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UNIVERSITY OF SOUTHAMPTON
FACULTY OF PHYSICAL AND APPLIED SCIENCES
Electronics and Computer Science

Switched linear differential systems

by

Jonathan C. Mayo-Maldonado

Thesis for the degree of Doctor of Philosophy

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ABSTRACT

FACULTY OF PHYSICAL AND APPLIED SCIENCES

Electronics and Computer Science

Doctor of Philosophy

SWITCHED LINEAR DIFFERENTIAL SYSTEMS

by **Jonathan C. Mayo-Maldonado**

In this thesis we study systems with switching dynamics and we propose new mathematical tools to analyse them. We show that the postulation of a global state space structure in current frameworks is restrictive and lead to potential difficulties that limit its use for the analysis of new emerging applications. In order to overcome such shortcomings, we reformulate the foundations in the study of switched systems by developing a trajectory-based approach, where we allow the use of models that are most suitable for the analysis of a each system. These models can involve sets of higher-order differential equations whose state space does not necessarily coincide.

Based on this new approach, we first study closed switched systems, and we provide sufficient conditions for stability based on LMIs using the concept of multiple higher-order Lyapunov function. We also study the role of positive-realness in stability of bimodal systems and we introduce the concept of positive-real completion. Furthermore, we study open switched systems by developing a dissipativity theory. We give necessary and sufficient conditions for dissipativity in terms of LMIs constructed from the coefficient matrices of the differential equations describing the modes. The relationship between dissipativity and stability is also discussed.

Finally, we study the dynamics of energy distribution networks. We develop parsimonious models that deal effectively with the variant complexity of the network and the inherent switching phenomena induced by power converters. We also present the solution to instability problems caused by devices with negative impedance characteristics such as constant power loads, using tools developed in our framework.

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Declaration of Authorship

I, **Jonathan C. Mayo-Maldonado** , declare that the thesis entitled *Switched linear differential systems* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: [39, 40, 41, 42, 43, 44, 45]

Signed:.....

Date:.....

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*¿Con qué he de irme?
¿Nada dejaré en pos de mi sobre la tierra?
¿Cómo ha de actuar mi corazón?
¿Acaso en vano venimos a vivir,
a brotar sobre la tierra?
Dejemos al menos flores,
Dejemos al menos cantos.*

–“*Un recuerdo que dejo*” – Netzahualcóyotl (1402-1472).

Notation

I_q	Identity matrix of dimension q .
A^\top	Transpose of the matrix A .
$\text{col}(A, B)$	If A, B are matrices with the same number of columns, it denotes the matrix obtained by stacking A over B .
\mathbb{R}	Set of real numbers.
\mathbb{N}	Set of natural numbers.
\mathbb{C}	Set of complex numbers.
\mathbb{C}^+	Set of complex numbers with positive real part.
\mathbb{C}^-	Set of complex numbers with negative real part.
j	Imaginary unit $\sqrt{-1}$.
$\bar{\lambda}$	Conjugate of the complex number λ .
$\mathbb{R}^{\mathbf{w}}$	Space of real vectors with \mathbf{w} dimension.
$\mathbb{R}^{m \times n}$	Space of $m \times n$ dimensional real matrices.
$\mathbb{R}^{\bullet \times n}$	Space of real matrices with n columns and a finite unspecified number of rows.
$\mathbb{R}^{m \times \infty}$	Space of real matrices with m rows and an infinite number of columns.
$\mathbb{R}[\xi]$	Ring of polynomials with real coefficients in the indeterminate ξ .
$\mathbb{R}[\zeta, \eta]$	Ring of polynomials with real coefficients in the indeterminates ζ and η .
$\mathbb{R}^{m \times n}[\xi]$	Ring of $m \times n$ polynomial matrices with real coefficients in the indeterminate ξ .
$\mathbb{R}^{m \times n}[\zeta, \eta]$	Ring of $m \times n$ polynomial matrices with real coefficients in the indeterminates ζ and η .
$\det(A)$	Determinant of a square matrix A .
$\deg(r)$	Determinant of a polynomial r .
$\text{rank}(R)$	Rank of a matrix R .
$\mathbf{n}(\mathfrak{B})$	McMillan degree of the behaviour \mathfrak{B} .
$\mathbf{m}(\mathfrak{B})$	Number of maximally free components of the behaviour \mathfrak{B} (i.e. input variables).
$\mathcal{L}^{\mathbf{w}}$	Class of linear differential behaviours associated to an

	external variable of dimension \mathbf{w} .
$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$	Set of infinitely differentiable functions from \mathbb{R} to \mathbb{R}^q .
$\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$	Set of locally integrable functions from \mathbb{R} to \mathbb{R}^q .
$\mathfrak{D}^\infty(\mathbb{R}, \mathbb{R}^q)$	Set of infinitely differentiable functions from \mathbb{R} to \mathbb{R}^q with compact support.
$\mathfrak{C}_p^\infty(\mathbb{R}, \mathbb{R}^q)$	Set of infinitely piece-wise differentiable functions from \mathbb{R} to \mathbb{R}^q .
$\mathfrak{D}_p^\infty(\mathbb{R}, \mathbb{R}^q)$	Set of piece-wise infinitely differentiable functions from \mathbb{R} to \mathbb{R}^q with compact support.
$\lim_{\tau \nearrow t} f(t)$	Limit of f as τ approaches t from the right.
$\lim_{\tau \searrow t} f(t)$	Limit of f as τ approaches t from the left.
$f(t^-)$	Value of $\lim_{\tau \nearrow t} f(\tau)$ for a function $f : [t - \epsilon, t) \rightarrow \mathbb{R}^\bullet$.
$f(t^+)$	Value of $\lim_{\tau \searrow t} f(\tau)$ for a function $f : (t, t + \epsilon] \rightarrow \mathbb{R}^\bullet$.
$\sigma_+(G)$	Nmber of positive eigenvalues of $G = G^\top \in \mathbb{R}^{\mathbf{m} \times \mathbf{m}}$.

Chapter 1

Introduction

In this chapter, we provide a general introduction to systems with *switching dynamics* and we briefly study their classical framework based on *state space representations*. We argue that although many remarkable contributions are rested on the state space setting, its conventional modelling approach is not necessarily well-grounded on physical considerations. This situation impedes the use of such framework in the analysis of new emerging applications. We justify this position by discussing the following issues:

- *State representations are not a given.* Such models have to be computed from sets of differential equations obtained from first principles, possibly of higher-order.
- *Lack of parsimony.* Switching between systems with different state space dimension requires the introduction of fictitious variables and equations, only to satisfy a global predetermined structure.
- *Lack of modularity.* Once a global structure is imposed, the modes in the underlying bank need to be altered every time a new mode of higher complexity is added.

In the following sections, we explain how the latter issues prompted us to reformulate the current foundations in the study of systems with switching dynamics.

1.1 Switching dynamics

In many disciplines such as physics and engineering, there exist dynamical systems whose laws change abruptly. Such systems are called *switched systems* and we find them very often in real-life situations and applications, e.g. thermostats, air traffic control systems, switched multi-controller systems, switching power converters, etc.. The abrupt changes

in the laws describing the system give rise to different *dynamic modes*, whose modelling from first principles frequently involves sets of differential equations. The activation of such modes is orchestrated by a *switching rule* that is considered, unless otherwise specified, as an unconstrained piecewise constant function of time, i.e. free to adopt the value of any element of an index set at any time instant.

When switching between dynamic modes, the trajectories of the system may be subject to *algebraic constraints*, which are enforced by physical principles. For example, although discontinuities on the system trajectories may occur at switching instants, *conservation principles* forbid trajectories that imply instantaneous changes in *conserved quantities* such as charge, flux, momentum, molar mass, volume, etc. (see [47]). Another well-known example of this type of constraints is the case of *state reset maps* in multi-controller systems, which are used to re-initialise a bank of controllers that are interconnected to a plant (see [24]).

One of the main problems studied in switched systems is stability: we are interested in determining under what conditions the trajectories of a switched system remain bounded. However, finding such conditions is not a straightforward task since for instance, it is well-known that the switching between stable systems may produce instability, see [34, 36]. Other subjects of interest are stabilisability, control, system identification, simulation, design, as well as the application of results on relevant disciplines, see e.g. the practical cases discussed in [66, 81, 98].

We now introduce several examples of physical systems with switching dynamics.

1.1.1 Coupling of masses in motion

Consider the mechanical system in Fig. 1.1, consisting of two masses with rigid mechanical couplers (of negligible length) attached to their contiguous ends. There exist two possible dynamic modes that describe the motion of the masses.

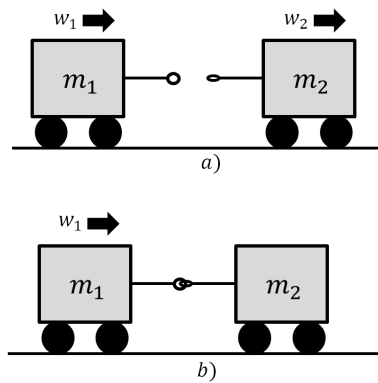


Figure 1.1: Coupling of masses in motion

In Fig. 1.1 (a) the masses are detached and move individually, possibly at different velocities. The positions w_1 and w_2 , with respect to a fixed point, are governed by the following laws

$$\text{Mode 1: } \begin{cases} m_1 \frac{d^2}{dt^2} w_1 = 0 \\ m_2 \frac{d^2}{dt^2} w_2 = 0 \end{cases}.$$

If at any instant of time t_k the condition $w_1(t_k) \geq w_2(t_k)$ is satisfied, the masses are coupled as in Fig. 1.1 (b). Consequently, the laws of the system are described by

$$\text{Mode 2: } \begin{cases} (m_1 + m_2) \frac{d^2}{dt^2} w_1 = 0 \\ w_1 - w_2 = 0 \end{cases}.$$

Note that when switching from Mode 1 to Mode 2, the masses are suddenly required to adopt the velocity of a single mass of magnitude $m_1 + m_2$. The new value of velocity must respect the *principle of conservation of momentum* (cf. [46]), which states that the sum of momenta before and after the switch is the same, i.e.

$$m_1 \frac{d}{dt} w_1(t_k^-) + m_2 \frac{d}{dt} w_2(t_k^-) = (m_1 + m_2) \frac{d}{dt} w_1(t_k^+).$$

Considering this equilibrium condition, we conclude that the only admissible value of velocity at the switching instant is

$$\frac{d}{dt} w_1(t_k^+) = \frac{m_1 \frac{d}{dt} w_1(t_k^-) + m_2 \frac{d}{dt} w_2(t_k^-)}{m_1 + m_2}.$$

1.1.2 DC-DC boost converter

In Fig. 1.2, we illustrate a *DC-DC boost converter* with a nominal resistive load R (see [10]). For practical purposes such as voltage/current/power regulation, we are particularly interested in the dynamics at the *input/output terminals*. Hence we focus our analysis on the variables $\text{col}(E, i, v)$.

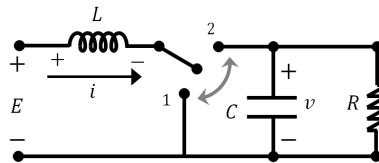


Figure 1.2: DC-DC boost converter

By means of a switching signal, we can arbitrarily induce two possible electrical configurations that occur when the switch is in position 1 or 2. We can thus derive the

following sets of equations.

$$\text{Mode 1: } \begin{cases} L \frac{d}{dt} i_L - E = 0 \\ C \frac{d}{dt} v + \frac{1}{R} v = 0 \end{cases} . \quad (1.1)$$

$$\text{Mode 2: } \begin{cases} L \frac{d}{dt} i + v - E = 0 \\ C \frac{d}{dt} v + \frac{1}{R} v - i = 0 \end{cases} . \quad (1.2)$$

By inspecting the circuit we conclude that the values of the magnetic flux and electric charge due to the inductor and capacitor remain unchanged at the switching instant, i.e. the internal charge/flux is not affected by external components. Thus the algebraic constraints to be satisfied at every switching instant t_k are

$$Cv(t_k^-) = Cv(t_k^+) , \quad Li(t_k^-) = Li(t_k^+) ;$$

which evidently implies that the trajectories of the current i through the inductor and the voltage v across the capacitor are continuous at switching instants.

1.1.3 Feedback multi-controller system

Consider the multi-controller system in Fig. 1.3, where the plant and controllers are described by the SISO transfer functions $\frac{n(s)}{d(s)}$ and $\frac{p_i(s)}{q_i(s)}$, $i = 1, \dots, N$, respectively.

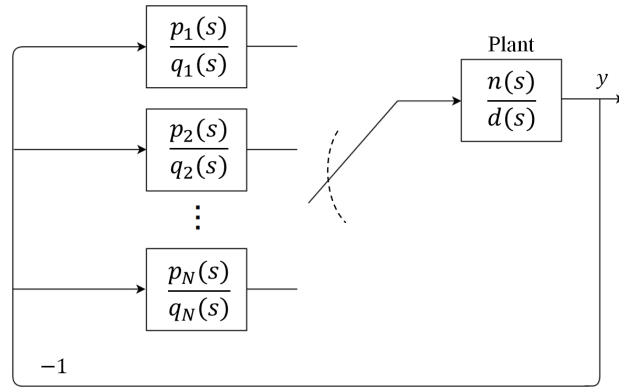


Figure 1.3: Feedback multi-controller system

When modelling the mode dynamics with respect to the output y , we obtain the following equations

$$\left(d \left(\frac{d}{dt} \right) q_i \left(\frac{d}{dt} \right) + n \left(\frac{d}{dt} \right) p_i \left(\frac{d}{dt} \right) \right) y = 0 , \quad i = 1, \dots, N .$$

Furthermore, in many cases we are interested in inducing certain “re-initialisation” of the controllers, which involves additional algebraic constraints at switching instants according to the application.

1.1.4 Energy distribution networks

Consider the *energy distribution network* (cf. [61, 89]) in Fig. 1.4, consisting of a DC-DC converter as in Sec. 1.1.2, feeding three types of loads represented by impedances.

- Z_N represents a nominal load, i.e. the load that is considered during the design stage of the converter and which remains connected in the implementation;
- Z_k , $k = 1, \dots, L$, represents a *switched impedance*, i.e. a finite amount of load that can be connected or disconnected arbitrarily and which is not necessarily known during the design stage, e.g. domestic/commercial (dis-)connectable loads, (dis-)connectable electric vehicles, etc.; and
- Z_{CPL} represents the negative impedance of a regulated switching power converter whose dynamics resemble a constant power load (CPL).

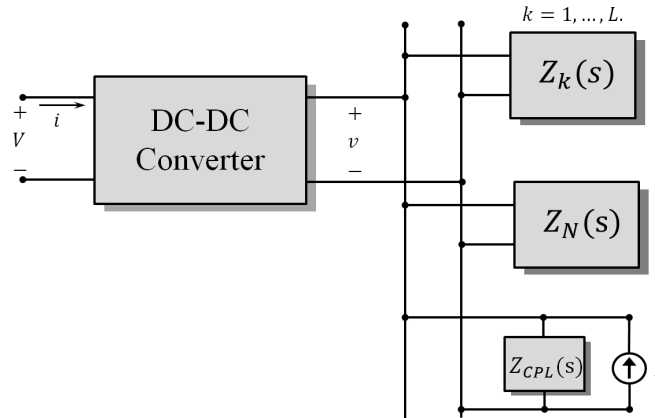


Figure 1.4: Energy distribution network

Note that the complexity of this system is neither initially bounded nor fixed, i.e. the McMillan degree associated to each impedance depends on their constitutive reactive elements which in the case of Z_k , $k = 1, \dots, L$, may change depending on the loads that are connected during certain intervals of time.

1.2 Switched state space systems

The traditional approach to switched (linear) systems is based on the state space framework (see [24, 34, 74]). In this case, the dynamical modes are uniformly represented

as

$$E_{s(t)} \frac{d}{dt} x = A_{s(t)} x + B_{s(t)} u ; \quad (1.3)$$

where $x(t) \in \mathbb{R}^n$ is the *state function*, $u(t) \in \mathbb{R}^m$ is the *input function*, $s : \mathbb{R} \rightarrow \{1, \dots, N\}$, with $N \in \mathbb{N}$, is a *switching signal*, and $E_i \in \mathbb{R}^{n \times n}$, $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$, with $i = 1, \dots, N$, are constant matrices associated with the laws of the i -th dynamical mode.

In many cases the matrices E_i , $i = 1, \dots, N$, are invertible and can be omitted for ease of exposition. However, in some cases E_i is singular and the resulting state space description is more properly called a *differential algebraic equation* (see [74]). When switching from the i -th to the j -th mode at t_k , the trajectory of the state may be continuous: $x(t_k^-) = x(t_k^+)$, or discontinuous: $x(t_k^-) \neq x(t_k^+)$. For the latter case, a *state reset map* $R_{i \rightarrow j} \in \mathbb{R}^{n \times n}$ can be used to specify a new value of the state, i.e. $x(t_k^+) = R_{i \rightarrow j} x(t_k^-)$.

Undoubtedly, the state space setting has played an important role in the study and understanding of switched systems, and many outstanding contributions rely on this framework, see e.g. [3, 7, 24, 35, 36]. The adoption of the state space approach to study switched systems is not a sheer coincidence, since there are special mathematical reasons that encourage its use, e.g. 1) the state plays a fundamental role in the concatenation of the system trajectories at switching instants, as illustrated with the use of state reset maps in [24]; 2) the use of state space models facilitates the computation of functionals using LMIs [36]; 3) there is a fundamental relationship between the state and the “memory” or “energy” of the system [77]. In spite of such appealing features, there exist also shortcomings in the state space setting.

The postulation of a global state space structure for each dynamical mode is rather restrictive. For instance, (1.3) indicates that abrupt changes in the laws of a switched system affect only its parameters, but the state remains unchanged. Note that in the previous examples in sections 1.1.1, 1.1.2, 1.1.3 and 1.1.4, this situation occurs rarely; in fact, only the model of the DC-DC converter in equations in (1.1)-(1.2) satisfy such premise. However if instead of restricting the applicability of such DC-DC converter to its elementary case (involving a resistor as load) and consider a more realistic scenario as in the energy distribution network in Sec. 1.1.4, such a global state space structure does not appear in a natural way; the slightest (dis-)connection of loads modify the dynamical complexity of the system. From these examples, we conclude that in general the state space of the dynamic modes of a physical switched system do not necessarily coincide.

Surprisingly, the issue of dynamic modes with different state spaces has been already identified and argued in subjects such as physics and computer sciences (see [11, 47, 63]), but has remained largely disregarded in system and control theory. In the following, we study the consequences derived from the restrictions of the state space setting in

the analysis of switched systems and we emphasise issues in *modelling*, *parsimony* and *modularity*.

1.2.1 Modelling from first principles

In the following we discuss two modelling approaches for physical systems: *tearing*, *zooming and linking* (see [85]), and *modelling of n-port networks* (see [48]). We also show that these methods lead in general to models based on higher-order differential equations.

Tearing, zooming and linking

When modelling a physical system, we are interested in studying the trajectories of certain *manifest variables*, that are crucial for analysis, control, simulation, etc. During the modelling process certain *latent variables* may appear, which are not necessarily of interest, but need to be introduced to represent the physical laws of the system in a convenient way. Modelling by tearing, zooming and linking embodies the general intuition of viewing a system as the interaction of subsystems of lower complexity, focusing on manifest variables.

This procedure consists of the following four steps, which are also depicted in Fig. 1.5.

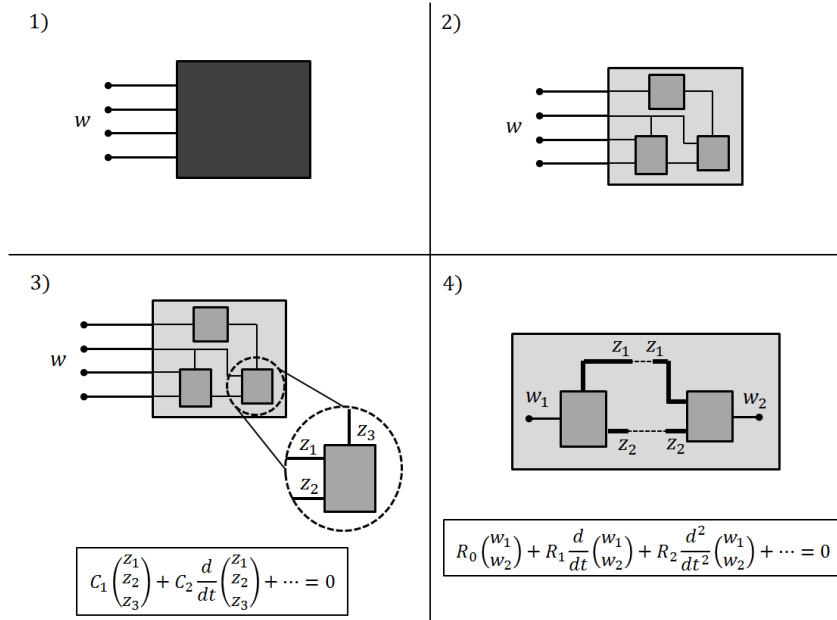


Figure 1.5: Modelling by tearing, zooming and linking

- 1) *Identifying relevant variables*: We view the system as a black-box where we identify the set of variables of interest denoted by w .

2) *Tearing*: we internally examine the black-box unveiling the system structure consisting of interconnections of smaller black-boxes corresponding to subsystems of lower complexity.

3) *Zooming*: we model the laws that describe the dynamics of the terminals of each subsystem.

If the complexity of the subsystem does not permit to model in a straightforward way the corresponding physical laws involving its terminals, we return to step 2) to decompose it into smaller subsystems.

4) *Linking*: we eliminate latent variables by matching the common terminals of the subsystems in a hierarchical way, starting from the smaller subsystems.

Note that during this *hierarchical modelling* procedure, the algebraic elimination of latent variables results in higher-order differential descriptions that involve only the variables of interest. \square

Modelling of n -port electrical networks

When we study systems consisting of interconnections of port-driven electrical networks, e.g. transmission lines with points of common coupling, filters, loads, etc., we are compelled to adopt, under some reasonable conditions, models based on the calculus of n -port *immitances* (such as impedances and admittances). These models greatly simplify the computation of complex models, see e.g. [48].

Models based on impedance matrices describe the input-output dynamics of the network in terms of the variables $V := \text{col}(v_1, \dots, v_n)$ and $I := \text{col}(i_1, \dots, i_n)$, corresponding respectively to the voltages across and currents through each port. The impedance $Z \in \mathbb{R}^{n \times n}(s)$ is a matrix transfer function with input I and output V .

In standard cases, Z can be computed by straightforward algebraic computations such as *series and parallel operations*, since any complex n -port impedance matrix Z consists of the interconnection of impedances of lower complexity. The most elementary components are 1-port impedances corresponding to inductors, resistors and capacitors:

$$Z_L(s) = Ls, \quad Z_R(s) = R, \quad Z_C(s) = \frac{1}{Cs}.$$

The inverse of an impedance, if exists, is equal to an admittance denoted by Y , i.e. $Y = Z^{-1}$.

Consider for instance the n -port networks in Fig. 1.6, whose terminals represent an n number of terminal pairs. The resultant n -port impedance/admittance due to series (Fig. 1.6 a) and parallel (Fig. 1.6 b) interconnections is computed as $Z = Z_1 + Z_2$ and $Y = Y_1 + Y_2$ respectively.

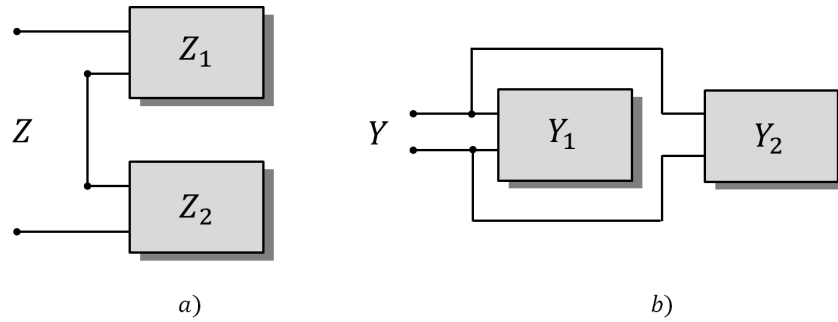


Figure 1.6: Series/parallel interconnection of impedances/admittances

The sum of impedances/admittances lead to complex descriptions of transfer functions that can be effectively translated to systems of higher-order differential equations.

Electrical networks are not the only type of networks than can be studied in this setting, since any physical system with ports involving *conjugate variables* such as voltage and current, force and velocity, pressure and change in volume, etc., can be modelled in this way (see [48]). Note for example the use of impedances in the analysis of mechanical networks in [50]. \square

The previous modelling approaches exhibit certain features that point out to some shortcomings in the classical state space framework. Consider for example the following remarks.

- *State representations are constructed from higher-order models.* Modelling by tearing, zooming and linking leads to higher-order descriptions involving a set of manifest variables. Note that this situation has been already exemplified in Sec. 1.1.1, 1.1.2 and 1.1.3, where mass displacements, input/output voltages and currents, and the output of a plant are selected as variables of interest for practical purposes. Hence, additional computations must be performed to derive first-order models.
- *The dimension of the state space vector is unsuitable in complex system analysis.* Sets of first-order differential equations is a special end result of a modelling procedure based on tearing, zooming and linking, where a vector of state variables is of interest. Although first-order representations can be always obtained in such a way or constructed from higher-order models, the resulting dimension of the state vector has been already determined to be unsuitable for the analysis of complex systems such as *smart grids*, see pp. 57-58 [90].
- *The use of state space models is not a fundamental requirement.* Physical systems with ports are conveniently modelled in terms of immittances, which are assembled from libraries of simpler sub-models, and cannot be described mathematically *right*

away in terms of first-order equations. In many cases such as in power systems, the analysis in terms of immittances is generally preferred over other methods, see e.g. [33, 48, 53, 72, 88]. The use of immittances is also a common approach for the study of stability of electrical networks with switching dynamics, see e.g. [61, 67, 82, 89].

For these reasons, the analysis of systems *directly* in higher-order terms is a sound mathematical approach for the study of physical systems.

1.2.2 Parsimony

We now show that the state space approach to switched systems scores low in parsimony, since it usually requires an increased number of variables and equations only to satisfy an imposed global structure. In order to illustrate this issue, consider the separately-excited DC motor (see [32], Sec. 2.4, p.78) in Fig. 1.7.

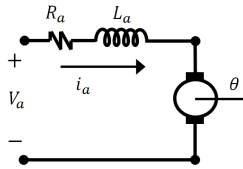


Figure 1.7: Separately excited DC motor (armature winding)

We select the armature current i_a and the rotor position θ as the variables of interest. The dynamics can be modelled using Newton's second law for rotatory masses and Kirchhoff's voltage law, obtaining

$$\text{Mode 1: } \begin{cases} J \frac{d^2}{dt^2} \theta + B_L \frac{d}{dt} \theta - L_{af} i_f i_a = 0 \\ L_a \frac{d}{dt} i_a - L_{af} i_f \frac{d}{dt} \theta + R_a i_a + V_a = 0 \end{cases} \quad (1.4)$$

where J is the rotor inertia, B_L the rotor viscous friction constant, L_{af} the mutual inductance, L_a the armature inductance, R_a the armature resistance, i_f the constant field winding current and V_a the voltage across the terminals of the armature winding that can be manipulated freely. To construct a minimal state space representation for the DC motor, the state variables can be chosen as

$$x := \begin{bmatrix} \theta \\ \frac{d}{dt} \theta \\ i_a \end{bmatrix}.$$

A switching phenomenon occurs when at an arbitrary time instant we connect e.g. a discharged capacitor C to the terminals of the armature winding. Mathematically, this

results in adding new equations to the existing ones, i.e. $C \frac{d}{dt}v + i_a = 0$, and $v = V_a$. Moreover, a minimal set of state variables for the new dynamic mode is

$$x' := \begin{bmatrix} \theta \\ \frac{d}{dt}\theta \\ i_a \\ v \end{bmatrix}^\top ;$$

which has a higher dimension than the previous x .

In practical situations we are not interested in monitoring the voltage dynamics of the attached capacitor, but in the overall dynamics that this new interconnection induces on the variables of the motor that we originally decided to be of interest. Consequently, using tearing, zooming and linking we obtain the following set of higher-order differential equations:

$$\begin{cases} J \frac{d^2}{dt^2}\theta + B_L \frac{d}{dt}\theta - L_{af}i_f i_a = 0 ; \\ L_{af}i_f C \frac{d^2}{dt^2}\theta - L_a C \frac{d^2}{dt^2}i_a - R_a C \frac{d}{dt}i_a + i_a = 0 . \end{cases} \quad (1.5)$$

Thus the variables of interest in (1.5) are the same that those in (1.4).

In the state space setting we construct representations using the largest state vector $x' := \left[\theta \quad \frac{d}{dt}\theta \quad i_a \quad v \right]^\top$ for both modes, by increasing the complexity of the simplest one. However, note that this situation not only undermines the possibility of obtaining parsimonious representations, but also deviates from a sound physical interpretation of the laws of the system, e.g. the voltage across the capacitor is included in the description of both modes, while such variable is only physically meaningful in one of them.

Note that the issue of augmented state space representations becomes more critical as the difference of complexity among dynamic modes is greater. Consequently in complex cases the state space setting becomes unsuitable for the study of real-life switched systems. Consider for example the case of the energy distribution network in Sec. 1.1.4, where the complexity of the switched system is not bounded, since it depends on the domestic, commercial and industrial loads that are arbitrarily (dis-)connected. In this case it is not helpful to postulate a *global state-space* as a starting point to model the network, since the slightest (dis-)connection of loads changes the dynamical complexity of the system. A global state space would be necessary only if *all* the possible loads were simultaneously connected; but this situation is not conceivable in practice.

The use of additional fictitious variables and augmented representations lead to unnecessary complications in the analysis of switched systems. For instance, if we demand a global state space structure for systems with different complexity, we decrease dramatically the efficiency of computational tools, see e.g. the discussions in [11, 63]. This

situation motivate the idea to study switched systems using directly higher-order models obtained from first principles, even if their state space does not coincide. Note for example that the dynamic modes (1.4) and (1.5) are parsimonious, since they display the strictly required level of complexity for their description.

1.2.3 Modularity

We now discuss modularity, i.e. the incremental development and combination of models. In order to illustrate this issue, consider for instance the scenario described in [53] of a DC-DC converter with impedances as loads, which is a special case of the energy distribution network in Fig. 1.4.

Consider the converter in Sec. 1.1.2 whose dynamic modes are described by (1.1)-(1.2) where the state can be selected as

$$x := \begin{bmatrix} i \\ v \end{bmatrix}.$$

As discussed in [53], during the design stage of a power converter only *nominal loads* are considered (see Fig. 1.2). However, in practical applications, it is natural to connect additional loads. Consequently it becomes of interest to study the dynamics induced by the new dynamic elements.

Consider for instance the interconnection of a series RL load at the output terminals of the converter, as depicted in Fig. 1.8.

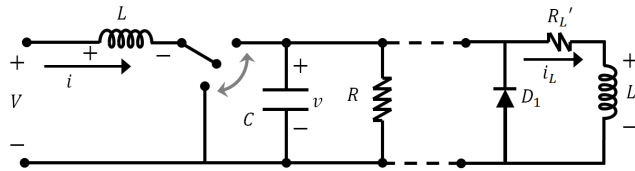


Figure 1.8: DC-DC boost converter

This situation induces two new dynamical modes (depending on the position of the switch), with state

$$x' := \begin{bmatrix} i \\ v \\ i_L \end{bmatrix}.$$

As argued in the previous section, in the state space setting the previously modelled dynamics (1.1)-(1.2) need to be rewritten to satisfy a global structure, by using the largest state vector. However, note that every time an additional load is connected,

this operation must be repeated. In other words, *in the state space setting dynamic mode descriptions depend on each other; consequently, new modes cannot be added to the underlying bank without altering the existing ones.*

Note that such inconvenient situation does not occur if we consider higher-order modelling: after selecting the variables of interest, each dynamic mode can be modelled individually and added to the underlying bank of the switched system in a *natural* incremental way. For example, when we consider the new load as in Fig. 1.8, we can model the new dynamic modes with respect to $\text{col}(E, i, v)$ as

$$\text{Mode 3: } \begin{cases} L \frac{d}{dt} i - E = 0 \\ L' C \frac{d^2}{dt^2} v + \left(R'_L C + \frac{L'}{R} \right) \frac{d}{dt} v + v = 0 \end{cases} \quad (1.6)$$

$$\text{Mode 4: } \begin{cases} L \frac{d}{dt} i + v - E = 0 \\ -L' \frac{d}{dt} i + L' C \frac{d^2}{dt^2} v + \left(R'_L C + \frac{L'}{R} \right) \frac{d}{dt} v + v = 0 \end{cases} \quad (1.7)$$

In this way, it does not matter how many new loads are connected: the models already constructed can be re-used, and new dynamics with incremental complexity can be considered.

1.3 Switched linear differential systems framework

As previously discussed, observation of physical switched systems suggest that the issues in modelling, parsimony and modularity, can be overcome if we deal directly with dynamic modes involving higher-order differential equations, whose associated state space does not necessarily coincide.

At this point it remains the question if further results (e.g. stability conditions, analysis using LMIs, applications, etc.) can be also obtained in an effective way when dealing with higher-order models, bypassing the need of state space representations. In this thesis we give answer to this question by developing *new* mathematical tools for modelling and analysis of switched systems. In order to do so, we use the theory and principles of behavioural system theory (see [55]) to develop a trajectory-based approach that permits to deal directly with sets of linear differential equations. Consequently, instead of requiring a particular modelling structure, we deal with models that are more natural for the application at hand, e.g. descriptions obtained from the modelling of immittances.

The collection of results derived from our new approach to switched systems embodies the *switched linear differential systems framework*. In this framework we deal with the issues of modelling, parsimony and modularity as follows.

- *Modelling.* Each dynamical mode is associated with a *mode behaviour*, i.e. the set of trajectories satisfying *higher-order linear differential equations* obtained directly from *first principles*. A *switching signal* determines which of the modes is active. Additionally, *gluing conditions* are introduced to specify the concatenability conditions of the trajectories at switching instants, e.g. charge/flux conservation principles, kinematic constraints, reset maps, etc.
- *Parsimony.* The mode behaviours do not necessarily share the same state space and their modelling does not require to satisfy a particular mathematical structure. Such freedom permits the use of equations featuring the strictly required level of complexity for each mode. Moreover, dynamic modes exhibiting a global state space is considered as a special elementary case.
- *Modularity.* Each dynamical mode is modelled individually, thus they can be added to the underlying bank in a *natural incremental way*. This feature greatly simplifies not only the modelling phase, but also the computations necessary for the study of e.g. stability.

Concepts of the behavioural setting such as linear differential systems [55], state maps [60] and quadratic differential forms [86], are crucial in the development of the main contributions in this thesis, since they provide suitable mathematical tools to deal with higher-order models and facilitate computations. The main results include a parsimonious modelling approach, stability conditions using multiple higher-order Lyapunov functions, new results concerning the notion of positive-realness in stability of switched systems, a dissipativity theory for open systems, and applications in analysis and stabilisation of energy distribution networks.

The main results in this thesis are translated into the computation of LMIs that can be easily constructed directly from the higher-order models describing the modes and the gluing conditions. Our framework constitutes a fundamental step towards the foundation of an extensive theory on which additional results in control, system identification, model order reduction, simulation, etc., can be rested.

1.4 Outline of the thesis

We now describe the contents of this thesis:

- *Chapter 2.* Some selected concepts of behavioural system theory are shown in this chapter. We present the most relevant information to deal with dynamical systems described by higher-order linear differential equations. This material is crucial for the developing of our new approach to switched to switched systems.

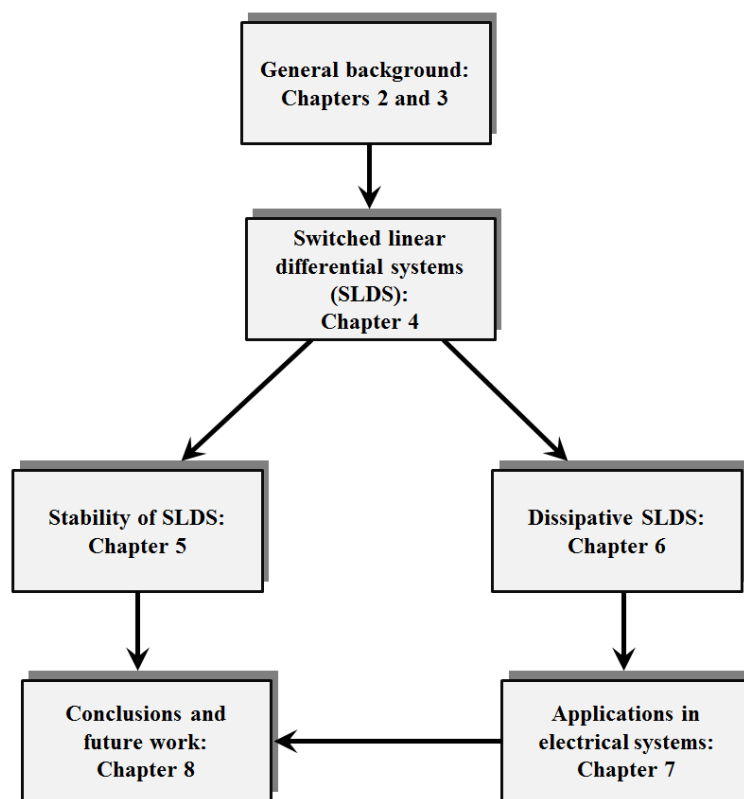
- *Chapter 3.* We review fundamental concepts and properties of quadratic differential forms (QDFs). We study QDFs as a mathematical tool to study Lyapunov stability and dissipativity under the behavioral setting. We also propose new results regarding the computation of *higher-order- Lyapunov functions* and *storage functions* using LMIs, that can be easily set-up from the linear differential equations describing the modes.

The following chapters contain the main results and contributions in this thesis.

- *Chapter 4.* In this chapter we introduce the switched linear differential systems approach. We present a modelling framework where the dynamic modes are not required to share the same state space. We introduce the concept of gluing conditions to specify algebraic constraints on the trajectories at switching instants.
- *Chapter 5.* We provide general stability conditions in terms of *multiple higher-order Lyapunov functions* and we show a method to compute them using LMIs. We also study the concept of *positive-realness* and *positive-real completion* as sufficient conditions for stability of bimodal switched differential systems. We also introduce results encompassing stability of switched systems with three modes in the underlying bank.
- *Chapter 6.* We develop a dissipativity theory that permits the study of switched differential systems in terms of energy. We give necessary and sufficient conditions for a switched system to be *dissipative*. The relationship between dissipativity and stability is also discussed.
- *Chapter 7.* We show the application of the switched linear differential systems framework in the modelling and analysis of power converters and energy distribution networks. We provide a solution to instability problems associated to constant power loads.
- *Chapter 8.* We provide some general conclusions and future research directions.

Appendix A contains all the proofs of the original results in thesis.

Finally, the following flow-diagram illustrates the content and organisation of the material of this thesis.



Chapter 2

Behavioural system theory

In this chapter we introduce general concepts of behavioural system theory that are instrumental for the development of the main results in this thesis. We concentrate on the study of linear time-invariant dynamical systems represented by differential equations. Further elaboration of the topics discussed in this chapter can be found in [55, 60].

2.1 Linear differential systems

Let us consider a dynamical system whose laws are described by a set of ordinary linear differential equations, i.e.

$$R_0 w + R_1 \frac{d}{dt} w + \dots + R_L \frac{d^L}{dt^L} w = 0, \quad (2.1)$$

where $R_i \in \mathbb{R}^{\bullet \times \mathfrak{w}}$, $i = 0, \dots, L$, and $w = \text{col}(w_1, \dots, w_{\mathfrak{w}})$ is the vector of *external variables*. Such equations can be expressed in a compact way as

$$R \left(\frac{d}{dt} \right) w = 0, \quad (2.2)$$

where $R(\xi) := R_0 + R_1 \xi + \dots + R_L \xi^L$. Note that (2.2) may contain also algebraic equations in addition to ordinary differential equations. If we adopt \mathfrak{C}^∞ as solution space of (2.2), then the *external behaviour* is defined as

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathfrak{w}}) \mid R \left(\frac{d}{dt} \right) w = 0 \right\}. \quad (2.3)$$

We denote with $\mathfrak{L}^{\mathfrak{w}}$ the set of external behaviours whose trajectories take their values in the signal space $\mathbb{R}^{\mathfrak{w}}$ as in (2.3). Note that according to (2.3), \mathfrak{B} consists of the set of trajectories in the *null space* or *kernel* of $R \left(\frac{d}{dt} \right)$, consequently we call equation (2.2) a *kernel representation* of the behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathfrak{w}}$, denoted by $\mathfrak{B} = \ker R \left(\frac{d}{dt} \right)$.

