To appear in the *International Journal of Control*
Vol. 00, No. 00, Month 20XX, 1–26

Gap Metric based Robustness Analysis of Nonlinear Systems with Full and Partial Feedback Linearization

A. Al-Gburi, C. T. Freeman, M. C. French

University of Southampton, United Kingdom
{abokal2005@yahoo.com/cf@ecs.soton.ac.uk/mcf@ecs.soton.ac.uk}

(*Received 00 Month 20XX; accepted 00 Month 20XX*)

This paper uses gap metric analysis to derive robustness and performance margins for feedback linearizing controllers. Distinct from previous robustness analysis, it incorporates the case of output unstructured uncertainties, and is shown to yield general stability conditions which can be applied to both stable and unstable plants. It then expands on existing feedback linearizing control schemes by introducing a more general robust feedback linearizing control design which classifies the system nonlinearity into stable and unstable components and cancels only the unstable plant nonlinearities. This is done in order to preserve the stabilizing action of the inherently stabilizing nonlinearities. Robustness and performance margins are derived for this control scheme, and are expressed in terms of bounds on the plant nonlinearities and the accuracy of the cancellation of the unstable plant nonlinearity by the controller. Case studies then confirm reduced conservatism compared with standard methods.

Keywords: Robustness analysis; Gap metric; Feedback linearization; Nonlinear systems

1. Introduction

Full state feedback techniques for continuous time nonlinear systems have been intensely discussed in the literature (e.g. in Isidori (1989), Sastry (1999) and Nijmeijer and van der Schaft (1990)). The feedback linearization approach of Isidori (1989), which builds on the earlier work of Brockett (1978); Hunt et al. (1983); Jakubczyk and Respondek (1980), is a well known state feedback technique where an exact linearization is performed to the system states via internal feedback. This approach has the attractive feature of yielding dynamics with transparent nominal performance properties. However not all classes of nonlinear systems can be linearized, in which case one of the many alternative methods must be employed to stablise the system (e.g. neural networks, fuzzy control, sliding mode control, backstepping, optimal schemes or combinations of the aforementioned). Given an admissible class of system, a major drawback of feedback linearization is that it relies strongly on exact knowledge of nonlinearities and an exact model of the nonlinear process, which is generally not available. Also, since exact feedback linearization cancels all nonlinearities, it may destroy inherently stabilizing nonlinearities that can be used to stabilize the plant (an example is given in Section 3). This problem was stated in, for example, Freeman and Kokotović (2008); Khalil (2002); Sepulchre et al. (1997).

Many approaches exist to add robustness to state feedback linearization, including applications to systems with structured or unstructured uncertainties. Most research to date is for systems with structured uncertainties, for example systems of the form $\dot{x} = f(x) + g(x)u + \kappa(x)$ where $\|\kappa(x)\| < M \quad \forall x$ and $M < \infty$. In the work presented in Spong and Vidyasagar (1987) a robust state feedback controller is designed to control a nonlinear robotic system. Assuming that the plant nonlinearities are bounded, the stability of this system was established using the small gain theorem. In Spong et al. (1984) a state feedback controller was designed for a robotic manipulator with structured bounded uncertainties. However, this controller was designed using Lyapunov's direct method and did not account for actuator saturation. To solve this problem, an optimal decision strategy was incorporated to realize a robust unsaturated controller. Meanwhile, Khalil (1994) used a state feedback controller to drive the states of the system to a region of attraction and then enlisted a servomechanism to recover robustness and asymptotic tracking properties. Kravaris (1987) proposed a robust nonlinear state feedback control design based on input-output linearization, and robustness of the closed loop system was guaranteed using Lyapunov based analysis.

Another approach is to combine feedback linearization with adaptive control, see e.g. Ortega and Spong (1989); Sastry and Isidori (1989). Here the adaptive controller adds robustness to feedback linearized systems by helping to achieve asymptotic exact cancellation of the system nonlinearity in the presence of parametric uncertainty. However, in this approach a matching condition is required to be placed on the uncertainty of the system. To overcome this problem, backstepping was introduced to adaptive nonlinear control. This was illustrated in Kanellakopoulos et al. (1991), while in Freeman and Kokotović (2008) a study of robust backstepping controller designs was carried out. More results can be found in Marino and Tomei (1996); Slotine and Hedrick (1993).

Although there has been limited research addressing robustness in the presence of unstructured uncertainties, several works have dealt with robustness of feedback linearizing controllers in the presence of input unstructured uncertainties (both additive and multiplicative). In these works the small gain theorem is combined with backstepping to deal with unstructured uncertainty. The analysis is based on the input to state stability (ISS) concept introduced by Sontag (1995) and presented in Jiang and Mareels (1997); Jiang et al. (1994); Krstić et al. (1996); Praly and Wang (1996). However, these designs require the unmodelled dynamics to have bounded ISS-gain. Subsequently, this condition was replaced with a strict passivity condition on the class of the unmodelled dynamics in Hamzi and Praly (2001); Janković et al. (1999). All small gain and strict passivity designs require the unstructured uncertainties to have relative degree zero. Finally, in Kokotović and Arcak (2001) the small gain and strict passivity conditions were relaxed by combining the dynamic nonlinear normalizing design of Krstić et al. (1996) with the backstepping scheme in Arcak et al. (1999).

Other works that address unstructured uncertainties include Taylor et al. (1989), where a robust state feedback linearization controller design is presented for nonlinear systems with parametric and multiplicative uncertainties. Here an adaptive parametric update law was introduced to accommodate large parameters uncertainties. Robust stability for this design was established using LaSalle's theorem. In Chao (1995) robust stability analysis was carried out for a multiple input multiple output (MIMO) nonlinear system under feedback linearization which has multiplicative unstructured uncertainty at the plant input. Meanwhile, Wang and Wen (2009) presented an approach to design robust backstepping controllers for MIMO systems with linear input unstructured uncertainty.

An important tool with which to analyze the robustness of unstructured uncertainties is the gap metric, however few works have applied it to analyze the robustness of feedback linearizing controllers. The gap metric was introduced in Zames and El-Sakkary (1980) to measure the size of coprime factor perturbations. Further development appeared in Georgiou (1988); Vidyasagar (1984), and Georgiou and Smith (1990) then established equivalence between the robust optimization problems associated with the gap metric and the normalized coprime factor perturbation. This paper also derived robust stability conditions for linear perturbed systems based on the gap metric. The gap metric was generalized to a nonlinear setting in Georgiou and Smith (1997), which also extended the associated robust stability conditions. The gap metric based robustness theorems of Georgiou and Smith (1990) and Georgiou and Smith (1997) both define a nominal model, P, and consider the true physical plant to be a perturbation to this model (both models being interpreted as points in the space of all possible plant models, $\mathcal{P}$). A controller, C, is assumed to stabilise the nominal model, and a scalar $\rho > 0$ is computed such that all plants with a gap metric less than or equal to ρ are proven to be stabilized by C. This robustness analysis hence defines a 'ball' of perturbed plants centered on P with radius ρ in the space $\mathcal{P}$ that are stabilized by C.

The only application of the gap metric for analysis of stability of feedback linearizing controllers is Al-Gburi et al. (2013), which employed the gap metric network result of Georgiou and Smith (1997) to analyze the stability of feedback linearizing controllers applied to affine nonlinear systems. Specifically, it addressed the pressing need for robustness analysis of feedback linearizing controllers operating in the presence of output unstructured uncertainties (inverse multiplication uncertainties), with potential for inclusion of more general classes. Apart from, Al-Gburi et al. (2013), this is currently lacking from the literature, which has focused primarily on input uncertainties. The current paper significantly extends Al-Gburi et al. (2013) by addressing both full and partial feedback linearization, where the latter approach allows inherently stabilizing nonlinearities to be preserved (where throughout the paper stability is defined with respect to the zero equilibrium point regulation problem). It focuses on a general class of affine nonlinear systems with output unstructured uncertainties and derives robust performance bounds using gap metric analysis.

This paper is organized as follows: Section 2 introduces the concepts and notation employed subsequently. In Section 3 a class of affine nonlinear systems with unstable linear and stable and unstable nonlinear components is introduced. Then Section 4 undertakes a robustness analysis for this system using the gap metric. In this case the system is assumed to have two nonlinear parts: an unstable nonlinear component canceled by control action, and a useful stable nonlinear component, whose control action is preserved in this approach. Stability conditions for these systems are derived using the gap metric network result introduced in Georgiou and Smith (1997)(Theorem (10)). Section 5 specializes this analysis to an unstable affine nonlinear system with only unstable nonlinear parts, with the controller in this case carrying out an inverting action to cancel all nonlinear system terms. In Section 6 conclusions are drawn and future work described.

In this paper we denote $\mathcal{L}_p^r[0,\infty)$ as the Lebesgue p-space of r-vector valued functions on $[0,\infty)$, with norm $\|\cdot\|_p$. We also denote the corresponding extended Lebesgue p-space as $\mathcal{L}_{p,e}^r[0,\infty)$, which is defined by $\mathcal{L}_{p,e}^r[0,\infty) := \{f(t) :\ 0 \le t < \infty \mid T_\tau f \in \mathcal{L}_p^r[0,\infty), \forall\, \tau > 0\}$ where T_τ is the truncation operator defined by $T_\tau f(t) = f(t)$ on $[0,\tau]$, and zero otherwise. To simplify the notation, we write $\mathcal{L}_p^r$ and $\mathcal{L}_{p,e}^r$ respectively. Systems are represented as operators acting on these spaces (note that this precludes the possibility of finite-time escape, however this is a standard assumption in robust stability literature, see e.g. Georgiou and Smith (1997)). Throughout this paper we also assume the various feedback configurations are well-posed, an assumption which is guaranteed under mild physically motivated conditions on the systems under consideration.

2. Stability Analysis for a Network System Using the Gap Metric

Consider the interconnection of three systems shown in Figure 1(a), which underpins the network analysis of Georgiou and Smith (1997). Let the external signals u_0, x_0, y_0 belong to signal spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ respectively, with each space assumed to take the form $\mathcal{L}_p^r$. Similarly, for $i = 1, 2, 3$ let u_i, x_i and y_i belong to the extended spaces $\mathcal{U}_e$, $\mathcal{X}_e$ and $\mathcal{Y}_e$ respectively, with each space taking the form $\mathcal{L}_{p,e}^r$ for compatible dimension r. The system operators, P_i, map between these spaces and are assumed to satisfy $P_i(0) = 0$ for $i = 1, 2, 3$. Denote $\mathcal{W} := \mathcal{U} \times \mathcal{X} \times \mathcal{Y}$ and $\mathcal{W}_e := \mathcal{U}_e \times \mathcal{X}_e \times \mathcal{Y}_e$.

