_H,u}\hat{u}_{H,t}=-\phi^b_{H,t}h_{S_H}-\phi^b_{F,t}f_{S_H}\\
&(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\hat{s}^b_{F,t}+\Omega'_{S_H,S_F}\hat{s}^b_{H,t}-\Omega_{s_F,u}\hat{u}_{F,t}=-\phi^b_{F,t}f_{S_F}-\phi^b_{H,t}h_{S_F}\\
&h_{S_H}\hat{s}^b_{H,t}+h_{S_F}\hat{s}^b_{F,t}=0\\
&f_{S_F}\hat{s}^b_{F,t}+f_{S_H}\hat{s}^b_{H,t}=0.
\end{aligned}
\tag{49}
$$

Alternatively, we can rewrite (49) as:

$$
\begin{aligned}
&\varpi_{1,b}\hat{y}^b_{H,t}+\varpi_{2,b}\hat{c}^b_{H,t}+\varpi_{8,b}\hat{c}^b_{F,t}-\varpi_{11,b}\hat{a}_{H,t}-\varpi_{12,b}\hat{\mu}_{H,t}=\phi^b_{H,t}\\
&\varpi_{5,b}\hat{y}^b_{F,t}+\varpi_{6,b}\hat{c}^b_{F,t}+\varpi_{9,b}\hat{c}^b_{H,t}-\varpi_{13,b}\hat{a}_{F,t}-\varpi_{14,b}\hat{\mu}_{F,t}=\phi^b_{F,t}\\
&\varpi_{2,b}\hat{y}^b_{H,t}+\varpi_{3,b}\hat{c}^b_{H,t}+\varpi_{9,b}\hat{y}^b_{F,t}+\varpi_{10,b}\hat{c}^b_{F,t}=-(\delta_b\phi^b_{H,t}+(1-\delta_b)\phi^b_{F,t})\\
&\varpi_{6,b}\hat{y}^b_{F,t}+\varpi_{7,b}\hat{c}^b_{F,t}+\varpi_{8,b}\hat{y}^b_{H,t}+\varpi_{10,b}\hat{c}^b_{H,t}=-(\delta_b\phi^b_{F,t}+(1-\delta_b)\phi^b_{H,t})\\
&\varpi_{4,b}\pi_{H,t}=0\\
&\hat{y}^b_{H,t}=\delta_b\hat{c}^b_{H,t}+(1-\delta_b)\hat{c}_{F,t}\\
&\hat{y}^b_{F,t}=\delta_b\hat{c}^b_{F,t}+(1-\delta_b)\hat{c}^b_{H,t}.
\end{aligned}
\tag{50}
$$

Since in equilibrium:

$$\hat{s}_{HF,t}=-\sigma(2\gamma_b-1)(\hat{c}_{F,t}-\hat{c}_{H,t}). \tag{51}$$

it can be shown that according to (50):[5]

$$
\begin{aligned}
&[1-\zeta_b(\varphi+1)]\,\widehat{mc}^{e,b}_{H,t}=\zeta_b(\varphi+1)\hat{\mu}_{H,t}+\kappa^b_H\hat{s}^b_{HF,t}\\
&-(\zeta_w-\zeta_b)(\varphi+1)\widehat{mc}^{e,b}_{F,t}=(\zeta_w-\zeta_b)(\varphi+1)\hat{\mu}_{F,t}+\kappa^b_F\hat{s}^b_{HF,t}\\
&\hat{s}^b_{HF,t}=\kappa^b_a(\hat{a}_{F,t}-\hat{a}_{H,t})+\kappa^b_\mu(\hat{\mu}_{F,t}-\hat{\mu}_{H,t})+\kappa^b_{\mu H}\hat{\mu}_{H,t}
\end{aligned}
\tag{52}
$$

where $\widehat{mc}^{e,b}_{H,t}$ and $\widehat{mc}^{e,b}_{F,t}$ are defined consistently with $\widehat{mc}^{e}_{H,t}\equiv(\sigma+\varphi)\,\hat{y}_{H,t}-\frac{\delta_b-\gamma_b}{2\gamma_b-1}\hat{s}_{HF,t}-(1+\varphi)\hat{a}_{H,t}$ and its foreign counterpart and

$$
\begin{aligned}
\kappa^b_H&\equiv\frac{(\gamma_b-\delta_b)\left(1-(2\zeta_b-\zeta_w)(\varphi+1)\right)}{2(2\gamma_b-1)}\\
\kappa^b_F&\equiv\frac{(\gamma_b-\delta_b)\left(1-(2\zeta_b-\zeta_w)(\varphi+1)\right)}{2(2\gamma_b-1)}\\
\kappa^b_a&\equiv\frac{4(\zeta_w-\zeta_b)(\varphi+1)^2\sigma(2\gamma_b-1)}{a^b}\left[1-\zeta_b(\varphi+1)\right]\\
\kappa^b_\mu&\equiv-\frac{4(\zeta_w-\zeta_b)(\varphi+1)\sigma(2\gamma_b-1)}{a^b}\left[1-\zeta_b(\varphi+1)\right]\\
\kappa^b_{\mu H}&\equiv-\frac{4(\zeta_w-\zeta_b)(\varphi+1)\sigma(2\gamma_b-1)}{a^b}
\end{aligned}
\tag{53}
$$

with:

$$a^b\equiv\left(\sigma(2\delta_b-1)+\varphi(2\gamma_b-1)\right)\left(1-\zeta_w(\varphi+1)\right)^2-(\sigma+\varphi)(2\delta_b-1)\left(1-(2\zeta_w-\zeta_w)(\varphi+1)\right)^2$$

The first two conditions in (52) determine the target for the fluctuations in the efficient marginal cost in areas H and F, whereas the other condition expresses the target for the terms-of-trade fluctuations as a function of the exogenous shocks.

[5]This result is not straightforward. A formal proof is available on request.