Definition 2.1. A *linear differential system* is a triple $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$, with time axis \mathbb{R} , signal space \mathbb{R}^w and behaviour $\mathfrak{B} \in \mathfrak{L}^w$.

As discussed in the previous chapter, when a dynamical system is modelled from first principles, it is normally necessary to introduce auxiliary variables (*latent variables*) in addition to those in which we are interested in (*manifest or external variables*). In such cases, we obtain the following *hybrid system* of linear, constant coefficient differential equations

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_L \frac{d^L}{dt^L} w = M_0 z + M_1 \frac{d}{dt} z + \cdots + M_{L'} \frac{d^{L'}}{dt^{L'}} z,$$

where $R_i \in \mathbb{R}^{\bullet \times w}$, $i = 0, \dots, L$; $M_j \in \mathbb{R}^{\bullet \times z}$, $j = 0, \dots, L'$; $w = \text{col}(w_1, \dots, w_w)$ and $z = \text{col}(z_1, \dots, z_z)$ are the vectors of *external*- and *latent*- variables respectively. Such equations can be expressed in a compact way as

$$R \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) z, \quad (2.4)$$

with $R \in \mathbb{R}^{\bullet \times w}[\xi]$ and $M \in \mathbb{R}^{\bullet \times z}[\xi]$. Similarly to the previous case, we can define a hybrid linear differential system as follows.

Definition 2.2. A *hybrid linear differential system* is a quadruple $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathbb{R}^z, \mathfrak{B}_f)$, where:

- \mathbb{R} , is the time axis.
- \mathbb{R}^w is the *manifest signal space*.
- \mathbb{R}^z is the *latent variable space*.
- $\mathfrak{B}_f := \left\{ (w, z) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^z) \mid R \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) z \right\}$ is called the *full behaviour* of the system.

The behavior defined as

$$\mathfrak{B} := \{ w : \mathbb{R} \rightarrow \mathbb{R}^w \mid \exists z : \mathbb{R} \rightarrow \mathbb{R}^z \text{ s.t. } (w, z) \in \mathfrak{B}_f \},$$

is called the *external behaviour* corresponding to (2.4), while the full behaviour \mathfrak{B}_f is sometimes called *internal behaviour*.

In the previous definitions we have adopted \mathfrak{C}^∞ as solution space of the linear differential equations that describe the laws of a dynamical system. Note however that \mathfrak{C}^∞ leaves out functions such as steps, sawtooth waves, etc., which are involved in the description and analysis of many physical situations.

In such cases we rather choose an *extended* solution space such as the set of *locally integrable functions*, that is large enough to accomodate most of the typical functions that we find in practice. This set consists of all functions $f : \mathbb{R} \rightarrow \mathbb{R}^q$ such that for all $a, b \in \mathbb{R}$ it holds that

$$\int_a^b \|f(t)\| dt < \infty ,$$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^q . The space of locally integrable functions $f : \mathbb{R} \rightarrow \mathbb{R}^q$ is denoted by $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$.

2.2 Controllability and observability

We introduce the concept of *controllability* of a linear differential system.

Definition 2.3. A linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exists a $t' \geq 0$ and $w \in \mathfrak{B}$, such that $w(t) = w_1(t)$ for $t < 0$ and $w(t + t') = w_2(t)$ for $t \geq 0$.

The property of controllability corresponds to the possibility of conveying any trajectory from the past to the future of any other trajectory in the behavior, with a certain finite time delay. The set of linear differential controllable behaviours is denoted by $\mathfrak{L}_{\text{cont}}^w$.

Controllability of a system represented by (2.2) can be characterised in terms of properties of the polynomial matrix R .

Theorem 2.4. Let $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ be a linear differential system with $\mathfrak{B} := \ker R \left(\frac{d}{dt} \right)$ and $R \in \mathbb{R}^{\bullet \times w}[\xi]$. \mathfrak{B} is controllable if and only if $R(\lambda)$ is full row rank for all $\lambda \in \mathbb{C}$.

Proof. See [55], Theorem 5.2.10. □

Controllable systems admit a special representation called *image representation*.

Theorem 2.5. Let $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ be a linear differential system. There exists a $z \in \mathbb{N}$ and $M \in \mathbb{R}^{w \times z}[\xi]$ such that $\mathfrak{B} = \{w \mid \exists z \text{ s.t. } w = M \left(\frac{d}{dt} \right) z\}$ if and only if \mathfrak{B} is controllable.

Proof. See [55], Theorem 6.6.1. □

Note that the image representation $w = M \left(\frac{d}{dt} \right) z$ is a special case of the hybrid representation in (2.4), where the polynomial matrix R is equal to the identity. The behaviors described by an image representation as in Theorem 2.5 are denoted by $\mathfrak{B} = \text{im } M \left(\frac{d}{dt} \right)$. Note that in this case, the latent variable z remains unconstrained, i.e. it can be any trajectory in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^z)$. This representation has important practical implications, for

instance in simulations any arbitrary time function $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^z)$ induces a trajectory $w \in \mathfrak{B}$.

Another commonly studied property in dynamical systems is *observability*. In the behavioral framework this property corresponds to the possibility of deducing part of the variables of the system from the remaining ones. In order to formalize this concept, the following definition is introduced.

Definition 2.6. A linear differential system with latent variables $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_f)$ is said to have observable latent variables if there exists a map $f : \mathbb{R} \rightarrow \mathbb{R}^w$ such that for all $(w, z) \in \mathfrak{B}_f$, $z = f(w)$.

The observability of a linear differential system can be characterized algebraically as follows.

Theorem 2.7. Let $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathbb{R}^z, \mathfrak{B})$ be a linear differential system with latent variables represented by (2.4). The variable z is observable from w if and only if $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$.

Proof. See [55], Theorem 5.3.3. □

Moreover, as it is shown in Sec. 4 of [84], if \mathfrak{B} is controllable it always admits an image representation such that the latent variable z is observable from w .

2.3 Inputs and outputs

Given $\mathfrak{B} \in \mathcal{L}^w$, it may be possible to choose some components of the external variable w freely, according to the following definition.

Definition 2.8. Let $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ be a linear differential system. Partition the signal space $\mathbb{R}^w = \mathbb{R}^{w_1} \times \mathbb{R}^{w_2}$, and correspondingly any trajectory $w \in \mathfrak{B}$ as $w = \text{col}(w_1, w_2)$ with $w_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1})$ and $w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2})$. This partition is called an *input/output* partition if:

- 1) w_1 is *free*, i.e. for all $w_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1})$, there exists $w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2})$ such that $\text{col}(w_1, w_2) \in \mathfrak{B}$.
- 2) w_1 is *maximally free*, i.e. given w_1 , none of the components of w_2 are free.

If 1) and 2) hold, we call w_1 an *input variable* and w_2 an *output variable*.

Remark 2.9. As discussed in Sec. 2.1, in Def. 2.8 we can also adopt the set of locally integrable functions as solution space. Then some trajectories can be (maximally) free on $\mathcal{L}_1^{\text{loc}}$. □

The maximal number of input variables is an invariant and is denoted by $\mathfrak{m}(\mathfrak{B})$. Once $\mathfrak{m}(\mathfrak{B})$ free variables have been chosen, the remaining components of w are *output variables*; evidently, the number $\mathfrak{p}(\mathfrak{B}) := \mathfrak{w} - \mathfrak{m}(\mathfrak{B})$ of output variables is also an invariant.

Under certain conditions, linear differential systems with inputs and outputs can be described using image form representations; we now elaborate on this feature. Let $\mathfrak{B} \in \mathfrak{L}^{\mathfrak{w}}$ be a controllable behaviour represented by an observable image representation $w = M\left(\frac{d}{dt}\right)z$. Let P be a permutation matrix such that

$$PM\left(\frac{d}{dt}\right) = \text{col}\left(U\left(\frac{d}{dt}\right), Y\left(\frac{d}{dt}\right)\right),$$

where $Y(\xi)U(\xi)^{-1}$ is a matrix of proper rational functions called *transfer function* (see [55], Sec. 3.3). This corresponds to a permutation of the elements of the external variable such that it can be rewritten as $Pw = (u, y)$, where $u = U\left(\frac{d}{dt}\right)z$ is an input variable, and $y = Y\left(\frac{d}{dt}\right)z$ is an output variable. Moreover, since z is observable from w , it follows that $\mathfrak{m}(\mathfrak{B}) = \mathfrak{z}$, i.e. the number of input variables is equal to the dimension of z (see [87], Sec. VI-A).

Example 2.1. Consider the 1-port electrical circuit in Fig. 2.1.

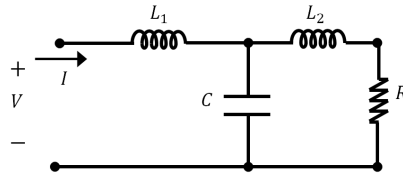


Figure 2.1: Port-driven electrical circuit

The 1-port impedance of the circuit is computed by series and parallel operations as

$$Z(s) = L_1 s + \frac{(L_2 s + R) \left(\frac{1}{C_1 s}\right)}{(L_2 s + R) + \left(\frac{1}{C_1 s}\right)} = \frac{L_1 L_2 C_1 s^3 + R L_1 C_1 s^2 + (L_1 + L_2)s + R}{L_2 C_1 s^2 + R C_1 s + 1}, \quad (2.5)$$

which corresponds to the input-output description

$$L_1 L_2 C_1 \frac{d^3}{dt^3} I + R L_1 C_1 \frac{d^2}{dt^2} I + (L_1 + L_2) \frac{d}{dt} I + R I = L_2 C_1 \frac{d^2}{dt^2} V + R C_1 \frac{d}{dt} V + V.$$

Let for simplicity $R = 1 \Omega$, $L_1 = L_2 = 1 H$ and $C = 1 F$, then

$$\underbrace{\begin{bmatrix} V \\ I \end{bmatrix}}_{=:w} = \underbrace{\begin{bmatrix} \frac{d^3}{dt^3} + \frac{d^2}{dt^2} + 2\frac{d}{dt} + 1 \\ \frac{d^2}{dt^2} + \frac{d}{dt} + 1 \end{bmatrix}}_{=:M\left(\frac{d}{dt}\right)} z,$$

where z is a latent variable corresponding to the current through the inductor L_2 . Since $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$, we conclude that the latent variable z is observable from w . \square

2.4 Autonomous systems and stability

We now study systems with *autonomous behaviour*, according to the following definition.

Definition 2.10. A behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathfrak{w}}$ is said to be autonomous if for all $(w_1, w_2) \in \mathfrak{B}$,

$$\{w_1(t) = w_2(t) \text{ for } t < 0\} \implies \{w_1 = w_2\}.$$

In an autonomous behaviour the future of every trajectory is completely determined by its past; there are no free components among its variables. Thus, if \mathfrak{B} is autonomous; $\mathfrak{m}(\mathfrak{B}) = 0$, or equivalently $\mathfrak{p}(\mathfrak{B}) = \mathfrak{w}$.

An explicit mathematical description of autonomous behaviours is provided in the following theorem.

Theorem 2.11. Let \mathfrak{B} be described in kernel form by $R\left(\frac{d}{dt}\right)w = 0$. Then \mathfrak{B} is autonomous iff R has full column rank. Moreover, if \mathfrak{B} is autonomous, there exists $R \in \mathbb{R}^{\mathfrak{w} \times \mathfrak{w}}[\xi]$ with $\det(R) \neq 0$ such that $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$.

Assume that R is a nonsingular matrix such that $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$, and denote $\lambda_i \in \mathbb{C}$, $i = 1, \dots, N$, the roots of $\det(R)$, each with multiplicity n_i . Then

$$\{w \in \mathfrak{B}\} \iff \{w = \sum_{i=0}^N \sum_{j=0}^{n_i-1} B_{i,j} t^j e^{\lambda_i t}\},$$

where the vectors $B_{i,j} \in \mathbb{C}^{\mathfrak{w}}$ satisfy:

$$\begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} R^{(0)}(\lambda_i) & \cdots & \cdots & \begin{pmatrix} n_i-1 \\ 0 \end{pmatrix} R^{(n_i-1)}(\lambda_i) \\ 0 & \ddots & & \begin{pmatrix} n_i-1 \\ 1 \end{pmatrix} R^{(n_i-2)}(\lambda_i) \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \begin{pmatrix} n_i-1 \\ n_i-1 \end{pmatrix} R^{(0)}(\lambda_i) \end{bmatrix} \begin{bmatrix} B_{i,0} \\ B_{i,1} \\ \vdots \\ B_{i,n_i-1} \end{bmatrix} = 0,$$

for $i = 1, \dots, N$, where $R^{(k)}(\xi)$ denotes the k -derivative of the polynomial matrix $R(\xi)$ with respect to ξ .

Proof. See Th. 3.2.16 of [55]. \square

We now introduce the notion of *stability* and *asymptotic stability* for autonomous systems.

Definition 2.12. A linear differential system $\Sigma = \{\mathbb{R}, \mathbb{R}^w, \mathfrak{B}\}$ is said to be *stable* if all the elements in \mathfrak{B} are bounded on \mathbb{R}^+ , this means that given $w \in \mathfrak{B}$ there exists an $\epsilon \in \mathbb{R}$ such that $\|w(t)\| \leq \epsilon$ for all $t \geq 0$. Σ is said to be *unstable* if it is not stable.

Definition 2.13. A linear differential system $\Sigma = \{\mathbb{R}, \mathbb{R}^w, \mathfrak{B}\}$ is said to be *asymptotically stable* if all elements of \mathfrak{B} approach to zero as $t \rightarrow \infty$, i.e.

$$\{w \in \mathfrak{B}\} \implies \left\{ \lim_{t \rightarrow \infty} w(t) = 0 \right\}.$$

According to the previous definition, (asymptotically) stable systems are autonomous, since none of the components of $w \in \mathfrak{B}$ is free, otherwise they could be chosen as not bounded or not going to zero. Stability of an autonomous behaviour can be characterised in terms of the properties of R , as shown in the following theorem.

Theorem 2.14. Let $\mathfrak{B} = \ker R(\frac{d}{dt})$, with $R \in \mathbb{R}[\xi]^{w \times w}$ nonsingular, be an autonomous behaviour. Then \mathfrak{B} is stable if and only if $\det R(\lambda) \neq 0$ for all $\lambda \notin \mathbb{C}^-$.

Proof. See [55], section 7.2. □

2.5 State space systems

Before introducing the notion of *state space system*, we briefly discuss the concepts of *concatenability* and of *state variable*. In order to do so, let us consider two trajectories w_1 and w_2 in \mathfrak{B} , then the *concatenation* of w_1 and w_2 is defined by

$$(w_1 \wedge w_2)(t) := \begin{cases} w_1(t) & t < 0 \\ w_2(t) & t \geq 0 \end{cases}.$$

Concatenation is closely related to the notion of state variable which is a special case of a hybrid linear differential system.

Definition 2.15. Let $\Sigma_s = (\mathbb{R}, \mathbb{R}^w, \mathbb{R}^z, \mathfrak{B}_f)$ be a hybrid linear differential system. The latent variable z is a *state* variable if

$$\begin{aligned} & \{(w_1, z_1), (w_2, z_2) \in \mathfrak{B}_f\} \text{ and } \{z_1(0) = z_2(0)\} \text{ and } \{z_1, z_2 \text{ continuous at } t = 0\} \\ & \implies \{(w_1 \wedge w_2) \in \mathfrak{B}\}. \end{aligned}$$

Such a system Σ_s is called *state space system*, and the latent variable z corresponds to the memory of the system that “splits” the past and future of a trajectory at zero. This variable is usually denoted by x .

We now introduce the concept of minimality for state space systems.

Definition 2.16. Let $\Sigma_s = (\mathbb{R}, \mathbb{R}^w, \mathbb{R}^n, \mathfrak{B}_f)$ be a state space system, and \mathfrak{B} be the external behaviour of Σ_s . Σ_s is *state minimal* if for any other state space system $\Sigma'_s = (\mathbb{R}, \mathbb{R}^w, \mathbb{R}^{n'}, \mathfrak{B}'_f)$ with the same external behaviour \mathfrak{B} , it follows that $n \leq n'$.

The minimal number of state variables necessary to represent \mathfrak{B} is an invariant called the *McMillan degree* of \mathfrak{B} , and is denoted by $n(\mathfrak{B})$.

2.6 State construction

A state variable for \mathfrak{B} can be computed as the image of a polynomial differential operator $X \left(\frac{d}{dt} \right)$ called *state map* (see [60]); we now review how to construct it. Further elaboration and methods regarding state space construction can be found in [60, 80].

2.6.1 State maps for autonomous systems

In this case, a state map acts on the external variable w . Let $\mathfrak{B} := \ker R \left(\frac{d}{dt} \right)$ with $R \in \mathbb{R}^{w \times w}[\xi]$ nonsingular, and consider the set defined by

$$\mathfrak{X}(R) := \{f \in \mathbb{R}^{1 \times w}[\xi] \mid fR^{-1} \text{ is strictly proper}\}, \quad (2.6)$$

then, $\mathfrak{X}(R)$ is a finite-dimensional subspace of the vector space $\mathbb{R}^{1 \times w}[\xi]$ over \mathbb{R} , (see [60], Proposition 8.4). A state map $X \in \mathbb{R}^{\bullet \times w}[\xi]$ is constructed by row vectors in $\mathfrak{X}(R)$. A minimal state map has dimension $n(\mathfrak{B}) = \deg(\det(R))$, as shown in Cor. 6.7 of [60]. We now illustrate the construction of minimal state maps for autonomous systems using the following algorithm.

Algorithm 1: Construction of a minimal state map for a kernel representation.

Input: $R \in \mathbb{R}^{w \times w}[\xi]$ with $\det(R) \neq 0$.

Output: $X \in \mathbb{R}^{\bullet \times w}[\xi]$ inducing, through $x = X \left(\frac{d}{dt} \right) w$, a state for the system described by $R \left(\frac{d}{dt} \right) w = 0$.

Step 1: Choose a set of generators $\{x_i\}_{i=1, \dots, n(\mathfrak{B})}$ so that they form a basis of (2.6).

Step 2: Return $X := \text{col}(x_i)_{i=1, \dots, n(\mathfrak{B})}$.

Given an autonomous behaviour $\mathfrak{B} := \ker R \left(\frac{d}{dt} \right)$ there exists (according to Th. 6.2 of [60]) $A \in \mathbb{R}^{\bullet \times \bullet}$, $B \in \mathbb{R}^{\bullet \times w}[\xi]$ such that

$$\xi X(\xi) = AX(\xi) + B(\xi)R(\xi) .$$

In addition, if we consider a behaviour $\mathfrak{B} \in \mathfrak{L}^w$, and $X \in \mathbb{R}^{\bullet \times w}[\xi]$ a state map for \mathfrak{B} acting on w , a polynomial differential operator $Y \in \mathbb{R}^{\bullet \times w}[\xi]$ is a linear *function of the state* of \mathfrak{B} if there exists a constant vector $C \in \mathbb{R}^{\bullet \times n}$ such that

$$Y(\xi) = CX(\xi) ,$$

which implies that the rows of $Y(\xi)$ belong to $\mathfrak{X}(R)$.

2.6.2 State maps for controllable systems

We now summarise a result concerning the construction of state maps for controllable systems, acting on a latent variable z . Let $w = M \left(\frac{d}{dt} \right) z$, with $M \in \mathbb{R}^{w \times z}[\xi]$, be an observable image representation of \mathfrak{B} . As discussed in Sec. 2.3, use a permutation matrix P to obtain an input-output partition $PM = (U, Y)$. Then, consider the set defined by

$$\mathfrak{X}(M) := \{r \in \mathbb{R}^{1 \times 1}[\xi] \mid rU^{-1} \text{ is strictly proper}\} . \quad (2.7)$$

which is a vector space $\mathbb{R}^{1 \times w}[\xi]$ over \mathbb{R} , (see [60], Proposition 8.4). X is a state map for \mathfrak{B} if and only if its rows span the vector space (2.7), and a *minimal* one if and only if its rows form a basis for (2.7) (see [60], Sec. 8). Moreover, since $w = M \left(\frac{d}{dt} \right) z$ is an observable image representation, then $\mathfrak{n}(\mathfrak{B}) = \deg(\det(U))$ (see Prop. 3.5.5 of [60]). The construction of state maps for controllable systems described by observable image form representations is summarised in the following algorithm.

Algorithm 2: Construction of a minimal state map for an observable image representation.

Input: $M \in \mathbb{R}^{w \times w}[\xi]$, such that $M(\lambda)$ is of column rank for all $\lambda \in \mathbb{C}$.

Output: $X \in \mathbb{R}^{\bullet \times w}[\xi]$ inducing, through $x = X \left(\frac{d}{dt} \right) z$, a state for the system described by $w = M \left(\frac{d}{dt} \right) z$.

Step 1: Partition M using a permutation matrix P , such that $PM = \text{col}(Y, U)$ where $\det(U) \neq 0$ and YU^{-1} is proper.

Step 1: Choose a set of generators $\{x_i\}_{i=1, \dots, \mathfrak{n}(\mathfrak{B})}$ so that they form a basis of (2.7).

Step 3: Return $X := \text{col}(x_i)_{i=1, \dots, \mathfrak{n}(\mathfrak{B})}$.

2.7 Equivalence of representations

We now introduce important concepts in the behavioural setting associated to the equivalence of representations in kernel form $R \left(\frac{d}{dt} \right) w = 0$.

In this analysis, the concept of *unimodular matrix* plays an important role. We call $V \in \mathbb{R}^{q \times q}[\xi]$ *unimodular*, if there exists $V' \in \mathbb{R}^{q \times q}[\xi]$ such that $V'(\xi)V(\xi) = I_q$, equivalently $\det(V) \neq 0$ is constant.

Theorem 2.17. *Let $\mathfrak{B}_1 := \ker R_1 \left(\frac{d}{dt} \right)$ with $R_1 \in \mathbb{R}^{q \times w}[\xi]$ and let $V \in \mathbb{R}^{q \times q}[\xi]$. Define $\mathfrak{B}_2 := \ker V R_2 \left(\frac{d}{dt} \right)$ with $R_2 \in \mathbb{R}^{q \times w}[\xi]$, then $\mathfrak{B}_2 \subseteq \mathfrak{B}_1$. Moreover, if V is unimodular, then $\mathfrak{B}_1 = \mathfrak{B}_2$.*

Proof. See [55], Theorem 2.5.4. □

Now we introduce the notion of *R-equivalence* for polynomial matrices.

Definition 2.18. Let $R \in \mathbb{R}^{w \times w}[\xi]$ be nonsingular. The polynomial matrices $F_1, F_2 \in \mathbb{R}^{\bullet \times w}[\xi]$ are said to be *R-equivalent* if there exists a polynomial matrix $P \in \mathbb{R}^{\bullet \times w}[\xi]$ such that $F_1 - F_2 = PR$.

Note that according to Definition 2.18, the polynomial matrices $F_1 \left(\frac{d}{dt} \right)$ and $F_2 \left(\frac{d}{dt} \right)$, are *equivalent along $\ker R \left(\frac{d}{dt} \right)$* in the sense that

$$F_1 \left(\frac{d}{dt} \right) w = F_2 \left(\frac{d}{dt} \right) w, \quad \text{for all } w \in \ker R \left(\frac{d}{dt} \right).$$

If a polynomial matrix $F \in \mathbb{R}^{\bullet \times w}$ is such that FR^{-1} is strictly proper, we say that F is *R-canonical*. Furthermore, FR^{-1} can be uniquely written as

$$FR^{-1} = S + P,$$

where $S \in \mathbb{R}^{\bullet \times w}(\xi)$ is a matrix of strictly proper rational functions, and $P \in \mathbb{R}^{\bullet \times w}[\xi]$. According to this, we present the following definition of *R-canonical representative*.

Definition 2.19. Consider $R \in \mathbb{R}^{w \times w}[\xi]$ nonsingular; $F, P \in \mathbb{R}^{\bullet \times w}[\xi]$ and $S \in \mathbb{R}^{\bullet \times w}(\xi)$ as previously defined. We call $SR \in \mathbb{R}^{\bullet \times w}[\xi]$, the *R-canonical representative* of F modulo R , denoted by $F \bmod R$.

2.8 Summary

We have introduced fundamental concepts of behavioural system theory that are frequently used in the rest of this thesis, in particular the concept of linear differential system, autonomous systems, controllable systems and state maps.

Chapter 3

Quadratic differential forms and dissipativity

In the previous section we introduced some general concepts that are instrumental for the study of the behaviour of the system variables. In some cases it is also crucial to study the properties of certain functionals of these variables. In the behavioural setting, quadratic differential forms play an important role, for instance, they allow us to use higher-order Lyapunov functions and formulate the concept of dissipativity. Let us start by introducing the theory and notation that will be frequently used in the following chapters.

3.1 Preliminary concepts

Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ be a real polynomial matrix in the indeterminates ζ and η ; then

$$\Phi(\zeta, \eta) = \sum_{k,j}^m \Phi_{k,j} \zeta^k \eta^j, \quad (3.1)$$

where $m \in \mathbb{N}$ and $\Phi_{k,j} \in \mathbb{R}^{w \times w}$ for all $k, j \in \{1, \dots, m\}$. Such a Φ induces a *quadratic differential form*

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w) \rightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$
$$Q_\Phi(w) := \sum_{k,j} \left(\frac{d^k w}{dt^k} \right)^\top \Phi_{k,j} \left(\frac{d^j w}{dt^j} \right).$$

We call $\Phi \in \mathbb{R}^{\mathbf{w} \times \mathbf{w}}[\zeta, \eta]$ *symmetric* if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$, and this is the case we deal with in the rest of this thesis. $\Phi(\zeta, \eta)$ can be identified with its *coefficient matrix*

$$\tilde{\Phi} := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,L} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,L} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{L,0} & \Phi_{L,1} & \cdots & \Phi_{L,L} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} ;$$

in the sense that

$$\Phi(\zeta, \eta) = \begin{bmatrix} I_{\mathbf{w}} & \zeta I_{\mathbf{w}} & \cdots & \zeta^L I_{\mathbf{w}} & \cdots \end{bmatrix} \tilde{\Phi} \begin{bmatrix} I_{\mathbf{w}} \\ \eta I_{\mathbf{w}} \\ \vdots \\ \eta^L I_{\mathbf{w}} \\ \vdots \end{bmatrix} .$$

Although $\tilde{\Phi}$ is infinite, only a finite number of its entries are nonzero, since the highest power of ζ and η in $\Phi(\zeta, \eta)$ is finite. Note that $\Phi(\zeta, \eta)$ is symmetric if and only if its coefficient matrix is symmetric, i.e. $\tilde{\Phi}^\top = \tilde{\Phi}$. In that case, $\tilde{\Phi}$ can be factored as

$$\tilde{\Phi} := \widetilde{M}^\top \Sigma_\Phi \widetilde{M} ,$$

where $\widetilde{M} \in \mathbb{R}^{\bullet \times \infty}$, with all but a finite number of elements equal to zero, and Σ_Φ a *signature matrix*. This decomposition leads, after premultiplication by $\begin{bmatrix} I_{\mathbf{w}} & \zeta I_{\mathbf{w}} & \cdots \end{bmatrix}$ and post-multiplication by $\text{col} \begin{bmatrix} I_{\mathbf{w}} & \eta I_{\mathbf{w}} & \cdots \end{bmatrix}$, to the factorisation

$$\Phi(\zeta, \eta) = M(\zeta)^\top \Sigma_\Phi M(\eta) ,$$

which, if we take \widetilde{M} surjective, is called a *canonical factorisation* of $\Phi(\zeta, \eta)$.

Expressing quadratic differential forms in terms of two-variable polynomial matrices allows some convenient computations. For instance the result of the *differentiation of a quadratic differential form* is expressed in terms of two-variable polynomial matrices and leads to the dot operator \bullet defined as

$$\bullet : \mathbb{R}^{\mathbf{w} \times \mathbf{w}}[\zeta, \eta] \rightarrow \mathbb{R}^{\mathbf{w} \times \mathbf{w}}[\zeta, \eta] ;$$

and

$$\dot{\Phi}(\zeta, \eta) := (\zeta + \eta)\Phi(\zeta, \eta) . \tag{3.2}$$

The two-variable polynomial matrix (3.2) induces

$$\frac{d}{dt}Q_{\Phi(\zeta,\eta)} := Q_{\dot{\Phi}(\zeta,\eta)} .$$

3.2 Equivalence of quadratic differential forms

An analogous concept of R -canonical representative given for one-variable polynomial matrices is now introduced for two-variable polynomial matrices.

Definition 3.1. Let $R \in \mathbb{R}^{w \times w}[\xi]$ be nonsingular. Two quadratic differential forms Q_{Φ_1}, Q_{Φ_2} are R -equivalent, i.e. equivalent along $\ker R\left(\frac{d}{dt}\right)$, if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for all $w \in \ker R\left(\frac{d}{dt}\right)$.

Among all QDFs that are equivalent to a given Q_{Φ} , there is exactly one which is R -canonical (see [86] p. 1716). Recall from the material in section 2.7 the computation of the R -canonical representative of one-variable polynomial matrices. The canonical representative of a two-variable polynomial matrix is computed as follows.

Consider $R \in \mathbb{R}^{w \times w}[\xi]$ nonsingular. Factorise

$$\Phi(\zeta, \eta) = M(\zeta)^{\top} N(\eta) .$$

Now compute the R -canonical representatives M' of M and N' of N . Then the R -canonical representative of $\Phi(\zeta, \eta)$ modulo R , denoted by $\Phi(\zeta, \eta) \bmod R$, is $M'(\zeta)^{\top} N'(\eta)$ (see [86], p. 1716).

3.3 Positivity, negativity and nonnegativity of a QDF

The notions of positivity, negativity and nonnegativity are crucial in many applications such as Lyapunov stability theory. We now introduce these concepts.

Definition 3.2. Given $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$, the quadratic differential form Q_{Φ} is said to be *nonnegative*, denoted by $Q_{\Phi} \geq 0$, if $Q_{\Phi}(w) \geq 0$ for all $w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^w)$; and *positive*, denoted by $Q_{\Phi} > 0$, if $Q_{\Phi}(w) \geq 0 \forall w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^w)$ and $[Q_{\Phi}(w) = 0] \implies [w = 0]$. We define $Q_{\Phi} < 0$ in an analogous manner.

Positivity and nonnegativity have the following algebraic characterisation.

Proposition 3.3. Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. $Q_{\Phi} \geq 0$ if and only if there exists $D \in \mathbb{R}^{w \times w}[\xi]$ such that

$$\Phi(\zeta, \eta) = D(\zeta)^{\top} D(\eta) .$$

Moreover, $Q_{\Phi} > 0$ if such D has the property that $D(\lambda)$ is of rank w for all $\lambda \in \mathbb{C}$.

Proof. See [86], p. 1712. □

In some cases, we are interested in studying positivity, negativity and nonnegativity along a particular behaviour.

Definition 3.4. Given $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$, the quadratic differential form Q_Φ is said to be *nonnegative along \mathfrak{B}* , denoted by $Q_\Phi \stackrel{\mathfrak{B}}{\geq} 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$; and *positive along \mathfrak{B}* , denoted by $Q_\Phi \stackrel{\mathfrak{B}}{>} 0$, if $Q_\Phi(w)$ is positive, i.e. $Q_\Phi(w) \geq 0 \ \forall w \in \mathfrak{B}$ and $\{Q_\Phi(w) = 0\} \implies \{w = 0\}$. We define $Q_\Phi \stackrel{\mathfrak{B}}{<} 0$ in an analogous manner.

The following results show how to test algebraically whether a QDF is positive or non-negative along the trajectories of a behaviour.

Proposition 3.5. Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ and let $\mathfrak{B} := \ker R \left(\frac{d}{dt} \right)$, with $R \in \mathbb{R}^{\bullet \times w}[\xi]$. Then

1. $Q_\Phi \stackrel{\mathfrak{B}}{\geq} 0$ iff there exists $Y \in \mathbb{R}^{\bullet \times w}[\zeta, \eta]$ and $D \in \mathbb{R}^{\bullet \times w}[\xi]$, such that

$$\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta) + Y(\zeta, \eta)^\top R(\eta) + R(\zeta)^\top Y(\zeta, \eta) .$$

2. $Q_\Phi \stackrel{\mathfrak{B}}{>} 0$, iff $Q_\Phi \stackrel{\mathfrak{B}}{\geq} 0$ and $\begin{bmatrix} D(\lambda) \\ R(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$.

Proof. See Prop. 3.5 of [86]. □

3.4 Reformulation of QDFs in terms of latent variables

The reformulation of QDFs in terms of latent variables is often used when dealing with controllable behaviours (see e.g. [78, 86, 87]), since it simplifies certain computations such as positivity tests. We now discuss this procedure.

Let \mathfrak{B} be a controllable behaviour, with an observable image representation

$$w = M \left(\frac{d}{dt} \right) z .$$

Let Q_Φ be a QDF induced by $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ acting on the external variable w . Define $\Phi' \in \mathbb{R}^{z \times z}[\zeta, \eta]$ as

$$\Phi'(\zeta, \eta) := M(\zeta)^\top \Phi(\zeta, \eta) M(\eta) .$$

It can be easily verified that if w and z satisfy $w = M \left(\frac{d}{dt} \right) z$, it follows that

$$Q_{\Phi'}(z) = Q_\Phi(w) .$$

In this way, it is possible to study the properties of a functional in terms of an unconstrained variable, e.g. note that since $Q_{\Phi'}(z) = Q_{\Phi}(w)$, it follows that $Q_{\Phi} \stackrel{\text{B}}{\geq} 0$ if and only if $Q_{\Phi'} \geq 0$ on $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^z)$.

3.5 Integrals of QDFs

We now introduce the *integral of a quadratic differential form*. In order to make sure that such integral exists, we assume that the trajectories on which the quadratic differential form operates are *infinitely differentiable functions with compact support*. The set of such trajectories is denoted by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$. Given a quadratic differential form Q_{Φ} , we define its integral as

$$\begin{aligned} \int Q_{\Phi} : \mathfrak{D}(\mathbb{R}, \mathbb{R}^w) &\rightarrow \mathbb{R} , \\ \int Q_{\Phi}(w) &:= \int_{-\infty}^{\infty} Q_{\Phi}(w) dt . \end{aligned}$$

We now define *average nonnegativity* and *average strict positivity* of a QDF.

Definition 3.6. Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. Q_{Φ} is called

1. *average nonnegative*, if $\int Q_{\Phi}(w) \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$;
2. *strictly average positive*, if there exists $\epsilon > 0$ such that for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$, it holds that

$$\int Q_{\Phi}(w) \geq \epsilon \int_{-\infty}^{\infty} \|w\|_2^2 dt ,$$

Average nonnegativity and average strict positivity of a QDF can be also tested by studying the properties of its two-variable polynomial matrix on the imaginary axis. In order to do so, we first introduce the ∂ operator, that permits to obtain one-variable polynomial matrices from two-variable polynomial ones. This operator is defined as follows

$$\partial : \mathbb{R}^{w \times w}[\zeta, \eta] \rightarrow \mathbb{R}^{w \times w}[\xi] ; \partial\Phi(\xi) := \Phi(-\xi, \xi) .$$

Now consider the following proposition.

Proposition 3.7. Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$, and obtain a symmetric canonical factorisation $\Phi(\zeta, \eta) := M(\zeta)^{\top} \Sigma_{\Phi} M(\eta)$. Then Q_{Φ} is

1. *average nonnegative* iff $\partial\Phi(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$;
2. *strictly average positive* iff there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$, it holds that

$$\partial\Phi(j\omega) \geq \epsilon M(-j\omega)^{\top} M(j\omega) .$$

Proof. See Prop. 5.2 of [86]. □

3.6 Lyapunov stability theory

We now study *Lyapunov stability* in terms of quadratic differential forms.

Theorem 3.8. Let $\mathfrak{B} \in \mathfrak{L}^w$. If there exists a two-variable polynomial matrix $\Psi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ such that $Q_\Psi \stackrel{\mathfrak{B}}{\geq} 0$ and $\frac{d}{dt}Q_\Psi \stackrel{\mathfrak{B}}{<} 0$, then \mathfrak{B} is asymptotically stable.

Proof. See Theorem 4.3 of [86]. □

The quadratic differential form Q_Ψ as in Theorem 3.8 is called a *Lyapunov function* for \mathfrak{B} . We now recall the following result from [86], that reduces the computation of quadratic Lyapunov functions to the solution of a two-variable polynomial equation called *polynomial Lyapunov equation (PLE)*.

Theorem 3.9. Let $\mathfrak{B} = \ker R\left(\frac{d}{dt}\right)$, with $R \in \mathbb{R}^{w \times w}[\xi]$ nonsingular. If \mathfrak{B} is asymptotically stable, for every $Q \in \mathbb{R}^{w \times w}[\xi]$ there exist $\Psi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ and $Y \in \mathbb{R}^{w \times w}[\xi]$ such that $Q_\Psi \geq 0$ and

$$(\zeta + \eta)\Psi(\zeta, \eta) = Y(\zeta)^\top R(\eta) + R(\zeta)^\top Y(\eta) - Q(\zeta)^\top Q(\eta). \quad (3.3)$$

If either one of Q or Y is R -canonical, then also the other and Ψ are R -canonical.

Moreover, if $\text{rank} \begin{bmatrix} R(\lambda) \\ Q(\lambda) \end{bmatrix} = w$ for all $\lambda \in \mathbb{C}$, then $Q_\Psi \stackrel{\mathfrak{B}}{>} 0$.

Proof. The result follows from Th. 4.8 and Th. 4.12 of [86]. □

Thus given an asymptotically stable behaviour, a quadratic Lyapunov function Q_Ψ can be computed by choosing some polynomial matrix Q and solving the PLE (3.3). Algebraic methods for solving it are illustrated in [52]; we now devise an LMI-based one that will be instrumental for some of the results developed in this thesis. We first introduce the following important result.

Proposition 3.10. Under the assumptions of Th. 3.9, define $n := \deg(\det(R))$ and let $X \in \mathbb{R}^{n \times w}[\xi]$ be a minimal state map for \mathfrak{B} . Write $R(\xi) = \sum_{j=0}^L R_j \xi^j$, with $R_j \in \mathbb{R}^{w \times w}$, $j = 0, 1, \dots, L$. There exists $\tilde{Y} \in \mathbb{R}^{w \times n}$, $\tilde{Q} \in \mathbb{R}^{w \times n}$ and $K \in \mathbb{R}^{n \times n}$ such that $Y(\xi) = \tilde{Y}X(\xi)$, $Q(\xi) = \tilde{Q}X(\xi)$ and $\Psi(\zeta, \eta) = X(\zeta)^\top KX(\eta)$. Moreover, there exist $X_j \in \mathbb{R}^{n \times w}$, with $j = 0, 1, \dots, L-1$, such that $X(\xi) = \sum_{j=0}^{L-1} X_j \xi^j$.

Proof. See Appendix A.1. □

Now we proceed to relate the PLE (3.3) with a constant *matrix* equation.

Proposition 3.11. *Under the assumptions of Th. 3.9 and Prop. 3.10, denote the coefficient matrices of $R(\xi)$ and $X(\xi)$ by*

$$\tilde{R} := \begin{bmatrix} R_0 & \dots & R_L \end{bmatrix}, \quad \tilde{X} := \begin{bmatrix} X_0 & \dots & X_{L-1} \end{bmatrix}.$$

Let $K = K^\top \in \mathbb{R}^{n \times n}$. The following statements are equivalent:

1. $\Psi(\zeta, \eta) := X(\zeta)^\top K X(\eta)$, R , Y , and Q satisfy (3.3);
2. There exists $K > 0$, such that

$$\begin{aligned} & \begin{bmatrix} 0_{w \times n} \\ \tilde{X}^\top \end{bmatrix} K \begin{bmatrix} \tilde{X} & 0_{n \times w} \end{bmatrix} + \begin{bmatrix} \tilde{X}^\top \\ 0_{w \times n} \end{bmatrix} K \begin{bmatrix} 0_{n \times w} & \tilde{X} \end{bmatrix} - \begin{bmatrix} \tilde{X}^\top \\ 0_{w \times n} \end{bmatrix} \tilde{Y}^\top \tilde{R} - \tilde{R}^\top \tilde{Y} \begin{bmatrix} \tilde{X} & 0_{n \times w} \end{bmatrix} \\ & + \begin{bmatrix} \tilde{X}^\top \\ 0_{w \times n} \end{bmatrix} \tilde{Q}^\top \tilde{Q} \begin{bmatrix} \tilde{X} & 0_{n \times w} \end{bmatrix} = 0. \end{aligned}$$

Proof. See Appendix A.1. □

This result permits the computation of Lyapunov functions for linear differential systems represented in kernel form in terms of easy-to-construct LMIs, by exploiting the properties of the coefficient matrices of QDFs.