The closed-loop operator H_{P_1,P_2,P_3} is defined as the mapping from external to internal signals

$$H_{P_1,P_2,P_3} : \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e \times \mathcal{W}_e : \begin{pmatrix} u_0 \\ x_0 \\ y_0 \end{pmatrix} \to \left(\begin{pmatrix} u_1 \\ x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} u_3 \\ 0 \\ y_3 \end{pmatrix} \right) \tag{1}$$

Introducing $\Pi_i : \mathcal{W}_e \times \mathcal{W}_e \times \mathcal{W}_e \to \mathcal{W}_e$ as the natural projection onto the i^{th} $(i = 1, 2, 3)$ component of $\mathcal{W}_e \times \mathcal{W}_e \times \mathcal{W}_e$ then allows the mapping from external signals to those associated with P_i to be

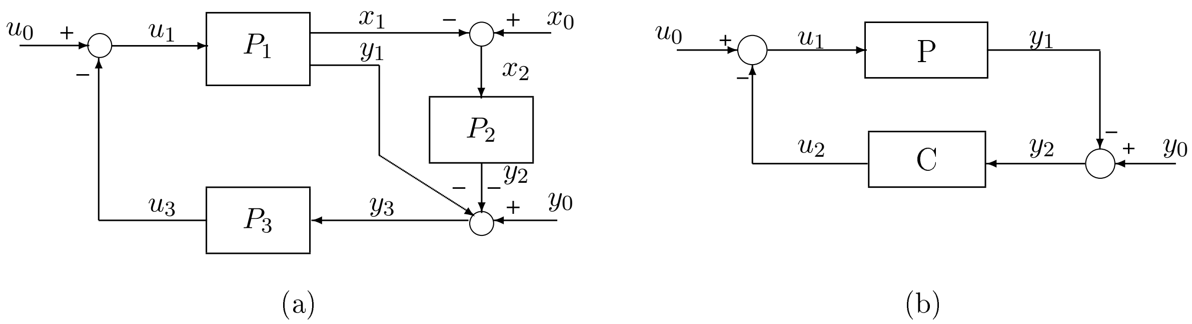

Figure 1. (a) Feedback interconnection of three subsystems $P_i, (i = 1, 2, 3)$, (b) Classical feedback system.

defined by

$$\Pi_{(i)} := \Pi_i H_{P_1,P_2,P_3} : \mathcal{W} \to \mathcal{W}_e : \begin{pmatrix} u_0 \\ x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} u_i \\ x_i \\ y_i \end{pmatrix}, \qquad (i = 1, 2, 3)$$

where $u_2 = x_3 = 0$. Noting that the well-posedness assumption means $\Pi_{(i)} 0 = 0$ for $i = 1, 2, 3$, we can now define the following concept of stability used in, for example, Georgiou and Smith (1997):

Definition 1: The closed-loop $[P_1, P_2, P_3]$ is termed gain stable about equilibrium point 0 if the induced norm of H_{P_1,P_2,P_3} is finite. This is equivalent to the requirement

$$\|\Pi_{(i)}\| = \sup_{\substack{w \in \mathcal{W} \\ \|w\| \neq 0}} \frac{\|\Pi_{(i)} w\|}{\|w\|} < \infty \qquad i = 1, 2, 3. \tag{2}$$

The analysis considered in this paper also uses the concept of the graph of a system, which is the set of all possible *bounded* input-output signals which are compatible with the system description. For the systems in Figure 1(a) these can be formally defined as:

$$\mathcal{G}_{P_1} = \left\{ \begin{pmatrix} u_1 \\ x_1 \\ y_1 \end{pmatrix} : \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = P_1 u_1, \|u_1\| < \infty, \left\| \begin{matrix} x_1 \\ y_1 \end{matrix} \right\| < \infty \right\} \subset \mathcal{W},$$

$$\mathcal{G}_{P_2} = \left\{ \begin{pmatrix} 0 \\ x_2 \\ y_2 \end{pmatrix} : y_2 = P_2 x_2, \|x_2\| < \infty, \|y_2\| < \infty \right\} \subset \mathcal{W},$$

$$\mathcal{G}_{P_3} = \left\{ \begin{pmatrix} u_3 \\ 0 \\ y_3 \end{pmatrix} : u_3 = P_3 y_3, \|u_3\| < \infty, \|y_3\| < \infty \right\} \subset \mathcal{W}.$$

The remainder of this subsection contains results from Georgiou and Smith (1997) which will be employed in subsequent analysis. We first assume that subsystems P_1, P_2 and P_3 in Figure 1(a) are all perturbed to produce systems P_1', P_2', P_3' respectively. System P_i' acts on the same spaces as P_i, and has a graph $\mathcal{G}_{P_i'}$ which is defined analogously to $\mathcal{G}_{P_i}$. Similarly, the feedback operator $H_{P_1',P_2',P_3'}$ is defined analogously to (1), and hence the projection operator from the external signals to those associated with perturbed system P_i' is given by

$$\Pi_{(i)}' := \Pi_i H_{P_1',P_2',P_3'} : \mathcal{W} \to \mathcal{W}_e : \begin{pmatrix} u_0 \\ x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} u_i \\ x_i \\ y_i \end{pmatrix}, \qquad (i = 1, 2, 3)$$

where $u_2 = x_3 = 0$. The gap metric is a measure of the mismatch between the nominal system P_i and the corresponding perturbed system P_i', and is defined as follows:

$$\vec{\delta}(P_i, P_i') = \begin{cases} \inf\{\|(\Phi - I)\mid_{\mathcal{G}_{P_i}}\| : \Phi \text{ is a causal, surjective map from } \mathcal{G}_{P_i} \text{ to } \mathcal{G}_{P_i'} \text{ with } \Phi(0) = 0\}, \\ \infty \text{ if no such operator } \Phi \text{ exists,} \end{cases}$$

$$\delta(P_i, P_i') = \max\left\{\vec{\delta}(P_i, P_i'), \vec{\delta}(P_i', P_i)\right\}.$$

The robust stability theorem can now be stated as:

Theorem 1: *Let the closed-loop arrangement $[P_1, P_2, P_3]$ be gain stable about equilibrium point 0. If*

$$\alpha := \sum_{i=1}^{3} \vec{\delta}(P_i, P_i')\|\Pi_{(i)}\| < 1 \tag{3}$$

then the closed-loop arrangement $[P_1', P_2', P_3']$ is gain stable and its internal signals are bounded as

$$\|\Pi_{(i)}'\| \leq \|\Pi_{(i)}\| \frac{1 + \vec{\delta}(P_i, P_i')}{1 - \alpha}. \tag{4}$$

Proof. See Georgiou and Smith (1997) (Theorem 10). □

Condition (3) specifies the maximum distance that each disturbed plant P_i' must lie from its corresponding nominal description P_i, as measured using the gap metric. It therefore defines a 'ball' in the space of all possible plants P_i', in which P_i' must lie in order to guarantee stability of the overall perturbed system $[P_1', P_2', P_3']$. Each plant P_i' $(i = 1, 2, 3)$ has its own ball, and the radius of each of the three balls depends on the radius of the other two.

While the closed-loop system arrangement $[P_1, P_2, P_3]$ of Figure 1(a) is required in subsequent analysis, the system structure used to define the problem considered in this paper is the classical feedback loop of Figure 1(b) where P is the plant and C is the controller. In this set-up, P' is the perturbed plant, while C is not perturbed (i.e. $C' = C$) since it is specified by the designer and therefore not subject to uncertainty. Theorem 1 can readily be specified to this case by setting $P_1 = \begin{pmatrix} 0 \\ P \end{pmatrix}$ with the corresponding perturbed system $P_1' = \begin{pmatrix} 0 \\ P' \end{pmatrix}$, $P_2 = P_2' = 0, P_3 = P_3' = C$. It follows that Theorem 1 simplifies to the following result:

Lemma 1: *For the system shown in Figure 1(b) with the uncertainty present only in model P with the corresponding perturbed representation P', let*

$$\Pi_{P//C} : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \tag{5}$$

if

$$\vec{\delta}(P, P')\|\Pi_{P//C}\| < 1, \tag{6}$$

then closed loop system $[P', C]$ *is gain stable about equilibrium point* 0 *and its internal signals are bounded as*

$$\|\Pi_{P'//C}\| \leq \|\Pi_{P//C}\| \frac{1 + \vec{\delta}(P, P')}{1 - \|\Pi_{P//C}\|\vec{\delta}(P, P')}.$$

Proof. See Georgiou and Smith (1997) (Theorem 1). □

3. Nonlinear Systems with Stable and Unstable Nonlinear Components

This section introduces an affine nonlinear system which incorporates both stable and unstable nonlinear components and develops a linearizing controller for this system. This system structure is considered since exact feedback linearization typically cancels all plant nonlinearities, and thereby destroys inherently stabilizing nonlinearities that can be usefully exploited to achieve stabilization. A motivating example (from Freeman and Kokotović (2008)) is given next to illustrate this point.

Example 1: Consider the single input, single output (SISO) system:

$$\begin{aligned} \dot{x} &= -x^3 + u + \omega x \\ y &= x, \end{aligned} \tag{7}$$

where u is an unconstrained control input, ω is a disturbance which takes values in the interval $[-1, 1]$. A robustly stabilizing feedback controller for this system is

$$u = x^3 - 2x.$$

This control law achieves feedback linearization, however, it wastefully cancels a beneficial nonlinearity x^3. Furthermore, the term x^3 in this control law adds positive feedback which increases the risk of instability in the control system if uncertainty is present.

The need for a control law that can classify the system nonlinearity into stable and unstable components and subsequently only cancel the unstable component is motivated by Freeman and Kokotović (2008) who introduced an “Inverse Optimal" design which replaced feedback linearization by robust backstepping, and achieved a form of worst case optimality. However, use of backstepping in this approach restricts the design and motivates the approach which is considered next.

3.1 *System Description*

Consider an affine nonlinear SISO system defined by the map

$$\mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} \quad : \quad u_1 \mapsto y, \quad \dot{x} = f(x) + g(x)u_1, \quad y = h(x), \tag{8}$$

where $x = (x_1, x_2, \cdots, x_n)^\top$, and the function $h : \mathbb{R}^n \to \mathbb{R}$ and vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ belong to $C^n(\mathbb{R}^n)$. The following definition is from Khalil (2002):

Definition 2: System (8) is feedback linearizable if it can be transformed into a linear (controllable) system via a state diffeomorphism and a subsequent invertible feedback transformation. Here a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a diffeomorphism if it is smooth, and if its inverse T^{-1} exists and is smooth.

A state diffeomorphism $x^* = T(x)$ transforms the state equation from $x-coordinates$ to $x^*-coordinates$. It is shown in Khalil (2002) that, for a system (8) with full relative degree, a suitable (local) diffeomorphism is given by the map

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_n(x) \end{bmatrix}, \tag{9}$$

where $L_f h(x) = \frac{\partial h}{\partial x} f(x)$ is the Lie derivative of $h(x)$ along $f(x)$, with $L_f^k h(x) = L_f L_f^{k-1} h(x)$ and $L_f^0 h(x) = h$. Applying diffeomorphism (9) realises the normal form of system (8), given by

$$\begin{aligned} P \ : \ & \mathcal{L}_{\infty,e} \to \mathcal{L}^n_{\infty,e} : u_1 \mapsto y_1, \\ & \dot{x}^* = Ax^* + B(f^*(x^*) + g^*(x^*)u_1), \\ & y_1 = (y, \dot{y}, \ldots, y^{(n-1)})^\top = x^*. \end{aligned} \tag{10}$$

Here $A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_1 & a_2 & \ldots & & a_{n-1} & a_n \end{pmatrix}, B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, $a_i \in \mathbb{R}$, and $f^*(x^*)$, $g^*(x^*)$ denote the normal form nonlinearities

$$f^*(x^*) \ := \ L_f^n h(T^{-1}(x^*)) : \mathbb{R}^n \to \mathbb{R}, \tag{11}$$

$$g^*(x^*) \ := \ L_g L_f^{n-1} h(T^{-1}(x^*)) : \mathbb{R}^n \to \mathbb{R}. \tag{12}$$

From (10), it follows immediately that the associated invertible feedback transformation control input

$$u_1 = \frac{1}{g^*(x^*)} \left[v_1 - f^*(x^*) \right]. \tag{13}$$

applied to system (10) will produce a linear mapping between input $v_1 \in \mathcal{L}_{\infty,e}$ and y_1. Since system (10) explicitly includes the mapping $u_1 \mapsto y$ within it, the same control input applied to system (8) produces a linear mapping between v_1 and y (comprising a chain of n integrators). The remainder of this article considers the robustness of feedback controllers applied to system (10), since it contains the dynamics of (8) within it.