2.2.3 The optimal policy

By conditions (50) and (51), we can rewrite the objective of the policy maker of the monetary union in area H in deviations from the welfare-relevant targets as:[6]

$$-\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t \frac{1}{2}E_0\left[\varpi_{15,b}(\tilde{y}^b_{H,t})^2+\varpi_{16,b}(\tilde{s}^b_{HF,t})^2+\varpi_{17,b}(\tilde{y}^b_{F,t})^2+\varpi_{4,b}(\pi_{H,t})^2\right]+t.o.c. \quad (54)$$

where:

$$\begin{aligned}
\varpi_{15,b} \equiv & \frac{1}{(2\delta_b-1)}\left[\delta_b\left(\varpi_{2,b}-\varpi_{6,b}\right)+\frac{1}{2}\left(\varpi_{3,b}-\varpi_{7,b}\right)-\left(1-\delta_b\right)\left(\varpi_{8,b}-\varpi_{9,b}\right)\right] \\
& +\varpi_{1,b}+\varpi_{2,w}+\frac{1}{2}\varpi_{3,w}+\varpi_{5,w}+\frac{1}{2}\varpi_{6,w}+\varpi_{7,w}+\varpi_{10,b} \\
\varpi_{16,b} \equiv & \frac{1}{\sigma^2(2\gamma_b-1)^2}\left[(1-\delta_b)(2\delta_b-1)(\varpi_{2,w}+\varpi_{5,w})+(1-\delta_b)\delta_b(\varpi_{3,w}+\varpi_{6,w})\right. \\
& \left.-(1-\delta_b)(2\delta_b-1)\varpi_{7,w}-(1-\delta_b)^2\varpi_{10,b}\right] \\
\varpi_{17,b} \equiv & -\frac{1}{(2\delta_b-1)}\left[\delta_b\left(\varpi_{2,b}-\varpi_{6,b}\right)+\frac{1}{2}\left(\varpi_{3,b}-\varpi_{7,b}\right)-\left(1-\delta_b\right)\left(\varpi_{8,b}-\varpi_{9,b}\right)\right] \\
& +\varpi_{5,b}+\varpi_{2,w}+\frac{1}{2}\varpi_{3,w}+\varpi_{5,w}+\frac{1}{2}\varpi_{6,w}+\varpi_{7,w}+\varpi_{10,b} \qquad (55)
\end{aligned}$$

with:

$$\begin{aligned}
\varpi_{2,w} &\equiv -\zeta_w\gamma_s\sigma \\
\varpi_{3,w} &\equiv (1-\tilde{\tau})(\sigma-1)+(\zeta_w\sigma^2+(1-\zeta_w\varphi))-(1-\gamma_s^2)\eta\sigma^2(\zeta_w+(1-\zeta_w\varphi)) \\
\varpi_{5,w} &\equiv -\zeta_w\sigma(\gamma_b-\gamma_s) \\
\varpi_{6,w} &\equiv \left(1-\gamma_s^2-2\left(1-\gamma_b\right)\gamma_b\right)\eta\sigma^2\left(\zeta_w+(1-\varphi\zeta_w)\right) \\
\varpi_{7,w} &\equiv -\sigma\left(1-\gamma_b\right)\zeta_w
\end{aligned}$$

Moreover, $\tilde{x}^b_t$ stands for the deviation of the variable $\hat{x}_t$ from its target $\hat{x}^b_t$. Before we formulate the optimal monetary policy problem, we should rearrange its constraints as:

$$\begin{aligned}
\tilde{y}^b_{H,t} &= \tilde{y}^b_{F,t}+\frac{(2\delta_b-1)}{\sigma(2\gamma_b-1)}\tilde{s}^b_{HF,t} \\
\pi_{H,t} &= \lambda\left[(\varphi+\sigma)\tilde{y}^b_{H,t}+\frac{(\gamma_b-\delta_b)}{(2\gamma_b-1)}\tilde{s}^b_{HF,t}\right]+\beta E_t\{\pi_{H,t+1}\}+\upsilon^b_{H,t} \\
\pi_{F,t} &= \lambda\left[(\varphi+\sigma)\tilde{y}^b_{F,t}-\frac{(\gamma_b-\delta_b)}{(2\gamma_b-1)}\tilde{s}^b_{HF,t}\right]+\beta E_t\{\pi_{F,t+1}\}+\upsilon^b_{F,t} \qquad (56)
\end{aligned}$$

for all t and where:

$$\begin{aligned}
\upsilon^b_{H,t} &\equiv \frac{\lambda}{1-\zeta_b(\varphi+1)}\left[\left(1+\kappa^b_H\kappa^b_{\mu H}\right)\hat{\mu}_{H,t}+\kappa^b_H\kappa^b_\mu(\hat{\mu}_{F,t}-\hat{\mu}_{H,t})+\kappa^b_H\kappa^b_a(\hat{a}_{F,t}-\hat{a}_{H,t})\right] \\
\upsilon^b_{F,t} &\equiv -\frac{\lambda}{(\zeta_w-\zeta_b)(\varphi+1)}\left[\kappa^b_F\kappa^b_{\mu H}\hat{\mu}_{H,t}+\kappa^b_F\kappa^b_\mu(\hat{\mu}_{F,t}-\hat{\mu}_{H,t})+\kappa^b_F\kappa^b_a(\hat{a}_{F,t}-\hat{a}_{H,t})\right]
\end{aligned}$$

[6] As for the case of the small open economy, first we show that conditions (50) determine the welfare-relevant target of the monetary union, i.e., that (45) can be rewritten in deviations from the allocation satisfying (50). Then, we use condition (51) to express (45) as a function of $\tilde{y}^b_{H,t}$, $\tilde{y}^b_{F,t}$ and $\tilde{s}^b_{HF,t}$ exclusively.

