Separability properties of graph products of groups

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Tuto práci věnuji svým rodičům, Ivaně a Milanovi, a mojí sestře Dáše.

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
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Doctor of Philosophy

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by Michal Ferov

Separability properties provide an algebraic analogue to the solvability of decision problems in groups. It is natural to ask whether a certain group property is preserved by some specific group-theoretic constructions. In this thesis we study the stability of certain separability properties under graph products, a natural generalisation of free and direct products.

This thesis consists of material published in:

[**15**] M. Ferov: On conjugacy separability of graph products of groups;

[**16**] M. Ferov: Separability properties of automorphisms of graph products of groups;

[**6**] F. Berlai, M. Ferov: Residual properties of graph products of groups.

In [**15**] we study conjugacy separability in graph products of groups. In particular, we show that the class of $\mathcal{C}$-hereditarily conjugacy separable groups is closed under taking arbitrary graph products whenever the class $\mathcal{C}$ is an extension closed variety of finite groups. As a consequence we show that the class of $\mathcal{C}$-conjugacy separable groups is closed under taking arbitrary graph products. In particular, we show that right angled Coxeter groups are hereditarily conjugacy separable and 2-hereditarily conjugacy separable, and we show that infinitely generated right angled Artin groups are hereditarily conjugacy separable and p-hereditarily conjugacy separable for every prime number p.

In [**16**] we study various properties of automorphisms of graph products of groups. In particular, we show that a graph product $\Gamma\mathcal{G}$ has non-inner pointwise inner automorphisms if and only if some vertex group corresponding to a central vertex has non-inner pointwise inner automorphisms. We use this result to study the residual finiteness of $\mathrm{Out}(\Gamma\mathcal{G})$. We show that if all vertex groups are finitely generated residually finite and the if vertex groups corresponding to central vertices satisfy a certain technical (yet natural) condition, then $\mathrm{Out}(\Gamma\mathcal{G})$ is residually finite. Finally, we generalise this result to graph products of residually p-finite groups to show that if $\Gamma\mathcal{G}$ is a graph product of finitely generated residually p-finite groups such that the vertex groups corresponding to central vertices satisfy the p-version of the technical condition, then $\mathrm{Out}(\Gamma\mathcal{G})$ is virtually residually p-finite. We use this result to prove bi-orderability of the Torelli groups of some graph products of finitely generated residually torsion-free nilpotent groups.

In [**6**] we study residual properties of graph products of groups. In particular, we prove that the class of residually-$\mathcal{C}$ groups is closed under taking graph products, provided that $\mathcal{C}$ is closed under taking subgroups and finite direct products, and that free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$. As a consequence, we show that local embeddability into various classes of groups is stable under graph products. In particular, we prove that

graph products of residually amenable groups are residually amenable, and that the class groups locally embeddable into amenable groups is closed under taking graph products.

Declaration of authorship

I, Michal Ferov, declare that the thesis entitled *Separability properties of graph products of groups* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- This thesis consists of material published in:
  [**15**] M. Ferov: On conjugacy separability of graph products of groups;
  [**16**] M. Ferov: Separability properties of automorphisms of graph products of groups;
  [**6**] F. Berlai, M. Ferov: Residual properties of graph products of groups.

Signed:..

Date:..

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Contents

CHAPTER 1

Introduction

Groups are very important in mathematics because they describe symmetries of mathematical objects and structures. Consequently, it is natural to ask questions about their behaviour. In the case of finite groups we can construct the Cayley table which fully captures the structure of the given group. Sometimes we are given an explicit description of how to work with the elements of the given group, for example we might be given a matrix representation, as in the case of groups of automorphisms of finite dimensional vector spaces, or we might be given a nice formula as in the case of additive group of points on an elliptic curve. However this is not always the case. Sometimes we are given only an abstract presentation of a group, i.e. we are given a set of generators and a description of the relations between them. More formally, we have a presentation $\langle X \| R\rangle$ where X is the set of generators and R is the set of relations. For example, the presentation of $\pi_1(\Sigma)$ where Σ is an orientable surface of genus 2 is $\langle x_1, y_1, x_2, y_2 \| [x_1, y_1][x_2, y_2] = 1\rangle$ and the presentation of the group of symmetries of a bi-infinite simplicial path is $\langle a, b \| a^2 = 1, a^{-1}ba = b^{-1}\rangle$. In these cases we are just working with words over the alphabet of the given generating symbols and we do not primarily know which actual elements of the given group these words represent. By a group word in an alphabet X we mean a finite string $g = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i \in \{-1, 1\}$. We will often just call this a *word*.

1.1. Decision problems in groups

In the beginning of twentieth century Max Dehn formulated the three fundamental decision problems for groups:

(1) *Word problem* - given a presentation $\langle X \| R\rangle$ of a group G and a word g in the generating alphabet X we ask: does g represent the trivial element in G? In other words: is $g = 1$ in G?
(2) *Conjugacy problem* - given a presentation $G = \langle X \| R\rangle$ of a group G and a pair of words g_1, g_2 in the generating alphabet X we ask: do the words g_1 and g_2 represent conjugate elements in G? In other words: is there $c \in G$ such that $g_1 = c^{-1} g_2 c$?
(3) *Isomorphism problem* - given a presentation $\langle X_1 \| R_1\rangle$ of a group G_2 and a presentation $\langle X_2 \| R_2\rangle$ of a group G_2 we ask: is G_1 isomorphic to G_2?

Given a presentation $\langle X \| R\rangle$ of a group G we will use the symbol $\equiv$ to denote that two words are identical, thus by $g_1 \equiv g_2$ with $g_1 = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ and $g_2 = y_1^{\beta_1} y_2^{\beta_2} \ldots y_m^{\beta_m}$ where $x_i, y_j \in X$ and $\alpha_i, \beta_j \in \{-1, 1\}$, we mean if $g_1 = g_2$ then $n = m$, $x_i = y_i$ and $\alpha_i = \beta_i$ for all $i = 1, 2, \ldots, n$. For the sake of simplicity we will always assume that words are freely reduced. We will use the symbol $=_G$ to denote that two words represent the same element in a group G. Often, if it is obvious from the context, we will omit the subscript. Lastly, we will use the symbol $\sim_G$ to denote that two words represent conjugate elements in G.

The word and conjugacy problems have been well studied. It is not difficult to show that there is a recursively presented group (meaning that the generating set X is finite and the set of relations R is recursively enumerable) in which the word problem is unsolvable. A result of Boone and Novikov [**24**, Theorem 7.2, page 225] in the late fifties showed that there is a finitely presented group with unsolvable word problem. It is easy to see that if we could solve the conjugacy problem then we would be able to solve the word problem, since $g \in G$ is conjugate to 1 in G if and only if g is trivial. However, the implication in the opposite direction does not hold: Miller [**10**] constructed a finite presentation of a group with solvable word problem and unsolvable conjugacy problem. By a theorem of Adian and Rabin [**24**, Theorem 4.1, page 192] the property of having solvable word problem cannot be algorithmically recognised. The same holds for the conjugacy problem.

However, there are group properties that imply the solvability of the word conjugacy problems, which behave 'nicely'.

1.2. Algorithms of Mal'cev-Mostowski type

We say that a group G is residually finite (RF) if for every $g \in G \setminus \{1\}$ there is a finite group H and a homomorphism $\pi : G \to H$ such that $\pi(g) \neq_H 1$.

Note that finite groups are recursively enumerable; for each $n \in \mathbb{N}$ we can simply generate all the Cayley tables of size $n \times n$ and then check whether a given table represents a group and whether the generated group is already in the list of groups that we have already generated. Thus we can use G_i to denote the i-th finite group. Further, let us note that if a finite group G_i is given by a Cayley table then it is very easy to solve the word problem simply by computing the actual value of g using the table.

If we are given a presentation $\langle X \| R \rangle$ of a finitely generated group G and a finite group G_i then the set $\mathrm{Hom}(G, G_i)$ is finite because every generator of G can be mapped only to finitely many elements of G_i. However it might not be easy to check whether a map $\pi : X \to G_i$ extends to a homomorphism. We need to check whether all the relations $r \in R$ are satisfied, meaning that when $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n} = r \in R$ we want to check whether $\pi(x_1)^{\epsilon_1} \pi(x_2)^{\epsilon_2} \dots \pi(x_n)^{\epsilon^n} = 1$ in G_i. In case when R is infinite, the naive method of checking for every $r \in R$ would not terminate, but if R is finite then the naive method always terminates. Therefore for a finitely presented group G and a finite group G_i, we can always list all the elements of $\mathrm{Hom}(G, G_i)$ in finite amount of time.

Note that given a recursive presentation $\langle X \| R \rangle$ of a group G we can enumerate all words in X that represent the trivial element in G. Let g be a word in X. Obviously $g =_G 1$ if and only if $g =_{F(X)} \prod_{i=1}^{n} u_i^{-1} r_i^{\epsilon_i} u_i$ for some $n \in \mathbb{N}$ where $r_i \in R$, $\epsilon_i = \pm 1$, u_i is a word in X and $F(X)$ is the free group over the alphabet X. Every word of this type can be generated by a sequence of transformations (or their inverses) of the following two types:

(1) insert a 'trivial'subword: xx^{-1} where $x \in X^{\pm}$,
(2) insert a relator or its inverse: r^{ϵ} where $r \in R$ and $\epsilon = \pm 1$.

More formally: let $g = ab$ be a word where a, b are words (possibly empty) in X. A transformation of the type (1) transforms the word $g = ab$ to the word $g' = ax^{-1}xb$, where $x \in X^{\pm}$. A transformation of the type (2) transforms the word $g = ab$ to $g'' = ar^{\epsilon}b$ where $r \in R$ and $\epsilon = \pm 1$. Obviously $g =_G g' =_G g''$. Since the set X is finite and the set R is recursively enumerable, we can enumerate all the possible sequences of transformations and thus we can enumerate all representatives of the identity element in G.

Therefore if we were given a word g in X we could naively go through all the representatives w_i of 1 in G and check whether $w_i \equiv g$. If $g =_G 1$ then this process will surely terminate, however if $g \neq_G 1$ then this naive method will never stop.

Let G_i denote the i-th finite group and let w_i denote the i-th representative of the trivial element in G.

Algorithm 1 Mal'cev(G, g)

$i \leftarrow 0$
$answer \leftarrow$ ""
while $answer =$ "" **do**
  $i \leftarrow i + 1$
  if $g \equiv w_i$ **then**
    $answer = YES$
  end if
  for all $\pi \in \mathrm{Hom}(G, G_i)$ **do**
    if $\pi(g) \neq_{G_i} 1$ **then**
      $answer = NO$
    end if
  end for
end while

It is obvious that for finitely presented groups this algorithm halts on every word g if and only if G is RF, hence we get equivalent definition of the property RF for finitely presented groups in terms of algorithms and solvability: a finitely presented group G is RF if and only if the word problem in G can be solved by the Mal'cev's algorithm.

This algorithm can be generalised to the conjugacy problem if we pose a stronger condition on the group G.

We say that a group G is *conjugacy separable* (CS for short) if for every pair $g_1, g_2 \in G$ such that $g_1 \not\sim_G g_2$ there is a finite group H and a homomorphism $\phi : G \to H$ such that $\phi(g_1) \not\sim_H \phi(g_2)$.

It is clear that CS implies RF as the identity element is conjugate only to itself, therefore the word problem in a finitely presented CS group can be solved by the algorithm 1. Hence, given three words g_1, g_2, c we can determine whether $c^{-1}g_1c =_G g_2$ by asking whether $c^{-1}g_1cg_2^{-1} =_G 1$. Let us also note that solving the conjugacy problem in a finite group given by a Cayley table is fairly easy, because we can simply try all possible candidates for the conjugating element.

Note that we can easily enumerate all group words over finite alphabet X, thus we can use c_i to denote the i-th word in X.

The naive method to determine whether $g_1 \sim_G g_2$ would simply go through all $c_i \in G$ and check whether $c^{-1}g_1c =_G g_2$. If g_1 and g_2 truly represent conjugate elements of G then this method will terminate. However, if $g_1 \not\sim_G g_2$ this algorithm will never stop.

Let G_i denote the i-th finite group and let c_i denote the i-th word in X.

Algorithm 2 Mostowski(G, g_1, g_2)

$i \leftarrow 0$
$answer \leftarrow$ ""
while $answer =$ "" **do**
  $i \leftarrow i + 1$
  if $c_i^{-1}g_1c_i =_G g_2$ **then**
    $answer = YES$
  end if
  for all $\pi \in \mathrm{Hom}(G, G_i)$ **do**
    if $\pi(g_1) \not\sim_{G_i} \pi(G_2)$ **then**
      $answer = NO$
    end if
  end for
end while

It is clear that for finitely presented groups this algorithm will halt on every input pair $g_1, g_2 \in G$ if and only if G is CS. Thus we get equivalent definition of property of CS for finitely presented groups in terms of algorithms and solvability: a finitely presented group G is CS if and only if the conjugacy problem in G can be solved by Mostowski's algorithm.

Unlike in the case of word problem, behaviour of conjugacy equivalence is much more complicated and less predictable. If a group G has solvable word problem, then every $H \leq G$ has solvable word problem. Conversely if $H \leq_{\text{f.i.}} G$ and H has solvable word problem, then G has solvable word problem (see [**4**]). These statements are not true for the conjugacy problem. Collins and Miller [**11**] have shown:

(i) there is a finitely presented group G with solvable conjugacy problem with $H \leq_{\text{f.i.}} G$ such that H has unsolvable conjugacy problem,
(ii) there is a finitely presented group G with unsolvable conjugacy problem with $H \leq_{\text{f.i.}} G$ such that H has solvable conjugacy problem.

Again, there is a partial analogue to this statement in terms of separability properties. Chagas and Zalesskii [**9**] have showed that there is a CS group G with $H \leq_{\text{f.i.}} G$ such that H is not CS. However, the group constructed by Chagas and Zalesskii was not finitely presented. Minasyan and Martino [**25**] showed that for every integer $n \in \mathbb{N}$ there is a finitely presented CS group G together with a subgroup $N \leq G$ such that $|G : N| = n$ and N is not CS.

We say that a group G is *hereditarily conjugacy separable* (HCS for short) if it is CS and every $H \leq_{\text{f.i.}} G$ is CS.

Goryaga [**18**] gave a partial CS analogue to (ii), i.e. he constructed a finitely generated group G with $H \leq_{\text{f.i.}} G$ such that H is CS but G is not. However, there are no known finitely presented examples.

1.3. Separability properties

Group properties like RF and CS are called *separability properties*, as they are described by whether certain subsets can be separated: we say that a subset $X \subseteq G$ is *separable* in G if for every $g \in G \setminus X$ there is a finite group F and a homomorphism $\varphi\colon G \to X$ such that $\varphi(g) \notin \varphi(X)$ in F. Clearly a group G is RF if and only if the singleton set $\{1\}$ is separable in G and G is CS if and only if for every $g \in G$ the conjugacy class $g^G = \{cgc^{-1} \mid h \in G\}$ is separable in G.

We say that a group is *cyclic subgroup separable* (CSC, or sometimes π_C) if for every $f, g \in G$ such that $f \notin \langle g \rangle$ in G there is a finite group F and a homomorphism $\varphi\colon G \to F$ such that $\varphi(g) \notin \langle \varphi(g) \rangle$ in F, i.e. as the name suggests, a group G is CSC if for every $g \in G$ the cyclic subgroup $\langle g \rangle \leq G$ is separable in G.

Similarly, a group is *locally extended residually finite* (LERF) if every finitely generated subgroup $K \leq G$ is separable in G.

A group is *double coset separable* (DCS) if for every pair of finitely generated subgroups $H, K \leq G$ and an arbitrary element $g \in G$ the double coset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in G.

In a way, separability properties provide an algebraic analogue to decision problems in groups. It is obvious that *the power problem* in a group G, i.e. the problem of deciding, given elements $f, g \in G$, whether f is a power of g can be solved by an algorithm of Mal'cev-Mostowski type if and only if G is π_C. Similarly, *the generalised word problem*, i.e. the problem of deciding, given elements $f, g_1, \ldots, g_n \in G$, whether $f \in \langle g_1, \ldots, g_n \rangle \leq G$, can be solved in finitely presented LERF groups.

As we already mentioned, it is easy to see that every CS group is RF. Similarly, every π_C group is RF and so on. The following diagram demonstrates the known implications between various separability properties.

$$\begin{tikzcd}
 & \mathrm{CS} \arrow[dl, Rightarrow] & \mathrm{HCS} \arrow[l, Rightarrow] & \\
\mathrm{RF} & & & \\
 & \pi_C \arrow[ul, Rightarrow] & \mathrm{LERF} \arrow[l, Rightarrow] & \mathrm{DCS} \arrow[l, Rightarrow]
\end{tikzcd}$$

The motivation behind separability properties is that we would like to be able to approximate groups by their finite quotients. Different separability properties tell us how precisely this approximation can be done, i.e. how much information about a group can be reconstructed simply by looking at its finite quotients.

CHAPTER 2

Pro-$\mathcal{C}$ topologies on groups

What if we were not interested in all finite groups? What if we were to consider, say, only finite 2-groups or finite nilpotent groups? Or what if we were not interested in finite groups at all? Let $\mathcal{C}$ be a class of groups. Note that we will always assume that classes of groups are closed under isomorphisms, i.e. if $C \in \mathcal{C}$ and $C' \cong C$ then $C' \in \mathcal{C}$ as well. For a group G we say that a subset $X \subset G$ is $\mathcal{C}$-separable in G if for every $g \in G \setminus X$ there is a group $C \in \mathcal{C}$ and a homomorphism $\gamma\colon G \to C$ such that $\gamma(g) \notin \gamma(X)$. We then say that a group G is *residually-$\mathcal{C}$* if the singleton set $\{1\}$ is $\mathcal{C}$-separable in G. Other separability properties can be generalised to $\mathcal{C}$ in a similar manner: a group G is $\mathcal{C}$-conjugacy separable ($\mathcal{C}$-CS) if the conjugacy class g^G is $\mathcal{C}$-separable in G for every $g \in G$.

Clearly, some classes are more difficult to work with than other. For example every RF group is *fully* residually finite, i.e. if G is a RF group, then for every finite subset $X \subseteq G \setminus \{1\}$ there is a finite group C and a homomorphism $\gamma\colon G \to C$ such that $1 \notin \gamma(X)$ in C and $\gamma \restriction_X$ is injective. However, it is not true that every residually free group is fully residually free. In this chapter we show that if the class $\mathcal{C}$ satisfies certain closure properties, then we can actually equip every group G with a topology that captures the notion of $\mathcal{C}$-separability. Amongst other things, this will allow us to use basic methods and terminology from point-set topology and thus significantly simplify our proofs.

Most of the statements in this chapter can be found in [**23**] and [**33**].

What closure properties do we require the class $\mathcal{C}$ to have? We will be considering the following properties:

(c1) subgroups: let $G \in \mathcal{C}$ and suppose that $H \leq G$; then $H \in \mathcal{C}$,
(c2) finite direct products: let $G_1, G_2 \in \mathcal{C}$; then $G_1 \times G_2 \in \mathcal{C}$,
(c3) quotients: let $G \in \mathcal{C}$ and let $N \trianglelefteq G$; then $G/N \in \mathcal{C}$,
(c4) extensions: let $Q, K \in \mathcal{C}$ and let G be a group such that the following sequence

$$1 \to K \to G \to Q \to 1$$

is exact; then $G \in \mathcal{C}$.

Let $\mathcal{C}$ be a class of groups and let G be a group. If $N \trianglelefteq G$ is such that $G/N \in \mathcal{C}$ then we say that N is a *co-$\mathcal{C}$* subgroup of G and that G/N is a $\mathcal{C}$-quotient of G. We will use $\mathcal{N}_\mathcal{C}(G) = \{N \trianglelefteq G \mid G/N \in \mathcal{C}\}$ to denote the set of all co-$\mathcal{C}$ subgroups of G. We want the system of cosets $\mathcal{B}_\mathcal{C}(G) = \{gN \mid g \in G, N \in \mathcal{N}_\mathcal{C}(G)\}$ to form a basis of open sets for a topology on G, thus we need the set $\mathcal{N}_\mathcal{C}(G)$ to be closed under intersections.

Lemma 2.1. *Let $\mathcal{C}$ be a class of groups. If $\mathcal{C}$ satisfies (c1) and (c2), then the set $\mathcal{N}_\mathcal{C}(G)$ is closed under intersections for every group G.*

PROOF. Let G be a group and let $N_1, N_2 \in \mathcal{N}_\mathcal{C}(G)$. Clearly $N_1 \cap N_2$ is a normal subgroup of G. By assumption $G/N_1, G/N_2 \in \mathcal{C}$ and thus $G/N_1 \times G/N_2 \in \mathcal{C}$ since the class $\mathcal{C}$ is closed under direct products. By a standard result we see that $G/(N_1 \cap N_2)$ embeds into $G/N_1 \times G/N_2$. Since we assume that the class $\mathcal{C}$ is closed under taking subgroups we get that $G/(N_1 \cap N_2) \in \mathcal{C}$ and we can conclude that $N_1 \cap N_2 \in \mathcal{N}_\mathcal{C}(G)$. □

Suppose that the system of cosets $\mathcal{B}_\mathcal{C}(G)$ forms a basis of open sets for a topology on a group G. This topology is called the *pro-$\mathcal{C}$ topology* on G and we will use pro-$\mathcal{C}(G)$ to refer to this topology. If $\mathcal{C}$ is the class of all finite groups this topology is the profinite topology $\mathcal{PT}(G)$, and if $\mathcal{C}$ is the class of all finite p-groups, where p is a prime number, this topology is referred to as pro-p topology.

We say that a subset $X \subseteq G$ is *$\mathcal{C}$-separable* or *$\mathcal{C}$-closed* in G if the subset X is closed in pro-$\mathcal{C}(G)$. In other words, a subset $X \subseteq G$ is $\mathcal{C}$-separable if for every $g \in G \setminus X$ there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $gN \cap XN = \emptyset$. Similarly we say that a subset $X \subseteq G$ is *$\mathcal{C}$-open* if it is open in pro-$\mathcal{C}(G)$.

2.1. Basic properties

Unless stated otherwise we will only assume that the class $\mathcal{C}$ satisfies (c1) and (c2).

If the class $\mathcal{C}$ satisfies (c1) and (c2) then the pro-$\mathcal{C}$ topology on G is well-defined for every group G by Lemma 2.1. Note that it would be enough to assume that the class $\mathcal{C}$ is closed under subdirect products: let G, N_1, N_2 be as in the proof of the Lemma 2.1, then $G/(N_1 \cap N_2)$ is a subdirect product of G/N_1 and G/N_2. Being closed under taking direct products and subgroups is a much stronger property. However, if we assume that the class $\mathcal{C}$ is also closed under taking subgroups, then we see that equipping a group G with pro-$\mathcal{C}$ topology is actually a faithful functor from the category of groups to the category of topological groups.

LEMMA 2.2. *Let G be a group. Then G, equipped with* pro-$\mathcal{C}(G)$*, is a topological group. Moreover, homomorphisms between groups are continuous maps and isomorphisms are homeomorphisms, thus preimages of $\mathcal{C}$-open/closed sets are $\mathcal{C}$-open/closed and isomorphic images of $\mathcal{C}$-open/closed sets are $\mathcal{C}$-open/closed.*

PROOF. To show that G is a topological group we need to show that maps

$$\begin{aligned} &i : G \to G && i(g) = g^{-1} \\ &\circ : G \times G \to G && \circ(g_1, g_2) = g_1 g_2 \end{aligned}$$

are continuous. Since $(gh)^{-1} = h^{-1}g^{-1}$ we have $i^{-1}(gH) = Hg^{-1}$ meaning that i is continuous because the preimage of an open set is an open set. Note that we do not consider the pro-$\mathcal{C}$ topology on $G \times G$, we consider the product topology. Thus the basic open sets in $G \times G$ are of the form $g_1H_1 \times g_2H_2$ where $g_1, g_2 \in G$ and $H_1, H_2 \in \mathcal{N}_\mathcal{C}(G)$. Now let $H \in \mathcal{N}_\mathcal{C}(G)$ and let $g \in G$. Then $\circ^{-1}(gH) = \{g_1H \times g_2H \mid g_1g_2 \in gH\}$ which is an union of open sets. This means that both $\circ$ and i are continuous and consequently G together with pro-$\mathcal{C}(G)$ is a topological group.

Let A, B be groups equipped with the pro-$\mathcal{C}$ topology and let $\phi : A \to B$ be a group homomorphism. Let $B' \in \mathcal{N}_\mathcal{C}(B)$ and $b \in B$. Set $A' = \phi^{-1}(B')$ and consider the map $\tilde{\phi}$: $A/A' \to B/B'$ given by $\tilde{\phi}(aA') = \phi(a)B$. Clearly $\tilde{\phi}$ is an monomorphism and thus A/A' is isomorphic to a subgroup of B/B'. Since we assume that the class $\mathcal{C}$ satisfies (c1) we see that $A' = \phi^{-1}(B) \in \mathcal{N}_\mathcal{C}(A)$. Also if $\phi(a_1) = b = \phi(a_2)$ for some $a_1, a_2 \in A$, then $a_1^{-1}a_2 \in \phi^{-1}(B')$, so preimages of cosets are cosets as well. Thus a preimage of a basic open set is a basic open set, therefore ϕ is a continuous map of topological spaces. $\square$

Note that if the class $\mathcal{C}$ was not closed under taking subgroups, then only epimorphisms would be continuous.

One of the consequences of the lemma we have just proved is that the pro-$\mathcal{C}$ topology on a group G is invariant under group translation: if $X \subseteq G$ is $\mathcal{C}$-closed in G then so are the sets gX and Xg for all $g \in G$.

LEMMA 2.3. *Let G be a group. Then the following are equivalent*

(i) $\{1\} \in \mathcal{N}_\mathcal{C}(G)$,
(ii) pro-$\mathcal{C}(G)$ *is the discrete topology on G,*
(iii) $G \in \mathcal{C}$.

PROOF. (i) $\Leftrightarrow$ (iii): since $G/\{1\} \cong G$ we see that $G \in \mathcal{C}$ if and only if $\{1\} \in \mathcal{N}_\mathcal{C}(G)$.

(i) $\Rightarrow$ (ii): note that a topology on a set is discrete if and only if all singleton sets are open. If $\{1\} \in \mathcal{N}_\mathcal{C}(G)$ then it is $\mathcal{C}$-open in G. Let $g \in G$ be arbitrary. Clearly $\{g\} = g\{1\}$ and therefore $\{g\}$ is $\mathcal{C}$-open in G as it is a translate of an $\mathcal{C}$-open set.

(i) $\Leftarrow$ (ii): assume that pro-$\mathcal{C}(G)$ is discrete. Then $X = G \setminus \{1\}$ is $\mathcal{C}$-closed and $1 \notin X$, thus there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $1N \cap X = N \cap X = \emptyset$. Clearly the only such N is $\{1\}$. $\square$

As we already said: a group G is *residually-$\mathcal{C}$* if for every $g \in G \setminus \{1\}$ there is a group $F \in \mathcal{C}$ and a group homomorphism $\pi\colon G \to F$ such that $\pi(g) \neq 1$ in F. Assuming that the class $\mathcal{C}$ satisfies (c1) and (c2), we equivalently can say that G is residually-$\mathcal{C}$ if for every $g \in G \setminus \{1\}$ there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $g \notin N$. Since $\mathcal{N}_\mathcal{C}(G)$ is closed under finite intersections we see that the notion of being residually-$\mathcal{C}$ and fully residually-$\mathcal{C}$ coincide.

LEMMA 2.4. *Let G be a group. Then the following are equivalent:*

(i) *G is residually-$\mathcal{C}$,*
(ii) *$\{1\}$ is $\mathcal{C}$-closed in G,*
(iii) $\bigcap_{N \in \mathcal{N}_\mathcal{C}(G)} N = \{1\}$,
(iv) pro-$\mathcal{C}(G)$ *is Hausdorff.*

PROOF. (i) $\Rightarrow$ (ii): let G be a residually-$\mathcal{C}$ group. Then for every $g \in G \setminus \{1\}$ there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $g \notin N$, thus $gN \cap \{1\} = \emptyset$. This means that $\{1\}$ is $\mathcal{C}$-closed in G.

(ii) $\Rightarrow$ (iii): suppose that $\{1\}$ is $\mathcal{C}$-closed in G. Let $g \in G \setminus \{1\}$ and assume that $g \in \bigcap_{N \in \mathcal{N}_\mathcal{C}(G)} N$. Since $\{1\}$ is closed in G there is an $N' \in \mathcal{N}_\mathcal{C}(G)$ such that $g \notin N'$, which is a contradiction as we assumed that $g \in N$ for all $N \in \mathcal{N}_\mathcal{C}(G)$.

(iii) $\Rightarrow$ (i): let $\bigcap_{N \in \mathcal{N}_\mathcal{C}(G)} N = \{1\}$ and let $g \in G \setminus \{1\}$. We see that $\{1\}$ is an intersection of $\mathcal{C}$-closed subsets and thus it is $\mathcal{C}$-closed in G. As $g \notin \{1\}$ there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $gN \cap \{1\} = \emptyset$, hence $g \notin N$ and we see that G is residually-$\mathcal{C}$.

(i) $\Rightarrow$ (iv): let $g_1, g_2 \in G$ be arbitrary such that $g_1 \neq g_2$. Then $g_1^{-1}g_2 \neq 1$ and thus there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $g_1^{-1}g_2 \notin N$. This means that $g_1N \cap g_2N = \emptyset$, therefore any two distinct elements of G can be separated in G and thus pro-$\mathcal{C}(G)$ is Hausdorff.

(iv) $\Rightarrow$ (i): let $g \in G \setminus \{1\}$ be arbitrary. Since pro-$\mathcal{C}(G)$ is Hausdorff we see that there are $N, N' \in \mathcal{N}_\mathcal{C}(G)$ such that $1N \cap gN' = \emptyset$. This means that $g \notin N$ and thus G is residually-$\mathcal{C}$. □

Being residually-$\mathcal{C}$ is clearly a hereditary property.

REMARK 2.5. *Let G be a group and let $H \leq G$. If G is residually-$\mathcal{C}$ then H is residually-$\mathcal{C}$.*

Let G be a group and assume that $H \leq G$. For an element $g \in G$ we will use g^H to denote $\{hgh^{-1} \mid h \in H\} \subseteq G$, the set of H-conjugates of H. The symbol $\sim_H$ will denote the relation of being H-conjugates, i.e. $f \sim_H g$ if and only if $f \in g^H$. We can then restate the definition of $\mathcal{C}$-conjugacy separability in terms of pro-$\mathcal{C}$ topologies: group G is $\mathcal{C}$-CS if the conjugacy class g^G is $\mathcal{C}$-closed in G for every $g \in G$.

2.2. $\mathcal{C}$-open and $\mathcal{C}$-closed subgroups

Let $H \leq G$ be such that there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $N \leq H$. Then clearly H is a union of cosets of N and hence H is $\mathcal{C}$-open in G as it is a union of $\mathcal{C}$-open subsets of G. It turns out that the opposite implication holds as well.

LEMMA 2.6 (Classification of $\mathcal{C}$-open subgroups). *Let G be a group and let $H \leq G$. Then H is $\mathcal{C}$-open in G if and only if there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $N \leq H$. Moreover, every $\mathcal{C}$-open subgroup is $\mathcal{C}$-closed in G and if $\mathcal{C}$ contains only finite groups, then H is of finite index in G.*

PROOF. The sufficiency holds trivially as discussed before the statement of the lemma.

Now let $H \in G$ be $\mathcal{C}$-open in G. Since $1 \in H$ then H must contain some open neighbourhood of 1, hence there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $N \leq H$. Obviously $G \setminus H$ is a union of cosets of N and thus $G \setminus H$ is $\mathcal{C}$-open subset of G. Hence H is $\mathcal{C}$-closed in G. Finally, if $\mathcal{C}$ is a class of finite groups then $|G : N| < \infty$ and $N \leq H$, hence we see that $|G : H| < \infty$. □

Lemma 2.6 implies that in the profinite topology, open subgroups of G are exactly subgroups of finite index and in the pro-p topology, p-open subgroups of G are exactly subnormal subgroups of finite index whose index is a power of p. This allows us to generalise the notion of hereditary conjugacy separability.

DEFINITION 2.7. *Let $\mathcal{C}$ be a class of finite groups. We say that a group G is $\mathcal{C}$-*hereditarily conjugacy separable *($\mathcal{C}$-HCS) if G is $\mathcal{C}$-CS and $H \leq G$ is $\mathcal{C}$-CS as well whenever H is $\mathcal{C}$-open in G.*

We have a classification of $\mathcal{C}$-open subgroups. What can be said about $\mathcal{C}$-closed subgroups?

LEMMA 2.8 (Classification of $\mathcal{C}$-closed subgroups). *Let G be a group and let $H \leq G$. Then H is $\mathcal{C}$-closed in G if and only if it is an intersection of $\mathcal{C}$-open subgroups of G.*

PROOF. Assume that $H = \bigcap_{H' \in \mathcal{H}} H'$ where $\mathcal{H} = \{H' \in G \mid H' \text{ is } \mathcal{C}\text{-open in } G\}$. Then H is clearly $\mathcal{C}$-closed in G as it is an intersection of $\mathcal{C}$-closed sets.

Now assume that H is $\mathcal{C}$-closed in G. Denote $\mathcal{H} = \{H' \leq G \mid H \leq H'$ and H' is $\mathcal{C}$-open$\}$ and let $\overline{H} = \bigcap_{H' \in \mathcal{H}} H'$. Clearly $H \leq \overline{H}$. Assume that $H \neq \overline{H}$, then there is some $g \in \overline{H} \setminus H$. As H is $\mathcal{C}$-closed in G there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $gN \cap H = \emptyset$. This is true if and only if $g \notin NH$. Since we assume that $N \in \mathcal{N}_{\mathcal{C}}(G)$, we see that NH is a union of cosets of H and thus it is a $\mathcal{C}$-open subgroup of G. Also $H \leq NH$, thus $NH \in \mathcal{H}$. However, $g \notin NH$ but $g \in \overline{H} = \bigcap_{H' \in \mathcal{H}} H'$, which is a contradiction. $\square$

CHAPTER 3

Graph products of groups

When studying group properties, it is natural to ask whether given properties are stable under various group-theoretic constructions. Our main focus is the study of separability properties in graph products, a natural generalisation of free and direct products.

3.1. Definition and normal form

By a graph we will always mean a simplicial graph, i.e. Γ is a tuple $(V\Gamma, E\Gamma)$, where $V\Gamma$ is a set and $E\Gamma \subseteq \binom{V\Gamma}{2}$. We call $V\Gamma$ the set of vertices of Γ and $E\Gamma$ the set of edges of Γ.

Let Γ be a graph and suppose that $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ a family of groups. The *graph product* $\Gamma\mathcal{G}$ of the family $\mathcal{G}$ with respect to the graph Γ is the quotient of the free product $*_{v\in V\Gamma} G_v$ modulo all the relations of the form

$$g_u g_v = g_v g_u \text{ for all } g_u \in G_u,\ g_v \in G_v \text{ whenever } \{u, v\} \in V\Gamma.$$

If the underlying graph Γ is finite then we say that $\Gamma\mathcal{G}$ is a *finite graph product.*

We will refer to the groups G_v as *vertex groups.*

Obviously, if the graph Γ is totally disconnected, i.e. $E\Gamma = \emptyset$, then $\Gamma\mathcal{G}$ is isomorphic to $*_{v\in V\Gamma} G_v$, the free product of the vertex groups, and if the graph Γ is complete, i.e. if $E\Gamma = \binom{\Gamma V}{2}$, then $\Gamma\mathcal{G}$ is isomorphic to $\amalg_{v\in V\Gamma}$, the direct product of the vertex groups.

If $G_v \cong \mathbb{Z}$, the infinite cyclic group, for every $v \in V\Gamma$ then we say that the corresponding graph product $\Gamma\mathcal{G}$ is a *right angled Artin group* (RAAG). Graph products of groups were first introduced by Green in her Ph.D. thesis [**21**] as a generalisation of RAAGs.

If we set $G_v = C_2$, the cyclic group of order two, for all $v \in V\Gamma$ we get another well known class of groups: right angled Coxeter groups (RACGs).

Let $G = \Gamma\mathcal{G}$ be a graph product. Then every $g \in G$ can be obtained as a product of sequence of generators $W \equiv (g_1, g_2, \dots, g_n)$ where each g_i belongs to some $G_{v_i} \in \mathcal{G}$. We will say that W is a *word* in G and the g_i are its *syllables.* The number of syllables is the *length* of a word.

Transformations of the three following types can be defined on words in graph products:

(T1) remove a syllable g_i if $g_i =_G 1$,
(T2) remove two consecutive syllables g_i, g_{i+1} belonging to the same vertex group G_v and replace them by a single syllable $g_i g_{i+1} \in G_v$,
(T3) interchange consecutive syllables $g_i \in G_u$ and $g_{i+1} \in G_v$ if $\{u, v\} \in E\Gamma$.

The last transformation is also called *syllable shuffling*. Note that the transformations (T1) and (T2) decrease the length of a word whereas (T3) preserves it. Thus by applying these transformations to a word W we will get a word W' of minimal length representing the same element in G.

For $v \in V\Gamma$ we define $\mathrm{link}_\Gamma(v)$ to be the the set of vertices adjacent to v in Γ, more precisely $\mathrm{link}_\Gamma(v) = \{u \in V\Gamma \mid \{u, v\} \in E\Gamma\}$.