Having described full linearizing feedback, we next derive an alternative control action which has scope to only partially cancel the nonlinearites in (10). In generalizing the standard form of (13), the aim is to design stabilizing controllers with improved robustness margins. To do this we first assume that $g^*(x^*)$ within (10) can be separated into components $g_s^*(x^*)$ and $g_u^*(x^*)$, such that

$$g^*(x^*) = g_u^*(x^*) g_s^*(x^*), \tag{14}$$

and similarly, function $f^*(x^*)$ can be separated into components $f_s^*(x^*)$ and $f_u^*(x^*)$, such that

$$f^*(x^*) = g_s^*(x^*) f_u^*(x^*) + f_s^*(x^*). \tag{15}$$

These functional forms enable P to be expressed equivalently as

$$\begin{aligned} P \; : \; & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e}^n : u_1 \mapsto y_1, \\ & \dot{x}^* = Ax^* + B\big(f_s^*(x^*) + g_s^*(x^*)(f_u^*(x^*) + g_u^*(x^*)u_1)\big), \\ & y_1 = (y, \dot{y}, \ldots, y^{(n-1)})^\top = x^*. \end{aligned} \tag{16}$$

The form (16) has effectively split the nonlinearities into two sets: $\{f_s^*(x^*), g_s^*(x^*)\}$ and $\{f_u^*(x^*), g_u^*(x^*)\}$. We desire to completely cancel the latter set by simply modifying the original control action (13) to a form with identical structure (but with $f^*(x^*)$ changed to $f_u^*(x^*)$ and $g^*(x^*)$ changed to $g_u^*(x^*)$):

$$u_1 = \frac{1}{g_u^*(x^*)} \left[v_1 - f_u^*(x^*) \right]. \tag{17}$$

The term v_1 must then be chosen to implement a stabilizing action. In the case of fully linearizing feedback (i.e. setting $f_s^*(x^*) = 0, g_s^*(x^*) = 1$ in (16)), this action typically would have the state feedback form $v_1 = -\tilde{c}^\top x^*$, with $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n)^\top$, to ensure overall stability (i.e. by manipulating the eigenvalues of the resulting state transition matrix $A - B\tilde{c}^\top$).

After applying control action (17) to system (16) we are left with a system that has identical structure to the original system (8) (but with $f^*(x^*)$ exchanged for $f_s^*(x^*)$ and $g^*(x^*)$ exchanged for $g_s^*(x^*)$):

$$\begin{aligned} \tilde{P} \; : \; & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e}^n : v_1 \mapsto y_1, \\ & \dot{x}^* = Ax^* + B(f_s^*(x^*) + g_s^*(x^*)v_1), \\ & y_1 = x^*. \end{aligned} \tag{18}$$

System (18) must now be stabilized using a control action that preserves nonlinearities $f_s^*(x^*)$ and $g_s^*(x^*)$. To obtain a useful definition of $f_s^*(x^*)$ and $g_s^*(x^*)$, we must therefore consider a simpler stabilizing control action for a system of the form (18) that falls short of employing full linearization. Such a control action is not unique, and here we consider the form $v_1 = \frac{1}{g_s^*(x^*)} c^\top x^*$ where $c = (c_1, \ldots, c_n)^\top$. By assuming this action, we can now propose a definition of the useful 'stabilizing' nonlinearities that we wish to preserve that has not previously appeared in the feedback linearization literature:

Definition 3: Function $g_s^* : \mathbb{R}^n \to \mathbb{R}$ and function $f_s^* : \mathbb{R}^n \to \mathbb{R}$ are assumed to stablise the system (18) with control action $v_1 = \frac{1}{g_s^*(x^*)} c^\top x^*$, about equilibrium 0.

Combining the two state feedback terms above yields

$$v_1 = -\tilde{c}^\top x^* + \frac{1}{g_s^*(x^*)} c^\top x^* \tag{19}$$

and substituting this into (17) gives the overall control signal

$$\begin{aligned} u_1 &= \frac{1}{g_u^*(x^*)}\left[-\tilde{c}^\top x^* + \frac{1}{g_s^*(x^*)}c^\top x^* - f_u^*(x^*)\right] \\ &= -\frac{1}{g_u^*(x^*)}\tilde{c}^\top x^* + \frac{1}{g^*(x^*)}c^\top x^* - \frac{1}{g_u^*(x^*)}f_u^*(x^*). \end{aligned} \tag{20}$$

Having derived the feedback control action, we must now express it in the form of operator C within the closed loop structure $[P, C]$ of Figure 1(b), which includes disturbances u_0 and y_0. This is achieved by defining

$$\begin{aligned} C &: \mathcal{L}_{\infty,e}^n \to \mathcal{L}_{\infty,e} : y_2 \mapsto u_2 \\ &\quad u_2 = \frac{1}{g^*(-y_2)}C_{Linear}y_2 + \frac{1}{g_u^*(-y_2)}f_u^*(-y_2) + \frac{1}{g_u^*(-y_2)}\tilde{C}y_2, \end{aligned} \tag{21}$$

where $y_2 = (y_{21}, \ldots, y_{2n})^\top$, and the operators

$$\begin{aligned} C_{Linear} &: \mathcal{L}_{\infty,e}^n \to \mathcal{L}_{\infty,e} : y_2 \mapsto v_2,\ v_2 = c^\top y_2, \\ \tilde{C} &: \mathcal{L}_{\infty,e}^n \to \mathcal{L}_{\infty,e} : y_2 \mapsto v,\ v = -\tilde{c}^\top y_2. \end{aligned}$$

The overall feedback system is shown in Figure 2. Note that representations (17) and (18) have

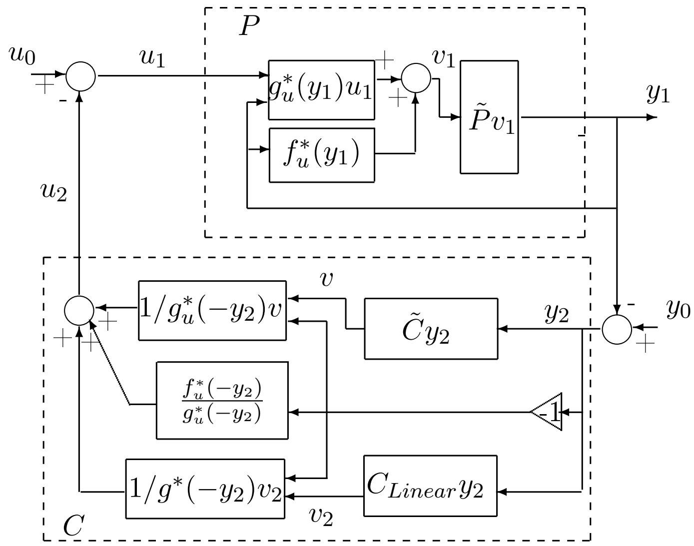

Figure 2. Nonlinear control system with stable/unstable plant nonlinearity

been used to write (10) equivalently as

$$\begin{aligned} P &: \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e}^n : u_1 \mapsto y_1, \\ &\quad y_1 = \tilde{P}(f_u^*(y_1) + g_u^*(y_1)u_1) \end{aligned} \tag{22}$$

$$\tag{23}$$

and the remaining closed loop equations of Figure 2 are:

$$\begin{aligned} &u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2, \quad v_1 = f_u^*(y_1) + g_u^*(y_1)u_1, \\ &v_2 = C_{Linear}y_2, \quad y_1 = \tilde{P}v_1, \quad v = \tilde{C}y_2. \end{aligned} \tag{24}$$

The system shown in Figure 2 is equivalent to the closed loop system $[P, C]$ of Figure 1(b) and hence Lemma 1 can be directly applied to provide a robust stability bound for the system. Unfortunately this requires calculation of the gain bound $\|\Pi_{P//C}\|$ which may not be straightforward. To overcome this difficulty, our approach is to introduce a simplified system with a gain bound which is readily available. By measuring the gap between the simplified and original system, robust performance bounds can then be derived for the original system of Figure 2. However, this approach requires both P and C to be modified, so Lemma 1 cannot be used since it only allows mismatch to exist in P. An alternative route is to extend Lemma 1 to include a $\delta(C, C')$ term, where C' is the simplified version of the controller, however, this would naturally lead to a conservative result.

Therefore, a new configuration of the system shown in Figure 2 will be used. In this configuration all the nonlinear terms will appear in a single block, which then allows it to be compared with a simplified version while confining the resulting mismatch to a single block. To do this it is necessary to employ the more general robust analysis framework of Theorem 1.

4. Robustness Analysis using the Gap Metric

This section undertakes robust stability analysis for the affine nonlinear system shown in Figure 2. The following assumptions on the forms of g^*, f_u^* and g_u^* appearing in equations (12), (15) and (14), respectively, are required in subsequent analysis:

Assumption 1: *Let $g^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the condition:*

$$\exists\, \varepsilon, \mathbf{D} \in \mathbb{R} \quad \textit{such that} \quad 0 < \varepsilon \leq |g^*(x)| \leq \mathbf{D} \;\; \forall \;\; x \in \mathbb{R}^n. \tag{25}$$

Assumption 2: *Let $f_u^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the condition:*

$$\exists\, \mathbf{B}_u \in \mathbb{R} \quad \textit{such that} \quad |f_u^*(x)| \leq \mathbf{B}_u|x| \;\; \forall \;\; x \in \mathbb{R}^n.$$

Assumption 3: *Let $g_u^* : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonlinear function, satisfying the condition:*

$$\exists\, \epsilon, \mathbf{D}_u \in \mathbb{R} \quad \textit{such that} \quad 0 < \epsilon \leq |g_u^*(x)| \leq \mathbf{D}_u \;\; \forall \;\; x \in \mathbb{R}^n.$$

The bounds in Assumptions 1-3 are of a form typically employed in robust analysis literature, for example see Arcak et al. (1999); Kokotović and Arcak (2001); Marino and Tomei (1996); Ortega and Spong (1989); Spong and Vidyasagar (1987). They are not overly restrictive and admit a large class of systems. These assumptions are required in the following analysis since exact linearization of the plant P nonlinearities cannot be achieved by the controller C in the presence of uncertainties. This uncertainty is hence contained by the terms ε, D on $g^*(x)$, $\mathbf{B}_u$ on $f_u^*(x)$ and $\epsilon, \mathbf{D}_u$ on $g_u^*(x)$. Applying the gap metric framework to the system of Figure 2 results in the following theorem:

Theorem 2: *Consider the nonlinear closed loop system shown in Figure 2 and given by (24). Let g^* satisfy Assumption 1, let f_u^* satisfy Assumption 2 and let g_u^* satisfy Assumption 3. Then this system has a robust stability margin.*

This section builds up the results needed to prove this theorem. In particular, the analysis culminates in Corollary 1 which generalises Theorem 2 to provide sufficient conditions for stability of the system.

This analysis considers the triple system configuration shown in Figure 1 and requires the 'network' result of Theorem 1. The route taken is as follows: Since the presence of nonlinear elements in multiple blocks in the system shown in Figure 2 leads to significant conservatism, a new system configuration is used so that Theorem 1 can be applied. In this configuration the unstable nonlinear component of the plant P and the nonlinear component of the controller C are considered to be included along with the nominal plant $\tilde{P}$ in the block P_3' and an external input x_0 is added to the system. Also the feedback input $x_0 - y_1$ is considered as an input, z_1, to the nonlinear components of the plant $f_u^*(z_1), g_u^*(z_1)$, $f_s^*(z_1)$ and $g_s^*(z_1)$, and the feedback input $-y_2$ is considered as an input z_2 to the nonlinear components $\frac{1}{g_u^*(z_2)} f_u^*(z_2)$, $\frac{1}{g^*(z_2)}$ and $\frac{1}{g_u^*(z_2)}$.

Accordingly, consider three signal spaces $\mathcal{U} = \mathcal{L}_{\infty,e}^n, \mathcal{X} = \mathcal{L}_{\infty,e}^n$ and $\mathcal{Y} = \mathcal{L}_{\infty,e}^n$, together with the following augmented signals; let $\hat{v}_2 = -v_2$ and let $u_1' = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top$ and let $u_2' = \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top$ also let the external input u_0 be changed to $u_0' = \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top$, where $d_2 = (d_{21}, \dots, d_{2n})$ and $d_3 = (d_{31}, \dots, d_{3n})$, also let $u_3' = u_0' - u_2' - u_1' = \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top - \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top - \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top = \begin{pmatrix} u_0 & d_0 + v & d_1 - \hat{v}_2 & d_2 - z_1 & d_3 - z_2 \end{pmatrix}^\top$, let $\tilde{v} = d_0 + v$, $\tilde{v}_2 = d_1 - \hat{v}_2$, $\tilde{z}_1 = d_2 - z_1$, $\tilde{z}_2 = d_3 - z_2$ then $u_3' = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$. Also let $x_0' = y_0$, $y_0' = x_0$, $y_3' = y_1$, $x_1' = x_1$, $x_2' = y_2$ and finally $y_1' = y_0' - y_3' = x_0 - y_1$. The resulting system, shown in Figure 3, has the structure of Figure 1(a) and equates to Figure 2 when $x_0 = d_0 = d_1 = d_2 = d_3 = 0$.