Then, the optimal policy problem of the big economy policy maker can be solved by minimizing the following Lagrangian:

$$\begin{aligned}
\mathcal{L}^b = &\sum_{t=0}^{\infty}\beta^t E_0 \left\{ \frac{1}{2}\left[\varpi_{15,b}(\tilde{y}^b_{H,t})^2 + \varpi_{16,b}(\tilde{s}^b_{HF,t})^2 + \varpi_{17,b}(\tilde{y}^b_{F,t})^2 + \varpi_{4,b}(\pi_{H,t})^2\right]\right. \\
&+ \psi^b_{1,t}\left[\pi_{H,t} - \lambda\left((\varphi+\sigma)\tilde{y}^b_{H,t} + \frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}^b_{HF,t}\right) - \upsilon^b_{H,t}\right] - \psi^b_{1,t-1}\pi_{H,t} \\
&+ \psi^b_{2,t}\left[\pi_{F,t} - \lambda\left((\varphi+\sigma)\tilde{y}^b_{F,t} - \frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}^b_{HF,t}\right) - \upsilon^b_{F,t}\right] - \psi^b_{2,t-1}\pi_{F,t} \\
&\left.+ \psi^b_{3,t}\left[\tilde{y}^b_{H,t} - \tilde{y}^b_{F,t} - \frac{2\delta_b-1}{\sigma(2\gamma_b-1)}\tilde{s}^b_{HF,t}\right]\right\} \qquad (57)
\end{aligned}$$

with respect to $\tilde{y}^b_{H,t}$, $\tilde{s}^b_{HF,t}$, $\tilde{y}^b_{F,t}$ and $\pi_{H,t}$. The corresponding first-order conditions are:

$$\begin{aligned}
\varpi_{15,b}\tilde{y}^b_{H,t} &= \psi^b_{1,t}\lambda(\varphi+\sigma) - \psi^b_{3,t} \\
\varpi_{16,b}\tilde{s}^b_{HF,t} &= \psi^b_{1,t}\lambda\frac{\gamma_b-\delta_b}{2\gamma_b-1} - \psi^b_{2,t}\lambda\frac{\gamma_b-\delta_b}{2\gamma_b-1} + \psi^b_{3,t}\frac{2\delta_b-1}{\sigma(2\gamma_b-1)} \\
\varpi_{17,b}\tilde{y}^b_{F,t} &= \psi^b_{2,t}\lambda(\varphi+\sigma) + \psi^b_{3,t} \\
\varpi_{4,b}\pi_{H,t} &= -\left(\psi^s_{1,t} - \psi^s_{1,t-1}\right). \qquad (58)
\end{aligned}$$

The solution to this problem determines the best response of the area H policy maker under regime B, given the state-contingent path of $\pi_{F,t}$. Notice that once the average-union variables are recovered, the region-specific variables can be found by using the corresponding equilibrium conditions. At the same time, the optimal best responses under both regime A and B can be determined by solving the symmetric problem for the monetary union in area F. Indeed, under this formulation, the optimal policy problem of the authority in area F is independent of whether in area H the small open economies are under monetary autonomy or share the same currency.

2.3 The case of cooperation

As a last step, we solve the optimal policy problem when there is one single policy maker for the entire economy.

2.3.1 The approximation of the welfare criterion

The average welfare of the world economy can be approximated to the second order as:

$$\begin{aligned}
\frac{1}{1-\tilde{\tau}}&\sum_{t=0}^{\infty}\beta^t E_0 \left\{ \int_0^{\frac{1}{2}} \left[\hat{s}^{i'}_t w_s - \frac{1}{2}\hat{s}^{i'}_t W_{s,s}\hat{s}^i_t + \hat{s}^{i'}_t W_{s,u}\hat{u}^i_t\right] di \right. \\
&\left. + \int_{\frac{1}{2}}^1 \left[\hat{s}^{i'}_t w_s - \frac{1}{2}\hat{s}^{i'}_t W_{s,s}\hat{s}^i_t + \hat{s}^{i'}_t W_{s,u}\hat{u}^i_t\right] di \right\} + t.i.p. \qquad (59)
\end{aligned}$$

where w_s, $W_{s,s}$ and $W_{s,u}$ were defined in the Appendix 2.1. It is easy to show that

$$\begin{aligned}
w_s &= (1-\varphi\zeta_w)h_{S_H} + (1-\varphi\zeta_w)\varphi f_{S_H} - \zeta_w k_{S_H} - \zeta_w r_{S_H} \\
w_s &= (1-\varphi\zeta_w)h_{S_F} + (1-\varphi\zeta_w)\varphi f_{S_F} - \zeta_w k_{S_F} - \zeta_w r_{S_F} \qquad (60)
\end{aligned}$$

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^{i\prime}\Omega^w_{s_H,s_H}\hat{s}_t^i di+\frac{1}{2}\int_{\frac{1}{2}}^1\hat{s}_t^{i\prime}\Omega^w_{s_F,s_F}\hat{s}_t^i di+\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega^w_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di\right.$$
$$+\int_{\frac{1}{2}}^1\hat{s}_t^i di'\Omega^w_{S_F,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di+2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega^w_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di-\int_0^{\frac{1}{2}}\hat{s}_t^{i\prime}\Omega^w_{s_H,u}\hat{u}_t^i di$$
$$\left.-\int_{\frac{1}{2}}^1\hat{s}_t^{i\prime}\Omega^w_{s_F,u}\hat{u}_t^i di\right]+t.i.p. \tag{61}$$

where:

$$\begin{aligned}
\Omega^w_{s_H,s_H} &\equiv W_{s,s}+(1-\varphi\zeta_w)H_{s_H,s_H}+(1-\varphi\zeta_w)F_{s_H,s_H}-\zeta_w K_{s_H,s_H}-\zeta_w R_{s_H,s_H}\\
\Omega^w_{s_F,s_F} &\equiv W_{s,s}+(1-\varphi\zeta_w)H_{s_F,s_F}+(1-\varphi\zeta_w)F_{s_F,s_F}-\zeta_w R_{s_F,s_F}-\zeta_w K_{s_F,s_F}\\
\Omega^w_{S_H,S_H} &\equiv \tfrac{1}{2}(1-\varphi\zeta_w)H_{S_H,S_H}+\tfrac{1}{2}(1-\varphi\zeta_w)F_{S_H,S_H}-\tfrac{1}{2}\zeta_w K_{S_H,S_H}-\tfrac{1}{2}\zeta_w R_{S_H,S_H}\\
\Omega^w_{S_F,S_F} &\equiv \tfrac{1}{2}(1-\varphi\zeta_w)H_{S_F,S_F}+\tfrac{1}{2}(1-\varphi\zeta_w)F_{S_F,S_F}-\tfrac{1}{2}\zeta_w K_{S_F,S_F}-\tfrac{1}{2}\zeta_w R_{S_F,S_F}\\
\Omega^w_{S_H,S_F} &\equiv \tfrac{1}{2}(1-\varphi\zeta_w)H_{S_H,S_F}+\tfrac{1}{2}(1-\varphi\zeta_w)F'_{S_F,S_H}-\tfrac{1}{2}\zeta_w K_{S_H,S_F}-\tfrac{1}{2}\zeta_w R'_{S_F,S_H}\\
\Omega^w_{s_H,u} &\equiv W_{s,u}-\zeta_w K_{s_H,u} \qquad \Omega^w_{s_F,u}\equiv W_{s,u}-\zeta_w R_{s_F,u}
\end{aligned}$$

with $\zeta_w=\frac{\tilde{\tau}}{\sigma+\varphi}$ and $\Omega^w_{s_H,s_H}$, $\Omega^w_{S_H,S_H}$, $\Omega^w_{S_H,S_F}$ and $\Omega^w_{s_H,u}$ respectively equal to:

$$\begin{bmatrix}
(1-\zeta_w(\varphi+1))\varphi & -\sigma\gamma_s\zeta_w & 0\\
-\sigma\gamma_s\zeta_w & (1-\tilde{\tau})(\sigma-1)+(\zeta_w\sigma^2+(1-\zeta_w\varphi))-(1-\gamma_s^2)\eta\sigma^2(\zeta_w+(1-\zeta_w\varphi)) & 0\\
0 & 0 & \frac{\varepsilon(1-\zeta_w(\varphi+1))}{\lambda}
\end{bmatrix}$$

$$\begin{bmatrix}
0 & -\sigma\left(\gamma_b-\gamma_s\right)\zeta_w & 0\\
-\sigma\left(\gamma_b-\gamma_s\right)\zeta_w & \left(1-\gamma_s^2-2\left(1-\gamma_b\right)\gamma_b\right)\eta\sigma^2\left(\zeta_w+\left(1-\varphi\zeta_w\right)\right) & 0\\
0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
0 & -\sigma\left(1-\gamma_b\right)\zeta_w & 0\\
-\sigma\left(1-\gamma_b\right)\zeta_w & 2\left(1-\gamma_b\right)\gamma_b\eta\sigma^2\left(\zeta_w+\left(1-\varphi\zeta_w\right)\right) & 0\\
0 & 0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
(1-\zeta_w(\varphi+1))(\varphi+1) & \zeta_w(\varphi+1)\\
0 & 0\\
0 & 0
\end{bmatrix}$$

At the same time, $\Omega^w_{s_F,s_F}=\Omega^w_{s_H,s_H}$, $\Omega^w_{S_F,S_F}=\Omega^w_{S_H,S_H}$ and $\Omega^w_{s_F,u}=\Omega^w_{s_H,u}$. Alternatively, (61) can be written as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\varpi_{1,w}\int_0^1(\hat{y}_t^i)^2di+\varpi_{2,w}\int_0^1\hat{c}_t^i\hat{y}_t^i di+\frac{1}{2}\varpi_{3,w}\int_0^1(\hat{c}_t^i)^2di+\frac{1}{2}\varpi_{4,w}\int_0^1(\pi_{i,t})^2di\right.$$
$$+2\varpi_{5,w}\left(\int_0^{\frac{1}{2}}\hat{y}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di+\int_{\frac{1}{2}}^1\hat{y}_t^i di\int_{\frac{1}{2}}^1\hat{c}_t^i di\right)+\varpi_{6,w}\left(\int_0^{\frac{1}{2}}\hat{c}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di+\int_{\frac{1}{2}}^1\hat{c}_t^i di\int_{\frac{1}{2}}^1\hat{c}_t^i di\right)$$
$$+2\varpi_{7,w}\left(\int_{\frac{1}{2}}^1\hat{y}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di+\int_0^{\frac{1}{2}}\hat{y}_t^i di\int_{\frac{1}{2}}^1\hat{c}_t^i di\right)+2\varpi_{8,w}\int_{\frac{1}{2}}^1\hat{c}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di-\varpi_{9,w}\int_0^1\hat{y}_t^i\hat{a}_t^i di$$
$$\left.-\varpi_{10,w}\int_0^1\hat{y}_t^i\hat{\mu}_t^i di\right]+t.i.p. \tag{62}$$

where:

$$\begin{aligned}
\varpi_{1,w} &\equiv [1-\zeta_w(\varphi+1)]\,\varphi \\
\varpi_{2,w} &\equiv -\zeta_w\gamma_s\sigma \\
\varpi_{3,w} &\equiv (1-\tilde{\tau})(\sigma-1)+(\zeta_w\sigma^2+(1-\zeta_w\varphi))-(1-\gamma_s^2)\eta\sigma^2(\zeta_w+(1-\zeta_w\varphi)) \\
\varpi_{4,w} &\equiv [1-\zeta_w(\varphi+1)]\frac{\varepsilon}{\lambda} \\
\varpi_{5,w} &\equiv -\zeta_w\sigma(\gamma_b-\gamma_s) \\
\varpi_{6,w} &\equiv \left(1-\gamma_s^2-2\left(1-\gamma_b\right)\gamma_b\right)\eta\sigma^2\left(\zeta_w+\left(1-\varphi\zeta_w\right)\right) \\
\varpi_{7,w} &\equiv -\sigma\left(1-\gamma_b\right)\zeta_w \\
\varpi_{8,w} &\equiv 2\left(1-\gamma_b\right)\gamma_b\eta\sigma^2\left(\zeta_w+\left(1-\varphi\zeta_w\right)\right) \\
\varpi_{9,w} &\equiv [1-\zeta_w(\varphi+1)]\left(\varphi+1\right) \\
\varpi_{10,w} &\equiv \zeta_w(\varphi+1) \qquad (63)
\end{aligned}$$

Condition (61) represents a purely quadratic approximation to the welfare criterion in (59).