For $1 \leq i < j \leq n$, we say that the syllables g_i, g_j can be joined together if they belong to the same vertex group and 'everything between them commutes'. More formally: $g_i, g_j \in G_v$ for some $v \in V\Gamma$ and for all $i < k < j$ we have $g_k \in G_{v_k}$ such that $v_k \in \mathrm{link}_\Gamma(v)$. In this case obviously the words $W \equiv (g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_{j-1}, g_j, g_{j+1}, \dots, g_n)$ and $W' \equiv (g_1, \dots, g_{i-1}, g_i g_j, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$ represent the same group element in G, but the word W' is strictly shorter than W. We say that a word $W \equiv (g_1, g_2, \dots, g_n)$ is *reduced* if it is either an empty word or if $g_i \neq 1$ for all i and no two distinct syllables can be joined together.

As Green proved in her Ph.D. thesis, the notion of being reduced and the notion of having minimal length coincide.

THEOREM 3.1 (Normal Form Theorem for graph products of groups). *Let $G = \Gamma\mathcal{G}$ be a graph product. Every element $g \in G$ can be represented by a reduced word. Moreover, if two reduced words W, W' represent the same element in the group G then W can be obtained from W' by a finite sequence of syllable shuffling. In particular, the length of a reduced word is minimal among all words representing g, and a reduced word represents the trivial element if and only if it is the empty word.*

As an immediate consequence of the Normal Form Theorem we see that every vertex group G_v embeds into G. We can actually say much more.

For any subset $A \subseteq V\Gamma$ we will denote the corresponding full subgraph by Γ_A: $V\Gamma_A = A$ and for $u, v \in A$ we have $\{u, v\} \in E\Gamma_A$ if and only if $\{u, v\} \in E\Gamma$. Suppose that $A \subseteq V\Gamma$ and let G_A denote the subgroup of G generated by all G_v, where $v \in A$. We can construct the restricted graph product $\mathcal{G}_A\Gamma_A$, where $\mathcal{G}_A = \{G_v \mid v \in A\}$. Let $\iota : \Gamma_A\mathcal{G}_A \to G$ be a homomorphism defined by $\iota(g_v) = g_v$.

Suppose that $W = (g_1, \dots, g_n)$ is a reduced word in $\Gamma_A\mathcal{G}_A$. Since Γ_A is a full subgraph we see that elements $g_u \in G_u$ and $g_v \in G_v$, where $u, v \in A$, commute in $\Gamma_A\mathcal{G}_A$ if and only if $\iota(g_u), \iota(g_v)$ commute in G, therefore W is reduced in $\Gamma_A\mathcal{G}_A$ if and only $\iota(W)$ is reduced in G. By the Normal Form Theorem we get that $\Gamma_A\mathcal{G}_A$ embeds into G. Thus every $A \subset V\Gamma$ determines a subgroup $G_A \leq G$. We will call subgroups of this type *full subgroups* of $\Gamma\mathcal{G}$ and we say that a full subgroup G_A is a *proper* full subgroup if A is a proper subset of $V\Gamma$. By definition $G_\emptyset = \{1\}$ is also a full subgroup corresponding to the empty subset of $V\Gamma$. We say that G_A is a *maximal* full subgroup if $|V\Gamma \setminus A| = 1$.

Let G be a group and suppose that $R \leq G$. We say that R is a *retract* of G if there exists a surjective homomorphism $\rho \colon G \to R$ such that $\rho \restriction_R = \mathrm{id}_R$. We say that the map ρ is the *retraction* corresponding to R.

It can be easily seen that full subgroups are retracts.

REMARK 3.2. *Let $G = \Gamma\mathcal{G}$ be a graph product of groups and let $G_A \leq G$ be a full subgroup. Then G_A is a retract in G with corresponding retraction map $\rho_A : G \to G_A$*

defined on generators of G as follows:

$$\rho_A(g) = \begin{cases} g & \text{if } g \in G_v \text{ and } v \in A, \\ 1 & \text{otherwise.} \end{cases}$$

PROOF. Define a map $\pi : G \to \Gamma_A\mathcal{G}_A$ on generators by $\pi(g) = g$ if $g \in G_v$ for some $v \in A$, and $\pi(g) = 1$ otherwise. Let $g_1 \in G_u$ and $g_2 \in G_v$ where $u, v \in V\Gamma$. If g_1, g_2 commute in G, then their images $\pi(g_1), \pi(g_2)$ commute in $\Gamma_A\mathcal{G}_A$. Thus all relations of $\Gamma\mathcal{G}$ hold in $\Gamma_A\mathcal{G}_A$. By von Dycks theorem [**36**, Theorem 2.2.1] π extends to a surjective homomorphism. As was stated before, $\Gamma_A\mathcal{G}_A$ embeds in $\Gamma\mathcal{G}$. Denote this embedding by $\iota : \Gamma_A\mathcal{G}_A \to \Gamma\mathcal{G}$. Obviously $\rho_A = \iota \circ \pi$ on all vertex groups, thus $\rho_A|_{G_A} = \mathrm{id}_{G_A}$. $\square$

In [**15**, Section 3] we study further properties of graph product of groups. In particular, we develop a theory of cyclically reduced elements, which allows us to state the conjugacy criterion for graph products.

3.2. Special amalgams and structure of graph products

DEFINITION 3.3. *Let A, H be groups and let $H \leq A$. Then we define $A \star_H C$, the* special amalgam *of A and C over H, to be the following free product with amalgamation:*

$$A *_H (H \times C).$$

As a very special case of free products with amalgamation, special amalgams do have a canonical normal form (see [**26**, Theorem 4.4, page 201]). In fact we can say much more about canonical normal form in the case of special amalgams.

Let $G = A \star_H C$. Obviously, every element $g \in G$ can be represented as a product $x_0c_1x_1 \dots c_nx_n$ where $x_i \in A$ for $i = 0, 1, \dots, n$ and $c_j \in C$ for $j = 1, \dots, n$. We say that $g = x_0c_1x_1 \dots c_nx_n$ is in a *reduced form* if $x_i \notin H$ for $i = 1, \dots, n-1$ and $c_j \neq 1$ for $j = 1, \dots, n$. In the first presented paper [**15**] we prove the following, using the normal form theorem for free products with amalgamation [**26**, Theorem 4.4]:

[**6**, Lemma 5.3]. *: Let $H \leq A, C$ be groups and let $G = A \star_H C$. Suppose that $g = x_0c_1x_1 \dots c_nx_n$, where $x_0, x_1, \dots, x_n \in A$ and $c_1, \dots, c_n \in C$, with $n \geq 1$ is in reduced form. Then $g \neq 1$ in G.*

Moreover, suppose that $f = y_0d_1y_1 \dots d_my_m$, where $y_0, y_1, \dots, y_m \in A$ and $d_1, \dots, d_m \in C$, is in reduced form with $m \geq 1$ as well and $f = g$. Then $m = n$ and $c_i = d_i$ for all $i = 1, \dots, n$.

Let $G = A *_{C_A = C_B} B$ be a free product with amalgamation and let $\alpha : A \to H$, $\beta : B \to H$ be group homomorphisms. By the universal property of free products with amalgamation, we see that α, β extend uniquely to a homomorphism $\phi : G \to H$ if and only if α and β coincide. In the case of special amalgams this functorial property can be strengthened.

REMARK 3.4 (Functorial property of special amalgams). *Let $G = A \star_C D$ be the special amalgam of groups A and D over C and let $\alpha : A \to A'$ and $\delta : D \to D'$ be homomorphisms of groups. Then α and δ extend uniquely to a homomorphism $\phi : G \to G'$, where*

$$G' = A' \star_{\alpha(C)} D',$$

such that $\phi(a) = \alpha(a)$ for all $a \in A$ and $\phi(d) = \delta(d)$ for all $d \in D$.

Proof. Let $\gamma = \alpha|_C$ be the restriction of α to C. Then by the functorial property of direct products there is unique homomorphism $\mu : C \times D \to \alpha(C) \times D'$ such that $\mu(c, d) = (\gamma(c), \delta(d))$. Since μ and α coincide on C, they can be uniquely extended to a homomorphism ϕ such that $\phi(a) = \alpha(a)$ for all $a \in A$ and $\phi(cd) = \mu(cd)$ for all $cd \in C \times D$. □

In [**15**, Section 5] we develop the theory of special amalgams. In particular, we give a complete description of centralisers and we state a conjugacy criterion for special amalgams.

The main reason why we are interested in special amalgams is that they naturally appear in graph products.

Remark 3.5. *Let $G = \Gamma\mathcal{G}$ be a graph product and suppose that $|V\Gamma| \geq 2$. Then for every $v \in V\Gamma$ there is a natural splitting of G as a special amalgam of full subgroups.*

Proof. Let $v \in V\Gamma$ be arbitrary. Let us denote $C = \operatorname{link}_\Gamma(v)$, $B = \{v\} \cup \operatorname{link}_\Gamma(v)$ and $A = V \setminus \{v\}$. Obviously $G_C \leq G_B$, $G_C \leq G_A$ and $G = \langle G_A, G_B \rangle$. It is easy to see that $G = G_A *_{G_C} G_B$. Note that graph Γ_B is reducible since v is connected to every other vertex, therefore $G_B = G_v \times G_C$. Consequently $G = G_A *_{G_C} (G_C \times G_v)$. □

CHAPTER 4

Results and methods

We are interested in studying the behaviour of separability properties under graph products. More formally: let P be a separability property and let $G = \Gamma\mathcal{G}$ be a graph product where all the vertex groups have the property P. Then we ask: does G have the property P as well?

Some separability properties are not preserved by graph products. Free groups are LERF by result of M. Hall Jr. [**22**]. However, by a result of Mihailova [**27**] we know that $F_2 \times F_2$ contains a finitely generated subgroup with unsolvable membership problem. Thus a direct product of LERF groups may not be LERF, as the membership problem can be solved for any finitely generated subgroup by an algorithm of Mal'cev-Mostowski type in finitely presented LERF groups. Since direct products are a special case of graph products we see that property LERF is not preserved under graph products.

In her Ph.D. thesis [**21**] Green proved that the class of RF groups is closed under taking graph products [**21**, Corollary 5.4] and that the same holds for the class of residually p-finite groups [**21**, Theorem 5.6].

4.1. Conjugacy separability

In the first presented paper

[**15**] *On conjugacy separability in graph products of groups*,

we study the behaviour of $\mathcal{C}$-conjugacy separability and $\mathcal{C}$-hereditary conjugacy separability in graph products. The aim of this section is to describe and explain the main ideas and methods we used in [**15**].

One can easily check that the class of $\mathcal{C}$-CS groups is closed under direct products for any class $\mathcal{C}$. Suppose that $G = G_1 \times G_2$ is a product of $\mathcal{C}$-CS groups and let $(f_1, f_2), (g_1, g_2) \in G$ be arbitrary such that $(f_1, f_2) \not\sim_G (g_1, g_2)$. Obviously, either $f_1 \not\sim_{G_1} g_1$ or $f_2 \not\sim_{G_2} g_2$. Without loss of generality we may assume that $f_1 \not\sim_{G_1} g_1$. As G_1 is $\mathcal{C}$-CS, we see that there is $L_1 \in \mathcal{N}_\mathcal{C}(G_1)$ such that $\lambda_1(f_1) \not\sim_{G/L} \lambda_1(g_1)$, where $\lambda_1 \colon G_1 \to G_1/L_1$ is the natural projection. Set $L = L_1 \times G_2$. Clearly, $L \in \mathcal{N}_\mathcal{C}(G)$ and for the natural projection $\lambda \colon G \to G/L$ we have $\lambda((f_1, g_1)) \not\sim_{G/L} \lambda((f_2, g_2))$.

However, showing that the class of $\mathcal{C}$-HCS groups is closed under direct products needs more work. In general, showing that a group G is $\mathcal{C}$-HCS directly, i.e. showing that every $\mathcal{C}$-open subgroup of G is $\mathcal{C}$-CS can turn out to be quite a strenuous task. We will often use a convenient workaround: instead of working with conjugacy in $\mathcal{C}$-quotients of G we will work with centralisers in $\mathcal{C}$-quotients.

Definition 4.1. *Let $\mathcal{C}$ be a class of groups satisfying (c1) and (c2). We say that a group G satisfies the $\mathcal{C}$-centraliser condition ($\mathcal{C}$-CC) if for every $g \in G$ and $K \in \mathcal{N}_\mathcal{C}(G)$*

there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and

$$C_{G/L}(\lambda(g)) \subseteq \lambda(C_G(g)K),$$

where $\lambda\colon G \to G/L$ is the natural projection.

Roughly speaking, if G has $\mathcal{C}$-CC, then for every element $g \in G$ we know that every projection $\gamma\colon G \to C$, where $C \in \mathcal{C}$, factors through some group $C' \in \mathcal{C}$ in which we have control over the size of the centraliser of the image of g.

Centraliser condition was introduced by Chagas and Zalesskii in [**9**], in the case where $\mathcal{C}$ is the class of all finite groups. However, their definition of the centraliser condition was given in terms of profinite completion. They showed that if a group G is CS and satisfies the centraliser condition, then G is HCS. Minasyan gave the definition in terms of subgroups of finite index and showed that for RF groups the definitions are equivalent. Minasyan also showed that the implication in the other direction holds as well: a CS group G is HCS if and only if it satisfies CC (see [**28**, Proposition 3.2]). Toinet proved that the same statement holds when $\mathcal{C}$ is the class of all finite p-groups for some $p \in \mathbb{P}$ (see [**40**, Proposition 3.6]). We show that the statement is true whenever the class $\mathcal{C}$ satisfies (c1), (c2) and (c4).

[**15**, Theorem 4.2]. *Suppose that $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2) and (c4). Let G be a group. Then the following are equivalent:*

(i) *G is $\mathcal{C}$-HCS,*
(ii) *G is $\mathcal{C}$-CS and satisfies $\mathcal{C}$-CC.*

We demonstrate the usefulness of [**15**, Theorem 4.2] in the proof of the following lemma.

Lemma 4.2. *Suppose that $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2) and (c4). Then class of $\mathcal{C}$-HCS groups is closed under taking direct products.*

Proof. Let $G = G_1 \times G_2$ be a direct product of $\mathcal{C}$-HCS groups. By the previous discussion, we see that G is $\mathcal{C}$-CS. We want to use [**15**, Theorem 4.2], so we need to show that G satisfies $\mathcal{C}$-CC. Let $g = (g_1, g_2) \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$ be arbitrary. Note that $C_G(g) = C_{G_1}(g_1) \times C_{G_2}(g_2)$. Set $K_1 = G_1 \cap K$ and $K_2 = K \cap G_2$. Both G_1, G_2 are $\mathcal{C}$-HCS, thus by [**15**, Theorem 4.2] we see that both satisfy $\mathcal{C}$-CC. We see that there are $L_1 \in \mathcal{N}_{\mathcal{C}}(G_1)$ and $L_2 \in \mathcal{N}_{\mathcal{C}}(G_2)$ such that $L_1 \leq K_1$, $L_2 \leq K_2$ and

$$\begin{aligned} C_{G_1/L_1}(\lambda_1(g_1)) &\subseteq \lambda_1(C_{G_1}(g_1)K_1), \\ C_{G_2/L_2}(\lambda_2(g_2)) &\subseteq \lambda_2(C_{G_2}(g_2)K_2), \end{aligned}$$

where $\lambda_1\colon G_1 \to G_1/L_1$ and $\lambda_2\colon G_2 \to G_2/L_2$ are the natural projections. Since the class $\mathcal{C}$ is closed under taking direct products, we see that $L = L_1 \times L_2 \in \mathcal{N}_{\mathcal{C}}(G)$. Obviously $L \leq K$. Let $\lambda\colon G \to G/L = G_1/L_1 \times G_2/L_2$ be the natural projection. We see that

$$\begin{aligned} C_{G/L}(\lambda(g)) = C_{G_1/L_1}(\lambda_1(g_1)) \times C_{G_2 \times L_2}(\lambda_2(g_2)) &\subseteq \lambda_1(C_{G_1}(g_1)K_1) \times \lambda_2(C_{G_2}(g_2)K_2) \\ &\subseteq \lambda(C_G(g)K) \end{aligned}$$

and thus $G_1 \times G_2$ satisfies $\mathcal{C}$-CC. By [**15**, Theorem 4.2] we see that $G_1 \times G_2$ is $\mathcal{C}$-HCS. $\square$

Note that in the case where $\mathcal{C}$ is the class of all finite groups, this was proved by Martino and Minasyan [**25**, Lemma 7.3].

It was proved by Stebe [**39**], and independently by Remeslennikov [**34**], that the class of CS groups is closed under taking free products. The main idea in Stebe's proof is that given two-non conjugate elements one can always construct a homomorphism to a free product of two finite groups such that the images are still not conjugate. Then one can use the fact that free products of finite groups are CS. In a way, we use a similar idea: constructing homomorphisms onto amalgams of finite groups. Dyer [**13**] proved that free-by-finite groups are CS and, in particular, amalgams of finite groups are CS. In his paper on p-conjugacy separability in RAAGs, Toinet [**40**, Theorem 4.2] proved that free-by-(finite p) groups are p-CS for every prime p. Recently, it was proved that these results can be generalised for any class of finite groups with sufficiently strong closure properties.

We say that a class $\mathcal{C}$ is an *extension closed variety of finite groups* if $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2), (c3) and (c4). The most obvious examples of extension closed varieties of groups are the following classes

- the class of all finite groups,
- the class of all finite solvable groups,
- the class of all finite p-groups, where p is a prime.

Ribes and Zalesskii [**35**, Theorem 3.2] proved that finitely generated free-by-$\mathcal{C}$ groups are $\mathcal{C}$-CS whenever $\mathcal{C}$ is an extension closed variety of finite groups.

Using the result of Ribes and Zalesskii one can easily show that the class of $\mathcal{C}$-CS groups is closed under taking free products whenever $\mathcal{C}$ is an extension closed variety of finite groups.

As the class of $\mathcal{C}$-CS groups is closed under taking free and direct products, it is natural to ask whether the class of CS groups is closed under taking graph products. Green herself proved that the class of CS groups is closed under taking *tree products*, i.e. where the underlying graph Γ is a tree. Minasyan [**28**, Theorem 1.1] proved that finitely generated RAAGs are HCS. Minasyan's method was later used by Toinet [**40**, Theorem 6.15] to show that finitely generated RAAGs are p-HCS. Minasyan's proof uses the fact that special HNN-extensions of finite groups are free-by-finite and hence by Dyer's result [**13**, Theorem 3] they are HCS. Similarly, Toinet uses the fact that special HNN-extensions of finite p-groups are free-by-(finite p) and hence by [**40**, Theorem 4.2] they are p-HCS. We use the result of Ribes and Zalesskii to prove the following generalisation of results of Minasyan and Toinet, whose proof is the most substantial part of [**15**].

[**15**, Theorem 6.1]. *Assume that $\mathcal{C}$ is an extension closed variety of finite groups. Then the class of $\mathcal{C}$-HCS groups is closed under taking finite graph products.*

The theorem is proved using [**15**, Theorem 4.2], hence we show that if G is a graph product of $\mathcal{C}$-HCS groups then G is $\mathcal{C}$-CS and satisfies $\mathcal{C}$-CC. Both these claims are proved by simultaneous induction on the number of vertices of the graph. We assume that the statement has been proved for all graphs Γ' with $|V\Gamma| < r$, then we take $\Gamma\mathcal{G}$ to be a graph product of $\mathcal{C}$-HCS groups such that $|V\Gamma| = r$. By the inductive hypothesis we see that every proper full subgroup is $\mathcal{C}$-HCS, thus by [**15**, Theorem 4.2] every proper full subgroup is $\mathcal{C}$-CS and satisfies $\mathcal{C}$-CC.

For conjugacy, the proof uses the ideas used by Stebe in [**39**]: we take $f, g \in G$ to be arbitrary such that $f \not\sim_G g$ and our aim is to construct a homomorphism onto a free-by-$\mathcal{C}$ (and thus $\mathcal{C}$-CS) group, which separates the conjugacy classes of f and g. We find a suitable splitting of G as an amalgam of its full subgroups $G = A \star_H C$ and, using the inductive hypothesis together with the conjugacy criterion for special amalgams, we show that there are groups $Q, S \in \mathcal{C}$ together with homomorphisms $\gamma_A \colon A \to Q$ and $\gamma_C \colon C \to S$, such that for the corresponding extension $\gamma\colon A \star_H C \to P = Q \star_{\gamma_A(H)} C$ we have $\gamma(f) \not\sim_P \gamma(g)$. Now, P is a special amalgam of $\mathcal{C}$-groups and thus it is free-by-$\mathcal{C}$ (see [**15**, Lemma 6.6]). Consequently, it is $\mathcal{C}$-HCS by the result of Ribes and Zalesskii. We see that there is a $\mathcal{C}$-group D and a homomorphism $\delta\colon P \to D$ such that $\delta(f) \not\sim_D \delta(f)$. If we set $\phi\colon G \to D$ to be given by $\phi = \delta \circ \gamma$ we see that $\phi(f) \not\sim_D \phi(g)$ and thus G is $\mathcal{C}$-CS.

Similarly, for the centraliser condition, we take $g \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$ to be arbitrary. Again, we consider a suitable splitting of G of as a special amalgam of its proper full subgroups $G = A \star_H C$. Using the inductive hypothesis together with the classification of centralisers in special amalgams, we show that there are groups $Q, S \in \mathcal{C}$ together with homomorphisms $\gamma_A \colon A \to Q$ and $\gamma_C \colon C \to S$ such that for corresponding extension $\gamma\colon A \star_H C \to P = Q \star_{\gamma_A(H)} C$ satisfies $C_P(\gamma(g)) \subseteq \gamma(C_G(g)K)$. Note that $\gamma(K) \in \mathcal{N}_{\mathcal{C}}(P)$. Now, P is a special amalgam of $\mathcal{C}$-groups and thus it is free-by-$\mathcal{C}$ (see [**15**, Lemma 6.6]) and, consequently, is $\mathcal{C}$-HCS by the result of Ribes and Zalesskii. By [**15**, Theorem 4.2] we see that P has $\mathcal{C}$-CC, hence there is $L' \in \mathcal{N}_{\mathcal{C}}(P)$ such that

$$C_{P/L'}(\lambda(\gamma(g))) \subseteq \lambda(C_P(\gamma(g)\gamma(K))).$$

By setting $L = \gamma^{-1}(L')$ we see that $C_{G/L}(\psi(g)) \subseteq \psi(C_G(g)K)$, where $\psi\colon G \to G/L$ is the natural projection, and therefore G has $\mathcal{C}$-CC.

In order to able to this, in [**15**, Section 5] we develop theory of special amalgams, most importantly the conjugacy criterion for special amalgams [**15**, Lemma 5.8, Lemma 5.11] and a classification of centralisers in special amalgams [**15**, Lemma 5.9, Lemma 5.10, Lemma 5.12]. The induction step is establised in [**15**, Lemma 6.12] and rest of [**15**, Section 6] is a case analysis dealing with all possible situations that might occur.

Note that every $\mathcal{C}$-group is trivially $\mathcal{C}$-HCS, so as an immediate consequence of [**15**, Theorem 6.1] we see that finite graph products of $\mathcal{C}$-groups are $\mathcal{C}$-HCS.

[**15**, Corollary 6.17]. *Assume that $\mathcal{C}$ is an extension closed variety of finite groups. Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of groups such that $G_v \in \mathcal{C}$ for all $v \in V\Gamma$. Then the group $G = \Gamma\mathcal{G}$ is $\mathcal{C}$-HCS.*

In [**15**, Section 3] we give a conjugacy criterion for graph products which we later use in [**15**, Section 7] along with Theorem [**15**, Theorem 6.1] to prove the following two theorems.

[**15**, Theorem 1.1]. *Let $\mathcal{C}$ be an extension closed variety of finite groups. Then the class of $\mathcal{C}$-CS groups is closed under arbitrary graph products.*

[**15**, Theorem 1.2]. *Let $\mathcal{C}$ be an extension closed variety of finite groups. Then the class of $\mathcal{C}$-HCS groups is closed under arbitrary graph products.*

Again, the idea behind the proof of [**15**, Theorem 1.1] is analogous to Stebe's proof for free products: finding suitable $\mathcal{C}$-quotients of vertex groups. Fist of all it is important to note that for every pair $f, g \in G$ the subset $S = \mathrm{supp}(f) \cup \mathrm{supp}(g)$ is finite, and f and g are conjugate in G_S, the full subgroup of G corresponding to S, if and only if f is conjugate to g in G. Thus we see that that we can without loss of generality assume that the corresponding graph Γ is actually finite. Using the conjugacy criterion for graph products (see [**15**, Lemma 3.12]) we show that if $G = \Gamma\mathcal{G}$ is a graph product of $\mathcal{C}$-CS groups then for every pair of elements $f, g \in G$ such that $f \not\sim_G g$ we can find a family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ together with a family of homomorphisms $\{\phi_v \colon G_v \to F_v\}$ such that for the canonical extension $\phi \colon G \to F = \Gamma\mathcal{F}$ we have $\phi(f) \not\sim_F \phi(g)$. Then by [**15**, Corollary 6.17] we see that F is $\mathcal{C}$-HCS. From that we immediately obtain that there is a group $C \in \mathcal{C}$ and a homomorphism $\gamma \colon F \to C$ such that $\gamma(\phi(f)) \not\sim_C \gamma(\phi(g))$ and we are done.

The idea behind the proof of [**15**, Theorem 1.2] is somewhat similar. Obviously, if $G = \Gamma\mathcal{G}$ is a graph product of $\mathcal{C}$-HCS groups then G is $\mathcal{C}$-CS by [**15**, Theorem 1.1]. We show that for every $g \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$ there is a finite graph Δ (actually a quotient of Γ) and a family of $\mathcal{C}$-HCS groups $\mathcal{D}$ together with a group homomorphism $\delta \colon G \to D = \Delta\mathcal{D}$ such that $\ker(\delta) \leq K$ and $C_D(\delta(g)) \subseteq \delta(C_G(g)K)$. Now, as D is a finite graph product of $\mathcal{C}$-HCS groups we see that D is $\mathcal{C}$-HCS by [**15**, Theorem 6.1] and D satisfies $\mathcal{C}$-CC by [**15**, Theorem 4.2]. Note that $\delta(K) \in \mathcal{N}_{\mathcal{C}}(D)$, thus there is $L' \in \mathcal{N}_{\mathcal{C}}(D)$ such that

$$C_{D/L}(\lambda(\delta(g))) \subseteq \lambda(C_D(\delta(g))\delta(K)),$$

where $\lambda \colon D \to D/L$ is the natural projection. Again, by composing $\gamma = \lambda \circ \delta$ we see that $C_{D/L}(\gamma(g)) \subseteq \gamma(C_G(g)K)$ and therefore G satisfies $\mathcal{C}$-CC. Again, we see that G is $\mathcal{C}$-HCS by [**15**, Theorem 4.2].

4.2. Residual finiteness of Out

In the second presented paper,

[**16**] *Separability properties of automorphisms of graph products of groups*,

we study residual properties of the groups of outer automorphisms of graph products of groups. By a classical result of Baumslag it is known that for finitely generated groups the property of being RF is passed to automorphism groups.

THEOREM 4.3 (Baumslag [**3**]). *Let G be a finitely generated RF group. Then* $\mathrm{Aut}(G)$ *is RF.*

PROOF. Let $\phi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}_G\}$ be arbitrary. As ϕ is not the identity there exists $g' \in G$ such that $\phi(g') \neq g'$ in G. By residual finiteness there is $N \trianglelefteq_{\text{f.i.}} G$ such that $\phi(g')N \cap g'N = \emptyset$. Now set

$$K = \bigcap_{\gamma \in \mathrm{Aut}(G)} \gamma^{-1}(N).$$

Note that $|G : \gamma^{-1}(N)| = |G : N|$ for every $\gamma \in \mathrm{Aut}(G)$. Since G is finitely generated, we see that for every $n \in \mathbb{N}$ there are only finitely many $H \leq G$ such that $|G : N| = n$. It follows that $K \trianglelefteq_{\text{f.i.}} G$ as it is an intersection of finitely many subgroups of finite index. Also we see that $\gamma(K) = K$ for every $\gamma \in \mathrm{Aut}(G)$, i.e. K is characteristic in

G. We see that the natural homomorphism $\kappa\colon G \to G/K$ induces a homomorphism $\tilde{\kappa}\colon \operatorname{Aut}(G) \to \operatorname{Aut}(G/K)$ given by

$$\tilde{\kappa}(\gamma)(gK) = \gamma(g)K$$

for every $\gamma \in \operatorname{Aut}(G)$, $g \in G$. Note that $\operatorname{Aut}(G/K)$ is a finite group. As $K \leq N$ we see that $\phi(g')K \neq g'K$ and thus $\tilde{\kappa}(\phi)(g'K) \neq gK$. Therefore $\tilde{\kappa}(\phi)$ is not the identity on G/K, and thus $\operatorname{Aut}(G)$ is residually finite. $\square$

Baumslag's proof uses two relatively simple observations:

(1) if a group G is finitely generated then for every $H \leq_{\text{f.i.}} G$ there is $K \trianglelefteq_{\text{f.i.}} G$ characteristic in G such that $K \leq H$;
(2) if $K \trianglelefteq G$ is characteristic then the natural projection $\kappa\colon G \to G/K$ induces a homomorphism $\tilde{\kappa}\colon \operatorname{Aut}(G) \to \operatorname{Aut}(G/K)$.

It is natural to ask whether something similar can be said about $\operatorname{Out}(G)$, the group of outer automorphisms of G. In general, the answer is negative: Bumagin and Wise [**7**] proved that for every finitely presented group O there is a finitely generated RF group G such that $\operatorname{Out}(G) \cong O$. If we set $O = BS(2,3)$, the Baumslag-Solitar group given by the presentation $\langle a, b \| ba^2b^{-1} = a^3\rangle$, then we see that there is a finitely generated RF group G such that $\operatorname{Out}(G)$ is isomorphic to $BS(2,3)$, which is known not to be RF. Note that G will not be finitely presented.

Another naturally arising question is: which properties does a finitely generated RF group G need to satisfy to ensure that $\operatorname{Out}(G)$ is RF? We say that $\phi \in \operatorname{Aut}(G)$ is *pointwise inner* if $\phi(g) \sim_G g$ for every $g \in G$. We say that a group G has *Grossman's property (A)* if it does not have nontrivial pointwise inner automorphisms, i.e. for every $\phi \in \operatorname{Aut}(G)$ if ϕ is pointwise inner, then $\phi \in \operatorname{Inn}(G)$.

THEOREM 4.4 (Grossman [**19**]). *Let G be a finitely generated CS group and suppose that G has Grossman's property (A). Then* $\operatorname{Out}(G)$ *is RF.*

PROOF. Let $\phi \in \operatorname{Aut}(G) \setminus \operatorname{Inn}(G)$ be arbitrary. As G has Grossman's property (A) we see that ϕ is not pointwise inner, i.e. there is a $g' \in G$ such that $\phi(g') \not\sim_G g'$. By conjugacy separability there is $N \trianglelefteq_{\text{f.i.}} G$ such that $\phi(g')^G N \cap g'^G N = \emptyset$ in G. Again, set

$$K = \bigcap_{\gamma \in \operatorname{Aut}(G)} \gamma^{-1}(N).$$

Again, as G is finitely generated, we see that $K \trianglelefteq_{\text{f.i.}} G$ is characteristic in G and thus the natural projection $\kappa\colon G \to G/K$ induces a homomorphism $\tilde{\kappa}\colon \operatorname{Aut}(G) \to \operatorname{Aut}(G/K)$ given by

$$\tilde{\kappa}(\gamma)(gK) = \gamma(g)K$$

for every $\gamma \in \operatorname{Aut}(G)$, $g \in G$. Note that $\tilde{\kappa}(\operatorname{Inn}(G)) \subseteq \operatorname{Inn}(G/K)$. As $K \leq N$ we see that $\phi(g')^G K \cap g'^G K = \emptyset$, i.e. $\phi(g')K \not\sim_{G/K} g'K$. We see that $\tilde{\kappa}(\phi) \notin \operatorname{Inn}(G/K)$ and thus $\operatorname{Inn}(G)$ is closed in $\operatorname{Aut}(G)$. It follows that $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ is RF. $\square$

Like Baumslag's proof, Grossman's proof uses two simple observations:

(1) if $O = A/I$ is a quotient of a group A by its normal subgroup I, then O is RF if and only if I is closed in A;

(2) if for every $\phi \in \mathrm{Aut}(G)\backslash\mathrm{Inn}(G)$ there is $K \trianglelefteq_{\text{f.i.}} G$ characteristic such that for the induced homomorphism $\tilde{\kappa}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/K)$ we have $\tilde{\kappa}(\phi) \notin \mathrm{Inn}(G/K)$ then $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$.

Groups satisfying the assumptions of Grossman's theorem will be called *Grossmanian groups*. That is, a group G is Grossmanian if G is a finitely generated CS group with Grossman's property (A).

The hypotheses of Grossman's theorem are sufficient but not necessary. For example, $G = \mathbb{Z} * SL_3(\mathbb{Z})$ is RF but not CS and following the results of Minasyan and Osin [**29**], one can show that G has Grossman's property (A) and $\mathrm{Out}(G)$ is RF. Grossman's property (A) is not necessary either: if G is finite, then $\mathrm{Out}(G)$ is finite (and therefore RF), but Burnside [**8**] gave examples of finite-p groups with nontrivial pointwise inner automorphisms. Perhaps the simplest infinite examples are virtually polycyclic groups: virtually polycyclic groups are CS by results of Formanek [**17**] and Remeslennikov [**32**] and groups of outer automorphisms of virtually polycyclic groups are RF by result of Wehrfritz [**41**], yet Segal [**38**] gave an example of a torsion-free polycyclic group with nontrivial pointwise inner automorphisms.

This motivates the following definition.

Definition 4.5. *Let $\mathcal{C}$ be a class of finite groups. We say that a group G is $\mathcal{C}$-*inner automorphism separable *($\mathcal{C}$-IAS) if for every $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ there is $K \in \mathcal{N}_{\mathcal{C}}(G)$ characteristic in G such that for the induced homomorphism $\tilde{\kappa}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/K)$ given by*

$$\tilde{\kappa}(\gamma)(gK) = \gamma(g)K$$

for every $\gamma \in \mathrm{Aut}(G)$, $g \in G$ we have $\tilde{\kappa}(\phi) \in \mathrm{Aut}(G/K) \setminus \mathrm{Inn}(G/K)$. In other words: G is $\mathcal{C}$-IAS if every non-trivial outer automorphism of G can be realised as a non-trivial outer automorphism of some $\mathcal{C}$-quotient of G.

If $\mathcal{C}$ is the class of all finite groups, then we will say that a group G is IAS to mean that G is $\mathcal{C}$-IAS. Similarly, we say that G is p-IAS to mean that G is $\mathcal{C}$-IAS in the case where $\mathcal{C}$ is the class of all finite p-groups.

Obviously, if G is IAS then $\mathrm{Out}(G)$ is RF. In [**16**, Lemma 7.2.] we show that if G is finitely generated p-IAS then $\mathrm{Out}(G)$ is virtually residually p-finite.

Another reason for introducing the $\mathcal{C}$-IAS property is the study of direct products. Residual finiteness of outer automorphisms of free products of finitely generated RF groups is well understood by the results of Minasyan and Osin in [**29**]. However, very little is known about residual finiteness of outer automorphisms of direct products of groups. In [**16**, Section 2] we study basic properties of $\mathcal{C}$-IAS groups and we show that the property of being $\mathcal{C}$-IAS is stable under taking direct products.

[**16**, Proposition 2.1]. *Let $\mathcal{C}$ be a class of finite groups satisfying (c1) and (c2). Let A, B be finitely generated $\mathcal{C}$-IAS residually-$\mathcal{C}$ groups. Then the group $A \times B$ is $\mathcal{C}$-IAS.*

As an immediate consequence we see that if A, B are finitely generated IAS RF groups then $A \times B$ is IAS and consequently $\mathrm{Out}(A \times B)$ is RF. Similarly, if A, B are finitely generated p-IAS residually p-finite groups, then $A \times B$ is p-group and $\mathrm{Out}(A \times B)$ is virtually residually p-finite.

In [**16**, Section 3] we show that groups satisfying a generalised version of Grossman's criterion are $\mathcal{C}$-IAS (see [**16**, Lemma 3.2]) and we give examples of such groups. We also show that all virtually polycyclic groups are IAS (see [**16**, Lemma 3.5]).

The use of Grossman's property (A) motivates [**16**, Section 5], where we study pointwise inner endomorphisms of graph products of groups.

For a graph Γ we say that a vertex $v \in V\Gamma$ is *central* if $\mathrm{link}(v) = V\Gamma \setminus \{v\}$, i.e. v is central in Γ if it is adjacent to all the vertices of Γ (apart from itself).

[**16**, Theorem 1.1]. *Let Γ be a finite graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial groups. Then the group $\Gamma\mathcal{G}$ has Grossman's property (A).*

Note that if the underlying graph Γ is irreducible and has at least two vertices, i.e. if the vertex set $V\Gamma$ cannot be written as a disjoint union $V\Gamma = X \dot{\cup} Y$ such that $\{x, y\} \in E\Gamma$ for all $x \in X$ and $y \in Y$, then [**16**, Theorem 1.1] can be recovered from the work of Minasyan and Osin [**30**] on acylindrical hyperbolicity of graph products of groups combined with the work of Antolín, Minasyan and Sisto [**2**] on commensurating endomorphisms of acylindricaly hyperbolic groups. However, results presented in [**30**] and [**2**] use geometric arguments, whereas our proof uses purely combinatorial methods based on properties of normal forms in graph products and the conjugacy criterion for graph products (see [**15**, Lemma 3.12]).