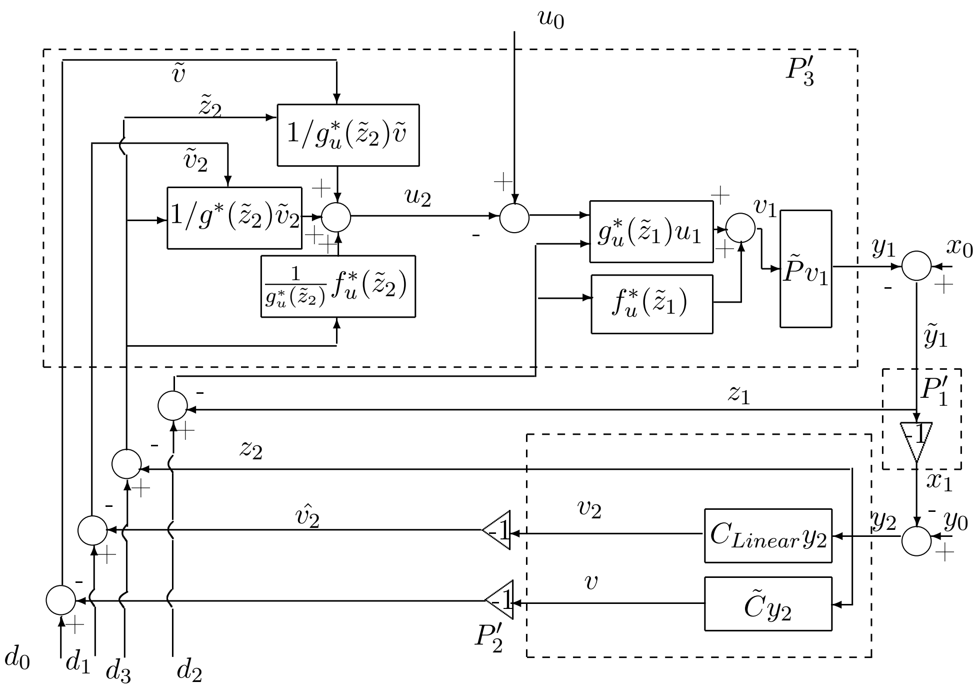

Figure 3. Augmented nonlinear system with stable and unstable nonlinear components

The nominal system configuration is taken to comprise the system components P_1, P_2, P_3 with nonlinearity $g_u^*(z_1)$ being replaced by the linear operator $\pi : (u_1, z_1) \mapsto v_1, v_1 = u_1$ and nonlinearity $\frac{1}{g^*(z_2)}$ being replaced by $\frac{1}{g_s^*(z_2)}$, also $\frac{1}{g_u^*(z_2)}$ being replaced by linear operator $\pi' : (v_2, z_2) \mapsto u_2, u_2 = v_2$, and setting $f_u^*(z_1) = f_u^*(z_2) = 0$. This configuration is shown in Figure 4. Note from the two systems shown in Figure 3 and Figure 4 that $P_1 = P_1'$ and $P_2 = P_2'$ which leads to:

$$\vec{\delta}(P_1, P_1') = 0, \quad \vec{\delta}(P_2, P_2') = 0.$$

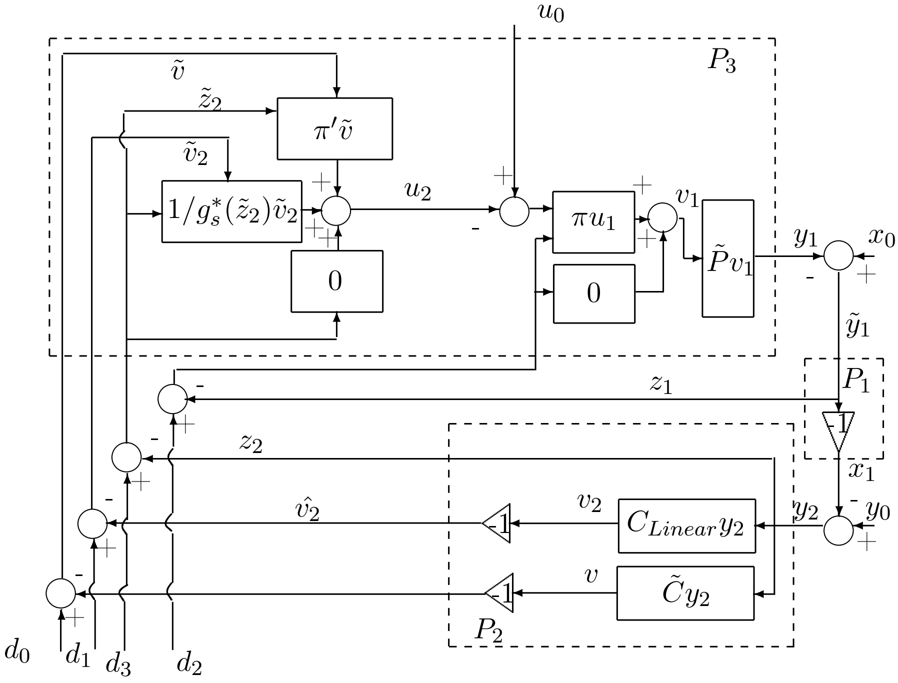

Figure 4. Augmented nonlinear system with only stable nonlinear components

Applying Theorem 1, robust stability condition (3) therefore becomes:

$$\vec{\delta}(P_3, P_3') < \|\Pi_{(3)}\|^{-1}, \tag{26}$$

in which the gap metric measures the difference between the nominal plant $P_3 : u_3' \mapsto y_3', y_3' = \tilde{P}\pi(u_0 - (\frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 + \pi'(\tilde{v}, \tilde{z}_2)), \tilde{z}_1) = \tilde{P}(u_0 - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 - \tilde{v})$ and the perturbed plant $P_3' : u_3' \mapsto y_3', y_3' = \tilde{P}\left(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)\left(u_0 - \left(\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right)\right)\right)$. The plants P_3 and P_3' are shown in Figure 5. It will be shown later in the proof of Theorem 2 that the stability margin for the system shown in Figure 3 is less than or equal to the stability margin corresponding to the original system shown in Figure 2. This is because the latter is a special case of the former. This means that stability condition (26) can also be applied to the original system of Figure 2 to yield robust performance bounds.

In the following two subsections, the two sides of inequality (26) will be considered, namely the gain $\|\Pi_{(3)}\|$ and the gap value $\delta(P_3, P_3')$. This requires the following closed loop operator definitions:

$$\begin{aligned} P_1' \quad : \quad & \mathcal{L}_{\infty,e}^n \to \mathcal{L}_{\infty,e}^{2n} : y_1' \mapsto (x_1', u_1'), x_1' = -y_1', \\ & u_1' = \begin{pmatrix} 0 & 0 & 0 & z_1 & 0 \end{pmatrix}^\top, z_1 = y_1', \end{aligned}$$

where $y_1' = \tilde{y}_1$, and

$$\begin{aligned} P_2' \quad : \quad & \mathcal{L}_{\infty,e}^n \to \mathcal{L}_{\infty,e}^{n+2} : x_2' \mapsto u_2', u_2' = \begin{pmatrix} 0 & -v & \hat{v}_2 & 0 & z_2 \end{pmatrix}^\top, \\ & z_2 = x_2', \hat{v}_2 = -C_{Linear}x_2', v = \tilde{C}x_2', \end{aligned}$$

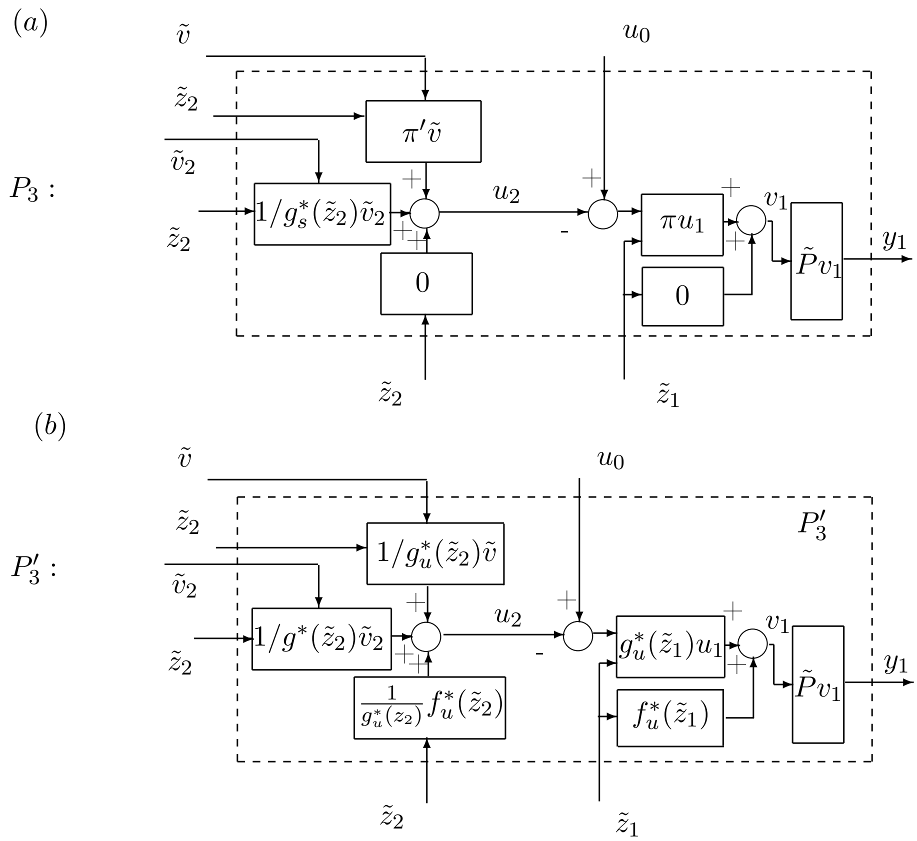

Figure 5. Nonlinear plant mapping:(a) unperturbed, (b) perturbed

and P_3' is given as

$$
\begin{aligned}
P_3' \quad &: \quad \mathcal{L}_{\infty,e}^{3n+3} \to \mathcal{L}_{\infty,e}^{n} : u_3' \mapsto y_3', \\
y_3' \quad &= \quad P_3' u_3', \\
&= \quad \tilde{P}v_1, \\
&= \quad \tilde{P}\left(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)\left(u_0 - \left(\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right)\right)\right).
\end{aligned}
\tag{27}
$$

For the configuration shown in Figure 4, $P_1 = P_1'$, $P_2 = P_2'$, and the operator

$$
\begin{aligned}
P_3 \quad &: \quad \mathcal{L}_{\infty,e}^{3n+3} \to \mathcal{L}_{\infty,e}^{n} : u_3' \mapsto y_3', \\
y_3' \quad &= \quad P_3 u_3', \\
&= \quad \tilde{P}(u_0 - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 - \tilde{v}).
\end{aligned}
\tag{28}
$$

4.1 *Gain Margin Derivation for the Unperturbed Augmented System without Nonlinearity*