3.7 Dissipativity theory

A concept that has played an important role in the study of dynamical systems is *dissipativity*, firstly introduced in a control and systems setting in [83]. This concept is useful in dealing with issues such as stability, stabilisability, design, control, model reduction and other important applications (see e.g. [78, 83, 87]). Dissipativity theory allows to study properties of a dynamical system in terms of energy as a generalised concept where the physical system energy is a special case.

In dissipativity theory we often require the integration of functionals, and as previously discussed we assume that such integrals act on trajectories of compact support. In order to ensure that such trajectories exist, we focus our analysis in the study of linear differential systems with *controllable* behaviours.

Definition 3.12. Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. \mathfrak{B} is called Φ -dissipative if for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$ it holds that

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0.$$

Moreover, Q_Φ is called the *supply rate*.

Definition 3.13. Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. \mathfrak{B} is called *strictly Φ -dissipative* if there exists $\epsilon > 0$ such that for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$

$$\int_{-\infty}^{\infty} Q_{\Phi}(w) dt \geq \epsilon \int_{-\infty}^{\infty} \|w\|_2^2 dt .$$

Although energy is flowing in and out of the system, the dissipativity property imposes that the average of the energy must be nonnegative when evaluated over $(-\infty, +\infty)$. Consequently, the total energy is strictly provided by the external world and no energy has been generated by the system itself.

Some dynamical systems have the property of energy storage, such a property is also associated to a QDF.

Definition 3.14. Let $\Psi, \Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. Q_{Ψ} is a *storage function* for \mathfrak{B} with respect to Q_{Φ} if for all $w \in \mathfrak{B}$ it holds that

$$\frac{d}{dt} Q_{\Psi}(w) \leq Q_{\Phi}(w) .$$

In some cases, the energy absorbed by a system is positive in any arbitrary interval of time, we call this special case of dissipativity as *half-line dissipativity*. The most common example of this concept is called passivity, see [48].

Definition 3.15. Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. \mathfrak{B} is *half-line Φ -dissipative* if for every $\tau \in \mathbb{R}$ and for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$ it holds that

$$\int_{-\infty}^{\tau} Q_{\Phi}(w) dt \geq 0 .$$

In the following lemma, we show an important consequence that follows when we deal with constant supply rates and the *liveness condition* is satisfied (see [87], sec. IV-B), i.e. when given $\Phi \in \mathbb{R}^{w \times w}$ and $w = \text{col}(u, y) \in \mathfrak{B}$, it holds that the number of components in the input variable u , denoted by $\mathfrak{m}(\mathfrak{B})$, equals the number of positive eigenvalues of Φ , denoted by $\sigma_+(\Phi)$.

Lemma 3.16. Let $\Phi \in \mathbb{R}^{w \times w}$ be a constant supply rate and $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$. Assume that $\sigma_+(\Phi) = \mathfrak{m}(\mathfrak{B})$. If \mathfrak{B} is half-line Φ -dissipative, then every storage function Q_{Ψ} for \mathfrak{B} is such that $Q_{\Psi} \stackrel{\mathfrak{B}}{\geq} 0$.

Proof. See Theorem 6.4 in [86]. □

A storage function satisfies the *dissipation inequality* $\frac{d}{dt} Q_{\Psi} \leq Q_{\Phi}$, which represents the property that the rate of change of the stored energy is never greater than the supply rate. We conclude that the latter fact accounts for a portion of the supplied energy being dissipated towards the environment.

Definition 3.17. Let $\Phi, \Delta \in \mathbb{R}^{w \times w}[\zeta, \eta]$ and let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ be Φ -dissipative. Q_Δ is a *dissipation function* for \mathfrak{B} with respect to Q_Φ if $Q_\Delta \geq 0$, and for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$, it holds that

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt = \int_{-\infty}^{\infty} Q_\Delta(w) dt .$$

Storage functions, supply rates and dissipation functions are associated as follows.

Proposition 3.18. Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$. The following statements are equivalent.

- \mathfrak{B} is Φ -dissipative.
- There exists a storage function Q_Ψ for \mathfrak{B} with respect to Q_Φ .
- There exists a dissipation function Q_Δ for \mathfrak{B} with respect to Q_Φ .

Moreover, there exists a one-to-one relation between Q_Φ , Q_Ψ and Q_Δ , defined by the dissipation equality

$$\frac{d}{dt} Q_\Psi = Q_\Phi - Q_\Delta .$$

If $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$, this equality holds true if and only if

$$(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta) .$$

Proof. See [77], Th. 4.3. □

Dissipativity can be also characterised in the frequency domain. In order to do so, we reformulate the supply rate in terms of the latent variable (see Sec. 3.4).

Proposition 3.19. Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ and let $\mathfrak{B} := \text{im } M\left(\frac{d}{dt}\right)$ where $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$. Define

$$\Phi'(\zeta, \eta) := M(\zeta)^\top \Phi(\zeta, \eta) M(\eta) .$$

Then \mathfrak{B} is Φ -dissipative, i.e. $\int Q_\Phi \geq 0$, iff $\partial\Phi(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

Proof. The proof follows directly from Prop. 3.7. □

Thus the inequality $\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0$ is equivalent with the condition $\partial\Phi(j\omega) \geq 0 \forall \omega \in \mathbb{R}$. Consequently, since $\int_{-\infty}^{\infty} Q_\Phi(w) dt = \int_{-\infty}^{\infty} Q_\Delta(w) dt$ and since $\partial\Phi(\xi)$ is para-Hermitian, i.e. $\partial\Phi(\xi) = \partial\Phi(-\xi)^\top$, we can compute a dissipation function by factoring

$$\Phi(-\xi, \xi) = D(-\xi)^\top D(\xi) =: \Delta(-\xi, \xi) .$$

with $D \in \mathbb{R}^{\bullet \times w}[\xi]$. This factorisation is called *polynomial spectral factorisation*, see [6].

According to Prop. 3.18, we can thus compute a storage function as

$$\Psi(\zeta, \eta) := \frac{\Phi(\zeta, \eta) - D(\zeta)^\top D(\eta)}{\zeta + \eta}.$$

Note that since the factorisation $\Phi(-\xi, \xi) = D(-\xi)^\top D(\xi)$ is not unique, it follows that there exists more than one storage function Ψ .

The set of all possible storage functions is bounded from above by the *required supply* Q_{Ψ_+} and from below by the *available storage*, Q_{Ψ_-} , which can be computed using two-variable polynomial matrices. Consider $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ such that $\partial \Phi(j\omega) > 0 \ \forall \ \omega \in \mathbb{R}$. Given standard results of polynomial spectral factorisation (see [6]), factorise

$$\begin{aligned} \partial \Phi(\xi) &= A(-\xi)^\top A(\xi), \\ \partial \Phi(\xi) &= H(-\xi)^\top H(\xi). \end{aligned}$$

with $A, H \in \mathbb{R}^{w \times w}[\xi]$, corresponding respectively to the cases when the roots of $\det(A)$ are *anti-Hurwitz* and those of $\det(H)$ *Hurwitz*. Then

$$\begin{aligned} \Psi_+(\zeta, \eta) &:= \frac{\Phi(\zeta, \eta) - A(\zeta)^\top A(\eta)}{\zeta + \eta}, \\ \Psi_-(\zeta, \eta) &:= \frac{\Phi(\zeta, \eta) - H(\zeta)^\top H(\eta)}{\zeta + \eta}. \end{aligned}$$

The set of storage functions is convex (see [83], Th. 3), i.e. if Ψ_1 and Ψ_2 are storage functions, so is

$$\Psi_\alpha := \alpha \Psi_1 + (1 - \alpha) \Psi_2, \quad \text{with } 0 \leq \alpha \leq 1.$$

The computation of storage functions can be also performed in terms of LMIs. Let us consider the case when the supply rate is constant. In the first result we show that the storage function is a quadratic function of the state, and the dissipation function a quadratic function of the state and the input.

Proposition 3.20. *Let $\Phi \in \mathbb{R}^{w \times w}$ and let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$. Define $\mathfrak{B} = \text{im } M \left(\frac{d}{dt} \right)$, where $M \in \mathbb{R}^{w \times z}[\xi]$. Fix a state map $X \in \mathbb{R}^{\bullet \times z}[\xi]$ for \mathfrak{B} . Assume that \mathfrak{B} is Φ -dissipative. Let $\Psi \in \mathbb{R}^{z \times z}[\zeta, \eta]$ and $\Delta \in \mathbb{R}^{z \times z}[\zeta, \eta]$ be as in Prop. 3.18. There exist real symmetric matrices K and Q of suitable sizes such that*

$$\Psi(\zeta, \eta) = X(\zeta)^\top K X(\eta),$$

and

$$\Delta(\zeta, \eta) = \begin{bmatrix} X(\zeta)^\top & M(\zeta)^\top \end{bmatrix} Q \begin{bmatrix} X(\eta) \\ M(\eta) \end{bmatrix}.$$

Proof. The proof follows directly from Th. 5.5 in [86]. \square

We now introduce results that provide conditions based on LMIs for the existence of a storage function for linear differential behaviours. For practical purposes, we consider the two-variable polynomial matrix version $\Phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta) + \Delta(\zeta, \eta)$ of the dissipation equality $Q_\Phi = \frac{d}{dt}Q_\Psi + Q_\Delta$ (see Prop. 3.18). We first introduce the following propositions that reveal important properties of coefficient matrices.

Proposition 3.21. *Let $M \in \mathbb{R}^{w \times z}[\xi]$ be defined as $M = \text{col}(U, Y)$, such that YU^{-1} is strictly proper. Let $X \in \mathbb{R}^{n(\mathfrak{B}) \times z}[\xi]$ be a minimal state map for $\text{im } M \left(\frac{d}{dt} \right)$. Write $M(\xi) = \sum_{i=0}^L M_i \xi^i$, with $M_i \in \mathbb{R}^{w \times z}$, $i = 0, \dots, L$; then there exist $X_i \in \mathbb{R}^{n(\mathfrak{B}) \times z}$, $i = 0, 1, \dots, L-1$, such that $X(\xi) = \sum_{i=0}^{L-1} X_i \xi^i$.*

Proof. See App. A.1. \square

Proposition 3.22. *Under the assumptions of Prop. 3.21, let $\Phi = \Phi^\top \in \mathbb{R}^{w \times w}$. Define*

$$\widetilde{M} := \begin{bmatrix} M_0 & \dots & M_L \end{bmatrix}, \quad \widetilde{X} := \begin{bmatrix} X_0 & \dots & X_{L-1} \end{bmatrix}.$$

Let $K = K^\top \in \mathbb{R}^{n(\mathfrak{B}) \times n(\mathfrak{B})}$. The following statements are equivalent:

- 1) $\Psi(\zeta, \eta) := X(\zeta)^\top K X(\eta)$ and $\Delta \in \mathbb{R}^{z \times z}[\zeta, \eta]$ satisfy $\Delta(\zeta, \eta) = M(\zeta)^\top \Phi M(\eta) - (\zeta + \eta)\Psi(\zeta, \eta)$;
- 2) $\widetilde{\Delta} := \widetilde{M}^\top \Phi \widetilde{M} - \begin{bmatrix} 0_{z \times n(\mathfrak{B})} \\ \widetilde{X}^\top \end{bmatrix} K \begin{bmatrix} \widetilde{X} & 0_{n(\mathfrak{B}) \times z} \end{bmatrix} - \begin{bmatrix} \widetilde{X}^\top \\ 0_{z \times n(\mathfrak{B})} \end{bmatrix} K \begin{bmatrix} 0_{n(\mathfrak{B}) \times z} & \widetilde{X} \end{bmatrix}.$

Proof. See App. A.1. \square

In the following lemma, we provide a dissipativity test for linear differential systems in terms of LMIs.

Lemma 3.23. *Under the assumptions of Prop. 3.21, let $\Phi \in \mathbb{R}^{w \times w}$ and define $\mathfrak{B} := \text{im } M \left(\frac{d}{dt} \right)$. Assume that \mathfrak{B} is Φ -dissipative. Then there exists $K = K^\top \in \mathbb{R}^{n(\mathfrak{B}) \times n(\mathfrak{B})}$ such that any of the statements 1) and 2) in Prop. 3.22 holds, and moreover $Q_\Delta \geq 0$ or equivalently $\widetilde{\Delta} \geq 0$.*

Proof. See App. A.1. \square

The results in Lemma 3.23 permit to transform the computation of storage functions into solving the expression 2) in Prop. 3.22 as an LMI, i.e. $\widetilde{\Delta} \geq 0$, involving coefficient matrices that can be straightforwardly set up from the equations describing the laws of the system.

3.8 Summary

In this chapter we have studied the main concepts of quadratic differential forms and dissipativity theory for the study of linear differential systems. We have also developed systems of LMIs that facilitate the computation of Lyapunov functions and storage functions directly from higher-order models.

Chapter 4

Switched linear differential systems

As discussed in Chap. 1, there exist established approaches to switched systems that deal with models in state space or descriptor form representations (see [24, 34, 74]) sharing a global state space, together with a supervisory system determining which of the modes is active. We argued that first-order representations are usually constructed from higher-order equations or transfer functions; thus it makes sense to try to work with higher-order representations directly. Moreover, the traditional state space modelling approach may not always be justified or advisable, and we offered some practical examples of problematic situations. These issues prompted us to develop a more general framework that permits the analysis of dynamic modes directly in higher-order terms.

4.1 Main definitions

In the *switched linear differential systems* (SLDS) framework, each dynamical mode is associated with a *mode behaviour*, the set of trajectories that satisfy the dynamical laws of that mode. A *switching signal* determines when a transition between dynamical modes occurs. At the switching instants the system trajectories must satisfy certain *gluing conditions*, that represent algebraic constraints enforced by physical principles.

Definition 4.1. A *switched linear differential system (SLDS)* Σ is a quadruple $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$ where

- $\mathcal{P} = \{1, \dots, N\} \subset \mathbb{N}$, is the set of *indices*;
- $\mathcal{F} = \{\mathfrak{B}_1, \dots, \mathfrak{B}_N\}$, with \mathfrak{B}_j a linear differential behaviour and $j \in \mathcal{P}$, is the *bank of behaviours*;

- $\mathcal{S} = \{s : \mathbb{R} \rightarrow \mathcal{P}\}$, with s piecewise constant and right-continuous, is the set of admissible *switching signals*; and
- $\mathcal{G} = \{(G_{k \rightarrow \ell}^-(\xi), G_{k \rightarrow \ell}^+(\xi)) \in \mathbb{R}^{\bullet \times w}[\xi] \times \mathbb{R}^{\bullet \times w}[\xi] \mid 1 \leq k, \ell \leq N, k \neq \ell\}$, is the set of *gluing conditions*.

The set of *switching instants* associated with $s \in \mathcal{S}$ is defined by

$$\mathbb{T}_s := \{t \in \mathbb{R} \mid s(t^-) \neq s(t^+)\} = \{t_1, t_2, \dots\},$$

where $t_i < t_{i+1}$.

We assume that for every $s \in \mathcal{S}$ and for every finite interval of \mathbb{R} , there exists only a finite number of switching instants. This is a conventional assumption in switched systems literature that prevent phenomena such as the *Zeno behaviour* (see e.g. [34, 68]).

The set of all admissible trajectories satisfying the laws of the mode behaviours and the gluing conditions is the *switched behaviour*, and is the central object of study in our framework.

Definition 4.2. Let $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$ be a SLDS, and let $s \in \mathcal{S}$. The *s-switched linear differential behaviour* \mathfrak{B}^s is the set of trajectories $w : \mathbb{R} \rightarrow \mathbb{R}^w$ that satisfy the following two conditions:

1. for all $t_i, t_{i+1} \in \mathbb{T}_s$, $w|_{[t_i, t_{i+1})} \in \mathfrak{B}_{s(t_i)}|_{[t_i, t_{i+1})}$;
2. w satisfies the gluing conditions \mathcal{G} at the switching instants for each $t_i \in \mathbb{T}_s$, i.e.

$$G_{s(t_{i-1}) \rightarrow s(t_i)}^+ \left(\frac{d}{dt} \right) w(t_i^+) = G_{s(t_{i-1}) \rightarrow s(t_i)}^- \left(\frac{d}{dt} \right) w(t_i^-). \quad (4.1)$$

The *switched linear differential behaviour* (SLDB) \mathfrak{B}^Σ of Σ is defined by $\mathfrak{B}^\Sigma := \bigcup_{s \in \mathcal{S}} \mathfrak{B}^s$.

The following example illustrates the concepts in Def. 4.1 and Def. 4.2, including the role of the gluing conditions in the concatenability of trajectories in the mode behaviours at switching instants.

Example 4.1. Let Σ be a SLDS as in Def. 1 with $\mathcal{P} = \{1, 2\}$ and

$$\mathcal{F} = \left\{ \mathfrak{B}_1 := \ker \left(\frac{d}{dt} + 2 \right), \mathfrak{B}_2 := \ker \left(\frac{d^2}{dt^2} + 1 \right) \right\}.$$

The set of gluing conditions is

$$\mathcal{G} := \{(G_{1 \rightarrow 2}^-(\xi), (G_{1 \rightarrow 2}^+(\xi)), (G_{2 \rightarrow 1}^-(\xi), G_{2 \rightarrow 1}^+(\xi))\} = \left\{ \left(\begin{bmatrix} 1 \\ \xi \end{bmatrix}, \begin{bmatrix} 1 \\ \xi \end{bmatrix} \right), (1, 1) \right\}.$$

Consider the switching signal

$$s(t) = \begin{cases} 1, & t \in [0, \frac{\pi}{2}); \\ 2, & t \in [\frac{\pi}{2}, \pi); \\ 1, & t \geq \pi. \end{cases}$$

When we switch from \mathfrak{B}_1 to \mathfrak{B}_2 at $t = \frac{\pi}{2}$, the gluing conditions impose

$$\begin{aligned} \lim_{t \searrow \frac{\pi}{2}} \left(\begin{bmatrix} 1 \\ \frac{d}{dt} \end{bmatrix} (k_2 \cos(t) + k_3 \sin(t)) \right) &= \lim_{t \nearrow \frac{\pi}{2}} \left(\begin{bmatrix} 1 \\ \frac{d}{dt} \end{bmatrix} k_1 e^{-2t} \right) \\ &= \begin{bmatrix} k_1 e^{-\pi} \\ -2k_1 e^{-\pi} \end{bmatrix}, \end{aligned}$$

and consequently $k_2 = 2k_1 e^{-\pi}$ and $k_3 = k_1 e^{-\pi}$. Then, when we switch from \mathfrak{B}_2 to \mathfrak{B}_1 at $t = \pi$, the gluing conditions impose

$$\lim_{t \nearrow \pi} (k_4 e^{-2t}) = k_4 e^{-2\pi} = \lim_{t \searrow \pi} (2k_1 e^{-\pi} \cos(t) + k_1 e^{-\pi} \sin(t)) = 2k_1 e^{-\pi};$$

and consequently $k_3 = 2k_1 e^{\pi}$. The switched trajectory $w \in \mathfrak{B}^s$ corresponding to the switching signal $s(t)$ is thus given by

$$w(t) = \begin{cases} k_1 e^{-t}, & t \in [0, \frac{\pi}{2}); \\ 2k_1 e^{-\pi} \cos(t) + k_1 e^{-\pi} \sin(t), & t \in [\frac{\pi}{2}, \pi); \\ 2k_1 e^{\pi} e^{-t}, & t \geq \pi. \end{cases}$$

Note that this corresponds to one trajectory in \mathfrak{B}^s according to Def. 2. Varying the parameter k_1 , the gluing conditions are automatically satisfied and we obtain all the trajectories in \mathfrak{B}^s . Moreover, \mathfrak{B}^Σ consists of all such trajectories \mathfrak{B}^s for all $s \in \mathcal{S}$. \square

It is important to emphasise that in our framework we allow different state spaces in the modes. However, the external variables (and thus their number) are the same for every dynamic mode: they have been chosen as variables of interest during the modelling state.

Example 4.2 (State space case). Note that the traditional switched linear state space systems fit also in our setting as a special case by defining the mode behaviours as $\mathfrak{B}_i := \{(x, u, y) \mid \frac{d}{dt}x = A_i x + B_i u, y = C_i x + D_i u\}$, $i = 1, \dots, N$. \square

4.2 Impulsive effects

Since $\mathfrak{B}_i \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$, $i = 1, \dots, N$ (see Sec. 2.1), it follows that the trajectories in \mathfrak{B}^Σ are *piecewise infinitely differentiable functions* from \mathbb{R} to \mathbb{R}^w with set denoted by $\mathfrak{C}_p^\infty(\mathbb{R}, \mathbb{R}^w)$. Switched systems whose trajectories belong to this solution space permit

impulsive effects in the sense of [24], [25], [91], i.e. allowing discontinuities on the trajectories at switching instants.

It has been pointed out in the literature that the presence of Dirac impulses (and their derivatives) may appear in the trajectories of the variables of a switched system. For example, in [74] a unifying, rigorous distributional framework for switched systems has been given, this approach encompasses also the detection of impulses directly from the equations. Similarly, for higher-order representations as in [76], the jumps and impulses induced by the system equations together with additional *impact maps* are used to specify the impulsive part of the behaviour.

In our framework, impulsive effects are *implicitly* defined by the gluing conditions and the mode dynamics involved in the transition (i.e. do not depend for example on the degree of differentiability of some input variable). Our position is that gluing conditions are a given; we take them at face value. Whether they imply impulses or not; and whether the latter is an important issue for the particular physical system at hand, are major modelling issues that we assume have been weighed carefully by the modeller (on this issue see also p. 749 of [15]). We elaborate on this modelling perspective in the following section.

4.3 Modelling of gluing conditions

We now give two examples of switched behaviours; besides exemplifying the definitions, they allow us to point out some important features of switched linear differential systems.

Example 4.3. Consider the electrical circuit in Fig. 4.1, where $C = 1\text{ F}$, $R = \frac{1}{2}\text{ }\Omega$ and w_1 and w_2 are voltages.

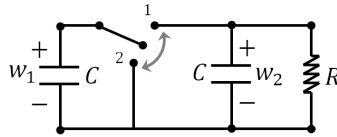


Figure 4.1: A switched electrical circuit

With the switch in position 1, the dynamical equations are

$$\begin{aligned} \frac{d}{dt}w_2 + w_2 &= 0 \\ w_1 - w_2 &= 0; \end{aligned} \tag{4.2}$$

when the switch is in position 2, the dynamical equations are

$$\begin{aligned}\frac{d}{dt}w_2 + w_2 &= 0 \\ w_1 &= 0.\end{aligned}\tag{4.3}$$

If we consider the voltage across the capacitors as the variables of interest, we then define

$$\mathfrak{B}_1 := \ker \begin{bmatrix} 0 & \frac{d}{dt} + 1 \\ 1 & -1 \end{bmatrix}, \quad \mathfrak{B}_2 := \ker \begin{bmatrix} 0 & \frac{d}{dt} + 1 \\ 1 & 0 \end{bmatrix}.$$

The switched behaviour consists of all piecewise smooth functions $\text{col}(w_1, w_2)$ that satisfy (4.2) or (4.3) depending on the position of the switch, and that at the switching instant satisfy the gluing conditions that follow from the principle of conservation of charge (see [47]), i.e. either $w_1(0^+) = \frac{1}{2}w_2(0^-)$, $w_2(0^+) = \frac{1}{2}w_2(0^-)$ (for a transition $\mathfrak{B}_2 \rightarrow \mathfrak{B}_1$) or $w_1(0^+) = 0$, $w_2(0^+) = w_2(0^-)$ (for a transition $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$). The corresponding matrices are

$$G_{2 \rightarrow 1}^- := \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, \quad G_{2 \rightarrow 1}^+ := I_2, \tag{4.4}$$

and

$$G_{1 \rightarrow 2}^- := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_{1 \rightarrow 2}^+ := I_2. \tag{4.5}$$

These gluing conditions imply that in any non-trivial case the value of w_1 jumps at the switching instant. \square

Example 4.4. Consider two behaviours respectively described by the equations

$$\begin{aligned}\frac{d}{dt}w_2 + w_2 &= 0 \\ w_1 - w_2 &= 0,\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}\frac{d}{dt}w_1 + \frac{d}{dt}w_2 + w_1 + w_2 &= 0 \\ w_1 &= 0.\end{aligned}\tag{4.7}$$

The gluing conditions for a transition $\mathfrak{B}_2 \rightarrow \mathfrak{B}_1$ are associated with the matrices

$$G_{2 \rightarrow 1}^- := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad G_{2 \rightarrow 1}^+ := I_2, \tag{4.8}$$

and for a transition $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ they are defined by

$$G_{1 \rightarrow 2}^- := \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad G_{1 \rightarrow 2}^+ := I_2; \tag{4.9}$$

i.e. in a switch $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ the new value of w_2 is the average of the old values of w_1 and w_2 . \square

Examples 4.3 and 4.4 offer the opportunity of making important conclusions regarding the modelling of gluing conditions.

Note that (4.3) and (4.7) describe the same set of solutions; indeed, the description (4.3) can be obtained from (4.7) by *unimodular* operations, which in the case of autonomous systems do not alter the solution set (see Th. 2.17, on equivalence of polynomial representations of switched systems, see also sect. 3 of [16]). Since (4.2) and (4.6) are the same, it follows that the dynamic modes are the same for both switched systems; thus the two switched behaviours are *different because the gluing conditions are*. It is proved later that these two switched systems also have different stability properties- that of Ex. 4.3 is stable under arbitrary switching signals, while the other is not. Stability arises from the *interplay* of mode dynamics and gluing conditions.

We conclude that gluing conditions should be defined on the basis of the physics of the system under study. Those for the system of Example 4.3 are meaningful for the particular physical system at hand. However, for *another* physical system whose modes happen to be described also by (4.6)-(4.7), the conditions (4.8)-(4.9) may also be physically plausible. In each case we assume that well-grounded physical considerations have been motivating the choice. A similar point of view is discussed in the *linear complementary framework* [13] and in *switched port-hamiltonian systems* [79], where concatenability is specified via *jump rules* for *passive systems*. A new set of initial conditions, after a transition between dynamic modes, corresponds to the solution of a quadratic program involving the energy stored in the switched system, and whose unique solution follows physical principles e.g. the principle of conservation of charge.

In the following we discuss important concepts associated to gluing conditions in the SLDS framework. We concentrate our analysis in two particular cases: *autonomous*-, and *controllable*- behaviours.

4.4 Switched autonomous behaviours

In the following we study switched linear differential systems with autonomous mode behaviours (see Sec. 2.4). We start by introducing the concept of *well-posedness* of gluing conditions, that will play a relevant role in our analysis.

4.4.1 Well-posedness of gluing conditions

In principle Def.s 4.1 and 4.2 do not restrict the gluing conditions; however, since we assume that the modes are autonomous, i.e. no external influences are applied to the

system between consecutive switching times, it is reasonable to require more. Namely, no different admissible trajectories should exist with the same past (i.e. same mode transitions at the same switching instants, and same restrictions from $t = -\infty$ up until a given switching instant \bar{t}). If such trajectories exist, then at \bar{t} the past “splits” in different futures; however, since no external inputs could trigger such a change, the past of a trajectory should uniquely define its future. These considerations lead to the concept of *well-posed* gluing conditions, which we now introduce.

Let $\mathfrak{B}_k =: \ker R_k \left(\frac{d}{dt} \right)$, with $R_k \in \mathbb{R}^{w \times w}[\xi]$ nonsingular, $k = 1, \dots, N$, for the modes. We also define $n_k := \deg(\det(R_k))$, $k = 1, \dots, N$, and we choose minimal state maps (see Sec. 2.6.1) $X_k \in \mathbb{R}^{n_k \times w}[\xi]$, $k = 1, \dots, N$. Moreover, given the material in Sec. 2.7, we know that every polynomial differential operator $G \left(\frac{d}{dt} \right)$ on \mathfrak{B}_k has a unique R_k -canonical representative $G' \left(\frac{d}{dt} \right)$, denoted by $G' = G \bmod R_k$, such that $G' \left(\frac{d}{dt} \right) w = G \left(\frac{d}{dt} \right) w$ for all $w \in \mathfrak{B}_k$. Now let $(G_{k \rightarrow \ell}^-, G_{k \rightarrow \ell}^+) \in \mathcal{G}$; then

$$(G_{k \rightarrow \ell}^- \bmod R_k, G_{k \rightarrow \ell}^+ \bmod R_\ell) ,$$

is equivalent to $(G_{k \rightarrow \ell}^-, G_{k \rightarrow \ell}^+)$, in the sense that the algebraic conditions imposed by the one pair are satisfied iff they are satisfied by the other. Moreover, since $G_{k \rightarrow \ell}^- \bmod R_k$ and $G_{k \rightarrow \ell}^+ \bmod R_\ell$ are R_k -, respectively R_ℓ -canonical, there exist constant matrices $F_{k \rightarrow \ell}^-$ and $F_{k \rightarrow \ell}^+$ of suitable dimensions such that $G_{k \rightarrow \ell}^-(\xi) \bmod R_k = F_{k \rightarrow \ell}^- X_k(\xi)$ and $G_{k \rightarrow \ell}^+(\xi) \bmod R_\ell = F_{k \rightarrow \ell}^+ X_\ell(\xi)$. We call

$$\mathcal{G}' := \{(F_{k \rightarrow \ell}^- X_k(\xi), F_{k \rightarrow \ell}^+ X_\ell(\xi)) \mid 1 \leq k, \ell \leq N, k \neq \ell\}$$

the *normal form* of \mathcal{G} .

Remark 4.3. The gluing conditions are algebraic constraints imposed to the trajectories of the external variable w and their derivatives at switching instants, and not on the state *per se*. However, in SLDS with autonomous mode behaviours, gluing conditions can be *always* rewritten in terms of the state. \square

Definition 4.4. Let Σ be a SLDS with $\mathfrak{B}_i = \ker R_i \left(\frac{d}{dt} \right)$ autonomous, $i = 1, \dots, N$. The normal form gluing conditions

$$\mathcal{G}' := \{(F_{k \rightarrow \ell}^- X_k(\xi), F_{k \rightarrow \ell}^+ X_\ell(\xi)) \mid 1 \leq k, \ell \leq N, k \neq \ell\} ,$$

are *well-posed* if for all $k, \ell = 1, \dots, N$, $k \neq \ell$, and for all $v_k \in \mathbb{R}^{n_k}$ there exists at most one $v_\ell \in \mathbb{R}^{n_\ell}$ such that $F_{k \rightarrow \ell}^- v_k = F_{k \rightarrow \ell}^+ v_\ell$.

Thus if a transition occurs between \mathfrak{B}_k and \mathfrak{B}_ℓ at t_j , and if an admissible trajectory ends at a “final state” $v_k = X_k \left(\frac{d}{dt} \right) w(t_j^-)$, then there exists at most one “initial state” for \mathfrak{B}_ℓ , defined by $X_\ell \left(\frac{d}{dt} \right) w(t_j^+) := v_\ell$, compatible with the gluing conditions.

Well-posedness implies that for all $k, \ell = 1, \dots, N$, $k \neq \ell$, $F_{k \rightarrow \ell}^+$ is full column rank, and consequently there exists a *re-initialisation map* $L_{k \rightarrow \ell} : \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_\ell}$ defined by

$$L_{k \rightarrow \ell} := F_{k \rightarrow \ell}^{+\star} F_{k \rightarrow \ell}^- ,$$

where $F_{k \rightarrow \ell}^{+\star}$ is a left inverse of $F_{k \rightarrow \ell}^+$. For all $t_j \in \mathbb{T}_s$ and all admissible $w \in \mathfrak{B}^\Sigma$ it holds that

$$\begin{aligned} [s(t_{j-1}) = k, s(t_j) = \ell] \text{ and } \left[G_{k \rightarrow \ell}^+ \left(\frac{d}{dt} \right) w(t_j^+) = G_{k \rightarrow \ell}^- \left(\frac{d}{dt} \right) w(t_j^-) \right] \\ \implies \left[X_\ell \left(\frac{d}{dt} \right) w(t_j^+) = L_{k \rightarrow \ell} \left(X_k \left(\frac{d}{dt} \right) w(t_j^-) \right) \right] . \end{aligned}$$

Note that the re-initialisation map is not uniquely determined unless $F_{k \rightarrow \ell}^+$ is nonsingular. In the rest of the paper, we assume well-posed gluing conditions with fixed re-initialisation maps.

The concept of well-posedness and switching between mode behaviours with different state spaces is illustrated in Fig. 4.2.

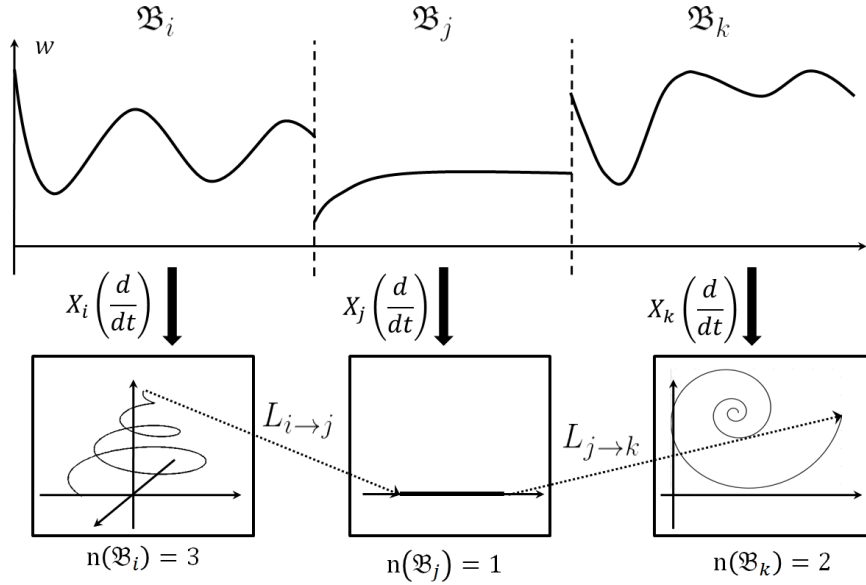


Figure 4.2: Example: well-posed gluing conditions

Example 4.5. Consider the multi-controller system depicted in Fig. 4.3. Depending on the value of a switching signal a plant Σ_P with two external variables, described by the differential equation $\frac{d}{dt}w_1 - w_1 - w_2 = 0$, is connected with one of two possible controllers Σ_{C_1} and Σ_{C_2} , described respectively by $-3\frac{d}{dt}w_1 - w_1 - \frac{d}{dt}w_2 = 0$ and $-2w_1 - w_2 = 0$. Depending on which controller is active, the resulting closed-loop behaviours are

$$\mathfrak{B}_1 := \ker \begin{bmatrix} \frac{d}{dt} - 1 & -1 \\ -3\frac{d}{dt} - 1 & -\frac{d}{dt} \end{bmatrix} ,$$

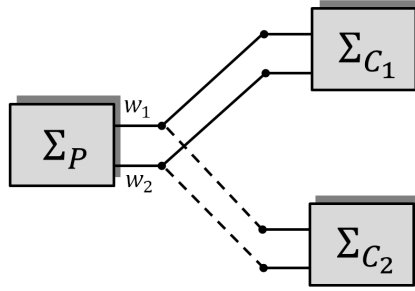


Figure 4.3: Plant/controller interconnection

and

$$\mathfrak{B}_2 := \ker \begin{bmatrix} \frac{d}{dt} - 1 & -1 \\ -2 & -1 \end{bmatrix}.$$

Note that \mathfrak{B}_1 and \mathfrak{B}_2 have different McMillan degrees (2 and 1, respectively). We define the gluing conditions for the SLDS associated with \mathfrak{B}_1 and \mathfrak{B}_2 by

$$G_{2 \rightarrow 1}^-(\xi) := \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad G_{2 \rightarrow 1}^+(\xi) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$G_{1 \rightarrow 2}^-(\xi) := \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad G_{1 \rightarrow 2}^+(\xi) := \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The rationale underlying our choice of gluing conditions is that any trajectory of \mathfrak{B}_1 is uniquely specified by the instantaneous values of $\text{col}(w_1, w_2)$, while a trajectory of \mathfrak{B}_2 is uniquely specified by the instantaneous value of w_1 . Moreover, when switching from the dynamics of \mathfrak{B}_1 to those of \mathfrak{B}_2 , we require that the values of w_1 before and after the switching instant coincide. In a switch from \mathfrak{B}_2 to \mathfrak{B}_1 , since the second differential equation describing \mathfrak{B}_2 yields $w_2 = -2w_1$ before the switch, we impose that $w_2(t_k^+) = w_2(t_k^-) = -2w_1(t_k^-)$. Moreover, note that a minimal state map for \mathfrak{B}_1 is $X_1(\xi) := I_2$, and a minimal state map for \mathfrak{B}_2 is $X_2(\xi) = \begin{bmatrix} 1 & 0 \end{bmatrix}$; and

$$G_{2 \rightarrow 1}^+ \mod R_1(\xi) = G_{2 \rightarrow 1}^+(\xi) = F_{2 \rightarrow 1}^+ X_1(\xi) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} I_2.$$

Similarly,

$$G_{1 \rightarrow 2}^+ \mod R_2(\xi) = G_{1 \rightarrow 2}^+(\xi) = F_{1 \rightarrow 2}^+ X_2(\xi) := 1 \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Consequently, these gluing conditions are well-posed. It can be verified in a similar way that the gluing conditions of Examples 4.3 and 4.4 are also well-posed. \square

4.4.2 Consistency of gluing conditions

The definition of well-posedness concerns *uniqueness* of an admissible “initial condition” v_ℓ in \mathfrak{B}_ℓ for a given “final condition” v_k in \mathfrak{B}_k . However, another important issue is *existence* of such admissible initial condition at the switching instant. Note for instance that it may happen that the gluing conditions cannot be satisfied by nonzero trajectories; they may not be “consistent” with the mode dynamics.

Example 4.6. Consider a SLDS with modes (4.6) and (4.7), and (well-posed) gluing conditions

$$G_{2 \rightarrow 1}^- := I_2, G_{2 \rightarrow 1}^+ := I_2 ,$$

$$G_{1 \rightarrow 2}^- := I_2, G_{1 \rightarrow 2}^+ := I_2 .$$

Note that $w \in \mathfrak{B}_1$ iff $w(t) = \alpha \operatorname{col}(e^{-t}, e^{-t})$, $\alpha \in \mathbb{R}$; and $w \in \mathfrak{B}_2$ iff $w(t) = \alpha \operatorname{col}(e^{-t}, 0)$, $\alpha \in \mathbb{R}$.

Since constant switching signals $\sigma_1 = 1$ and $\sigma_2 = 2$ are admissible, it follows that $\mathfrak{B}^\Sigma \supset \mathfrak{B}_i$, $i = 1, 2$. However, no genuine switched trajectory exists besides the zero one, since the gluing conditions cannot be satisfied by nonzero trajectories of either of the behaviours. \square

We now discuss the an algebraic characterisation of consistent gluing conditions in our framework.

Denote the roots of $\det R_k(\xi)$ by $\lambda_{k,i}$, $i = 1, \dots, n_k$. Assume for ease of exposition that the algebraic multiplicity of $\lambda_{k,i}$ equals the dimension of $\ker R_k(\lambda_{k,i})$. It follows from Th. 2.11 that $w \in \mathfrak{B}_k$ iff there exist $\alpha_{k,i} \in \mathbb{C}$, $i = 1, \dots, n_i$ such that

$$w = \sum_{i=1}^{n_k} \alpha_{k,i} w_{k,i} e^{\lambda_{k,i} t} ,$$

where $w_{k,i} \in \mathbb{C}^w$ is such that $R_k(\lambda_{k,i})w_{k,i} = 0$, and the $w_{k,i}$ associated with equal $\lambda_{k,i}$ are linearly independent. Note that the $\alpha_{k,i}$ associated to conjugate $\lambda_{k,i}$ are conjugate.