It can be easily seen that a finite direct product of groups has Grossman's property (A) if and only if all direct factors have Grossman's property (A). Combining this simple fact with [**16**, Theorem 1.1] we show:

[**16**, Corollary 1.2]. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial groups. The group $G = \Gamma\mathcal{G}$ has Grossman's property (A) if and only if all vertex groups corresponding to central vertices of Γ have Grossman's property (A).*

Note that [**16**, Corollary 1.2] generalises a result of Minasyan [**28**, Proposition 6.9], which states that finitely generated RAAGs have Grossman's property(A).

In [**16**, Section 6] we study $\mathcal{C}$-conjugacy distinguishable pairs in graph products. For $f, g \in G$ such that $f \not\sim_G$ we say that the pair (f, g) is *$\mathcal{C}$-conjugacy distinguishable* ($\mathcal{C}$-CD) if there exist a group $C \in \mathcal{C}$ and a homomorphism $\gamma \colon G \to C$ such that $\gamma(f) \not\sim_C \gamma(g)$. Clearly, a group G is $\mathcal{C}$-CS if for every pair $f, g \in G$ we have the following dichotomy: either $f \sim_G g$ or the pair (f, g) is $\mathcal{C}$-CD in G. Note that the conjugacy classes f^G and g^G do not need to be $\mathcal{C}$-closed in G, we just require that f^G can be separated from g^G in some $\mathcal{C}$-quotient of G.

If $\mathcal{C}$ is an extension closed variety of finite groups, then graph products of $\mathcal{C}$-groups are $\mathcal{C}$-CS by [**15**, Corollary 6.17]. Using this result, by constructing suitable homomorphisms from a graph product of residually-$\mathcal{C}$ groups to graph products of $\mathcal{C}$-groups, we show that most pairs of elements in a graph product of a graph products of residually-$\mathcal{C}$ groups are indeed $\mathcal{C}$-CD (see [**16**, Lemma 6.7]). This allows us to prove the following:

[**16**, Proposition 6.2]. *Let Γ be a finite simplicial graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually-$\mathcal{C}$ groups. Then the group $G = \Gamma\mathcal{G}$ is $\mathcal{C}$-IAS.*

The proof of [**16**, Proposition 6.2] is led by contradiction: let $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ be arbitrary and suppose that for every $g \in G$ the pair $(\phi(g), g)$ is not $\mathcal{C}$-CD in G. Using the fact that almost all pairs of elements are actually $\mathcal{C}$-CD (see [**16**, Lemma 6.7]) we then show that $\phi(g) \sim_G g$ for every $g \in G$ and thus ϕ is pointwise inner. However, the graph Γ does not contain central vertices and thus by [**16**, Theorem 1.1] G has Grossman's property (A). This means that ϕ must be inner, which is a contradiction with our assumptions. It follows that there must be an element $g \in G$ such that the pair $(\phi(g), g)$ is $\mathcal{C}$-CD. Using a generalisation of Grossman's criterion (see [**16**, Lemma 3.2]) we then show that $\Gamma\mathcal{G}$ is indeed $\mathcal{C}$-IAS.

Using the fact that the class of $\mathcal{C}$-IAS groups is closed under taking direct products we extend [**16**, Proposition 6.2] to graph products with central vertices.

[**16**, Corollary 6.8]. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually-$\mathcal{C}$ groups such that the group G_v is $\mathcal{C}$-IAS whenever the vertex v is central in Γ. Then the group $G = \Gamma\mathcal{G}$ is $\mathcal{C}$-IAS.*

Applying [**16**, Proposition 6.2] and [**16**, Corollary 6.8] to the class of all finite groups we immediately obtain the following two results:

[**16**, Theorem 1.3]. *Let Γ be a finite graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated RF groups. Then the group $\Gamma\mathcal{G}$ is IAS and consequently* $\mathrm{Out}(\Gamma\mathcal{G})$ *is RF.*

[**16**, Corollary 1.4]. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be family of non-trivial finitely generated RF groups. Assume that G_v is IAS whenever v is central in Γ. Then the group $\Gamma\mathcal{G}$ is IAS and consequently* $\mathrm{Out}(\Gamma\mathcal{G})$ *is RF.*

Note that [**16**, Theorem 1.3] generalises result of Minasyan and Osin in [**29**] on residual finiteness of outer automorphism groups of free products of finitely generated RF groups.

Similarly, applying [**16**, Proposition 6.2] and [**16**, Corollary 6.8] together with Lemma [**16**, Lemma 7.2] to the class of all finite p-groups we immediately get the following:

[**16**, Theorem 1.5]. *Let Γ be a finite graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually p-finite groups. Then the group $\Gamma\mathcal{G}$ is p-IAS,* $\mathrm{Out}_p(\Gamma\mathcal{G})$ *is residually p-finite and* $\mathrm{Out}(\Gamma\mathcal{G})$ *is virtually residually p-finite.*

[**16**, Corollary 1.6]. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually p-finite groups. Assume that G_v is p-IAS whenever v is central in Γ. Then the group $\Gamma\mathcal{G}$ is p-IAS and consequently* $\mathrm{Out}_p(\Gamma\mathcal{G})$ *is residually p-finite and* $\mathrm{Out}(\Gamma\mathcal{G})$ *is virtually residually p-finite.*

Note that if G is a group, then $\mathrm{Out}_p(G) \leq \mathrm{Out}(G)$ denotes the set of all outer automorphisms of G that act trivially on the first mod-p homology of G.

4.3. Residual properties and local embeddability

In the third presented paper

[**6**] with F. Berlai, *Residual properties of graph products of groups*,

we study residual properties of graph products, especially residual amenability. In the first presented paper [**15**, Lemma 6.8], the second named author proved that if $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2) and (c4) then the class of residually-$\mathcal{C}$ groups is closed under graph products. The proof of [**15**, Lemma 6.8] heavily relies on the fact that if $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2) and (c4) then free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$ (see [**15**, Lemma 2.9]).

We prove the following generalisation of [**15**, Lemma 6.8]:

[**6**, Theorem A]. *Let $\mathcal{C}$ be a class of groups satisfying (c1) and (c2). Assume that free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$. Then the class of residually-$\mathcal{C}$ groups is closed under taking graph products.*

The main idea of the proof is to use induction on the number of vertices of the underlying graph.

First, we show that if $\mathcal{C}$ is a class of groups satisfying (c1) and (c2) such that free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$, then a special amalgam $A \star_B C$ of residually-$\mathcal{C}$ groups is residually-$\mathcal{C}$ if and only if the amalgamated subgroup B is $\mathcal{C}$-closed in A (see [**6**, Proposition 4.2]).

For the induction, we assume that [**6**, Theorem A] has been proved for all graphs Γ' with $|V\Gamma'| < r$. We then argue that if $G = \Gamma\mathcal{G}$ is a graph product of residually-$\mathcal{C}$ groups with $|V\Gamma| = r$, then G splits as a special amalgam $G = A \star_B C$, where $B \leq A, C$ are some proper full subgroups of G. By the inductive hypothesis we see that A, B, C are residually-$\mathcal{C}$. Also, B is a full subgroup (and thus a retract) of A, hence B is $\mathcal{C}$-closed in G. Theorem A then follows by [**6**, Proposition 4.2] as G is a special amalgam of residually-$\mathcal{C}$ groups over a $\mathcal{C}$-closed subgroup.

However, determining for which classes $\mathcal{C}$ can we say that free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$ is quite difficult. This was studied by the first named author in [**5**], where he gave some sufficient conditions which the class $\mathcal{C}$ needs to satisfy to ensure that free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$ (see [**5**, Lemma 3.3]). In particular, he showed that free-by-$\mathcal{C}$ groups are residually-$\mathcal{C}$ whenever $\mathcal{C}$ is one following classes:

(1) amenable groups,
(2) elementary amenable groups,
(3) solvable groups.

Combining [**6**, Theorem A] with previous results of Berlai we show the following:

[**6**, Corollary A]. *The class of residually amenable groups is closed under taking graph products. The same is true for residually elementary amenable groups.*

In [**6**, Section 5] we study local embeddability of graph products. Let G_1, G_2 be groups and suppose that $K \subseteq G_1$. We say that a function $\varphi\colon G_1 \to G_2$ is a K*-almost-homomorphism* if the following hold:

(i) $\varphi\restriction_K$ is injective,
(ii) $\varphi(kk') = \varphi(k)\varphi(k')$ for all $k, k' \in K$.

Let $\mathcal{C}$ be a class of groups. We say that a group G is *locally embeddable into* $\mathcal{C}$ (LE-$\mathcal{C}$) if for every subset $K \subseteq G$ such that $|K| < \infty$ there is a group $C \in \mathcal{C}$ and a K-almost-homomorphism $\varphi\colon G \to C$.

In a way, being LE-$\mathcal{C}$ is similar to being residually-$\mathcal{C}$ but weaker, as we do not require the approximating maps to be homomorphisms. It is obvious that every residually-$\mathcal{C}$ group is LE-$\mathcal{C}$ but in general the reverse implication does not hold.

It can be easily seen that every finitely presented LE-$\mathcal{C}$ is residually-$\mathcal{C}$: let G be a LE-$\mathcal{C}$ group given by a finite presentation $\langle X \| R\rangle$ and let $g \in G \setminus \{1\}$ be arbitrary. Let l_R be the length of longest relator in R, i.e. $l_R = \max_{r\in R}\{|r|\}$, and set $l = \max\{l_R, |g|\}$. Let $K = B_l(1, d_X) \subseteq G$ be the closed ball of radius l with respect to d_X centred around the identity, where d_X denotes the word metric on G with respect to the generating set X. Note that $g \in K$. By assumption, there is a $\mathcal{C}$-group C and a K-almost-homomorphism $\varphi\colon G \to C$. Now consider the restriction $\varphi_B = \varphi\restriction_K$. Clearly, the map φ_B respects all the relations in R and thus by the theorem of von Dyck (see [**36**, Theorem 2.2.1]) the function φ_B extends to a homomorphism $\overline{\varphi}\colon G \to C$. Obviously, $\overline{\varphi}(g) \neq 1$ in C as $g \in K$ and therefore G is residually-$\mathcal{C}$.

An obvious example of a group which is locally embeddable into finite, but not residually finite, would be $FSym$, the group of finitely supported bijections on an infinite set. However, $FSym$ is infinitely generated. Examples of finitely generated groups that are locally embeddable into the class of amenable groups but are not residually amenable were given in [**14**, Theorem 3].

The first named author proved that if the class of residually-$\mathcal{C}$ groups is closed under taking free products, then the class of LE-$\mathcal{C}$ groups is closed under taking free products as well (see [**5**, Theorem 1.5]). We generalise this result to graph products of groups.

[**6**, Theorem B]. *Let $\mathcal{C}$ be a class of groups, suppose that $\mathcal{C}$ is closed under taking subgroups, finite direct products and that graph products of residually-$\mathcal{C}$ groups are residually-$\mathcal{C}$. Then the class of LE-$\mathcal{C}$ groups is closed under graph products.*

The main idea behind the proof of [**6**, Theorem B] is as follows: using properties of the normal form for graph products together with local embeddability of the vertex groups, we show that for every finite subset $K \subseteq \Gamma\mathcal{G}$ there is a family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ together with a K-almost-homomorphism $\varphi\colon \Gamma\mathcal{G} \to \Gamma\mathcal{F}$. We then use the fact that graph products of $\mathcal{C}$-groups are residually-$\mathcal{C}$ by assumption.

We combine the known results about residual properties of graph products of groups presented in [**21**, Corollary 5.4, Theorem 5.6], [**15**, Lemma 6.8] and our new result [**6**, Theorem B] to obtain the following:

[**6**, Corollary B]. *Let $\mathcal{C}$ be one of the following classes:*
(1) *finite groups,*
(2) *finite p-groups,*
(3) *solvable groups,*
(4) *finite solvable groups,*
(5) *elementary amenable groups,*
(6) *amenable groups.*

Then the class of LE-$\mathcal{C}$ groups is closed under graph products.

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ON CONJUGACY SEPARABILITY OF GRAPH PRODUCTS OF GROUPS

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Abstract. We show that the class of $\mathcal{C}$-hereditarily conjugacy separable groups is closed under taking arbitrary graph products whenever the class $\mathcal{C}$ is an extension closed variety of finite groups. As a consequence we show that the class of $\mathcal{C}$-conjugacy separable groups is closed under taking arbitrary graph products. In particular, we show that right angled Coxeter groups are hereditarily conjugacy separable and 2-hereditarily conjugacy separable, and we show that infinitely generated right angled Artin groups are hereditarily conjugacy separable and p-hereditarily conjugacy separable for every prime number p.

Contents

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1. Introduction

Let G be a group. We say that G is *residually finite* (RF) if for every non-trivial element $g \in G$ there exists a a finite group F and a group homomorphism $\phi\colon G \to F$ such that $\phi(g) \neq 1$ in F. We say that the group G is *conjugacy separable* (CS) if for every pair of elements $f, g \in G$ such that f is not conjugate to g in G there is a finite group F and a homomorphism $\phi\colon G \to F$ such that $\phi(f)$ is not conjugate to $\phi(g)$ in F.

Group properties of this type are called *separability properties*. In this paper we will be dealing with conjugacy separability and its various generalisations.

1.1. **Motivation.** Separability properties provide an algebraic analogue to solvability of decision problems for finitely presented groups. Mal'cev [15] proved that finitely presented residually finite groups have solvable word problem. Similarly, Mostowski [20] showed that finitely presented conjugacy separable groups have solvable conjugacy problem.

The following classes of groups are known to be conjugacy separable: virtually free groups (Dyer [7]), virtually polycyclic groups (Formanek [8], Remeslennikov [22]), virtually surface groups (Martino [16]), limit groups (Chagas and Zalesskii [6]), finitely generated right angled Artin groups (Minasyan [18]), even Coxeter groups whose diagram does not contain $(4,4,2)$-triangles (Caprace and Minasyan [4]), finitely presented residually free groups (Chagas and Zalesskii [5]), one-relator groups with torsion (Minasyan and Zalesskii [19]) and fundamental groups of compact orientable 3-manifolds (Hamilton, Wilton and Zalesskii [12]).

Conjugacy separability is similar to residual finiteness but is much stronger. It can be easily seen that every CS group is RF, but the implication in the opposite direction does not hold. Perhaps the easiest example of a RF group which is not CS was given by Stebe [25] and independently by Remeslenikov [21] when they proved that $\mathrm{SL}_3(\mathbb{Z})$ is not CS.

It is easy to see that being residually finite is a hereditary property: if a group G is RF then every $H \leq G$ is residually finite as well. Unlike residual finiteness, conjugacy separability does not behave well with respect to subgroups, not even of finite index. Martino and Minasyan [17] showed that for every integer $m \geq 2$ there exists a finitely presented conjugacy separable group T that contains a subgroup $S \leq T$ such that $|T : S| = m$ and S is not CS. We say that a group G is *hereditarily conjugacy separable* (HCS) if G is conjugacy separable and if $H \leq G$ is of finite index in G then H is CS as well.

Let $\mathcal{C}$ be a class of finite groups (we will always assume that classes of finite groups are closed under isomorphisms) and let G be a group. We say that G is *residually-$\mathcal{C}$* if for every non-trivial $g \in G$ there is a group $F \in \mathcal{C}$ and a homomorphism $\phi\colon G \to F$ such that $\phi(g)$ is non-trivial in F. Similarly, we say that G is *$\mathcal{C}$-conjugacy separable* ($\mathcal{C}$-CS) if for every tuple $f, g \in G$ such that f is not conjugate to g in G there is a group $F \in \mathcal{C}$ and a homomorphism $\phi\colon G \to F$ such that $\phi(f)$ is not conjugate to $\phi(g)$ in F. We say that G is *$\mathcal{C}$-hereditarily conjugacy separable* ($\mathcal{C}$-HCS) if it is $\mathcal{C}$-CS and every subgroup $H \leq G$, open in pro-$\mathcal{C}$ topology, is $\mathcal{C}$-CS (H is open in pro-$\mathcal{C}$ topology if and only if there is $K \trianglelefteq G$ such that $K \leq H$ and $G/K \in \mathcal{C}$ - see Section 2). If the class $\mathcal{C}$ satisfies certain closure properties we can equip the group G with the so called pro-$\mathcal{C}$ topology and use basic terminology and methods from point-set topology to significantly simplify our proofs. Basic properties of pro-$\mathcal{C}$ topologies on groups, their connection to closure properties of the class $\mathcal{C}$ and definitions of residually-$\mathcal{C}$, $\mathcal{C}$-CS and $\mathcal{C}$-HCS groups in terms of pro-$\mathcal{C}$ topologies are given in Section 2.

We say that a class of finite groups $\mathcal{C}$ is an *extension closed variety of finite groups* if it is closed under taking subgroups, direct products, quotients and extensions. The most common examples of extension closed varieties of finite groups would be the class of all finite p-groups, where p is a prime number, the class of all finite soluble groups or the class of all finite groups.

In this paper we study the behaviour of $\mathcal{C}$-(hereditary) conjugacy separability under group constructions, where the class $\mathcal{C}$ is an extension closed variety of finite groups. It is easy to see that a direct product of $\mathcal{C}$-CS groups is again a conjugacy separable group, similarly for hereditary conjugacy separability. It was proved by Stebe [26] and independently by Remeslennikov [21] that the class of CS groups is closed under taking free products and using this result one can show that a free product of HCS groups is again an HCS group. In his paper [29] Toinet proved a specialised version of Dyer's theorem: free-by-(finite p) groups are p-CS. This result was generalised by Ribes and Zalesskii [28]: finitely generated free-by-$\mathcal{C}$ groups are $\mathcal{C}$-CS whenever $\mathcal{C}$ is an extension closed variety of finite groups. Using the result of Ribes and Zalesskii one could easily generalise the result of Stebe and Remeslennikov to $\mathcal{C}$-CS and $\mathcal{C}$-HCS groups. Our aim is to show that the property of being $\mathcal{C}$-(H)CS is stable under graph products, group theoretic construction naturally generalising direct and free products in the category of groups.

1.2. **Statement of the results.** By a graph we will always mean a simplicial graph: i.e. graph Γ is a tuple $(V\Gamma, E\Gamma)$, where $V\Gamma$ is a set and $E\Gamma \subseteq \binom{V\Gamma}{2}$. We call $V\Gamma$ the set of vertices of Γ and $E\Gamma$ the set of edges of Γ.

Let Γ be a graph and let $\mathcal{G} = \{G_v | v \in V\Gamma\}$ be a family of groups indexed by the vertices of Γ. The *graph product* $\Gamma\mathcal{G}$ is the quotient of the free product $*_{v\in V\Gamma} G_v$ obtained by adding all the relations of the form

$$g_u g_v = g_v g_u \text{ for all } g_u \in G_u, g_v \in G_v \text{ such that } \{u, v\} \in E\Gamma.$$

The groups G_v will be usually referred to as *vertex groups.*

Clearly if Γ is a complete graph then $\Gamma\mathcal{G}$ is equal to the direct product $\prod_{v\in V\Gamma} G_v$ and if Γ is the totally disconnected graph, meaning that $E\Gamma = \emptyset$, the resulting graph product is equal to the free product $*_{v\in V\Gamma} G_v$. We say that the group $\Gamma\mathcal{G}$ is a *finite graph product* if the corresponding graph Γ is finite.

If $G_v = \mathbb{Z}$, the additive group of integers, for all $v \in V\Gamma$, then we are talking about *right angled Artin groups* (RAAGs), and if $G_v = C_2$, the cyclic group of order 2, we are talking about *right angled Coxeter groups* (RACGs). In a way, RAAGs sit between free groups and free abelian groups. Since both free abelian groups and free groups are CS it is natural to ask whether RAAGs are CS as well. The positive answer to this question was given by Minasyan [18], when he proved that finitely generated RAAGs are HCS. Toinet [29] modified Minasyan's proof and showed that finitely generated RAAGs are p-HCS for every prime number p. The main results of this paper are the following two theorems.

Theorem 1.1. *Assume that $\mathcal{C}$ is an extension closed variety of finite groups. Then the class of $\mathcal{C}$-CS groups is closed under taking arbitrary graph products.*

Theorem 1.2. *Let $\mathcal{C}$ be an extension closed variety of finite groups. Then the class of $\mathcal{C}$-HCS groups is closed under taking arbitrary graph products.*

Note that we do not impose any restrictions on the cardinality of the corresponding graph, i.e. $|V\Gamma|$ can be any cardinal.

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2. Pro-$\mathcal{C}$ topologies on groups

In this section we will explain basic properties of pro-$\mathcal{C}$ topologies on groups. In the profinite (or pro-p) case these are standard results and are part of mathematical folklore. We include this section in order to make this paper self-contained and readers familiar with pro-$\mathcal{C}$ topologies on groups can skip it.

What closure properties do we require the class $\mathcal{C}$ to have? We will be considering the following ones:

(c1) subgroups: let $G \in \mathcal{C}$ and suppose that $H \leq G$; then $H \in \mathcal{C}$,
(c2) finite direct products: let $G_1, G_2 \in \mathcal{C}$; then $G_1 \times G_2 \in \mathcal{C}$,
(c3) quotients: let $G \in \mathcal{C}$ and let $N \trianglelefteq G$; then $G/N \in \mathcal{C}$,
(c4) extensions: let $K, Q \in \mathcal{C}$ and let G be a group such that the following sequence

$$1 \to K \to G \to Q \to 1$$

is exact; then $G \in \mathcal{C}$.

Let $\mathcal{C}$ be a class of finite groups and let G be a group. If $N \trianglelefteq G$ is such that $G/N \in \mathcal{C}$ then we say that N is a *co-$\mathcal{C}$* subgroup of G. We will use $\mathcal{N}_{\mathcal{C}}(G) = \{N \trianglelefteq G \mid G/N \in \mathcal{C}\}$ to denote the set of all co-$\mathcal{C}$ subgroups of G. We want the system of cosets $\mathcal{B}_{\mathcal{C}}(G) = \{gN \mid g \in G, N \in \mathcal{N}_{\mathcal{C}}(G)\}$ to form a basis of open sets for a topology on G, thus we need the set $\mathcal{N}_{\mathcal{C}}(G)$ to be closed under intersections. It can be easily seen that if $\mathcal{C}$ satisfies (c1) and (c2), then the set $\mathcal{N}_{\mathcal{C}}(G)$ is closed under intersections for every group G.

Suppose that the system of cosets $\mathcal{B}_{\mathcal{C}}(G)$ forms a basis of open sets for a topology on a group G. This topology is called the *pro-$\mathcal{C}$ topology* on G and we will use pro-$\mathcal{C}(G)$ to refer to this topology. If $\mathcal{C}$ is the class of all finite groups this topology is the profinite topology $\mathcal{PT}(G)$ and if $\mathcal{C}$ is the class of all finite p-groups, where p is a prime number, this topology is referred to as pro-p topology and is denoted by pro-$p(G)$.

We say that a subset $X \subseteq G$ is *$\mathcal{C}$-separable* or *$\mathcal{C}$-closed* in G if the subset X is closed pro-$\mathcal{C}(G)$. In other words, a subset $X \subseteq G$ is $\mathcal{C}$-separable if for every $g \in G \setminus X$ there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $gN \cap X = gN \cap XN = \emptyset$. Similarly we say that a subset $X \subseteq G$ is *$\mathcal{C}$-open* if it is open in pro-$\mathcal{C}(G)$.

2.1. Basic properties.

Unless stated otherwise we will only assume assume that the class $\mathcal{C}$ satisfies (c1) and (c2).

If the class $\mathcal{C}$ satisfies (c1) and (c2) then the pro-$\mathcal{C}$ topology on G is well-defined for every group G. Note that it would be enough to assume that the class $\mathcal{C}$ is closed under subdirect products. However, if we assume that the class $\mathcal{C}$ is also closed under taking subgroups we see that equipping a group G with pro-$\mathcal{C}$ topology is actually a faithfull functor from the category of groups to the category of topological groups: homomorphisms between groups are continuous maps with respect to the corresponding pro-$\mathcal{C}$ topologies and isomorphisms are homeomorphisms, thus preimages of $\mathcal{C}$-open/closed sets are $\mathcal{C}$-open/closed and isomorphic images of $\mathcal{C}$-open/closed sets are $\mathcal{C}$-open/closed.

Obviously, the pro-$\mathcal{C}$ topology on a group G is invariant under group translation: if $X \subseteq G$ is $\mathcal{C}$-closed in G then so are the sets gX and Xg for all $g \in G$.

The following lemma will help us to shorten and simplify proofs.

Lemma 2.1. *Let G be a group. Then $X \subseteq G$ is $\mathcal{C}$-closed in G if and only if for every $g \in G \setminus X$ there is a group F and a homomorphism $\phi : G \to F$, such that $\phi(g) \notin \phi(X)$ and $\phi(X)$ is $\mathcal{C}$-closed in F.*

Proof. Suppose X is $\mathcal{C}$-closed in G. Clearly if we take $F = G$ and $\phi = \mathrm{id}_G$ then $\phi(X) = X$ is C-closed in G and $\phi(g) = g \notin \phi(X) = X$ for all $g \in G \setminus X$.

Let $X \subseteq G$ and suppose that for every $g \in G \setminus X$ there is a group homomorphism $\phi_g \colon G \to F_g$ such that $\phi_g(g) \notin \phi_g(X)$ and $\phi_g(X)$ is $\mathcal{C}$-closed in F_g. We see that the set $\phi_g^{-1}(\phi_g(X))$ is $\mathcal{C}$-closed in G as it is a preimage of a $\mathcal{C}$-closed set. Clearly $X = \bigcap_{g \in G \setminus X} \phi_g^{-1}(\phi_g(X))$ and thus X is $\mathcal{C}$-closed in G as it is intersection of $\mathcal{C}$-closed sets. $\square$

As we already said: a group G is *residually-$\mathcal{C}$* if for every $g \in G \setminus \{1\}$ there is a group $F \in \mathcal{C}$ and a group homomorphism $\pi \colon G \to F$ such that $\pi(g) \neq 1$ in F or, equivalently, we say that G is residually-$\mathcal{C}$ if for every $g \in G \setminus \{1\}$ there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $g \notin N$. Assuming that the class $\mathcal{C}$ satisfies (c1) and (c2) one can easily check that the following are equivalent:

(1) G is residually-$\mathcal{C}$,
(2) $\{1\}$ is $\mathcal{C}$-closed in G,
(3) $\bigcap_{N \in \mathcal{N}_{\mathcal{C}}(G)} N = \{1\}$,
(4) pro-$\mathcal{C}$(G) is Hausdorff.

Being residually-$\mathcal{C}$ is clearly a hereditary property.

Remark 2.2. *Let G be a group and let $H \leq G$. If G is residually-$\mathcal{C}$ then H is residually-$\mathcal{C}$.*

Let G be a group and assume that $H \leq G$. For an element $g \in G$ we will use g^H to denote $\{hgh^{-1} \mid h \in H\} \subseteq G$, the set of H-conjugates of H. The symbol $\sim_H$ will denote the relation of being H-conjugates, i.e. $f \sim_H g$ if and only if $f \in g^H$. We can then restate the definition of $\mathcal{C}$-conjugacy separability in terms of pro-$\mathcal{C}$ topologies: group G is $\mathcal{C}$-CS if the conjugacy class g^G is $\mathcal{C}$-closed in G for every $g \in G$.

Note that a specialised version of Lemma 2.1 is that a group G is $\mathcal{C}$-CS if and only if for every tuple of elements $f, g \in G$ such that $f \not\sim_G g$ there is a $\mathcal{C}$-CS group H and a homomorphism $\phi \colon G \to H$ such that $\phi(f) \not\sim_H \phi(g)$.

2.2. **$\mathcal{C}$-open and $\mathcal{C}$-closed subgroups.** Let $H \leq G$ and suppose that there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $N \leq H$. Then clearly H is a union of cosets of N and hence H is $\mathcal{C}$-open in G as it is a union of $\mathcal{C}$-open subsets of G. It was proved by Hall [11, Theorem 3.1] that the opposite implication holds as well.

Lemma 2.3 (Classification of $\mathcal{C}$-open subgroups)**.** *Let G be a group and let $H \leq G$. Then H is $\mathcal{C}$-open in G if and only if there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $N \leq H$. Moreover, every $\mathcal{C}$-open subgroup is $\mathcal{C}$-closed in G and is of finite index in G.*

Lemma 2.3 allows us to restate the definition of $\mathcal{C}$-hereditary conjugacy separability in terms of pro-$\mathcal{C}$ topology: a group G is $\mathcal{C}$-HCS if G is $\mathcal{C}$-CS and $H \leq G$ is $\mathcal{C}$-CS as well whenever H is $\mathcal{C}$-open in G.

Obviously, an intersection of $\mathcal{C}$-open subgroups is a $\mathcal{C}$-closed subgroup. It was proved by Hall [11, Theorem 3.3] that all $\mathcal{C}$-closed subgroups arise in this way.

Lemma 2.4 (Classification of $\mathcal{C}$-closed subgroups)**.** *Let G be a group and let $H \leq G$. Then H is $\mathcal{C}$-closed in G if and only if it is an intersection of $\mathcal{C}$-open subgroups of G.*

2.3. **Restrictions of pro-$\mathcal{C}$ topologies.** Imagine the following situation: let G be a group and let H be its subgroup. The pro-$\mathcal{C}$ topology induces a subspace topology on H, say $\mathcal{T}$. However, this topology might not necessarily be the same as pro-$\mathcal{C}$ topology on H: pro-$\mathcal{C}(H)$ will usually be finer than $\mathcal{T}$. For example, if G is F_2, the free group on 2 generators and H is $[G, G]$, the commutator subgroup of G, then H contains only countably many subgroups that are open in $\mathcal{T}$; however, as H is infinitely generated free group, $\mathcal{N}_{\mathcal{C}}(H)$ is uncountable.

For $H \leq G$ we say that pro-$\mathcal{C}(H)$ is *induced* by pro-$\mathcal{C}(G)$ if pro-$\mathcal{C}(H)$ coincides with the subspace topology on H induced by pro-$\mathcal{C}(G)$.

If $H \leq G$ and $X \subseteq H$ is $\mathcal{C}$-closed in G then clearly X is $\mathcal{C}$-closed in H. However, the implication in the other direction does not hold: let G be the Baumslag-Solitar group BS(2,3) given by the presentation $\langle a, b \| ba^2b^{-1} = a^3 \rangle$. It is well known that this group is not residually finite. Take $H = \langle a \rangle \leq G$. Clearly $H \cong (\mathbb{Z}, +)$, which is a residually finite group. Thus the singleton set $\{1\}$ is closed in the profinite topology on H but it is not closed in the profinite topology on G as G is not residually finite.

Definition 2.5. *Let G be a group and let $H \leq G$ be its subgroup. We say that the pro-$\mathcal{C}(H)$ is a restriction of pro-$\mathcal{C}(G)$ if for all $X \subseteq H$ we have that X is $\mathcal{C}$-closed in H if and only if it is $\mathcal{C}$-closed in G.*

One can easily check that for $H \leq G$ pro-$\mathcal{C}(H)$ is a restriction of pro-$\mathcal{C}(G)$ if and only if pro-$\mathcal{C}(H)$ is induced by pro-$\mathcal{C}(G)$ and H is $\mathcal{C}$-closed in G.

One type of subgroup on which the pro-$\mathcal{C}$ topologies behave well are retracts. Let G be a group and let $R \leq G$. We say that R is a *retract* of G if there is a homomorphism $\rho \colon G \to R$ such that $\rho\restriction_R = \mathrm{id}_R$. We will refer to ρ as to the *retraction* corresponding to R. We will often abuse the notation and consider ρ as an endomorphism of G as well. One could then equivalently say that R is a retract of G if and only if there is $\rho \colon G \to G$ such that $\rho \circ \rho = \rho$ and R is the image of ρ.

Note that if $R \leq G$ is a retract then G can be viewed as semidirect product $G = \ker(\rho) \rtimes R$, where $\rho \colon G \to R$ is the corresponding retraction. One can easily show the following by using the proof of [27, Lemma 3.1.5].

Lemma 2.6. *Let G be a residually-$\mathcal{C}$ group and let $R \leq G$ be a retract. Then R is $\mathcal{C}$-closed in G and pro-$\mathcal{C}(R)$ is induced by pro-$\mathcal{C}(G)$, hence pro-$\mathcal{C}(R)$ is a restriction of pro-$\mathcal{C}(G)$.*

So far we have only been using assumptions that the class $\mathcal{C}$ satisfies (c1) and (c2). From now on we will also require the class $\mathcal{C}$ to satisfy (c4). The proof the next lemma follows from the proof of [27, Lemma 3.1.4].

Lemma 2.7. *Suppose that $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2) and (c4). Let G be a group and let $H \leq G$ be $\mathcal{C}$-open in G. Then pro-$\mathcal{C}(H)$ is a restriction of pro-$\mathcal{C}(G)$.*

As we mentioned earlier, the property of being residually-$\mathcal{C}$ is passed to subgroups. Obviously, the implication in the opposite direction does not hold: the group BS(2,3) contains a residually finite subgroup but BS(2,3) is not residually finite. However, the property of being residually-$\mathcal{C}$ is passed upwards from $\mathcal{C}$-open subgroups.

Corollary 2.8. *Let $\mathcal{C}$ be a class of finite groups satisfying (c1), (c2) and (c4). Let G be a group and let $H \leq G$ be $\mathcal{C}$-open in G. If H is residually-$\mathcal{C}$ then G is residually-$\mathcal{C}$ as well.*

Proof. The singleton set $\{1\}$ is $\mathcal{C}$-closed in H as H is residually-$\mathcal{C}$. By Lemma 2.7 we see that the singleton set $\{1\}$ is $\mathcal{C}$-closed in G as the pro-$\mathcal{C}(H)$ is a restriction of pro-$\mathcal{C}(G)$. Hence, we see that the group G is residually-$\mathcal{C}$. □

The structure of classes of finite groups that satisfy (c1) and (c2) only can be quite wild. However, what if we also require the class $\mathcal{C}$ to satisfy (c4)? Suppose that $\mathcal{C}$ is a class of finite groups satisfying (c1), (c2) and (c4) and suppose that there is a nontrivial group F such that $F \in \mathcal{C}$. Let $n = |F|$ and let p be a prime number such that p divides n. Clearly there is $g \in F$ such that $\mathrm{ord}(g) = p$ and thus F contains C_p, the cyclic group of size p as a subgroup and thus $C_p \in \mathcal{C}$ as well. We see that the class $\mathcal{C}$ contains all finite p-groups as every finite p-group can be constructed from C_p by a series of extensions. It is well known fact (see [10]) that free groups are residually-p for every prime number p and therefore free groups are residually-$\mathcal{C}$ whenever the class $\mathcal{C}$ satisfies (c1), (c2) and (c4) and contains at least one nontrivial group.

Lemma 2.9. *Let $\mathcal{C}$ be a class of finite groups satisfying (c1), (c2) and (c4) and assume that $\mathcal{C}$ contains a nontrivial group. Then free-by-$\mathcal{C}$ groups are residually-C.*

Proof. Let G be a free-by-$\mathcal{C}$ group. By assumption there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that N is free. Clearly N is $\mathcal{C}$-open in G. By previous argumentation we know that N is residually-C and hence G is residually-$\mathcal{C}$ by Corollary 2.8. □

3. Graph products of groups

Let Γ be a simplicial graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of groups indexed by $V\Gamma$. Recal that the graph product $\Gamma\mathcal{G}$ is the quotient of the free product $\ast_{v \in V\Gamma} G_v$ obtained by adding relations of the form

$$g_u g_v = g_v g_u \text{ for all } g_u \in G_u, g_v \in G_v \text{ such that } \{u, v\} \in E\Gamma.$$

For $v \in V\Gamma$ we define the link(v) to be the the set of vertices adjacent to v in Γ (excluding v itself), more precisely $\mathrm{link}(v) = \{u \in V\Gamma \mid \{u, v\} \in E\Gamma\}$. For a subset $A \subseteq V\Gamma$ we define $\mathrm{link}(A) = \bigcap_{v \in A} \mathrm{link}(v)$.

For $v \in V\Gamma$ we define the star(v) to be the the set of vertices adjacent to v in Γ including v, more precisely $\mathrm{star}(v) = \{u \in V\Gamma \mid \{u,v\} \in E\Gamma\} \cup \{v\}$. For a subset $A \subseteq V\Gamma$ we define $\mathrm{star}(A) = \bigcap_{v\in A} \mathrm{star}(v)$.

Let $G = \Gamma\mathcal{G}$ be a graph product. Then every $g \in G$ can be obtained as a product of a sequence of generators $W \equiv (g_1, g_2, \dots, g_n)$ where each g_i belongs to some $G_{v_i} \in \mathcal{G}$. We will say that W is a *word* in G and the g_i are its *syllables*. The number of syllables is the *length* of a word.

Transformations of the three following types can be defined on words in graph products:

(T1) remove a syllable g_i if $g_i =_{G_v} 1$, where $v \in V\Gamma$ and $g_i \in G_v$,
(T2) remove two consecutive syllables g_i, g_{i+1} belonging to the same vertex group G_v and replace them by a single syllable $g_i g_{i+1} \in G_v$,
(T3) interchange consecutive syllables $g_i \in G_u$ and $g_{i+1} \in G_v$ if $\{u,v\} \in E\Gamma$.

The last transformation is also called *syllable shuffling*. Note that the transformations (T1) and (T2) decrease the length of a word whereas (T3) preserves it. Thus by applying these transformations to a word W we will eventually get a word W' of minimal length representing the same element in G.