In the RHS of inequality (26), the parallel projection $\Pi_{(3)}$ is the mapping from the external signals (u_0', x_0', y_0') to the internal signals $(u_3', 0, y_3')$ for the linearized configuration of the system shown in Figure 4. In satisfying Definition 3, the stabilizing role of nonlinearities f_s^* and g_s^* and control action C_{Linear} implies the existence of a finite $\|\Pi_{(3)}\|$. However this bound may still not be straightforward to compute. Therefore in this section the linear operator $\|\Pi_{(3)}\|$ is explicitly computed for the case where the stabilizing terms are omitted by the designer, i.e. $g_s^*(\cdot) = 1$, $f_s^*(\cdot) = 0$ are selected. This means that $\frac{1}{g_s^*(\tilde{z}_2)}$ is replaced by π and $\tilde{P}$ is replaced by linear operator $P_L : v_1 \mapsto y_1$, $\dot{x}^* = Ax^* + Bv_1$, $y_1 = x^*$. In this linear system configuration $\tilde{v}_2$ is dependent on $\tilde{v}$ and therefore introducing $\tilde{v}_c = \tilde{v} + \tilde{v}_2$, writing u_3' as $u_3' = \begin{pmatrix} u_0 & \tilde{v}_c & \tilde{z}_1 & \tilde{z}_2 \end{pmatrix}^\top$, $C_L = C_{Linear}$, the relation

$$\begin{pmatrix} u_3' \\ 0 \\ y_3' \end{pmatrix} = \Pi_{(3)} \begin{pmatrix} u_0' \\ x_0' \\ y_0' \end{pmatrix}, \qquad \|\Pi_{(3)}\| = \sup_{\|u_0',x_0',y_0'\| \neq 0} \frac{\|u_3', 0, y_3'\|}{\|u_0', x_0', y_0'\|},$$

can be written as

$$\|\Pi_{(3)}\| = \sup_{\|u_0',x_0',y_0'\| \neq 0} \frac{\|u_3', y_3'\|}{\|u_0', x_0', y_0'\|} = \sup_{\|u_0',x_0',y_0'\| \neq 0} \frac{\|u_3', P_3 u_3'\|}{\|u_0', x_0', y_0'\|}. \tag{29}$$

To find expressions for u_3' and $P_3 u_3'$ in terms of u_0', x_0' and y_0', we start with $P_3 u_3'$ as follows:

$$P_3 u_3' = P_L(u_0 - \tilde{v}_2 - \tilde{v}), \tag{30}$$

Let $(I - C_L P_L - \tilde{C} P_L) = G$. Using (30) and routine analysis $P_3 u_3'$ can be expressed as:

$$P_3 u_3' = P_L \begin{pmatrix} I + G^{-1}(C_L P_L + \tilde{C} P_L) & -G^{-1} & -G^{-1} & -G^{-1}(C_L + \tilde{C}) & -G^{-1}(C_L + \tilde{C}) \end{pmatrix} * \begin{pmatrix} u_0 & d_0 & d_1 & x_0' & y_0' \end{pmatrix}^\top.$$

Since $I + (I - C_L P_L - \tilde{C} P_L)^{-1}(C_L P_L + \tilde{C} P_L) = (I - C_L P_L - \tilde{C} P_L)^{-1} = G^{-1}$ and defining

$$\zeta = \Big(P_L(G^{-1} \quad -G^{-1} \quad -G^{-1} \quad 0 \quad 0 \quad -G^{-1}(C_L + \tilde{C}) \quad -G^{-1}(C_L + \tilde{C}))\Big), \tag{31}$$

produces

$$P_3 u_3' = \zeta \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{pmatrix}^\top. \tag{32}$$

Next u_3' is found to be:

$$u_3' = \Lambda \begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{pmatrix}^\top,$$

where $\Lambda =$

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ -G^{-1}(C_L P_L + \tilde{C} P_L) & G^{-1} & G^{-1} & 0 & 0 & G^{-1}(C_L + \tilde{C}) & G^{-1}(C_L + \tilde{C}) \\ P_L G^{-1} & -P_L G^{-1} & -P_L G^{-1} & I & 0 & -P_L G^{-1}(C_L + \tilde{C}) & -(I + P_L G^{-1}(C_L + \tilde{C})) \\ P_L G^{-1} & -P_L G^{-1} & -P_L G^{-1} & 0 & I & -(I - P_L G^{-1}(C_L + \tilde{C})) & -(I + P_L(I - G^{-1}(C_L + \tilde{C}))) \end{pmatrix} \tag{33}$$

and using (29) and defining $Q = \begin{pmatrix} \Lambda \\ \zeta \end{pmatrix}$ we arrive at:

$$\begin{aligned} \|\Pi_{(3)}\| &= \sup_{\|u_0',x_0',y_0'\| \neq 0} \frac{\|u_3', P_3 u_3'\|}{\|u_0', x_0', y_0'\|}, \\ &\leq \sup_{\|u_0',x_0',y_0'\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u_0 & d_0 & d_1 & d_2 & d_3 & x_0' & y_0' \end{matrix} \right\|}{\|u_0', x_0', y_0'\|}. \end{aligned}$$

Since $\begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 \end{pmatrix}^\top = u_0'$, then:

$$\begin{aligned} \|\Pi_{(3)}\| &\leq \sup_{\|u_0', x_0', y_0'\| \neq 0} \frac{\|Q\| \left\| \begin{matrix} u_0' & x_0' & y_0' \end{matrix} \right\|}{\|u_0', x_0', y_0'\|}, \\ &= \|Q\|. \end{aligned}$$

The components of $\|\Pi_{(3)}\|$ are hence the closed loop transfer functions of the linear system $[P_L, C_L + \tilde{C}]$, confirming that $\|\Pi_{(3)}\|$ is finite. Hence from (26) the gap between P_3 and P_3' must satisfy

$$\vec{\delta}(P_3, P_3') < \|Q\|^{-1}. \tag{34}$$

4.2 *Gap Metric Derivation for a Nonlinear System with Stable and Unstable Nonlinearities*

To find $\vec{\delta}(P_3, P_3')$ within inequality (26) the following approach is developed. Note that in this analysis $\tilde{P}$ is a potentially unstable nonlinear plant. First, the graphs for P_3 and P_3' are defined as:

$$\mathcal{G}_{\tilde{P}} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} : y = \tilde{P}u, \|u\| < \infty, \|y\| < \infty \right\}, \tag{35}$$

$$\mathcal{G}_{P_3} := \left\{ \begin{matrix} \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : y_3' = \tilde{P}(u_0 - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v}_2 - \tilde{v}), \\ \left\| \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \right\| < \infty \end{matrix} \right\}, \tag{36}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{matrix} \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : y_3' = \tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - \\ (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))), \left\| \begin{pmatrix} u_0 & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \right\| < \infty \end{matrix} \right\}. \tag{37}$$

To establish a bound on the gap between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$, a surjective map Φ is required between their graphs. First, consider the nonlinear component of the plant P_3' shown in Figure 5(b). For this component the following lemma is used to define Φ:

Lemma 2: *Let g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively, and consider the nonlinear component of plant P_3' shown in Figure 5(b), where:*

$$v_1 = \left(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1) \left(u_0 - \left(\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right) \right) \right). \tag{38}$$

Then

$$\|\tilde{v}_2\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty,$$

and

$$\|v_1\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

Proof. We will first prove that:

$$\|\tilde{v}_2\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|v_1\| < \infty.$$

By Assumption 2, f_u^* is bounded and hence $\|f_u^*(\tilde{z}_1)\|, \|f_u^*(\tilde{z}_2)\| < \infty$. By Assumption 3, g_u^* is bounded and hence $\|g_u^*(\tilde{z}_1)\|, \|1/g_u^*(\tilde{z}_2)\| < \infty$, and using Assumption 2 $\left\|\frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2)\right\| < \infty$. From Assumption 1, g^* is bounded so that $\|1/g^*(\tilde{z}_2)\| < \infty$ and since $\|\tilde{v}_2\| < \infty$, $\|\tilde{v}\| < \infty$, $\|u_0\| < \infty$,

$$\begin{aligned}
\|v_1\| &= \left\| f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})) \right\|, \\
&\leq \|f_u^*(\tilde{z}_1)\| + \|g_u^*(\tilde{z}_1)\| \left(\|u_0\| + \left\| \frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 \right\| + \left\| \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) \right\| + \left\| \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right\| \right) \\
&\leq \|f_u^*(\tilde{z}_1)\| + \|g_u^*(\tilde{z}_1)\|\|u_0\| + \|g_u^*(\tilde{z}_1)\| \left\| \frac{1}{g^*(\tilde{z}_2)} \right\| \|\tilde{v}_2\| + \|g_u^*(\tilde{z}_1)\| \left\| \frac{1}{g_u^*(\tilde{z}_2)} \right\| \|f_u^*(\tilde{z}_2)\| \\
&\quad + \|g_u^*(\tilde{z}_1)\| \left\| \frac{1}{g_u^*(\tilde{z}_2)} \right\| \|\tilde{v}\| < \infty
\end{aligned}$$

as required. Next we will prove that:

$$\|v_1\| < \infty, \|\tilde{v}\| < \infty, \|u_0\| < \infty \Rightarrow \|\tilde{v}_2\| < \infty.$$

By (38):

$$\tilde{v}_2 = -g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right).$$

Since g_u^*, f_u^*, and g^* are all bounded functions, and since $\|v_1\| < \infty$, $\|\tilde{v}\| < \infty$, $\|u_0\| < \infty$, then:

$$\begin{aligned}
\|\tilde{v}_2\| &= \left\| -g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right) \right\|, \\
&\leq \|g^*(\tilde{z}_2)\| (\left\| \frac{1}{g_u^*(\tilde{z}_1)} \right\| (\|v_1\| + \|f_u^*(\tilde{z}_1)\|) + \|u_0\| + \left\| \frac{1}{g_u^*(\tilde{z}_2)} \right\| \|f_u^*(\tilde{z}_2)\| + \left\| \frac{1}{g_u^*(\tilde{z}_2)} \right\| \|\tilde{v}\|) \\
&< \infty.
\end{aligned}$$

as required. $\square$

The graphs for P_3 and P_3' can be represented using coprime factorization functions as shown in the following proposition:

Proposition 1: *Let $\tilde{P}$ be the potentially unstable system given by (18), let g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively, for the systems P_3 and P_3' given by Figure 5 and (28) and (27), respectively. Then the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ satisfy:*

$$\mathcal{G}_{P_3} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \\ \tilde{v}_2 = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - v_1), v_n, \tilde{z}_1, \tilde{z}_2, u_0, \tilde{v} \in \mathcal{U} \end{array} \right\}, \tag{39}$$

$$\mathcal{G}_{P_3'} := \left\{ \begin{array}{c} \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top : \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \\ \tilde{v}_2 = -g^*(\tilde{z}_2) \left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v} \right), \\ v_n, \tilde{z}_1, \tilde{z}_2, u_0, \tilde{v} \in \mathcal{U} \end{array} \right\}. \tag{40}$$

where M, N form a right coprime factorization of $\tilde{P}$ i.e. $\tilde{P} = NM^{-1}$ and are given as:

$$\begin{aligned} M &: v_n \mapsto v_1, \\ \dot{x}^* &= A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), \\ v_1 &= -\frac{1}{g_s^*(x^*)} C_{Linear} x^* + v_n, \end{aligned} \tag{41}$$

and

$$\begin{aligned} N &: v_n \mapsto y_1, \\ \dot{x}^* &= A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), \quad y_1 = x^*. \end{aligned} \tag{42}$$

Proof. To show that $\mathcal{G}_{P_3'}$ given by (40) is equivalent to that of (37), denote the set defined by (40) as $\mathcal{A}$. First we prove that $\mathcal{A} \subset \mathcal{G}_{P_3'}$. Let $\begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{A}$, i.e $\begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - u_0 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}\right)$ where $v_n \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}, u_0 \in \mathcal{U}, \tilde{v} \in \mathcal{U}$. Since $u_0 \in \mathcal{U}, \tilde{v} \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}$ and $\tilde{z}_2 \in \mathcal{U}$ we have $\|u_0\| < \infty, \|\tilde{v}\| < \infty, \|\tilde{z}_1\| < \infty$ and $\|\tilde{z}_2\| < \infty$, respectively. Since $v_1 = Mv_n$, since M is as given in (41) where $\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), v_1 = -\frac{1}{g_s^*(x^*)} C_{Linear} x^* + v_n$, since f_s^* and g_s^* are as defined in Definition 3 and A_c is stable, then M is a bounded operator, since $v_n \in \mathcal{U}$ it follows that $\|v_1\| < \infty$. In the same way, since $y_3' = Nv_n$, since N is as given in (42) where $\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), y_1 = x^*$, since f_s^* and g_s^* are as defined in Definition 3 and A_c is stable, then N is a bounded operator, since $v_n \in \mathcal{U}$ it follows that $\|y_3'\| < \infty$. Since $\|v_1\| < \infty$, $\|u_0\| < \infty$ and $\|\tilde{v}\| < \infty$, and $v_1 = f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))$, it follows from Lemma 2 (second statement), that $\|\tilde{v}_2\| < \infty$. Also given that $y_3' = Nv_n = NM^{-1}v_1 = \tilde{P}v_1$, it follows that $y_3' = \tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$. Thus we conclude that $\mathcal{A} \subset \mathcal{G}_{P_3'}$.