Define

$$V_i := \begin{bmatrix} X_i(\lambda_{i,1})w_{i,1} & \dots & X_i(\lambda_{i,n_i})w_{i,n_i} \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}, \quad i = 1, \dots, N .$$

and

$$\alpha_i := \begin{bmatrix} \alpha_{i,1} & \dots & \alpha_{i,n_i} \end{bmatrix}^\top, \quad i = k, \ell ;$$

and consider a switch from \mathfrak{B}_k to \mathfrak{B}_ℓ at $t = 0$. The gluing conditions require that

$$G_{k \rightarrow \ell}^-(w)(0^-) = F_{k \rightarrow \ell}^- V_k \alpha_k = F_{k \rightarrow \ell}^+ V_\ell \alpha_\ell = G_{k \rightarrow \ell}^+(w)(0^+) .$$

Such α_i , $i = k, \ell$ exist if and only if

$$\text{im } F_{k \rightarrow \ell}^- V_k \subseteq \text{im } F_{k \rightarrow \ell}^+ V_\ell$$

Standard arguments in ordinary differential equations show that V_k and V_ℓ are nonsingular; consequently the consistency property can be equivalently stated as

$$\text{im } F_{k \rightarrow \ell}^- \subseteq \text{im } F_{k \rightarrow \ell}^+.$$

Remark 4.5. The problem whether a given “initial condition” is consistent or not with the mode dynamics has been also studied in the switched DAE’s framework (see Ch. 4 of [74]); algorithms are stated that from the matrices describing a mode compute “consistency projectors” whose image is the subspace of consistent initial values. \square

4.5 Switched controllable behaviours

The SLDS framework also considers *open systems*, i.e. systems with inputs and outputs. We now consider an important case when the mode behaviours in the bank are *controllable*.

As discussed in Sec. 2.2, *controllable* mode behaviours can be described using *observable* image representations $w = M_j \left(\frac{d}{dt} \right) z_j$, $j = 1, \dots, N$. It follows that every trajectory of the *latent variable* z_j corresponds to a unique trajectory of the external variable w when the j -th mode is active.

We illustrate this modelling approach in the following example.

Example 4.7. Consider the *high-voltage switching power converter* presented in [8] and depicted in Fig. 4.4 a). For practical purposes such as voltage/current/power regulation, we are particularly interested in the dynamics at the input/output terminals. Consequently we define the external variable (the set of variables of interest) as $w := \text{col}(E, i_L, v_2, i_o)$.

By means of a switching signal, we can arbitrarily induce two possible electrical configurations that occur when the transistor is in either *closed* (see Fig. 4.4 b)) or *open* (see Fig. 4.4 c)) operation. Considering a standard modelling of input/output impedances for each case, we can derive the following physical laws describing the dynamics of the power converter. For simplicity we consider $L = 1H$, $C_1 = C_2 = 1F$, $R_L = 1\Omega$ and $R = 1\Omega$. It can be verified that the corresponding mode behaviours \mathfrak{B}_j , $i = 1, 2$, are *controllable* and thus can be described by image representations $w = M_j \left(\frac{d}{dt} \right) z_j$, $j = 1, 2$,

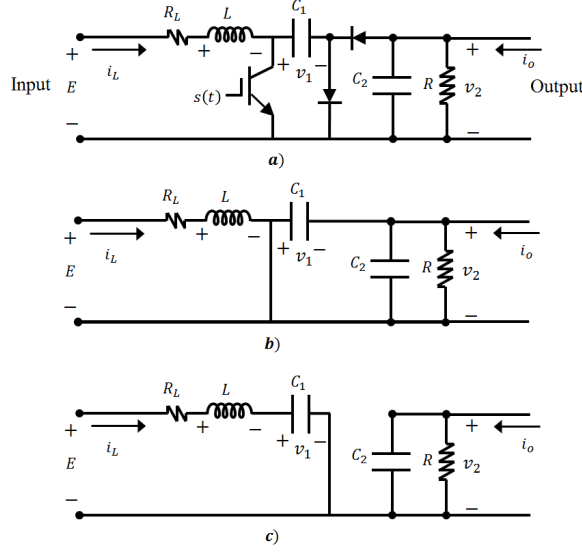


Figure 4.4: High-voltage switching power converter

where

$$M_1 \left(\frac{d}{dt} \right) := \begin{bmatrix} \frac{d}{dt} + 1 & 0 \\ 0 & 2\frac{d}{dt} + 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad M_2 \left(\frac{d}{dt} \right) := \begin{bmatrix} \frac{d^2}{dt^2} + \frac{d}{dt} + 1 & 0 \\ 0 & \frac{d}{dt} + 1 \\ \frac{d}{dt} & 0 \\ 0 & 1 \end{bmatrix};$$

and $z_1 := \text{col}(i_L, v_2)$, $z_2 := \text{col}(v_1, v_2)$. Moreover, since $M_j(\lambda)$, $j = 1, 2$, are full column rank for all $\lambda \in \mathbb{C}$ we conclude that the latent variables z_j , $j = 1, 2$ are observable from w . \square

4.5.1 Gluing conditions in terms of latent variables

In order to facilitate computations when dealing with controllable mode behaviours, we rather reformulate the gluing conditions in terms of latent variables.

According to Def.s 4.1 and 4.2, the gluing conditions are algebraic constraints acting on the external variables at switching instants; however, when dealing with controllable mode behaviours we can reformulate them in terms of latent variables in the following manner. Define

$$\overline{G}_{s(t_{i-1}) \rightarrow s(t_i)}^+ \left(\frac{d}{dt} \right) := \left(G_{s(t_{i-1}) \rightarrow s(t_i)}^+ M_{s(t_i)} \right) \left(\frac{d}{dt} \right),$$

and

$$\overline{G}_{s(t_{i-1}) \rightarrow s(t_i)}^- \left(\frac{d}{dt} \right) := \left(G_{s(t_{i-1}) \rightarrow s(t_i)}^- M_{s(t_i-1)} \right) \left(\frac{d}{dt} \right),$$

with $s \in \mathcal{S}$. Consequently, if w and z_j are related by $w = M_j \left(\frac{d}{dt} \right) z_j$, the gluing conditions in (4.1) can be equivalently written as

$$\overline{G}_{s(t_{i-1}) \rightarrow s(t_i)}^+ \left(\frac{d}{dt} \right) z_{s(t_i)}(t_i^+) = \overline{G}_{s(t_{i-1}) \rightarrow s(t_i)}^- \left(\frac{d}{dt} \right) z_{s(t_{i-1})}(t_i^-) .$$

Example 4.8 (Cont'd from Ex. 4.7). At switching instants the physical laws impose constraints to the system trajectories. By inspecting the circuits in Fig. 4.4 and using the *principle of conservation of charge* (see [47]), we find the following conditions at switching instants.

- When switching from \mathfrak{B}_1 to \mathfrak{B}_2 at t_i :

$$\begin{aligned} i_L(t_i^+) &= i_L(t_i^-) , \\ E(t_i^+) - i_L(t_i^+) - \frac{d}{dt} i_L(t_i^+) &= v_2(t_i^-) , \\ v_2(t_i^+) &= v_2(t_i^-) . \end{aligned} \tag{4.10}$$

- When switching from \mathfrak{B}_2 to \mathfrak{B}_1 at t_i :

$$\begin{aligned} i_L(t_i^+) &= i_L(t_i^-) , \\ 2v_2(t_i^+) &= E(t_i^-) - i_L(t_i^-) - \frac{d}{dt} i_L(t_i^-) + v_2(t_i^-) . \end{aligned} \tag{4.11}$$

Consequently, the gluing conditions are defined as

$$\begin{aligned} G_{1 \rightarrow 2}^+ \left(\frac{d}{dt} \right) &:= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 - \frac{d}{dt} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad G_{1 \rightarrow 2}^- \left(\frac{d}{dt} \right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \\ G_{2 \rightarrow 1}^+ \left(\frac{d}{dt} \right) &:= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} ; \quad G_{2 \rightarrow 1}^- \left(\frac{d}{dt} \right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 - \frac{d}{dt} & 1 \end{bmatrix} . \end{aligned}$$

Hence, the equations in (4.10)-(4.11) can be compactly written as

$$\begin{aligned} G_{1 \rightarrow 2}^+ \left(\frac{d}{dt} \right) w(t_i^+) &= G_{1 \rightarrow 2}^- \left(\frac{d}{dt} \right) w(t_i^-) ; \\ G_{2 \rightarrow 1}^+ \left(\frac{d}{dt} \right) w(t_i^+) &= G_{2 \rightarrow 1}^- \left(\frac{d}{dt} \right) w(t_i^-) . \end{aligned}$$

These gluing conditions can be reformulated them in terms of latent variables using $M_1 \left(\frac{d}{dt} \right)$ and $M_2 \left(\frac{d}{dt} \right)$ as follows.

$$\overline{G}_{1 \rightarrow 2}^- \left(\frac{d}{dt} \right) := (G_{1 \rightarrow 2}^- M_1) \left(\frac{d}{dt} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^\top ,$$

$$\begin{aligned}\overline{G}_{1 \rightarrow 2}^+ \left(\frac{d}{dt} \right) &:= (G_{1 \rightarrow 2}^+ M_2) \left(\frac{d}{dt} \right) = \begin{bmatrix} \frac{d}{dt} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top, \\ \overline{G}_{2 \rightarrow 1}^- \left(\frac{d}{dt} \right) &:= (G_{2 \rightarrow 1}^- M_2) \left(\frac{d}{dt} \right) = \begin{bmatrix} \frac{d}{dt} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \\ \overline{G}_{2 \rightarrow 1}^+ \left(\frac{d}{dt} \right) &:= (G_{2 \rightarrow 1}^+ M_1) \left(\frac{d}{dt} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

□

4.5.2 Well-definedness and well-posedness

We now discuss the concept of well-posedness of gluing conditions for *open* SLDS. In contrast with the case of switched autonomous behaviours in Sec. 4.4.1, in the case of dynamic modes with inputs and outputs, well-posedness requires the gluing conditions to be also *well-defined*.

Definition 4.6. Let Σ be a SLDS and let $X_j \in \mathbb{R}^{n(\mathfrak{B}_j) \times z}[\xi]$, induce minimal state maps for $\mathfrak{B}_j := \text{im } M_j \left(\frac{d}{dt} \right)$, $j = 1, \dots, N$. The gluing conditions are *well-defined* if there exist constant matrices $F_{j \rightarrow k}^-$ and $F_{j \rightarrow k}^+$, with $j, k = 1, \dots, N$, $j \neq k$, such that $\overline{G}_{j \rightarrow k}^-(\xi) = F_{j \rightarrow k}^- X_j(\xi)$ and $\overline{G}_{j \rightarrow k}^+(\xi) = F_{j \rightarrow k}^+ X_k(\xi)$, with $j, k = 1, \dots, N$, $j \neq k$. □

Well-definedness implies that gluing conditions are linear functions of the state of the corresponding modes before and after the switch. Consequently, they do not impose restrictions to the trajectories of the input variables, since the latter must be *maximally free* (see Sec. 2.3).

Definition 4.7. Let Σ be a SLDS with $\mathfrak{B}_j := \text{im } M_j \left(\frac{d}{dt} \right)$, $j = 1, \dots, N$. The well-defined gluing conditions $\mathcal{G} := \{(F_{j \rightarrow k}^- X_j(\xi), F_{j \rightarrow k}^+ X_k(\xi))\}_{j,k=1,\dots,N, j \neq k}$ are *well-posed* if for all $k, j = 1, \dots, N$ with $k \neq j$, there exists a *re-initialisation* map $L_{j \rightarrow k} : \mathbb{R}^{n(\mathfrak{B}_j)} \rightarrow \mathbb{R}^{n(\mathfrak{B}_k)}$ such that given a switching signal $s \in \mathcal{S}$ such that $s(t_{i-1}) = j$ and $s(t_i) = k$; for all $t_i \in \mathbb{T}_s$ and all admissible $w \in \mathfrak{B}^\Sigma$ with associated latent variable trajectories, it holds that $X_j \left(\frac{d}{dt} \right) z_j(t_i^+) = L_{j \rightarrow k} X_k \left(\frac{d}{dt} \right) z_k(t_i^-)$.

As in the case of autonomous behaviours, well-posedness implies that if a transition occurs between \mathfrak{B}_j and \mathfrak{B}_k at t_i , and if an admissible trajectory ends at a “final state” $v_j := X_j \left(\frac{d}{dt} \right) z_j(t_i^-)$, then there exists at most one “initial state” for \mathfrak{B}_k , defined by $X_k \left(\frac{d}{dt} \right) z_k(t_i^+) =: v_k$, compatible with the gluing conditions. In other words, for all $j, k = 1, \dots, N$, $j \neq k$, $F_{j \rightarrow k}^+$ is full column rank, and consequently a *re-initialisation map* can be defined as $L_{j \rightarrow k} := F_{j \rightarrow k}^{+*} F_{j \rightarrow k}^-$, where $F_{j \rightarrow k}^{+*}$ is a left inverse of $F_{j \rightarrow k}^+$.

4.6 Summary

We developed a framework for the modelling of closed linear switched systems in which the dynamical modes are not required to be described in a global state-space form. Pivotal in our approach is the concept of gluing conditions, that impose concatenation constraints on the system trajectories at the switching instants.

Chapter 5

Stability of SLDS

In this chapter we study stability of SLDS with autonomous behaviours. We provide sufficient conditions for stability based on LMIs for systems with general gluing conditions. We also study the role of positive-realness in providing sufficient polynomial-algebraic conditions for stability of two-modes switched systems with special gluing conditions.

5.1 Lyapunov stability

We call a SLDB \mathfrak{B}^Σ (and by extension, the SLDS Σ) *asymptotically stable* if

$$\lim_{t \rightarrow +\infty} w(t) = 0 \text{ for all } w \in \mathfrak{B}^\Sigma.$$

It follows from this definition and the fact that arbitrary switching signals are considered, that in an asymptotically stable SLDS, all mode behaviours \mathfrak{B}_i must be asymptotically stable and consequently autonomous (see Sec. 2.4).

Asymptotic stability for linear differential behaviours can be proved by producing a *higher-order Lyapunov function*, i.e. a quadratic differential form (QDF) Q_Ψ such that $Q_\Psi \stackrel{\mathfrak{B}}{\geq} 0$ and $\frac{d}{dt}Q_\Psi \stackrel{\mathfrak{B}}{<} 0$, see Sec. 3.6. In this chapter we give a sufficient condition for stability of SLDS in terms of quadratic *multiple Lyapunov functions* (MLFs).

Theorem 5.1. *Let Σ be a SLDS (see Def. 4.1). Assume that there exist QDFs Q_{Ψ_i} , $i = 1, \dots, N$ such that*

1. $Q_{\Psi_i} \stackrel{\mathfrak{B}_i}{\geq} 0$, $i = 1, \dots, N$;
2. $\frac{d}{dt}Q_{\Psi_i} \stackrel{\mathfrak{B}_i}{<} 0$, $i = 1, \dots, N$;
3. $\forall w \in \mathfrak{B}^\Sigma$ and $\forall t_j \in \mathbb{T}_s$, $Q_{\Psi_{s(t_{j-1})}}(w)(t_j^-) \geq Q_{\Psi_{s(t_j)}}(w)(t_j^+)$.

Then Σ is asymptotically stable.

Proof. See Appendix A.2. □

Conditions 1 and 2 of Th. 5.1 are equivalent to Q_{Ψ_i} being a Lyapunov function for \mathfrak{B}_i , $i = 1, \dots, N$. Condition 3 requires that the value of the multiple functional associated to Q_{Ψ_i} , $i = 1, \dots, N$, does not increase at the switching instants.

Remark 5.2. QDFs act on \mathcal{C}^∞ -functions, while the trajectories of a SLDS are non-differentiable; however, this mismatch in differentiability is irrelevant to Th. 5.4 and the other results developed in this thesis. Indeed, we only use the calculus of QDFs as an *algebraic* tool, taking into account its value before and after the switch and the properties of their coefficient matrices. □

We now recall Ex. 4.3 and Ex. 4.4 in the previous section.

Example 5.1. The SLDS in Ex. 4.3 is stable. An MLF is (Q_{Ψ_1}, Q_{Ψ_2}) , where

$$\Psi_1(\zeta, \eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \Psi_2(\zeta, \eta) ,$$

induces the QDFs $Q_{\Psi_1}(w) = w_2^2 = Q_{\Psi_2}(w)$. Their derivatives along \mathfrak{B}_1 and \mathfrak{B}_2 equal $-2w_2 \frac{d}{dt} w_2 = -2w_2^2$; due to the gluing conditions, the value of the MLF is the same before and after the switch.

For the system in Ex. 4.4, straightforward computations show that since the only R_i -canonical quadratic functionals for \mathfrak{B}_i are of the form

$$\Psi_i(\zeta, \eta) = c \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} , \quad i = 1, 2 \quad \text{for } c > 0 ,$$

no quadratic multiple Lyapunov functions for the SLDS exist. In fact, an argument analogous to that of pp. 126-ff. of [74] proves that the system is unstable. □

Remark 5.3. The existence of an MLF is only a sufficient condition for asymptotic stability: the class of quadratic Lyapunov functionals is not *universal* (see [1], Corollary 4.3 and Remark 4.1 p. 457). The class of *polyhedral Lyapunov functions* (PLFs) is universal for linear systems with structured uncertainties; in [92] PLFs are applied to linear switched systems in state space form, and a numerical procedure to overcome the complexity of PLF computations is illustrated, see pp. 1021-1022 *ibid.* □

We now describe a procedure, based on results of Prop. 3.11, to compute MLFs as in Th. 5.1 for SLDS using LMIs. For ease of exposition we assume that for each root of $\det R_k(\xi)$ the algebraic multiplicity coincides with the geometric multiplicity.

Theorem 5.4. Let Σ be a SLDS (see Def. 4.1), with $\mathfrak{B}_k = \ker R_k \left(\frac{d}{dt} \right)$ asymptotically stable, $k = 1, \dots, N$ and $R_k \in \mathbb{R}^{w \times w}[\xi]$ nonsingular. Let $X_k \in \mathbb{R}^{n \times w}[\xi]$ be a minimal state map for \mathfrak{B}_k . Write $R_k(\xi) = \sum_{i=0}^{L_k} R_{k,i} \xi^i$, and denote the coefficient matrix of $R_k(\xi)$ by

$$\tilde{R}_k := \begin{bmatrix} R_{k,0} & \dots & R_{k,L_k} \end{bmatrix},$$

and that of $X_k(\xi)$ by

$$\tilde{X}_k := \begin{bmatrix} X_{k,0} & \dots & X_{k,L_k-1} \end{bmatrix}.$$

Denote the roots of $\det R_k(\xi)$ by $\lambda_{k,i}$, $i = 1, \dots, n_k$. Assume that the algebraic multiplicity of $\lambda_{k,i}$ equals the dimension of $\ker R_k(\lambda_{k,i})$. Let $w_{k,i} \in \mathbb{C}^w$ be such that $R_k(\lambda_{k,i})w_{k,i} = 0$, with the $w_{k,i}$ associated with equal $\lambda_{k,i}$ linearly independent. Define $V_k \in \mathbb{C}^{n_k \times n_k}$ by

$$V_k := \begin{bmatrix} X_k(\lambda_{k,1})w_{k,1} & \dots & X_k(\lambda_{k,n_k})w_{k,n_k} \end{bmatrix}, \quad k = 1, \dots, N,$$

Denote by $L_{k \rightarrow \ell}$, $k, \ell = 1, \dots, N$, $k \neq \ell$, the re-initialisation maps of Σ .

There exist $\bar{K}_k \in \mathbb{R}^{n_k \times n_k}$, $\bar{Y}_k \in \mathbb{R}^{w \times n_k}$, $k = 1, \dots, N$ such that

$$\begin{aligned} \tilde{\Phi}_k &:= \begin{bmatrix} 0_{w \times n} \\ \tilde{X}_k^\top \end{bmatrix} \bar{K}_k \begin{bmatrix} \tilde{X}_k & 0_{n \times w} \end{bmatrix} + \begin{bmatrix} \tilde{X}_k^\top \\ 0_{w \times n} \end{bmatrix} \bar{K}_k \begin{bmatrix} 0_{n \times w} & \tilde{X}_k \end{bmatrix} - \begin{bmatrix} \tilde{X}_k^\top \\ 0_{w \times n} \end{bmatrix} \bar{Y}_k^\top \tilde{R}_k \\ &\quad - \tilde{R}_k^\top \bar{Y}_k \begin{bmatrix} \tilde{X}_k & 0_{n \times w} \end{bmatrix} \leq 0. \end{aligned} \quad (5.1)$$

Moreover, there exist $\bar{F}_k \in \mathbb{R}^{n_k \times n_k}$ such that $\tilde{\Phi}_k = \begin{bmatrix} \tilde{X}_k^\top \\ 0_{w \times n} \end{bmatrix} \bar{F}_k \begin{bmatrix} \tilde{X}_k & 0_{n \times w} \end{bmatrix}$, $k = 1, \dots, N$.

Moreover, if for $k, \ell = 1, \dots, N$, $\ell \neq k$, it holds that

$$\bar{F}_k < 0 \quad \text{and} \quad V_k^* \bar{K}_k V_k \geq V_k^* L_{k \rightarrow \ell}^\top \bar{K}_\ell L_{k \rightarrow \ell} V_k, \quad (5.2)$$

then Σ is asymptotically stable, and $(X_k(\zeta)^\top \bar{K}_k X_k(\eta))_{k=1, \dots, N}$ induces an MLF.

Proof. See Appendix A.2. □

Th. 5.4 reduces the computation of quadratic MLFs as in Th. 5.1, to the solution of a system of structured LMIs (5.1)-(5.2), a straightforward matter for standard LMI solvers.

Remark 5.5. Th. 5.4 and the associated LMI-based procedure to find an MLF assume that the $\lambda_{k,i}$ and associated directions $w_{k,i}$ are known. If one wants to avoid such pre-computations, a weaker (i.e. more conservative) sufficient condition for the existence of a multiple Lyapunov function can be obtained by solving (5.1) together with $\bar{F}_k < 0$ and $\bar{K}_k \geq L_{i \rightarrow \ell}^\top \bar{K}_\ell L_{i \rightarrow \ell}$ in place of (5.2). □

Remark 5.6. For state-space switched systems, $R_k(\xi) = \xi I_n - A_k$ and $X_k(\xi) = I_n$, $k = 1, \dots, N$, straightforward computations yield that in (5.1) $\bar{Y}_k = \bar{K}_k$; with the first

condition in (5.2) we obtain the matrix Lyapunov equations

$$A_k^\top \bar{K}_k + \bar{K}_k A_k < 0.$$

The second condition in (5.2) reduces to the classical condition on the reset maps (see e.g. Cor. 2.2 of [51]). For the case of switched DAE's, see Sec. 6.3 of [75]. \square

5.2 Example: Boost converter with multiple loads

We now illustrate the application of Th. 5.4 in a realistic setting.

Example 5.2. Some *source converters* used in distributed power systems (see e.g. [53]) consist of a traditional DC-DC boost converter coupled with a (dis-)connectable load, see Fig. 5.1.

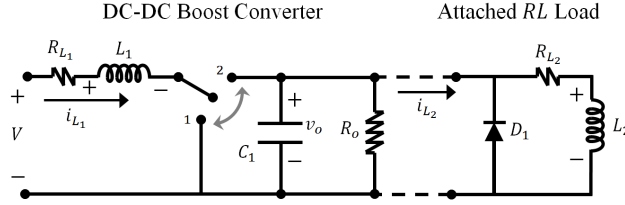


Figure 5.1: Source converter

We take $w = \text{col}(i_{L1}, v_o)$ as the external variable. In order to deal with autonomous behaviours, we set the input voltage $V = 0$. From standard circuit modelling we conclude that the modes are given by $\mathfrak{B}_i = \ker R_i \left(\frac{d}{dt} \right)$, $i = 1, \dots, 4$ where

$$R_1(\xi) := \begin{bmatrix} L_1 \xi + R_{L1} & 0 \\ 0 & C_1 \xi + \frac{1}{R_o} \end{bmatrix},$$

$$R_2(\xi) := \begin{bmatrix} L_1 \xi + R_{L1} & 1 \\ -1 & C_1 \xi + \frac{1}{R_o} \end{bmatrix},$$

$$R_3(\xi) := \begin{bmatrix} L_1 \xi + R_{L1} & 0 \\ 0 & L_2 C_1 \xi^2 + \left(R_{L2} C_1 + \frac{L_2}{R_o} \right) \xi + \frac{R_{L2}}{R_o} + 1 \end{bmatrix},$$

$$R_4(\xi) := \begin{bmatrix} L_1 \xi + R_{L1} & 1 \\ -L_2 \xi - R_{L2} & L_2 C_1 \xi^2 + \left(R_{L2} C_1 + \frac{L_2}{R_o} \right) \xi + \frac{R_{L2}}{R_o} + 1 \end{bmatrix}.$$

$\mathfrak{B}_1, \mathfrak{B}_2$ correspond to the switch in 1 and 2 respectively and the RL load disconnected, and $\mathfrak{B}_3, \mathfrak{B}_4$ to the modes for the switch in position 1 and 2 and the load connected. The

gluing conditions derived from physical considerations are

$$\begin{aligned}
(I_2, I_2) &= (G_{1 \rightarrow 2}^+(\xi), G_{1 \rightarrow 2}^-(\xi)) = (G_{2 \rightarrow 1}^+(\xi), G_{2 \rightarrow 1}^-(\xi)) \\
&= (G_{3 \rightarrow 1}^+(\xi), G_{3 \rightarrow 1}^-(\xi)) = (G_{3 \rightarrow 2}^+(\xi), G_{3 \rightarrow 2}^-(\xi)) \\
&= (G_{4 \rightarrow 1}^+(\xi), G_{4 \rightarrow 1}^-(\xi)) = (G_{4 \rightarrow 2}^+(\xi), G_{4 \rightarrow 2}^-(\xi)) ; \\
\\
(G_{1 \rightarrow 3}^+(\xi), G_{1 \rightarrow 3}^-(\xi)) &:= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -C_1\xi - \frac{1}{R_o} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right) =: (G_{2 \rightarrow 3}^+(\xi), G_{2 \rightarrow 3}^-(\xi)) ; \\
\\
(G_{1 \rightarrow 4}^+(\xi), G_{1 \rightarrow 4}^-(\xi)) &:= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -C_1\xi - \frac{1}{R_o} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right) =: (G_{2 \rightarrow 4}^+(\xi), G_{2 \rightarrow 4}^-(\xi)) ; \\
\\
(G_{3 \rightarrow 4}^+(\xi), G_{3 \rightarrow 4}^-(\xi)) &:= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -C_1\xi - \frac{1}{R_o} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -C_1\xi - \frac{1}{R_o} \end{bmatrix} \right) =: (G_{4 \rightarrow 3}^-(\xi), G_{4 \rightarrow 3}^+(\xi)) .
\end{aligned}$$

The following polynomial differential operators induce state maps for \mathfrak{B}_k , $k = 1, \dots, 4$:

$$X_1(\xi) = X_2(\xi) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad X_3(\xi) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -C_1\xi - \frac{1}{R_o} \end{bmatrix} ; \quad X_4(\xi) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -C_1\xi - \frac{1}{R_o} \end{bmatrix} .$$

They can be derived by physical considerations or automatically, using the procedures in [60]. Proceeding as in Sec. 4.4.1, we compute the re-initialisation maps

$$\begin{aligned}
L_{1 \rightarrow 2} &= L_{2 \rightarrow 1} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \\
\\
L_{1 \rightarrow 3} &= L_{1 \rightarrow 4} = L_{2 \rightarrow 3} = L_{2 \rightarrow 4} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} ; \\
\\
L_{3 \rightarrow 4} &= L_{4 \rightarrow 3} := I_3 ; \\
\\
L_{3 \rightarrow 1} &= L_{3 \rightarrow 2} = L_{4 \rightarrow 1} = L_{4 \rightarrow 2} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} .
\end{aligned}$$

In order to obtain a numeric solution, we set the parameters $L_1 = 100\mu F$, $R_{L_1} = 0.01\Omega$, $C_1 = 100\mu F$, $R_o = 2\Omega$, $R_{L_2} = 0.02\Omega$, $L_2 = 100\mu F$.

We obtain the characteristic frequencies $\lambda_{1,1} = -5000$, $\lambda_{1,2} = -100$, $\lambda_{2,1} = -2550 + j9695.2 = \overline{\lambda_{2,2}}$, $\lambda_{3,1} = -2600 + j9707.7 = \overline{\lambda_{3,2}}$, $\lambda_{3,3} = -100$, $\lambda_{4,1} = -149.94$, $\lambda_{4,2} =$

$-2575 + j13933 = \overline{\lambda_{4,3}}$. The V -matrices of Th. 5.4 are

$$V_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

$$V_2 = \begin{bmatrix} 0.70711 & 0.70711 \\ 0.17324 - j0.68556 & 0.17324 + j0.68556 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0.16971 + j0.68644 & 0.16971 - j0.68644 & 0 \\ 0.62564 - j0.32949 & 0.62564 + j0.32949 & 0 \end{bmatrix};$$

$$V_4 = \begin{bmatrix} 0.70796 & 0.08739 + j0.49199 & 0.08739 - j0.49199 \\ 0.00353 & 0.70711 & 0.70711 \\ 0.70625 & -0.08407 - j0.49323 & -0.17147 + j0.98522 \end{bmatrix}.$$

Using standard LMI solvers for the LMIs (5.1), (5.2) we obtain

$$\overline{K}_1 = \overline{K}_2 = \begin{bmatrix} 0.00123 & -0.00002 \\ -0.00002 & 0.00112 \end{bmatrix};$$

$$\overline{K}_3 = \overline{K}_4 = \begin{bmatrix} 0.00123 & -0.00002 & 0 \\ -0.00002 & 0.00112 & 0 \\ 0 & 0 & 0.00121 \end{bmatrix}.$$

Applying Th. 5.4 we conclude that $(X_k(\zeta)^\top K_k X_k(\eta))_{k=1,\dots,4}$ induces an MLF.

To illustrate the modularity of our modelling framework, assume that the source converter can also be connected to yet another RC load as depicted in Fig. 5.2.

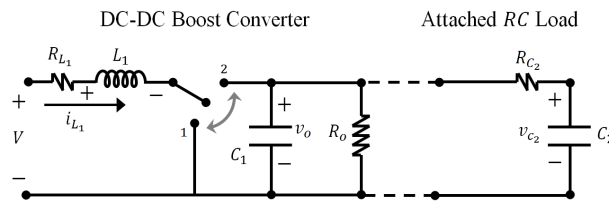


Figure 5.2: Source converter with an RC load

This results in two additional behaviours in \mathcal{F} , namely $\mathfrak{B}_i = \ker R_i \left(\frac{d}{dt} \right)$, $i = 5, 6$, where

$$R_5(\xi) := \begin{bmatrix} L_1\xi + R_{L_1} & 0 \\ 0 & R_{C_2}C_1C_2\xi^2 + \left(\frac{R_{C_2}C_2}{R_o} + C_1 + C_2 \right) \xi + \frac{1}{R_o} \end{bmatrix}$$

$$R_6(\xi) := \begin{bmatrix} L_1\xi + R_{L_1} & 1 \\ -R_{C_2}C_2\xi - 1 & R_{C_2}C_1C_2\xi^2 + \left(\frac{R_{C_2}C_2}{R_o} + C_1 + C_2 \right) \xi + \frac{1}{R_o} \end{bmatrix}.$$

We choose as state maps for \mathfrak{B}_5 and \mathfrak{B}_6

$$X_5(\xi) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & R_{C_2}C_1\xi + \frac{R_{C_2}}{R_o} + 1 \end{bmatrix} ; \quad X_6(\xi) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -R_{C_2} & R_{C_2}C_1\xi + \frac{R_{C_2}}{R_o} + 1 \end{bmatrix} ,$$

corresponding to the re-initialisation maps

$$\begin{aligned} L_{5 \rightarrow 6} = L_{6 \rightarrow 5} &:= I_3 ; \\ L_{1 \rightarrow 5} = L_{1 \rightarrow 6} = L_{2 \rightarrow 5} = L_{2 \rightarrow 6} &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} ; \\ L_{5 \rightarrow 1} = L_{5 \rightarrow 2} = L_{6 \rightarrow 1} = L_{6 \rightarrow 2} &:= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} . \end{aligned}$$

Given the values $R_{C_2} = 1\Omega$, $C_2 = 100\mu F$, in order to compute an MLF for $\mathcal{F} := \{\mathfrak{B}_k\}_{k=1,\dots,6}$ we only need to add two LMIs to those set up previously; the solution is

$$\begin{aligned} \overline{K}_1 = \overline{K}_2 &= \begin{bmatrix} 0.00127 & -0.00002 \\ -0.00002 & 0.00126 \end{bmatrix} ; \\ \overline{K}_3 = \overline{K}_4 &= \begin{bmatrix} 0.00127 & -0.00002 & 0 \\ -0.00002 & 0.00126 & 0 \\ 0 & 0 & 0.00131 \end{bmatrix} ; \\ \overline{K}_5 = \overline{K}_6 &= \begin{bmatrix} 0.00127 & -0.00002 & 0 \\ -0.00002 & 0.00126 & 0 \\ 0 & 0 & 0.00382 \end{bmatrix} . \end{aligned}$$

□

Remark 5.7. We have chosen a purposely straightforward example (with $V = 0$ and the interconnection of passive loads) that though simple, it is helpful to illustrate the computation of MLFs, as well as some advantages in the analysis of SLDS such as modularity, i.e. additional dynamic modes (associated with new loads) are incrementally modelled adding them to the existing description. Note that the set of LMIs that are set-up for the stability analysis is also modularly augmented. In chapter 7 we revisit this type of problems in a more challenging scenario, where V is a free input and loads with negative impedance characteristics may cause instability. □

Remark 5.8 (State space case). In contrast with the SLDS framework, in the traditional state space setting we would consider for the case in Fig. 5.1 dynamic modes $\frac{d}{dt}x = A_i x$, with $A_i \in \mathbb{R}^{3 \times 3}$, $i = 1, \dots, 4$. Since a global augmented state space $x := [i_{L_1} \quad v_o \quad i_{L_2}]^\top$ is used, then every mode contains the highest possible complexity, even the modes

in which the RL load is not connected. When the new RC load is considered after performing the stability analysis, the previous state space x is augmented to include a new state variable, thus we define e.g. $x' := \begin{bmatrix} i_{L_1} & v_o & i_{L_2} & v_{c_2} \end{bmatrix}^\top$. In order to construct the same type of representation for every mode, we write new modes $\frac{d}{dt}x' = A'_k x'$, with $A'_k \in \mathbb{R}^{4 \times 4}$, $k = 1, \dots, 8$. Consequently, the previously modelled matrices A_i , $i = 1, \dots, 4$ need to be inflated with zeros and all the previously constructed LMIs need to be set-up again. \square

5.3 Positive-realness and Lyapunov functions

Positive-realness has played an important role in the study of switched systems. For instance it is well-known that if an open-loop transfer function of a system is positive-real, then all stable closed-loop systems obtained from it by state feedback share a common quadratic Lyapunov function (see Sec. 2.3.2 of [34] and [64, 65]).

The polynomial Lyapunov equation (PLE) resemblance to the dissipation equality (see Th. 3.9 and Prop. 3.18) underlies the results of this section, aimed at connecting positive-realness and stability of two dynamic modes (see [64, 65] in the classical setting). We begin by recalling the definition of strict positive-real rational function (note that this definition is not universally accepted; cf. [71], Th. 2.1.).

Definition 5.9. $G \in \mathbb{R}^{w \times w}(\xi)$ is *strictly positive-real* if it is analytic in \mathbb{C}_+ and $G(-j\omega)^\top + G(j\omega) > 0 \forall \omega \in \mathbb{R}$.

We now relate the PLE (3.3) with strict positive-realness of an associated transfer function.

Proposition 5.10. *Let $N, D \in \mathbb{R}^{w \times w}[\xi]$. Assume that D and N are Hurwitz, and that ND^{-1} is strictly proper and strictly positive real. There exist $Q \in \mathbb{R}^{\bullet \times w}[\xi]$ such that*

$$D(-\xi)^\top N(\xi) + N(-\xi)^\top D(\xi) = Q(-\xi)^\top Q(\xi) ;$$

moreover $\text{rank } \text{col}(D(\lambda), Q(\lambda)) = w$ for all $\lambda \in \mathbb{C}$, and QD^{-1} is strictly proper. Define

$$\Psi(\zeta, \eta) := \frac{D(\zeta)^\top N(\eta) + N(\zeta)^\top D(\eta) - Q(\zeta)^\top Q(\eta)}{\zeta + \eta} .$$

Then $\Psi(\zeta, \eta)$ is a D -canonical Lyapunov function for $\ker D \left(\frac{d}{dt} \right)$, and $\Psi(\zeta, \eta) \bmod N$ is a Lyapunov function for $\ker N \left(\frac{d}{dt} \right)$.

Proof. See Appendix A.2. \square

Thus if Ψ is a suitable storage function of the system with transfer function ND^{-1} , associated with a supply rate induced by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and with dissipation rate $Q(\zeta)^\top Q(\eta)$, then it is also a Lyapunov function for $\ker D\left(\frac{d}{dt}\right)$ and (after the “mod” operation) also for $\ker N\left(\frac{d}{dt}\right)$ (on dissipativity and Lyapunov stability see also [58]).

Remarkably, it turns out that such storage functions also induce a *MLF* for an *SLDS* with modes $\ker N\left(\frac{d}{dt}\right)$, $\ker D\left(\frac{d}{dt}\right)$, and special gluing conditions, naturally associated with the “mod” operation. This result is elaborated in the following section.

5.4 Stability of standard SLDS

In the following, we consider SLDSs where $\mathcal{F} = (\ker R_1\left(\frac{d}{dt}\right), \ker R_2\left(\frac{d}{dt}\right))$, with $R_j \in \mathbb{R}^{w \times w}[\xi]$, $j = 1, 2$ nonsingular. We assume that $R_2 R_1^{-1}$ is strictly proper; this implies that the state space of $\ker R_2\left(\frac{d}{dt}\right)$ is included in that of $\ker R_1\left(\frac{d}{dt}\right)$, as we presently show.

Lemma 5.11. *Let $\mathfrak{B}_i = \ker R_i\left(\frac{d}{dt}\right)$, $i = 1, 2$. Assume that $R_1, R_2 \in \mathbb{R}^{w \times w}[\xi]$ are nonsingular, and that $R_2 R_1^{-1}$ is strictly proper. Let $n_i := \deg(\det(R_i))$; then $n_2 < n_1$. There exist $X'_1 \in \mathbb{R}^{(n_1-n_2) \times w}[\xi]$, $X_2 \in \mathbb{R}^{n_2 \times w}[\xi]$ such that $X_2\left(\frac{d}{dt}\right)$ is a minimal state map for \mathfrak{B}_2 , and*

$$X_1\left(\frac{d}{dt}\right) := \text{col}\left(X_2\left(\frac{d}{dt}\right), X'_1\left(\frac{d}{dt}\right)\right), \quad (5.3)$$

is a minimal state map for \mathfrak{B}_1 . Moreover, $\exists \Pi \in \mathbb{R}^{(n_1-n_2) \times n_2}$ such that $X'_1(\xi) \bmod R_2 = \Pi X_2(\xi)$.

Proof. See Appendix A.2. □

Example 5.3. If $w = 1$, $R_2 R_1^{-1}$ is strictly proper iff $n_1 = \deg(R_1) > \deg(R_2) = n_2$. A state map for \mathfrak{B}_1 is $\text{col}(\xi^k)_{k=0, \dots, n_1-1}$, whose first n_2 elements form a basis for the state space of \mathfrak{B}_2 . The rows of Π consist of the coefficients of the polynomials $\xi^k \bmod R_2(\xi)$, $k = n_2, \dots, n_1 - 1$. □

In the rest of this chapter we consider *standard SLDS*, defined as follows.