For $1 \le i < j \le n$ we say that syllables g_i, g_j can be *joined together* if they belong to the same vertex group and 'everything in between commutes with them'. More formally: $g_i, g_j \in G_v$ for some $v \in V\Gamma$ and for all $i < k < j$ we have $g_k \in G_{v_k}$ such that $v_k \in \mathrm{link}(v)$. In this case obviously the words $W \equiv (g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_{j-1}, g_j, g_{j+1}, \dots, g_n)$ and $W' \equiv (g_1, \dots, g_{i-1}, g_i g_j, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$ represent the same group element in G, but the word W' is strictly shorter than W.

We say that a word $W \equiv (g_1, g_2, \dots, g_n)$ is Γ-*reduced* if it is either an empty word or if $g_i \neq 1$ for all i and no two distinct syllables can be joined together. To avoid any confusion with the terminology of special amalgams (see Section 5) we will be using the Greek letter Γ to emphasise that we are considering sequences and elements (cyclically) reduced in the context of graph products and not in the context of special amalgams.

As it turns out, the notion of being Γ-reduced and the notion of having minimal length coincide: the following theorem was proved by E. Green [9, Theorem 3.9].

Theorem 3.1 (Normal Form Theorem)**.** *Let $G = \Gamma\mathcal{G}$ be a graph product. Every element $g \in G$ can be represented by a Γ-reduced word. Moreover, if two Γ-reduced words W, W' represent the same element in the group G then W can be obtained from W' by a finite sequence of syllable shuffling. In particular, the length of a Γ-reduced word is minimal among all words representing g, and a Γ-reduced word represents the trivial element if and only if it is the empty word.*

Thanks to Theorem 3.1 the following definitions make sense. Let $g \in G$ and let $W = (g_1, \dots, g_n)$ be a Γ-reduced word representing g. We define the *length* of g in G to be $|g| = n$ and the *support* of g in G to be

$$\mathrm{supp}(g) = \{v \in V\Gamma | \exists i \in \{1, \dots, n\} \text{ such that } g_i \in G_v \setminus \{1\}\}.$$

We define $\mathrm{FL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there is a Γ-reduced word W that represents the element g and starts with a syllable from G_v. Similarly we define $\mathrm{LL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there is a Γ-reduced word W that represents the element g and ends with a syllable from G_v. Note that $\mathrm{FL}(g) = \mathrm{LL}(g^{-1})$.

Let $x, y \in G$ and let $W_x \equiv (x_1, \ldots, x_n)$ be a Γ-reduced expression for x and let $W_y \equiv (y_1 \ldots, y_m)$ be a Γ-reduced expression for y. We say that the product xy is aΓ-*reduced product* if the word $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is Γ-reduced. Obviously, xy is a Γ-reduced product if and only if $|xy| = |x| + |y|$. Or equivalently we could say that xy is Γ-reduced product if and only if $\mathrm{LL}(x) \cap \mathrm{FL}(y) = \emptyset$. We can naturally extend this definition: for $g_1, \ldots, g_n \in G$ we say that the product $g_1 \ldots g_n$ is Γ-*reduced* if $|g_1 \ldots g_n| = |g_1| + \cdots + |g_n|$.

3.1. **Full and parabolic subgroups.** Let Γ be a graph. For any subset $A \subseteq V\Gamma$ we will denote the corresponding full subgraph by Γ_A: $V\Gamma_A = A$ and for $u, v \in A$ we have $\{u, v\} \in E\Gamma_A$ if and only if $\{u, v\} \in E\Gamma$.

Let $A \subseteq V\Gamma$ and let G_A denote the subgroup of G generated by all G_v, where $v \in A$. Using Theorem 3.1 one can easily check that G_A is isomorphic to the graph product $\Gamma_A \mathcal{G}_A$, where $\mathcal{G}_A = \{G_v \mid v \in A\}$. We see that every $A \subset V\Gamma$ induces a subgroup $G_A \leq G$. We will call subgroups of this type *full subgroups* of $\Gamma\mathcal{G}$, and we say that a full subgroup G_A is a *proper* full subgroup if A is a proper subset of $V\Gamma$. We say that a full subgroup $G_A \leq G$ is *maximal* if $|V\Gamma \setminus A| = 1$. By definition $G_\emptyset = \{1\}$ is also a full subgroup corresponding to the empty subset of $V\Gamma$.

It can be easily seen that full subgroups are actually retracts.

Remark 3.2. *Let $G = \Gamma\mathcal{G}$ be a graph product of groups and let $G_A \leq G$ be a full subgroup. Then G_A is a retract in G with corresponding retraction map $\rho_A : G \to G_A$ defined on generators of G as follows:*

$$\rho_A(g) = \begin{cases} g & \text{if } g \in G_v \text{ and } v \in A, \\ 1 & \text{otherwise.} \end{cases}$$

Let $A, B \subseteq V\Gamma$ be arbitrary. Let $G_A, G_B \leq G$ be the corresponding full subgroups and let $\rho_A, \rho_B \in \mathrm{End}(G)$ be the corresponding retractions. We see that ρ_A and ρ_B commute: $\rho_A \circ \rho_B = \rho_B \circ \rho_A$. It follows that $G_A \cap G_B = G_{A \cap B}$ and $\rho_A \circ \rho_B = \rho_{A \cap B}$. This result can be generalised and strengthened.

Let $K \leq G$. We say that K is *parabolic* subgroup of G if K is conjugate to a full subgroup, i.e., if there are $A \subseteq V\Gamma$ and $g \in G$ such that $K = gG_Ag^{-1}$. As it turns out, the intersection of parabolic subgroups is again a parabolic subgroup. The following theorem was proved in [1, Corollary 3.6].

Theorem 3.3. *Let $G = \Gamma\mathcal{G}$ be a graph product and let $K, L \leq G$ such that $K = gG_Ag^{-1}$ and $H = fG_Bf^{-1}$, where $A, B \subseteq V\Gamma$ and $f, g \in G$. Then there is $h \in G$ and $C \subseteq A \cap B$ such that $K \cap L = hG_Ch^{-1}$.*

As an easy consequence of Theorem 3.3 we get the following lemma.

Lemma 3.4. *Let $g \in G = \Gamma\mathcal{G}$ and suppose that $|V\Gamma| < \infty$ and $g \neq 1$. Then there is a maximal full subgroup A of G such that $g \notin A^G$.*

Proof. Let $g \in G \setminus \{1\}$ and assume that the statement of the lemma does not hold for g, thus for every maximal full subgroup A_v, where $A_v = G_{V\Gamma \setminus \{v\}}$ for some $v \in V\Gamma$, there is $h_v \in G$ such that $g \in h_v A_v h_v^{-1}$. We see that $g \in \bigcap_{v \in V\Gamma} h_v A_v h_v^{-1}$. By Theorem 3.3 we see that there are $h \in G$ and $C \subseteq \bigcap_{v \in V\Gamma} V\Gamma \setminus \{v\}$ such that $g \in hG_Ch^{-1}$. However,

$\bigcap_{v \in V\Gamma} V\Gamma \setminus \{v\} = \emptyset$ and thus we see that $g = 1$, which is a contradiction because we assumed that $g \neq 1$. □

The following theorem was proved in [1, Proposition 3.10].

Theorem 3.5. *Let X be a subset of the graph product $G = \Gamma\mathcal{G}$ such that at least one of the following conditions holds:*
 (i) *the graph Γ is finite;*
 (ii) *the subgroup $\langle X \rangle \leq G$ is finitely generated.*
Then there exists a unique minimal parabolic subgroup of G containing X.

Suppose that a subset $X \subseteq G$ is contained in a minimal parabolic subgroup of G. Then this subgroup will be called the *parabolic closure* of X and will be denoted by $\mathrm{Pc}_\Gamma(X)$.

For a subset $X \subseteq G$ and a subgroup H we will use $N_H(X)$ to denote $\{g \in G \mid gX = Xg\}$, the H-normaliser of X in G. The following characterisation of normalisers of parabolic subgroups was given in [1, Proposition 3.13].

Theorem 3.6. *Let K be a nontrivial parabolic subgroup of the graph product $G = \Gamma\mathcal{G}$. Choose $f \in G$ and $S \in V\Gamma$ such that $K = fG_Sf^{-1}$ and $G_s \neq \{1\}$ for all $s \in S$. Then $N_G(K) = fG_{S \cup \mathrm{link}(S)}f^{-1}$; in particular the normaliser $N_G(K)$ is a parabolic subgroup of G.*

For a subset $X \subseteq G$ and a subgroup H we will use $C_H(X)$ to denote $\{g \in G \mid gx = xg\}$, the H-centraliser of X in G. Centralisers in graph products were fully described by Barkauskas in [2]. We give simple a lemma describing centralisers of elements in terms of certain special subgroups and centralisers in full subgroups.

Lemma 3.7. *Let $G = \Gamma\mathcal{G}$ be a graph product of groups and let $g \in G$ be arbitrary. Suppose that there is $A \subseteq G$ such that $\mathrm{Pc}_\Gamma(\langle g \rangle) = G_A$. Then $C_G(g) = C_{G_A}(g)G_{\mathrm{link}(A)}$.*

Proof. Clearly $C_G(g) \leq N_G(\langle g \rangle)$. Since $G_A = \mathrm{Pc}_\Gamma(\langle g \rangle)$ we see by [1, Lemma 3.12] that $N_G(\langle g \rangle) \subseteq N_G(G_A)$. By Theorem 3.6 we see that $N_G(G_A) = G_A \cdot G_{\mathrm{link}(A)}$ and thus $C_G(g) \subseteq G_A \cdot G_{\mathrm{link}(A)}$. We can then write $C_G(g) = C_G(g) \cap G_A \cdot G_{\mathrm{link}(A)}$. Note that $G_{\mathrm{link}(A)} \leq C_G(G_A)$ and thus $G_{\mathrm{link}(A)} \leq C_G(g)$. This means that $C_G(g) \cap G_A \cdot G_{\mathrm{link}(A)} = (C_G(g) \cap G_A)G_{\mathrm{link}(A)}$ and we see that $C_G(g) = C_{G_A}(g)G_{\mathrm{link}(A)}$. □

3.2. **Cyclically reduced elements and conjugacy in graph products.** Let $g \in G$, let $W \equiv (g_1, \dots, g_n)$ be a Γ-reduced expression for g. We say that a sequence $W' = (g_{j+1}, \dots, g_n, g_1, \dots, g_j)$, where $j \in \{1, \dots, n-1\}$, is a *cyclic permutation* of W. We say that the element $g' \in G$ is a *cyclic permutation* of g if g' can be expressed by a cyclic permutation of some Γ-reduced expression for g.

Let $W \equiv (g_1, \dots, g_n)$ be some reduced expression in G. We say that W is Γ-*cyclically reduced* if all cyclic permutations of W are Γ-reduced. We would like to extend this definition to elements of G. However, to achieve that we first need to show that this property does not depend on the choice of Γ-reduced expression.

Lemma 3.8. *Let $g \in G$ be arbitrary and let $W \equiv (g_1, \dots g_n)$ be some Γ-reduced expression for g. If W is Γ-cyclically reduced then all Γ-reduced expressions representing g are Γ-cyclically reduced.*

Proof. Assume that $W = (g_1, \dots, g_n)$ is Γ-cyclically reduced sequence and let $i \in \{1, \dots, n-1\}$ be arbitrary such that $[g_i, g_{i+1}] = 1$. Consider the expression $W' = (g_1, \dots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \dots, g_n)$. Obviously W' is a Γ-reduced expression for g as well. Let W'' be some cyclic permutation of W'. Then there are three cases to consider:

(i) $W'' = (g_{j+1}, \dots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \dots, g_n, g_1, \dots, g_j)$ for some $j < i$,
(ii) $W'' = (g_i, g_{i+2}, \dots, g_n, g_1, \dots, g_{i-1}, g_{i+1})$,
(iii) $W'' = (g_{j+1}, \dots, g_n, g_1, \dots, g_{i-1}, g_{i+1}, g_i, g_{i+2}, \dots, g_j)$ for some $j > i$.

Consider the case (i) first. The expression W'' can be rewritten to the expression $V = (g_{j+1}, \dots, g_n, g_1, \dots, g_j)$ by swapping the syllables g_i and g_{i+1}. We see that V is Γ-reduced as it is a cyclic permutation of W and W is Γ-cyclically reduced by assumption. It follows by Theorem 3.1 that W'' is Γ-reduced as it represents the same element as V and both W'' and V are of the same length. The case (iii) can be dealt with similarly.

For case (ii) we see that the segment $(g_{i+2}, \dots, g_n, g_1, \dots, g_{i-1})$ is Γ-reduced as it is a segment of a cyclic permutation of W and W is Γ-cyclically reduced. Suppose that the sequence W'' is not Γ-reduced. Suppose that g_i can be joined with g_k, where $k \in \{i+2, \dots, n\}$. If this was the case then the syllable g_i could have been joined with g_k in W which is a contradiction with our assumption that W is Γ-reduced. Suppose that the syllable g_i can be joined with g_l, where $l \in \{1, \dots, i-1\}$. Since the syllables g_i and g_{i+1} commute we see that g_i and g_l could be joined in the expression $P = (g_i, g_{i+1}, \dots, g_n, g_1, \dots, g_{i-1})$. However, P is a cyclic permutation of W and therefore P is Γ-reduced as W is Γ-cyclically reduced by assumption. By a similar argumentation we can show that the syllable g_{i+1} cannot be joined with any of the syllables $g_1, \dots, g_{i-1}$ or $g_{i+1}, \dots, g_n$. Clearly g_i cannot be joined with g_{i+1} as we assume that W is Γ-reduced. Therefore we see that W'' is Γ-reduced.

We have shown that the property of being Γ-cyclically reduced is preserved by transformation (T3). By Theorem 3.1 every Γ-reduced expression for g can be obtained from W by a finite sequence of transformation of type (T3). Hence all Γ-reduced expressions for g are Γ-cyclically reduced. □

As a direct consequence of this lemma we see that a Γ-reduced expression $W = (g_1, \dots, g_n)$ is Γ-cyclically reduced if and only if the following condition is satisfied: let $i, j \in \{1 \dots, n\}$ be such that g_i can be shuffled to the beginning of W and g_j can be shuffled to the end of W and g_i and g_j belong to the same vertex group; then $i = j$.

Definition 3.9. *Let $g \in G$ be arbitrary. We say that g is Γ-cyclically reduced if either g is trivial or some Γ-reduced word representing g is Γ-cyclically reduced.*

Note that $\mathrm{FL}(g) \cap \mathrm{LL}(g) \neq \emptyset$ does not necessarily mean that g is not Γ-cyclically reduced. Suppose that $\mathrm{supp}(g) \cap \mathrm{star}(\mathrm{supp}(g)) \neq \emptyset$. Then there is $v \in \mathrm{supp}(g)$ such that it is connected with all the other vertices in $\mathrm{supp}(g)$. This means that there is $i \in \{1, \dots, n\}$ such that the syllable g_i commutes with all the other syllables and can be shuffled to both ends of g, thus $v \in \mathrm{FL}(g) \cap \mathrm{LL}(g)$.

Definition 3.10 (P-S decomposition)**.** *Let $g \in G$. We define $S(g) = \mathrm{supp}(g) \cap \mathrm{star}(\mathrm{supp}(g))$. Similarly, we define $P(g) = \mathrm{supp}(g) \setminus S(g)$. Obviously g uniquely factorises as a Γ-reduced product $g = s(g)p(g)$ where $\mathrm{supp}(s(g)) = S(g)$ and $\mathrm{supp}(p(g)) = P(g)$. We call this factorisation the P-S decomposition of g.*

Note that $\mathrm{FL}(g) = S(g)\dot\cup\,\mathrm{FL}(p(g))$, $\mathrm{LL}(g) = S(g)\dot\cup\,\mathrm{LL}(p(g))$ and $S(p(g)) = \emptyset$. Another simple observation is that if g' is a cyclic permutation of g then g' can be uniquely factorised as $s(g)p'$, where p' is a cyclic permutation of $p(g)$.

Lemma 3.11. *Let $g \in G$. Then the following are equivalent:*

(i) *g is Γ-cyclically reduced,*
(ii) $(\mathrm{FL}(g) \cap \mathrm{LL}(g)) \setminus S(g) = \emptyset$,
(iii) $\mathrm{FL}(p(g)) \cap \mathrm{LL}(p(g)) = \emptyset$,
(iv) *$p(g)$ is Γ-cyclically reduced.*

Proof. (i) $\Rightarrow$ (ii): assume that g is Γ-cyclically reduced. Let $(g_1, \dots, g_n)$ be some Γ-reduced expression for g. Without loss of generality we may assume that $s(g) = g_1 \dots g_s$ and $p(g) = g_{s+1} \dots g_n$, where $s = |S(g)|$. Suppose that $v \in (\mathrm{FL}(g) \cap \mathrm{LL}(g)) \setminus S(g)$. Then there are $1 \leq i < j \leq n$ such that $g_i, g_j \in G_v$. Since $v \in \mathrm{FL}(g)$ we see that g_i can be shuffled to beginning of g. Similarly g_j can be shuffled to the end of g and hence

$$W = (g_i, g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n, g_j)$$

is also a Γ-reduced expression for g. However, the expression

$$W' = (g_j, g_i, g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$$

is not reduced which is a contradiction as W' is a cyclic permutation of W and g is Γ-cyclically reduced.

(ii) $\Rightarrow$ (iii): suppose that $(\mathrm{FL}(g) \cap \mathrm{LL}(g)) \setminus S(g) = \emptyset$. As mentioned before, $\mathrm{FL}(g) = \mathrm{FL}(p(g))\dot\cup S(g)$ and $\mathrm{LL}(g) = \mathrm{LL}(p(g))\dot\cup S(g)$ and therefore $\mathrm{FL}(p(g)) \cap \mathrm{LL}(p(g)) = \emptyset$.

(iii) $\Rightarrow$ (iv): if $\mathrm{FL}(p(g)) \cap \mathrm{LL}(p(g)) = \emptyset$ then clearly $p(g)$ is Γ-cyclically reduced.

(iv) $\Rightarrow$ (i): assume that $p(g)$ is Γ-cyclically reduced and there is $v \in \mathrm{FL}(p(g)) \cap \mathrm{LL}(p(g))$. Let $(p_1, \dots, p_m)$ be a Γ-reduced expression for $p(g)$. Suppose that there are $1 \leq i < j \leq m$ such that $p_i, p_j \in G_v$. This is clearly a contradiction since $p(g)$ is Γ-cyclically reduced by assumption. This means that there is $i \in \{1, \dots, n\}$ such that the expression $(p_1, \dots, p_n)$ can be rewritten by shuffling to $(p_i, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ and to $(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n, p_i)$ as well. This means that p_i commutes with all the other syllables from $p(g)$ and hence the vertex v is adjacent to all the vertices in $P(g) \setminus \{v\}$. But since v is also connected to all the vertices in $S(g)$ by the definition of $S(g)$ we see that $v \in S(g)$. This is a contradiction as $v \in \mathrm{supp}(p(g)) \subseteq P(g)$, hence we may assume that $\mathrm{FL}(p(g)) \cap \mathrm{LL}(p(g)) = \emptyset$. As stated before, $\mathrm{FL}(g) = \mathrm{FL}(p(g))\dot\cup S(g)$ and $\mathrm{LL}(g) = \mathrm{LL}(p(g))\dot\cup S(g)$. Since $\mathrm{FL}(p(g)) \cap \mathrm{LL}(p(g)) = \emptyset$ we see that $\mathrm{FL}(g) \cap \mathrm{LL}(g) = S(g)$. Let $W = (g_1, \dots, g_m)$ be a Γ-reduced expression for g. Suppose that there are $i, j \in \{1, \dots, n\}$ such that g_i can be shuffled to the beginning of W, g_j can be shuffled to the end of g and g_i and g_i belong to the same vertex group. Since $\mathrm{FL}(g) \cap \mathrm{LL}(g) = S(g)$ we see that $g_i, g_j \in G_s$ for some $s \in S(g)$ as W is Γ-reduced. This means that $i = j$ and consequently that g is Γ-cyclically reduced. $\square$

Lemma 3.12 (Conjugacy criterion for graph products)**.** *Let x, y be Γ-cyclically reduced elements of $G = \Gamma\mathcal{G}$. Then $x \sim_G y$ if and only if the all of the following are true:*

(i) $|x| = |y|$ *and* $\mathrm{supp}(x) = \mathrm{supp}(y)$,
(ii) *$p(x)$ is a cyclic permutation of $p(y)$,*
(iii) $s(y) \in s(x)^{G_{S(x)}}$.

Proof. The "if" part of the claim holds trivially.

Let $x, y \in G$ be Γ-cyclically reduced such that $x \sim_G y$. Without loss of generality we will assume that $|x| \geq |y|$. Let $X \subseteq G$ denote the set of all cyclic permutations of x. Clearly $X^{G_{S(x)}} \subseteq x^G$. Pick $x' \in X^{G_{S(x)}}$ such that the corresponding $g' \in G$, where $g'x'g'^{-1} = y$, is of minimal length. First, we show by induction on $|\operatorname{LL}(g') \cap \operatorname{FL}(x')|$ that there are elements $x'' \in X^{G_{S(x)}}$ and $g'' \in G$ such that $|g''| = |g'|$, $g''x''g''^{-1} = y$ and the product $g''x''$ is Γ-reduced. If $|\operatorname{LL}(g') \cap \operatorname{FL}(x')| = 0$ then clearly the product $g'x'$ is Γ-reduced and the claim holds for $g'' = g'$ and $x'' = x'$. Suppose that $|\operatorname{LL}(g) \cap \operatorname{FL}(x')| = c > 0$ and that the statement holds for all $c' < c$. Let $(g_1, \ldots, g_k)$ be a Γ-reduced expression for g' and let $(x_1, \ldots, x_n)$ be a Γ-reduced expression for x'. Without loss of generality we may assume that g_k and x_1 belong to the same vertex group, say G_v. Then

$$\begin{aligned} y &= g_1 \ldots g_k x_1 \ldots x_n g_k^{-1} \ldots g_1^{-1} \\ &= g_1 \ldots g_{k-1}(g_k x_1) x_2 \ldots x_n x_1 (g_k x_1)^{-1} g_{k-1}^{-1} \ldots g_1^{-1}. \end{aligned}$$

Obviously $g_k \neq x_1^{-1}$ as otherwise we could replace x' by $x_2 \ldots x_n x_1$, a cyclic permutation of x, and g' by $g_1 \ldots g_{k-1}$. Clearly $x_2 \ldots x_n x_1 \in X^{G_{S(X)}}$ and

$$g_1 \ldots g_{k-1} x_2 \ldots x_n x_1 g_{k-1}^{-1} \ldots g_1^{-1} = y$$

which is a contradiction with our choice of x' and g' as $|g_1 \ldots g_{k-1}| < |g'|$. If $v \in S(x)$ then $g_k x g_k^{-1} \in X^{G_{S(x)}}$ and again we have a contradiction with our choice of x' and g'. We see that $v \notin S(x)$ and thus $\operatorname{LL}(g') = \operatorname{LL}(g'x_1)$ and also $v \notin \operatorname{FL}(x_2 \ldots x_n x_1)$. Note that if g_i can be shuffled to the end of g' then $[g_i, g_k] = 1$ and necessarily $\{u, v\} \in E\Gamma$, where $g_i \in G_u$. If $w \in \operatorname{FL}(x_2 \ldots x_n x_1) \setminus \operatorname{FL}(x')$ then we see that $\{v, w\} \notin E\Gamma$ hence $w \notin \operatorname{LL}(g'x_1)$. From this we can conclude that $v \notin \operatorname{LL}(g'x_1) \cap \operatorname{FL}(x_2 \ldots x_n) \subseteq \operatorname{LL}(g') \cap \operatorname{FL}(x')$ hence $\operatorname{LL}(g'x_1) \cap \operatorname{FL}(x_2 \ldots x_n x_1)$ is a proper subset of $\operatorname{LL}(g') \cap \operatorname{FL}(x')$. Now we can use induction hypothesis and we are done.

We have $g''x'' = yg''$. Since $g''x''$ is a Γ-reduced product we see that $|g''x''| = |g''| + |x''| = |g| + |x| = n + k$. Also $|yg''| \leq |y| + |g| = m + k$, where $m = |y|$. However, we assumed that $|x| \geq |y|$ and thus we see that $n = m$ and consequently yg'' is a Γ-reduced product as well. Let $(y_1, \ldots, y_n)$ be some Γ-reduced expression for y and suppose that $(x_1, \ldots, x_n)$ and $(g_1, \ldots, g_k)$ are Γ-reduced expressions for x'' and g''. We have

$$g_1 \ldots g_k x_1 \ldots x_n g_k^{-1} \ldots g_1^{-1} = y_1 \ldots y_n.$$

The expression $(g_1, \ldots, g_k, x_1, \ldots, x_n, g_k^{-1}, \ldots, g_1^{-1})$ cannot be Γ-reduced by Theorem 3.1. Assume that the syllable g_k^{-1} can be joined up with g_j for some $j \in \{1, \ldots, k\}$. But then by definition $[g_k^{-1}, x_i] = 1$ for all $i = 1, \ldots, n$. Clearly

$$g_1 \ldots g_{k-1} x_1 \ldots x_n g_{k-1}^{-1} \cdots g_1^{-1} = y_1 \ldots y_n.$$

which is a contradiction with the minimality of $|g|$. Since $g''x''$ is a Γ-reduced product we then see that the expression $(x_1, \ldots, x_n, g_k^{-1}, \ldots g_1^{-1})$ is not Γ-reduced. Without loss of generality we may assume that g_k and x_n belong to the same vertex group. Assume that $g_k \neq x_n$. Then we have

$$g_1 \ldots g_k x_1 \ldots x_{n-1}(x_n g_k^{-1}) = y_1 \ldots y_n g_1 \ldots g_{k-1}.$$

From the construction of x'' and g'' we see that $(g_1, \ldots, g_n, x_1, \ldots, x_{n-1}, x_n g_k^{-1})$ is a Γ-reduced expression and so is $(y_1, \ldots, y_n, g_1, \ldots, g_{k-1})$. However, this is a contradiction with Theorem 3.1 as both of these expressions represent the same group element, but they are not of the same length. Hence we see that $g_k = x_n$

$$y_1 \ldots y_n = g_1 \ldots g_{k-1} x_n x_1 \ldots x_{n-1} g_{k-1}^{-1} \ldots g_1^{-1}$$

which is a contradiction as we could replace $x_n x_1 \ldots x_{n-1}$, a cyclic permutation of x'' and thus element of $X^{G_{S(x)}}$ and g'' by $g_1 \ldots g_{k-1}$ and get a shorter conjugator. We see that unless $g = 1$ we always get a contradiction. It follows that $y = x'' \in X^{G_{S(x)}}$ and consequently $\mathrm{supp}(x) = \mathrm{supp}(y)$, $s(y) \in s(x)^{G_{S(x)}}$ and $p(x)$ is a cyclic permutation of $p(y)$. □

4. $\mathcal{C}$-centraliser conditions and $\mathcal{C}$-conjugacy separability

In this section we will assume that the class $\mathcal{C}$ satisfies (c1), (c2) and (c4), i.e. we will assume that the class $\mathcal{C}$ is closed under taking subgroups, direct products and extensions.

Definition 4.1. *We say that a group G satisfies the $\mathcal{C}$-centraliser condition ($\mathcal{C}$-CC) if for every $K \in \mathcal{N}_{\mathcal{C}}(G)$ and every $g \in G$ there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and*

$$C_{G/L}(\psi(g)) \subseteq \psi(C_G(g)K) \text{ in } G/L,$$

where $\psi\colon G \to G/L$ is the natural projection.

Centraliser condition was introduced by Chagas and Zalesskii in [5] in case when $\mathcal{C}$ is the class of all finite groups. However, their definition of centraliser condition was given in terms of profinite completion. They showed that if group G is conjugacy separable and satisfies centraliser condition then G is HCS. Minasyan gave the definition in terms of subgroups of finite index and showed that for residually finite groups the definitions are equivalent. Minasyan also showed that the implication in the other direction holds as well: CS group G is HCS if and only if it satisfies CC (see [18, Proposition 3.2]). Toinet proved that the same statement holds when $\mathcal{C}$ is the class of all finite p-groups for some $p \in \mathbb{P}$ (see [29, Proposition 3.6]). We show that the statement is true whenever the class $\mathcal{C}$ satisfies (c1), (c2) and (c4).

Theorem 4.2. *Let G be a group. Then the following are equivalent:*

(i) *G is $\mathcal{C}$-HCS,*
(ii) *G is $\mathcal{C}$-CS and satisfies $\mathcal{C}$-CC.*

Before we proceed with the proof of Theorem 4.2 we need to define two more conditions.

Definition 4.3. *Let G be a group and let $H \leq G$ and $g \in G$. We say that the pair (H, g) satisfies the $\mathcal{C}$-centraliser condition in G ($\mathcal{C}$-CC$_G$) if for every $K \in \mathcal{N}_{\mathcal{C}}(G)$ there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and*

$$C_{\psi(H)}(\psi(g)) \subseteq \psi(C_H(g)K) \text{ in } G/L,$$

where $\psi\colon G \to G/L$ is the natural projection.

Note that a group G satisfies $\mathcal{C}$-CC if and only if the pair (G, g) has $\mathcal{C}$-CC$_G$ for every $g \in G$.

Definition 4.4. *Let G be a group and let $H \leq G$ be a subgroup. We say that that H satisfies the $\mathcal{C}$-centraliser condition in G ($\mathcal{C}$-CC_G) if the pair (H, g) satisfies $\mathcal{C}$-CC for every $g \in G$.*

Very often our proofs require case by case analysis. To keep the our proofs simple we will use the following lemma, which is a centraliser condition analogue of Lemma 2.1.

Lemma 4.5. *Let G be a group and let $H \leq G$ and $g \in G$. Then the pair (H, g) satisfies $\mathcal{C}$-CC_G if and only if for every $K \in \mathcal{N}_{\mathcal{C}}(G)$ there is a group F and a surjective homomorphism $\phi : G \to F$, such that* $\ker(\phi) \leq K$*, the pair $(\phi(H), \phi(g))$ satisfies $\mathcal{C}$-CC_F and*

$$C_{\phi(H)}(\phi(g)) \subseteq \phi(C_H(g)K) \text{ in } F.$$

Proof. Assume that the pair (H, g) has $\mathcal{C}$-CC_G, thus for every $K \in \mathcal{N}_{\mathcal{C}}(G)$ there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and

$$C_{\psi(H)}(\psi(g)) \subseteq \psi(C_H(g)K) \text{ in } G/L,$$

where $\psi \colon G \to G/L$ is the natural projection. Then we can take $\phi = \mathrm{id}_G$ and the statement clearly holds.

To prove sufficiency let $K \in \mathcal{N}_{\mathcal{C}}(G)$ be arbitrary. By assumption there is a group F and a homomorphism $\phi \colon G \to F$ such that $\ker(\phi) \leq K$, $\phi(K) \in \mathcal{N}_{\mathcal{C}}(F)$ and the pair $(\phi(H), \phi(g))$ satisfies $\mathcal{C}$-CC_F, thus there is $L' \in \mathcal{N}_{\mathcal{C}}(F)$ such that $L' \leq \phi(K)$ and

$$C_{\zeta(\phi(H))}(\zeta(\phi(g))) \subseteq \zeta(C_{\phi(H)}(\phi(g))\phi(K)) \text{ in } F/L', \tag{1}$$

where $\zeta \colon F \to F/L'$ is the natural projection. Define $\psi \colon G \to F/L'$ to be given by $\psi = \zeta \circ \phi$. Set $L = \phi^{-1}(L')$. As $L' \leq \phi(K)$ and $\ker(\phi) \leq K$ we get that $L = \phi^{-1}(L') \leq K$. We see that $\phi^{-1}(L') = \ker(\psi) \in \mathcal{N}_{\mathcal{C}}(G)$. Since $C_{\phi(H)}(\phi(g)) \subseteq \phi(C_H(g)K)$ in F we see that

$$\zeta(C_{\phi(H)}(\phi(g))) \subseteq \zeta(\phi(C_H(g)K)) = \psi(C_H(g)K)$$

and thus the equation (1) can be rewritten to

$$C_{\psi(H)}(\psi(g)) \subseteq \psi(C_H(g)K) \text{ in } G/L.$$

Since K was arbitrary we see that the pair (H, g) satisfies $\mathcal{C}$-CC_G. □

In order to be able to prove Theorem 4.2 we will need the following three statements. All the proofs in this chapter (except for Lemma 4.5) closely follow those given in [18, Section 3].

Lemma 4.6. *Let G be a group, let $H \leq G$ and $g \in G$. Assume that the pair (G, g) satisfies $\mathcal{C}$-CC_G and the conjugacy class g^G is $\mathcal{C}$-closed in G. If the double coset $C_G(g)H$ is $\mathcal{C}$-closed in G then the set g^H is also $\mathcal{C}$-closed in G.*

Proof. Let $y \in G \setminus g^H$ be arbitrary.

If $y \notin g^G$ then there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $\phi(y) \notin \phi(g^G)$ in G/L, where $\phi \colon G \to G/L$ is the natural projection, therefore $\phi(y) \notin \phi(g^H)$.

Assume that $y \in g^G \setminus g^H$, thus $y = zgz^{-1}$ for some $z \in G \setminus H$. Suppose $C_G(g) \cap z^{-1}H$ is nonempty, thus there is $f \in C_G(g)$ such that $zf \in H$. Then $g = fgf^{-1}$ and thus $y = zgz^{-1} = (zf)g(zf)^{-1} \in g^H$ which is a contradiction as we assume that $y \notin g^H$, thus $C_G(g) \cap z^{-1}H = \emptyset$, in other words $z^{-1} \notin C_G(g)H$. Since $C_G(g)H$ is $\mathcal{C}$-closed in G by assumption, there is $K \in \mathcal{N}_{\mathcal{C}}(G)$ such that $\{z^{-1}\} \cap C_G(g)HK = \emptyset$. Since the pair (G, g) has $\mathcal{C}$-CC_G by assumption, there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and

$$C_{G/L}(\phi(g)) \subseteq \phi(C_G(g)K),$$

where $\phi\colon G \to G/L$ is the natural projection.

Suppose that $\phi(y) \in \phi(g^H)$, thus there is some $h \in H$ such that $\phi(y) = \phi(zgz^{-1}) = \phi(hgh^{-1})$. We see that $\phi(z^{-1}h) \in C_{G/L}(\phi(g))$, thus

$$\phi(z^{-1}) \in C_{G/L}(\phi(g))\phi(H) \subseteq \phi(C_G(g)K)\phi(H) = \phi(C_G(g)HK).$$

This means that $z^{-1} \in C_G(g)HKL = C_G(g)HK$ as $L \leq K$. But that is a contradiction with the construction of K.

He have showed that for arbitrary $y \in G \setminus g^H$ there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $\phi(y) \notin \phi(g^H)$, hence the set g^H is $\mathcal{C}$-closed in G. □

Corollary 4.7. *Let G be a $\mathcal{C}$-CS group satisfying $\mathcal{C}$-CC and let $H \leq G$ such that $C_G(h)H$ is $\mathcal{C}$-closed in G for every $h \in H$. Then H is $\mathcal{C}$-CS. Moreover, for every $h \in H$ the set h^H is $\mathcal{C}$-closed in G.*

Lemma 4.8. *Let G be a group and suppose that $H \leq G$, $g \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$. If the set $g^{H\cap K}$ is $\mathcal{C}$-closed in G then there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and*

$$C_{\phi(H)}(\phi(g)) \subseteq \phi(C_H(G)K) \text{ in } G/L,$$

where $\phi\colon G \to G/L$ is the natural projection.

Proof. Denote $k = |H : (H \cap K)| < \infty$. Then $H = \bigsqcup_{i=1}^{k} z_i(H \cap K)$ for some $z_1, \ldots, z_k \in H$. If necessary we can renumber the elements z_i so that there is $l \in \{0, 1, \ldots, k-1\}$ such that $z_i^{-1}gz_i \notin g^{H\cap K}$ if $1 \leq i \leq l$ and $z_i^{-1}gz_i \in g^{H\cap K}$ if $l < i \leq k$. Since $g^{H\cap K}$ is $\mathcal{C}$-closed in G there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $\phi(z_i^{-1}gz_i) \notin \phi(g^{H\cap K})$ in G/L, where $\phi\colon G \to G/L$ is the natural projection, for all $i = 1, 2, \ldots, l$. Note that by replacing L with $L \cap K$ we may assume that $L \leq K$.

Let $\overline{x} \in C_{\phi(H)}(\phi(g))$ be arbitrary. Clearly $\overline{x} = \phi(x)$ for some $x \in H$ and thus $\phi(x^{-1}gx) = \phi(g)$ in G/L, therefore $x^{-1}gx \in gL$ in G. As $x \in H$ there is $i \in \{1, 2, \ldots, k\}$ and $y \in H \cap K$ such that $x = z_iy$, thus $x^{-1}gx = y^{-1}z_i^{-1}gz_iy$. As a consequence we see that $z_i^{-1}gz_i \in ygLy^{-1} = ygy^{-1}L \subseteq g^{H\cap K}L$. This means that $\phi(z_i^{-1}gz_i) \in \phi(g^{H\cap K})$ in G/L and thus from construction of L we see that $l < i \leq k$, therefore $z_i^{-1}gz_i \in g^{H\cap K}$ and there is some $u \in H \cap K$ such that $z_i^{-1}xz_i = ugu^{-1}$. We see that $z_iu \in C_H(g)$ and since $x = z_iy = z_iuu^{-1}y$ we see that $x \in C_H(g)(H \cap K) \subseteq C_H(g)K$. This means that $\overline{x} \in \phi(C_H(g)K)$ in G/L. Since $\overline{x} \in C_{\phi(H)(\phi(g))}$ was arbitrary we see that

$$C_{\phi(H)}(\phi(g)) \subseteq \phi(C_H(g)K) \text{ in } G/L,$$

which concludes the proof. □

Now we are ready to prove the main statement of this chapter.