Next we prove that $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Let $\begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}$. Then we have $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$ and $y_3' = \tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$. We need to show that $\begin{pmatrix} v_1 \\ y_3' \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} v_n, \tilde{v}_2 = -g^*(\tilde{z}_2)(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$ and $v_n \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}, \tilde{v} \in \mathcal{U}, u_0 \in \mathcal{U}$. Here $u_0 \in \mathcal{U}, \tilde{v} \in \mathcal{U}, \tilde{z}_1 \in \mathcal{U}, \tilde{z}_2 \in \mathcal{U}$ follow from the definition of $\mathcal{G}_{P_3'}$ and since $\|y_3'\| < \infty$ then $\|\tilde{P}(f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))\| < \infty$, also given that $\|u_0\|, \|\tilde{v}\|, \|\tilde{v}_2\| < \infty$ and defining $v_1 = f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})))$, by Lemma 2 first statement, it follows that $\|v_1\| < \infty$, this leads to $\begin{pmatrix} v_1 \\ \tilde{P}v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ y_3' \end{pmatrix} \in \mathcal{G}_{\tilde{P}}$. Now any element in $\mathcal{G}_{\tilde{P}}$ can be written in the form $\begin{pmatrix} M \\ N \end{pmatrix} v_n$ for some $v_n \in \mathcal{U}$. So let:

$$\begin{pmatrix} M \\ N \end{pmatrix} v_n = \begin{pmatrix} v_1 \\ \tilde{P}v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ y_3' \end{pmatrix}.$$

This leads to $\mathcal{G}_{P_3'} \subset \mathcal{A}$. Hence $\mathcal{G}_{P_3'} = \mathcal{A}$. To show that $\mathcal{G}_{P_3}$ given in (36) is equivalent to that given in (39), set $g^*(\tilde{z}_2) = g_s^*(\tilde{z}_2)$, $g_u^*(\tilde{z}_1) = \pi$, $\frac{1}{g_u^*(\tilde{z}_2)} = \pi'$ and $f_u^*(\tilde{z}_1) = f_u^*(\tilde{z}_2) = 0$. In this case $\mathcal{G}_{P_3}$ follows as a special case, as required. $\square$

The map Φ between $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ can now be defined using the following proposition:

Proposition 2: *Let $\tilde{P}$ be the potentially unstable system given by (18), let g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively, for the systems P_3 and P_3' given by Figure 5 and (28) and (27), respectively. Then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by:*

$$\Phi \begin{pmatrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - Mv_n) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ Nv_n \end{pmatrix} = \begin{pmatrix} u_0 \\ \tilde{v} \\ -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(Mv_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ Nv_n \end{pmatrix}. \quad (43)$$

Furthermore this map is surjective.

Proof. First we need to prove that if

$$x = \begin{pmatrix} u_0'' & \tilde{v}' & g_s^*(\tilde{z}_2')(u_0'' - Mv_n') & \tilde{z}_1' & \tilde{z}_2' & Nv_n' \end{pmatrix}^\top \in \mathcal{G}_{P_3},$$

then $\Phi(x) \in \mathcal{G}_{P_3'}$. Since $x \in \mathcal{G}_{P_3}$ then $\|u_0''\|, \|\tilde{v}'\|, \|\tilde{v}_2'\| = \|g_s^*(\tilde{z}_2')(u_0'' - \tilde{v}' - v_1')\| = \|g_s^*(\tilde{z}_2')(u_0'' - \tilde{v}' - Mv_n')\|, \|\tilde{z}_1'\|, \|\tilde{z}_2'\|, \|y_3''\| < \infty$, $\begin{pmatrix} v_1' \\ y_3'' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n', \tilde{v}_2' = g_s^*(\tilde{z}_2')(u_0'' - \tilde{v}' - v_1')$. Let $y = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top = \Phi(x)$. We need to show that $\tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(Mv_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))\right)$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$, $v_n, \tilde{z}_1, \tilde{z}_2, u_0 \in \mathcal{U}$. It follows from (43) that $u_0 = u_0''$, $\tilde{v} = \tilde{v}'$, $\tilde{z}_1 = \tilde{z}_1'$, $\tilde{z}_2 = \tilde{z}_2'$, $y_3' = y_3''$, $\tilde{v}_2 = -g^*(\tilde{z}_2')\left(\frac{1}{g_u^*(\tilde{z}_1')}(Mv_n' - f_u^*(\tilde{z}_1')) - (u_0'' - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2') - \frac{1}{g_u^*(\tilde{z}_2')\tilde{v}'}))\right)$, then $\|u_0\|$, $\|\tilde{v}\|$, $\|\tilde{z}_1\|$, $\|\tilde{z}_2\|$, $\|y_3'\| < \infty$. Since $v_1' = u_0'' - \tilde{v}' - \frac{1}{g_s^*(\tilde{z}_2')}\tilde{v}_2'$, then by Proposition 1 (39) there exist $v_n' \in \mathcal{U}$ such that $\begin{pmatrix} v_1' & y_3'' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n'$. It follows that $y_3'' = Nv_n' = NM^{-1}v_1'$. Now let $v_1 = v_1'$, and note that $y_3' = y_3'' = NM^{-1}v_1' = NM^{-1}v_1$, then there exists $v_n = v_n'$ such that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Since $v_1 = Mv_n$, with M given in (41) where $\dot{x}^* = A_c x^* + B(f_s^*(x^*) + g_s^*(x^*)v_n), v_1 = -\frac{1}{g_s^*(x^*)}C_{Linear}x^* + v_n$, and since f_s^* and g_s^* are as defined in Definition 3 and A_c is stable, then M is a bounded operator, since $v_n \in \mathcal{U}$ it follows that $\|v_1\| < \infty$. Using Lemma 2 (second statement) as $\tilde{v}_2 = -g^*(\tilde{z}_2)(\frac{1}{g_u^*(\tilde{z}_1)}(Mv_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})) = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right)$ and since $\|u_0\|, \|\tilde{v}\|, \|v_1\| < \infty$ then $\|\tilde{v}_2\| < \infty$. Then:

$$\begin{aligned} \tilde{v}_2 &= -g^*(\tilde{z}_2')\left(\frac{1}{g_u^*(\tilde{z}_1')}(v_1' - f_u^*(\tilde{z}_1')) - (u_0'' - \frac{1}{g_u^*(\tilde{z}_2')}f_u^*(\tilde{z}_2') - \frac{1}{g_u^*(\tilde{z}_2')}\tilde{v}')\right), \\ &= -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right), \end{aligned}$$

and hence y equals the RHS of (43) as required. Next, to prove that Φ is surjective, for each $y \in \mathcal{G}_{P_3'}$ then there exists $x \in \mathcal{G}_{P_3}$ such that $\Phi(x) = y$. Accordingly, let us choose an element:

$$y = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top \in \mathcal{G}_{P_3'}.$$

where $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\|, \|y_3'\| < \infty$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$,

$\tilde{v}_2 = -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(v_1 - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right), v_n \in \mathcal{U}$. Let

$$x = \begin{pmatrix} u_0 \\ \tilde{v} \\ \frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}\tilde{v}_2 + g_s^*(\tilde{z}_2)\left(1 - g_u^*(\tilde{z}_1)\right)u_0 - g_s^*(\tilde{z}_2)\left(1 - \frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}\right)\tilde{v} + g_s^*(\tilde{z}_2)\left(\frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - f_u^*(\tilde{z}_1)\right) \\ \tilde{z}_1 \\ \tilde{z}_2 \\ y_3' \end{pmatrix},$$

we need to show that $x \in \mathcal{G}_{P_3}$ i.e. $v_n, \tilde{v}, \tilde{z}_1, \tilde{z}_2, \tilde{v}_2' = \frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}\tilde{v}_2 + g_s^*(\tilde{z}_2)\left(1 - g_u^*(\tilde{z}_1)\right)u_0 - g_s^*(\tilde{z}_2)\left(1 - \frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}\right)\tilde{v} + g_s^*(\tilde{z}_2)\left(\frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - f_u^*(\tilde{z}_1)\right) \in \mathcal{U}$, $\tilde{v}_2' = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - v_1)$, $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Since $v_1 = f_u^*(\tilde{z}_1) + g_u^*(\tilde{z}_1)(u_0 - (\frac{1}{g^*(\tilde{z}_2)}\tilde{v}_2 + \frac{1}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) + \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v}))$ then $\tilde{v}_2' = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - v_1)$, Since $\|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$ then $\tilde{z}_1, \tilde{z}_2 \in \mathcal{U}$. From Assumption 3, g_u^*, $\frac{1}{g_u^*}$ are bounded functions, and it follows from Assumption 2 that $g_s^*(\tilde{z}_2)$ is a bounded function. From Assumption 2, f_u^* is also a bounded function. Since $\|u_0\|, \|\tilde{v}_2\|, \|\tilde{v}\|, \|\tilde{z}_1\|, \|\tilde{z}_2\| < \infty$, it follows that:

$$\begin{aligned} \|\tilde{v}_2'\| &\leq \left\|\frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}\tilde{v}_2\right\| + \left\|g_s^*(\tilde{z}_2)\left(1 - g_u^*(\tilde{z}_1)\right)u_0\right\| - \left\|g_s^*(\tilde{z}_2)\left(1 - \frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}\right)\tilde{v}\right\| \\ &\quad + \left\|g_s^*(\tilde{z}_2)\left(\frac{g_u^*(\tilde{z}_1)}{g_u^*(\tilde{z}_2)}f_u^*(\tilde{z}_2) - f_u^*(\tilde{z}_1)\right)\right\| \\ &\leq \|g_s^*(\tilde{z}_2)\|\left(1 + \|g_u^*(\tilde{z}_1)\|\right)\|u_0\| + \|g_s^*(\tilde{z}_2)\|\left(1 + \|g_u^*(\tilde{z}_1)\|\left\|\frac{1}{g_u^*(\tilde{z}_2)}\right\|\right)\|\tilde{v}\| \\ &\quad + \|g_s^*(\tilde{z}_2)\|\left(\|g_u^*(\tilde{z}_1)\|\left\|\frac{1}{g_u^*(\tilde{z}_2)}\right\|\|f_u^*(\tilde{z}_2)\| + \|f_u^*(\tilde{z}_1)\|\right) + \|g_u^*(\tilde{z}_1)\|\left\|\frac{1}{g_u^*(\tilde{z}_2)}\right\|\|\tilde{v}_2\| < \infty. \end{aligned}$$

Then by Proposition 1 (40) there exists $v_n \in \mathcal{U}$ such that $\begin{pmatrix} v_1 & y_3' \end{pmatrix}^\top = \begin{pmatrix} M & N \end{pmatrix}^\top v_n$. Hence $x \in \mathcal{G}_{P_3}$ is such that $\Phi(x) = \begin{pmatrix} u_0 & \tilde{v} & \tilde{v}_2 & \tilde{z}_1 & \tilde{z}_2 & y_3' \end{pmatrix}^\top$ as required. $\square$

Using the previous results, a bound on the gap, $\vec{\delta}(P_3, P_3')$, between P_3 and P_3' appearing in the inequality (26) can now be established using the following theorem:

Theorem 3: *Let $\tilde{P}$ be the potentially unstable system given by (18), let g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively. Let P_3 and P_3' be given by Figure 5 and equations (28) and (27), respectively. Then a bound on the gap between P_3 and P_3' is*

$$\vec{\delta}(P_3, P_3') \;\leq\; F_{\delta 2}, \tag{44}$$

where

$$F_{\delta 2} = \left\|\mathbf{D}\left|1 - \frac{1}{\epsilon}\right|, \left|\frac{\mathbf{D}_u}{\epsilon} - 1\right|, \left|\frac{\mathbf{D}_u}{\epsilon} - 1\right|\frac{\mathbf{D}}{\epsilon}, \frac{\mathbf{DB}_u}{\epsilon}, \frac{\mathbf{DB}_u}{\epsilon}\right\| \tag{45}$$