Definition 5.12. Let $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$ be a SLDS with $\mathcal{F} = (\ker R_1\left(\frac{d}{dt}\right), \ker R_2\left(\frac{d}{dt}\right))$, where $R_j \in \mathbb{R}^{w \times w}[\xi]$ is nonsingular, $j = 1, 2$. Assume that $R_2 R_1^{-1}$ is strictly proper. Let $n_j := \deg(\det(R_j))$, $j = 1, 2$, and let $X'_1 \in \mathbb{R}^{(n_1-n_2) \times w}[\xi]$, $X_2 \in \mathbb{R}^{n_2 \times w}[\xi]$ and $\Pi \in \mathbb{R}^{(n_1-n_2) \times n_2}$ be as in Lemma 5.11. Σ is a *standard SLDS* if the gluing conditions are

$$\begin{aligned} (G_{2 \rightarrow 1}^-(\xi), G_{2 \rightarrow 1}^+(\xi)) &:= (\text{col}(X_2(\xi), \Pi X_2(\xi)), \text{col}(X_2(\xi), X'_1(\xi))) , \\ (G_{1 \rightarrow 2}^-(\xi), G_{1 \rightarrow 2}^+(\xi)) &:= (X_2(\xi), X_2(\xi)) . \end{aligned}$$

Remark 5.13. It is straightforward to check that the gluing conditions in Def. 5.12 are well-posed according to Def. 4.4. Note also that the state space of \mathfrak{B}_2 is contained in that of \mathfrak{B}_1 ; however, at any time the state used for the description of the system is that of the active dynamics, and not a global one. \square

Example 5.4. Assume that R_1 and R_2 in Ex. 5.3 are monic, and that $n_1 = n_2 + 1$. Denote $R_2(\xi) =: \sum_{j=0}^{n_1-1} R_{2,j} \xi^j$, and define $S(\xi) := \begin{bmatrix} 1 & \dots & \xi^{n_1-2} \end{bmatrix}^\top$. The gluing conditions of the standard SLDS are

$$(G_{2 \rightarrow 1}^-(\xi), G_{2 \rightarrow 1}^+(\xi)) = \left(\text{col}(S(\xi), -\sum_{j=0}^{n_1-2} R_{2,j} \xi^j), \text{col}(S(\xi), \xi^{n_1-1}) \right),$$

and

$$(G_{1 \rightarrow 2}^-(\xi), G_{1 \rightarrow 2}^+(\xi)) = (S(\xi), S(\xi)).$$

In a switch $\mathfrak{B}_2 \rightarrow \mathfrak{B}_1$, to obtain “initial conditions” uniquely specifying $w \in \mathfrak{B}_1$, we need to define the value of $\frac{d^{n_1-1}}{dt^{n_1-1}} w$ after the switch. Standard gluing conditions stipulate that it coincides with $\frac{d^{n_1-1}}{dt^{n_1-1}} w = -\sum_{i=0}^{n_1-2} R_{2,i} \frac{d^i}{dt^i} w$, since before the switch $w \in \mathfrak{B}_2$. In a switch $\mathfrak{B}_1 \rightarrow \mathfrak{B}_2$, we project the vector of derivatives characteristic of $w \in \mathfrak{B}_1$ down onto the shorter vector of derivatives of $w \in \mathfrak{B}_2$. \square

Consider the following example of a realistic scenario.

Example 5.5. Consider the basic multi-controller system in Fig. 5.3, where the plant described by the transfer function

$$\frac{n(\xi)}{d(\xi)} := \frac{(\xi+1)(\xi+4)}{(\xi-2)(\xi+3)},$$

is interconnected to stabilising switched controllers described by

$$\frac{p_1(\xi)}{q_1(\xi)} := \frac{K_D \xi^2 + K_P \xi + K_I}{\xi},$$

and

$$\frac{p_2(\xi)}{q_2(\xi)} := \frac{K'_P \xi + K'_I}{\xi},$$

where $K_D = 1$, $K_P = 25$, $K_I = 150$, $K'_P = 1$ and $K'_I = 33$.

By selecting the output variable w as the variable of interest, we can model the mode behaviours as $\mathfrak{B}_i := \ker r_i \left(\frac{d}{dt} \right)$, $i = 1, 2$, with

$$\begin{aligned} r_1(\xi) &:= 600 + 844\xi + 280\xi^2 + 31\xi^3 + \xi^4, \\ r_2(\xi) &:= 132 + 163\xi + 39\xi^2 + 2\xi^3. \end{aligned}$$

Furthermore, in many cases we are interested in determining a re-initialisation for the controllers at switching instants that guarantees the continuity of the external variable,

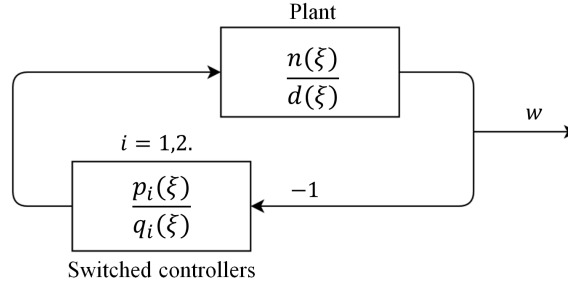


Figure 5.3: Multi-controller system with two dynamic modes

i.e. $w(t_j^+) = w(t_j^-)$ for all $t_j \in \mathbb{T}_s$ (see e.g. the *bumpless transfer* problem in [49]). We model such requirements via gluing conditions, i.e. when switching from \mathfrak{B}_1 to \mathfrak{B}_2 at t_j we require that

$$\begin{bmatrix} w(t_j^+) \\ \frac{d}{dt}w(t_j^+) \\ \frac{d^2}{dt^2}w(t_j^+) \end{bmatrix} = \begin{bmatrix} w(t_j^-) \\ \frac{d}{dt}w(t_j^-) \\ \frac{d^2}{dt^2}w(t_j^-) \end{bmatrix}.$$

On the other hand, when we switch from \mathfrak{B}_2 to \mathfrak{B}_1 we require that

$$\begin{bmatrix} w(t_j^+) \\ \frac{d}{dt}w(t_j^+) \\ \frac{d^2}{dt^2}w(t_j^+) \\ \frac{d^3}{dt^3}w(t_j^+) \end{bmatrix} = \begin{bmatrix} w(t_j^-) \\ \frac{d}{dt}w(t_j^-) \\ \frac{d^2}{dt^2}w(t_j^-) \\ -66w(t_j^-) - \frac{163}{2}\frac{d}{dt}w(t_j^-) - \frac{39}{2}\frac{d^2}{dt^2}w(t_j^-) \end{bmatrix}.$$

The rationale underlying this choice of gluing conditions is that at switching instants any trajectory of \mathfrak{B}_1 and \mathfrak{B}_2 is uniquely specified by the instantaneous values of w and its derivatives, respecting the laws imposed by the mode behaviors and requiring that the value of $w(t_j^+)$ and $w(t_j^-)$ coincide. Since

$$X_1(\xi) := \begin{bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{bmatrix}, \quad X_2(\xi) := \begin{bmatrix} 1 \\ \xi \\ \xi^2 \end{bmatrix},$$

induce state maps for \mathfrak{B}_1 and \mathfrak{B}_2 respectively, note that the proposed gluing conditions are standard in the sense of Def. 5.12, where

$$\Pi = \begin{bmatrix} -66 & -\frac{163}{2} & -\frac{39}{2} \end{bmatrix}.$$

□

Remark 5.14. Standard gluing conditions describe concatenability specifications when switching between mode behaviours with different state space dimension as in Ex. 5.5. In more complex cases, e.g. a multivariable version of Ex. 5.5, standard gluing conditions

can be computed using Lemma 5.11. Standard gluing conditions also appear in switched electrical systems, see e.g. the example of the energy distribution network shown in Sec. V of [40]. \square

We now prove that a standard SLDS where $R_2 R_1^{-1}$ is strictly positive real admits a multiple Lyapunov function induced by (Ψ_1, Ψ_2) where Ψ_1 is a storage function for $R_2 R_1^{-1}$, and $\Psi_2 = \Psi_1 \bmod R_2$. The following result can be considered as the multivariable generalisation of the early results about stability of scalar behaviours shown in [62].

Theorem 5.15. *Let Σ be a standard SLDS (see Def. 5.12), with R_1 and R_2 Hurwitz. Assume that $R_2 R_1^{-1}$ is strictly proper and strictly positive-real. Define*

$$\Phi(\zeta, \eta) := R_1(\zeta)^\top R_2(\eta) + R_2(\zeta)^\top R_1(\eta). \quad (5.4)$$

There exists $Q \in \mathbb{R}^{\bullet \times \mathbf{w}}[\xi]$ such that $\Phi(-\xi, \xi) = Q(-\xi)^\top Q(\xi)$, $\text{rank col}(R_1(\lambda), Q(\lambda)) = \mathbf{w}$ for all $\lambda \in \mathbb{C}$ and $Q R_1^{-1}$ is strictly proper. Define

$$\Psi_1(\zeta, \eta) := \frac{\Phi(\zeta, \eta) - Q(\zeta)^\top Q(\eta)}{\zeta + \eta}. \quad (5.5)$$

Then Ψ_1 is R_1 -canonical. Define $\Psi_2 := \Psi_1 \bmod R_2$; then (Ψ_1, Ψ_2) induces an MLF for Σ .

Proof. See Appendix A.2. \square

Example 5.6 (Contd from Ex. 5.5). Note that since $r_1(-j\omega)r_2(j\omega) + r_2(-j\omega)r_1(j\omega) > 0$ for all $\omega \in \mathbb{R}$, we conclude that $\frac{r_2(\xi)}{r_1(\xi)}$ is strictly positive real. Consequently, using Th. 5.15 we also conclude that the standard SLDS is asymptotically stable under arbitrary switching signals. \square

Th. 5.15 yields two approaches to computing an MLF for a standard SLDS. The first is algebraic and consists of a polynomial spectral factorisation of Φ in (5.4), and the computation of Ψ_1 from (5.5). The second, based on LMIs, arises from the proof of Th. 5.15. We state it in the following results.

Proposition 5.16. *Under the assumptions of Th. 5.15, define $n_1 := \deg(\det(R_1))$ and let $X_1 \in \mathbb{R}^{n_1 \times \mathbf{w}}[\xi]$ be a minimal state map for \mathfrak{B}_1 . Write $R_1(\xi) = \sum_{j=0}^L R_{1,j} \xi^j$, with $R_{1,j} \in \mathbb{R}^{\mathbf{w} \times \mathbf{w}}$, $j = 0, 1, \dots, L$. There exists $\tilde{R}_2 \in \mathbb{R}^{\mathbf{w} \times n_1}$, $\tilde{Q} \in \mathbb{R}^{\bullet \times n_1}$ and $K \in \mathbb{R}^{n_1 \times n_1}$ such that $R_2(\xi) = \tilde{R}_2 X_1(\xi)$, $Q(\xi) = \tilde{Q} X_1(\xi)$ and $\Psi_1(\zeta, \eta) = X_1(\zeta)^\top K X_1(\eta)$. Moreover, there exist $X_{1,j} \in \mathbb{R}^{n_1 \times \mathbf{w}}$, with $j = 0, 1, \dots, L-1$, such that $X_1(\xi) = \sum_{j=0}^{L-1} X_{1,j} \xi^j$*

Proof. See Prop. 3.10. \square

Proposition 5.17. *Under the assumptions of Th. 5.15 and Prop. 5.16, denote the coefficient matrices of $R_1(\xi)$ and $X_1(\xi)$ by*

$$\tilde{R}_1 := \begin{bmatrix} R_{1,0} & \dots & R_{1,L} \end{bmatrix}, \quad \tilde{X}_1 := \begin{bmatrix} X_{1,0} & \dots & X_{1,L-1} \end{bmatrix}.$$

Let $K = K^\top \in \mathbb{R}^{n_1 \times n_1}$. The following statements are equivalent:

1. $\Psi(\zeta, \eta) := X_1(\zeta)^\top K X_1(\eta)$, $R_i(\xi)$, $i = 1, 2$, and $Q(\xi)$ satisfy (5.5);
2. There exists $K > 0$, such that

$$\begin{aligned} & \begin{bmatrix} 0_{\mathbf{w} \times n_1} \\ \tilde{X}_1^\top \end{bmatrix} K \begin{bmatrix} \tilde{X}_1 & 0_{n_1 \times \mathbf{w}} \end{bmatrix} + \begin{bmatrix} \tilde{X}_1^\top \\ 0_{\mathbf{w} \times n_1} \end{bmatrix} K \begin{bmatrix} 0_{n_1 \times \mathbf{w}} & \tilde{X}_1 \end{bmatrix} \\ & - \begin{bmatrix} \tilde{X}_1^\top \\ 0_{\mathbf{w} \times n_1} \end{bmatrix} \tilde{R}_2^\top \tilde{R}_1 - \tilde{R}_1^\top \tilde{R}_2 \begin{bmatrix} \tilde{X}_1 & 0_{n_1 \times \mathbf{w}} \end{bmatrix} + \begin{bmatrix} \tilde{X}_1^\top \\ 0_{\mathbf{w} \times n_1} \end{bmatrix} \tilde{Q}^\top \tilde{Q} \begin{bmatrix} \tilde{X}_1 & 0_{n_1 \times \mathbf{w}} \end{bmatrix} = 0. \end{aligned}$$

Consequently, $(\Psi, \Psi \bmod R_2)$ induces a multiple Lyapunov function for Σ .

Proof. See Appendix A.2. □

Remark 5.18. If $\mathbf{w} = 1$ the proof of Th. 5.15 simplifies considerably; see [59] for details. □

Remark 5.19. Theorem 5.15 holds also if $R_2 R_1^{-1}$ is *bi-proper*, i.e. proper and with a proper inverse; note that in this case the state spaces of \mathfrak{B}_1 and of \mathfrak{B}_2 coincide. Let $X \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ be a state map for \mathfrak{B}_1 ; the standard gluing conditions are

$$(G_{1 \rightarrow 2}^-(\xi), G_{1 \rightarrow 2}^+(\xi)) = (X(\xi), X(\xi)) = (G_{2 \rightarrow 1}^-(\xi), G_{2 \rightarrow 1}^+(\xi)).$$

It is straightforward to check that e.g. the largest storage function for $R_2 R_1^{-1}$ yields a MLF. For $\mathbf{w} = 1$ this is shown in [62]. □

Remark 5.20. In the state-space framework it is well-known that if the open-loop transfer function of a system is positive-real, then all closed-loop systems obtained from it by state feedback share a common quadratic Lyapunov function (see sect. 2.3.2 of [34] and [64, 65]). Th. 5.15 offers a new perspective on the relation between positive-realness and stability: in our framework, the different dynamical regimes do not arise from closing the loop around some fixed plant, and positive-realness arises from the interplay of the mode dynamics. □

For standard SLDS, positive-realness of $R_2 R_1^{-1}$ is a sufficient condition for stability. This assumption is rather restrictive and we now show how to relax it. To this purpose we introduce the concept of *positive-real completion*.

5.5 Positive-real completions

We now study the role of positive real completions in stability of standard SLDS.

Definition 5.21. Let $R_i \in \mathbb{R}^{w \times w}[\xi]$, $i = 1, 2$ be nonsingular and $R_2 R_1^{-1}$ strictly proper. $M \in \mathbb{R}^{w \times w}[\xi]$ is a *strictly positive-real completion* of $R_2 R_1^{-1}$ if $M R_2 R_1^{-1}$ is strictly positive-real.

Remark 5.22. A positive-real completion can be regarded as the multivariable version of the “passivation” technique used for open SISO systems in Sec. 3 of [31]. We will show that in the context of SLDS, positive-real completions provide a less conservative stability condition than that of Th. 5.15, i.e. for the case when $R_2 R_1^{-1}$ is not positive-real. \square

Remark 5.23. Strictly- positive-real completions are not unique; e.g. the rational function $\frac{mr_2}{r_1}$ with $r_1(\xi) := (\xi + 1)(\xi + 3)(\xi + 6)$ and $r_2(\xi) := \xi + 2$ is positive-real with m equal to $\xi + 4$, $\xi + 5$ or many other. Note also that not every pair of Hurwitz matrices has a strictly- positive-real completion, for example the polynomials

$$\begin{aligned} r_1(\xi) &:= 2523677 + 435616\xi + 81559\xi^2 + 7000\xi^3 + 603\xi^4 + 24\xi^5 + \xi^6, \\ r_2(\xi) &:= 65 + 46\xi + 26\xi^2 + 6\xi^3 + \xi^4. \end{aligned}$$

\square

We now show that if an MLF exists, then a positive-real completion can be found.

Theorem 5.24. Let Σ be a standard SLDS (see Def. 5.12). If $\{\Psi_1, \Psi_1 \bmod R_2\}$ induces an MLF for Σ such that $(\zeta + \eta)\Psi_1(\zeta, \eta) \bmod R_1 = -Q(\zeta)^\top Q(\eta)$ with $\text{rank } Q(j\omega) = w$ for all $\omega \in \mathbb{R}$ and $Q R_1^{-1}$ strictly proper, then there exists a strictly positive-real completion $M \in \mathbb{R}^{w \times w}[\xi]$ for $R_2 R_1^{-1}$.

Proof. See Appendix A.2. \square

In the following result we establish general conditions for the existence of a Lyapunov function for a standard SLDS using positive-real completions.

Theorem 5.25. Let Σ be a standard SLDS as in Def. 5.12. Let \tilde{R}_1 and \tilde{X}_1 be as in Prop. 5.17. Define $Y := M R_2$ with $M \in \mathbb{R}^{w \times w}[\xi]$ such that $Y R_1^{-1}$ is strictly proper. There exist $Y_j \in \mathbb{R}^{w \times w}$, with $j = 0, 1, \dots, L-1$, such that $Y(\xi) = \sum_{j=0}^{L-1} Y_j \xi^j$. Denote $\tilde{Y} := \begin{bmatrix} Y_0 & \dots & Y_{L-1} \end{bmatrix}$. If there exists $K > 0$ with $K \in \mathbb{R}^{n_1 \times n_1}$ such that

$$\begin{bmatrix} 0_{w \times n} \\ \tilde{X}_1^\top \end{bmatrix} K \begin{bmatrix} \tilde{X}_1 & 0_{n \times w} \end{bmatrix} + \begin{bmatrix} \tilde{X}_1^\top \\ 0_{w \times n} \end{bmatrix} K \begin{bmatrix} 0_{n \times w} & \tilde{X}_1 \end{bmatrix} - \begin{bmatrix} \tilde{Y}^\top \\ 0_{w \times w} \end{bmatrix} \tilde{R}_1 - \tilde{R}_1^\top \begin{bmatrix} \tilde{Y} & 0_{w \times w} \end{bmatrix} \leq 0, \quad (5.6)$$

then M is a strictly positive-real completion of $R_2 R_1^{-1}$. Moreover, define $\Psi_1(\zeta, \eta) := X_1(\zeta)^\top K X_1(\eta)$. If K partitioned as

$$K := \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^\top & K_{22} \end{bmatrix},$$

with $K_{11} \in \mathbb{R}^{n_2 \times n_2}$, $K_{12} \in \mathbb{R}^{n_2 \times (n_1 - n_2)}$ and $K_{22} \in \mathbb{R}^{(n_1 - n_2) \times (n_1 - n_2)}$; is such that $K_{12} = -\Pi^\top K_{22}$, then $\{\Psi_1, \Psi_1 \bmod R_2\}$ induces a multiple Lyapunov function for Σ .

Proof. See Appendix A.2. □

5.5.1 Computation of positive-real completions

Th. 5.25 establishes general conditions for stability of standard SLDS in terms of LMIs. Although positive-real completions are instrumental for the computation of multiple Lyapunov functions, they are not necessarily known *a priori*; we can compute them by using the LMI (5.6). In order to do so, let $M(\xi) = \sum_{j=0}^N M_j \xi^j$, i.e. $M(\xi)$ is written in terms of unspecified parameters, with $N \leq L - 1$. Write

$$\tilde{Y}^\top := \underbrace{\begin{bmatrix} R_{2,0} & 0 & 0 & \cdots & 0 \\ R_{2,1} & R_{2,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \end{bmatrix}}_{=: \tilde{T}} \underbrace{\begin{bmatrix} M_0 \\ M_1 \\ \vdots \end{bmatrix}}_{=: \tilde{M}}; \quad (5.7)$$

where $\tilde{T} \in \mathbb{R}^{L \times \bullet}$ is a block *Töplitz matrix* (see [30], Sec. 8.3.1) containing the coefficients $R_{2,j}$ of $R_2(\xi)$; and $\tilde{M} \in \mathbb{R}^{\bullet \times w}$ contains the unknown coefficients of $M(\xi)$. The LMI (5.6) with \tilde{Y} as in (5.7) can be solved using standard LMI solvers. On the other hand, if (5.6) has no solution, we conclude that the pair R_1, R_2 does not have a positive-real completion, see remark 5.23.

Example 5.7 (Cont'd from Ex. 5.5). We cannot expect that the positive-real property will be satisfied when considering any pair of stabilising controllers, however this does not imply that a Lyapunov function does not exist. For instance, redefine the transfer function

$$\frac{p_2(\xi)}{q_2(\xi)} := 10,$$

corresponding to an elementary negative-feedback proportional controller. It follows that the resulting $r_2(\xi) := 34 + 51\xi + 11\xi^2$ is also Hurwitz; however,

$$\frac{r_2(\xi)}{r_1(\xi)} := \frac{34 + 51\xi + 11\xi^2}{600 + 844\xi + 280\xi^2 + 31\xi^3 + \xi^4},$$

is not positive-real.

We proceed to test stability by constructing the LMI (4) introduced in Th. 2, p. 7. In order to do so, we define

$$X_1(\xi) := \begin{bmatrix} 1 \\ \xi \\ \xi^2 \\ \xi^3 \end{bmatrix}, \quad X_2(\xi) := \begin{bmatrix} 1 \\ \xi \end{bmatrix},$$

then we consider standard gluing conditions as in Def. 3, where

$$\Pi := \begin{bmatrix} -\frac{34}{11} & -\frac{51}{11} \\ \frac{1734}{121} & \frac{2227}{121} \end{bmatrix}.$$

Now we solve the LMI (4) with the constraints $K > 0$ and $\Psi_{12} = -\Pi^\top \Psi_{22}$, using the coefficient matrices

$$\tilde{X}_1 := I_4, \quad \tilde{R}_1 := \begin{bmatrix} 600 & 844 & 280 & 31 & 1 \end{bmatrix},$$

and a matrix involving the coefficients of the positive-real completion:

$$\tilde{Y}^\top := \begin{bmatrix} 34 & 0 \\ 51 & 34 \\ 11 & 51 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix},$$

which is constructed according to eq. (5) in remark 6 on p. 8. It can be verified using standard LMI solvers that there exist several solutions for K , e.g.

$$K := \begin{bmatrix} 9526.35 & 10338.06 & 2330.56 & 152.62 \\ 10338.06 & 16723.12 & 3819.00 & 265.71 \\ 2330.56 & 3819.00 & 1238.73 & 104.54 \\ 152.62 & 265.71 & 104.54 & 11.89 \end{bmatrix};$$

corresponding to the positive real completion $m(\xi) = 4.48 + 1.08\xi$ whose coefficients are also determined by solving the LMI. Consequently, the asymptotic stability under arbitrary switching between stabilising controllers is proved. \square

5.6 Stability of SLDS with three behaviours

In this section we analyse important consequences of the existence of positive-real completions. The following lemma will be instrumental for this aim.

Lemma 5.26. *Let $\mathfrak{B}_i := \ker R_i \left(\frac{d}{dt} \right)$, $i = 1, 2$, be as in Def. 5.12. Let $M \in \mathbb{R}^{w \times w}[\xi]$ be such that $MR_2R_1^{-1}$ is strictly proper. Define $\mathfrak{B}_3 := \ker R_3 \left(\frac{d}{dt} \right)$ where $R_3(\xi) :=$*

$M(\xi)R_2(\xi)$ and $n_j := \deg(\det(R_j))$, $j = 1, 2, 3$. There exist $X_2 \in \mathbb{R}^{n_2 \times w}[\xi]$, $X'_3 \in \mathbb{R}^{(n_3 - n_2) \times w}[\xi]$ and $X'_1 \in \mathbb{R}^{(n_1 - n_3) \times w}[\xi]$ such that

1. $X_1 := \begin{bmatrix} X_2 & X'_3 & X'_1 \end{bmatrix}^\top$ is a minimal state map for \mathfrak{B}_1 .
2. X_2 is a minimal state map for \mathfrak{B}_2 .
3. $X_3 := \begin{bmatrix} X_2 & X'_3 \end{bmatrix}^\top$ is a minimal state map for \mathfrak{B}_3 .

Moreover, there exist Π_j , $j = 1, 2, 3$, of appropriate sizes, such that $\text{col}(X'_3(\xi), X'_1(\xi)) \bmod R_2 = \Pi_1 X_2(\xi)$; $X'_3(\xi) \bmod R_2 = \Pi_2 X_2(\xi)$; and $X'_1(\xi) \bmod R_3 = \Pi_3 X_3(\xi)$.

Proof. See Appendix A.2. □

In the following, we show a sufficient condition for the asymptotic stability of a SLDS with *three* behaviours.

Theorem 5.27. *Let Σ be a standard SLDS as in Def. 5.12. Assume that there exists M and K satisfying the conditions of Th. 5.25. Define $R_3 := MR_2$, $\mathfrak{B}_i := \ker R_i \left(\frac{d}{dt} \right)$, $i = 1, 2, 3$; and let X_i , $i = 1, 2, 3$ be as in Lemma 5.26. Consider a SLDS Σ' with $\mathcal{F}' = \{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3\}$ and gluing conditions*

$$\begin{aligned} (G_{2 \rightarrow 1}^-(\xi), G_{2 \rightarrow 1}^+(\xi)) &:= (\text{col}(X_2(\xi), \Pi_1 X_2(\xi)), \text{col}(X_3(\xi), X'_1(\xi))) , \\ (G_{1 \rightarrow 2}^-(\xi), G_{1 \rightarrow 2}^+(\xi)) &:= (X_2(\xi), X_2(\xi)) , \\ (G_{3 \rightarrow 1}^-(\xi), G_{3 \rightarrow 1}^+(\xi)) &:= (\text{col}(X_3(\xi), \Pi_3 X_3(\xi)), \text{col}(X_3(\xi), X'_1(\xi))) , \\ (G_{1 \rightarrow 3}^-(\xi), G_{1 \rightarrow 3}^+(\xi)) &:= (X_3(\xi), X_3(\xi)) , \\ (G_{2 \rightarrow 3}^-(\xi), G_{2 \rightarrow 3}^+(\xi)) &:= (\text{col}(X_2(\xi), \Pi_2 X_2(\xi)), \text{col}(X_2(\xi), X'_3(\xi))) , \\ (G_{3 \rightarrow 2}^-(\xi), G_{3 \rightarrow 2}^+(\xi)) &:= (X_2(\xi), X_2(\xi)) , \end{aligned}$$

with Π_i , $i = 1, 2, 3$ as in Lemma 5.26.

Define $\Psi(\zeta, \eta) := X_1(\zeta)^\top K X_1(\eta)$, then $\{\Psi_1 \bmod R_i\}_{i=1,2,3}$ induces a multiple Lyapunov function for \mathcal{F}' .

Proof. See Appendix A.2. □

Example 5.8 (Cont'd from Ex. 5.7). In this case, we proved that the Lyapunov function obtained by applying Th. 2 is also a Lyapunov function for a SLDS with an extended bank including a third behaviour $\mathfrak{B}_3 := \ker r_3 \left(\frac{d}{dt} \right)$, where $r_3(\xi) := m(\xi)r_2(\xi)$; and standard gluing conditions. Note that supported on these results, we can straightforwardly compute the transfer function of an additional stabilising controller

$$\frac{p_3(\xi)}{q_3(\xi)} := \frac{44.8 + 10.8\xi}{4.48 + 1.08\xi}$$

Then an extended bumpless transfer strategy with three controllers is obtained. \square

Another consequence of the notion of positive-real completion is given in the following theorem where we prove stability of parameter dependent families of SLDS with three behaviours.

Theorem 5.28. *Let Σ be a standard SLDS as in Def. Theorem 5.27. Assume that there exist strictly positive-real completions M_1 and M_2 of $R_2 R_1^{-1}$, each one associated to a Lyapunov function for Σ as in Th. 5.25. Then, the polynomial matrix*

$$M_\alpha := \alpha M_1 + (1 - \alpha) M_2, \quad 0 \leq \alpha \leq 1,$$

is also a strictly positive-real completion. Moreover, define a SLDS Σ' with

$$\mathcal{F}'_\alpha := \left\{ \ker R_1 \left(\frac{d}{dt} \right), \ker R_2 \left(\frac{d}{dt} \right), \ker R_{3,\alpha} \left(\frac{d}{dt} \right) \right\},$$

where $R_{3,\alpha} := M_\alpha R_2$, and with gluing conditions as in Th. 5.27. Then Σ' is asymptotically stable.

Proof. See Appendix A.2. \square

Theorem 5.28 shows that the existence of two separate completions allows to establish the stability of a whole family of parameter-dependent SLDS with three behaviors \mathfrak{F}_α . This result also shows that the asymptotic stability of a completion established in Theorem 5.27 is robust: perturbations of a given completion, parametrized by α as in Theorem 5.28, also result in a stable SLDS.

5.7 Summary

We provided results regarding stability using multiple higher-order Lyapunov functions for general SLDS with arbitrary gluing conditions. We also studied stability of a special class of SLDS using the concept of positive-realness and positive-real completion.

Chapter 6

Dissipative switched linear differential systems

Dissipativity and its special case *passivity* have been studied extensively in general settings such as impulsive, discontinuous and hybrid systems (see e.g. [20, 21, 22, 23, 54, 93]), as well as in the switched systems setting (see e.g. [5, 17, 38, 94, 95, 96]). In [18], the role of passivity for stability of switched systems has been also studied considering dynamical modes with Hamiltonian structure. In [96], novel definitions of dissipative switched systems are presented involving the use of *cross-supply rates*. This approach also encompasses important results (e.g. stability, passivity, L_2 -gain) associated to dissipative nonlinear systems with infinitely differentiable trajectories. In [23], another definition of dissipativity is presented where the use of *connective supply rates* characterises the energy change of inactive modes. More recently, in [38, 37], the notion of *decomposable dissipativity* is introduced for discrete-time switched systems.

In this chapter, we give definitions of dissipativity of switched linear differential systems. Furthermore, we provide sufficient conditions for dissipativity based on systems of LMIs for arbitrary switching signals and involving the computation of multiple storage functions. Such systems of LMIs can be set up straightforwardly from the equations of the mode dynamics and the gluing conditions. We also study the relationship between dissipativity and stability of switched systems by studying *passive systems*.

6.1 Preliminaries

In order to formulate the concept of dissipativity and the main results obtained in this chapter, we need to establish a few standard assumptions that are made partly for convenience of exposition, these initial considerations are now enlisted.

1. *Compact support trajectories.* In dissipativity theory we often require the integration of functionals acting on $w \in \mathfrak{B}^\Sigma$. In order to avoid convergence issues, and ensure that such integrals exist, we assume that they involve *piecewise* infinitely differentiable trajectories of compact support whose set is denoted by $\mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$. For this reason we introduce the notation $\mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$.
2. *Inputs and outputs.* We consider dynamical modes sharing the same external variable and admitting the same input-output partition $w = \text{col}(u, y)$ (see Sec. 2.3). Here we consider that the selection of the manifest variables is the same for every mode, and moreover, during the switching between dynamic modes, inputs are not transformed into outputs and vice versa. Note that this consideration includes e.g. the case of physical systems with *ports* and *conjugate variables* (see [48]) e.g. mechanical, electrical, thermodynamical systems, etc; among many other situations.
3. *Controllability and observability.* We consider switched linear differential systems with *controllable* mode behaviours, i.e. $\mathfrak{B}_j \in \mathfrak{L}_{\text{cont}}^w$, $j = 1, \dots, N$, described by *observable* image form representations $w = M_j \left(\frac{d}{dt} \right) z_j$, $j = 1, \dots, N$, with $M_j \in \mathbb{R}^{w \times z}[\xi]$. Controllability ensures that compact support trajectories exist (see assumption 2 above). Observability ensures that every trajectory of the latent variable z_j corresponds to a unique trajectory of the external variable w when the j -th mode is active, which in many instances simplifies computations (see e.g. Sec. 3.4).

6.2 Dissipative SLDS

Our concept of dissipativity is fundamentally based on that for linear differential systems discussed in Sec. 3.7. In order to introduce the main definition and results about dissipative SLDS, we introduce the following notation.

Let $s \in \mathcal{S}$ be a fixed but otherwise arbitrary switching signal, whose associated set of switching instants is $\mathbb{T}_s := \{t_1, t_2, \dots, t_n, \dots\}$. We denote by $|\mathbb{T}_s|$ the total number of switching instants in \mathbb{T}_s .

1. If $|\mathbb{T}_s| = \infty$, define

$$\int Q_\Phi(w) := \int_{-\infty}^{t_1^-} Q_\Phi(w) dt + \int_{t_1^+}^{t_2^-} Q_\Phi(w) dt + \dots + \int_{t_n^+}^{t_{n+1}^-} Q_\Phi(w) dt + \dots ;$$

and

$$\int \|w\|_2^2 := \int_{-\infty}^{t_1^-} \|w\|_2^2 dt + \int_{t_1^+}^{t_2^-} \|w\|_2^2 dt + \dots + \int_{t_n^+}^{t_{n+1}^-} \|w\|_2^2 dt + \dots .$$

2. If $0 < |\mathbb{T}_s| < \infty$, then define

$$\int Q_\Phi(w) := \int_{-\infty}^{t_1^-} Q_\Phi(w) dt + \sum_{k=2}^{|\mathbb{T}_s|} \int_{t_{k-1}^+}^{t_k^-} Q_\Phi(w) dt + \int_{t_{|\mathbb{T}_s|}^+}^{\infty} Q_\Phi(w) dt ;$$

and

$$\int \|w\|_2^2 := \int_{-\infty}^{t_1^-} \|w\|_2^2 dt + \sum_{k=2}^{|\mathbb{T}_s|} \int_{t_{k-1}^+}^{t_k^-} \|w\|_2^2 dt + \int_{t_{|\mathbb{T}_s|}^+}^{\infty} \|w\|_2^2 dt .$$

3. If $|\mathbb{T}_s| = 0$, i.e. no switching takes place, then

$$\int Q_\Phi(w) := \int_{-\infty}^{+\infty} Q_\Phi(w) dt ;$$

and

$$\int \|w\|_2^2 := \int_{-\infty}^{+\infty} \|w\|_2^2 dt .$$

Moreover, given a trajectory $w \in \mathfrak{B}^\Sigma$, we denote the switching signal associated to it (see Def. 4.2) as s_w .

Definition 6.1. Let Σ be a SLDS and let Q_Φ be a QDF. Σ is Φ -dissipative if for all $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$ it holds

$$\int Q_\Phi(w) \geq 0 ;$$

and *strictly* Φ -dissipative if there exists $\epsilon > 0$ such that for all $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$, it holds

$$\int Q_\Phi(w) \geq \epsilon \int \|w\|_2^2 .$$

In the previous definition, the quadratic differential form Q_Φ can be interpreted as power, consequently, its integral over the real line measures the energy that is being supplied to, or flows out from the SLDS. If the net flow of energy is nonnegative then we call the SLDS Φ -dissipative.

The definition of dissipativity is not uniform in the literature for switched/hybrid systems. For instance, in [96] *multiple-* and *cross-supply rates* are considered to characterise the energy change of inactive modes for the case when they share the same state space. A similar concept is used in [23], where *connective supply rates* are used. These definitions permit the modelling of dynamical modes with different inputs, which is a suitable approach in cases such as multi-controller control systems. In our definition, we consider the use of a *main supply rate* acting on a fixed external variable for modes that do not necessarily share the same state-space. This definition is most suitable for the study of switched systems whose variables of interest are fixed, consequently the modes interchange energy with the environment in the same manner for every mode e.g. by means of ports.

If a SLDS is dissipative, then every dynamic mode in the bank is also dissipative.

Proposition 6.2. *Let Σ be a SLDS. If Σ is (strictly) Φ -dissipative according to Def. 6.1. Then \mathfrak{B}_i , $i = 1, \dots, N$, is a (strictly) Φ -dissipative linear differential behaviour.*

Proof. See Appendix A.3. □

Following standard results in dissipative linear differential systems, if an SLDS is dissipative each dynamic mode can be associated with a storage function.

Proposition 6.3. *Let Σ be a (strictly) Φ -dissipative SLDS. For all $i \in \mathcal{P}$ there exists a QDF Q_{Ψ_i} that is a storage function for \mathfrak{B}_i . Let $a < b$, then for all $w \in \mathfrak{B}^\Sigma$ with $s_w(t) = i$ for $t \in [a, b]$, it holds that*

$$\int_a^b Q_{\Phi}(w) dt \geq Q_{\Psi_i}(w)(b) - Q_{\Psi_i}(w)(a) .$$

Proof. See Appendix A.3. □

6.3 Multiple storage functions

As discussed in the literature (see e.g. [23, 96]), the use of a global storage function for all dynamical modes of a dissipative switched system is not only conservative but also not supported by physical considerations. Note for instance that physical switched systems may have different ways to store energy depending on the mode that is active.

Example 6.1 (Cont'd from Ex. 4.7). Consider the electrical circuit in Fig. 4.4. Following first principles, the stored energy for each mode is given respectively by the QDFs

$$Q_{\Psi_1}(w) := \frac{1}{2}Li_L^2 + \frac{1}{2}(C_1 + C_2)v_2^2$$

and

$$Q_{\Psi_2}(w) := \frac{1}{2}Li_L^2 + \frac{1}{2}C_1(E + R_L i_L - L \frac{d}{dt} i_L)^2 + \frac{1}{2}C_2 v_2^2 .$$

□

When switching between modes, the trajectories of w are in general subject to algebraic constraints modelled via the gluing conditions. Consequently, the transition between storage functions becomes of interest in dissipative systems. The second law of thermodynamics prevents stored energy in a dissipative system to increase at switching instants, since the process of dissipation cannot be reversed and energy is strictly provided by external sources characterised by the supply rate. Consequently any change in the physical stored energy is necessarily accounted as energy losses. This point of view has been

elaborated in [47] where the analysis of a wide variety of physical systems exhibiting discontinuities is presented; the same principle is also discussed in [14, 19, 73]. This energy condition is also used for a definition of passivity for hybrid systems in Prop. 1 of [93], where the nonincreasing condition for *multiple Lyapunov functions* introduced in [3] is used for multiple storage functions. Here we illustrate such condition for dissipative systems from a physical point of view using the power converter in Fig. 4.4.

Example 6.2 (Cont'd from Ex. 6.1). Let us compute the changes in stored energy of the circuit at a switching instant t_i . Taking into account the gluing conditions in Ex. 4.7 and after some straightforward computations, the change in stored energy when switching respectively from \mathfrak{B}_1 to \mathfrak{B}_2 and vice versa can be computed respectively as

$$Q_{\Psi_1}(w)(t_i^-) - Q_{\Psi_2}(w)(t_i^+) = 0 ,$$

i.e. there is no loss; and

$$Q_{\Psi_2}(w)(t_i^-) - Q_{\Psi_1}(w)(t_i^+) = \frac{1}{4} \left(E(t_i^-) + i_L(t_i^-) - \frac{d}{dt} i_L(t_i^-) - v_2(t_i^-) \right)^2 .$$

Evidently the latter quantity is nonnegative implying that the circuit loses energy. \square

Definition 6.4. Let Σ be a SLDS and let $s \in \mathcal{S}$. An N -tuple $(Q_{\Psi_1}, \dots, Q_{\Psi_N})$ is a *multiple storage function* for Σ with respect to Q_Φ if

- 1) $\frac{d}{dt} Q_{\Psi_i} \stackrel{\mathfrak{B}_i}{\leq} Q_\Phi, i = 1, \dots, N.$
- 2) $\forall w \in \mathfrak{B}^\Sigma$ s.t. $s = s_w$ and $\forall t_k \in \mathbb{T}_s$, it holds

$$Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0 .$$

Remark 6.5. In condition 1) of Def. 6.4 we require each mode behaviour to be Φ -dissipative which is equivalent to Q_{Ψ_i} satisfying the dissipation inequality for the i -th mode. In condition 2) we require that the storage function does not increase when we switch from one mode to another: switching cannot increase the amount of stored energy in the system. \square

Theorem 6.6. Let Σ be a SLDS and let Q_Φ be a QDF. Assume that there exists a multiple storage function as in Def. 6.4. Then Σ is Φ -dissipative.

Proof. See Appendix A.3. \square

In Th. 6.6, we proved that the existence of a multiple storage function as in Def. 6.4 is a sufficient condition for dissipativity. In the classical theory for linear differential behaviours, dissipativity is actually *equivalent* to the existence of a storage function (see Prop. 3.18). In the following we show that if Φ is a *constant* matrix, then strict Φ -dissipativity implies the existence of a multiple storage function for SLDS.

Theorem 6.7. *Let $\Phi \in \mathbb{R}^{w \times w}$ and let Σ be a strictly Φ -dissipative SLDS with \mathcal{G} well-defined and well-posed, and with mode behaviours \mathfrak{B}_k , $k = 1, \dots, N$. There exist storage functions Q_{Ψ_i} , $i = 1, \dots, N$, for the linear differential behaviours \mathfrak{B}_k , $i = 1, \dots, N$, with respect to Q_Φ , such that for all $t_k \in \mathbb{T}_s$ and for all $i, j \in \mathcal{P}$, $i \neq j$, it holds that*

$$Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0.$$

Consequently, $(Q_{\Psi_1}, \dots, Q_{\Psi_N})$ is a multiple storage function for Σ .