Proof of Theorem 4.2. (i) $\Leftarrow$ (ii): let $H \leq G$ be a $\mathcal{C}$-open subgroup of G. By Lemma 2.3 we see that H is of finite index in G, hence the double coset $C_G(h)H$ is a finite union of $\mathcal{C}$-closed sets and thus is $\mathcal{C}$-closed in G. By Corollary 4.7 we see that H is $\mathcal{C}$-CS.

(i) $\Rightarrow$ (ii): assume that G is $\mathcal{C}$-HCS. Let $g \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$ be arbitrary. Let $H = \langle g \rangle K$ and note that $g^K = g^H = g^{H \cap K}$. Clearly H is $\mathcal{C}$-open in G by Lemma 2.3 and thus it is $\mathcal{C}$-CS. Since $g \in H$ and H is $\mathcal{C}$-CS we see that g^H is $\mathcal{C}$-closed in H. By Lemma 2.7 we see that $g^H = g^{H \cap K}$ is $\mathcal{C}$-closed in G. By previous lemma there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and

$$C_{\phi(H)}(\phi(g)) \subseteq \phi(C_H(G)K) \text{ in } G/L,$$

where $\phi \colon G \to G/L$ is the natural projection. Since $g \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$ were arbitrary we see that G has $\mathcal{C}$-CC. $\square$

Note that we used the fact the class $\mathcal{C}$ is closed under extensions only in the proof of Theorem 4.2 when we used Lemma 2.7. All the other statements in this chapter require only (c1) and (c2).

5. Special amalgams

In order to be able to understand certain properties of graph products we will turn our attention to *special amalgams*. The following section is a close analogue of [18, Section 7].

Definition 5.1. *Let A, C be groups and let $H \leq A$. Then we define $A \star_H C$, the* special amalgam *of A and C over H, to be the following free product with amalgamation:*

$$A *_H (H \times C) \text{ given by presentation } \langle A, C \| [h, c] = 1 \ \forall h \in H, \forall c \in C \rangle,$$

where $[h, c] = hch^{-1}c^{-1}$.

The main reason why we are interested in special amalgams is that they naturally appear in graph products.

Remark 5.2. *Let $G = \Gamma\mathcal{G}$ be a graph product and suppose that $|V\Gamma| \geq 2$. Then for every $v \in V\Gamma$ there is a natural splitting of $G = G_A \star_{G_C} G_B$ as a special amalgam of full subgroups, where $A = V \setminus \{v\}$, $B = star(v)$ and $C = \mathrm{link}(v)$.*

Proof. Let $v \in V\Gamma$ be arbitrary, set $A, C \subseteq V\Gamma$ as in the statement of the remark and let $B = \mathrm{star}(v)$. Obviously $G_C \leq G_B$, $G_C \leq G_A$ and $G = \langle G_A, G_B \rangle$. By looking at the presentations it is easy to see that $G \cong G_A *_{G_C} G_B$. Note that the vertex v is central in the graph Γ_B therefore $G_B = G_v \times G_C$. Consequently $G \cong G_A *_{G_C} (G_C \times G_v) = G_A \star_{G_C} G_v$. $\square$

There are two extreme cases that can occur. If $v \in V\Gamma$ is an isolated vertex, i.e. v is not connected to any other vertex, we see that $G_C = \{1\}$ and $G = G_A * G_v$. On the other side, if $\mathrm{link}(v) = V\Gamma \setminus \{v\}$, i.e. if v is central in Γ, we see that $G = G_B = G_A \times G_v$.

5.1. **Normal form and functorial property.** Let $G = A \star_H C$. Obviously every element $g \in G$ can be represented as a product $x_0 c_1 x_1 \dots c_n x_n$ where $x_i \in A$ for $i = 0, 1, \dots, n$ and $c_j \in C$ for $j = 1, \dots, n$. We say that $g = x_0 c_1 x_1 \dots c_n x_n$ is in a *reduced form* if $x_i \notin H$ for $i = 1, \dots, n-1$ and $c_j \neq 1$ for $j = 1, \dots, n$. By using the normal form theorem for free products with amalgamation [14, Theorem 4.4] we can prove the following.

Lemma 5.3. *Let $H \leq A, C$ be groups and let $G = A \star_H C$. Suppose that $g = x_0 c_1 x_1 \dots c_n x_n$, where $x_0, x_1, \dots, x_n \in A$ and $c_1, \dots, c_n \in C$, with $n \geq 1$ is in reduced form. Then $g \neq 1$ in G.*

Moreover, suppose that $f = y_0 d_1 y_1 \dots d_m y_m$, where $y_0, y_1, \dots, y_m \in A$ and $d_1, \dots, d_m \in C$, is in reduced form with $m \geq 1$ as well and $f = g$. Then $m = n$ and $c_i = d_i$ for all $i = 1, \dots, n$.

Proof. The first assertion of the lemma follows directly from normal form theorem for free products with amalgamation. Now, assume that $g = x_0 c_1 x_1 \dots c_n x_n$, where $x_i \in A$ for $i = 0, 1, \dots, n$ and $c_i \in C$ for $i = 1, 2, \dots, n$, $f = y_0 d_1 y_1 \dots d_m y_m$, where $y_i \in A$ for $i = 0, 1, \dots, m$ and $d_i \in C$ for $i = 1, 2, \dots, m$, are both in reduced form and $f = g$. We will proceed by induction on $m + n$.

In case $m + n = 2$ we see that $m = n = 1$ and thus $g = x_0 c_1 x_1$, $f = y_0 d_1 y_1$. By the assumption we have that $y_1^{-1} d_1^{-1} y_0^{-1} x_0 c_1 x_1 = 1$. This product cannot be in a reduced form and thus $y_0^{-1} x_0 \in H$. Then we see that

$$y_1^{-1} d_1^{-1} y_0^{-1} x_0 c_1 x_1 = y_1^{-1} (d_1^{-1} c_1)(y_0^{-1} x_0 x_1) = 1.$$

Again, this product is not reduced and thus $d_1 = c_1$ and we are done.

Now assume that $m + n = K > 2$. Then clearly

$$y_m^{-1} d_m^{-1} \dots y_1^{-1} d_1^{-1} y_0^{-1} x_0 c_1 x_1 \dots c_n x_n = 1 \tag{2}$$

and thus the left hand side of (2) is not reduced. Since both f, g were in reduced form we see that $y_0^{-1} x_0 \in H$ and therefore $d_1^{-1} y_0^{-1} x_0 c_1 = d_1^{-1} c_1 y_0^{-1} x_0$ and thus we can rewrite (2) to

$$y_m^{-1} d_m^{-1} \dots y_1^{-1} d_1^{-1} c_1 y_0^{-1} x_0 x_1 \dots c_n x_n = 1.$$

Without loss of generality we may assume that $n \geq m$ and thus $x_1 \notin H$. Since $x_1 \notin H$ we see that $y_0^{-1} x_0 x_1 \notin H$ and thus $d^{-1} c_1 = 1$ and we can rewrite (2) to

$$y_m^{-1} d_m^{-1} \dots y_1^{-1} y_0^{-1} x_0 x_1 \dots c_n x_n = 1.$$

Since both f and g were in reduced form we see that $f_1 = g_1$, where $f_1 = (x_0 x_1) c_2 x_2 \dots c_n x_n$ and $g_1 = (y_0 y_1) d_2 y_2 \dots y_m d_m$ are in reduced form as well. Thus by induction hypothesis we get that $m = n$ and $d_i = c_i$ for $i = 2, \dots, n$. □

The above lemma shows that if $g = x_0 c_1 x_1 \dots c_n x_n$ is reduced then $c_1, \dots, c_n$ are given uniquely. We will call them the *consonants* of g. Denote $|g|_C = n$ and we will call $|g|_C$ the *consonant length* of g.

Special amalgams are useful because they have a functorial property.

Remark 5.4. *Let H, A, C, Q, S be groups such that $H \leq A$ and let $\psi_A \colon A \to B$, $\psi_C \colon C \to S$ be group homomorphisms. Then by universal property of amalgamated free products ψ_A, ψ_C uniquely extend to a homomorphism $\psi : G \to P$, where $G = A \star_H C$ and $P = Q \star_{\psi(H)} S$, such that*

$$\psi(g) = \begin{cases} \psi_A(a) & \text{if } g = a \text{ for some } a \in A, \\ \psi_C(c) & \text{if } g = c \text{ for some } c \in C. \end{cases}$$

Lemma 5.5. *With notation as stated in Remark 5.4,* $\ker(\psi) = \langle\langle \ker(\psi_A), \ker(\psi_C) \rangle\rangle^G$.

Proof. Let's use $N = \langle\langle \ker(\psi_A), \ker(\psi_C) \rangle\rangle^G$. Obviously $N \leq \ker(\psi)$, thus we need to show the opposite inclusion.

Let $\phi : G \to G/N$ be the natural projection, thus $N = \ker(\phi)$. Let $\theta : G/N \to P$ be a homomorhism such that $\ker(\theta) = \phi(\ker(\psi))$ and $\psi = \theta \circ \phi$. Note that $\ker(\psi_A) = \ker(\psi) \cap A$ and $\ker(\psi) \cap C = \ker(\psi_C)$ thus it makes sense to define $\xi_A \colon \psi(A) \to \phi(A)$ be the homomorphism given by $\xi_A(\psi(a)) = \phi(a)$ and $\xi_C \colon \psi(C) \to \phi(C)$ be the homomorphism given by $\xi_C(\psi(c)) = \phi(c)$. Clearly ξ_A, ξ_C are isomorphisms. Let $h \in H$ and $c \in C$, then $[\xi_A(\psi(h)), \xi_C(\psi(c))] = [\phi(h), \phi(c)] = \phi([h, c]) = 1$. Therefore by von Dyck's theorem the homomorphisms ξ_A, ξ_C extend to a homomorphism $\xi \colon P \to G/L$ defined on the generators of P by

$$\xi(p) = \begin{cases} \phi(a) & \text{if } p = \psi_A(a) \text{ for some } a \in A, \\ \phi(c) & \text{if } p = \psi_C(c) \text{ for some } c \in C. \end{cases}$$

Then clearly $\xi \circ \theta \colon G/L \to G/L$ is the identity as it is defined on the generators of G/L by following:

$$\xi \circ \theta(q) = \begin{cases} \xi(\theta(\phi(a))) = \xi(\psi(a)) = \phi(a) & \text{if } q = \phi(a) \text{ for some } a \in A, \\ \xi(\theta(\phi(c))) = \xi(\psi(c)) = \phi(c) & \text{if } q = \phi(c) \text{ for some } c \in C. \end{cases}$$

It follows that ξ is injective and thus $\ker(\theta) = \{1\} = \phi(\ker(\psi))$. Therefore $\ker(\psi) \leq \ker(\phi)$. Altogether we see that $\ker(\psi) = \langle\langle \ker(\psi_A), \ker(\psi_C) \rangle\rangle^G$. $\square$

5.2. **Cyclically reduced elements and conjugacy.** From now on let $H \leq A, C$ be groups and let G denote $A \star_H C$, the special amalgam of A and C along H.

Definition 5.6. *Let $g = c_1x_1 \dots c_nx_n$, where $x_i \in A$ and $c_i \in C$ for $i = 1, \dots, n$. We say that g is* cyclically reduced *if $c_1x_1 \dots c_nx_n$ is a reduced expression and if $n \geq 2$ then $x_n \notin H$. We will say that an element $p \in G$ is a* prefix *of g if $p = c_1x_1 \dots c_lx_l$ for some $0 \leq l \leq n$ and that $s \in G$ is a* suffix *of g if $s = c_{n-m}x_{n-m} \cdots c_nx_n$ for some $-1 \leq m \leq n - 1$.*

Note that we define prefix and suffix only for cyclically reduced elements.

Lemma 5.7. *Let $g = c_1x_1 \dots c_nx_n$ and $f = d_1y_1 \dots d_ny_n$, where $x_i, y_i \in A$ and $c_i, d_i \in C$ for $i = 1, 2, \dots, n$, be cyclically reduced elements of G such that $n \geq 1$ and $x_n \notin H$. Assume that $ugu^{-1} = f$ for some $u \in G$. Let $u = z_0e_1z_1 \dots e_mz_m$, where $z_i \in A$ and $e_j \in C$ for $i = 0, 1, \dots, m$ and $j = 1, \dots, m$, be a reduced expression. Then exactly one of the following is true*

- a) *$m = 0$ and $u \in H$,*
- b) *$m \geq 1$, $z_m \in H$ and there is a prefix p of g such that $u = hp^{-1}g^{-l}$ for some $h \in H$ and $l \in \mathbb{N}_0$,*
- c) *$m \geq 1$, $x_nz_m^{-1} \in H$ and there is a suffix s of g such that $u = hsg^l$ for some $h \in H$ and $l \in \mathbb{N}_0$.*

Proof. If $m = 0$, then $u = z_0$ and thus

$$y_n^{-1}d_n^{-1} \dots y_1^{-1}d_1^{-1}z_0c_1x_1 \dots c_nx_nz_0^{-1} = 1.$$

This product clearly cannot be reduced and therefore z_0 must belong to H.

Now suppose $m \geq 1$. Then since $f = ugu^{-1}$ we get

$$z_0 e_1 z_1 \dots e_m z_m c_1 x_1 \dots c_n x_n z_m^{-1} e_m^{-1} \dots z_1^{-1} e_1^{-1} z_0^{-1} = d_1 y_1 \dots d_n y_n.$$

Right hand side of this equation is shorter than left hand side and right hand side is reduced by assumption, therefore left hand side cannot be reduced and thus we see that either $z_m \in H$ or $x_n z_m^{-1} \in H$. Since $x_n \notin H$ we see that exactly one of these two possibilities may happen.

First suppose that $z_m \in H$. Then $e_m z_m = z_m e_m$ and thus we have

$$z_0 e_1 z_1 \dots e_{m-1}(z_{m-1} z_m)(e_m c_1) x_1 \dots c_n x_n z_m^{-1} e_m^{-1} \dots z_1^{-1} e_1^{-1} z_0^{-1} = d_1 y_1 \dots d_n y_n. \tag{3}$$

Again, left hand side cannot be reduced. Since $z_m \in H$ we cannot have $z_{m-1} z_m \in H$ as that would make $z_{m-1} \in H$ which would contradict our assumption that u is reduced. Thus we must have that $e_m = c_1^{-1}$. Denote $h_1 := z_{m-1} z_m x_1 \in A$. Then

$$z_m^{-1} e_m^{-1} z_{m-1}^{-1} = e_m^{-1} z_m^{-1} z_{m-1}^{-1} = c_1 z_m^{-1} z_{m-1}^{-1} = c_1 x_1 x_1^{-1} z_m^{-1} z_{m-1}^{-1} = c_1 x_1 h_1^{-1}$$

Since $z_{m-1} z_m = h_1 x_1^{-1}$ we get

$$(z_0 e_1 z_1 \dots e_{m-1} h_1)(c_2 x_2 \dots c_n x_n c_1 x_1)(z_0 e_1 z_1 \dots e_{m-1} h_1)^{-1} = d_1 y_1 \dots d_n y_n.$$

Set $u' = z_0 e_1 z_1 \dots e_{m-1} h_1$ and $g' = c_2 x_2 \dots c_n x_n c_1 x_1$. On the left hand side of the equation (3) we have

$$z_0 e_1 z_1 \dots e_{m-1} h_1 c_2 x_2 \dots c_n x_n c_1 x_1 h_1^{-1} e_{m-1}^{-1} \dots z_1^{-1} e_1^{-1} z_0^{-1}$$

and since this expression has longer consonant length than the right hand side of the equation we see that it cannot be reduced and therefore $h_1 = z_{m-1} z_m x_1 \in H$. If $m = 1$ we get that $u = h_1 (c_1 x_1)^{-1}$ and the lemma is proved.

Now suppose that $m = M > 1$ and that the statement has been already proved for all $u \in G$ such that $|u|_C < M$, thus we can use the induction hypothesis for f, g' and u' as $|u'|_C = |u|_C - 1$.

We have $u' g' u'^{-1} = f$,

$$z_0 e_1 z_1 \dots e_{m-1} h_1 c_2 x_2 \dots c_n x_n c_1 x_1 h_1^{-1} e_{m-1}^{-1} \dots z_1^{-1} e_1^{-1} z_0^{-1} = d_1 y_1 \dots d_n y_n \tag{4}$$

Since we have already shown that $h_1 \in H$ we can use induction hypothesis and by b) we see that there is a prefix p' of g' such that $z_0 e_1 z_1 \dots z_{m-2} e_{m-1} h_1 = h p'^{-1} g'^{-l}$ for some $h \in H$ and $l \in \mathbb{N}$. As a result of this we have

$$\begin{aligned}
u &= z_0 e_1 z_1 \dots e_{m-1} z_{m-1} e_m z_m \\
&= z_0 e_1 z_1 \dots e_{m-1} z_{m-1} z_m c_1^{-1} \\
&= z_0 e_1 z_1 \dots e_{m-1} h_1 x_1^{-1} c_1^{-1} \\
&= h p'^{-l} g'^{-l} x_1^{-1} c_1^{-1} = h p'^{-1} x_1^{-1} c_1^{-1} g^{-l} = h (c_1 x_1 p')^{-1} g^{-l}.
\end{aligned}$$

Now two possibilities can occur. As p' is a prefix of g' we see that $p' = c_2 x_2 \dots c_k x_k$ where $2 \leq k \neq n$. Then either $c_1 x_1 p'$ is a prefix of g or $c_1 x_1 p' = g$. Either way we are done.

In case $x_n z_m^{-1} \in H$ we can proceed analogously. $\square$

Let $g = c_1x_1 \dots c_nx_n$, where $x_1, \dots, x_n \in A$ and $c_1, \dots, c_n \in C$ with h, be a cyclically reduced element of G. Then we say that $g' \in G$ is a *cyclic permutation* of g if $g' = c_mx_m \dots c_nx_nc_1x_1 \dots c_{m-1}x_{m-1}$ for some $1 \leq m \leq n$. Equivalently, g' is a cyclic permutation of g if there is f, a prefix (or a suffix) of g , such that $g = x^{-1}gx$ (or $g' = xgx^{-1}$).

5.3. **Centralisers and a conjugacy criterion.** Recall that $G = A \star_H C$. The following lemma is a special version of [13, Chapter IV, Theorem 2.8].

Lemma 5.8 (Conjugacy criterion for special amalgams). *Let $g = c_1x_1 \dots c_nx_n$ and $f = d_1y_1 \dots d_my_m$, where $x_1, \dots, x_n, y_1, \dots y_m \in A$, and $c_1, \dots, c_n, d_1, \dots, d_m \in C$, be cyclically reduced elements of G with $n \geq 1$. Then $g \notin A^G$. If $f \sim_G g$ then $m = n$ and there is $g' \in G$, a cyclic permutation of g, such that $f \sim_H g'$.*

Clearly every cyclically reduced element has only finitely many cyclic permutations. Lemma 5.8 motivates us to give a sufficient and necessary condition for whether two cyclically reduced elements of G are conjugate by some element of H.

Lemma 5.9. *Suppose $g = cx \in G$ such that $c \in C \setminus \{1\}$ and $x \in H$. Then*

$$C_G(g) = C_C(c)C_H(x) \cong C_C(c) \times C_H(x).$$

Proof. Obviously $f \in C_G(g)$ if and only if $fgf^{-1} = g$. Let $f \in C_G(g)$ and let $f = z_0e_1z_1 \dots e_mz_m$, where $z_0, z_1, \dots, z_m \in A$ and $e_1, \dots, e_m \in C$, be the reduced expression for f. Then

$$z_0e_1z_1 \dots e_mz_mcxz_m^{-1}e_m^{-1} \dots z_1^{-1}e_1^{-1}z_0^{-1} = cx. \tag{5}$$

If $m = 0$ we get that $z_0cxz_0^{-1}x^{-1}c^{-1} = 1$ thus $xz_0^{-1}x^{-1} \in H$, therefore $z_0 \in H$ and $z_0c = cz_0$. Consequently we get $z_0xz_0^{-1}x^{-1} = 1$ and thus $z_0 \in C_H(x)$.

Assume $m \geq 1$. Then either $xz_m^{-1} \in H$ or $z_m \in H$. Since $x \in H$ we see that both must be true, thus $z_mc = cz_m$ and $e_mz_mcxz_m^{-1}e_m^{-1} = (e_mce_m^{-1})(z_mxz_m^{-1})$ and thus (5) rewrites to

$$z_0e_1z_1 \dots e_{m-1}z_{m-1}(e_mce_m^{-1})(z_mxz_m^{-1})z_{m-1}^{-1}e_{m-1}^{-1} \dots z_1^{-1}e_1^{-1}z_0^{-1} = cx.$$

Since $c \neq 1$ we see that $e_mce_m^{-1} \neq 1$. Left hand side cannot be reduced and therefore either $z_{m-1} \in H$ or $(z_mxz_m^{-1})z_{m-1}^{-1} \in H$. Since $z_mxz_m^{-1} \in H$ we see that both must be true. If $m \geq 2$ this contradicts $z_{m-1} \notin H$.

Thus we may assume that $m = 1$ and consequently $f = z_0ez_1$ with $z_0, z_1 \in H$. Therefore

$$\begin{aligned} g^{-1}fgf^{-1} &= x^{-1}c^{-1}z_0ez_1cxz_1^{-1}e^{-1}z_0^{-1} \\ &= x^{-1}(c^{-1}ec)(z_0z_1xz_1^{-1})e^{-1}z_0^{-1} = 1. \end{aligned}$$

Since $m \geq 1$ we see that $e \neq 1$ and consequently $c^{-1}ec \neq 1$. This leaves us with $z_0z_1xz_1^{-1} \in H$. As $z_1xz_1^{-1} \in H$ and $z_0 \in H$ we see that

$$1 = g^{-1}fgf^{-1} = (c^{-1}ece^{-1}) \cdot (x^{-1}(z_0z_1)x(z_0z_1)^{-1}) \in CH.$$

This gives us that $e \in C_C(c)$ and $z_0z_1 \in C_H(x)$. Altogether we see that $f = ez_0z_1 \in C_C(c) \times C_H(x)$. $\square$

Lemma 5.10. *Let $H \leq A, C$ be groups and let $G = A \star_H C$. Suppose that $g = c_1 x_1 \dots c_n x_n$, $c_i \in C$, $x_i \in A$ for $i = 1, \dots, n$, is cyclically reduced in G and $n \geq 1$.*

If $x_n \in H$ then $n = 1$ and $C_G(g) = C_C(c_1) \times C_H(x_1) \leq G$.

If $x_n \notin H$, let $\{p_1, \dots, p_k\}$, where $1 \leq k \leq n+1$, be the set of all prefixes of g satisfying $p_i^{-1} g p_i \in g^H$. For each $i = 1, \dots, k$ choose $h_i \in H$ such that $h_i p_i^{-1} g p_i h_i^{-1} = g$ and define finite subset $\Omega \subseteq G$ by $\Omega = \{h_i p_i^{-1} | i = 1, \dots, k\}$.

Then $C_G(g) = C_H(g) \langle g \rangle \Omega$.

Proof. If $x_n \in H$ then g is cyclically reduced in G if and only if $n = 1$. Then the claim follows from the previous lemma. Suppose $x_n \in A \setminus H$. Let $u \in C_G(g)$ thus $g = ugu^{-1}$. Then by Lemma 5.7 we know that there are $h \in H$ and $l \in \mathbb{N}$ such that either there is a prefix p of g such that $u = hp^{-1}g^{-l}$ or there is a suffix s of g such that $u = hsg^l$. In the second case there is a prefix p of g such that $s = p^{-1}g$ and thus $u = hsg^l = hp^{-1}g^{l+1}$. Thus without loss of generality we may assume that $u = hp^{-1}g^l$ for some prefix p of g, $h \in H$ and $l \in \mathbb{Z}$.

We see that $g = hp^{-1}g^l g g^{-l} p h^{-1} = hp^{-1} g p h^{-1}$, therefore $p \in \{p_1, \dots, p_k\}$ and thus there is $h_i \in H$ such that $h_i p^{-1} g p h_i^{-1} = g = hp^{-1} g p h^{-1}$. This yields $hh_i^{-1} \in C_H(g)$, thus

$$u = hp_i^{-1} g^l \in C_H(g) h_i p_i^{-1} g^l \subseteq C_H(g) \Omega \langle g \rangle.$$

Since $\Omega \subseteq C_G(g)$ we see that $C_H(g) \Omega \langle g \rangle = C_H(g) \langle g \rangle \Omega$.

So it has been proven that $C_G(g) \subseteq C_H(g) \langle g \rangle \Omega$. Inclusion in the opposite direction is obvious. $\square$

Lemma 5.11. *Let $H \leq A, C$ be groups and let $G = A \star_H C$. Suppose $B \leq A$ $f, g \in G$ are arbitrary. Let $g = x_0 c_1 x_1 \dots c_n x_n$, $f = y_0 d_1 y_1 \dots y_m d_m$, where $x_0, \dots, x_n, y_0, \dots, y_m \in A$ and $c_1, \dots, c_n, d_1, \dots, d_m \in C$, be reduced expressions for g and f respectively and assume that $n \geq 1$. Then $f \in g^B$ if and only if all of the following conditions are met*

(i) *$m = n$ and $c_i = d_i$ for $i = 1, \dots, n$,*
(ii) *$y_0 y_1 \dots y_n \in (x_0 x_1 \dots x_n)^B$ in A,*
(iii) *for every $b_0 \in B$ such that $y_0 y_1 \dots y_n = b_0 (x_0 x_1 \dots x_n) b_0^{-1}$ we have $I \neq \emptyset$ where*

$$I = b_0 C_B(x_0 x_1 \dots x_n) \cap y_0 H x_0^{-1} \cap (y_0 y_1) H (x_0 x_1)^{-1} \cap \dots \cap (y_0 y_1 \dots y_{n-1}) H (x_0 x_1 \dots x_{n-1})^{-1}.$$

Proof. Suppose (i) - (iii) hold. Let $b_0 \in B$ be such that $y_0 y_1 \dots y_n = b_0 (x_0 x_1 \dots x_n) b_0^{-1}$ and let $b \in I$. Clearly $y_0 y_1 \dots y_n = b (x_0 x_1 \dots x_n) b^{-1}$ as $b \in b_0 C_B(x_0 x_1 \dots x_n)$. We want to show that $f^{-1} b g b^{-1} = 1$. Since (i) holds we can write

$$f^{-1} b g b^{-1} = y_n^{-1} c_n^{-1} \dots y_1^{-1} c_1^{-1} y_0^{-1} b x_0 c_1 x_1 \dots c_n x_n b^{-1}. \tag{6}$$

Since $b \in y_0 H x_0^{-1}$ we see that $y_0^{-1} b x_0 \in H$ and thus

$$y_1^{-1} c_1^{-1} y_0^{-1} b x_0 c_1 x_1 = y_1^{-1} y_0^{-1} b x_0 x_1,$$

therefore we can rewrite (6) to

$$f^{-1} b g b^{-1} = y_n^{-1} c_n^{-1} \dots y_2^{-1} c_2 (y_0 y_1)^{-1} b (x_0 x_1) c_2 x_2 \dots c_n x_n b^{-1}.$$

Again, since $b \in (y_0 y_1) H (x_0 x_1)^{-1}$ we see that $(y_0 y_1)^{-1} b (x_0 x_1) \in H$ and thus (6) rewrites to

$$f^{-1} b g b^{-1} = y_n^{-1} c_n^{-1} \dots y_3^{-1} c_3 (y_0 y_1 y_2)^{-1} b (x_0 x_1 x_2) c_3 x_3 \dots c_n x_n b^{-1}.$$

By repeating this argument n-times we rewrite (6) to

$$f^{-1}bgb^{-1} = (y_0y_1\dots y_n)^{-1}b(x_0x_1\dots x_n)b^{-1}.$$

Which is equal to 1 by assumption, thus $f \in g^B$.

Now assume $f \in g^B$, so there is $b \in B$ such that

$$y_0d_1y_1\dots d_my_m = b(x_0c_1x_1\dots c_nx_n)b^{-1}.$$

By Lemma 5.3 we see that $m = n$ and $c_i = d_i$ for $i = 1,\dots,n$, thus we've established (i). There is a natural retraction $\rho : G \to A$ defined by $\rho(a) = a$ for all $a \in A$ and $\rho(c) = 1$ for all $c \in C$. By applying this retraction we establish (ii). Let b_0 be an arbitrary element of B such that $y_0y_1\dots y_n = b_0(x_0x_1\dots x_n)b_0^{-1}$. Obviously $b \in b_0C_B(x_0x_1\dots x_n)$. From assumptions we have that

$$y_n^{-1}c_n^{-1}\dots y_1^{-1}c_1^{-1}(y_0^{-1}bx_0)c_1x_1\dots c_nx_nb^{-1} = 1 \tag{7}$$

By Lemma 5.3 this expression cannot be reduced thus $y_0^{-1}bx_0 \in H$, therefore $b \in y_0Hx_0^{-1}$. This means that we can rewrite (7) as

$$y_n^{-1}c_n^{-1}\dots y_2^{-1}c_2^{-1}(y_1^{-1}y_0^{-1}bx_0x_1)c_2x_2\dots c_nx_nb^{-1} = 1.$$

Again, by Lemma 5.3 we see that $(y_0y_1)^{-1}b(x_0x_1) \in H$ or equivalently $b \in (y_0y_1)H(x_0x_1)^{-1}$. By applying this step n-times we establish (iii), thus $I \neq \emptyset$. □

Lemma 5.12. *Let $H \leq A, C$ be groups and let $G = A \star_H C$, suppose $B \leq A$. Let $g \in G$ and $g = x_0c_1x_1\dots c_nx_n$, where $x_0,\dots,x_n \in A$ and $c_1,\dots,c_n \in C$, be a reduced expression of g with $n \geq 1$. Then $C_B(g) = I$, where*

$$I = C_B(x_0x_1\dots x_n)\cap x_0Hx_0^{-1}\cap(x_0x_1)H(x_0x_1)^{-1}\cap\dots\cap(x_0x_1\dots x_{n-1})H(x_0x_1\dots x_{n-1})^{-1}.$$

Proof. Clearly $g \sim_G g$ and thus by previous lemma for any $b_0 \in B$ such that $b_0(x_0x_1\dots x_n)b_0^{-1} = x_0x_1\dots x_n$ we have $I \neq \emptyset$, where

$$I = b_0C_B(x_0x_1\dots x_n) \cap x_0Hx_0^{-1} \cap (x_0x_1)H(x_0x_1)^{-1} \cap \dots \cap (x_0x_1\dots x_{n-1}).$$

We can set $b_0 = 1$. Now take $b \in I$ by argumentation analogous to the proof of the previous lemma we see that $bgb^{-1} = g$ and thus $I \subseteq C_B(g)$.

Let $b \in C_B(g)$. Then $bgb^{-1} = g$. By the previous lemma we see that $b \in I$. Therefore $I = C_G(g)$. □

6. Finite graph products of $\mathcal{C}$-HCS groups

From now on we will assume that the class $\mathcal{C}$ is an extension closed variety of finite groups, i.e. $\mathcal{C}$ satisfies (c1), (c2), (c3) and (c4). The main result of this section is the following generalisation of [18, Theorem 1.1] and [29, Theorem 6.15].

Theorem 6.1. *Assume that $\mathcal{C}$ is an extension closed variety of finite groups. Then the class of $\mathcal{C}$-HCS groups is closed under taking finite graph products.*

6.1. **Some auxiliary statements.** The following two statements were proved first by Minasyan in [18, Lemma 5.6 and 5.7] in case when $\mathcal{C}$ is the class of all finite groups. Later in his paper [29] Toinet proved them in case when $\mathcal{C}$ is the class of all finite p-groups for some prime number p. The proofs can easily be generalised for the case when the class $\mathcal{C}$ is an extension closed variety of finite groups and we leave them to the reader.

Lemma 6.2. *Let G be a group and let $A, B \leq G$ be retracts of G with corresponding retractions $\rho_A, \rho_B \in \mathrm{End}(G)$ such that $\rho_A \circ \rho_B = \rho_B \circ \rho_A$. Let x be an element of G and let $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in G$. Suppose that the pair $(A \cap B, \alpha)$ satisfies $\mathcal{C}$-CC in G. Then for any $K \in \mathcal{N}_\mathcal{C}(G)$ there exists $M \in \mathcal{N}_\mathcal{C}(G)$ such that $M \leq K$, $\rho_A(M) \subseteq M$, $\rho_B(M) \subseteq M$ and $\phi(A) \cap \phi(xBx^{-1}) \subseteq \phi(A \cap xBx^{-1})\phi(K)$ in G/M, where $\phi\colon G \to G/M$ is the natural epimorphism.*

Lemma 6.3. *Let G be a group and let $A, B \leq G$ be retracts of G with corresponding retractions $\rho_A, \rho_B \in \mathrm{End}(G)$ such that $\rho_A \circ \rho_B = \rho_B \circ \rho_A$. Consider arbitrary elements $x, g \in G$. Denote $D = xBx^{-1} \leq G$ and $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in G$. Suppose that the conjugacy classes $\alpha^{A\cap B}$ and $g^{A\cap D}$ are $\mathcal{C}$-closed in G, and the pair $(A \cap B, \alpha)$ satisfies $\mathcal{C}$-CC_G. Then the double coset $C_A(g)D$ is $\mathcal{C}$-closed in G.*

Dyer [7, Theorem 3] proved that free-by-finite groups are CS. In [29, Theorem 4.2] Toinet proved that free-by-(finite-p) groups are p-CS. Ribes and Zalesskii generalised these results (see [28, Section 3, Theorem 3.2]) to the following.

Theorem 6.4. *Let $\mathcal{C}$ be an extension closed variety of finite groups and let G be finitely generated free-by-$\mathcal{C}$ group. Then G is $\mathcal{C}$-CS.*

Clearly, every $\mathcal{C}$-open subgroup of a finitely generated free-by-$\mathcal{C}$ group is finitely generated free-by-$\mathcal{C}$ group as well. We can state the following corollary as an immediate consequence of Theorem 6.4.

Corollary 6.5. *Let G be finitely generated free-by-$\mathcal{C}$ group. Then G is $\mathcal{C}$-HCS.*

The following simple lemma will be crucial for our proofs.

Lemma 6.6. *Suppose that $\mathcal{C}$ is a class of groups satisfying (c2). Let $Q, S \in \mathcal{C}$ and suppose that $R \leq Q$. Then $G = Q \star_R S$ is free-by-$\mathcal{C}$.*

Proof. Let $\sigma\colon G \to Q \times S$ be the epimorphism defined on the generators of G as follows:

$$\sigma(q) = q \text{ for all } q \in Q,$$
$$\sigma(s) = s \text{ for all } s \in S.$$

Clearly $\ker(\sigma) \in \mathcal{N}_\mathcal{C}(G)$ as $\mathcal{C}$ is closed under taking direct products. We want to show that $\ker(\sigma)$ is a free group. From the definition of σ we see that $\ker(\sigma) \cap R \times S = \{1\}$. Let T be the Bass-Serre tree for $Q \star_R S = Q *_R (R \times S)$ and consider the induced action of $\ker(\sigma)$ on T. By a standard result of Bass-Serre theory (see [3, Theorem 12.1]) we know that the stabiliser of a vertex v has to be conjugate either into Q or $R \times S$, but since $\ker(\sigma)$ is normal and does not intersect any of the factors we see that $\ker(\sigma)$ acts freely on T and thus it is free. As a consequence we see that G is free-by-$\mathcal{C}$. $\square$

Combining corollary 6.5 together with Lemma 6.6 we immediately get the following.

Corollary 6.7. *Suppose that $\mathcal{C}$ is an extension closed variety of finite groups. Let $Q, S \in \mathcal{C}$ and assume that $R \leq Q$. Then $G = Q \star_R S$ is $\mathcal{C}$-HCS.*

6.2. **Proof of Theorem 6.1.** Before we proceed to the proof we first prove a weaker statement. This was first proved by Green in [9] both for the case when $\mathcal{C}$ is the class of all finite groups and for the case when $\mathcal{C}$ is the class of all finite p-groups for some prime number p.

Lemma 6.8. *Let $\mathcal{C}$ be a class of finite groups satisfying (c1), (c2) and (c4). Then the class of residually-$\mathcal{C}$ groups is closed under taking graph products.*

Proof. First, we show the statement holds for all finite graph products. The proof will by done by induction on $|V\Gamma|$. If $|V\Gamma| = 1$ we see that $\Gamma\mathcal{G} = G_v$ and G_v is residually-$\mathcal{C}$ by assumption.

Now assume that the statement has been proved for all graph products $\Gamma\mathcal{G}$ such that $|V\Gamma| \leq r$. Let $G = \Gamma\mathcal{G}$ be such that $|V\Gamma| = r + 1$ and let $g \in G \setminus \{1\}$ be arbitrary. Pick $v \in V\Gamma$ and denote $C = G_v$, $H = G_{\mathrm{link}_\Gamma(v)}$ and $A = G_{V\Gamma\setminus\{v\}}$. Then clearly $G = A *_H C$ by Remark 5.2. By the induction hypothesis we get that A, C, H are residually-$\mathcal{C}$. Let $g = x_0c_1x_1 \dots c_nx_n$, where $x_0, x_1, \dots, x_n \in A$ and $c_1, \dots, c_n \in C$, be a reduced expression for g in G. There are two cases to consider: either $n = 0$ or $n \geq 1$. If $n = 0$ then $g = x_0 \in A \setminus \{1\}$ and we can use the fact that A is a retract in G, thus we can consider the canonical retraction $\rho_A \colon G \to A$. Then $\rho_A(x_0) = x_0$ and A is residually-$\mathcal{C}$ by induction hypothesis.