Proof. Using Proposition 2, since $\tilde{P}$ is a potentially unstable system and since g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively, then there exists a surjective map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (43). It follows that the gap between P_3 and P_3' is bounded as:

$$\vec{\delta}(P_3, P_3') \leq \sup_{x \in \mathcal{G}_{P_3} \setminus \{0\}} \frac{\|(\Phi - I)x\|}{\|x\|},$$

$$\leq \sup_{\left\| \begin{matrix} u_0 \\ \tilde{v} \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \\ \tilde{z}_1 \\ \tilde{z}_2 \end{matrix} \right\| \neq 0} \frac{\left\| \begin{matrix} -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(M v_n - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_s^*(\tilde{z}_2)}\tilde{v})\right) - \\ g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \end{matrix} \right\|}{\|u_0, \tilde{v}, g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n), \tilde{z}_1, \tilde{z}_2\|},$$

$$\leq \sup_{\| u_0 \; \tilde{v} \; s \; \tilde{z}_1 \; \tilde{z}_2 \| \neq 0} \frac{\left\| -g^*(\tilde{z}_2)\left(\frac{1}{g_u^*(\tilde{z}_1)}(u_0 - \tilde{v} - \frac{1}{g_s^*(\tilde{z}_2)} s - f_u^*(\tilde{z}_1)) - (u_0 - \frac{1}{g_u^*(\tilde{z}_2)} f_u^*(\tilde{z}_2) - \frac{1}{g_u^*(\tilde{z}_2)}\tilde{v})\right) - s \right\|}{\|u_0, \tilde{v}, s, \tilde{z}_1, \tilde{z}_2\|},$$

$$\leq \sup_{\| u_0 \; \tilde{v} \; s \; \tilde{z}_1 \; \tilde{z}_2 \| \neq 0} \frac{\left\| g^*(\tilde{z}_2)\left(1 - \frac{1}{g_u^*(\tilde{z}_1)}\right) u_0 + \left(\frac{g_u^*(\tilde{z}_2)}{g_u^*(\tilde{z}_1)} - 1\right)(g_s^*(\tilde{z}_2)\tilde{v} + s) + \frac{g^*(\tilde{z}_2)}{g_u^*(\tilde{z}_1)} f_u^*(\tilde{z}_1) - g_s^*(\tilde{z}_2) f_u^*(\tilde{z}_2) \right\|}{\|u_0, \tilde{v}, s, \tilde{z}_1, \tilde{z}_2\|}, \tag{46}$$

$$\leq \left\| \mathbf{D} \left| 1 - \frac{1}{\epsilon} \right|, \left| \frac{\mathbf{D}_u}{\epsilon} - 1 \right|, \left| \frac{\mathbf{D}_u}{\epsilon} - 1 \right| \frac{\mathbf{D}}{\epsilon}, \frac{\mathbf{D}\mathbf{B}_u}{\epsilon}, \frac{\mathbf{D}\mathbf{B}_u}{\epsilon} \right\| \tag{47}$$

where $s = g_s^*(\tilde{z}_2)(u_0 - \tilde{v} - M v_n) \in \mathcal{U}$ since $\|g_s^* \tilde{z}_2)\|, \|u_0\|, \|\tilde{v}\| < \infty$ and $\|M\| < \infty$. □

Theorem 3 states that an upper bound on the gap depends on the nonlinear system components and on how exact the inversion of the unstable part of the plant nonlinearity is, within the nonlinear component of the controller. Moreover, Theorem 3 directly leads to the following proposition which establishes robust stability of the system shown in Figure 3.

Proposition 3: *Consider the nonlinear closed loop system $[P_1', P_2', P_3']$ shown in Figure 3. Let g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively, then $[P_1', P_2', P_3']$ has a robust stability margin.*

Proof. Since g^*, f_u^*, g_u^* satisfy Assumptions 1, 2, 3 respectively, then by Lemma 2, and Proposition 1 for the systems P_3 and P_3' given by Figure 5 and (28) and (27), respectively, the graphs $\mathcal{G}_{P_3}$ and $\mathcal{G}_{P_3'}$ can be expressed using (39) and (40), respectively. Using Proposition 2, then there exists a map $\Phi : \mathcal{G}_{P_3} \to \mathcal{G}_{P_3'}$ given by (43). This leads to the presence of a finite gap value between the perturbed and the unperturbed configurations of this system given by inequality (47). Hence $[P_1', P_2', P_3']$ shown in Figure 3 is stable for all P_3' satisfying $F_{\delta 2} < \|\Pi_{(3)}\|^{-1}$. From (4) the corresponding robust stability margin is

$$\|\Pi'_{(3)}\|^{-1} \geq \|\Pi_{(3)}\|^{-1} \frac{1 - \vec{\delta}(P_3, P_3')}{1 + \vec{\delta}(P_3, P_3')}. \tag{48}$$

□

The main result of Theorem 2 is an extension of Theorem 4 which is established next.

Theorem 4: *Consider the nonlinear closed loop system shown in Figure 2 and given by (24). Let g^* satisfy Assumption 1, let f_u^* satisfy Assumption 2, and let g_u^* satisfy Assumption 3. Then this system has a robust stability margin $b_{P,C}$ which satisfies the inequality*

$$b_{P,C} \geq \|\Pi'_{(3)}\|^{-1}. \tag{49}$$

Proof. Recall that $\|\Pi'_{(3)}\|^{-1}$ is the stability margin for the system $[P'_1, P'_2, P'_3]$ shown in Figure 3, and let $b_{P,C} = \|\Pi_{P//C}\|^{-1}$ be a stability margin for the system shown in Figure 2. Then

$$
\begin{aligned}
\left\|\Pi'_{(3)}\right\| &= \sup_{\|u'_0,x'_0,y'_0\|\neq 0} \frac{\left\|\Pi'_{(3)}\begin{pmatrix} u'_0 \\ x'_0 \\ y'_0 \end{pmatrix}\right\|}{\|u'_0, x'_0, y'_0\|}, \\
&= \sup_{\|u_0,d_0,d_1,d_2,d_3,y_0,x_0\|\neq 0} \frac{\left\|\Pi'_{(3)}\begin{pmatrix} u_0 & d_0 & d_1 & d_2 & d_3 & y_0 & x_0 \end{pmatrix}^{\top}\right\|}{\|u_0, d_0, d_1, d_2, d_3, y_0, x_0\|}, \\
&\geq \sup_{\|u_0,0,0,0,0,y_0,x_0\|\neq 0} \frac{\left\|\Pi'_{(3)}\begin{pmatrix} u_0 & 0 & 0 & 0 & 0 & y_0 & x_0 \end{pmatrix}^{\top}\right\|}{\|u_0, 0, 0, 0, 0, y_0, x_0\|}, \\
&\geq \sup_{\|u_0,y_0,0\|\neq 0} \frac{\left\|\Pi_{P//C}\begin{pmatrix} u_0 & y_0 & 0 \end{pmatrix}^{\top}\right\|}{\|u_0, y_0, 0\|}, \\
&= \|\Pi_{P//C}\|.
\end{aligned}
$$

This leads us to

$$b_{P,C} = \|\Pi_{P//C}\|^{-1} \geq \|\Pi'_{(3)}\|^{-1}.$$

Therefore the existence of a stability margin for the system shown in Figure 3 guarantees the existence of a stability margin for the system $[P, C]$ shown in Figure 2. Also, since g^*, f^*_u, g^*_u satisfy Assumptions 1, 2, 3 respectively, then by Proposition 3, the nonlinear closed loop system $[P'_1, P'_2, P'_3]$ shown in Figure 3, has a robust stability margin. This leads to the conclusion that the system $[P, C]$ given by Figure 2 and (24) also has a robust stability margin as required. □

Theorems 3 and 4 also give rise to the following corollary:

Corollary 1: *Consider the nonlinear closed loop system shown in Figure 2 and given by (24). Let g^* satisfy Assumption 1, let f^*_u satisfy Assumption 2, and let g^*_u satisfy Assumption 3. Then this system is stable if*

$$F_{\delta 2} < \|\Pi_{(3)}\|^{-1}. \tag{50}$$

Proof. From Theorem 3 inequality (44), $\vec{\delta}(P_3, P'_3) \leq F_{\delta 2}$ and it follows that, if (50) holds, then we have:

$$\vec{\delta}(P_3, P'_3) \leq F_{\delta 2} < \|\Pi_{(3)}\|^{-1}$$

which guarantees stability of closed loop system $[P'_1, P'_2, P'_3]$, with the associated stability margin (48). Then using Theorem 4 it follows that closed loop system $[P, C]$ is stable with gain margin

$$b_{P,C} \geq \|\Pi_{(3)}\|^{-1} \frac{1 - \vec{\delta}(P_3, P'_3)}{1 + \vec{\delta}(P_3, P'_3)} \tag{51}$$

□

4.3 *Illustrative Example 1*

In this example, we consider the SISO system described in (7) and compare the robust stability condition derived in Section 4 with that produced using the small gain theorem. Hence we have:

$$\begin{aligned} P &: \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} : u_1 \mapsto y_1, \\ \dot{x} &= -x^3 + u_1 + \omega x \\ y_1 &= x, \end{aligned} \tag{52}$$

where $-1 \leq \omega \leq 1$. Comparison with (16) yields $g^*(\cdot) = g_s^*(\cdot) = g_u^*(\cdot) = 1$, $f_s^*(\cdot) = -x^3$, $f_u^*(\cdot) = 0$, $A = \omega$, and $B = 1$. Therefore, Assumptions 1 and 3 are satisfied with $\epsilon = \varepsilon = 1$ and $\mathbf{D} = \mathbf{D}_u = 1$, $f_u^*(\cdot) = 0$ and Assumption 2 is satisfied with $\mathbf{B}_u = 0$. In this case the feedback linearizing controller (21) for this system is given as:

$$\begin{aligned} C &: \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} : y_2 \mapsto u_2 \\ &\quad u_2 = -2y_2. \end{aligned} \tag{53}$$

Applying the stability condition of Corollary 1 to the closed loop system $[P, C]$, we have

$$F_{\delta 2} = 0.$$

Therefore condition (50) is always satisfied and the system $[P, C]$ is always stable. Next, applying the small gain stability condition to the system $[P, C]$ corresponds to

$$\|P\|\|C\| < 1 \tag{54}$$

and since $\|C\| = 2$, the stability condition reduces to

$$\|P\| < \frac{1}{2}. \tag{55}$$

This stability condition shows that the small gain criterion cannot be satisfied for all ω, and hence this result is far more limited than the gap metric result derived in Section 4. In the next section the results will be specified to an unstable affine nonlinear system with an unstable nonlinear component. An example of this type is the system $\dot{x} = x^2 + u, y = x$ where x^2 is an unstable nonlinear component which is desired to be canceled by control action.

5. Specification to the Case of Full Feedback Linearization

This section considers a special case of the affine nonlinear system considered in Section 3. Here either the nonlinear components $g^*(z_1)$ and $f^*(z_1)$ are unstable and cannot be divided into a stable and unstable components, or the designer chooses not to employ such separation.