Proof. See Appendix A.3. □

Derived from strict dissipativity and the fact that for constant supply rates, storage functions are quadratic functions of the state (see Prop. 3.20), we can construct an LMI equivalent with condition 2) in Def. 6.4.

Lemma 6.8. *Under the assumptions of Th. 6.7, denote by z_i , $i = 1, \dots, N$, the unique latent variable trajectories associated with the external variable, i.e. $w = M_i \left(\frac{d}{dt} \right) z_i$, $i = 1, \dots, N$. Let $X_i \in \mathbb{R}^{n(\mathfrak{B}_i) \times z}[\xi]$ induce minimal state maps for \mathfrak{B}_i , $i = 1, \dots, N$, and let $Q_{\Psi'_i}(z_i) = Q_{\Psi_i}(w)$, $i = 1, \dots, N$. Let $L_{i \rightarrow j} \in \mathbb{R}^{n(\mathfrak{B}_j) \times n(\mathfrak{B}_i)}$ for all $i, j \in \mathcal{P}$, $i \neq j$, be the re-initialisations maps. There exist $K_i = K_i^\top \in \mathbb{R}^{n(\mathfrak{B}_i) \times n(\mathfrak{B}_i)}$, such that $\Psi'_i(\zeta, \eta) = X_i(\zeta)^\top K_i X_i(\eta)$, $i = 1, \dots, N$.*

Moreover, the following conditions are equivalent: for all $w \in \mathfrak{B}^\Sigma$, $t_k \in \mathbb{T}_s$ and $i, j \in \mathcal{P}$, $i \neq j$,

- 1) $Q_{\Psi_i}(w)(t_k^-) \geq Q_{\Psi_j}(w)(t_k^+)$.
- 2) $Q_{\Psi'_i}(z_i)(t_k^-) \geq Q_{\Psi'_j}(z_j)(t_k^+)$.
- 3) $K_i \geq L_{i \rightarrow j}^\top K_j L_{i \rightarrow j}$.

Proof. See Appendix A.3. □

An important consequence of Lemma 6.8 is the following result.

Proposition 6.9. *Under the assumptions of Th. 6.7 and Lemma 6.8, if the re-initialisation maps $L_{i \rightarrow j}$ associated to the switching between \mathfrak{B}_i to \mathfrak{B}_j , for all $i, j \in \mathcal{P}$ and $i \neq j$, are the identity, there exists Q_Ψ such that $(Q_\Psi, Q_\Psi, \dots, Q_\Psi)$ is a multiple storage function for Σ with respect to Q_Φ .*

Proof. See Appendix A.3. □

As a special case of Th. 6.7, Prop. 6.9 can be interpreted in the following way. If the mode behaviours share the same state space and the state trajectories are continuous at switching instants, strict dissipativity implies the existence of a *common storage function* Q_Ψ for *open* systems. This result is analogous to the *converse Lyapunov theorem* (see Th. 2.2 of [34], p. 25), where asymptotic stability implies the existence of a *common Lyapunov function* for *closed* systems under analogous conditions. Note that in such case, the “asymmetric” converse implication in our results arises as well, i.e. the existence of a *quadratic common Lyapunov function* implies that the switched system is *stable* (not necessarily *asymptotically stable*, see e.g. [34], Ex. 2.1). However, stability does not imply the existence of a quadratic common Lyapunov function, see e.g. the counterexample in [34], Sec. 2.1.5.; this is only true for asymptotic stability according to the converse Lyapunov theorem.

6.4 Half-line dissipativity

When the energy absorbed by a SLDS is positive in any arbitrary interval of time, we call such SLDS *half-line dissipative*.

In order to introduce the definition and results regarding half-line dissipativity, we use the following notation. Let $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$ and $\tau \in \mathbb{R}$. Let $s = s_w \in \mathcal{S}$ whose associated set of switching instants is $\mathbb{T}_s := \{t_1, t_2, \dots, t_n, \dots\}$; we define

$$\int_{-\infty}^{\tau} Q_\Phi(w) dt := \int_{-\infty}^{t_1^-} Q_\Phi(w) dt + \sum_{k=2}^n \int_{t_{k-1}^+}^{t_k^-} Q_\Phi(w) dt + \int_{t_n^+}^{\tau} Q_\Phi(w) dt ; \quad (6.1)$$

where $n = \max\{k \mid t_k \in \mathbb{T}_s, \text{ and } t_k \leq \tau\}$.

Definition 6.10. Let Q_Φ be a QDF. A SLDS Σ is *half-line Φ -dissipative* if for every $\tau \in \mathbb{R}$ and for all $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$, it holds that

$$\int_{-\infty}^{\tau} Q_\Phi(w) dt \geq 0 .$$

Half-line dissipativity appears very frequently in physical systems. For instance, in n -port driven electrical circuits we can select a external variable $w := \text{col}(V, I)$ consisting of a vector of voltages $V := \text{col}(V_1, \dots, V_n)$ and currents $I := \text{col}(I_1, \dots, I_n)$ across and through the ports. We say that the circuit is passive if for the supply rate defined as $Q_\Phi(w) := V^\top I$, it follows that for all τ and for all the trajectories of w with compact support $\int_{-\infty}^{\tau} Q_\Phi(w) dt \geq 0$ (cf. the classical definition in [48]).

Proposition 6.11. *Let Σ be a SLDS. If Σ is half-line Φ -dissipative, then $\mathfrak{B}_i, i = 1, \dots, N$, are half-line Φ -dissipative linear differential behaviours.*

Proof. See Appendix A.3 □

Consider now the following proposition regarding half-line dissipativity of SLDS. We consider the case when the *liveness condition* is satisfied (see [87], sec. IV-B), namely, given $\Phi \in \mathbb{R}^{w \times w}$ and $w = \text{col}(u, y) \in \mathfrak{B}^\Sigma$, the number of components in the input u , denoted by $\mathfrak{m}(\mathfrak{B}^\Sigma)$, equals the number of positive eigenvalues of Φ , denoted by $\sigma_+(\Phi)$.

Theorem 6.12. *Let Σ be a SLDS and let $\Phi \in \mathbb{R}^{w \times w}$. Assume that $\sigma_+(\Phi) = \mathfrak{m}(\mathfrak{B}^\Sigma)$. If there exists a multiple storage function as in Def. 6.4, then Σ is half-line Φ -dissipative.*

Proof. See Appendix A.3. □

6.5 Computation of multiple storage functions

In this section, we develop procedures based on LMIs to compute multiple storage functions. The following theorem follows from the results in Prop. 3.21, Prop. 3.22, and Lemma 3.23.

Theorem 6.13. *Let $\Phi \in \mathbb{R}^{w \times w}$ and let Σ be a SLDS with \mathcal{G} well-defined and well-posed. Let $\mathfrak{B}_k := \text{im } M_k \left(\frac{d}{dt} \right)$, with $M_k \in \mathbb{R}^{w \times z}[\xi]$, $k = 1, \dots, N$, be strictly Φ -dissipative. Let $X_k \in \mathbb{R}^{n(\mathfrak{B}_k) \times z}[\xi]$ be a minimal state map for \mathfrak{B}_k , $i = 1, \dots, N$, and let $L_{i \rightarrow j} \in \mathbb{R}^{n(\mathfrak{B}_j) \times n(\mathfrak{B}_i)}$ for all $i, j \in \mathcal{P}$, $i \neq j$, be the re-initialisations maps of Σ . Denote the coefficient matrix of $M_k(\xi)$ by*

$$\widetilde{M}_k := \begin{bmatrix} M_{k,0} & \dots & M_{k,L_k} \end{bmatrix} ;$$

then that of $X_k(\xi)$ can be written as

$$\widetilde{X}_k := \begin{bmatrix} X_{k,0} & \dots & X_{k,L_k-1} \end{bmatrix} .$$

There exist $K_k = K_k^\top \in \mathbb{R}^{n(\mathfrak{B}_k) \times n(\mathfrak{B}_k)}$, $k = 1, \dots, N$, such that

$$\widetilde{M}_k^\top \Phi \widetilde{M}_k - \begin{bmatrix} 0_{z \times n(\mathfrak{B}_k)} \\ \widetilde{X}_k^\top \end{bmatrix} K_k \begin{bmatrix} \widetilde{X}_k & 0_{n(\mathfrak{B}_k) \times z} \end{bmatrix} - \begin{bmatrix} \widetilde{X}_k^\top \\ 0_{z \times n(\mathfrak{B}_k)} \end{bmatrix} K_k \begin{bmatrix} 0_{n(\mathfrak{B}_k) \times z} & \widetilde{X}_k \end{bmatrix} \geq 0 . \quad (6.2)$$

Moreover, if for $k, j = 1, \dots, N$, $k \neq j$, it holds that

$$K_k - L_{k \rightarrow j}^\top K_j L_{k \rightarrow j} \geq 0 , \quad (6.3)$$

then $(\Psi_k(\zeta, \eta) := X_k(\zeta)^\top K_k X_k(\eta))_{k=1, \dots, N}$ induces a multiple storage function for Σ , and Σ is Φ -dissipative.

Proof. See Appendix A.3. □

Theorem 6.13 reduces the computation of multiple storage functions to the solution of a system of LMIs (6.2)-(6.3), a straightforward matter for standard LMI solvers.

Example 6.3. Consider the switched electrical circuit in Fig. 6.1. The switching occurs when at an arbitrary instant of time, the inductor L_2 is connected. We select $w := \text{col}(V, i_1)$ as the external variables. For simplicity we consider $C_1 = 1 F$, $L_1 = 1 H$, and $L_2 = 1 H$.

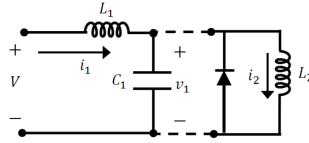


Figure 6.1: Switched electrical circuit with two modes

- *Mode behaviours:* The (controllable) mode behaviours \mathfrak{B}_i , $i = 1, 2$, are described by the observable image representations $w = M_i \left(\frac{d}{dt} \right) z_i$, $i = 1, 2$, with

$$M_1 \left(\frac{d}{dt} \right) := \begin{bmatrix} \frac{d^2}{dt^2} + 1 \\ \frac{d}{dt} \end{bmatrix}, \quad M_2 \left(\frac{d}{dt} \right) := \begin{bmatrix} \frac{d^3}{dt^3} + 2\frac{d}{dt} \\ \frac{d^2}{dt^2} + 1 \end{bmatrix};$$

and $z_1 := v_1$, $z_2 := i_2$.

- *Gluing conditions:* We consider the state maps acting on the latent variables induced by

$$X_1(\xi) := \begin{bmatrix} 1 & \xi \end{bmatrix}^\top, \quad X_2(\xi) := \begin{bmatrix} \xi & \xi^2 & 1 \end{bmatrix}^\top.$$

The physics of the circuit imposes that for every $t_k \in \mathbb{T}_s$, the gluing conditions can be expressed as $X_2 \left(\frac{d}{dt} \right) i_2(t_k^+) = L_{1 \rightarrow 2} X_1 \left(\frac{d}{dt} \right) v_1(t_k^-)$ and $X_1 \left(\frac{d}{dt} \right) v_1(t_k^+) = L_{2 \rightarrow 1} X_2 \left(\frac{d}{dt} \right) i_2(t_k^-)$, where

$$L_{1 \rightarrow 2}^\top := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =: L_{2 \rightarrow 1}.$$

- *LMI conditions:* Define

$$\Phi := \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

corresponding to the supply rate $Q_\Phi(w) = Vi$. Based on Th. 6.13, we construct the LMIs (6.2) and (6.3) for this case. Then using standard LMI solvers we obtain

$$K_1 := \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}; \quad K_2 := \begin{bmatrix} 0.500 & 0 & 0 \\ 0 & 0.500 & 0 \\ 0 & 0 & 0.577 \end{bmatrix}.$$

Thus, $(X_i(\zeta)^\top K_i X_i(\eta))_{i=1,2}$ induces a multiple storage function for the SLDS. Note that the system is thus dissipative according to Prop. 6.6 and in fact half-line dissipative according to Th. 6.12, since $\sigma_+(\Phi) = \mathfrak{m}(\mathfrak{B}^\Sigma) = 1$. \square

The result of Th. 6.13 permits to deduce a further result regarding the computation of multiple storage functions. If the LMIs (6.2)-(6.3) are feasible, then they may have more than one solution; i.e. several values of the “optimisation variable” K_i , $i = 1, \dots, N$, may satisfy the same constraints (see [2]), concluding that a multiple storage function is not necessarily unique. Moreover, the set of all possible multiple storage functions is a convex set.

Proposition 6.14. *Let Σ be a Φ -dissipative SLDS. Let the N -tuples $Q_\Psi := (Q_{\Psi_1}, \dots, Q_{\Psi_N})$ and $Q_{\Psi'} := (Q_{\Psi'_1}, \dots, Q_{\Psi'_N})$ be multiple storage functions for Σ . Then, for all $0 \leq \alpha \leq 1$, the N -tuple*

$$\alpha Q_\Psi + (1 - \alpha) Q_{\Psi'} ,$$

is a multiple storage function for Σ .

Proof. See Appendix A.3. □

6.6 Passivity

In this section we study passive SLDS, i.e. SLDS which are dissipative with respect to the positive-real supply rate

$$\Phi := \frac{1}{2} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} .$$

In particular, we are interested in determining under which circumstances a multiple storage function is also a *multiple Lyapunov function* as in Th. 5.1 for switched autonomous behaviours.

In a dissipative SLDS the external variables include inputs, consequently the modes are not autonomous. However, we can associate to Σ an autonomous SLDS as follows.

Definition 6.15. Let $\Sigma := \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$ be a SLDS with switched behaviour \mathfrak{B}^Σ and $w = \text{col}(u, y)$. The *unforced SLDS* Σ_{aut} associated to Σ is defined as $\Sigma_{\text{aut}} := \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$, with switched behaviour

$$\mathfrak{B}_{\text{aut}}^\Sigma := \{w = \text{col}(u, y) \in \mathfrak{B}^\Sigma \mid u = 0\} .$$

Note that $\mathfrak{B}_{\text{aut}}^\Sigma$ is not empty, since it contains at least the zero trajectory $w = 0$. The following proposition deals with asymptotic stability of unforced SLDS as in Def. 6.15.

Theorem 6.16. *Let Σ and Σ_{aut} be as in Def. 6.15 and let $\Phi := \frac{1}{2} \begin{bmatrix} 0 & I_z \\ I_z & 0 \end{bmatrix}$. Assume that Σ is strictly Φ -dissipative, then Σ_{aut} is asymptotically stable.*

Proof. See Appendix A. □

6.7 Summary

We developed a theory of dissipativity for switched systems in which the dynamical modes are not described in state space form, and do not necessarily share a common state space. We provided necessary and sufficient conditions for the existence of multiple storage functions, and a method to compute them using sets of LMIs. We studied the notion of passivity as a special case, as well as its relationship with stability.

Chapter 7

An SLDS approach to energy distribution networks

In recent years, the development of a new paradigm of energy generation and distribution systems has become a pressing research question. Issues such as the urge to reduce CO₂ emissions, the compelling advantages of renewable energy generation and the undesirable power losses in complex transmission lines, have motivated the development of distributed energy generation systems based on renewable energies [90]. However, the intermittent nature of renewable energies is reflected in the characteristics of the voltages/currents (e.g. amplitude and frequency) provided by transducers, prompting to regulate such variables to satisfy the nominal requirements of the the loads. In order to achieve voltage/current/frequency regulation and distribution of electricity, interconnections of power converters are implemented; however, their interaction can display unstable behaviors (see [61, 89, 98]). A common example of this issue is the *negative impedance instability* produced by current/voltage controlled converters behaving as *constant power loads* (see [57]).

In this chapter we discuss the modelling and analysis of energy distribution networks consisting of interconnections of DC-DC switching power converters and multiple (dis-) connectable loads, see [61]). We start by reviewing the classical modelling framework of power converters, and we show that such approach is not general enough to be applied to the analysis of new emerging topologies; consequently, a paradigm-shift is required.

We demonstrate that the SLDS framework is suitable to accommodate the models of switching power converters and energy distribution systems. Moreover we show that the results provided in Chap. 6, regarding dissipative SLDS, can be *directly* applied to solve the problem of negative impedance stability in energy distribution networks. In particular, we introduce a systematic method to design stabilising filters in terms of bilinear- and linear- matrix inequalities that can be easily constructed from higher-order models obtained from first principles.

7.1 Traditional approach to DC-DC converters

In this thesis we have exemplified the modelling of DC-DC switching power converters, see e.g. Sec. 1.1.2 and Sec. 4.5. In such examples we adopted the switched systems perspective to model the constitutive dynamic modes of the converters separately. There exist however a different modelling approach that is widespread in the power electronics literature called *state space averaging* (see [10]), which has been widely adopted in the literature as the starting point for analysis and control (see [66]). In this section, we discuss the limitations of such approach and explain how the SLDS framework provides more general tools that can be used in the analysis of more challenging situations and applications. For ease of exposition, we concentrate our analysis on bimodal systems.

The piecewise linear dynamics of power electronics devices with two dynamic modes, constant inputs, and ideal switches, are usually modelled using the following structure (see [66])

$$\frac{d}{dt}x = A_u x + B_u ; \quad u = 0, 1 ; \quad (7.1)$$

where $x(t) \in \mathbb{R}^n$ is called the state function; $A_u \in \mathbb{R}^{n \times n}$, $B_u \in \mathbb{R}^{n \times 1}$ are the matrices that describe the physical laws of the dynamic modes. A switching signal $s : \mathbb{R} \rightarrow \{1, 0\}$ determines the value of u , which is a binary index term that denotes which of the two modes is active due to the position of physical switches such as diodes and transistors.

Now consider a traditional *pulse width modulation*, which is based on a periodical switching signal defined as

$$s(t) := \begin{cases} 0, & t_k \leq t < t_k + DT; \\ 1, & t_k + DT \leq t < t_k + T. \end{cases} \quad (7.2)$$

with $t_0 := -\infty$, $t_{k+1} = t_k + T$, $k = 0, 1, 2, \dots$; and where $D \in [0, 1]$ is called *duty cycle* and T is the *switching period*.

As discussed in [10], we can *approximate* the dynamics of the switched linear system into “averaged” ones by involving the *duty cycle* D in the description of the system (7.1). This action is equivalent to approximate a switched linear system into a “unified” bilinear one where the so-called current and voltage ripples of the converter are neglected. The *averaging technique* allows us to obtain the following structure

$$\frac{d}{dt}x_{av} = [DA_0 + (1 - D)A_1] x_{av} + DB_0 + (1 - D)B_1 ; \quad (7.3)$$

where $x_{av}(t) \in \mathbb{R}^n$ is an *averaged state function*.

Such approximation is justified in the following lemma where it is shown that trajectories of the original state space switched system with switching signal (7.2), and its averaged approximation remain close to each other.

Lemma 7.1. *Consider the state space switched system in (7.1) with switching signal (7.2). Define the average state space system (7.3). Then there exists $\varepsilon > 0$ such that for every compact time interval it holds that*

$$\|x(t) - x_{\text{av}}(t)\| \leq \varepsilon T .$$

Proof. See Sec. 2.8 of [68]. □

The structure (7.3) is usually the starting point for the dynamic analysis and control of DC-DC converters, since it can be derived almost directly from the switched linear system structure (7.1), and it allows to apply a wide number of nonlinear control techniques (see for instance the compendium of controllers in [66]).

Remark 7.2. The piecewise linear modelling in the traditional approach to power converters is a special case in the SLDS framework, where the mode behaviours are $\mathfrak{B}_i := \{x \mid \frac{d}{dt}x = A_i x + B_i\}$, $i = 1, 2$; and the gluing conditions are such that $x(t_k^+) = x(t_k^-)$ for every $t_k \in \mathbb{T}_s$. Moreover, note that (7.2) is a very specific admissible switching signal in \mathcal{S} , which means that only a subset of the switched behaviour \mathfrak{B}^Σ in Def. 4.2 is usually taken into account. □

Unfortunately, though useful in many instances, the traditional averaging technique have some significant disadvantages. Some of them are inherited from the original switched state space structure on which it is rested (see Sec. 1.2). Other relevant limitations of this approach are the strong assumptions that cannot be always satisfied in practice.

We now enlist the main assumptions of the traditional averaging technique, and we discuss the resulting difficulties and limitations.

- 1) *First-order models.* The requirement of a this type of models undermines the possibility of using advantageous techniques such as the modelling of loads as impedances, that leads naturally to higher-order descriptions.
- 2) *Global state space.* As discussed in Sec. 1.2, this approach scores low in parsimony and modularity and is unsuitable for complex cases.
- 3) *PWM switching signal.* The averaging approximation demands the switching between dynamic modes to be orchestrated by the switching signal (7.2). However, as shown in Sec. 5.2, in practice dynamic modes are not only induced by the position of physical switches, but also by arbitrary (dis-)connection of loads.
- 4) *Continuous trajectories and switching instants.* The averaging technique cannot be applied in general if the trajectories of the state are discontinuous at switching instants (see the counter-example in [27]).

In the following sections we show that there exist modern implementations where none of the previous assumptions are satisfied, and consequently the traditional averaging technique cannot be used to study important characteristics in some energy distribution networks such as stability. Note also that in the SLDS framework these assumptions are not necessary, i.e. 1) systems with higher-order models are allowed, 2) dynamic models can be associated to different state spaces, 3) arbitrary switching signals are permitted; and 4) gluing conditions may imply discontinuities at switching instants.

Remark 7.3. The development of less restrictive averaging techniques is still an open research direction, see e.g. [26] and [56], where some additional special cases have been presented. \square

In order to show the generality of our modelling framework, we now study a family of power converters with discontinuous trajectories, whose dynamics can be easily modelled as an SLDS.

7.2 Switched-capacitor DC-DC converters

DC-DC power converters with switched-capacitors exhibit highly desirable features in energy distribution systems such as high-voltage gains, high-efficiency, and transformerless profiles, (see [28]). However, their analysis is severely limited by the way in which they are currently modelled, since the resulting mathematical descriptions have not been directly linked to advanced control algorithms and stability analysis, as in the case of other traditional topologies (see e.g. [66]).

In switched-capacitor converters parallel connections between capacitors induce discontinuities on the voltages across them at switching instants, demanding the introduction of non-trivial concatenability conditions in their modelling. In order to avoid to deal with such discontinuous behavior, models with non-ideal switches involving parasitic resistances have been proposed (see e.g. [12]); however, they involve highly-nonlinear equations for which currently unavailable dynamic analysis tools are required to solve control and stability problems. Consequently, we propose to study this type of converters using the SLDS framework, since it not only provides a suitable and general modelling approach, but also the application of the mathematical tools for analysis that have been developed in the previous sections.

In order to illustrate the modelling procedure, we consider the *Two-phase Fibonacci SC converter* depicted in Fig. 7.1, corresponding to a simplified version¹ of the SC converter in Fig. 1(b) of [70]. The converter in Fig. 7.1(a) has two possible modes depending on the position of the group of switches “1” and “2” illustrated by blocks and

¹For ease of exposition, the third “Fibonacci cell” of the SC converter in Fig. 1(b) of [70] has been omitted, and a parallel RC load is considered.

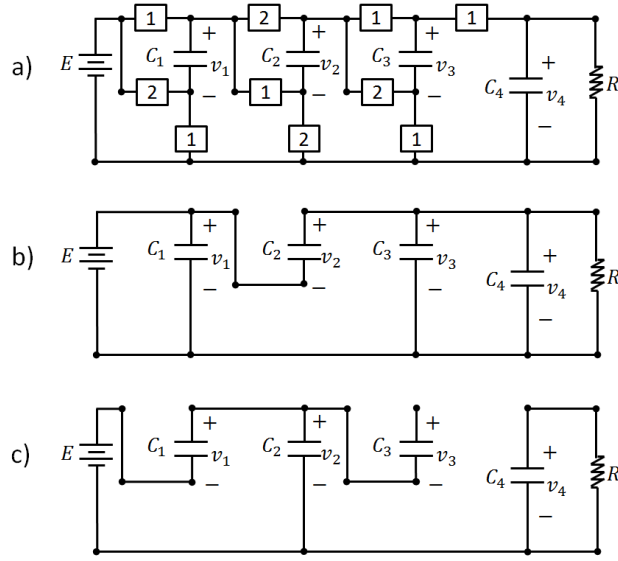


Figure 7.1: Fibonacci switched-capacitor converter.

whose operation is complementary. Fig. 7.1(b) and Fig. 7.1(c) show the two possible equivalent circuits of the converter.

We define $w := \text{col}(E, v_1, v_2, v_3, v_4)$ as the set of variables of interest. Now consider the case in Fig. 7.1(b), we obtain the following mode behaviours $\mathfrak{B}_i = \ker R_i \left(\frac{d}{dt} \right)$, $i = 1, 2$, with

$$R_1(\xi) := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & C_2\xi & 0 & (C_3 + C_4)\xi + \frac{1}{R} \end{bmatrix},$$

$$R_2(\xi) := \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & C_1\xi & C_2\xi & 0 & 0 \\ 0 & 0 & 0 & C_3\xi & 0 \\ 0 & 0 & 0 & 0 & C_4\xi + \frac{1}{R} \end{bmatrix}.$$

Moreover, the gluing conditions are not trivial, since the voltages across capacitors are discontinuous and need to be uniquely specified at switching instants. In order to model them, we consider the principle of conservation of charge [47], and the algebraic constraints imposed by the active modes. When we switch from \mathfrak{B}_2 to \mathfrak{B}_1 , the total charge in the capacitors that exhibits a redistribution due to parallel connections must be the same before and after every switching instant t_k , i.e.

$$C_2v_2(t_k^-) + C_3v_3(t_k^-) + C_4v_4(t_k^-) = C_2v_2(t_k^+) + C_3v_3(t_k^+) + C_4v_4(t_k^+).$$

Additionally the algebraic constraints, due to parallel connections among capacitors, when \mathfrak{B}_1 is active dictate that

$$\begin{aligned} v_1(t_k^+) &= E(t_k^+) , \\ v_2(t_k^+) &= v_3(t_k^+) - v_1(t_k^+) , \\ v_3(t_k^+) &= v_4(t_k^+) . \end{aligned}$$

Similarly, when we switch from \mathfrak{B}_1 to \mathfrak{B}_2 , the physical redistribution of charge establishes that for every switching instant t_s we have that

$$\begin{aligned} C_1 v_1(t_s^-) + C_2 v_2(t_s^-) &= C_1 v_1(t_s^+) + C_2 v_2(t_s^+) , \\ C_3 v_3(t_k^-) &= C_3 v_3(t_k^+) , \\ C_4 v_4(t_k^-) &= C_4 v_4(t_k^+) . \end{aligned}$$

Moreover, note that \mathfrak{B}_2 imposes the algebraic constraint

$$v_2(t_s^+) - v_1(t_s^+) = E(t_s^+) .$$

We have shown that the modelling of power converters with discontinuous trajectories can be accommodated as a special case in our framework. In the following section, we illustrate a general approach to model energy distribution networks where any type of switching power converter with controllable mode behaviours is considered.

7.3 Modelling of energy distribution networks

We now consider the modelling of energy distribution networks consisting of switching power converters and multiple (dis-) connectable loads as in Fig. 1.4. Once again, for ease of exposition we consider bimodal DC-DC converters whose dynamic modes are controllable, and consequently represented by

$$w = M_u \left(\frac{d}{dt} \right) z_u , \quad u = 0, 1 ; \quad (7.4)$$

where u is a binary index term that denotes the *on/off* operation of complementary ideal switches. Note that the auxiliary variables z_0 and z_1 , are not necessarily the same for each mode. In order to facilitate the illustration of our modelling approach we start with a simple case, where the auxiliary variables are the same for each mode and the voltages/currents of the converter are continuous at switching instants.

Example 7.1. Consider for instance the DC-DC boost converter in Fig. 7.2. We select the conjugate variables (input/output voltages and currents) as the variables of interest, i.e. $w = \text{col}(E, i_o, i_L, v)$.

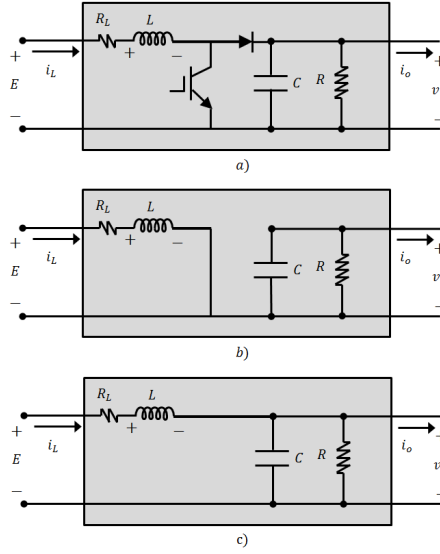


Figure 7.2: DC-DC boost converter and its electrical configurations

When the transistor is closed (see Fig. 7.2b), we obtain the following dynamic equations

$$\text{Mode 1} \begin{cases} E = R_L i_L + L \frac{d}{dt} i_L \\ i_o = -C \frac{d}{dt} v - \frac{1}{R} v \end{cases} ;$$

when the transistor is open (see Fig. 7.2c), we obtain

$$\text{Mode 0} \begin{cases} E = R_L i_L + L \frac{d}{dt} i_L + v \\ i_o = i_L - C \frac{d}{dt} v - \frac{1}{R} v \end{cases} .$$

In both cases we can select $z_1 = z_0 = \text{col}(i_L, v)$, then the mode dynamics of the boost converter can be written as in (7.4), where

$$M_1 \left(\frac{d}{dt} \right) := \begin{bmatrix} R_L + L \frac{d}{dt} & 0 \\ 0 & C \frac{d}{dt} + \frac{1}{R} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ;$$

$$M_0 \left(\frac{d}{dt} \right) := \begin{bmatrix} R_L + L \frac{d}{dt} & 1 \\ -1 & C \frac{d}{dt} + \frac{1}{R} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

We now consider a more challenging scenario, e.g. the energy distribution in Fig. 1.4. The calculus of impedances facilitates our analysis, for instance the energy distribution

network in Fig. 1.4 can be simplified by computing Z_{T_k} , $k = 1, \dots, L$, as

$$\frac{1}{Z_{T_k}(s)} = \frac{1}{Z_k(s)} + \frac{1}{Z_N(s)} + \frac{1}{Z_{CPL}(s)}; \quad k = 1, \dots, L.$$

The simplified network is depicted in Fig. 7.3.

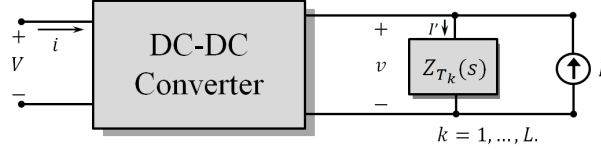


Figure 7.3: Simplification of the energy distribution network in Fig. 1.4

The dynamic modes arising from the the interconnection of the DC-DC converter and the multiple (dis-) connectable loads can be obtained by algebraic manipulations, preserving an image form representation.

Proposition 7.4. *Consider the energy distribution network in Fig. 1.4. Assume that the dynamical modes of the switching power converter can be described in image form by $w = M_j \left(\frac{d}{dt} \right) z_j$, where $M_j \in \mathbb{R}^{4 \times 2}[s]$; $z_j = \text{col}(z_{1,j}, z_{2,j}) \in \mathfrak{C}_p^\infty(\mathbb{R}, \mathbb{R}^2)$; $j = 1, 2$; and $w := \begin{bmatrix} V & I & i & v \end{bmatrix}^\top$. Let $z_k \in \mathfrak{C}_p^\infty(\mathbb{R}, \mathbb{R})$, $k = 1, \dots, L$; then there exist $\widehat{M}_{j,k} \in \mathbb{R}^{4 \times 2}[s]$ such that the mode behaviors can be described by image representations*

$$\begin{bmatrix} V \\ I \\ i \\ v \end{bmatrix} = \widehat{M}_{j,k} \left(\frac{d}{dt} \right) \begin{bmatrix} z_{1,j} \\ z'_k \end{bmatrix}, \quad (7.5)$$

with $j = 1, 2$, and $k = 1, \dots, L$.

Proof. See App. A.4 □

We now discuss the properties of well-definedness and well-posedness of gluing conditions (see Sec. 4.5.2).

Proposition 7.5. *Assume that switching among the modes described by (7.5) does not involve short-circuiting of voltage sources, or open-circuiting of current sources. Then the gluing conditions are well-defined.*

Proof. See Appendix A.4. □

Note that the requirement in Prop. 7.5 is reasonable from a practical point of view, since constraints on the inputs such as short-circuiting are not desirable in practice.

Well-posed gluing conditions (see Def. 4.6) guarantee that after a switching instant only one initial state for the new dynamical regime is specified from the final state of the previous one. Such property holds since the switching cannot cause any increase in the total amount of charge or flux stored in the system and thus the value of the state after the switch is uniquely determined by the state before the switch, on this issue see [14, 47]. In the rest of this chapter we assume that *the gluing conditions are well-posed*.

Example 7.2. Consider the energy distribution network in Fig. 7.3, where the DC-DC converter is that of Fig. 4.4. Let $n_k, d_k \in \mathbb{R}[s]$, $k = 1, \dots, L$, define $Z_{T_k}(s) := \frac{n_k(s)}{d_k(s)}$, $k = 1, \dots, L$. The mode dynamics with $w := \text{col}(E, I, i_L, v)$ are described by $w = M_{j,k} \left(\frac{d}{dt} \right) z_k$, where $z_1 := \text{col}(i_1, z'_k)$, $z_2 := \text{col}(v_1, z'_k)$, $k = 1, \dots, L$, and $j = 1, 2$.

$$M_{1,k} \left(\frac{d}{dt} \right) := \begin{bmatrix} R_L + L \frac{d}{dt} & 0 \\ 0 & d_k \left(\frac{d}{dt} \right) + (C_1 + C_2) \frac{d}{dt} n_k \left(\frac{d}{dt} \right) \\ 1 & 0 \\ 0 & n_k \left(\frac{d}{dt} \right) \end{bmatrix};$$

$$M_{2,k} \left(\frac{d}{dt} \right) := \begin{bmatrix} LC_1 \frac{d^2}{dt^2} + R_L C_1 \frac{d}{dt} + 1 & 0 \\ 0 & d_k \left(\frac{d}{dt} \right) + C_2 \frac{d}{dt} n_k \left(\frac{d}{dt} \right) \\ C_1 \frac{d}{dt} & 0 \\ 0 & n_k \left(\frac{d}{dt} \right) \end{bmatrix};$$

with $k = 1, \dots, L$. The gluing conditions can be obtained by defining the impedances Z_{T_k} , $k = 1, \dots, L$ and following the procedure exemplified in Sec. 7.2. \square

Remark 7.6. As illustrated in Ex. 7.2, each mode can be modelled independently, i.e. we compute the laws of each two-port network that depends on the mode of operation of the converter and the model of the switched impedance Z_{T_k} , $1, \dots, L$. It can be easily verified that the McMillan degree of each mode behaviour is not fixed and depends on the degree of the denominator of Z_{T_k} , $1, \dots, L$. However each mode exhibits only the required level of complexity to describe each dynamic mode. This is in sharp contrast with the traditional approach where the dynamic modes are represented by $\frac{d}{dt}x = A_i x$, with $A_i \in \mathbb{R}^{n \times n}$, i.e. considering a global state space and where n is the highest possible McMillan degree. The latter approach results in more complex dynamic models (with more variables and more equations), which has an impact also on the complexity of stability analysis, simulation, control, etc. Moreover, there is no compelling reason to resort to such non-parsimonious approach if we can study the dynamic properties of the network directly in higher-order terms, as shown in the following section.

7.4 Stabilization by passive damping

In energy distribution networks the interconnected subsystems (e.g. generators, switching power converters, loads, etc.) are designed separately and their stability is thus

assured by design. However, their interaction in the overall system can result in unstable behaviours (see [61, 89]). In some cases, such interactions can be accurately modelled assuming that some components behave as *constant power loads*, leading to *negative-impedance instability* [9].

To deal with instability of energy distribution networks we use *passive damping* (see e.g. [4]), where a passive load (filter) is interconnected to the system in order to guarantee stability.

We consider the case where the energy distribution network is unstable due to the presence of constant power loads (see [57]). We proceed to design a filter that guarantees stability when interconnected to the converter, see Fig. 7.4.

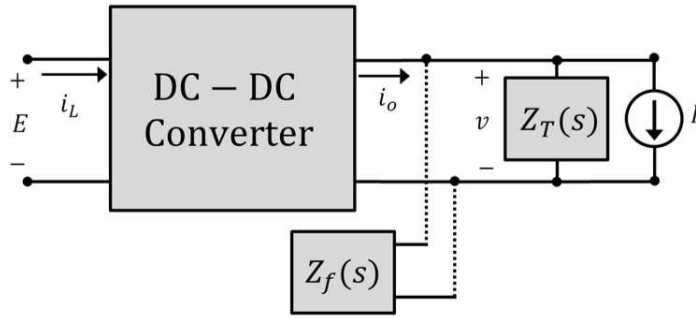


Figure 7.4: Energy distribution network with a stabilising filter

We consider now consider the impedance $Z_T(s)$ and the filter as an additional load in the array depicted in Fig. 7.4. The impedance function of the filter is given by

$$Z_f(s) = \frac{p(s)}{q(s)} ; \quad (7.6)$$

with an associated image form representation

$$\begin{bmatrix} i_f \\ v \end{bmatrix} = \begin{bmatrix} p(\frac{d}{dt}) \\ q(\frac{d}{dt}) \end{bmatrix} z' , \quad (7.7)$$

whose parameters need be computed. The interconnection of impedances (7.6) and $Z_T(s)$ in Fig. 7.4 yields

$$Z_{int}(s) := \frac{Z_T(s)Z_f(s)}{Z_T(s) + Z_f(s)} = \frac{n(s)}{d(s)} . \quad (7.8)$$

The first step in our procedure is to obtain image representations $w = M_k(\frac{d}{dt})z_k$, $i = 1, \dots, N$, describing each mode as in Prop. 7.4, and exemplified in Ex. 7.2. Similarly, we model the corresponding gluing conditions and compute re-initialisation maps as in Def. 4.6.

The second step in our procedure is the setting up of a system of matrix inequalities corresponding to the conditions of Th. 6.13. To make explicit the linear dependence on the parameters of Z_{int} , in the following we write $M_k(s)$ and their corresponding state maps $X_k(s)$ respectively as $M_{k,\tilde{n},\tilde{d}}(s)$ and $X_{k,\tilde{n},\tilde{d}}(s)$, where \tilde{n}, \tilde{d} are the coefficient matrices of the numerator and denominator of Z_{int} , that also involve the coefficients of the passive filter:

$$\begin{aligned} & \widetilde{M}_{k,\tilde{n},\tilde{d}}^\top \Phi \widetilde{M}_{k,\tilde{n},\tilde{d}} - \begin{bmatrix} 0_{m \times n(\mathfrak{B}_k)} \\ \widetilde{X}_{k,\tilde{n},\tilde{d}}^\top \end{bmatrix} K_k \begin{bmatrix} \widetilde{X}_{k,\tilde{n},\tilde{d}} & 0_{n(\mathfrak{B}_k) \times m} \end{bmatrix} \\ & - \begin{bmatrix} \widetilde{X}_{k,\tilde{n},\tilde{d}}^\top \\ 0_{m \times n(\mathfrak{B}_k)} \end{bmatrix} K_k \begin{bmatrix} 0_{n(\mathfrak{B}_k) \times m} & \widetilde{X}_{k,\tilde{n},\tilde{d}} \end{bmatrix} \geq 0, \quad k = 1, \dots, N, \\ & K_k - L_{k \rightarrow j}^\top K_j L_{k \rightarrow j} \geq 0, \quad k, j = 1, \dots, N, \quad k \neq j. \end{aligned} \quad (7.9)$$

The third step is to formalise the requirement that the filter is passive. Define

$$\Phi' := \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M'(s) := \begin{bmatrix} p(s) \\ q(s) \end{bmatrix}, \quad X'(s) := \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\deg(p)-1} \end{bmatrix}, \quad (7.10)$$

and denote the coefficient matrices of M' and X' by $\widetilde{M}'_{\tilde{p},\tilde{q}}$ and \widetilde{X}' , respectively. With these positions, it follows from the positive-real lemma that $\frac{q}{p}$ is positive-real if and only if there exists $K' = K'^\top \in \mathbb{R}^{\deg(p) \times \deg(p)}$ such that

$$\begin{aligned} & \widetilde{M}'_{\tilde{p},\tilde{q}}{}^\top \Phi' \widetilde{M}'_{\tilde{p},\tilde{q}} - \begin{bmatrix} 0_{1 \times \deg(p)} \\ \widetilde{X}'^\top \end{bmatrix} K' \begin{bmatrix} \widetilde{X}' & 0_{\deg(p) \times 1} \end{bmatrix} \\ & - \begin{bmatrix} \widetilde{X}'^\top \\ 0_{1 \times \deg(p)} \end{bmatrix} K' \begin{bmatrix} 0_{\deg(p) \times 1} & \widetilde{X}' \end{bmatrix} \geq 0. \end{aligned} \quad (7.11)$$

If values of the parameters \tilde{p} and \tilde{q} exist such that the matrix inequalities (7.9),(7.11) are satisfied for some K_k , $k = 1, \dots, N$ and K' , then the interconnection of Fig. 7.4 is passive, and consequently i/o stable. Moreover, the filter $\frac{q}{p}$ can be implemented using only resistors, capacitors, inductors and transformers (see [48]).