Suppose that $n \geq 1$. Clearly, H is a retract of A and therefore by Lemma 2.6 we see that H is $\mathcal{C}$-closed in A. This means that there is a group $Q \in \mathcal{C}$ and an epimorphism $\alpha \colon A \to Q$ such that $\alpha(x_i) \notin \alpha(H)$ for whenever $x_i \notin H$. Similarly since C is residually-$\mathcal{C}$ by assumption as it is a vertex group we see that there is a group $S \in \mathcal{C}$ and an epimorphism $\gamma \colon C \to S$ such that $\gamma(c_i) \neq 1$ in S for all $i = 1, \dots, n$. Let $\phi \colon G \to P$, where $P = Q *_{\alpha(H)} S$, be the canonical extension of α and γ (see Remark 5.4). We see that

$$\phi(g) = \alpha(x_0)\gamma(c_1)\alpha(x_1)\dots\gamma(c_n)\alpha(x_n),$$

is a reduced expression and thus $\phi(g)$ nontrivial in P is by Lemma 5.3. By Lemma 6.6 we see that P is free-by-$\mathcal{C}$ and thus P is residually-$\mathcal{C}$ by Lemma 2.9.

We have showed that both in case if $n = 0$ and if $n \geq 1$ we can separate g from $\{1\}$. Using Lemma 2.1 we see that $\{1\}$ is $\mathcal{C}$-closed in G and thus G is residually-$\mathcal{C}$.

Now, assume that the graph Γ is infinite and let $g \in G \setminus \{1\}$ be arbitrary. Obviously, $S = \mathrm{supp}(g)$ is finite. Let G_S be the full subgroup corresponding to S and let $\rho_S \colon G \to G_S$ be the canonical retraction. Clearly, $\rho_S(g) = g \neq 1$ and G_S is residually-$\mathcal{C}$ as it is a finite graph product of residually-$\mathcal{C}$ groups. Using Lemma 2.1 we se that $\{1\}$ is $\mathcal{C}$-closed in G and thus G is residually-$\mathcal{C}$. ☐

The main idea of the proof of Theorem 6.1 is somewhat similar to the proof of Lemma 6.8. However, significantly more work needs to be done. To be able to prove Theorem 6.1 we will need the following two lemmas.

Lemma 6.9. *Let Γ be a finite graph and let $G = \Gamma\mathcal{G}$ be a graph product where G_v is $\mathcal{C}$-HCS for all $v \in V\Gamma$. Then all full subgroups of G satisfy $\mathcal{C}$-CC_G.*

Lemma 6.10. *Let Γ be a finite graph and let $G = \Gamma\mathcal{G}$ be a graph product where G_v is $\mathcal{C}$-HCS for all $v \in V\Gamma$. Then for all $g \in G$ and all full subgroups $B \leq G$ the set g^B is $\mathcal{C}$-separable in G.*

Lemmas 6.9 and 6.10 will be proved simultaneously by induction on $|V\Gamma|$. If $|V\Gamma| = 1$ we see that both lemmas hold trivially as $G = G_v$ which is $\mathcal{C}$-HCS by assumption. Now assume that the two lemmas are true for all $\Gamma\mathcal{G}$ where $|V\Gamma| \leq r$.

To be able to control conjugacy classes and centralisers in special amalgam $A \star_H C$ we need to be able to control intersections of conjugates of the amalgamated group H inside A as stated in Lemmas 5.10 and 5.11. In terms of our setting with graph products this means that we need to be able to control intersections of conjugates of full subgroups. This is established in Lemma 6.12. The rest of Section 6 is a case analysis dealing with all possible situations that might occur and shows that in all of the cases we can construct a suitable homomorphism from our graph product onto a special amalgam groups belonging to the class $\mathcal{C}$ which is a free-by-$\mathcal{C}$ group and thus by Corollary 6.7 is $\mathcal{C}$-CS group.

Remark 6.11. *Let G be a group and let $H, F \leq G$, $b, x, y \in G$ be arbitrary. If $bH \cap xFy \neq \emptyset$ then for any $a \in bH \cap xFy$ we have $bH \cap xFy = a(H \cap y^{-1}Fy)$.*

Proof. Let $a \in bH \cap xFy$. Since $a \in bH$ we see that $aH = bH$. Since $a \in xFy$ we have $a = xfy$ for some $f \in F$. Thus we can write

$$bH \cap xFy = aH \cap aa^{-1}xFy = aH \cap ay^{-1}f^{-1}x^{-1}xFy = aH \cap ay^{-1}Fy = a(H \cap y^{-1}Fy).$$

□

The following statements and their proofs very closely follow the contents of [18, Section 8].

Lemma 6.12. *Let G be a graph product and assume that every full subgroup $B \leq G$ satisfies $\mathcal{C}$-CC_G and for each $g \in G$ the conjugacy class g^B is $\mathcal{C}$-closed in G.*

Let $A_1, \ldots, A_n \leq G$ be full subgroups of G, let A_0 be a conjugate of a full subgroup of G and let $b, x_0, x_i, y_i \in G$ for $i = 1, \ldots, n$. Then for any $K \in \mathcal{N}_{\mathcal{C}}(G)$ there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and

$$\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n} \overline{x}_i\overline{A}_i\overline{y}_i \subseteq \psi\left(\left(bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_iA_iy_i\right)K\right) \text{ in } G/L,$$

where $\psi: G \to G/L$ is the natural projection and $\overline{b} = \psi(b)$, $\overline{A}_i = \psi(A_i)$ $\overline{x}_i = \psi(x_i)$, $i = 0, \ldots, n$ and $\overline{y}_j = \psi(y_j)$, $j = 1, \ldots, n$.

Proof. We will proceed by induction on n. If $n = 0$ then we just want $\overline{b}C_{\overline{A}_0}(\overline{x}_0) \subseteq \psi(bC_{A_0}(x_0)K)$. By assumption $A_0 = hAh^{-1}$ for some $h \in G$ and $A \leq G$ and thus the pair $(A, h^{-1}gh)$ has $\mathcal{C}$-CC_G. We can consider $\phi_{h^{-1}}$, the inner automorphism of G given by h^{-1}. Obviously $C_{\phi_{h^{-1}}(H)}(\phi(g)) = \phi_{h^{-1}}(C_H(g)) \subseteq \phi_{h^{-1}}(C_H(g)K)$ in G for any $K \in \mathcal{N}_{\mathcal{C}}(G)$ since $\phi_{h^{-1}} \in \mathrm{Aut}(G)$. Since $\ker(\phi_{h^{-1}}) = \{1\}$ we can use Lemma 4.5 to see that the pair (A_0, g) has $\mathcal{C}$-CC_G as well, thus there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $C_{\overline{A}_0}(\overline{x}_0) \subseteq \psi(C_{A_0}(x_0)K)$ in G/L. This is equivalent to $\overline{b}C_{\overline{A}_0}(\overline{x}_0) \subseteq \psi(bC_{A_0}(x_0)K)$.

Base of the induction: let $n = 1$. First suppose that $bC_{A_0}(x_0) \cap x_1A_1y_1 = \emptyset$. This is equivalent to $x_1 \notin bC_{A_0}(x_0)y_1^{-1}A_1$. Since $A_0 = hAh^{-1}$ then $C_{A_0}(x_0) = hC_A(g)h^{-1}$ where $g = h^{-1}x_0h$. Thus $x_1 \notin bhC_A(g)h^{-1}y_1^{-1}A_1(y_1h)(y_1h)^{-1}$. Set $D = (y_1h)^{-1}A_1(y_1h)$. Thus we have $x_1 \notin bhC_A(g)Dh^{-1}y_1^{-1}$.

By theorem 3.3 we see that $A \cap A_1$ and $A \cap D$ are conjugates of full subgroups. Thus for arbitrary $f \in G$ we have that $f^{A \cap D}$ (or $f^{A_1 \cap A}$) and the pair $(A \cap D, f)$ (or $(A_1 \cap A, f)$) has $\mathcal{C}$-CC$_G$ for all $f \in G$, thus by Lemma 6.3 we see that the double coset $C_A(g)D$ is $\mathcal{C}$-separable in G. Then $bhC_A(g)Dh^{-1}y^{-1}$ is $\mathcal{C}$-separable as well. Equivalently $bC_{A_0}(x_0)y_1^{-1}A_1$ is $\mathcal{C}$-closed in G and thus there is $N \in \mathcal{N}_\mathcal{C}(G)$ such that $x_1 \notin (bC_{A_0}(x_0)y_1^{-1}A_1)N$. By replacing $K \cap N$ we can assume $N \leq K$. Since the pair (A_0, x_0) has $\mathcal{C}$-CC$_G$ there is $L \in \mathcal{N}_\mathcal{C}(G)$ such that $L \leq N$ and $C_{\overline{A}_0}(\overline{x}_0) \subseteq \psi\left(C_{A_0}(x_0)N\right) \subseteq \psi\left(C_{A_0}(x_0)K\right)$ in G/L where $\psi : G \to G/L$ is the natural projection and $\overline{A}_0 = \psi(A_0)$, $\overline{x}_0 = \psi(x_0)$.

This means that $\psi^{-1}(\overline{b})C_{\overline{A}_0}(\overline{x}_0)\overline{y}_1^{-1}\overline{A}_1) \subseteq bC_{A_0}(x_0)y_1^{-1}A_1N$, where $\overline{b} = \psi(b)$, and thus from construction of N we see that $x_1 \notin \psi^{-1}(\overline{b}C_{\overline{A}_0}(\overline{x}_0)\overline{y}_1^{-1}\overline{A}_1)$ which concludes that $\overline{x}_1 \notin \overline{b}C_{\overline{A}_0}(\overline{x}_0)\overline{y}_1^{-1}\overline{A}_1$, thus $\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \overline{x}_1\overline{A}_1\overline{y}_1 = \emptyset$ and therefore

$$\emptyset = \overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \overline{x}_1\overline{A}_1\overline{y}_1 \subseteq \psi\left((bC_{A_0}(x_0) \cap x_1A_1y_1)K\right).$$

Now suppose $bC_{A_0}(x_0)x_1A_1y_1 \neq \emptyset$. By Remark 6.11 we see that $bC_{A_0}(x_0)x_1A_1y_1 = a(C_{A_0}(x_0) \cap y_1^{-1}A_1y_1)$, where $a \in bC_{A_0}(x_0) \cap x_1A_1y_1$. Clearly $C_{A_0}(x_0) \cap y_1^{-1}A_1y_1 = C_E(x_0)$, where $E = A_0 \cap y^{-1}A_1y$. By Theorem 3.3 we see that E is a conjugate of some full subgroup of G and thus the pair (E, x_0) has $\mathcal{C}$-CC$_G$ and therefore there is $M \in \mathcal{N}_\mathcal{C}(G)$ such that $M \leq K$ and

$$C_{\varphi(E)}(\varphi(x_0)) \subseteq \varphi(C_E(x_0)K) = \varphi\left((C_{A_0}(x_0) \cap y_1^{-1}A_1y_1)K\right) \text{ in } G/M, \tag{8}$$

where $\varphi : G \to G/M$ is the natural projection. However, we need to have control over $\overline{a}(C_{\overline{A}_0}(\overline{x}_0) \cap \overline{y}_1^{-1}\overline{A}_1\overline{y}_1)$. The full subgroups A, A_1 are retract whose corresponding retractions commute and their intersection, $A \cap A_1$, has $\mathcal{C}$-CC$_G$ because it is a full subgroup. Set $x = y_1h$. By Lemma 6.2 there is $L \in \mathcal{N}_\mathcal{C}(G)$ such that $L \leq M$ and

$$\psi(A) \cap \psi(xA_1x^{-1}) \subseteq \psi(A \cap xA_1x^{-1})\psi(M) \text{ in } G/L$$

where $\psi\colon G \to G/L$ is the natural projection. It can be easily checked that

$$\overline{A}_0 \cap \overline{y}_1^{-1}\overline{A}_1\overline{y}_1 = \overline{hAh}^{-1} \cap \overline{h}\left(\overline{y}_1\overline{h}\right)^{-1}\overline{A}_1\left(\overline{y}_1\overline{h}\right)\overline{h}^{-1} = \overline{h}(\overline{A} \cap \left(\overline{y}_1\overline{h}\right)^{-1}\overline{A}_1\left(\overline{y}_1\overline{h}\right))\,\overline{h}^{-1},$$

where $\overline{h} = \psi(h)$. Thus

$$\begin{aligned} \overline{A}_0 \cap \overline{y}_1{-}1\overline{A}_1\overline{y}_1 &\subseteq \psi(h)\left[\psi\left(A \cap (y_1h)^{-1}A_1(y_1h)\right)\psi(M)\right]\psi(h)^{-1} \\ &= \psi(hAh^{-1} \cap y_1^{-1}Ay_1)\psi(M) \\ &= \psi(E)\psi(M) = \psi(EM). \end{aligned} \tag{9}$$

Since $\psi(a) = \overline{a} \in \overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \overline{x}_1\overline{A}_1\overline{y}_1$ we can use Remark 6.11 and write

$$\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \overline{x}_1\overline{A}_1\overline{y}_1 = \overline{a}(C_{\overline{A}_0}(\overline{x}_0) \cap \overline{y}_1^{-1}\overline{A}_1\overline{y}_1).$$

Again, $C_{\overline{A}_0}(\overline{x}_0) \cap \overline{y}_1^{-1}\overline{A}_1\overline{y}_1 = C_{\overline{A}_0 \cap \overline{y}_1^{-1}\overline{A}_1\overline{y}_1}(\overline{x}_0)$. Since $\overline{A}_0 \cap \overline{y}_1^{-1}\overline{A}_1\overline{y}_1 \subseteq \psi(EM)$ we get that

$$\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \overline{x}_1\overline{A}_1\overline{y}_1 \subseteq \overline{a}C_{\psi(EM)}(\overline{x}_0). \tag{10}$$

Let $\varphi\colon G \to G/M$ be the natural projection. Since $L \leq M$ there is unique homomorphism $\xi\colon G/L \to G/M$ such that $\varphi = \xi \circ \psi$. Clearly $\psi(M) = \ker(\xi)$ and thus

$\xi(\psi(EM)) = \xi(\psi(E)\psi(M)) = \xi(\psi(E)) = \varphi(E)$, also $\xi(\overline{x}_0) = \varphi(x_0)$. Therefore for arbitrary $z \in C_{\psi(EM)}(\overline{x}_0)$ in G/L we have

$$\xi(z) \in C_{\varphi(E)}(\varphi(x_0)) \subseteq \varphi(C_E(x_0)K) = \xi(\psi(C_E(x_0)K)).$$

Altogether this means that $z \in \psi(C_E(x_0)K)\ker(\xi) = \psi(C_E(x_0)K)\psi(M) = \psi(C_E(x_0)K)$. Thus we get $C_{\psi(EM)}(\overline{x}_0) \subseteq \psi(C_E(x_0)K)$. Combined with (10) we get

$$\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \overline{x}_1\overline{A}_1\overline{y}_1 \subseteq \overline{a}\psi(C_E(x_0)K) = \psi(aC_E(x_0)K) = \psi((bC_{A_0}(x_0) \cap x_1A_1y_1)K).$$

Now suppose $n > 1$ and that the result has been proved for all $m \leq n-1$.

If $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = \emptyset$ then by induction there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq K$ and

$$\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n-1} \overline{x}_iA_i\overline{y}_i \subseteq \psi\left(\left(bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i\right)K\right) = \emptyset \text{ in } G/L.$$

Clearly $\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n} \overline{x}_iA_iy_i \subseteq \overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n-1} \overline{x}_iA_iy_i = \emptyset$ and thus we are done.

Now suppose that $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i \neq \emptyset$ in G. Then for any $a \in bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i$ we can use Remark 6.11 to see that $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = aC_E(x_0)$ where $E = A_0 \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i$.

Then $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_iA_iy_i = aC_E(x_0) \cap x_nA_ny_n$ thus by using the base case of the induction we can get that there is $M \in \mathcal{N}_{\mathcal{C}}$ such that $M \leq K$ and

$$\phi(a)C_{\phi(E)}(\phi(x_0)) \cap \phi(x_nA_ny_n) \subseteq \phi((aC_E(x_0) \cap x_nA_ny_n)K) \text{ in } G/M \tag{11}$$

where $\phi\colon G \to G/M$ is the natural projection. Also by induction hypothesis there is $L \in \mathcal{N}_{\mathcal{C}}(G)$ such that $L \leq M \leq K \leq G$ and

$$\overline{b}C_{\overline{A}_0}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x}_i\overline{A}_i\overline{y}_i \subseteq \psi\left(\left(bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i\right)M\right) \text{ in } G/L.$$

Note that $\ker(\psi) = L \leq M = \ker(\phi)$. Therefore

$$\begin{aligned}
\psi^{-1}\left(\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n} \overline{x}_i\overline{A}_i\overline{y}_i\right) &= \psi^{-1}\left(\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n-1} \overline{x}_i\overline{A}_i\overline{y}_i\right) \cap \psi^{-1}\left(\overline{x}_n\overline{A}_n\overline{y}_n\right) \\
&\subseteq \left(bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i\right)M \cap (x_nA_ny_n)L \\
&\subseteq aC_E(x_0)M \cap x_nA_ny_nM \\
&\subseteq \phi^{-1}\left[\phi(a)C_{\phi(E)}(\phi(x_0))\right] \cap \phi^{-1}\left[\phi(x_nA_ny_n)\right] \\
&= \phi^{-1}\left[\phi(a)C_{\phi(E)}(\phi(x_0)) \cap \phi(x_nA_nx_n)\right]
\end{aligned}$$

Using (11) we get

$$\phi^{-1}\left[\phi(a)C_{\phi(E)}(\phi(x_0)) \cap \phi(x_nA_nx_n)\right] \subseteq \left(aC_E(x_0) \cap x_nA_ny_n\right)K.$$

Finally, this leads us to

$$\psi^{-1}\left(\overline{b}C_{\overline{A}_0}(\overline{x}_0) \cap \bigcap_{i=1}^{n} \overline{x}_i\overline{A}_i\overline{y}_i\right) \subseteq \left(bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_iA_iy_i\right)K.$$

Which concludes the proof of the lemma. □

From now on we assume that Lemmas 6.9 and 6.10 are hold for all graph products $\Gamma\mathcal{G}$ such that $|V\Gamma| \leq r$. Let $G = \Gamma\mathcal{G}$ be a graph product such that $|V\Gamma| = r+1$. Let $v \in V\Gamma$ be arbitrary and set $A = G_{V\Gamma\setminus\{v\}}$, $H = G_{\mathrm{link}(v)}$ and $C = G_v$. Then by Remark 5.2 we see that $G = A \star_H C$. Also, suppose that $B \leq A$ is a full subgroup of A.

Lemma 6.13. *Let $g \in G\setminus A$ and $f \in G\setminus g^B$. Then there are homomorphisms $\psi_A \colon A \to Q$ and $\psi_C \colon C \to S$ where $Q, S \in \mathcal{C}$ such that for the corresponding extension $\psi \colon G \to P$, where $P = Q \star_{\psi_A(H)} S$, we have $\psi(f) \notin \psi(g)^{\psi(B)}$.*

Proof. Let $g = x_0c_1x_1 \ldots c_nx_n$, where $x_0, x_1, \ldots, x_n \in A$ and $c_1, \ldots, c_n \in C$, and $f = y_0d_1y_1 \ldots y_md_m$, where $y_0, y_1, \ldots, y_m \in A$ and $d_1, \ldots, d_m \in C$, be the reduced expressions for g and f respectively. Since $g \notin A$ we see that $n \geq 1$. We have to consider four separate cases.

Case 1: suppose $n \neq m$. Since G is residually-$\mathcal{C}$ by Lemma 6.8 (and A as well) and H is a full subgroup of A, H is a retract in A and thus is $\mathcal{C}$-closed in A by Lemma 2.6. Thus there is $L \in \mathcal{N}_{\mathcal{C}}(A)$ such that $\psi_A(x_i) \notin \psi_A(H)$ whenever $x_i \notin H$ and $\psi_A(y_j) \notin \psi_A(H)$ whenever $y_j \notin H$, where $\psi_A \colon A \to A/L$ is the natural projection. Since C is a vertex group we have that C is residually-$\mathcal{C}$ by assumption and thus there is $M \in \mathcal{N}_{\mathcal{C}}(C)$ such that $\psi_C(c_i) \neq 1$ and $\psi_C(d_j) \neq 1$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ where $\psi_C \colon C \to C/M$ is the natural projection. Then for the corresponding extension $\psi : A \star_H C \to A/L \star_{\psi_A(H)} C/M$ we have

$$\psi(g) = \psi_A(x_0)\psi_C(c_1)\psi_A(x_1) \ldots \psi_C(c_n)\psi_A(x_n),$$
$$\psi(f) = \psi_A(y_0)\psi_C(d_1)\psi_A(y_1) \ldots \psi_C(d_m)\psi_A(y_m).$$

These are again reduced expressions and $n \neq m$. Then by Lemma 5.11 we see that $\psi(g) \notin \psi(f)^{\psi(B)}$.

Case 2: $n = m$ and $c_j \neq d_j$ for some j. Again by argumentation analogous to previous case we see that there are $L \in \mathcal{N}_{\mathcal{C}}(A)$ and $M \in \mathcal{N}_{\mathcal{C}}(C)$ such that $\psi_A(x_i) \notin \psi_A(H)$ whenever $x_i \notin H$, $\psi_A(y_i) \notin \psi_A(H)$ whenever $y_i \notin H$ and $\psi_C(c_j) \neq \psi_C(d_j)$ where $\psi_A \colon A \to A/L$ $\psi_C \colon C \to C/M$ are the corresponding natural projections. Then for the corresponding extension $\psi : A \star_H C \to A/L \star_{\psi_A(H)} C/M$ we have

$$\psi(g) = \psi_A(x_0)\psi_C(c_1)\psi_A(x_1) \ldots \psi_C(c_n)\psi_A(x_n),$$
$$\psi(f) = \psi_A(y_0)\psi_C(d_1)\psi_A(y_1) \ldots \psi_C(d_n)\psi_A(y_n).$$

These are again reduced expressions and $\psi(c_j) \neq \psi(d_j)$. Then again by Lemma 5.11 we see that $\psi(g) \notin \psi(f)^{\psi(B)}$.

Case 3: $n = m$, $c_i = d_i$ for $i = 1, 2, \ldots, n$ and $x_0x_1 \ldots x_n \notin (y_0y_1 \ldots y_n)^B$. Since $x = x_0x_1 \ldots x_n \in A$ and B is a full subgroup of A we see that x^B is $\mathcal{C}$-closed in G by Lemma 6.10 and therefore there is $L \in \mathcal{N}_{\mathcal{C}}(A)$ such that $\psi_A(x) \notin \psi_A(g)^{\psi_A(B)}$, where $\psi_A \colon A/L$ is the natural projection. Since C is a vertex group we know by assumption that it is residually-$\mathcal{C}$ and thus there is $M \in \mathcal{N}_{\mathcal{C}}(C)$ such that $\psi_C(c_i) \neq 1 \neq \psi_C(d_i)$ for $i = 1, 2, \ldots, n$, where $\psi_C \colon C \to C/M$ is the natural projection. By extending $\psi_A \colon A \to Q = A/L$ and $\psi_C \colon C \to S = C/M$ to $\psi \colon A \star_H C \to Q \star_{\psi_A(H)} S$ we get that $\psi(f) \notin \psi(g)^{\psi(B)}$ by Lemma 5.11.

Case 4: Now we assume that condition (i) and (ii) from conjugacy criterion for special amalgams are satisfied and (iii) is not. Namely: let $b_0 \in B$ be such that $bxb^{-1} = y$ and $I = \emptyset$ where

$$I = b_0 C_B(x) \cap x_0 H x_0^{-1} \cap (y_0 y_1) H (x_0 x_1)^{-1} \cap \cdots \cap (y_0 y_1 \dots y_{n-1}) H (x_0 x_1 \dots x_{n-1})^{-1},$$

where $x = x_0 x_1 \dots x_n$. By assumption H is a full subgroup of A and thus it is $\mathcal{C}$-closed in A by Lemma 2.6, hence there is $K \in \mathcal{N}_{\mathcal{C}}(A)$ such that $x_i K \cap H = \emptyset = y_i K \cap H$ for all $i = 0, 1, \dots, n$. We assume that Lemmas 6.9 and 6.10 are true for A and thus assumptions of Lemma 6.12 are true for A. Therefore we can use Lemma 6.12 to see that there is $L \in \mathcal{N}_{\mathcal{C}}(A)$ such that $L \leq K$ and

$$\overline{b}_0 C_{\overline{B}}(\overline{x}) \cap \overline{y}_0 H \overline{x}_0^{-1} \cap (\overline{x}_0 \overline{x_1}) \overline{H} (\overline{y}_0 \overline{y}_1)^{-1} \cap \cdots \cap (\overline{x}_0 \overline{x}_1 \dots \overline{x}_{n-1}) \overline{H} (\overline{y}_0 \overline{y}_1 \dots \overline{y}_{n-1})^{-1} \subseteq \psi_A(IK),$$

where $\psi_A \colon A \to A/L$ is the natural projection and $\overline{x} = \psi_A(x)$, $\overline{x}_i = \psi_A(x_i)$, $\overline{y}_i = \psi_A(y_i)$ for $i = 0, 1, \dots n-1$, $\overline{H} = \psi_A(H)$ and $\overline{B} = \psi_A(B)$. Note that since $I = \emptyset$ we have $\psi_A(IK) = \emptyset$. Also since C is a vertex group we know it is $\mathcal{C}$-HCS and thus residually-$\mathcal{C}$, hence there is $M \leq \mathcal{N}_{\mathcal{C}}(C)$ such that $\psi_C(x_i) \neq 1 \neq \psi_C(y_i) \neq 1$ for $i = 1, 2, \dots, n$, where $\psi_C \colon C \to C/M$ is the natural projection. Therefore if we extend $\psi_A \colon A \to Q = A/L$ and $\psi_C \colon C \to S = C/M$ to $\psi \colon A \star_H C \to Q \star_{\psi_A(H)} S$ we see that $\psi(f) \notin \psi(g)^{\psi(B)}$ by Lemma 5.11 as the condition (iii) is not true for $\psi(f)$ and $\psi(g)$. □

Lemma 6.14. *Suppose that $g_0, f_0, f_1, \dots, f_n \in G$ are arbitrary and that the products $g_0 = c_1 x_1 \dots c_m x_m$, where $c_1, \dots, c_m \in C$ and $x_1, \dots, x_m \in A$, and $f_0 = d_1 y_1 \dots d_k y_k$, where $d_1, \dots, d_k \in C$ and $y_1, \dots, y_k \in A$, are cyclically reduced in G. If $f_j \notin g^H$ for all $j = 1, 2, \dots, m$ then there are groups $Q, S \in \mathcal{C}$ and a epimorphisms $\psi_A \colon A \to Q$ and $\psi_C \colon C \to S$ such that for the extension $\psi \colon A \star_H C \to P$, where $P = Q \star_{\psi_A(H)} S$, all of the following are true:*

(i) *$\psi(f_i) \notin \psi(g_0)^{\psi(H)}$ in P for all $i = 1, 2, \dots, n$,*

(ii) *the products*

$$\psi(g_0) = \psi_C(c_1)\psi_A(x_1) \dots \psi_C(c_m)\psi_A(x_m),$$
$$\psi(f_0) = \psi_C(d_1)\psi_A(y_1) \dots \psi_C(d_k)\psi_A(y_k)$$

are cyclically reduced in P.

Proof. We set $B := H$ which is a full subgroup of A thus we can apply Lemma 6.13 to pairs $(g_0, f_1), \dots, (g_0, f_n)$ to obtain $L_1, \dots, L_n \in \mathcal{N}_{\mathcal{C}}(A)$ and $M_1, \dots M_n \in \mathcal{N}_{\mathcal{C}}(C)$ with corresponding natural projections $\alpha_i \colon A \to A/L_i$ and $\gamma \colon C \to C/M_i$, such that $\psi_i(f_i) \notin \psi_i(g_0)^{\psi_i(H)}$ in P_i, where $\psi_i \colon A \star_H C \to A/L_i \star_{\alpha_i(H)} C/M_i$, for $i = 1, 2, \dots, n$. Note that since H is a retract of A and A is residually-$\mathcal{C}$ we get that H is $\mathcal{C}$-closed in A by Lemma 2.6. Thus there is $K \in \mathcal{N}_{\mathcal{C}}(A)$ such that $x_i K \cap H = \emptyset$ whenever $x_i \notin H$ and $y_j K \cap H = \emptyset$ whenever $y_j \notin H$. Also by the same argumentation there is $M' \in \mathcal{N}_{\mathcal{C}}(C)$ such that $c_i \notin M'$ and $d_j \notin M'$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$.

Set $L = K \cap \bigcap_{i=1}^n L_i$ and $M = M' \cap_{i=1}^n M_i$. Let $\psi_A \colon A \to A/L$ and $\psi_C \colon C \to C/M$ be the natural projections and let $\psi \colon A \star_H C \to A/L \star_{\psi_A(H)} C/M$. Clearly, this is the map we are looking for. □

Lemma 6.15. *Let $K \in \mathcal{N}_{\mathcal{C}}(G)$, $B \leq A$ be a full subgroup of A (and thus of G). Let $g \in G \setminus A$ be an element with reduced form $g = x_0 c_1 x_1 \dots c_n x_n$, where $x_0, x_1, \dots, x_n \in A$*

and $c_1, \dots, c_n \in C$, *such that* $n \geq 1$. *Then there are groups* $Q, S \in \mathcal{C}$ *and epimorphisms* $\psi_A \colon A \to Q$, $\psi_C \colon C \to S$ *with the corresponding extension* $\psi \colon A \star_H C \to P$, *where* $P = Q \star_{\psi_A(H)} S$, *such that the following are true:*

(i) $C_{\psi(B)}(\psi(g)) \subseteq \psi(C_B(g)K)$,

(ii) $\ker(\psi_A) \leq A \cap K$, $\ker(\psi_C) \leq C \cap K$ *and* $\ker(\psi) \leq K$.

Proof. Since A is residually-$\mathcal{C}$ and H is a retract in A we see that H is $\mathcal{C}$-closed in A by Lemma 2.6. Thus there is $M_1 \in \mathcal{N}_\mathcal{C}(A)$ such that $x_i M_1 \cap H = \emptyset$ for all $i = 1, 2, \dots, n-1$. We may replace M_1 by $M_1 \cap (A \cap K)$ to ensure that $M_1 \leq A \cap K$. By Lemma 5.12 we have $C_B(g) = I$ where

$$I = C_B(x) \cap x_0 H x_0^{-1} \cap (x_0 x_1) H (x_0 x_1)^{-1} \cap \dots \cap (x_0 \dots x_{n-1}) H (x_0 \dots x_{n-1})^{-1}$$

and $x = x_0 x_1 \dots x_n$. Since A is a graph product with less than n vertices we may assume that both Lemmas 6.9 and 6.10 hold for A. Thus we can use Lemma 6.12 to show the there is $L_1 \in \mathcal{N}_\mathcal{C}(A)$ such that $L_1 \leq M_1 \leq K \cap A$ and

$$J = C_{\overline{B}}(\overline{x}) \cap \overline{x}_0 \overline{H} \overline{x}_0^{-1} \cap \dots \cap (\overline{x}_0 \dots \overline{x}_{n-1}) \overline{H} (\overline{x}_0 \dots \overline{x}_{n-1})^{-1} \subseteq \psi_A(IM_1) \text{ in } A/L_1,$$

where $\psi_A \colon A \to Q = A/L_1$ is the natural projection and $\overline{x} = \psi_A(x)$, $\overline{x}_i = \psi_A(x_i)$, for $i = 0, 1, \dots n-1$, $\overline{H} = \psi_A(H)$ and $\overline{B} = \psi_H(B)$.

Since C is $\mathcal{C}$-HCS it is also residually-$\mathcal{C}$ and thus there is $Z \in \mathcal{N}_\mathcal{C}(C)$ such that $Z \leq C \cap K$ and $\psi_C(c_i) \neq 1$ in C/Z for $i = 1, 2, \dots, n$, where $\psi_C \colon C \to S = C/Z$ is the natural projection.

Let $P = Q \star_{\psi_A(H)} S$ and let $\psi \colon A \star_H C \to P$ be the canonical extension of ψ_A and ψ_C to G. Now

$$\psi(g) = \psi_A(x_0)\psi_C(c_1)\psi_A(x_1) \dots \psi_C(c_n)\psi(x_n),$$

thus $C_{\overline{B}}(g) = J$ in P by Lemma 5.12. This means that

$$C_{\psi(B)}(\psi(g)) \subseteq \psi(IM_1) = \psi(C_B(g)M_1),$$

therefore the first assertion of the lemma holds. Note that $\ker(\psi_A) = L_1 \leq M_1 \leq K \cap A$ and $\ker(\psi_C) \leq K \cap C$. Since $\ker(\psi) = \langle\langle \ker(\psi_A) \ker(\psi_C) \rangle\rangle^G$ by Lemma 5.5 we see that $\ker(\psi) \leq K$ and thus the second assertion holds as well. □

Lemma 6.16. *Let* $K \in \mathcal{N}_\mathcal{C}(G)$ *and let* $g = c_1 x_1 \dots c_n x_n$, *where* $c_1, \dots c_n \in C$ *and* $x_1, \dots, x_n \in A$, *be a cyclically reduced element of* G *with* $n \geq 1$. *Then there are homomorphisms* $\psi_A \colon A \to Q$ *and* $\psi_C \colon C \to S$, *where* $Q, S \in \mathcal{C}$, *with a corresponding extension* $\psi \colon G \to P$, *where* $P = Q \star_{\psi_A(H)} S$, *such that the following is true*

(i) $\ker(\psi_A) \leq A \cap K$, $\ker(\psi_C) \leq C \cap K$ *and* $\ker(\psi) \leq K$,

(ii) $C_P(\psi(g)) \subseteq \psi(C_G(g)K)$ *in* P.

Proof. We need to consider two separate cases: $x_n \in H$ or $x_n \notin H$.

Suppose $x_n \in H$. Then by Lemma 5.10 we see that $n = 1$ and $C_G(c_1 x_1) = C_C(c_1) \times C_H(x_1)$. Since A is a graph product with less than n vertices we can use the induction hypothesis of Lemma 6.10 to find $L \in \mathcal{N}_\mathcal{C}(A)$ such that $L \leq K \cap A$ and $C_{\alpha(H)}(\alpha(x_1)) \subseteq \alpha(C_H(x_1)(K \cap A))$, where $\alpha \colon A \to Q = A/L$ is the natural projection. Since C is a vertex group we assume that it is $\mathcal{C}$-HCS and therefore it satisfies $\mathcal{C}$-CC by Theorem 4.2. This means that there is $M \in \mathcal{N}_\mathcal{C}(C)$ such that $M \leq K \cap C$ and $C_S(\gamma(c_1)) \subseteq \gamma(C_C(c_1)(K \cap C))$, where $S = C/M$ and $\gamma \colon C \to S$ is the canonical projection. Note that since A is residually-$\mathcal{C}$ by Lemma 6.8 and C is residually-$\mathcal{C}$ by assumption these maps can be chosen so that $\alpha(x_1) \neq 1$ in Q and $\gamma(c_1) \neq 1$ in S.

Let $\psi\colon A \star_H C \to P$, where $P = Q \star_{\alpha(H)} S$, be the canonical extension of α and γ to G. Since $\psi(g) = \gamma(c_1)\alpha(x_1)$ is again reduced by Lemma 5.10 we see that $C_P(g) = C_S(\gamma(c_1)) \times C_P(\alpha)$. From the construction of the maps α and γ we see that $C_P(\psi(g)) \subseteq \psi(C_G(g)K)$.

Suppose $x_n \notin H$. Let $\{p_1, p_2, \dots p_{n+1}\}$ be the set of prefixes of g and assume there is $1 \leq m < n$ such that $p_i^{-1} g p_i \notin g^H$ if $i \leq m$ and $p_i^{-1} g p_i \in g^H$ if $i > m$. For every $i > m$ there is $h_i \in H$ such that $h_i p_i^{-1} g p_i h^{-1} = g$ and thus $h_i p_i^{-1} \in C_G(g)$. Set

$$\Omega = \{h_i p_i^{-1} \mid i > m\}. \tag{12}$$

Let $f_i = p_i^{-1} g p_i$ for $i = 1, 2, \dots, m$. Clearly $f_1, f_2, \dots, f_m \notin g^H$. Let $f_0 = g$. Using Lemma 6.14 we see that there are $\mathcal{C}$-groups Q_1 and S_1 and epimorphism $\alpha_1\colon A \to Q_1$ and $\gamma_1\colon C \to S_1$ such that if we take the corresponding extension $\psi_1\colon A \star_H C \to Q_1 \star_{\alpha_1(H)} S_1$ the element $\psi_1(f_0) = \gamma(c_1)\alpha(x_1) \dots \gamma(c_n)\alpha(x_n)$ is cyclically reduced and $\psi_i(f_i) \notin \psi(g)^{\psi(H)}$ for $i = 1, 2, \dots, m$. Let $L_1 = \ker(\alpha_1)$ and $M_1 = \ker(\gamma_1)$. Note that $L_1 \in \mathcal{N}_{\mathcal{C}}(A)$ and $M_1 \in \mathcal{N}_{\mathcal{C}}(C)$.