2.3.2 The welfare-relevant target

The welfare-relevant target of the cooperative policy maker can be determined by maximizing (61) subject to (21) for all $i \in [0, \frac{1}{2})$ and their foreign counterparts. The Lagrangian associated with this problem can be written as:

$$\begin{aligned}
L^w = \sum_{t=0}^{\infty}\beta^t E_0 \Bigg[& \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^{i,w\prime}\Omega^w_{s_H,s_H}\hat{s}_t^{i,w}di + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i,w\prime}\Omega^w_{s_F,s_F}\hat{s}_t^{i,w}di + \int_0^{\frac{1}{2}}\hat{s}_t^{i,w}di'\Omega^w_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^{i,w}di \\
&+ \int_{\frac{1}{2}}^1 \hat{s}_t^{i,w}di'\Omega^w_{S_F,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^{i,w}di + 2\int_0^{\frac{1}{2}}\hat{s}_t^{i,w}di'\Omega^w_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^{i,w}di \\
&- \int_0^{\frac{1}{2}}\hat{s}_t^{iw\prime}\Omega^w_{s_H,u}\hat{u}_t^i di - \int_{\frac{1}{2}}^1 \hat{s}_t^{iw\prime}\Omega^w_{s_F,u}\hat{u}_t^i di \\
&+ \int_0^{\frac{1}{2}}\phi^w_{i,t}\left(\hat{s}_t^{i,w\prime}g_s + \int_0^{\frac{1}{2}}\hat{s}_t^{i,w}di'g_{S_H} + \int_{\frac{1}{2}}^1\hat{s}_t^{i,w}di'g_{S_F}\right)di \\
&+ \int_{\frac{1}{2}}^1\phi^w_{i,t}\left(\hat{s}_t^{i,w\prime}g_s + \int_{\frac{1}{2}}^1\hat{s}_t^{i,w}di'g_{S_H} + \int_0^{\frac{1}{2}}\hat{s}_t^{i,w}di'g_{S_F}\right)di\Bigg] \qquad (64)
\end{aligned}$$

where the superscript w indicates the target of the cooperative policy market. By integrating the first-order conditions of L^w with respect to $\hat{s}_t^{i,w\prime}$ and $\phi^w_{i,t}$, we obtain:

$$\begin{aligned}
&(\Omega^w_{s_H,s_H}+\Omega^w_{S_H,S_H})\hat{s}^w_{H,t}+\Omega^w_{S_H,S_F}\hat{s}^w_{F,t}-\Omega^w_{s_H,u}\hat{u}_{H,t} = -\phi^w_{H,t}(g_s+\frac{1}{2}g_{S_H})-\phi^w_{F,t}\frac{1}{2}g_{S_F} \\
&(\Omega^w_{s_F,s_F}+\Omega^w_{S_F,S_F})\hat{s}^w_{F,t}+\Omega^{w\prime}_{S_H,S_F}\hat{s}^w_{H,t}-\Omega^w_{s_F,u}\hat{u}_{F,t} = -\phi^w_{F,t}(g_s+\frac{1}{2}g_{S_H})-\phi^w_{H,t}\frac{1}{2}g_{S_F} \\
&(g_s+\frac{1}{2}g_{S_H})\hat{s}^w_{H,t}+\frac{1}{2}g_{S_F}\hat{s}^w_{F,t}=0 \\
&(g_s+\frac{1}{2}g_{S_H})\hat{s}^w_{F,t}+\frac{1}{2}g_{S_F}\hat{s}^w_{H,t}=0. \qquad (65)
\end{aligned}$$

Alternatively:

$$\begin{aligned}
&\varpi_{1,w}\hat{y}^w_{H,t}+(\varpi_{2,w}+\varpi_{5,w})\hat{c}^w_{H,t}+\varpi_{7,w}\hat{c}^w_{F,t}-\varpi_{9,w}\hat{a}_{H,t}-\varpi_{10,w}\hat{\mu}_{H,t}=\phi^w_{H,t} \\
&\varpi_{1,w}\hat{y}^w_{F,t}+(\varpi_{2,w}+\varpi_{5,w})\hat{c}^w_{F,t}+\varpi_{7,w}\hat{c}^w_{H,t}-\varpi_{9,w}\hat{a}_{F,t}-\varpi_{10,w}\hat{\mu}_{F,t}=\phi^w_{F,t} \\
&(\varpi_{2,w}+\varpi_{5,w})\hat{y}^w_{H,t}+(\varpi_{3,w}+\varpi_{6,w})\hat{c}^w_{H,t}+\varpi_{7,w}\hat{y}^w_{F,t}+\varpi_{8,w}\hat{c}^w_{F,t}=-(\delta_b\phi^w_{H,t}+(1-\delta_b)\phi^w_{F,t})
\end{aligned}$$

$$(\varpi_{2,w}+\varpi_{5,w})\hat{y}^w_{F,t}+(\varpi_{3,w}+\varpi_{6,w})\hat{c}^w_{F,t}+\varpi_{7,w}\hat{y}^w_{H,t}+\varpi_{8,w}\hat{c}^w_{H,t}=-(\delta_b\phi^w_{F,t}+(1-\delta_b)\phi^w_{H,t})$$
$$\varpi_{4,w}\pi_{H,t}=0$$
$$\varpi_{4,w}\pi_{F,t}=0$$
$$\hat{y}^w_{H,t}=\delta_b\hat{c}^w_{H,t}+(1-\delta_b)\hat{c}^w_{F,t}$$
$$\hat{y}^w_{F,t}=\delta_b\hat{c}^w_{F,t}+(1-\delta_b)\hat{c}^w_{H,t} \tag{66}$$