Remark 7.7. The McMillan degree n of the stabilizing passive filter (assuming it exists) is not known a priori. To start our procedure, some value of n should be decided upon, and a solution attempted to the matrix inequalities (7.9),(7.11). If no solution exists for the current value of n , the latter should be increased, and the procedure repeated. \square

Note that inequalities (7.9) and (7.11) are *bilinear* in the coefficients of the polynomials p and q of the passive filter. However, sub-optimal solution of the system of bilinear matrix inequalities (BMIs) can be found as shown in the following example.

7.5 Example: High-voltage DC-DC converter

We consider the implementation in Fig 7.5, with $R_L = 0.1\Omega$; $L = 880\mu H$; $C_1 = C_2 = 220\mu F$; $R = 500\Omega$. According to (7.6) we define the impedance of the filter $Z_f(s) := \frac{p(s)}{q(s)}$ with $p(s) = a_0s + a_1$ and $q(s) = 1$, for which the a -parameters will be computed.

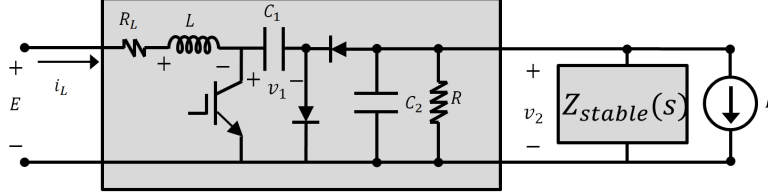


Figure 7.5: DC-DC converter with a passive filter and a constant power load.

We consider the total impedance as a constant power load, i.e. $Z_T(s) = -R_{CP}$ with $-R_{CP} = -300\Omega$. Considering (7.8), we obtain $n(s) = 300(a_0 + a_1s)$ and $d(s) = 300 - a_0 - a_1s$. We thus substitute $n(\frac{d}{dt})$ and $d(\frac{d}{dt})$ in the dynamic models computed in Ex. 7.2. Define state maps for each dynamical mode acting respectively on the latent variables z_1 and z_2 as

$$X_1\left(\frac{d}{dt}\right) := \begin{bmatrix} 1 & 0 \\ 0 & n\left(\frac{d}{dt}\right) \\ 0 & d\left(\frac{d}{dt}\right) \end{bmatrix}, \quad X_2 := \begin{bmatrix} C_1 \frac{d}{dt} & 0 \\ 1 & 0 \\ 0 & n\left(\frac{d}{dt}\right) \end{bmatrix},$$

then for every $t_k \in \mathbb{T}_s$, the gluing conditions can be expressed as $X_2\left(\frac{d}{dt}\right)z_2(t_k^+) = L_{1 \rightarrow 2}X_1\left(\frac{d}{dt}\right)z_1(t_k^-)$ and $X_1\left(\frac{d}{dt}\right)z_1(t_k^+) = L_{2 \rightarrow 1}X_2\left(\frac{d}{dt}\right)z_2(t_k^-)$, where

$$L_{1 \rightarrow 2} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_{2 \rightarrow 1} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C_1}{C_1+C_2} & \frac{C_2}{C_1+C_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now solve simultaneously the bilinear matrix inequalities (7.9) and (7.11) using standard solvers such as Yalmip. We thus obtain a solution $a_0 = 377$, $a_1 = 293 \times 10^{-6}$, $b_2 = 377$. Finally, the realization of the filter with impedance $Z_f(s) = 293 \times 10^{-6}s + 377$ is shown in Fig. 7.6.

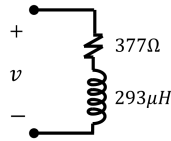


Figure 7.6: Realisation of the stabilising filter.

Remark 7.8. A common assumption in the known approaches to the constant load problem is to model the destabilizing load as a negative resistance (see e.g. [4]). In our methodology we do not restrict the class of load models; for example we admit frequency-domain models of as differences of squares ($P(-s)^\top P(s) - N(-s)^\top N(s)$). This may lead to less conservative solutions of the constant load problem, for example leading to the design of filters that induce a smaller power loss than those arrived at by conventional means. \square

In some special cases they can be made linear by fixing the values of certain parameters, for example by deciding a priori what the poles/zeros of the filter should be. We illustrate this feature in the following section.

7.6 Example: DC-DC Boost converter

Consider the DC-DC power converter in Fig. 7.7 which is interconnected to a constant power load.

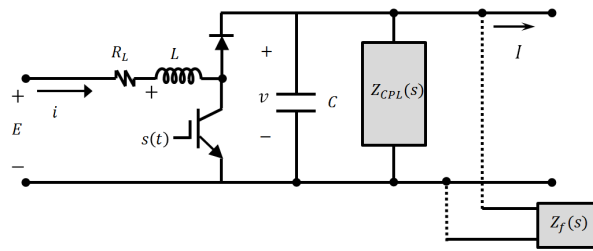


Figure 7.7: DC-DC converter with a stabilising filter

The filter is a 1-port impedance function $Z(s) = \frac{q(s)}{p(s)}$, with $p, q \in \mathbb{R}[s]$, or equivalently an image representation

$$\begin{bmatrix} v \\ i_f \end{bmatrix} = \underbrace{\begin{bmatrix} p\left(\frac{d}{dt}\right) \\ q\left(\frac{d}{dt}\right) \end{bmatrix}}_{=: M'\left(\frac{d}{dt}\right)} z' ,$$

where z' is a latent variable associated to the internal dynamics of the filter and whose physical meaning can be determined after the realisation. Similarly, the dynamics of the negative impedance are $Z_{CPL}(s) = -R$. Applying fundamental current and voltage laws, the mode dynamics when the transistor is closed and the diode open are described by

$$\begin{aligned} V &= L \frac{d}{dt} i_1 + R_L i_1 , \\ I &= C \frac{d}{dt} v - \frac{1}{R} v + q\left(\frac{d}{dt}\right) z' . \end{aligned}$$

When the transistor is open and the diode is closed, the mode dynamics are described by

$$\begin{aligned} V &= L \frac{d}{dt} i_1 + R_L i_1 + v, \\ I &= -i_1 + C \frac{d}{dt} v - \frac{1}{R} v + q \left(\frac{d}{dt} \right) z'. \end{aligned}$$

By selecting the external and latent variables as $w := \begin{bmatrix} V & I & i_1 & v \end{bmatrix}^\top$ and $z := \begin{bmatrix} i_1 & z' \end{bmatrix}^\top$, the mode dynamics can be modelled using image form representations $w = M_k \left(\frac{d}{dt} \right) z$, $k = 1, 2$, where

$$\begin{aligned} M_1(s) &:= \begin{bmatrix} Ls + R_L & 0 \\ 0 & -\left(\frac{1}{R} - Cs\right)p(s) + q(s) \\ 1 & 0 \\ 0 & p(s) \end{bmatrix} \\ M_2(s) &:= \begin{bmatrix} Ls + R_L & p(s) \\ -1 & -\left(\frac{1}{R} - Cs\right)p(s) + q(s) \\ 1 & 0 \\ 0 & p(s) \end{bmatrix}. \end{aligned} \quad (7.12)$$

Note that as described in Prop. 7.4, the dependence of M_1 and M_2 on the unknown parameters of the filter is linear.

State maps for \mathfrak{B}_i , $i = 1, 2$ and the filter can be computed e.g. as

$$\begin{aligned} X_1(s) &:= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s & s^2 \end{bmatrix}^\top =: X_2(s), \\ X'(s) &:= \begin{bmatrix} 1 & s \end{bmatrix}^\top. \end{aligned} \quad (7.13)$$

In this particular case the matrices \widetilde{X}_i , $i = 1, 2$ (the coefficient matrices of state maps for the dynamical modes) and the coefficient matrices \widetilde{M}_i , $i = 1, 2$, do not depend on the unknown coefficients associated to the filter. It follows from the physics of the switched circuit that the re-initialisation maps are $L_{1 \rightarrow 2} = L_{2 \rightarrow 1} = I_4$.

It is straightforward to see that the matrix inequalities (7.9) and (7.11) associated with the models (7.12) and the choice of the state maps (7.13) are bilinear in the coefficients of $p(s)$ and $q(s)$. Given the particular structure of our problem, if we fix the coefficients of the polynomial p , linear matrix inequalities occur, depending only on the coefficients of $q(s)$. Fixing p is tantamount to fixing the poles of the filter; this is often a reasonable choice, since filter performance specifications such as time response, characteristic frequencies etc., are directly related to the position of the poles of the filter in the complex plane (see [88]).

Now assume that $R_L = 0.01\Omega$, $L = 500\mu H$, $C = 47\mu F$ and $-R = -300\Omega$. Define $p(s) := 7392000 + 87200s + 160s^2$, and let $p(s) := a_2s^2 + a_1s + a_0$. Using standard LMI solvers for such choice of q , the coefficients of the polynomial p resulting from the solution of the LMIs, yield $p(s) = 14000 + 270s + s^2$. Consequently the filter described by the impedance

$$Z(s) = \frac{7392000 + 87200s + 160s^2}{14000 + 270s + s^2}$$

stabilizes the interconnection. Note that such filter can be physically realised using any suitable method of passive circuit synthesis (see [88]); for example, Fig. 7.8 depicts a realization of $Z(s)$, with $R_1 = 160\Omega$, $R_2 = 287.75\Omega$, $R_3 = 80.24\Omega$, $C_1 = 22.72\mu F$ and $C_2 = 136.105\mu F$.

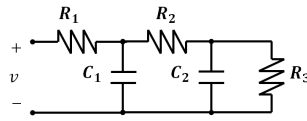


Figure 7.8: Filter realisation.

7.7 Summary

We introduced a modelling approach for energy distribution networks based on the switched linear differential framework. We also introduce a stabilisation method for switching power converters feeding potential destabilisers such as constant power loads.

Chapter 8

Conclusions and future work

In Chapter 4, 5 and 6 we introduced a new framework for the study of switched linear systems using concepts of behavioural system theory. In the following, we summarise the main contributions developed in this thesis.

Chapter 4:

- We proposed a trajectory-based approach to switched linear differential systems whose dynamic modes do not necessarily share the same state space. We also introduced the concept of gluing conditions, interpreted as physical equilibrium conditions that specify the concatenation of the trajectories of the dynamic modes at switching instants.
- Further analysis has been given for two main cases, namely, SLDS with mode behaviours that are 1) autonomous; and 2) controllable. We showed that in the autonomous case, *normal form* gluing conditions which are written in terms of the state, can be *always* derived from the general gluing conditions that act on the external variable. We also discuss that in the case of systems with inputs and outputs, gluing conditions can be written in terms of the state if they are *well-defined*, which means that the gluing conditions do not impose constraints to the inputs.
- We introduced the concept of well-posedn gluing conditions as those specifying a unique set of “initial conditions” for the external variable trajectory at switching instants. This property permits the computation of re-initialisation maps that can be interpreted as a linear map between the state spaces of the dynamic modes before and after switching instants.

Chapter 5:

- In Th. 5.1 we introduced a trajectory-based sufficient condition for asymptotic stability of SLDS. This condition is based on the existence of a *multiple higher-order Lyapunov function* described in terms of N -tuples of quadratic differential forms.
- In Th. 5.4 we introduced a set of LMIs whose solution, if it exists, leads directly to the computation of multiple higher-order Lyapunov functions. This set of LMIs can be set-up *directly* from the higher-order equations describing the laws of the dynamic modes. This result corroborates the intuition that state space representations are not a fundamental requirement in the analysis of switched systems, since methods such as the computation of functionals based on LMIs can be also performed directly for higher-order models and the gluing conditions.
- We studied the role of positive-realness in the SLDS setting for bimodal systems. In Th. 5.15, we showed that positive-realness is a sufficient condition for the existence of a multiple higher-order Lyapunov function, and consequently for asymptotic stability of a special class of SLDS (“standard” ones) whose modes have different state space dimension and special gluing conditions. In Th. 5.15 we show a different perspective when compared to existing results (e.g. [34, 64, 65]) on the relation between positive-realness and stability: the dynamical regimes do not arise from closing the loop around some fixed plant. In our case, positive-realness arises from the interplay of the mode dynamics, i.e. the construction of a rational matrix involving the two modes
- In Prop. 5.17 we provided an LMI version of Th. 5.15, that facilitates the stability test and permits the computation of a multiple Lyapunov function.
- We relaxed the positive-real condition by using the concept of *positive-real completions*. In Th. 5.24 we showed the fundamental role of positive-real completions in the study of stability of standard SLDS, since we proved that the existence of a positive-real completion is a *necessary* condition for the existence of a multiple Lyapunov function.
- In 5.25, we provided an LMI-based test for stability using positive-real completions. We exemplified the application of Prop. 5.17 in Ex. 5.7, where we study the bumpless transfer problem in a multi-controller system.
- In Theorem 5.27, we showed that the existence of a positive-real completion for a standard SLDS leads to the stability of a SLDS with three modes. Moreover, we proved that the Lyapunov function in Theorem 5.24 is also a Lyapunov function for the new SLDS with three behaviors.

- In Theorem 5.28, we showed the robustness of the results in Theorem 5.27 against perturbations on the positive-real completion; and characterized a family of asymptotically stable SLDS with three behaviors.

Chapter 6:

- In this chapter we developed a dissipativity framework for SLDS. This approach enables the study of the properties of SLDS in terms of energy as a general concept where the physical system energy is a special case. We introduced the definition a (strictly) dissipative SLDS, as well as the concept of *multiple higher-order storage function*.
- In Th. 6.6, we showed that the existence of a multiple storage function is a sufficient condition for dissipativity, and in Th. 6.7 we showed that is also necessary in the case of strictly dissipative SLDS. A special case of Th. 6.7 can be interpreted as an analogous result to the *converse Lyapunov theorem* (see Th. 2.2 of [34], p. 25), where asymptotic stability implies the existence of a *common Lyapunov function* for *closed* systems under analogous conditions.
- In Th. 6.13, we showed that multiple storage functions can be obtained by setting up standard LMIs *directly* from the linear differential equations describing the mode dynamics of dissipative SLDS, and the gluing conditions.
- In Th. 6.16, we showed the relationship between a special case of dissipativity, called passivity, and stability of SLDS. In particular, we showed that a multiple storage function for an open SLDS is a Lyapunov function for an unforced version of such system.
- We exemplified that our main results can be directly applied to study physical systems as in Ex. 6.3. In such example, we showed that the computation of multiple storage functions can be reduced to the straightforward set-up and computation of LMIs, where our technical results can be directly applied using a minimal amount of effort, notation and technical concepts.

Chapter 7:

- In this chapter we reviewed the traditional modelling approach to switching power converters, and we showed some of its limitations. We also showed that the dynamics of such power converters can be easily accommodated in the SLDS framework since the gluing conditions permit the modelling of the algebraic conditions imposed at switching instants.

- We also showed that our modelling approach can be also applied to the analysis of complex networks involving switched power converters and multiple (dis-) connectable loads.
- We showed that the dissipativity framework for SLDS in Chap. 6 can be used to solve the problem of negative impedance instability in energy distribution networks. In order to do so, we introduced a systematic method to design stabilising filters, based on the LMI conditions introduced in Th. 6.13.
- We exemplified our procedure by computing stabilising filters for a DC-DC boost converter and a high-voltage DC-DC converter. Our method can be regarded as a modification of the *output impedance* of the power converter (see Sec. 7 of [98]), in our case achieved by adding a passive filter to the output stage of the converter. Note also that our method coincides with the “physical interpretation” of the output impedance control based on a feedback scheme provided in Sec. 8 of [98], where an admittance-like function is associated with controller gains. \square

The results developed in this thesis contribute to some essential issue in the switched linear differential systems framework, that can be used to solve other relevant problems in switched systems. In the following, we point out some future research directions.

- *Structural properties of SLDS.* In this thesis we have study autonomy, controllability and observability as properties derived from the individual dynamic modes of SLDS. However, the fundamental question remains of how to define these properties in a global sense, i.e. in terms of the trajectories of the switched behaviour \mathfrak{B}^Σ . As typical of behavioural system theory, it is also necessary to derive algebraic characterisations of such properties. An example of an important consequence of such characterisation is the design of controllers and observers (see e.g. [69] and [97]).
- *Control.* The concept of control by interconnection in the behavioural setting (see [87]) can be set-up also for SLDS. Moreover, the dissipativity framework can play an important role in this problem since in the case of linear differential systems it has been proved in [58] that interconnecting a plant with a stabilising controller is equivalent to imposing the requirement of dissipativity to the plant with respect to a dynamic supply rate induced by the controller. Extension of these results to the case of SLDS are to be expected.
- *Differential variational inequalities.* We have studied the application of the SLDS framework to the study of energy distribution networks. However, complex switching power converters such as inverters, involve many switches, leading to a large number of mode dynamics. To model such systems as switched systems would be impractical and innatural, considering the combinatorial number of possible modes

and the fact that switching is state-dependent. This motivates the development of compact and natural modelling techniques for complex power systems components, using differential variational inequalities, which is a general higher-order case of the linear complementarity framework (see [14]).

- *Applications.* Further applications of dissipativity and control can be envisioned for the implementation of energy distribution networks, in particular in those problems where the power exchange plays a fundamental role. Examples of this situation are (see [66, 98]): power factor correction, maximum power point tracking, power balancing, active filtering, virtual impedance design, etc.
- *Other developments.* System identification, model order reduction, simulation, the study of SLDS with mode behaviours represented by hybrid, non-image representations are also pressing research directions in our framework.

Appendix A

Proofs

A.1 Proofs of Chapter 3

Proof of Prop. 3.10. The existence of \tilde{Y} and \tilde{Q} follow from the fact that YR^{-1} and QR^{-1} are strictly proper (due to the R -canonicity of Y and Q) and that the rows of $X(\xi)$ are a basis of the vector space over \mathbb{R} defined by $\{f \in \mathbb{R}^{1 \times w}[\xi] \mid fR^{-1} \text{ is strictly proper}\}$. Using this argument and the fact that Ψ is R -canonical, we also conclude that K exists. The claim that the highest degree present in X is less than that in R , follows from the strict properness of XR^{-1} and Lemma 6.3-10 of [29]. \square

Proof of Prop. 3.11. Define $S_L(\xi) := \begin{bmatrix} I_w & \xi I_w & \cdots & \xi^L I_w \end{bmatrix}$, the equivalence of statements 1) and 2) follows from the equalities

$$X(\xi) = \begin{bmatrix} \tilde{X} & 0_{n \times w} \end{bmatrix} S_L(\xi), \quad \xi X(\xi) = \begin{bmatrix} 0_{n \times w} & \tilde{X} \end{bmatrix} S_L(\xi), \quad \text{and} \quad R(\xi) = \tilde{R} S_L(\xi),$$

and Th. 3.9. \square

Proof of Prop. 3.21. To prove that the degree of $X(\xi)$ is less than the degree of $M(\xi)$, note that since YU^{-1} and XU^{-1} are strictly proper (see Sec. 2.6.2), we can apply Lemma 6.3-10 of [29] and conclude that the highest degree of each entry in X is less than the highest degree present in M . \square

Proof of Prop. 3.22. To prove the equivalence of statements 1) and 2) let us define $S_L(\xi) := \begin{bmatrix} I_z & \xi I_z & \cdots & \xi^L I_z \end{bmatrix}$. The equivalence follows from the equalities $X(\xi) = \begin{bmatrix} \tilde{X} & 0_{n(\mathfrak{B}) \times z} \end{bmatrix} S_L(\xi)$, $\xi X(\xi) = \begin{bmatrix} 0_{n(\mathfrak{B}) \times z} & \tilde{X} \end{bmatrix} S_L(\xi)$, and $M(\xi) = \tilde{M} S_L(\xi)$. \square

Proof of Lemma 3.23. Since \mathfrak{B} is Φ -dissipative, then there exists a storage function $\Psi(\zeta, \eta)$, moreover according to Prop. 3.20, there exists K such that $\Psi(\zeta, \eta) = X(\zeta)^\top K X(\eta)$.

To prove that statements 1) and 2) in Prop. 3.21 hold, it is enough to recall from Prop. 3.18 that there exists a dissipation function $\Delta(\zeta, \eta)$ such that $\Delta(\zeta, \eta) = M(\zeta)^\top \Phi M(\eta) - (\zeta + \eta)\Psi(\zeta, \eta)$.

To prove the final claim define $S_L(\xi) := \begin{bmatrix} I_z & \xi I_z & \cdots & \xi^L I_z \end{bmatrix}$. Factorise $\Delta(\zeta, \eta) = S_L(\zeta)^\top \tilde{\Delta} S_L(\eta)$, with $\tilde{\Delta} = \tilde{\Delta}^\top \in \mathbb{R}^{(L+1)z \times (L+1)z}$. Then it follows from the definition of dissipation function that $Q_\Delta \geq 0$ and consequently $\tilde{\Delta} \geq 0$. \square

A.2 Proofs of Chapter 5

Proof of Th. 5.1. Let $s \in \mathcal{S}$ be a switching signal, and from $\{Q_{\Psi_1}, \dots, Q_{\Psi_N}\}$ define the “switched functional” Q_Λ acting on \mathfrak{B}^Σ by $Q_\Lambda(w)(t) := Q_{\Psi_{s(t)}}(w)(t)$. Observe that in every interval $[t_{j-1}, t_j)$ Q_Λ is nonnegative, continuous and strictly decreasing, since $Q_{\Psi_{s(t_{j-1})}}$ satisfies conditions 1)–2). Moreover, for every admissible trajectory the value of Q_Λ does not increase at switching instants (condition 3)). It follows from standard arguments (see e.g. Th. 4.1 of [91]) that Σ is asymptotically stable. \square

Proof of Th. 5.4. Solutions $\overline{K}_k, \overline{Y}_k$ to (5.1) exist because of Th. 3.9 and Prop. 3.11. Multiply (5.1) on the left by $S_L(\zeta)^\top$ defined as in the proof of Prop. 3.11 and on the right by $S_L(\eta)$, and define $\Psi_k(\zeta, \eta) := X_k(\zeta)^\top \overline{K}_k X_k(\eta)$ and $Y_k(\xi) := \overline{Y}_k X_k(\xi)$ to obtain

$$(\zeta + \eta)\Psi_k(\zeta, \eta) - Y_k(\zeta)^\top R_k(\eta) - R_k(\zeta)^\top Y_k(\eta) = \Phi_k(\zeta, \eta).$$

Since Y_k is R -canonical, it follows from Th. 3.9 that also $\Phi_k(\zeta, \eta)$ is, and consequently \overline{F}_k exist as claimed. Now observe that the first inequality in (5.2) is equivalent with $V_k^\top \overline{F}_k V_k < 0$ and thus it implies $Q_{\Phi_k}(w) = \frac{d}{dt} Q_{\Psi_k}(w) < 0$ for all $w \in \mathfrak{B}_k$. Applying Th. 3.9 we conclude that Q_{Ψ_k} is a Lyapunov function for \mathfrak{B}_k . The second LMI in (5.2) implies condition 3. of Th. 5.1. \square

Proof of Prop. 5.10. From the strict positive-realness of ND^{-1} (see Def. 5.9) and the fact that D is Hurwitz conclude that $N(-j\omega)^\top D(j\omega) + D(-j\omega)^\top N(j\omega) > 0$ for all $\omega \in \mathbb{R}$. The existence of Q then follows from standard arguments in polynomial spectral factorisation. That Ψ is a polynomial matrix follows from Th. 3.1 of [86]. Since $\text{rank col}(D(\lambda), Q(\lambda)) = \mathbf{w}$ for all $\lambda \in \mathbb{C}$, $\frac{d}{dt} Q_\Psi(w) < 0$ for all $w \in \ker D\left(\frac{d}{dt}\right)$, $w \neq 0$. Apply Th. 3.9 to conclude that $Q_\Psi(w) > 0$ for all nonzero $w \in \ker D\left(\frac{d}{dt}\right)$. This proves that Ψ is a Lyapunov function for $\ker D\left(\frac{d}{dt}\right)$. That Ψ is D -canonical and QD^{-1} strictly proper, follow from strict properness of ND^{-1} and Th. 3.9.

We prove the second part of the claim. Use Prop. 4.10 of [86] to conclude that since Ψ is D -canonical, it is also ≥ 0 . Denote $\Psi' := \Psi \bmod N$. Since $Q_\Psi(w) = Q_{\Psi'}(w)$ for all $w \in \ker N\left(\frac{d}{dt}\right)$, it follows that $Q_{\Psi'} \geq 0$ also along $\ker N\left(\frac{d}{dt}\right)$. We now show that

$\frac{d}{dt}Q_{\Psi'}$ is negative along $\ker N\left(\frac{d}{dt}\right)$. To do so it suffices to show that $\text{col}(Q(\lambda), N(\lambda)) = \mathbf{w}$ for all $\lambda \in \mathbb{C}$. Assume by contradiction that there exists $\lambda \in \mathbb{C}$ and a corresponding $v \in \mathbb{C}^{\mathbf{w}}$, $v \neq 0$, such that $Q(\lambda)v = 0$ and $N(\lambda)v = 0$. Substitute $\zeta = -\lambda$, $\eta = \lambda$ in the PLE, obtaining

$$D(-\lambda)^\top N(\lambda) + N(-\lambda)^\top D(\lambda) = Q(-\lambda)^\top Q(\lambda).$$

Multiply on the right by v ; it follows that $N(-\lambda)^\top D(\lambda)v = 0$. Since N is Hurwitz, this implies $D(\lambda)v = 0$, but this contradicts the assumption $\text{rank col}(D(\lambda), Q(\lambda)) = \mathbf{w}$. \square

Proof of Lemma 5.11. That $n_2 < n_1$ follows from $R_2 R_1^{-1}$ being strictly proper.

To prove the claim on X_1 defined by (5.3), define

$$\mathfrak{X}_i(R_i) := \{f \in \mathbb{R}^{1 \times \mathbf{w}}[\xi] \mid f R_i^{-1} \text{ is strictly proper}\}, \quad i = 1, 2;$$

we now show that $\mathfrak{X}_2 \subset \mathfrak{X}_1$. Observe that $f R_2^{-1} \cdot R_2 R_1^{-1} = f R_1^{-1}$; since both $f R_2^{-1}$ and $R_2 R_1^{-1}$ are strictly proper, so is their product. Consequently, $f \in \mathfrak{X}_1$ (see sec. 2.6.1). Arrange the vectors of a basis for \mathfrak{X}_2 in $X_2 \in \mathbb{R}^{n_2 \times \mathbf{w}}[\xi]$; then $X_2\left(\frac{d}{dt}\right)$ is a state map for \mathfrak{B}_2 . Complete X_2 with $X'_1 \in \mathbb{R}^{(n_1 - n_2) \times \mathbf{w}}[\xi]$ to form a basis of \mathfrak{X}_1 ; this defines a state map for \mathfrak{B}_1 . Since each row of $X'_1 \bmod R_2$ belongs to \mathfrak{X}_2 , it can be written as a linear combination of the rows of X_2 . This proves that Π exists. \square

Proof of Theorem 5.15. The existence of $Q \in \mathbb{R}^{\bullet \times \mathbf{w}}[\xi]$ and the R_1 -canonicity of Ψ_1 follow from Prop. 5.10. To prove that Ψ_1 and $\Psi_2 := \Psi_1 \bmod R_2$ yield an MLF we show that:

$$\mathbf{C1.} \quad Q_{\Psi_1} \stackrel{\mathfrak{B}_1}{\geq} 0 \text{ and } \frac{d}{dt}Q_{\Psi_1} \stackrel{\mathfrak{B}_1}{<} 0;$$

$$\mathbf{C2.} \quad Q_{\Psi_2} \stackrel{\mathfrak{B}_2}{\geq} 0 \text{ and } \frac{d}{dt}Q_{\Psi_2} \stackrel{\mathfrak{B}_2}{<} 0;$$

C3. The multiple functional associated with Ψ_1 and Ψ_2 does not increase at switching instants.

Conditions **C1** and **C2** follow from Prop. 5.10. To prove **C3**, we first define the *coefficient matrices* of Ψ_1 and Ψ_2 . Since Ψ_1 is R_1 -canonical, it can be written as $X_1(\zeta)^\top K_1 X_1(\eta)$ for some coefficient matrix $K_1 \in \mathbb{R}^{n_1 \times n_1}$. Since $Q R_1^{-1}$ is strictly proper, it follows (see Th. 3.9) that $Q_{\Psi_1} \stackrel{\mathfrak{B}}{>} 0$ and since X_1 is a minimal state map for \mathfrak{B}_1 it follows that $K_1 > 0$. Note that $\text{col}(X_2(\xi), X'_1(\xi)) \bmod R_2 = \text{col}(X_2(\xi) \bmod R_2, X'_1(\xi) \bmod R_2) = \text{col}(X_2(\xi), \Pi X_2(\xi))$. Consequently (see Prop. 4.9 of [86]),

$$\Psi_1(\zeta, \eta) \bmod R_2 = \begin{bmatrix} X_2(\zeta)^\top & X_2(\zeta)^\top \Pi^\top \end{bmatrix} K_1 \begin{bmatrix} X_2(\eta) \\ \Pi X_2(\eta) \end{bmatrix},$$

from which it follows that the coefficient matrix of Ψ_2 is $K_2 = \text{col}(I_{n_2}, \Pi)^\top K_1 \text{col}(I_{n_2}, \Pi)$.

We prove **C3** showing that K_1 and K_2 satisfy some structural properties. We begin proving the following linear algebra result.

Lemma A.1. *Let $\Pi \in \mathbb{R}^{(n_1-n_2) \times n_2}$, and $K_1 = K_1^\top \in \mathbb{R}^{n_1 \times n_1}$. Assume $K_1 > 0$, and define*

$$K_2^e := \begin{bmatrix} I_{n_2} & \Pi^\top \\ (0_{n_1-n_2 \times n_2}) & (0_{n_1-n_2 \times n_1-n_2}) \end{bmatrix} K_1 \begin{bmatrix} I_{n_2} & (0_{n_2 \times n_1-n_2}) \\ \Pi & (0_{n_1-n_2 \times n_1-n_2}) \end{bmatrix}.$$

Then $K_1 \geq K_2^e$ if and only if there exist $K_{11} \in \mathbb{R}^{n_2 \times n_2}$, $K_{12} \in \mathbb{R}^{n_2 \times (n_1-n_2)}$ and $K_{22} \in \mathbb{R}^{(n_1-n_2) \times (n_1-n_2)}$ such that

$$K_1 = \begin{bmatrix} K_{11} & -\Pi^\top K_{22} \\ -K_{22}\Pi & K_{22} \end{bmatrix}.$$

Proof of Lemma A.1. Partition $K_1 =: \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^\top & K_{22} \end{bmatrix}$, with $K_{11} \in \mathbb{R}^{n_2 \times n_2}$, $K_{12} \in \mathbb{R}^{n_2 \times (n_1-n_2)}$ and $K_{22} \in \mathbb{R}^{(n_1-n_2) \times (n_1-n_2)}$. Straightforward manipulations show that $K_1 \geq K_2^e$ iff

$$\begin{bmatrix} -(K_{12} + \Pi^\top K_{22})K_{22}^{-1}(K_{12}^\top + K_{22}\Pi) & 0 \\ 0 & K_{22} \end{bmatrix} \geq 0.$$

Now $K_{22} > 0$, since $K_1 > 0$; thus the inequality holds iff $K_{12}^\top = -K_{22}\Pi$. \square

We aim to show that Lemma A.1 holds for the coefficient matrix of Ψ_1 and the Π arising from the standard gluing conditions. To this purpose we first prove the following result.

Lemma A.2. *Define $L := \lim_{\xi \rightarrow \infty} \xi X_1'(\xi)R_1(\xi)^{-1}$; then $L \in \mathbb{R}^{(n_1-n_2) \times \mathfrak{w}}$. Moreover, partition K_1 as $K_1 =: \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^\top & K_{22} \end{bmatrix}$, with $K_{11} \in \mathbb{R}^{n_2 \times n_2}$, $K_{12} \in \mathbb{R}^{n_2 \times (n_1-n_2)}$ and $K_{22} \in \mathbb{R}^{(n_1-n_2) \times (n_1-n_2)}$. Then $R_2(\xi) = L^\top (K_{12}^\top X_2(\xi) + K_{22}X_1'(\xi))$.*

Proof of Lemma A.2. That the limit is finite follows from $X_1'R_1^{-1}$ being strictly proper. To prove the rest, recall from Sec. 2.6.1 that there exist $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $F_1 \in \mathbb{R}^{n_1 \times \mathfrak{w}}$ such that

$$\xi X_1(\xi) = A_1 X_1(\xi) + F_1(\xi)R_1(\xi). \quad (\text{A.1})$$

Multiply both sides of (A.1) by R_1^{-1} , and take the limit for $\xi \rightarrow \infty$. Since $R_2 R_1^{-1}$ is strictly proper and $X_2(\xi)$ is a state map for \mathfrak{B}_2 , it follows that $\lim_{\xi \rightarrow \infty} \xi X_2(\xi)R_1(\xi)^{-1} = 0_{n_2 \times \mathfrak{w}}$. Moreover, $\lim_{\xi \rightarrow \infty} X_1(\xi)R_1(\xi)^{-1} = 0_{n_1 \times \mathfrak{w}}$. Consequently F_1 is constant, and

$$F_1 = \lim_{\xi \rightarrow \infty} \text{col}(0_{n_2 \times \mathfrak{w}}, \xi X_1'(\xi)R_1(\xi)^{-1}) = \text{col}(0_{n_2 \times \mathfrak{w}}, L).$$

The claim on R_2 now follows from Prop. 4.3 of [52]. \square

From Lemma A.2 and the fact that R_2 is square and nonsingular, it follows that L^\top is of full row rank, and consequently $n_1 - n_2 \geq \mathfrak{w}$. We now prove that L is square, thus nonsingular.

Lemma A.3. $\deg(\det(R_1)) - \deg(\det(R_2)) = n_1 - n_2 = \mathfrak{w}$, and consequently L is nonsingular.

Proof of Lemma A.3. We prove the first part of the claim, well-known in the scalar case, but for whose multivariable version we have failed to find a proof in the literature.

Let $U \in \mathbb{R}^{\mathfrak{w} \times \mathfrak{w}}[\xi]$ be a unimodular matrix such that $R'_1 := R_1 U$ is column reduced (see sect. 6.3.2 of [29]); define $R'_2 := R_2 U$. Observe that $R'_2 R'^{-1}_1 = R_2 R^{-1}_1$; moreover $n_1 = \deg(\det(R'_1)) = \deg(\det(R_1))$ and $n_2 = \deg(\det(R_2)) = \deg(\det(R'_2))$. Thus w.l.o.g. we prove the claim for $R'_2 R'^{-1}_1$. Define $\mathfrak{X}'_1(R_1) := \{f \in \mathbb{R}^{1 \times \mathfrak{w}}[\xi] \mid f R'^{-1}_1 \text{ is strictly proper}\}$ and similarly \mathfrak{X}'_2 ; it is straightforward to see that \mathfrak{X}'_i equals \mathfrak{X}_i defined as in Lemma 5.11, $i = 1, 2$. Denote the degree of the i -th column of R'_1 by δ_i^1 and that of the i -th column of R'_2 by δ_i^2 , $i = 1, \dots, \mathfrak{w}$; strict properness yields $\delta_i^1 > \delta_i^2$, $i = 1, \dots, \mathfrak{w}$. A basis for \mathfrak{X}'_1 is $e_i \xi^k$, $k = 1, \dots, \delta_k^1 - 1$, $i = 1, \dots, \mathfrak{w}$, where e_i is the i -th vector of the canonical basis for $\mathbb{R}^{1 \times \mathfrak{w}}$. A straightforward argument proves that these vectors can be arranged in a matrix $X(\xi) = \text{col}(X_2(\xi), X'_1(\xi))$ so that the n_2 rows of X_2 span \mathfrak{X}'_2 and those of X'_1 span its complement in \mathfrak{X}'_1 . Permute the rows of X'_1 so that $e_i \xi^{\delta_i^1 - 1}$, $i = 1, \dots, \mathfrak{w}$, are its last \mathfrak{w} rows.

An analogous of (A.1) holds for R'_1 ; given the arrangement of the basis vectors for \mathfrak{X}'_1 , it is straightforward to verify that the last \mathfrak{w} rows of L contain the inverse of the highest column coefficient matrix of R_1 , while its first $n_1 - n_2 - \mathfrak{w}$ rows are equal to zero, i.e. $L^\top = \begin{bmatrix} 0_{(n_1 - n_2 - \mathfrak{w}) \times \mathfrak{w}} & L'^\top \end{bmatrix}$, with $L' \in \mathbb{R}^{\mathfrak{w} \times \mathfrak{w}}$ nonsingular.

Now let Ψ'_1 be a storage function for $R'_2 R'^{-1}_1$ with the same properties as Ψ_1 in the statement of Th. 5.15; we denote with K'_{ij} , $i, j = 1, 2$ the block submatrices arising from a partition of its coefficient matrix K'_1 as in Lemma A.2. Use the formula for $R'_2(\xi)$ established in Lemma A.2 to conclude that

$$R'_2(\xi) = L'^\top K'^\top_{12} X_2(\xi) + L'^\top \begin{bmatrix} K''_{22} & K'''_{22} \end{bmatrix} X'_1(\xi),$$

where $K'^\top_{12} \in \mathbb{R}^{\mathfrak{w} \times n_2}$, $\begin{bmatrix} K''_{22} & K'''_{22} \end{bmatrix} \in \mathbb{R}^{\mathfrak{w} \times (n_1 - n_2)}$, and K'''_{22} has \mathfrak{w} columns. $K'_1 > 0$ implies $K'''_{22} > 0$; thus the highest column coefficient matrix of $R_2(\xi)$ is $L' K'''_{22}$ and it is nonsingular. Thus also $R'_2(\xi)$ is column reduced; moreover, its column degrees are $\delta_i^1 - 1$, $i = 1, \dots, \mathfrak{w}$. From this it follows that

$$\deg \det(R'_2) = \sum_{i=1}^{\mathfrak{w}} (\delta_i^1 - 1) = \left(\sum_{i=1}^{\mathfrak{w}} \delta_i^1 \right) - \mathfrak{w} = n_1 - \mathfrak{w}.$$

The claim is proved. \square

We resume the proof of Th. 5.15. From the formula for $R_2(\xi)$ proved in Lemma A.2 it follows that

$$\begin{aligned} 0 &= R_2(\xi) \bmod R_2 \\ &= L^\top \left(K_{12}^\top X_2(\xi) + K_{22} X_1'(\xi) \right) \bmod R_2 \\ &= L^\top \left(K_{12}^\top + K_{22} \Pi \right) X_2(\xi). \end{aligned} \quad (\text{A.2})$$

The rows of $X_2(\xi)$ are linearly independent over \mathbb{R} , since X_2 is a minimal state map. Consequently (A.2) implies $L^\top (K_{12}^\top + K_{22} \Pi) = 0$, and since L is nonsingular by Lemma A.3, we conclude that $K_{12}^\top + K_{22} \Pi = 0$. Thus the coefficient matrix of K_1 is structured as in Lemma A.1.