By Lemma 6.15 there are $\mathcal{C}$-groups Q_2, S_2 and an epimorphism $\alpha_2\colon A \to Q_2$, $\gamma_2\colon C \to S_2$ such that if we take the corresponding extension $\psi_2\colon A \star_H C \to Q_2 \star_{\alpha_2(H)} S_2$ we have $\ker(\alpha_2) \leq K \cap A$, $\ker(\gamma_2) \leq K \cap C$, $\ker(\psi_2) \leq K$ and

$$C_{\psi_2(H)}(\psi_2(g)) \subseteq \psi_2(C_H(g)K) \text{ in } Q_2 \star_{\alpha_2(H)} S_2.$$

Take $L = \ker(\alpha_1) \cap \ker(\alpha_2) \trianglelefteq A$ and let $M = \ker(\gamma_1) \cap \ker(\gamma_2) \trianglelefteq C$. Let $\alpha\colon A \to Q = A/L$ and $\gamma\colon C \to S = C/M$ be the natural projections. Note that $L \in \mathcal{N}_{\mathcal{C}}(A)$ and $M \in \mathcal{N}_{\mathcal{C}}(C)$. Let $\psi\colon G \to P$ be the corresponding extension, where $P = Q \star_{\alpha(H)} S$. We immediately get

(1) $\ker(\alpha) \leq K \cap A$ and $\ker(\psi) \leq K$,
(2) $C_{\psi(H)}(\psi(g)) \subseteq \psi(C_H(g)K)$ in P,
(3) $\psi(f_1), \dots, \psi(f_m) \notin \psi(g)^{\psi(H)}$,
(4) the element $\psi(g) = \gamma(c_1)\alpha(x_1) \dots \gamma(c_n)\alpha(x_n)$ is cyclically reduced in P.

Since we assume that $x_n \notin H$ we get by Lemma 5.10 that

$$C_G(g) = C_H(g)\langle g\rangle\Omega,$$

where Ω is given by (12). Also by Lemma 5.10 we see that in P we have

$$C_P(\psi(g)) = C_{\psi(H}(\psi(g))\langle\psi(g)\rangle\tilde{\Omega}$$

where $\tilde{\Omega} = \{\overline{h_i}\overline{p_i}^{-1} \mid i > m\}$ such that $\overline{p}_i$ is a prefix of $\psi(g)$ such that $\overline{p}_i^{-1}\psi(g)\overline{p}_i \in \psi(g)^{\psi(H)}$ and $\overline{h_i} \in \psi(H)$ such that $\overline{h}_i\overline{p}_i\psi(g)\overline{p_i}\overline{h}_i^{-1} = \psi(H)$. Since $\psi(f_i) = \psi(p_i)^{-1}\psi(g)\psi(p_i) \notin \psi(g)^{\psi(H)}$ when $i \leq m$, therefore we can conclude that $\psi(p_i)^{-1}\psi(g)\psi(p_i) \in \psi(g)^{\psi(H)}$ if and only if $p_i^{-1} g p_i \in g^H$. Altogether we see that $\tilde{\Omega} = \psi(\Omega)$.

Finally, $\langle\psi(g)\rangle = \psi(\langle g\rangle)$. From this we see that

$$C_P(\psi(g)) \subseteq \tilde{\psi}(C_G(g)K)$$

and thus the lemma holds. $\square$

Proof of Lemma 6.9. We will proceed by induction on $|V\Gamma|$. If $|V\Gamma| = 0$ then $G = \{1\}$ and the statement holds trivially. Now suppose that the statement holds for all graph products $\Gamma\mathcal{G}$ with $|V\Gamma| \leq r-1$. Let $G = \Gamma\mathcal{G}$ where $|V\Gamma| = r$. There are two cases to be distinguished: $B \neq G$ and $B = G$.

Suppose B is a proper full subgroup of G. Then we can pick a maximal proper full subgroup $A \leq G$ such that $B \leq A$. If $g \in A$ then g^B is $\mathcal{C}$-closed in A by induction hypothesis and thus it is $\mathcal{C}$-closed in G by Lemma 2.6 as A is a retract in G and G is residually-$\mathcal{C}$ by Lemma 6.8. Suppose that $g \in G \setminus A$ and let $f \in G \setminus g^B$ be arbitrary. By Lemma 6.13 there are $\mathcal{C}$-groups Q, S and epimorphism $\alpha\colon A \to Q$, $\gamma\colon C \to S$ with the corresponding extension $\psi\colon G \to Q \star_{\alpha(H)} S$ such that $\psi(f) \notin \psi(g)^{\psi(B)}$ in P. Since P is a special amalgam of (finite) $\mathcal{C}$-groups we see that it is residually-$\mathcal{C}$ by Lemma 6.7. Since $|Q| < \infty$ we see that $\psi(B)$ is a finite subset of Q and therefore $\psi(g)^{\psi(B)}$ is finite and thus is $\mathcal{C}$-closed in P. By Lemma 2.1 we see that g^B is $\mathcal{C}$-closed in G.

Now suppose $B = G$. If $g = 1$ then $1^G = \{1\}$ is $\mathcal{C}$-separable in G since it is finite subset of G and G is residually-$\mathcal{C}$. Let's assume $g \neq 1$. Then by Lemma 3.4 there is a maximal full subgroup $A \leq G$ such that $g \notin A^G$. Then G naturally splits as $G = A \star_H C$ where H is a full subgroup of A and C is a vertex group. Then g is a conjugate to some cyclically reduced element of G, say g_0. Suppose $g_0 = c_1x_1 \ldots c_nx_n$, where $x_1, \ldots, x_n \in A$ and $c_1, \ldots, c_n \in C$, is the cyclically reduced expression for g_0 Note that $g^G = g_0^G$. Let $f \in G \setminus g^G$. There are two sub-cases to consider: $f \notin A^G$ and $f \in A^G$.

Suppose $f \notin A^G$. Let f_0 be a cyclically reduced element of G conjugate to f, thus $f^G = f_0^G$. Let $f_0 = d_1y_1 \ldots d_my_m$, where $y_1, \ldots, y_m \in A$ and $d_1, \ldots, d_m \in C$, be the reduced expression for f_0 and let $f_1, f_2, \ldots f_m$ denote the set of all of its cyclic permutations. Clearly $f_i \notin g^H$ for all i since $f \notin g^G$. Then by Lemma 6.14 there are groups $Q, S \in \mathcal{C}$ and epimorphisms $\alpha\colon A \to Q$, $\gamma\colon C \to S$ with corresponding extension $\psi\colon G \to P$, where $P = Q \star_{\alpha(H)} S$, such that $\psi(f_1), \psi(f_2), \ldots \psi(f_m) \notin \psi(g)^{\psi(H)}$ and $\psi(f_0) = \gamma(d_1)\alpha(x_1) \ldots \gamma(d_m)\alpha(x_m)$ is cyclically reduced in P. Since $\psi(f_1), \ldots, \psi(f_m)$ are all the cyclic permutations of $\psi(f_0)$ we can conclude that $\psi(f) \notin \psi(g)^P$ by Lemma 5.8.

Assume that $f \in A^G$. By Lemma 6.14 there are groups $Q, S \in \mathcal{C}$ and projections $\alpha\colon A \to Q$, $\gamma\colon C \to S$ with extension $\psi\colon G \to P$, where $P = Q \star_{\alpha(H)} S$, such that $\psi(g_0) = \gamma(c_1)\alpha(x_1) \ldots \gamma(c_n)\alpha(x_n)$ is cyclically reduced in P. Since $n \geq 1$ by Lemma 5.8 we see that $\psi(g_0) \notin Q^P = \psi(A^G)$. As we assume that $f \in A^G$ we see that $\psi(f^G) = \psi(f)^P \subseteq \psi(A^G)$. We see that $\psi(g_0) \notin \psi(f)^P$ and hence $\psi(f) \not\sim_P \psi(g)$.

Either way, in both cases when $f \notin A^G$ and $f \in A^G$ we have found a homomorphism ψ that separates f from g^G in an amalgam of $\mathcal{C}$-groups which is $\mathcal{C}$-HSC by Lemma 6.7. Thus by Lemma 2.1 we see that g^G is $\mathcal{C}$-closed in G. □

Proof of Lemma 6.10. Again, we proceed by induction on $|V\Gamma|$. If $|V\Gamma| = 0$ then $G = \{1\}$ and the statement holds trivially. Now suppose that the statement holds for all Γ with $|V\Gamma| \leq r-1$. Let $G = \Gamma\mathcal{G}$ where $|V\Gamma| = r$. Let $K \in \mathcal{N}_{\mathcal{C}}(G)$ be arbitrary. There are two cases to be distinguished: $B \neq G$ and $B = G$.

Suppose B is a proper full subgroup of G. Then there is a maximal full subgroup $A \leq G$ such that $B \leq A$. Clearly A is a graph product with $r-1$ vertices and therefore the statement holds for A. Let $K_A = K \cap A$ and let $K_C = K \cap C$. Obviously G splits as $G = A \star_H C$, where H is a full subgroup of A and C is a vertex group. We consider two separate sub-cases: $g \in A$ and $g \in G \setminus A$.

Assume that $g \in A$. By induction we see that the pair (B, g) has $\mathcal{C}$-CC_A in A. Thus there is a $\mathcal{C}$-group Q such that $L_1 = \ker(\alpha) \leq K_A$, and

$$C_{\alpha(B)} \subseteq \alpha(C_B(g)K_A) \text{ in } Q,$$

where $\alpha\colon A \to Q$ is the natural projection. Let $\rho_A\colon G \to A$ be the canonical retraction of G onto A and set $L = \rho_A^{-1}(L_1) \cap K$. Clearly $L \in \mathcal{N}_\mathcal{C}(G)$ and $\rho_A(L) = L_1 \leq K_A$. Let $\phi\colon G \to R = G/L$ be the natural projection. Note that $\ker(\alpha) = \ker(\phi) \cap A$ in G, thus we may assume that $Q \leq R$ and $\phi \restriction A = \alpha$. Then $\alpha(K_A) = \phi(K_A) \subseteq \phi(K)$ in R. Since $g \in A$, B is a full subgroup of A and $C_{\alpha(B)} \subseteq \alpha(C_B(g)K_1)$ in Q, we get that

$$C_{\phi(B)}(\phi(g)) = C_{\alpha(B)}(\alpha(g)) \subseteq \alpha(C_B(g)K_A) \subseteq \phi(C_B(g)K) \text{ in } R.$$

Thus we see that if $g \in A$ then the pair (B, g) has $\mathcal{C}$-CC_G.

Now suppose that $g \in G \setminus A$. Let $g = x_0c_1x_1 \dots c_nx_n$, where $x_0, \dots, x_n \in A$ and $c_1, \dots, c_n \in C$, be a reduced expression for g. By Lemma 6.15 we can find $\mathcal{C}$-groups Q, S and epimorphisms $\alpha\colon A \to Q$, $\gamma\colon C \cap S$ with corresponding extension $\psi\colon G \to P = Q \star_{\alpha(H)} S$ such that $\ker(\alpha) \leq K_A$, $\ker(\gamma) \leq K_C$, $\ker(\psi) \leq K$ and

$$C_{\psi(B)}(\psi(g)) \subseteq \psi(C_B(g)K).$$

P is a special amalgam of $\mathcal{C}$-groups and thus is residually-$\mathcal{C}$ by Corollary 6.7. Since Q is finite we see that $\psi(K) \cap \psi(B) \leq \psi(B) \leq Q$ is finite, thus $\psi(g)^{\psi(B)\cap\psi(K)}$ is $\mathcal{C}$-closed in P. By Lemma 4.8 one obtains $\xi\colon P \to R$, where $R \in \mathcal{C}$ such that $\ker(\xi) \leq \psi(K)$ and

$$C_{\xi(\psi(B))}(\xi(\psi(B))) \subseteq \xi(C_{\psi(B)}(\psi(g))\psi(K)) \text{ in } R.$$

Take $\phi\colon G \to R$ to be defined as $\phi = \xi \circ \psi$. Obviously, ϕ is the map we are looking for.

We are left with the last remaining case, when $B = G$. We may assume $g \in G \setminus \{1\}$ as the pair $(G, 1)$ has $\mathcal{C}$-CC_G trivially. By Lemma 3.4 there is a maximal full subgroup $A \leq G$ such that $g \notin A^G$. Then G naturally splits as $G = A \star_H C$, where $H \leq A$ is a full subgroup of A and C is a vertex group in G. There is $z \in G$ such that $g_0 = zgz^{-1}$ is cyclically reduced in G. Let $g_0 = c_1x_1 \dots c_nx_n$, where $x_i \in A$ and $c_i \in C$ for $i = 1, \dots, n$, be a reduced expression for g_0. Since $g \notin A^G$ we see that $n \geq 1$. By Lemma 6.16 there are $\mathcal{C}$-groups Q, S and epimorphisms $\alpha\colon A \to Q$, $\gamma\colon C \to S$ with a corresponding extension $\psi\colon G \to P$, where $P = Q \star_{\alpha(H)} S$, such that $\ker(\alpha) \leq K \cap A$, $\ker(\gamma) \leq K \cap C$ and

$$C_P(\psi(g)) \subseteq \psi(C_G(g)K) \text{ in } P.$$

Since P is special amalgam of $\mathcal{C}$ groups we see that it is $\mathcal{C}$-HCS by Corollary 6.7 and thus the pair $(\psi(G), \psi(g))$ satisfies $\mathcal{C}$-CC_P in P. Note that in every case the homomoprhism ψ was constructed so that $\ker(\psi) \leq K$ thus by Lemma 4.5 we see get that the pair (g, G) has $\mathcal{C}$-CC_G in G. $\square$

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let $G = \Gamma\mathcal{G}$ be a graph product such that $|V\Gamma| < \infty$ and G_v is $\mathcal{C}$-HCS for all $v \in V\Gamma$. Note that G is a full subgroup of itself and thus by Lemma 6.9 we see that the pair (G, g) has $\mathcal{C}$-CC_G for every $g \in G$ and thus G satisfies $\mathcal{C}$-CC. By

Lemma 6.10 we see that the set g^G is $\mathcal{C}$-closed in G for every $g \in G$, hence G is $\mathcal{C}$-CS. Finally using Theorem 4.2 we get that G is $\mathcal{C}$-HCS. $\square$

Note that every group from the class $\mathcal{C}$ is $\mathcal{C}$-HCS. Then as an immediate consequence of the Theorem 6.1 we get that graph products of groups belonging to an extension closed variety of finite groups $\mathcal{C}$ are $\mathcal{C}$-HCS.

Corollary 6.17. *Assume that $\mathcal{C}$ is an extension closed variety of finite groups. Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of groups such that $G_v \in \mathcal{C}$ for all $v \in V\Gamma$. Then the group $G = \Gamma\mathcal{G}$ is $\mathcal{C}$-HCS.*

7. Infinite graphs and $\mathcal{C}$-CS groups

Again, we will assume that the class $\mathcal{C}$ is an extension closed variety of finite groups.

7.1. **Graph products of $\mathcal{C}$-CS groups.** Before we proceed we mention one important property of graph products: they are functorial.

Remark 7.1. *Let Γ be a graph and let $\mathcal{G} = \{G_v | v \in V\Gamma\}$ and $\mathcal{F} = \{F_v | v \in V\Gamma\}$ be two families of groups indexed by vertices of $V\Gamma$. Assume that for every $v \in V\Gamma$ there is a homomorphism $\phi_v \colon G_v \to F_v$. Then there is a unique homomorphism $\phi \colon G \to F$, where $G = \Gamma\mathcal{G}$ and $F = \Gamma\mathcal{F}$ such that $\phi \restriction_{G_v} = \phi_v$ for all $v \in V\Gamma$*

We will use Corollary 6.17 to show that the class of $\mathcal{C}$-CS groups is closed under graph products. The main idea is to construct a suitable map onto a finite graph product of groups belonging to the class $\mathcal{C}$. First we need to show that we can always find such a homomorphism that preserves length and support of a given element.

Lemma 7.2. *Let $G = \Gamma\mathcal{G}$ be a graph product such that G_v is residually-$\mathcal{C}$ for every $v \in V\Gamma$ and let $g \in G$. Then there is $\mathcal{F} = \{F_v | v \in V\Gamma\}$, a family of $\mathcal{C}$-groups indexed by $V\Gamma$, and a homomorphism $\phi_v \colon G_v \to F_v$ for every $v \in V\Gamma$ such that for the corresponding extension $\phi \colon G \to F$ (given by Remark 7.1), where $F = \Gamma\mathcal{F}$, all of the following are true:*

(i) $|g| = |\phi(g)|$,
(ii) $\mathrm{supp}(g) = \mathrm{supp}(\phi(g))$,
(iii) *If g is Γ-cyclically reduced in G then $\phi(g)$ is Γ-cyclically reduced in F.*

Proof. Let $(g_1, \ldots, g_n)$ be a Γ-reduced expression for g in G. For every $v \in V\Gamma$ let $I_v = \{i \mid g_i \in G_v\} \subseteq \mathrm{supp}(g)$ be the set of indices such that the corresponding syllables belong to G_v. Since I_v is finite and G_v is residually-$\mathcal{C}$ for every v by assumption there is $F_v \in \mathcal{C}$ and a homomorphism $\phi_v \colon G_v \to F_v$ such that $\phi_v(g_i) \neq 1$ in F_v for all $i \in I_v$. By Remark 7.1 we have the corresponding unique extension $\phi \colon G \to F$, where $F = \Gamma\mathcal{F}$. Clearly, $(\phi_{v_1}(g), \ldots, \phi_{v_n}(g_n))$ is a Γ-reduced expression for $\phi(g)$ therefore $|g| = |\phi(g)|$ and $\mathrm{supp}(g) = \mathrm{supp}(\phi(g))$.

Suppose that g is Γ-cyclically reduced in G. Obviously $\mathrm{FL}(g) = \mathrm{FL}(\phi(g))$, $\mathrm{LL}(g) = \mathrm{LL}(\phi(g))$ and $S(g) = S(\phi(g))$ and thus $(\mathrm{FL}(\phi(g)) \cap \mathrm{LL}(\phi(g))) \setminus S(\phi(g)) = (\mathrm{FL}(g) \cap \mathrm{LL}(g)) \setminus S(g) = \emptyset$ by Lemma 3.11 because g is Γ-cyclically reduced and therefore $\phi(g)$ is Γ-cyclically reduced in F again by Lemma 3.11. $\square$

In fact we are can generalise the previous lemma to any finite number of given elements.

Corollary 7.3. *Let $f, g \in G$ be Γ-cyclically reduced in G and assume that $f \neq g$. Then there is $\mathcal{F} = \{F_v | v \in V\Gamma\}$, a family of $\mathcal{C}$-groups indexed by $V\Gamma$, and a homomorphism $\phi_v \colon G_v \to F_v$ for every $v \in V\Gamma$ such that for the corresponding extension $\phi \colon G \to F$, where $F = \Gamma\mathcal{F}$, all of the following are true:*

(i) $|g| = |\phi(g)|$ *and* $\operatorname{supp}(g) = \operatorname{supp}(\phi(g))$,
(ii) $|f| = |\phi(f)|$ *and* $\operatorname{supp}(f) = \operatorname{supp}(\phi(f))$,
(iii) $\phi(f), \phi(g)$ *are Γ-cyclically reduced in F,*
(iv) $\phi(f) \neq \phi(g)$ *in F.*

Proof. We use Lemma 7.2 on g,f and gf^{-1} to obtain three corresponding families $\mathcal{F}^f$, $\mathcal{F}^g$ and $\mathcal{F}^{fg^{-1}}$. For every $v \in V\Gamma$ we set $K_v = \ker(\phi_v^f) \cap \ker(\phi_v^g) \cap \ker(\phi_v^{fg^{-1}})$ and define $\phi_v \colon G_v \to F_v$, where $F_v = G_v/K_v$. Clearly the family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v | v \in V\Gamma\}$ together with homomorphisms $\phi_v \colon G_v \to F_v$ and the extension $\phi \colon G \to \Gamma\mathcal{F}$ has all the claimed properties. □

The proof of the following remark is left as a simple exercise for the reader.

Remark 7.4. *Suppose that $\mathcal{C}$ is a class of finite groups satisfying (c1) and (c2). Then the class of $\mathcal{C}$-CS groups is closed under taking direct products.*

Proof of Theorem 1.1. Let $g \in G$ be arbitrary and let $f \in G$ such that $f \not\sim_G g$. Note that the set of vertices $X = \operatorname{supp}(g) \cup \operatorname{supp}(f) \subseteq V\Gamma$ is finite and $\rho_X(f) = f \not\sim_{G_X} \rho_X(g) = g$, where $\rho_X \colon G \to G_X$ is the canonical retraction corresponding to the full subgroup $G_X \leq G$ given by the set $X \subseteq V\Gamma$. Hence without loss of generality we may assume that $|V\Gamma| < \infty$. Let $f_0, g_0 \in G$ be Γ-cyclically reduced elements of G such that $f_0 \sim_G f$ and $g_0 \sim_G g$. Clearly $f_0 \not\sim_G g_0$.

By Lemma 3.12 we have three possibilities to consider:

(i) $\operatorname{supp}(g_0) \neq \operatorname{supp}(f_0)$ or $|g_0| \neq |f_0|$,
(ii) $p(f_0)$ is not a cyclic permutation of $p(g_0)$,
(iii) $s(f_0) \notin s(g_0)^{G_{S(g_0)}}$.

Assume that either $\operatorname{supp}(g_0) \neq \operatorname{supp}(f_0)$ or $|f_0| \neq |g_0|$. Then we can use Corollary 7.3 to obtain a family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v | v \in V\Gamma\}$ and a homomorphism $\phi_v \colon G_v \to F_v$ for every $v \in V\Gamma$ such that for the corresponding extension $\phi \colon G \to \Gamma\mathcal{F}$ we have either $\operatorname{supp}(\phi(f_0)) \neq \operatorname{supp}(\phi(g_0))$ or $|\phi(f_0)| \neq |\phi(g_0)|$ respectively. By Lemma 3.12 we see that $\phi(f_0) \not\sim_{\Gamma\mathcal{F}} \phi(g_0)$ and hence $\phi(f) \not\sim_{\Gamma\mathcal{F}} \phi(g)$. Note that $\Gamma\mathcal{F}$ is a finite graph product of groups belonging to the class $\mathcal{C}$ and thus by Corollary 6.17 we see that the group $\Gamma\mathcal{F}$ is $\mathcal{C}$-HCS.

Assume that $\operatorname{supp}(g_0) = \operatorname{supp}(f_0)$ and $|g_0| = |f_0|$. Suppose that $p(f_0)$ is not a cyclic permutation of $p(g_0)$. Let $\{p_1, \ldots, p_m\} \subset G$ be the set of all cyclic permutations of $p(g_0)$ including $p(g_0)$. Then $p_i \neq p(f_0)$ for $i = 1, \ldots, m$ and we can use Corollary 7.3 for each pair $p(f_0), p_i$, where $1 \leq i \leq m$, to obtain a family of $\mathcal{C}$-groups $\mathcal{F}_i = \{F_v^i | v \in V\Gamma\}$ with homomorphisms $\phi_v^i \colon G_v \to F_v^i$ for all $v \in V\Gamma$. For every $v \in V\Gamma$ set $K_v = \bigcap_{i=1}^m \ker(\phi_v^i)$ and denote $F_v = G_v/K_v$. Set $\mathcal{F} = \{F_v | v \in V\Gamma\}$ and let $\phi_v \colon G_v \to F_v$ be the natural projection corresponding to v. Let $\phi \colon G \to \Gamma\mathcal{F}$ be the natural extension. Note that $p(\phi(f_0)) = \phi(p(f_0))$ and $p(\phi(g_0)) = \phi(p(g_0))$. Clearly the set $C = \{\phi(p_1), \ldots, \phi(p_m)\}$ is the set of all cyclic permutations of $p(\phi(g_0))$ and we see that $p(\phi(f_0)) \notin C$ and thus $p(\phi(f_0))$ is not a cyclic permutation of $p(\phi(g_0))$. By Lemma 3.12 we see that

$\phi(f_0) \not\sim_{\Gamma\mathcal{F}} \phi(g_0)$ and thus $\phi(f) \not\sim_{\Gamma\mathcal{F}} \phi(g)$. Again, by Corollary 6.17 we see that the group $\Gamma\mathcal{F}$ is $\mathcal{C}$-HCS.

Now assume that $\mathrm{supp}(g_0) = \mathrm{supp}(f_0)$, $|g_0| = |f_0|$, $p(f_0)$ is a cyclic permutation of $p(g_0)$. Since $\mathrm{supp}(f_0) = \mathrm{supp}(g_0)$ we see that $S(g_0) = S(f_0)$. Denote $S = S(g_0)$ and assume that $s(f_0) \notin s(g_0)^{G_S}$. Note that

$$G_S = \prod_{v \in S} G_v$$

is a direct product of $\mathcal{C}$-CS groups and thus it is a $\mathcal{C}$-CS group by Remark 7.4. Consider the retraction $\rho_S\colon G \to G_S$. Clearly $\rho_S(f_0) = s(f_0)$ and $\rho_S(g_0) = s(g_0)$. Therefore $\rho_S(g_0) \notin \rho_S(g_0)^{G_S}$ by assumption and consequently $\rho_S(f) \notin \rho_S(g)^{G_S}$.

In each of the cases we have constructed a homomorphism onto a $\mathcal{C}$-CS group, such that the images of f and g were not conjugate. Then by Lemma 2.1 we see that the conjugacy class g^G is $\mathcal{C}$-closed in G. As g was arbitrary we see that G is $\mathcal{C}$-CS. $\square$

7.2. **Infinite graph products of $\mathcal{C}$-HCS groups.** The idea of the proof of Theorem 1.2 somewhat similar to the proof of Theorem 1.1. In the proof of Theorem 1.1 we started with a possibly infinite graph Γ and showed that we can always retract to a full subgroup G_A given by a finite set of vertices $A \subseteq V\Gamma$ and thus we were able to use Corollary 6.17. In the proof of Theorem 1.2 we start with a graph product $\Gamma\mathcal{G}$, where Γ is an infinite graph, we show that for every $g \in G$ we can construct a finite graph product $\Delta\mathcal{D}$ of $\mathcal{C}$-HCS groups and a homomorphism $\delta\colon \Gamma\mathcal{G} \to \Delta\mathcal{D}$ such that $C_{\Delta\mathcal{D}}(\delta(g)) = \delta(C_G(g))$.

Proof of Theorem 1.2. Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of groups such that the group G_v is $\mathcal{C}$-HCS for every $v \in V\Gamma$. Let $G = \Gamma\mathcal{G}$. By Theorem 1.1 we see that the group G is $\mathcal{C}$-CS so we need to show that G satisfies $\mathcal{C}$-CC. Clearly, G satisfies $\mathcal{C}$-CC if and only if for every $g \in G$ the pair (G, g) satisfies $\mathcal{C}$-CC$_G$. Let $g \in G$ and $K \in \mathcal{N}_{\mathcal{C}}(G)$ be arbitrary. Pick $g' \in G$ such that $g \sim_G g'$ and g' is Γ-cyclically reduced. By Lemma 4.5 we see that the pair (G, g) has $\mathcal{C}$-CC$_G$ if and only if the pair (G, g') has $\mathcal{C}$-CC$_G$. Denote $A = \mathrm{supp}(g')$.

Let $\varphi\colon G \to G/K$ be the natural projection. Define a family of groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$, where $F_v = G_v$ if $v \in A$ and $F_v = \varphi(G_v)$ otherwise. For every $v \in V\Gamma$ we have a group homomorphism $\phi_v\colon G_v \to F_v$ where $\phi_v = \mathrm{id}_{G_v}$ if $v \in A$ and $\phi_v = \varphi\restriction_{G_v}$ otherwise. By Remark 7.1 there is a unique group homomorphism $\phi\colon \Gamma\mathcal{G} \to \Gamma\mathcal{F}$ such that $\phi\restriction_{G_v} = \phi_v$ for every $v \in V\Gamma$. Denote $F = \Gamma\mathcal{F}$. Note that $\ker\phi = \langle\langle \ker(\phi_v) \rangle\rangle^G_{v \in V\Gamma}$. Clearly if $v \in A$ then $\ker(\phi_v) = \{1\}$ and if $v \in V\Gamma \setminus A$ then $\ker(\phi_v) = K \cap G_v$. It follows that $\ker(\phi) \le K$ and hence there is a unique homomorphism $\overline{\phi}\colon F \to G/K$ such that $\varphi = \overline{\phi} \circ \phi$.

Set $A' = V\Gamma \setminus A$. Define equivalence $\approx_1$ on A' as follows: $u \approx_1 v$ if $\mathrm{link}(u) \cap A = \mathrm{link}(v) \cap A$. Define equivalence $\approx_2$ on A' as follows: $u \approx_K v$ if $\overline{\phi}(F_u) = \overline{\phi}(F_v)$ in G/K. Now let $\approx$ be the equivalence relation on A' obtained as intersection of $\approx_1$ and $\approx_2$, i.e. $u \approx v$ if $\mathrm{link}(u) \cap A = \mathrm{link}(v) \cap A$ and $\overline{\phi}(F_u) = \overline{\phi}(F_v)$ in G/K. Note that $|A'/\approx_2| \le 2^{|A|}$ and $|A'/\approx_2| \le 2^{|G/K|}$, therefore we see that $|A'/\approx| < \infty$.

Define a graph Δ with vertex set $V\Delta = A \cup (A'/\approx)$. Note that $V\Delta$ is finite. For $u, v \in A$ we set $\{u, v\} \in E\Delta$ if and only if $\{u, v\} \in E\Gamma$, for $u \in A$ and $[x]_\approx \in A'/\approx$ we set $\{u, [x]_\approx\} \in E\Delta$ if and only if there is $x_0 \in [x]_\approx$ such that $\{u, x\} \in E\Gamma$. Similarly for $[x]_\approx, [y]_\approx \in A'/\approx$ we set $\{[x]_\approx, [y]_\approx\} \in E\Delta$ if and only if there are $x_0 \in [x]_\approx$ and $y_0 \in [y]_\approx$ such that $\{x_0, y_0\} \in E\Gamma$. Note that the natural map from $V\Gamma$ to $V\Delta$ actually extends to a graph morphism from Γ to Δ.

To every vertex in $v \in V\Delta$ we assign a vertex group D_v in the following way: if $v \in A$ then $D_v = G_v$; if $v = [v_0]_\approx$ for some $v_0 \in A'$ then $D_v = \varphi(G_{v_0}) = \overline{\phi}(F_v)$. This leads to a family of groups $\mathcal{D} = \{D_v \mid v \in V\Delta\}$. For every $v \in V\Gamma$ we define a group homomorphism $\overline{\varphi}_v \colon F_v \to D_{x_v}$, where $x_v = v$ if $v \in A$ and $x_v = [v]_\approx$ otherwise. If $v \in A$ then $\overline{\varphi}_v = \mathrm{id}_{G_v}$ and if $v \in V\Gamma \setminus A$ then $\overline{\varphi}_v = \overline{\phi}\restriction_{F_v}$. By a theorem of von Dyck (see [24, footnote 2, page 346]) the family of group homomorphisms $\{\overline{\varphi}_v \mid v \in V\Gamma\}$ extends to a homomorphism $\overline{\varphi} \colon F \to D$, where $D = \Delta\mathcal{D}$ is the corresponding graph product.

Let $x, y \in F$ be arbitrary. It is obvious that if $\overline{\varphi}(x) = \overline{\varphi}(y)$ then $\overline{\phi}(x) = \overline{\phi}(y)$ and thus $\ker(\overline{\varphi}) \leq \ker(\overline{\phi}) = \phi(K)$. We see that there is unique homomorphism $\overline{\delta} \colon D \to G/K$ such that $\overline{\phi} = \overline{\delta} \circ \overline{\varphi}$. Denote $\delta = \overline{\delta} \circ \overline{\phi}$. The following commutative diagram illustrates the situation.

$$
\begin{tikzcd}
G = \Gamma\mathcal{G} \arrow[dr, "\phi"] \arrow[drr, bend left, "\delta"] \arrow[ddr, "\varphi"'] & & \\
& F = \Gamma\mathcal{F} \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\phi}"] & D = \Delta\mathcal{D} \arrow[dl, "\overline{\delta}"] \\
& G/K &
\end{tikzcd}
$$

Clearly $\ker(\delta) = \phi^{-1}(\ker(\overline{\varphi})) \leq K$.

Now we need to show that $C_{\delta(G)}(\delta(g)) \subseteq \delta(C_G(g)K)$. One can easily check that $\mathrm{Pc}_\Gamma(\langle g' \rangle) = G_A$ and therefore by Lemma 3.7 we have $C_G(g') = C_{G_A}(g')G_{\mathrm{link}(A)}$. Denote $\delta_A = \delta\restriction_{G_A}$. From the construction of δ it is easy to see that $\delta_A \colon G_A \to D_A$ is an isomorphism. Let $P = \mathrm{Pc}_\Delta(\delta(\langle g' \rangle))$. As D_A is a full (and hence parabolic) subgroup of D and $\delta(g') \in P$ we see that $P \leq D_A$ due to minimality of P. By [1, Lemma 3.7] we see that P is actually parabolic in $\Delta_A\mathcal{D}_A = D_A$. Let $P' = \delta_A(P)^{-1} \leq G_A \leq G$. From the construction of the map δ we see that P' is parabolic in G_A (and thus in G) and that $g' \in P'$. Since $\mathrm{Pc}_\Gamma(\langle g' \rangle) = G_A$ and $g' \in P'$ we see that $G_A \leq P'$ and therefore $P' = G_A$. This means that $P = D_A$. We see that $\mathrm{Pc}_\Delta(\langle \delta(g') \rangle) = \mathrm{Pc}_\Delta(\delta(\langle g' \rangle)) = D_A$ and hence by Lemma 3.7 we get that $C_D(\delta(g')) = C_{D_A}(\delta(g'))D_{\mathrm{link}(A)}$.

Again, since $\delta\restriction_{G_A}$ is an isomorphism we see that $\delta(C_{G_A}(g')) = C_{D_A}(\delta(g')) = C_{\delta(G_A)}(\delta(g'))$. From the construction of the equivalence $\approx$ we see that for every $v \in V\Gamma$ we have $[v]_\approx \in \mathrm{link}(A)$ in Δ if and only if $v \in \mathrm{link}(A)$ in Γ and hence $\delta(G_{\mathrm{link}(A)}) = D_{\mathrm{link}(A)}$. We see that

$$C_D(\delta(g')) = C_{D_A}(\delta(g'))D_{\mathrm{link}(A)} = \delta(C_{G_A}(g')G_{\mathrm{link}(A)}) = \delta(C_G(g')) \subseteq \delta(C_G(g')K).$$

For every $v \in V\Delta$ the group D_v is either an infinite $\mathcal{C}$-HCS group or belongs to the class $\mathcal{C}$. By Theorem 6.1 we see that the group D is $\mathcal{C}$-HCS and hence D satisfies $\mathcal{C}$-CC by Theorem 4.2. Consequently, the pair $(D, \delta(g'))$ satisfies $\mathcal{C}$-CC in D. By Lemma 4.5 we see that the pair (G, g) satisfies $\mathcal{C}$-CC_G for any $g \in G$ and therefore G satisfies $\mathcal{C}$-CC. We have proved that G is $\mathcal{C}$-CS and satisfies $\mathcal{C}$-CC, hence by Theorem 4.2 we see that G is $\mathcal{C}$-HCS. $\square$

7.3. **Some corollaries.** Applying Theorem 1.2 to the most obvious types of extension closed varieties of finitely presented groups we immediately get that the class of HCS groups is closed under taking finite graph products, similarly for p-HCS and (finite solvable)-HCS.

We can also extend the results of Minasyan (see [18]) and Toinet (see [29]) to infinitely generated right angled Artin groups.

Corollary 7.5. *Infinitely generated RAAGS are HCS and p-HCS for every prime number p.*

In [4, Theorem 1.2] Caprace and Minasyan proved that finitely generated RACGs are CS. By applying Theorem 1.2 to RACGs once in the context of the class of all finite groups and once in the context of all finite 2-groups we get following strengthening of the mentioned result.

Corollary 7.6. *Arbitrary (possibly infinitely generated) right angled Coxeter groups are HCS and 2-HCS.*

The statement of Corollary 7.6 can be compared with the following example: the group $G = \mathrm{FSym}(X)$ of finitary permutations of an infinite set X is an infinitely generated Coxeter group, but it is not even residually finite. Clearly being right angled is a strong requirement.

As we mentioned in the introductory section virtually polycyclic groups are HCS, thus we can state the following corollary.

Corollary 7.7. *Let Γ be any graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of groups such that the group G_v is virtually polycyclic for every $v \in V\Gamma$. Then the group $G = \Gamma\mathcal{G}$ is HCS.*

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SEPARABILITY PROPERTIES OF AUTOMORPHISMS OF GRAPH PRODUCTS OF GROUPS

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Abstract. We study properties of automorphisms of graph products of groups. We show that graph product $\Gamma\mathcal{G}$ has non-trivial pointwise inner automorphisms if and only if some vertex group corresponding to a central vertex has non-trivial pointwise inner automorphisms. We use this result to study residual finiteness of $\mathrm{Out}(\Gamma\mathcal{G})$. We show that if all vertex groups are finitely generated residually finite and the vertex groups corresponding to central vertices satisfy certain technical (yet natural) condition, then $\mathrm{Out}(\Gamma\mathcal{G})$ is residually finite. Finally, we generalise this result to graph products of residually p-finite groups to show that if $\Gamma\mathcal{G}$ is a graph product of finitely generated residually p-finite groups such that the vertex groups corresponding to central vertices satisfy the p-version of the technical condition then $\mathrm{Out}(\Gamma\mathcal{G})$ is virtually residually p-finite. We use this result to prove bi-orderability of Torelli groups of some graph products of finitely generated residually torsion-free nilpotent groups.