It can be shown that the conditions above imply that:

$$[1-\zeta_w(\varphi+1)]\,\widehat{mc}^{e,w}_{H,t}=\zeta_w(\varphi+1)\hat{\mu}_{H,t}$$
$$[1-\zeta_w(\varphi+1)]\,\widehat{mc}^{e,w}_{F,t}=\zeta_w(\varphi+1)\hat{\mu}_{F,t}$$
$$\hat{s}^w_{HF,t}=\kappa^w_a(\hat{a}_{F,t}-\hat{a}_{H,t})+\kappa^w_\mu(\hat{\mu}_{F,t}-\hat{\mu}_{H,t}) \tag{67}$$

where $\widehat{mc}^{e,w}_{H,t}$ and $\widehat{mc}^{e,w}_{F,t}$ are defined as above and:

$$\kappa^w_a\equiv\frac{(2\delta_b-1)(\varphi+1)}{(2\delta_b-1)\varphi+(2\gamma_b-1)\sigma}$$
$$\kappa^w_\mu\equiv\frac{(2\delta_b-1)\zeta_w(\varphi+1)}{(1-\zeta_w(\varphi+1))\,[(2\delta_b-1)\varphi+(2\gamma_b-1)\sigma]} \tag{68}$$

The first two conditions in (67) determine the target of the cooperative policy maker for the fluctuations of the efficient marginal cost. The other conditions are the market clearing conditions and the condition that expresses the target of the fluctuations in the terms of trade as a function of the underlying shocks of the model. By using (25), (26) and (51) and the first-order conditions of L^w, we can express (62) in deviations from the welfare-relevant target as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t\frac{1}{2}E_0\left[\varpi_{11,w}(\int_0^{\frac{1}{2}}(\tilde{s}^w_{iH,t})^2di+\int_{\frac{1}{2}}^1(\tilde{s}^w_{iF,t})^2di)+\varpi_{12,w}((\tilde{y}^w_{H,t})^2+(\tilde{y}^w_{F,t})^2)\right.$$
$$\left.+\varpi_{13,w}(\tilde{s}^w_{HF,t})^2+\varpi_{4,w}\int_0^1(\pi_{i,t})^2di\right]+t.i.p. \tag{69}$$

where:

$$\varpi_{11,w}\equiv\frac{1}{\sigma^2\gamma_s^2}\left[\delta_s^2\varpi_{1,w}+2\delta_s\varpi_{2,w}+\varpi_{3,w}\right]$$
$$\varpi_{12,w}\equiv\frac{1}{2}\varpi_{1,w}+\varpi_{2,w}+\frac{1}{2}\varpi_{3,w}+\varpi_{5,w}+\frac{1}{2}\varpi_{6,w}+\varpi_{7,w}+\frac{1}{2}\varpi_{8,w}$$
$$\varpi_{13,w}\equiv\frac{1}{\sigma^2(2\gamma_b-1)^2}\left[(2\delta_b-1)\,(1-\delta_b)(\varpi_{2,w}+\varpi_{3,w}+\varpi_{5,w})+\delta_b\left((1-\delta_b)\varpi_{6,w}\right.\right.$$
$$\left.\left.-\,(2\delta_b-1)\right)\varpi_{7,w}-\frac{1}{2}\left((1-\delta_b)^2+\delta_b^2\right)\varpi_{8,w}\right] \tag{70}$$

Given (69), the objective of the world monetary union policy maker can be written as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^tE_0\left[\frac{1}{2}\varpi_{12,w}((\tilde{y}^w_{H,t})^2+(\tilde{y}^w_{F,t})^2)+\frac{1}{2}\varpi_{13,w}(\tilde{s}^w_{HF,t})^2+\frac{1}{2}\varpi_{4,w}(\pi^2_{H,t}+\pi^2_{F,t})\right]$$
$$+\,t.o.c. \tag{71}$$

In condition (71), similarly to the case of a monetary union in areas H and F, we consider the difference between region-specific and average-area variables as terms out of control of the world monetary union policy maker. This assumption does not affect the choice of the optimal policy. Then, the *timelessly* optimal monetary policy can be retrieved by maximizing (71) subject to the following sequence of constraints:

$$\begin{aligned}
\tilde{y}^w_{H,t} &= \tilde{y}^w_{F,t} + \frac{1-\delta_b}{\sigma(2\gamma_b-1)}\tilde{s}^w_{HF,t} \\
\pi_{H,t} &= \lambda\left[(\varphi+\sigma)\tilde{y}^w_{H,t} + \frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}^w_{HF,t}\right] + \beta E_t\{\pi_{H,t+1}\} + \upsilon^w_{H,t} \\
\pi_{F,t} &= \lambda\left[(\varphi+\sigma)\tilde{y}^w_{F,t} - \frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}^w_{HF,t}\right] + \beta E_t\{\pi_{F,t+1}\} + \upsilon^w_{F,t} \\
\pi_{F,t} - \pi_{H,t} &= \Delta\tilde{s}^w_{HF,t} + \upsilon^w_{HF,t}
\end{aligned} \tag{72}$$

for all t and the constraints on $\pi_{H,0}$ and $\pi_{F,0}$ that render the policy timelessly optimal. In addition in (72) we define

$$\begin{aligned}
\upsilon^w_{H,t} &\equiv \frac{\lambda\mu^H_t}{1-\zeta_w(\varphi+1)} \\
\upsilon^w_{F,t} &\equiv \frac{\lambda\mu^F_t}{1-\zeta_w(\varphi+1)} \\
\upsilon^w_{HF,t} &\equiv \kappa^w_a(\Delta\hat{a}_{F,t} - \Delta\hat{a}_{H,t}) + \kappa^w_\mu(\Delta\hat{\mu}_{F,t} - \Delta\hat{\mu}_{H,t})
\end{aligned}$$

and $\tilde{x}^w_t \equiv \hat{x}_t - \hat{x}^w_t$ so that $\tilde{x}^w_t$ represents the gap of the variable $\hat{x}_t$ from its target $\hat{x}^w_t$. The associated Lagrangian can be written as:

$$\begin{aligned}
\mathcal{L}^w = \sum_{t=0}^{\infty}\beta^t E_0\Bigg\{ & \frac{1}{2}\left[\varpi_{12,w}((\tilde{y}^w_{H,t})^2 + (\tilde{y}^w_{F,t})^2) + \varpi_{13,w}(\tilde{s}^w_{HF,t})^2 + \varpi_{4,w}(\pi^2_{H,t} + \pi^2_{F,t})\right] \\
&+ \psi^w_{1,t}\left[\pi_{H,t} - \lambda\left((\varphi+\sigma)\tilde{y}^w_{H,t} + \frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}^w_{HF,t}\right) - \upsilon^w_{H,t}\right] - \psi^w_{1,t-1}\pi_{H,t} \\
&+ \psi^w_{2,t}\left[\pi_{F,t} - \lambda\left((\varphi+\sigma)\tilde{y}^w_{F,t} - \frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}^w_{HF,t}\right) - \upsilon^w_{F,t}\right] - \psi^w_{2,t-1}\pi_{F,t} \\
&+ \psi^w_{3,t}\left[\tilde{y}^w_{H,t} - \tilde{y}^w_{F,t} - \frac{2\delta_b-1}{\sigma(2\gamma_b-1)}\tilde{s}^w_{HF,t}\right] \\
&+ \psi^w_{4,t}\left[\pi_{F,t} - \pi_{H,t} - \Delta\tilde{s}^w_{HF,t} - \upsilon^w_{HF,t}\right]\Bigg\}.
\end{aligned} \tag{73}$$

The first-order conditions of $\mathcal{L}^w$ with respect to $\tilde{y}^w_{H,t}$, $\tilde{s}^w_{HF,t}$, $\tilde{y}^w_{F,t}$, $\pi_{H,t}$ and $\pi_{F,t}$ are:

$$\begin{aligned}
\varpi_{12,w}\tilde{y}^w_{H,t} &= \psi^w_{1,t}\lambda(\varphi+\sigma) - \psi^w_{3,t} \\
\varpi_{13,w}\tilde{s}^w_{FH,t} &= \psi^w_{1,t}\lambda\frac{\gamma_b-\delta_b}{2\gamma_b-1} - \psi^w_{2,t}\lambda\frac{\gamma_b-\delta_b}{2\gamma_b-1} + \psi^w_{3,t}\frac{2\delta_b-1}{\sigma(2\gamma_b-1)} + \psi^w_{4,t} - \beta E_t\left\{\psi^w_{4,t+1}\right\} \\
\varpi_{12,w}\tilde{y}^w_{F,t} &= \psi^w_{2,t}\lambda(\varphi+\sigma) + \psi^w_{3,t} \\
\varpi_{4,w}\pi_{H,t} &= -\left(\psi^w_{1,t} - \psi^w_{1,t-1}\right) + \psi^w_{4,t} \\
\varpi_{4,w}\pi_{F,t} &= -\left(\psi^w_{2,t} - \psi^w_{2,t-1}\right) - \psi^w_{4,t}.
\end{aligned} \tag{74}$$

3 Parametrization of shocks

As anticipated in the main text, we assume that:

$$\begin{aligned}
\hat{a}^i_{t+1} &= \rho_a\hat{a}^i_t + \varepsilon^i_{a,t} \\
\hat{\mu}^i_{t+1} &= \rho_\mu\hat{\mu}^i_t + \varepsilon^i_{\mu,t}
\end{aligned} \tag{75}$$

where $\varepsilon^i_{a,t}$ and $\varepsilon^i_{\mu,t}$ are white noise innovations with zero mean and standard deviation equal to σ_a and σ_μ respectively. Moreover, the innovations to productivity and mark-up shocks can be decomposed into purely idiosyncratic and common components, i.e.,:

$$\varepsilon^i_{a,t} = \eta_{a,t} + \eta^i_{a,t}$$
$$\varepsilon^i_{\mu,t} = \eta_{\mu,t} + \eta^i_{\mu,t}$$

with $\eta^i_{a,t} \equiv \varepsilon^i_{a,t} - \eta_{a,t}$ and $\eta^i_{\mu,t} \equiv \varepsilon^i_{\mu,t} - \eta_{\mu,t}$ being the idiosyncratic components. We assume that:

$$Cov\left\{\eta^i_{a,t}, \eta^j_{a,t}\right\} = \begin{cases} \sigma^2_{a,\eta^i} & i = j \\ 0 & i \neq j \end{cases}$$

and

$$Cov\left\{\eta^i_{\mu,t}, \eta^j_{\mu,t}\right\} = \begin{cases} \sigma^2_{\mu,\eta^i} & i = j \\ 0 & i \neq j. \end{cases}$$

Notice that as a consequence:

$$Cov\left\{\varepsilon^i_{a,t}, \varepsilon^j_{a,t}\right\} = \begin{cases} \sigma^2_{a,\eta} + \sigma^2_{a,\eta^i} & i = j \\ \sigma^2_{a,\eta} & i \neq j \end{cases}$$

and

$$Cov\left\{\varepsilon^i_{\mu,t}, \varepsilon^j_{\mu,t}\right\} = \begin{cases} \sigma^2_{\mu,\eta} + \sigma^2_{\mu,\eta^i} & i = j \\ \sigma^2_{\mu,\eta} & i \neq j. \end{cases}$$