We now show that this structure implies that condition **C3** holds. Consider first a switch from \mathfrak{B}_1 to \mathfrak{B}_2 at t_k . Taking the standard gluing conditions into account, $Q_{\Psi_1}(w)(t_k^-) \geq Q_{\Psi_2}(w)(t_k^+)$ if and only if

$$\begin{aligned} &\begin{bmatrix} X_2(\frac{d}{dt})w(t_k^-) \\ X_1'(\frac{d}{dt})w(t_k^-) \end{bmatrix}^\top K_1 \begin{bmatrix} X_2(\frac{d}{dt})w(t_k^-) \\ X_1'(\frac{d}{dt})w(t_k^-) \end{bmatrix} - \begin{bmatrix} X_2(\frac{d}{dt})w(t_k^+) \\ \Pi X_2(\frac{d}{dt})w(t_k^+) \end{bmatrix}^\top K_1 \begin{bmatrix} X_2(\frac{d}{dt})w(t_k^+) \\ \Pi X_2(\frac{d}{dt})w(t_k^+) \end{bmatrix} \\ &= \begin{bmatrix} X_2(\frac{d}{dt})w(t_k^-) \\ X_1'(\frac{d}{dt})w(t_k^-) \end{bmatrix}^\top \left(K_1 - \begin{bmatrix} I_{n_2} & \Pi^\top \\ 0 & 0 \end{bmatrix} K_1 \begin{bmatrix} I_{n_2} & 0 \\ \Pi & 0 \end{bmatrix} \right) \begin{bmatrix} X_2(\frac{d}{dt})w(t_k^-) \\ X_1'(\frac{d}{dt})w(t_k^-) \end{bmatrix} \geq 0. \end{aligned} \quad (\text{A.3})$$

Since the matrix between brackets is semidefinite positive (see Lemma A.1), (A.3) is satisfied.

It is straightforward to check that in a switch from \mathfrak{B}_2 to \mathfrak{B}_1 the value of the multifunctional is the same before and after the switch. The theorem is proved. \square

Proof of Th. 5.17. The proof follows readily from Prop. 3.11 and Th. 5.15. \square

Proof of Th. 5.24. W.l.o.g. assume that Q_Ψ is R_1 -canonical; then by Lemma 5.11, given a minimal state map $X_1(\frac{d}{dt})$ for \mathfrak{B}_1 as in (5.3) there exists $K = K^\top \in \mathbb{R}^{n_1 \times n_1}$ such that $\Psi(\zeta, \eta) = X_1(\zeta)^\top K X_1(\eta)$. Partition K as

$$K =: \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^\top & K_{22} \end{bmatrix},$$

where $K_{11} \in \mathbb{R}^{n_2 \times n_2}$, $K_{12} \in \mathbb{R}^{n_2 \times (n_1 - n_2)}$ and $K_{22} \in \mathbb{R}^{(n_1 - n_2) \times (n_1 - n_2)}$. At a switch from \mathfrak{B}_1 to \mathfrak{B}_2 at t_k the inequality (A.3) holds in particular for a switching signal $s(t) = 1$ for $t \leq t_k$, $s(t) = 2$ for $t > t_k$. Since for every choice of $v \in \mathbb{R}^{n_1}$ there exists a trajectory $w \in \mathfrak{B}_1|_{(-\infty, 0]}$ s.t. $(X_1(\frac{d}{dt})w)(0^-) = v$, using Lemma A.1 we conclude that (A.3) holds,

then $K_{12}^\top + K_{22}\Pi = 0$. Consequently,

$$K = \begin{bmatrix} K_{11} & -\Pi K_{22} \\ -K_{22}\Pi & K_{22} \end{bmatrix} = \begin{bmatrix} K' & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Pi^\top \\ -I_{n_1-n_2} \end{bmatrix} K_{22} \begin{bmatrix} \Pi & -I_{n_1-n_2} \end{bmatrix}, \quad (\text{A.4})$$

where $K' := K_{11} - \Pi^\top K_{22}\Pi$. Pre- and post-multiply (A.4) by $X_1(\zeta)^\top$ and $X_1(\eta)$ to obtain

$$\Psi(\zeta, \eta) = \underbrace{X_2(\zeta)^\top K' X_2(\eta)}_{=: \Psi'(\zeta, \eta)} + X_1(\zeta)^\top \begin{bmatrix} \Pi^\top \\ -I_{(n_1-n_2)} \end{bmatrix} K_{22} \begin{bmatrix} \Pi & -I_{(n_1-n_2)} \end{bmatrix} X_1(\eta). \quad (\text{A.5})$$

Since Ψ_1 is a Lyapunov function for $\ker R_1 \left(\frac{d}{dt} \right)$, there exists $V \in \mathbb{R}^{w \times w}[\xi]$ such that

$$(\zeta + \eta)\Psi_1(\zeta, \eta) = -Q(\zeta)^\top Q(\eta) + V(\zeta)^\top R_1(\eta) + R_1(\zeta)^\top V(\eta).$$

We now show that there exists $M \in \mathbb{R}^{w \times w}[\xi]$ such that $V = MR_2$.

From Prop. 4.3 of [52] it follows that

$$V(\xi) = \lim_{\mu \rightarrow \infty} \mu R_1(\mu)^{-\top} \Psi_1(\mu, \xi);$$

substituting (A.5) in this expression we obtain

$$\begin{aligned} V(\xi) = \lim_{\mu \rightarrow \infty} & \left(\mu R_1(\mu)^{-\top} X_2(\mu)^\top K' X_2(\eta) \right. \\ & \left. + \mu R_1(\mu)^{-\top} X_1(\mu)^\top \begin{bmatrix} \Pi^\top \\ -I_{(n_1-n_2)} \end{bmatrix} K_{22} \begin{bmatrix} \Pi & -I_{(n_1-n_2)} \end{bmatrix} X_1(\eta) \right). \end{aligned}$$

Since $R_2 R_1^{-1}$ is strictly proper, the first term goes to zero. Now $\begin{bmatrix} \Pi & -I_{n_1-n_2} \end{bmatrix} X_1(\xi) = -X_1'(\xi) + \Pi X_2(\xi)$ and consequently

$$\begin{aligned} V(\xi) &= -\mu R_1(\mu)^{-\top} X_1'^\top(\mu) K_{22} \begin{bmatrix} \Pi & -I_{(n_1-n_2)} \end{bmatrix} X_1(\xi) \\ &\quad + \lim_{\mu \rightarrow \infty} \underbrace{\mu R_1(\mu)^{-\top} X_2(\mu)^\top \Pi^\top}_{\rightarrow 0} K_{22} \begin{bmatrix} \Pi & -I_{(n_1-n_2)} \end{bmatrix} X_1(\xi) \\ &= - \begin{bmatrix} 0_{(n_1-n_2) \times w} & L'^\top \end{bmatrix} K_{22} \begin{bmatrix} \Pi & -I_{(n_1-n_2)} \end{bmatrix} X_1(\xi), \end{aligned}$$

where $L' \in \mathbb{R}^{w \times w}$ is a nonsingular matrix, as proved in Lemma A.2 and A.3. That V has the right factor R_2 follows from the following argument. Observe that

$$\begin{bmatrix} \Pi & -I_{(n_1-n_2)} \end{bmatrix} \begin{bmatrix} X_2(\xi) \\ X_1'(\xi) \end{bmatrix} = X_1'(\xi) \bmod R_2 - X_1'(\xi).$$

Write $X'_1(\xi)R_2(\xi)^{-1} = P(\xi) + S(\xi)$, with $S(\xi)$ a strictly proper polynomial matrix and $P \in \mathbb{R}^{(n_1-n_2) \times \mathbf{w}}[\xi]$; then

$$\Pi X_2(\xi) - X'_1(\xi) = X'_1(\xi) - P(\xi)R_2(\xi) - X'_1(\xi) = -P(\xi)R_2(\xi) .$$

This proves that

$$V(\xi) = \begin{bmatrix} 0_{(n_1-n_2) \times \mathbf{w}} & L'^\top \end{bmatrix} K_{22} P(\xi) R_2(\xi) =: M(\xi) R_2(\xi) .$$

Finally, the equality

$$(\zeta + \eta)\Psi_1(\zeta, \eta) = -Q(\zeta)^\top Q(\eta) + R_2(\zeta)^\top M(\zeta)^\top R_1(\eta) + R_1(\zeta)^\top M(\eta) R_2(\eta) ,$$

together with $\text{rank } Q(j\omega) = \mathbf{w}$ for all $\omega \in \mathbb{R}$ and R_1 being Hurwitz, prove strict positive-realness of $MR_2R_1^{-1}$. That $MR_2R_1^{-1}$ is strictly proper follows from QR_1^{-1} being strictly proper and Th. 3.9. This concludes the proof. \square

Proof of Th. 5.25. The first claim follows directly from Prop. 5.17 and the second from Lemma A.1 in the proof of Th. 5.15. \square

Proof of Lemma 5.26. We know from Lemma 5.11 that the polynomial row vectors that form a basis for the state space of \mathfrak{B}_2 are contained in that of \mathfrak{B}_1 . We can arrange such vectors that form a minimal state map X_2 in the first n_2 -rows of X_1 . Moreover, since $R_3R_1^{-1}$ is strictly proper, we apply the same argument to arrange in the first n_3 -rows, the vectors that form a basis for the state space of \mathfrak{B}_3 including those in X_2 and the additional $(n_3 - n_2)$ -vectors, denoted by X'_3 . The existence of Π_i , $i = 1, 2, 3$ follows from the same argument used in Lemma 5.11. \square

Proof of Th. 5.27. In order to show that $\{\Psi \bmod R_i\}_{i=1,2,3}$ induces a multiple Lyapunov function for \mathcal{F}' , we prove the following statements:

$$\text{S1. } Q_\Psi \stackrel{\mathfrak{B}_1}{\geq} 0 \text{ and } \frac{d}{dt}Q_\Psi \stackrel{\mathfrak{B}_1}{<} 0.$$

$$\text{S2. } Q_\Psi \stackrel{\mathfrak{B}_2}{\geq} 0 \text{ and } \frac{d}{dt}Q_\Psi \stackrel{\mathfrak{B}_2}{<} 0.$$

$$\text{S3. } Q_\Psi \stackrel{\mathfrak{B}_3}{\geq} 0 \text{ and } \frac{d}{dt}Q_\Psi \stackrel{\mathfrak{B}_3}{<} 0.$$

Moreover, we prove that the value of Q_Ψ does not increase when switching between:

$$\text{S4. } \mathfrak{B}_1 \text{ and } \mathfrak{B}_2;$$

$$\text{S5. } \mathfrak{B}_1 \text{ and } \mathfrak{B}_3;$$

S6. \mathfrak{B}_3 and \mathfrak{B}_2 .

Note that statements **S1** and **S2** and **S4** hold, since $\{\Psi, \Psi \bmod R_2\}$ is a multiple Lyapunov function for a standard SLDS with mode behaviours $\{\mathfrak{B}_1, \mathfrak{B}_2\}$.

The validity of statement **S5** follows from Th. 5.15, since $R_3 R_1^{-1}$ is strictly positive-real.

The proof of **S3**, follows from defining $\Psi_3(\zeta, \eta) := \Psi(\zeta, \eta) \bmod R_3$ and applying the same arguments used in the proof of Th. 5.25 for $\Psi(\zeta, \eta) \bmod R_2$.

It now remains to prove **S6**. Consider the following lemma.

Lemma A.4. *Let X_1, R_2, R_3 and $\Pi_i, i = 1, 2, 3$, be as in the theorem, then*

$$(X_1 \bmod R_3) \bmod R_2 = X_1 \bmod R_2 .$$

Moreover, considering the partition $\Pi_3 := \begin{bmatrix} \Pi'_3 & \Pi''_3 \end{bmatrix}$ with $\Pi'_3 \in \mathbb{R}^{(n_1-n_3) \times n_2}$ and $\Pi''_3 \in \mathbb{R}^{(n_1-n_3) \times (n_3-n_2)}$, it follows that $\Pi_1 = \begin{bmatrix} \Pi_2 \\ \Pi'_3 + \Pi''_3 \Pi_2 \end{bmatrix}$.

Proof. To prove the first claim, let $P_2, P_3 \in \mathbb{R}^{n_1 \times w}[\xi]$ be the non strictly proper part of $X_1 R_2^{-1}$ and $X_1 R_3^{-1}$ respectively. The claim follows from the computations $X_1 \bmod R_2 = X_1 - P_2 R_2$; and $(X_1 \bmod R_3) \bmod R_2 = (X_1 - P_3 M R_2) \bmod R_2 = X_1 - P_3 M R_2 - (P_2 - P_3 M) R_2 = X_1 - P_2 R_2$. The second claim is easily proved by computing $X_1 \bmod R_2$ in terms of Π_1 and $(X_1 \bmod R_3) \bmod R_2$ in terms of Π_3 and Π_2 according to Lemma 5.26, then factorise X_2 . \square

Taking the gluing conditions into account and using Lemma A.4, we conclude that when we switch from \mathfrak{B}_3 to \mathfrak{B}_2 , the condition $Q_\Psi(w)(t_i^-) - Q_\Psi(w)(t_i^+) \geq 0$ is equivalent with

$$Q_{\Psi \bmod R_3}(w)(t_i^-) - Q_{(\Psi \bmod R_3) \bmod R_2}(w)(t_i^+) \geq 0 ; \quad (\text{A.6})$$

In the following, we aim to express condition (A.6) in terms of an LMI. In order to do so, factorise $Q_\Psi = X_1 \left(\frac{d}{dt}\right)^\top K X_1 \left(\frac{d}{dt}\right)$ with

$$K := \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12}^\top & K_{22} & K_{23} \\ K_{13}^\top & K_{23}^\top & K_{33} \end{bmatrix} , \quad (\text{A.7})$$

with $K_{11} \in \mathbb{R}^{n_2 \times n_2}$, $K_{12} \in \mathbb{R}^{n_2 \times (n_3-n_2)}$, $K_{13} \in \mathbb{R}^{n_2 \times (n_1-n_3)}$, $K_{22} \in \mathbb{R}^{(n_3-n_2) \times (n_3-n_2)}$, $K_{23} \in \mathbb{R}^{(n_3-n_2) \times (n_1-n_3)}$ and $K_{33} \in \mathbb{R}^{(n_1-n_3) \times (n_1-n_3)}$. From the results of Lemma 6.7 and Lemma A.4, since the Lyapunov function Q_Ψ does not increase when switching

from \mathfrak{B}_1 to \mathfrak{B}_2 , it follows that

$$\begin{bmatrix} K_{12}^\top \\ K_{13}^\top \end{bmatrix} = - \begin{bmatrix} K_{22} & K_{23} \\ K_{23}^\top & K_{33} \end{bmatrix} \Pi_1 = - \begin{bmatrix} K_{22} & K_{23} \\ K_{23}^\top & K_{33} \end{bmatrix} \begin{bmatrix} \Pi_2 \\ \Pi'_3 + \Pi''_3 \Pi_2 \end{bmatrix},$$

and consequently

$$K_{12}^\top = -(K_{22}\Pi_2 + K_{23}\Pi'_3 + K_{23}\Pi''_3\Pi_2), \quad (\text{A.8})$$

$$K_{13}^\top = -(K_{23}\Pi_2 + K_{33}\Pi'_3 + K_{33}\Pi''_3\Pi_2). \quad (\text{A.9})$$

We now express the entries of the coefficient matrix of $Q_{\Psi \bmod R_3}$ in terms of those of $Q_{\Psi \bmod R_2}$ as in (A.7), according to the following lemma.

Lemma A.5. *Let $Q_\Psi = X_1 \left(\frac{d}{dt}\right)^\top K X_1 \left(\frac{d}{dt}\right)$ and $\Pi_3 := \begin{bmatrix} \Pi'_3 & \Pi''_3 \end{bmatrix}$, be as previously defined. Factorise $Q_{\Psi \bmod R_3} = X_3 \left(\frac{d}{dt}\right)^\top \tilde{K} X_3 \left(\frac{d}{dt}\right)$ with $\tilde{K} = \tilde{K}^\top \in \mathbb{R}^{n_3 \times n_3}$. Consider the partition*

$$\tilde{K} := \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{12}^\top & \tilde{K}_{22} \end{bmatrix}, \quad (\text{A.10})$$

with $\tilde{K}_{11} \in \mathbb{R}^{n_2 \times n_2}$, $\tilde{K}_{12} \in \mathbb{R}^{n_2 \times (n_3 - n_2)}$ and $\tilde{K}_{22} \in \mathbb{R}^{(n_3 - n_2) \times (n_3 - n_2)}$. Then

$$\begin{aligned} \tilde{K}_{11} &= (K_{11} + \Pi'_3 K_{13}^\top + K_{13} \Pi'_3 + \Pi_3'^\top K_{33} \Pi'_3), \\ \tilde{K}_{12} &= (K_{12} + \Pi_3'^\top K_{23}^\top + K_{13} \Pi''_3 + \Pi_3'^\top K_{33} \Pi''_3), \\ \tilde{K}_{22} &= (K_{22} + \Pi_3''^\top K_{23}^\top + K_{23} \Pi'_3 + \Pi_3''^\top K_{33} \Pi''_3). \end{aligned}$$

Proof. Following the same procedure as in the proof of Lemma 6.7 and considering the partitions (A.7) and (A.10), we conclude that the coefficient matrix (A.10) can be computed as

$$\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{12}^\top & \tilde{K}_{22} \end{bmatrix} = \begin{bmatrix} I_{n(\mathfrak{B}_2)} & 0 \\ 0 & I_{(n_3 - n_2)} \\ \Pi'_3 & \Pi''_3 \end{bmatrix}^\top \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12}^\top & K_{22} & K_{23} \\ K_{13}^\top & K_{23}^\top & K_{33} \end{bmatrix} \begin{bmatrix} I_{n(\mathfrak{B}_2)} & 0 \\ 0 & I_{(n_3 - n_2)} \\ \Pi'_3 & \Pi''_3 \end{bmatrix};$$

The desired equalities follow by inspection. \square

Now we return to the proof of Th. 5.27. Note that from inequality (A.6) we can obtain

$$\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{12}^\top & \tilde{K}_{22} \end{bmatrix} - \begin{bmatrix} I_{n_2} & \Pi_2^\top \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{12}^\top & \tilde{K}_{22} \end{bmatrix} \begin{bmatrix} I_{n_2} & 0 \\ \Pi_2 & 0 \end{bmatrix} \geq 0.$$

Arguing as in Lemma 6.7, this inequality holds if and only if $\tilde{K}_{12}^\top + \tilde{K}_{22}\Pi_2 = 0$, or equivalently from Lemma A.5, the condition is satisfied if and only if

$$K_{12}^\top + \Pi_3''^\top K_{13}^\top + K_{23} \Pi'_3 + \Pi_3''^\top K_{33} \Pi'_3 = -(K_{22} + \Pi_3''^\top K_{23}^\top + K_{23} \Pi'_3 + \Pi_3''^\top K_{33} \Pi''_3) \Pi_2.$$

Substituting (A.8) in the latter equation we obtain (A.9), therefore we conclude that the condition $\tilde{K}_{12}^\top = -\tilde{K}_{22}\Pi_2$ is satisfied. Consequently the value of Q_Ψ does not increase when switching from \mathfrak{B}_3 to \mathfrak{B}_2 . It is a matter of straightforward verification to check that when switching from \mathfrak{B}_2 to \mathfrak{B}_3 the value of Q_Ψ remains the same. This concludes the proof of the theorem. \square

Proof of Th. 5.28. To prove that M_α , with $0 \leq \alpha \leq 1$, is a strictly positive-real completion define $G_1(\xi) := M_1(\xi)R_2(\xi)R_1(\xi)^{-1}$ and $G_2(\xi) := M_2(\xi)R_2(\xi)R_1(\xi)^{-1}$. It follows that

$$\begin{aligned} & (M_\alpha(-j\omega)R_2(-j\omega)R_1(-j\omega)^{-1})^\top + M_\alpha(j\omega)R_2(j\omega)R_1(j\omega)^{-1} \\ &= ((\alpha M_1(-j\omega) + (1-\alpha)M_2(-j\omega))R_2(-j\omega)R_1(-j\omega)^{-1})^\top \\ & \quad + (\alpha M_1 + (1-\alpha)M_2)R_2(j\omega)R_1(j\omega)^{-1} \\ &= \alpha G_1(-j\omega)^\top + (1-\alpha)G_2(-j\omega)^\top + \alpha G_1(j\omega) + (1-\alpha)G_2(j\omega) \\ &= \alpha (G_1(-j\omega)^\top + G_1(j\omega)) + (1-\alpha) (G_2(-j\omega)^\top + G_2(j\omega)) . \end{aligned}$$

The claim follows from the fact that G_1 and G_2 are strictly positive-real. We now prove that Σ' is asymptotically stable. Let $K_i := \begin{bmatrix} K_{i,11} & -\Pi^\top K_{i,22} \\ -K_{i,22}\Pi & K_{i,22} \end{bmatrix} > 0$, $i = 1, 2$ be as in Th. 5.25, corresponding to the solution of (5.6) using the positive-real completion M_i , $i = 1, 2$, respectively. Following straightforward computations, it can be proved that

$$\begin{aligned} \Psi_\alpha(\zeta, \eta) &:= \alpha \Psi_1(\zeta, \eta) + (1-\alpha) \Psi_2(\zeta, \eta) \\ &= \frac{R_1(\zeta)^\top M_\alpha(\eta)R_2(\eta) + R_2(\zeta)^\top M_\alpha(\zeta)^\top R_1(\eta) - Q_\alpha(\zeta)^\top Q_\alpha(\eta)}{\zeta + \eta} , \end{aligned}$$

for some $Q_\alpha \in \mathbb{R}^{\bullet \times w}[\xi]$. It follows that the convex combination of the two-variable polynomial matrices Ψ_1 and Ψ_2 yields the coefficient matrix

$$K_\alpha := \begin{bmatrix} K_{\alpha,11} & -\Pi^\top K_{\alpha,22} \\ -K_{\alpha,22}\Pi & K_{\alpha,22} \end{bmatrix} > 0 ,$$

as in Th. 5.25, then Q_{Ψ_α} is a Lyapunov function for Σ . Finally, to conclude the proof apply Th. 5.27. \square

A.3 Proofs of Chapter 6

Proof of Prop. 6.2. Let $i \in \{1, \dots, N\}$; since Σ is (strictly) Φ -dissipative and a constant switching signal $s(t) = i$ for all t is admissible in \mathcal{S} , then it necessarily follows that $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0$ (respectively $\exists \epsilon > 0$ s.t. $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq \epsilon \int_{-\infty}^{+\infty} \|w\|_2^2 dt$) for all $w \in \mathfrak{B}_i$ of compact support, i.e. \mathfrak{B}_i is (strictly) Φ -dissipative. \square

Proof of Prop. 6.3. Since \mathfrak{B}_i is (strictly) Φ -dissipative, according to Prop. 6.2, the existence of Q_{Ψ_i} , $i = 1, \dots, N$, is guaranteed (see Prop. 3.18). Now integrate the inequality $\frac{d}{dt}Q_{\Psi_i} \leq Q_{\Phi}$ between a and b , for all $w \in \mathfrak{B}_i \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$. \square

Proof of Th. 6.6. We consider the three possible cases, i.e. **A)** $|\mathbb{T}_s| = \infty$, **B)** $0 < |\mathbb{T}_s| < \infty$ and **C)** $|\mathbb{T}_s| = 0$. Let $t_0 := -\infty$. Use Prop. 6.3 and the fact that $\lim_{t \rightarrow \pm\infty} w(t) = 0$ for all $w \in \mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^w)$ to obtain the following expressions for cases **A)** and **B)**, where $s = s_w$:

A)

$$\begin{aligned} \int Q_{\Phi}(w) \geq & (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+)) + \dots \\ & + (Q_{\Psi_{s(t_{n-1})}}(w)(t_n^-) - Q_{\Psi_{s(t_n)}}(w)(t_n^+)) + \dots \end{aligned}$$

B)

$$\begin{aligned} \int Q_{\Phi}(w) \geq & (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+)) \\ & + \sum_{k=2}^{|\mathbb{T}_s|-1} (Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+)) \\ & + (Q_{\Psi_{s(|\mathbb{T}_s|-1)}}(w)(t_{|\mathbb{T}_s|}^-) - Q_{\Psi_{s(|\mathbb{T}_s|)}}(w)(t_{|\mathbb{T}_s|}^+)) . \end{aligned}$$

Since $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0$, $\forall t_k \in \mathbb{T}_s$, we conclude that in both cases $\int Q_{\Phi}(w) \geq 0$.

Finally the claim for **C)** when no switching takes place, i.e. $s(t) = i$ for all t , follows readily from the existence of a storage function Q_{Ψ_i} (see Prop. 6.3) and the standard result quoted in Prop. 3.18. \square

Proof of Th. 6.7. The existence of storage functions Q_{Ψ_i} , $i = 1, \dots, N$, follows from Prop. 6.2 and Prop. 3.18. To prove the rest of the claim let us introduce first the following lemma.

Lemma A.6. *Let $\Phi \in \mathbb{R}^{w \times w}$ and let Σ be a strictly Φ -dissipative SLDS with \mathcal{G} well-posed. Consider two behaviours $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{F}$, described by the observable image representations $w = M_i \left(\frac{d}{dt} \right) z_i$, $i = 1, 2$, respectively. Consider the switching signal*

$$s(t) := \begin{cases} 1, & t \leq 0, \\ 2, & t > 0. \end{cases}$$

Let $X_i \in \mathbb{R}^{n(\mathfrak{B}_i) \times z}[\xi]$, $i = 1, 2$, be minimal state maps for \mathfrak{B}_i , $i = 1, 2$; and let $L_{1 \rightarrow 2} \in \mathbb{R}^{n(\mathfrak{B}_2) \times n(\mathfrak{B}_1)}$ be the corresponding re-initialisation map when switching from \mathfrak{B}_1 to \mathfrak{B}_2

at zero. Select a fixed but otherwise arbitrary final state v_1 , corresponding to the unique initial state $v_2 := L_{1 \rightarrow 2} v_1$.

There exists $A_1, H_2 \in \mathbb{R}^{z \times z}[\xi]$ such that $\det(A_1)$ and $\det(H_2)$ are respectively anti-Hurwitz and Hurwitz polynomials; and

$$M_1(-\xi)^\top \Phi M_1(\xi) = A_1(-\xi)^\top A_1(\xi) ,$$

and

$$M_2(-\xi)^\top \Phi M_2(\xi) = H_2(-\xi)^\top H_2(\xi) .$$

There exist unique latent variable trajectories $z_1, z_2 : \mathbb{R} \rightarrow \mathbb{R}^z$ such that $A_1 \left(\frac{d}{dt} \right) z_1 = 0$, $X_1 \left(\frac{d}{dt} \right) z_1(0^-) = v_1$; and $H_2 \left(\frac{d}{dt} \right) z_2 = 0$, $X_2 \left(\frac{d}{dt} \right) z_2(0^+) = L_{1 \rightarrow 2} v_1$. Consequently, the external variable trajectory defined by

$$w(t) := \begin{cases} M_1 \left(\frac{d}{dt} \right) z_1, & t \leq 0 , \\ M_2 \left(\frac{d}{dt} \right) z_2, & t > 0 ; \end{cases}$$

belongs to \mathfrak{B}^Σ . Moreover, the final/initial state of w at zero is v_1 and $L_{1 \rightarrow 2} v_1$ respectively.

Proof. The existence of $A_1, H_2 \in \mathbb{R}^{z \times z}[\xi]$ satisfying the conditions in the first claim follows directly from standard results in polynomial spectral factorization (see Sec. 3.7).

To verify that the second claim holds true, it is enough to prove that the state space of each mode behaviour equals the state space associated to its supply rate spectral factor, i.e. to prove that $\mathfrak{X}(M_1 \left(\frac{d}{dt} \right)) = \mathfrak{X}(A_1 \left(\frac{d}{dt} \right))$ and $\mathfrak{X}(M_2 \left(\frac{d}{dt} \right)) = \mathfrak{X}(H_2 \left(\frac{d}{dt} \right))$. In order to do so, we recall from Prop. 6.2 that if Σ is strictly Φ -dissipative, it follows that every behaviour in the bank is also strictly Φ -dissipative. Since there exists $\epsilon > 0$ such that

$$\int Q_\Phi(w) \geq \epsilon \left(\int_{-\infty}^{0^-} \|w\|_2^2 dt + \int_{0^+}^{\infty} \|w\|_2^2 dt \right) ;$$

using Prop. 3.7 we conclude that $\partial \Phi'_i(j\omega) := M_i(-j\omega)^\top \Phi M_i(j\omega) > \epsilon M_i(-j\omega)^\top M_i(j\omega) \forall \omega \in \mathbb{R}$, $i = 1, 2$. Select a submatrix $U_i \in \mathbb{R}^{z \times z}[\xi]$ of M_i of maximal determinantal degree, then $M_i U_i^{-1}$, $i = 1, 2$ is a proper rational matrix. Consequently

$$\lim_{\omega \rightarrow \infty} U_i(-j\omega)^{-\top} \partial \Phi_i(j\omega) U_i(j\omega)^{-1} = M_{i\infty}^\top \Phi M_{i\infty} > \epsilon M_{i\infty}^\top M_{i\infty} , \quad i = 1, 2 ,$$

with $M_{i\infty} := \lim_{\omega \rightarrow \infty} M_i(j\omega) U_i(j\omega)^{-1}$. It is easy to check that $M_{i\infty}$ contains I_z as a submatrix, and consequently $M_{i\infty}^\top M_{i\infty} > 0$, implying that

$$\lim_{\omega \rightarrow \infty} U_i(-j\omega)^{-\top} \Phi_i(-j\omega, j\omega) U_i(j\omega)^{-1} , \quad i = 1, 2 ,$$

is invertible and therefore

$$U_1(-\xi)^{-\top} \Phi_1(-\xi, \xi) U_1(\xi)^{-1} = U_1(-\xi)^{-\top} A_1(-\xi)^\top A_1(\xi) U_1(\xi)^{-1},$$

as well as

$$U_2(-\xi)^{-\top} \Phi_2(-\xi, \xi) U_2(\xi)^{-1} = U_2(-\xi)^{-\top} H_2(-\xi)^\top H_2(\xi) U_2(\xi)^{-1},$$

have a proper inverse. Considering (2.6) and (2.7) in Sec. 2.6, we conclude that $\mathfrak{X}(M_1(\frac{d}{dt})) = \mathfrak{X}(A_1(\frac{d}{dt}))$ and $\mathfrak{X}(M_2(\frac{d}{dt})) = \mathfrak{X}(H_2(\frac{d}{dt}))$.

Consequently, the trajectories¹ $z_1 \in \ker A_1(\frac{d}{dt})$ and $z_2 \in \ker H_2(\frac{d}{dt})$ are such that $X_1(\frac{d}{dt}) z_1(0^-) = v_1$ and $X_2(\frac{d}{dt}) z_2(0^+) = L_{1 \rightarrow 2} v_1$. Finally, since the latent variables z_1 and z_2 are observable, they correspond to a unique trajectory $w \in \mathfrak{B}^\Sigma$ defined as in the Lemma with final/initial state v_1 and $L_{1 \rightarrow 2} v_1$ respectively. The lemma is proved. \square

We now prove the claim of Th. 6.7 by contradiction. Let $X_i^{\mathfrak{n}(\mathfrak{B}_i) \times \mathbf{z}}[\xi]$ be minimal state maps for \mathfrak{B}_i , $i = 1, \dots, N$ and $L_{j \rightarrow k} \in \mathbb{R}^{\mathfrak{n}(\mathfrak{B}_k) \times \mathfrak{n}(\mathfrak{B}_j)}$ with $j, k = 1, \dots, N$ the re-initialisation maps. Let w.l.o.g. $i = 1$, $j = 2$ and assume that there exists a final/initial state v_1 and $L_{1 \rightarrow 2} v_1$ for $w \in \mathfrak{B}^\Sigma$ respectively, such that $Q_{\Psi_1}(w)(0^-) < Q_{\Psi_2}(w)(0^+)$.

Construct latent variable trajectories $z_1, z_2 : \mathbb{R} \rightarrow \mathbb{R}^{\mathbf{z}}$ as in Lemma A.6 corresponding to an admissible switched trajectory $w \in \mathfrak{B}^\Sigma$. For this trajectory it holds that

$$\int Q_\Phi(w) = \int_{-\infty}^{0^-} Q_{\Phi'_1}(z_1) dt + \int_{0^+}^{\infty} Q_{\Phi'_2}(z_2) dt = Q_{\Psi'_1}(z_1)(0^-) - Q_{\Psi'_2}(z_2)(0^+) < 0;$$

which contradicts the fact that Σ is strictly Φ -dissipative.

Note that it follows automatically from the latter results that there exists an N -tuple $(Q_{\Psi_1}, \dots, Q_{\Psi_N})$ that satisfies the conditions 1) and 2) in Def 6.4. The theorem is proved. \square

Proof of Lemma 6.8. The fact that $\Psi'_i(\zeta, \eta)$, $i = 1, \dots, N$, can be factorised as $X_i(\zeta) K_i X_i(\eta)$, $i = 1, \dots, N$, follows from Prop. 3.20.

The equivalence of conditions 1) and 2) follows from the fact that $w = M_i(\frac{d}{dt}) z_i$, $i = 1, \dots, N$, and the standard reformulation of QDFs in terms of latent variables, see Sec. 3.4. We now prove the equivalence of conditions 2) and 3). Use Lemma A.6 in the proof of Th. 6.7 to conclude that since Σ is strictly Φ -dissipative, then the final/initial states at switching instants corresponding to z_i and z_j are arbitrary. Use the factorisations $\Psi'_i(\zeta, \eta) = X_i(\zeta) K_i X_i(\eta)$, $i = 1, \dots, N$, to conclude that $v_i^\top K_i v_i \geq v_j^\top K_j v_j$ for all $i, j \in \mathcal{P}$,

¹Note that z_i , $i = 1, 2$ are not trajectories with compact support, however an approximation argument can be used to complete the proof of the claim.

$i \neq j$. Then use the re-initialisation map to conclude that $v_i^\top K_i v_i \geq v_i^\top L_{i \rightarrow j}^\top K_j L_{i \rightarrow j} v_i$, which is equivalent to condition 3). \square

Proof of Prop. 6.9. The proof follows from the fact that the re-initialisation maps are also the identity, and since $K_i \geq K_j$ and $K_j \geq K_i$ for all $i, j \in \mathcal{P}$. Consequently, $K_i = K_j$ and $Q_{\Psi_i} = Q_{\Psi_j}$ for all $i, j \in \mathcal{P}$. \square

Proof of Prop. 6.11. The proof of the proposition follows readily from the same argument used in Prop. 6.2. \square

Proof of Th. 6.12. Define $t_0 := -\infty$. Using Prop. 6.3 and equation (6.1), it follows that since $\lim_{t \rightarrow -\infty} (v)(t) = 0$, we obtain

$$\begin{aligned} \int^\tau Q_\Phi(w) \geq & (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+)) + \sum_{k=2}^n (Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+)) \\ & + Q_{\Psi_{s(t_n)}}(w)(\tau) . \end{aligned}$$

Note that $Q_{\Psi_{s(t_{j-1})}}(w)(t_j^-) - Q_{\Psi_{s(t_j)}}(w)(t_j^+) \geq 0$, for every $t_j \in \mathbb{T}_s$. Moreover, since every mode has the same partition $w = \text{col}(u, y)$, it follows that $\sigma_+(\Phi) = \mathfrak{m}(\mathfrak{B}_i)$, $i = 1, \dots, N$. Use lemma 3.16 to conclude that $Q_{\Psi_{s(t_n)}}(w)(\tau) \geq 0$, consequently $\int^\tau Q_\Phi(w) \geq 0$. \square

Proof of Th. 6.13. To prove the first part of the claim note that the degree of X_k cannot exceed that of M_k , $k = 1, \dots, N$, because of the same argument used in Prop. 3.21. Moreover solutions K_k , $k = 1, \dots, N$, for the LMIs (6.2) exist because of the fact that \mathfrak{B}_k , $i = 1, \dots, N$, is strictly Φ -dissipative and Lemma 3.23. Moreover, according to Lemma 3.23 and Prop. 3.22 if the LMIs (6.2) hold, $\Psi_k(\zeta, \eta)$ induces a storage function for \mathfrak{B}_k , $k = 1, \dots, N$. Finally, note that due to Lemma 6.8, the LMIs (6.3) imply condition 2) in Def. 6.4, then using Th. 6.6 we conclude that Σ is Φ -dissipative. \square

Proof of Prop. 6.14. Since $Q_\Phi \stackrel{\mathfrak{B}_i}{\geq} \frac{d}{dt} Q_{\Psi_i}$ and $Q_\Phi \stackrel{\mathfrak{B}_i}{\geq} \frac{d}{dt} Q_{\Psi'_i}$, $i = 1, \dots, N$, it follows from standard results regarding dissipative systems (see Sec. 3.7) that

$$Q_\Phi \stackrel{\mathfrak{B}_i}{\geq} \frac{d}{dt} \left(\alpha Q_{\Psi_i} + (1 - \alpha) Q_{\Psi'_i} \right) , \quad i = 1, \dots, N .$$

Moreover, to show that condition 2) in Def. 6.4 is satisfied, let $s \in \mathcal{S}$ and note that since $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) \geq Q_{\Psi_{s(t_k)}}(w)(t_k^+)$ and $Q_{\Psi'_{s(t_{k-1})}}(w)(t_k^-) \geq Q_{\Psi'_{s(t_k)}}(w)(t_k^+)$ for every $t_k \in \mathbb{T}$, it follows that

$$\alpha Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) + (1 - \alpha) Q_{\Psi'_{s(t_{k-1})}}(w)(t_k^-) - \alpha Q_{\Psi_{s(t_k)}}(w)(t_k^+) - (1 - \alpha) Q_{\Psi'_{s(t_k)}}(w)(t_k^+) \geq 0 .$$

\square

Proof of Th. 6.16. Since Σ is strictly Φ -dissipative, it follows from Th. 6.7 that there exists a multiple storage function $Q_{\Psi} := (Q_{\Psi_1}, \dots, Q_{\Psi_N})$ for Σ . Note that since only the trajectories $\mathfrak{B}_{\text{aut}}^{\Sigma} \subseteq \mathfrak{B}^{\Sigma}$ are permitted for Σ_{aut} according to Def. 6.15, it necessarily follows that the trajectories of its mode behaviours are also restricted as $\mathfrak{B}'_i := \{w = \text{col}(u, y) \in \mathfrak{B}_i \mid u = 0\}$, $i = 1, \dots, N$. We now show that Σ_{aut} is asymptotically stable by showing that Q_{Ψ} satisfies the conditions 1)-3) in Th. 5.1.

- C1.** The fact that $Q_{\Psi_i} \stackrel{\mathfrak{B}'_i}{\geq} 0$, $i = 1, \dots, N$, follows directly from Lemma 3.16, i.e. since $\mathfrak{B}'_i \subseteq \mathfrak{B}_i$ and $Q_{\Psi_i} \stackrel{\mathfrak{B}_i}{\geq} 0$, $i = 1, \dots, N$, then $Q_{\Psi_i} \stackrel{\mathfrak{B}'_i}{\geq} 0$, $i = 1, \dots, N$.
- C2.** In order to prove that $\frac{d}{dt}Q_{\Psi_i}$ decreases along \mathfrak{B}'_i , $i = 1, \dots, N$, use Prop. 6.2 to show that \mathfrak{B}_i , $i = 1, \dots, N$, is strictly Φ -dissipative and consequently there exists $\epsilon_i > 0$ such that $Q_{\Phi}(w) \geq \frac{d}{dt}Q_{\Psi_i}(w) + \epsilon_i \|w\|_2^2$, $i = 1, \dots, N$. Since, for every trajectory $\text{col}(0, y) \in \mathfrak{B}'_i$ it follows that $Q_{\Phi}(w) = 0$, then $\frac{d}{dt}Q_{\Psi_i}(w) \stackrel{\mathfrak{B}'_i}{\leq} -\epsilon_i \|w\|_2^2 < 0$, for every $w \neq 0$.
- C3.** Finally, note that the non increasing condition at switching instants 3) in Th. 5.1 is equivalent to condition 2) in Def. 6.4.

□

A.4 Proofs of Chapter 7

Proof of Prop. 7.4. The impedance Z_{T_k} , $k = 1, \dots, L$, is described by a one-port, and consequently can also be represented in observable image representation by $M' \in \mathbb{R}^{2 \times 1}[s]$ with external variables $w' := \begin{bmatrix} I' & v \end{bmatrix}^{\top}$ and a one-dimensional latent variable denoted by z'_k . It follows from the elimination theorem (see Sec. 6 of [55]) that after the elimination of the latent variable $z_{2,j}$, $j = 1, 2$, the interconnection of this one-port with the switching power converter has a number $2L$ of dynamic modes that can be described as two-ports, corresponding to the image representations (7.5). □

Proof of Prop. 7.5. If switching between modes does not involve short- or open-circuiting sources, no constraints on the input variables of the system are imposed at the switching instants. Consequently, the gluing conditions only impose constraints on the output variables of the modes, which are linear functions of the state variables. The claim follows. □

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