This corresponds to setting $f_u^*(z_1) = f^*(z_1)$, $f_s^*(z_2) = 0$, $g_u^*(z_1) = g^*(z_1)$ and $g_s^*(z_2) = 1$ in Subsection 3.1. The following stability condition then follows from Theorem 2 and Corollary 1:

Theorem 5: *Consider the nonlinear closed loop system shown in Figure 2 and given by (24). Let $f_u^*(z_1) = f^*(z_1)$, $f_s^*(z_2) = 0$, $g_u^*(z_1) = g^*(z_1)$ and $g_s^*(z_2) = \pi$, suppose $g_u^*(z)$ and $f_u^*(z)$ satisfy Assumptions 2 and 3, respectively. Then this system is stable if*

$$F_{\delta 3} < \|Q\|^{-1}, \tag{56}$$

where $Q = \begin{pmatrix} \Lambda \\ \zeta \end{pmatrix}$ *with* ζ, Λ *given by (31), (33) respectively, and*

$$F_{\delta 3} = \left\| \mathbf{D}_u \left| 1 - \frac{1}{\epsilon} \right|, \left| \frac{\mathbf{D}_u}{\epsilon} - 1 \right|, \frac{\mathbf{D}_u \mathbf{B}_u}{\epsilon}, \mathbf{B}_u \right\|. \tag{57}$$

Proof. This follows directly by substituting $f_u^*(z_1) = f^*(z_1)$, $f_s^*(z_2) = 0$, $g_u^*(z_1) = g^*(z_1)$ and $g_s^*(z_2) = \pi$ into equation (44), and noting that $g^*(z)$ and $f^*(z)$ satisfy Assumptions 1 and 2, respectively. Here (46) is employed to produce a tighter bound, using $\mathbf{B}_u = 0$. In addition, since $f_s^* = 0$, $g_s^* = 1$, we can use the result $\|\Pi_{(3)}\| \leq \|Q\|$ established in (34) so that satisfying (56) guarantees that $F_{\delta 3} < \|\Pi_{(3)}\|^{-1}$. □

Theorem 5 demonstrates that an upper bound on the gap depends on the nonlinear input component of the controller and how exact the inversion of the plant nonlinearity is, within the nonlinear component of the controller (equation (56)).

5.1 *Illustrative Example 2*

In this example, we compare the robust stability condition derived in Section 4 with that produced using the small gain theorem. Consider the system

$$\begin{aligned} P \ : \ & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} : u_1 \mapsto y_1, \\ & \dot{x}_1 = x_2, \\ & \dot{x}_2 = -x_1 - 2x_2 + (1 + \delta \sin(x_1))u_1, \\ & y_1 = x_1, \end{aligned}$$

where $0 < \delta < 1$. Comparison with (10) yields $g^*(\cdot) = (1 + \delta \sin(x_1))$, $f^*(\cdot) = 0$. Now $0 < 1 - \delta \leq g^*(\cdot) \leq 1 + \delta$ and therefore Assumption 1 is satisfied with $\epsilon = 1 - \delta$ and $\mathbf{D} = \mathbf{D}_u = 1 + \delta$. Choosing $c - \tilde{c} = 0.4$, the feedback linearizing controller (21) for this system is given as:

$$\begin{aligned} C \ : \ & \mathcal{L}_{\infty,e} \to \mathcal{L}_{\infty,e} : y_2 \mapsto u_2 \\ & u_2 = \frac{1}{g^*(-y_2)}(c^\top - \tilde{c}^\top)y_2 = \frac{0.4}{(1 + \delta \sin(-y_2))} y_2. \end{aligned}$$

Applying the small gain stability condition to system $[P, C]$ corresponds to

$$\frac{\mathbf{D}_u}{\epsilon} \|\tilde{P}\| \|\tilde{C} + C_{Linear}\| < 1 \tag{58}$$

where linear system component bounds $\|\tilde{P}\| = 1$ and $\|\tilde{C} + C_{Linear}\| = 0.4$. Hence the stability condition reduces to

$$0.4 \Big(\frac{1 + \delta}{1 - \delta} \Big) < 1. \tag{59}$$

Next, we apply the stability condition of Theorem 5. Here

$$F_{\delta 3} = \left\| (1 + \delta) \left(\frac{1}{1 - \delta} - 1 \right), \left(\frac{1 + \delta}{1 - \delta} - 1 \right) \right\| = \frac{\delta}{1 - \delta} \|1 + \delta, 2\| = \frac{\delta}{1 - \delta} \sqrt{\delta^2 + 2\delta + 5}$$

and application of (31), (33) results in $\|Q\| = 1.0079$. Therefore condition (56) gives

$$1.0079\Big(\frac{\delta}{1-\delta}\Big)\sqrt{\delta^2 + 2\delta + 5} < 1. \tag{60}$$

The two conditions (59) and (60) have been plotted in Figure 6, where $f_1(\delta) = 0.4\big(\frac{1+\delta}{1-\delta}\big)$ and $f_2(\delta) = 1.0079\big(\frac{\delta}{1-\delta}\big)\sqrt{\delta^2 + 2\delta + 5}$. We note that for small δ more plant uncertainty can be tolerated

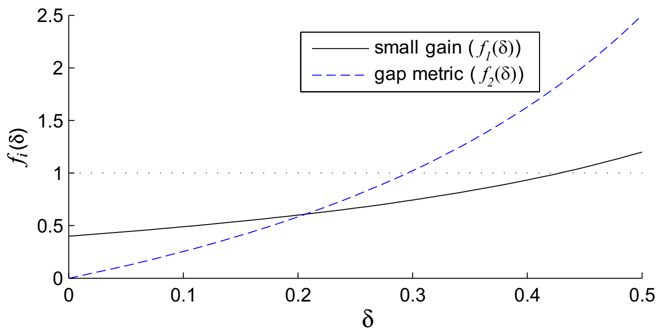

Figure 6. Comparison of stability conditions.

without affecting system stability. This plot shows that for $0 < \delta < 0.207$ the gap metric gives a superior stability condition (smaller δ) than the small gain theorem since it indicates a greater level of robustness, however this is reversed for higher values of δ. For $\delta > 0.43$ both conditions are violated.

6. Conclusions

This paper has addressed the lack of robust stability results in the literature for systems that contain output unstructured uncertainty under feedback linearization. In order to do this, it was first shown how gap metric robust stability analysis can be applied to unstable affine systems. A novel control law was then introduced which classifies the system nonlinearity into stable and unstable components. This controller preserves the stabilizing role of the inherently stabilizing nonlinearities in the plant and cancels only the unstable nonlinear component of the plant. Robust performance margins for the closed loop system were derived and shown to depend on bounds on the plant nonlinearities and the accuracy of the cancellation of the unstable plant nonlinearity by the controller. It was then confirmed that the approach yields less conservative results than other methods (such as the small gain theorem).

These results can be further extended in the following directions: Improving the gain bounds for the results given, which are obtained under strong assumptions on boundedness of nonlinearities to give the simplest global results, but can be generalized to local and semi-global results in the absence of such assumptions. Furthermore, generalization can be carried out for the robustness analysis undertaken to cover more system classes, such as partially linearizable systems, including many important topics related to the feedback linearization method such as internal states and zero dynamics.

References

Al-Gburi, A., French, M., and Freeman, C.T. (2013), "Robustness Analysis of Nonlinear Systems with Feedback Linearizing Control," in *52nd IEEE Conference on Decision and Control*, pp. 3055–3060.

Arcak, M., Seron, M., Braslavsky, J., and Kokotović, P.V. (1999), "Robustification of backstepping against input unmodeled dynamics," in *Proceedings of the 38th IEEE Conference on Decision and Control*, pp. 2495–2500.

Brockett, R.W. (1978), "Feedback invariants for nonlinear systems," in *Proceedings of the IFAC World Congress*, Helsinki, Finland, pp. 1115–1120.

Chao, A.I., *Stability Robustness to Unstructured Uncertainty for Nonlinear Systems Under Feedback Linearization*, Massachusetts Institute of Technology, Department of Mechanical Engineering (1995).

Freeman, R., and Kokotović, P., *Robust nonlinear control design: state-space and Lyapunov techniques*, Springer (2008).

Georgiou, T.T. (1988), "On the computation of the gap metric," *Systems and Control Letters*, 11, 253–257.

Georgiou, T.T., and Smith, M.C. (1990), "Optimal robustness in the gap metric," *IEEE Transactions on Automatic Control*, 35, 673–686.

Georgiou, T.T., and Smith, M.C. (1997), "Robustness Analysis of Nonlinear Feedback Systems: An Input-Output Approach," *IEEE Transactions on Automatic Control*, 42, 1200–1221.

Hamzi, B., and Praly, B. (2001), "Ignored input dynamics and a new characterization of control Lyapunov functions," *Automatica*, 37, 831–841.

Hunt, L.R., Su, R., and Meyer, G. (1983), "Global transformations of nonlinear systems," *IEEE Transactions On Automatic Control*, 28, 24–31.

Isidori, A., *Nonlinear Control Systems*, second ed., Springer-Verlag (1989).

Jakubczyk, B., and Respondek, W. (1980), "On linearization of ontrol systems," *Bulletin de l'Academie Polonaise des Sciences Série des Sciences Mathématiques*, 28, 517–522.

Janković, M., Sepulchre, R., and Kokotović, P.V. (1999), "CLF based designs with robustness to dynamic input uncertainties," *Systems and Control Letters*, 37, 45–54.

Jiang, Z.P., and Mareels, I. (1997), "A small-gain control method for nonlinear cascaded systems with dynamic uncertainties," *IEEE Transactions on Automatic Control*, 42, 292–308.

Jiang, Z.P., Teel, A.R., and Praly, L. (1994), "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals and Systems*, 7, 95–120.

Kanellakopoulos, I., Kokotović, P.V., and Morse, A.S. (1991), "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Transactions on Automatic Control*, 36, 1241–1253.

Khalil, H.K. (1994), "Robust servomechanism output feedback controllers for feedback linearizable systems," *Automatica*, 30, 1587–1599.

Khalil, H.K., *Nonlinear Systems*, third ed., Prentice Hall (2002).

Kokotović, P.V., and Arcak, M. (2001), "Constructive nonlinear control: a historical perspective," *Automatica*, 37, 637–662.

Kravaris, C. (1987), "On the Internal Stability and Robust Design of Globally Linearizing Control Systems," in *American Control Conference*.

Krstić, M., Sun, J., and Kokotović, P.V. (1996), "Robust control of nonlinear systems with input unmodeled dynamics," *IEEE Transactions on Automatic Control*, 41, 913–920.

Marino, R., and Tomei, P., *Nonlinear Control Design: Geometric, Adaptive and Robust*, Prentice Hall International (UK) Ltd. (1996).

Nijmeijer, H., and van der Schaft, A., *Nonlinear Dynamical Control Systems*, Springer-Verlag New York, Inc., USA (1990).

Ortega, R., and Spong, M.W. (1989), "Adaptive motion control of rigid robots: A tutorial," *Automatica*, 25, 877–888.

Praly, L., and Wang, Y. (1996), "Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability," *Mathematics of Control, Signals and Systems*, 9, 1–33.

Sastry, S.S., *Nonlinear Systems: Analysis, Stability and Control*, Springer, New York, USA (1999).

Sastry, S.S., and Isidori, A. (1989), "Adaptive control of linearizable systems," *IEEE Transactions on Automatic Control*, 34, 1123–1131.

Sepulchre, R., Janković, M., and Kokotović, P., *Constructive Nonlinear Control*, Springer-Verlag (1997).

Slotine, J.J., and Hedrick, J.K. (1993), "Robust input-output feedback linearization," *International J Control*, 57, 1133–1139.

Sontag, E.D. (1995), "On the Input-to-State Stability Property," *Systems and Control Letters*, 24, 351–359.

Spong, M.W., Thorp, J.S., and Kleinwaks, J.M. (1984), "The control of robot manipulators with bounded input: Part II: Robustness and disturbance rejection," in *Proceedings of 23rd IEEE Conference on Decision*

and Control, pp. 1047–1052.
Spong, M.W., and Vidyasagar, M. (1987), "Robust linear compensator design for nonlinear robotic control," *IEEE Journal of Robotics and Automation*, 3, 345–351.
Taylor, D.G., Kokotović, P., Marino, R., and Kannellakopoulos, I. (1989), "Adaptive regulation of nonlinear systems with unmodeled dynamics," *IEEE Transactions on Automatic Control*, 34, 405–412.
Vidyasagar, M. (1984), "The graph metric for unstable plants and robustness estimates for feedback stability," *IEEE Transactions on Automatic Control*, 29, 403–418.
Wang, W., and Wen, C. (2009), "A new approach to design robust backstepping controller for MIMO nonlinear systems with input unmodeled dynamics," in *Proceedings of the 48th Conference on Decision and Control*, pp. 2468–2473.
Zames, G., and El-Sakkary, A. (1980), "Unstable systems and feedback: The gap metric," *Proceedings of the Allerton Conferencce*.