4 The desired steady-state levels under incomplete financial markets

In this appendix we formulate the optimal policy problem to find the optimal subsidy and the corresponding desired levels of steady-state output under incomplete financial markets. Under financial autarky the good market equilibrium condition implies:

$$Y^i_t = \alpha_s \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} \left[C^i_t + 2(\alpha_b - \alpha_s)\mathcal{Q}^{\eta}_{i,H}\Upsilon_{H,t} + 2(1-\alpha_b)\mathcal{Q}^{\eta}_{i,F}\Upsilon_{F,t}\right] \tag{76}$$

for all i and with $Y^i_t \equiv \left[\int_0^1 y_t(h^i)^{\frac{\varepsilon-1}{\varepsilon}} dh^i\right]^{\frac{\varepsilon}{\varepsilon-1}}$, $\Upsilon_{H,t} \equiv \int_0^{\frac{1}{2}} \mathcal{Q}^{-\eta}_{j,H} C^j_t dj$ and $\Upsilon_{F,t} \equiv \int_{\frac{1}{2}}^1 \mathcal{Q}^{-\eta}_{j,F} C^j_t dj$. Moreover, $\mathcal{Q}_{i,H} \equiv \frac{P^*_{H,t}}{\mathcal{E}_{Hi,t}P_{C^i,t}}$ for all $i \in [0,1]$, $\mathcal{Q}_{i,F} \equiv \frac{P^*_{F,t}}{\mathcal{E}_{Fi,t}P_{C^i,t}}$ for all $i \in [0,1]$. At the same time, $\mathcal{Q}_{i,H}$, $\mathcal{Q}_{i,F}$ can be determined as:

$$\mathcal{Q}_{i,H} = \left[\frac{\alpha_b S^{1-\eta}_{i,H} + (1-\alpha_b)S^{1-\eta}_{i,F}}{\alpha_s + (\alpha_b - \alpha_s)S^{1-\eta}_{iH} + (1-\alpha_b)S^{1-\eta}_{iF}}\right]^{\frac{1}{1-\eta}} \qquad i \in [0, \frac{1}{2})$$
$$\mathcal{Q}_{i,H} = \left[\frac{\alpha_b S^{1-\eta}_{i,H} + (1-\alpha_b)S^{1-\eta}_{i,F}}{\alpha_s + (\alpha_b - \alpha_s)S^{1-\eta}_{iF} + (1-\alpha_b)S^{1-\eta}_{iH}}\right]^{\frac{1}{1-\eta}} \qquad i \in [\frac{1}{2}, 1] \tag{77}$$

while $\frac{P_{i,t}}{P_{C^i,t}}$ can be written as:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\alpha_s + (\alpha_b - \alpha_s)S^{1-\eta}_{iH} + (1-\alpha_b)S^{1-\eta}_{iF}\right]^{-\frac{1}{1-\eta}} \qquad i \in [0, \frac{1}{2}). \tag{78}$$

Symmetric conditions apply to $\mathcal{Q}_{i,F}$ and $\frac{P_{i,t}}{P_{C^i,t}}$ with $i \in [\frac{1}{2}, 1]$. In addition, under financial autarky the value of production of each small open economy should be equal to the domestic expenditure for home produced goods i.e.,:

$$\frac{P_{i,t}}{P_{C^i,t}} Y^i_t = C^i_t \tag{79}$$

for all i.

In order to find the desired level of output at the steady state under cooperation, we maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\int_0^1 \left(\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} \right) di \right] \tag{80}$$

with respect to C_t^i, Y_t^i, $S_{i,H}$, $S_{i,F}$, $S_{H,F}$ and $S_{F,H}$ for all $i \in [0,1]$, subject to (76), (79) and their foreign counterparts and taking into account that: 1) $S_{H,F} = \frac{S_{i,F}}{S_{i,H}}$ for all $i \in [0, \frac{1}{2})$, $S_{F,H} = \frac{S_{i,H}}{S_{i,F}}$ for all $i \in [\frac{1}{2}, 1]$ and $S_{H,F} = \frac{1}{S_{F,H}}$; 2) $\mathcal{Q}_{i,H}$, $\mathcal{Q}_{i,F}$ and $P_{i,t}/P_{C^i,t}$ are determined according to (77), (78) and their foreign analogues.

Consider now the case of the small open economy i. Under financial autarky, the desired steady state from the small open policy maker's perspective can be retrieved by maximizing:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} \right]$$

with respect to C_t^i, Y_t^i, $S_{i,H}$ and $S_{i,F}$ subject to (76) and (79) and where $\mathcal{Q}_{i,H}$, $\mathcal{Q}_{i,F}$ and $P_{i,t}/P_{C^i,t}$ are determined consistently with to (77), its foreign akin and (78), while – differently from the case of cooperation – $\Upsilon_{H,t}$ and $\Upsilon_{F,t}$ are taken as given. Moreover, $S_{H,F} = \frac{S_{i,F}}{S_{i,H}}$ for all $i \in [0, \frac{1}{2})$.

In the case of the monetary union policy maker in area H, the desired level of steady-state output can be determined by maximizing:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[2 \int_0^{\frac{1}{2}} \left(\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} \right) di \right] \tag{81}$$

C_t^i, Y_t^i, $S_{i,H}$, $S_{i,F}$, $S_{H,F}$ and $S_{F,H}$ for all $i \in [0,1]$ and subject to:

$$\frac{P_{i,t}}{P_{C^i,t}} = \frac{(1-\tilde{\tau})}{A_t^{i\,\varphi+1}} \frac{Y_t^{i\,\varphi}}{C_t^{i\,-\sigma}} \tag{82}$$

for all $i \in [\frac{1}{2}, 1]$, the constraints (76), (79) and their foreign counterparts.[7] In addition the authority of the monetary union takes into account that: 1) $S_{H,F} = \frac{S_{i,F}}{S_{i,H}}$ for all $i \in [0, \frac{1}{2})$, $S_{F,H} = \frac{S_{i,H}}{S_{i,F}}$ for all $i \in [\frac{1}{2}, 1]$ and $S_{H,F} = \frac{1}{S_{F,H}}$; 2) $\mathcal{Q}_{i,H}$, $\mathcal{Q}_{i,F}$ and $P_{i,t}/P_{C^i,t}$ can be recovered from (77), (78) and their foreign counterparts.

Finally, it can be shown that if the steady state is symmetric once we allow for trade in one riskless international bond, the desired levels of output at the steady state are the same as those resulting from the optimal policy problems just stated.

[7]Condition (82) implicitly states that the policy maker of Area H takes as given the strategy $\tilde{\tau}$ chosen by a symmetric policy maker in Area F.