Contents

1. Introduction and motivation

1.1. **Motivation.** Recall that a group G is *residually finite* (RF) if for every $g \in G \setminus \{1\}$ there is a finite group F and a homomorphism $\varphi\colon G \to F$ such that $\varphi(g) \neq 1$ in F. The main motivation to study residually finite groups is that they can be approximated by their finite quotients. In case of finitely presented groups this approximation can be used to solve the word problem: Mal'cev [11] constructed an algorithm that uniformly solves the word problem in the class of finitely presented RF groups.

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Baumslag [2] proved that if G is a finitely generated RF group then $\mathrm{Aut}(G)$, the group of automorphisms of G, is RF as well. One could ask whether this result can be generalised to $\mathrm{Out}(G) \cong \mathrm{Aut}(G)/\mathrm{Inn}(G)$, the group of outer automorphisms? Negative answer to this question was provided by Bumagin and Wise in [3] when they proved that for every finitely presented group O there is a finitely generated residually finite group G such that $\mathrm{Out}(G) \cong O$. The question that naturally arises is: what properties does a finitely generated RF group G need to satisfy to ensure that $\mathrm{Out}(G)$ is RF as well?

1.2. **Grossman's criterion.** An automorphism $\phi \in \mathrm{Aut}(G)$ is *pointwise inner* if $\phi(g)$ is conjugate to g for every $g \in G$. Let $\mathrm{Aut}_{\mathrm{PI}}(G)$ denote the set of all pointwise inner automorphisms of G. We say that a group G has *Grossman's property (A)* if every pointwise inner automorphism of G is inner, i.e. if $\mathrm{Aut}_{\mathrm{PI}}(G) = \mathrm{Inn}(G)$. We say that G is *conjugacy separable* (CS) if for every tuple $f, g \in G$ such that f is not conjugate to g there exists a finite group F and a homomorphism $\varphi\colon G \to F$ such that $\varphi(f)$ is not conjugate to $\varphi(g)$. Grossman [8, Theorem 1] proved that if G is a finitely generated CS group with Grossman's property (A) then $\mathrm{Out}(G)$ is residually finite. We call groups that satisfy this criterion *Grossmanian groups*, i.e. a group G is Grossmanian if G is a finitely generated CS group with Grossman's property (A). However, these are not necessary conditions; see Section 2 for a discussion.

1.3. **Statement of results.** Another natural question to ask is how does the property of having a residually finite group of outer automorphisms behave under group theoretic constructions? In this paper we study the case of graph products of groups, which naturally generalise the notion of free products and direct products in the category of groups.

Let Γ be a simplicial graph, i.e. $V\Gamma$ is a set and $E\Gamma \subseteq \binom{V\Gamma}{2}$, and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial groups. The group $\Gamma\mathcal{G}$, the *graph product* of the family $\mathcal{G}$ with respect to Γ, is the quotient of the free product $*_{v\in V\Gamma} G_v$ modulo relations of the form

$$g_u g_v = g_v g_u \quad \forall g_u \in G_u, \forall g_v \in G_v \text{ whenever } \{u, v\} \in E\Gamma.$$

The groups G_v are called *vertex groups.* In this study we will be considering only finite graph products, i.e. $|V\Gamma| < \infty$. Clearly, if Γ is totally disconnected then the graph product $\Gamma\mathcal{G}$ is equal to $*_{v\in V\Gamma} G_v$, the free product of the vertex groups, and similarly, if Γ is complete then $\Gamma\mathcal{G}$ is equal to $\times_{v\in V\Gamma} G_v$, the direct product of the vertex groups. Note that we will always assume that the vertex groups are non-trivial, i.e. $G_v \neq \{1\}$ for all $v \in V\Gamma$.

In the case when all vertex groups are infinite cyclic we are talking about *right angled Artin groups* (RAAGs). If all vertex groups are cyclic of order two then we are talking about *right angled Coxeter groups* (RACGs). We quickly list some partial results on residual finiteness of outer automorphisms of graph products of residually finite groups. In [13] Minasyan showed that if G is a finitely generated RAAG then G is CS and has Grossman's property (A), hence $\mathrm{Out}(G)$ is RF by Grossmans criterion. Independently of Minasyan, Charney and Vogtmann [5] proved that outer automorphism groups of finitely generated RAAGs are RF. In [4] Carette proved that if G is a finitely generated RACG then $\mathrm{Out}(G)$ is RF.

Following the results presented in [14] residual finiteness of outer automorphism groups of free products of finitely generated RF groups is well understood. However, it is not known whether the class of finitely generated RF groups with RF group of outer automorphisms is closed under direct products (see Question 8.1). To overcome this obstacle we introduce the following class of groups.

We will say that a group G is *inner automorphism separable* (IAS) if every non-trivial outer automorphism of G can be realised as a non-trivial outer automorphism of some finite quotient of G (see Section 2 for formal definition). Obviously, if G is IAS then $\mathrm{Out}(G)$ is RF. In Section 2 we show that the class of IAS groups is closed under taking direct products (see Corollary 2.5). In Section 3 we give examples of IAS groups. In particular, we show that Grossmanian groups are IAS (see Corollary 3.3) and that virtually polycyclic groups are IAS (see Lemma 3.5).

Let Γ be a graph. We say that a vertex $v \in V\Gamma$ is *central* in Γ if $\{u, v\} \in E\Gamma$ for all $u \in V\Gamma \setminus \{v\}$, i.e. v is central if it is adjacent to all the vertices of Γ (apart from itself).

In Section 5 we study pointwise inner automorphisms of graph products and we prove the following theorem.

Theorem 1.1. *Let Γ be a finite graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial groups. Then the group $\Gamma\mathcal{G}$ has Grossman's property (A).*

As a consequence, we give the following characterisation of graph products with Grossman's property (A).

Corollary 1.2. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial groups. The group $G = \Gamma\mathcal{G}$ has Grossman's property (A) if and only if all vertex groups corresponding to central vertices of Γ have Grossman's property (A).*

In Section 6 we study separability of conjugacy classes in graph products of residually finite groups and using Theorem 1.1 we prove the following.

Theorem 1.3. *Let Γ be a finite graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated RF groups. Then the group $\Gamma\mathcal{G}$ is IAS and, consequently,* $\mathrm{Out}(\Gamma\mathcal{G})$ *is RF.*

Note that this theorem generalises the result of Minasyan and Osin in [14] on residual finiteness of outer automorphism groups of free products of finitely generated RF groups.

Combining Theorem 1.3 with Corolary 2.5 we obtain the following.

Corollary 1.4. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be family of non-trivial finitely generated RF groups. Assume that G_v is IAS whenever v is central in Γ. Then the group $\Gamma\mathcal{G}$ is IAS and, consequently,* $\mathrm{Out}(\Gamma\mathcal{G})$ *is RF.*

Next, combining Corollary 1.4 with Lemma 3.5 (virtually polycyclic groups are IAS) we get the following.

Corollary 1.5. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of virtually polycyclic groups. Then the group $\Gamma\mathcal{G}$ is IAS and* $\mathrm{Out}(\Gamma\mathcal{G})$ *is RF.*

Suppose that $p \in \mathbb{N}$ is a prime number. In Section 7 we prove a p-analogue to Theorem 1.3:

Theorem 1.6. *Let Γ be a finite graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually p-finite groups. Then the group $\Gamma\mathcal{G}$ is p-IAS, $\mathrm{Out}_p(\Gamma\mathcal{G})$ is residually p-finite and $\mathrm{Out}(\Gamma\mathcal{G})$ is virtually residually p-finite.*

See Section 7 for definitions of residually p-finite groups, p-IAS groups and Out_p. As a matter of fact, we show that the class of finitely generated residually p-finite p-IAS groups is closed under direct products (See Lemma 7.3) and using that we prove a p-analogue to Corollary 1.4:

Corollary 1.7. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually p-finite groups. Assume that G_v is p-IAS whenever v is central in Γ. Then the group $\Gamma\mathcal{G}$ is p-IAS and, consequently, $\mathrm{Out}_p(\Gamma\mathcal{G})$ is residually p-finite and $\mathrm{Out}(\Gamma\mathcal{G})$ is virtually residually p-finite.*

Let G be a group. The *Torelli group* of G, $\mathrm{Tor}(G) \leq \mathrm{Out}(G)$, consists of all outer automorphisms of G that act trivially on the abelianisation of G; see Section 7 for the formal definition of $\mathrm{Tor}(G)$. Finally, we use Theorem 1.6 to establish bi-orderability for Torelli groups of certain graph products of residually torsion-free nilpotent groups.

Theorem 1.8. *Let Γ be a finite graph without central vertices Γ and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial residually torsion-free nilpotent groups. Then $\mathrm{Tor}(G)$ is residually p-finite for every prime number p and is bi-orderable.*

2. Direct products of $\mathcal{C}$-IAS groups and Baumslag's method

Let G be a group and suppose that $H \leq G$; we will use $H \leq_{\text{f.i.}} G$ to denote that $|G : H| < \infty$. Similarly, we will use $N \trianglelefteq_{\text{f.i.}} G$ to denote that $N \trianglelefteq G$ and $|G : N| < \infty$.

2.1. **Pro-$\mathcal{C}$ topologies on groups.** Let G be a group and let $\mathcal{C}$ be a class of finite groups. If $F \in \mathcal{C}$ then we say that F is a *$\mathcal{C}$-group*. We say that $N \trianglelefteq G$ is a *co-$\mathcal{C}$* subgroup of G if $G/N \in \mathcal{C}$ and we say that G/N is a *$\mathcal{C}$-quotient* of G. We will use $\mathcal{N}_{\mathcal{C}}(G) = \{N \trianglelefteq G \mid G/N \in \mathcal{C}\}$ to denote the set of co-$\mathcal{C}$ subgroups of G. In this paper we will always assume that the class $\mathcal{C}$ satisfies the following closure properties:

(c1) subgroups: let $G \in \mathcal{C}$ and $H \leq G$; then $H \in \mathcal{C}$,
(c2) finite direct products: let $G_1, G_2 \in \mathcal{C}$; then $G_1 \times G_2 \in \mathcal{C}$.

In this case one can easily check that for every group G the system of subsets $\mathcal{B}_{\mathcal{C}} = \{gN \mid g \in G, N \in \mathcal{N}_{\mathcal{C}}(G)\} \subseteq \mathcal{P}(G)$ forms a basis of open sets for a topology on G. This topology is called the *pro-$\mathcal{C}$ topology* on G and we will use pro-$\mathcal{C}(G)$ when referring to it. If $\mathcal{C}$ is the class of all finite groups then the corresponding group topology is called the *profinite topology* on G and is denoted $\mathcal{PT}(G)$. If $\mathcal{C}$ is the class of all finite p-groups, where p is a prime number, then the corresponding group topology is referred to as *pro-p* topology on G and is denoted as pro-$p(G)$.

We say that a subset $X \subseteq G$ is *$\mathcal{C}$-closed* or *$\mathcal{C}$-separable* in G if it is closed in pro-$\mathcal{C}(G)$; $\mathcal{C}$-open subsets of G are defined analogically. One can show that if the class $\mathcal{C}$ satisfies (c1) and (c2) then, equipping a group with its pro-$\mathcal{C}$ topology, is actually a faithful functor from the category of groups to the category of topological groups, i.e. group homomorphisms are continuous with respect to corresponding pro-$\mathcal{C}$ topologies and group isomorphisms are homeomorphisms.

We say that a group G is residually-$\mathcal{C}$ if for every $g \in G \setminus \{1\}$ there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $g \notin N$. One can easily check that for a group G the following are equivalent:

- G is residually-$\mathcal{C}$,
- $\{1\}$ is $\mathcal{C}$-closed in G,
- $\bigcap_{N\in\mathcal{N}_{\mathcal{C}}(G)} N = \{1\}$,
- pro-$\mathcal{C}(G)$ is Hausdorff.

2.2. **$\mathcal{C}$-IAS groups.** We say that a group G is *$\mathcal{C}$-inner automorphism separable* ($\mathcal{C}$-IAS) if for every $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ there is $K \in \mathcal{N}_{\mathcal{C}}(G)$ characteristic in G such that for the homomorphism $\tilde{\kappa}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/K)$ given by

$$\tilde{\kappa}(\psi)(gK) = \psi(g)K$$

for every $\psi \in \mathrm{Aut}(G)$ and $g \in G$ we have $\tilde{\kappa}(\phi) \notin \mathrm{Inn}(G/K)$. In other words, group G is $\mathcal{C}$-IAS if every outer automorphism of G can be realised as non-trivial outer automorphism of some $\mathcal{C}$-quotient of G.

The main idea of Baumslag's elegant proof presented in [2], which was later used by Grossman in [8], was to use the fact that if G is a finitely generated group then for every $K \leq_{\text{f.i.}} G$ there is $L \leq_{\text{f.i.}} G$ such that $L \leq K$ and L is characteristic in G. The main goal of this section is to adapt Baumslag's method to prove the following.

Proposition 2.1. *Let $\mathcal{C}$ be a class of finite groups satisfying (c1) and (c2). Let A, B be finitely generated $\mathcal{C}$-IAS residually-$\mathcal{C}$ groups. Then the group $A \times B$ is $\mathcal{C}$-IAS.*

For a group G we will use $\mathrm{End}(G)$ to denote the set of all endomorphisms of G. Similarly, for groups A, B we will use $\mathrm{Hom}(A, B)$ denote the set of all homomorphisms from A to B. Note that $\mathrm{Hom}(A, A) = \mathrm{End}(A)$ for every group A.

Now let A, B be groups and let $\phi \in \mathrm{End}(A \times B)$ be arbitrary. Set $\phi_A = \phi \restriction_{A\times\{1\}}$ and $\phi_B = \phi \restriction_{\{1\}\times B}$. Obviously, for $a \in A, b \in B$ we have $\phi((a, b)) = \phi_A((a, 1))\phi_B((1, b))$. It is easy to see that there are uniquely given $\alpha \in \mathrm{End}(A)$ and $\gamma \in \mathrm{Hom}(A, B)$ such that $\phi_A((a, 1)) = (\alpha(a), \gamma(a))$. Similarly, there are uniquely given $\delta \in \mathrm{Hom}(B, A)$ and $\beta \in \mathrm{End}((1, B))$ such that $\phi_B((1, b)) = (\delta(b), \beta(b))$. We sum up this simple observation in the following simple remark, which will be crucial for proving Proposition 2.1.

Remark 2.2. *Let A, B be groups. For every $\phi \in \mathrm{End}(A \times B)$ there are uniquely given $\alpha \in \mathrm{End}(A)$, $\gamma \in \mathrm{Hom}(A, B)$ and $\beta \in \mathrm{End}(B)$, $\delta \in \mathrm{Hom}(B, A)$ such that $\phi((a, b)) = (\alpha(a)\delta(b), \gamma(a)\beta(b))$ for all $a \in A$ and $b \in B$.*

Let A, B be groups, suppose that $K_A \trianglelefteq A$, $K_B \trianglelefteq B$ and let $\psi_A\colon A \to A/K_A$, $\psi_B\colon B \to B/K_B$ be the corresponding natural projections. Clearly, the map

$$\tilde{\psi}_{A,B}\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A/K_A, B/K_B)$$

given by

$$\tilde{\psi}_{A,B}(\phi)(aK_A) = \phi(a)K_B,$$

for all $a \in A$ and $\phi \in \mathrm{Hom}(A, B)$ is well defined if and only if $\phi(K_A) \subseteq K_B$ for every $\phi \in \mathrm{Hom}(A, B)$, or equivalently, if $K_A \subseteq \phi^{-1}(K_B)$ for every $\phi \in \mathrm{Hom}(A, B)$. We use this observation together with Remark 2.2 to adapt Baumslag's method to direct products.

Lemma 2.3. *Let A, B be finitely generated groups and let $K_A \in \mathcal{N}_{\mathcal{C}}(A)$, $K_B \in \mathcal{N}_{\mathcal{C}}(B)$ be arbitrary. Then there are $L_A \in \mathcal{N}_{\mathcal{C}}(A)$, $L_B \in \mathcal{N}_{\mathcal{C}}(B)$ such that all of the following hold:*

(1) *$L_A \leq K_A$ and $L_B \leq K_B$,*
(2) *L_A is fully characteristic in A,*
(3) *L_B is fully characteristic in B,*
(4) *$L_A \subseteq \gamma^{-1}(L_B)$ for all $\gamma \in \mathrm{Hom}(A, B)$,*
(5) *$L_B \subseteq \delta^{-1}(L_A)$ for every $\delta \in \mathrm{Hom}(B, A)$*
(6) *$L_A \times L_B$ is fully characteristic in $A \times B$.*

Proof. Set $k = \max\{|A : K_A|, |B : K_B|\}$ and denote

$$\mathcal{L}_A = \{M \in \mathcal{N}_{\mathcal{C}}(A) \mid |A : M| \leq k\},$$
$$\mathcal{L}_B = \{N \in \mathcal{N}_{\mathcal{C}}(B) \mid |B : N| \leq k\}.$$

As A is finitely generated, we see that for every $n \in \mathbb{N}$ there are only finitely many $H_A \leq A$ such that $|A : H_A| = n$, hence we see that $\mathcal{L}_A$ is a finite subset of $\mathcal{N}_{\mathcal{C}}(A)$. By a similar argument we see that $\mathcal{L}_B$ is a finite subset of $\mathcal{N}_{\mathcal{C}}(B)$. Now set $L_A = \bigcap_{M \in \mathcal{L}_A} M$ and $L_B = \bigcap_{N \in \mathcal{L}_B} N$. As $\mathcal{L}_A$ is a finite subset of $\mathcal{N}_{\mathcal{C}}(A)$ we see that $L_A \in \mathcal{N}_{\mathcal{C}}(A)$ and by an analogous argument we see that $L_B \in \mathcal{N}_{\mathcal{C}}(B)$.

Let $\alpha_0 \in \mathrm{Hom}(A, A)$ be arbitrary. Note that $\alpha_0^{-1}(M) \in \mathcal{N}_{\mathcal{C}}(A)$ and $|A : \alpha_0^{-1}(M)| \leq |A : M|$ for every $M \in \mathcal{N}_{\mathcal{C}}(A)$. Thus if $M \in \mathcal{L}_A$ then $\alpha_0^{-1}(M) \in \mathcal{L}_A$. We see that

$$\alpha_0^{-1}(L_A) = \alpha_0^{-1}\left(\bigcap_{M \in \mathcal{L}_A} M\right) = \bigcap_{M \in \mathcal{L}_A} \alpha_0^{-1}(M) \supseteq \bigcap_{M \in \mathcal{L}_A} M = L_A.$$

and thus $L_A \subseteq \alpha_0^{-1}(L_A)$ for every $\alpha_0 \in \mathrm{Hom}(A, A)$, i.e. L_A is fully characteristic in A.

Similarly, for $\gamma_0 \in \mathrm{Hom}(A, B)$ we have $\gamma_0^{-1}(N) \in \mathcal{N}_{\mathcal{C}}(A)$ and $|A \colon \gamma_0^{-1}(N)| \leq |B : N|$ for every $N \in \mathcal{N}_{\mathcal{C}}(B)$ and thus if $N \in \mathcal{L}_B$ then $\gamma_0^{-1}(N) \in \mathcal{L}_A$. We see that

$$\gamma_0^{-1}(L_B) = \gamma_0^{-1}\left(\bigcap_{N \in \mathcal{L}_B} N\right) = \bigcap_{N \in \mathcal{L}_B} \gamma_0^{-1}(N) \supseteq \bigcap_{M \in \mathcal{L}_A} M = L_A$$

and thus $L_A \subseteq \gamma_0^{-1}(L_B)$ for every $\gamma_0 \in \mathrm{Hom}(A, B)$.

Using analogous arguments one can easily check that $L_B \subseteq \beta_0^{-1}(L_B) \in \mathrm{Hom}(B, B)$, i.e. L_B is fully characteristic in B, and $L_B \subseteq \delta_0^{-1}(L_A)$ for every $\delta_0 \in \mathrm{Hom}(B, A)$.

Now, let $\phi \in \mathrm{End}(A \times B)$ be arbitrary. Following Remark 2.2 we see that there are uniquely given $\alpha \in \mathrm{Hom}(A, A)$, $\beta \in \mathrm{Hom}(B, B)$, $\gamma \in \mathrm{Hom}(A, B)$ and $\delta \in \mathrm{Hom}(A, B)$ such that $\phi((a, b)) = (\alpha(a)\delta(b), \beta(b)\gamma(a))$ for all $a \in A$, $b \in B$. Note that $\alpha(L_A) \leq L_A$, $\beta(L_B) \leq L_B$, $\gamma(L_A) \leq L_B$ and $\delta(L_B) \leq L_A$. We see that

$$\phi(L_A \times L_B) \subseteq \alpha(L_A)\delta(L_B) \times \beta(L_B)\gamma(L_A) \subseteq L_A \times L_B$$

and hence $L_A \times L_B$ is fully characteristic in $A \times B$. $\square$

We say that a homomorphism $\alpha\colon A \to B$, where A, B are groups, is *trivial* if $\alpha(a) = 1$ for all $a \in A$. Before we proceed to the proof of Proposition 2.1 we state one simple observation.

Remark 2.4. *Let A, B be groups and let $\phi \in \mathrm{Aut}(A \times B)$. If $\phi \in \mathrm{Inn}(A \times B)$ then $\phi(A) \subseteq A$ and $\phi(B) \subseteq B$.*

Now we are ready to prove Proposition 2.1

Proof. Let $\phi \in \mathrm{Aut}(A \times B)$ be arbitrary such that $\phi \notin \mathrm{Inn}(A \times B)$. Following Remark 2.4 we see that there are two disjoint cases:

(i) either $\phi(A) \nsubseteq A$ or $\phi(B) \nsubseteq B$,
(ii) $\phi(A) \subseteq A$ and $\phi(B) \subseteq B$.

Following Remark 2.2 we see that there are $\alpha \in \mathrm{End}(A)$, $\delta \in \mathrm{Hom}(B, A)$, $\gamma \in \mathrm{Hom}(A, B)$ and $\beta \in \mathrm{End}(B)$ such that $\phi(a, b) = (\alpha(a)\delta(b), \gamma(a)\beta(b))$ for all $a \in A$, $b \in B$.

Suppose that (i) is the case. This means that either γ is non-trivial or δ is non-trivial. Without loss of generality we may assume that δ is non-trivial, i.e. there is $b_0 \in B \setminus \{1\}$ such that $\delta(b_0) \in A \setminus \{1\}$. As both A, B are residually-$\mathcal{C}$ there are $K_A \in \mathcal{N}_{\mathcal{C}}(A)$ and $K_B \in \mathcal{N}_{\mathcal{C}}(B)$ such that $\delta(b_0) \notin K_A$ and $b_0 \notin K_B$. By Lemma 2.3 we see that there are $L_A \in \mathcal{N}_{\mathcal{C}}(A)$ and $L_B \in \mathcal{N}_{\mathcal{C}}(B)$ such that $L_A \leq K_A$, $L_B \leq K_B$, L_A is fully characteristic in A, L_B is fully characteristic in B, $L_A \subseteq \gamma^{-1}(L_B)$ for all $\gamma \in \mathrm{Hom}(A, B)$, $L_B \in \delta^{-1}(L_A)$ for every $\delta \in \mathrm{Hom}(B, A)$ and $L_A \times L_B$ is fully characteristic in $A \times B$. We see that the natural projections $\psi_A \colon A \to A/L_A$, $\psi_B \colon B \to B/L_B$ induce maps

$$\begin{aligned}
\tilde{\psi}_A \colon \mathrm{Hom}(A, A) &\to \mathrm{Hom}(A/L_A, A/L_A),\\
\tilde{\psi}_B \colon \mathrm{Hom}(B, B) &\to \mathrm{Hom}(B/L_B, B/L_B),\\
\tilde{\psi}_{A,B} \colon \mathrm{Hom}(A, B) &\to \mathrm{Hom}(A/L_A, B/L_B),\\
\tilde{\psi}_{B,A} \colon \mathrm{Hom}(B, A) &\to \mathrm{Hom}(B/L_B, A/L_A).
\end{aligned}$$

Let $\psi \colon A \times B \to (A \times B)/(L_A \times L_B) = A/L_A \times B/B_L$ be the natural projection. Clearly, for the induced homomorphism $\tilde{\psi} \colon \mathrm{Aut}(A \times B) \to \mathrm{Aut}(A/L_A \times B/L_B)$ we have

$$\begin{aligned}
\tilde{\psi}(\phi)(aL_A, bL_B) &= \Big(\tilde{\psi}_A(\alpha)(aL_A)\tilde{\psi}_{B,A}(\delta)(bL_B), \tilde{\psi}_{A,B}(\gamma)(aL_A)\tilde{\psi}_B(\beta)(bL_B)\Big)\\
&= (\alpha(a)\delta(b)L_A, \gamma(a)\beta(b)L_B)
\end{aligned}$$

for all $a \in A$, $b \in B$. Note that $b_0 \notin L_B$ and $\delta(b_0) \notin L_A$, thus $b_0 L_B$ is not the identity in B/L_B and $\delta(b_0)L_A$ is not the identity in A/L_A. As $\tilde{\psi}_{B,A}(\delta)(b_0 L_B) = \delta(b_0)L_A$ we see that $\tilde{\psi}_{B,A}(\delta)$ is not trivial and consequently $\tilde{\psi}(\phi)(B/L_B) \nsubseteq B/L_B$. This means that $\tilde{\psi}(\phi) \notin \mathrm{Inn}(A/L_A \times B/L_B)$.

Suppose that (ii) is the case. This means that δ, γ are trivial and either $\alpha \in \mathrm{Aut}(A) \setminus \mathrm{Inn}(A)$ or $\beta \in \mathrm{Aut}(B) \setminus \mathrm{Inn}(B)$. Without loss of generality we may assume that $\alpha \notin \mathrm{Inn}(A)$. Since A is $\mathcal{C}$-IAS by assumption we see that there is $K_A \in \mathcal{N}_{\mathcal{C}}(A)$ characteristic in A such that for the induced homomorphism $\tilde{\kappa} \colon \mathrm{Aut}(A) \to \mathrm{Aut}(A/L_A)$ we have $\tilde{\kappa}(\alpha) \notin \mathrm{Inn}(A/K_A)$. Note that $B \in \mathcal{N}_{\mathcal{C}}(B)$, thus we can set $K_B = B$ and use Lemma 2.3 to obtain $L_A \in \mathcal{N}_{\mathcal{C}}(A)$ and $L_B \in \mathcal{N}_{\mathcal{C}}(B)$ with the desired properties. We see that for the homomorphism $\tilde{\psi} \colon \mathrm{Aut}(A \times B) \to \mathrm{Aut}(A/L_A \times B/L_B)$ induced by the natural projection $\psi \colon A \times B \to A/L_A \times B/L_B$ we have $\tilde{\psi}(\phi)(aL_A, bL_B) = (\alpha(a)L_A, \beta(b)L_B)$ for all $a \in A$, $b \in B$. Since $L_A \leq K_A$ we see that $\tilde{\psi}_A(\alpha) \notin \mathrm{Inn}(A/L_A)$ and thus $\tilde{\psi}(\phi) \notin \mathrm{Inn}(A/L_A \times B/L_B)$.

In each case we were able to realise the automorphism $\phi \in \mathrm{Aut}(A \times B) \setminus \mathrm{Inn}(A \times B)$ as non-inner automorphism of a $\mathcal{C}$-quotient of $A \times B$ and hence we see that the group $A \times B$ is $\mathcal{C}$-IAS. □

Applying Proposition 2.1 to the class of all finite groups we get the following corollary.

Corollary 2.5. *Let A, B be finitely generated groups RF groups and suppose that both A and B are IAS. Then $A \times B$ is IAS and, consequently,* $\mathrm{Out}(A \times B)$ *is RF.*

3. Examples of $\mathcal{C}$-IAS groups and Grossmans method

Let G be a group and suppose that $H \leq G$. For $g \in G$ we will use g^H to denote $\{hgh^{-1} \mid h \in H\}$, the H-conjugacy class of g. For $f, g \in G$ we will use $f \sim_H g$ to denote that $f \in g^H$. Suppose that $f \not\sim_G g$. We say that the pair (f, g) is *$\mathcal{C}$-conjugacy distinguishable* ($\mathcal{C}$-CD) in G if there is a group $F \in \mathcal{C}$ and a homomorphism $\phi \colon G \to F$ such that $\phi(f) \not\sim_F \phi(g)$. Equivalently, the pair (f, g) is $\mathcal{C}$-CD in G if there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $f^G N \cap gN = \emptyset$ in G. Clearly, the group G is $\mathcal{C}$-CS if for all $f, g \in G$ the pair (f, g) is $\mathcal{C}$-CD whenever $f \not\sim_G g$. It is easy to see that the conjugacy class g^G is $\mathcal{C}$-separable in G if the pair (f, g) is $\mathcal{C}$-CD for every $f \in G \setminus g^G$.

To simplify our proofs we will often use the following remark.

Remark 3.1. *Let G be a group and let $f, g \in G$ such that $f \not\sim_G g$. The pair (f, g) is $\mathcal{C}$-CD if and only if there is a group F and a homomorphism $\phi \colon G \to F$ such that $\phi(f) \not\sim_F \phi(g)$ and the pair $(\phi(f), \phi(g))$ is $\mathcal{C}$-CD in F.*

The following lemma uses an adaptation of the method that Grossman used to prove [8, Theorem 1].

Lemma 3.2. *Let G be a finitely generated group and assume that for every $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ there is an element $g \in G$ such that $\phi(g) \not\sim_G g$ and the pair $(\phi(g), g)$ is $\mathcal{C}$-CD in G. Then the group G is $\mathcal{C}$-IAS.*

Proof. Take any $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$. By assumption, there is $g \in G$ such that $\phi(g) \not\sim_G g$ and the pair $(\phi(g), g)$ is $\mathcal{C}$-CD. There is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $\phi(g)N \cap g^G N = \emptyset$. Set

$$K = \bigcap_{\varphi \in \mathrm{Aut}(G)} \varphi^{-1}(N).$$

Obviously, K is characteristic in G. Also, $|G : \varphi^{-1}(N)| \leq |G : N|$ for every $\varphi \in \mathrm{Aut}(G)$. As G is finitely generated we see that K is actually an intersection of finitely many co-$\mathcal{C}$ subgroups of G and thus $K \in \mathcal{N}_{\mathcal{C}}(G)$. Let $\kappa \colon G \to G/K$ be the natural projection and let $\tilde{\kappa} \colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/K)$ be the induced homomorphism. As $K \leq N$ we see that $\phi(g)K \cap g^G K = \emptyset$. This means that $\tilde{\kappa}(\phi)(gK) = \phi(g)K \not\sim_{G/K} gK$ and thus $\tilde{\kappa}(\phi) \notin \mathrm{Inn}(G/K)$. We see that G is $\mathcal{C}$-IAS. □

We say that a group G is *$\mathcal{C}$-Grossmanian* if G is finitely generated, $\mathcal{C}$-CS group with Grossman's property (A).

Corollary 3.3. *If G is a $\mathcal{C}$-Grossmanian group then G is $\mathcal{C}$-IAS.*

Proof. Let $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ be arbitrary. As G has Grossman's property (A) we see that there is $g \in G$ such that $\phi(g) \not\sim_G g$. As G is $\mathcal{C}$-CS we see that the pair $(\phi(g), g)$ is $\mathcal{C}$-CD in G. The group G is $\mathcal{C}$-IAS by Lemma 3.2. □

As mentioned in the introduction, applying Corollary 3.3 to the class of all finite groups we see that Grossmanian groups are IAS.

We say that a group G satisfies the *centraliser condition* (CC) if for every $g \in G$ and $K \trianglelefteq_{\text{f.i.}} G$ there is $L \trianglelefteq_{\text{f.i.}} G$ such that $L \leq K$ and

$$C_{G/L}(\psi(g)) \leq \psi\left(C_G(g)K\right) \text{ in } G/L$$

where $\psi \colon G \to G/L$ is the natural projection.

We say that a group G is *hereditarily conjugacy separable* (HCS) if G is CS and for every $H \leq_{\text{f.i.}} G$ we have that H is CS as well. The following theorem was proved by Minasyan in [13, Proposition 3.2].

Theorem 3.4. *Let G be a group. Then the following are equivalent:*

(a) *G is HCS;*
(b) *G is CS and satisfies CC.*

Recall that a group is IAS if it is $\mathcal{C}$-IAS in the case when $\mathcal{C}$ is the class of all finite groups. Before we proceed to utilise Minasyan's theorem to show that virtually polycyclic groups are IAS we will need one more definition: we say that a group G is *double coset separable* if for every pair of finitely generated subgroups $H, K \leq G$ and an arbitrary element $g \in G$ the subset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in $\mathcal{PT}(G)$. Virtually polycyclic groups are double coset separable by [15] and conjugacy separable by [6, 16]. As every subgroup of a virtually polycyclic group is a virtually polycyclic group we see that virtually polycyclic groups are actually HCS.

Lemma 3.5. *Virtually polycyclic groups are IAS.*

Proof. Let G be a virtually polycyclic group and let $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ be arbitrary.

If ϕ is not pointwise inner then there is $g \in G$ such that $\phi(g) \not\sim g$. As stated before, virtually polycyclic groups are CS, thus there is $N \trianglelefteq_{\text{f.i.}} G$ such that $\phi(g)N \cap g^G N = \emptyset$. As G is finitely generated we might without loss of generality assume that N is actually characteristic in G. Let $\nu \colon G \to G/N$ be the natural projection. Using the same argument as in the proof of Lemma 3.2 we see that for the induced homomorphism $\tilde{\nu} \colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$ we have $\tilde{\nu}(\phi) \in \mathrm{Aut}(G/N) \setminus \mathrm{Inn}(G/N)$.

So, we can further suppose that ϕ is pointwise inner. Let $\{g_1, \ldots, g_n\} \subseteq G$ be some generating set for G. By assumption for every $i \in \{1, \ldots, n\}$ there is $c_i \in G$ such that $\phi(g_i) = c_i g_i c_i^{-1}$. Clearly there is no $c \in G$ such that $c g_i c^{-1} = c_i g_i c_i^{-1}$ for all $i = 1, \ldots, n$ because otherwise the automorphism ϕ would be inner. Equivalently, ϕ is not inner if and only if

$$c_1 C_G(g_1) \cap \cdots \cap c_n C_G(g_n) = \emptyset \text{ in } G. \tag{1}$$

Set $\overline{G} = G^n$, where G^n is the n-fold direct product of G, and let $\overline{D} = \{(g, \ldots, g) \mid g \in G\} \leq \overline{G}$ be the diagonal subgroup of $\overline{G}$. Clearly, the condition (1) holds if and only if $\overline{c} \notin C_{\overline{G}}(\overline{g})\overline{D}$ in $\overline{G}$, where $\overline{g} = (g_1, \ldots, g_n) \in \overline{G}$ and $\overline{c} = (c_1, \ldots, c_n) \in \overline{G}$. Note that $\overline{G}$ is a virtually polycyclic group and thus it is double coset separable. Every subgroup of a virtually polycyclic subgroup is virtually polycyclic and thus it finitely generated. Hence $C_{\overline{G}}(\overline{g}) \leq \overline{G}$ is finitely generated. By double coset separability of $\overline{G}$ we see that there is $N \trianglelefteq \overline{G}$ such that $|\overline{G} : N| < \infty$ and $\overline{c}N \cap C_{\overline{G}}(\overline{g})\overline{D} = \emptyset$. Let $\iota_j : G \to \overline{G}$ be the injection of G onto the j-th coordinate group of $\overline{G}$ for $j = 1, \ldots, n$ and set

$$K = \iota_1^{-1}\left(\iota_1(G) \cap N\right) \cap \cdots \cap \iota_n^{-1}\left(\iota_n(G) \cap N\right) \leq G.$$

Let $\overline{K} = K^n \leq \overline{G}$ be the n-fold direct product of K. Note that $K \trianglelefteq_{\text{f.i.}} G$, $\overline{K} \trianglelefteq_{\text{f.i.}} \overline{G}$ and $\overline{K} \leq N$ thus $\overline{c}\overline{K} \cap C_{\overline{G}}(\overline{g})\overline{D} = \emptyset$. This is equivalent to

$$c_1 C_G(g_1)K \cap \cdots \cap c_n C_G(g_n)K = \emptyset \text{ in } G.$$

Virtually polycyclic groups are hereditarily conjugacy separable and thus, by Theorem 3.4, they satisfy CC. We see that for every $i \in \{1, \ldots, n\}$ there is $L_i \trianglelefteq_{\text{f.i.}} G$ such that $L_i \leq K$ and

$$C_{G/L_i}(\psi_i(g_i)) \subseteq \psi_i(C_G(g_i)K) \text{ in } G/L_i,$$

where $\psi_i \colon G \to G/L_i$ is the natural projection. As G is finitely generated we may without loss of generality assume that L_i is actually characteristic in G. Set $L = L_1 \cap \cdots \cap L_n$. Clearly, L is characteristic in G, $L \leq K$ and for every $i \in \{1, \ldots, n\}$ we have

$$C_{G/L}(\psi(g_i)) \subseteq \psi(C_G(g_i)K) \text{ in } G/L,$$

where $\psi \colon G \to G/L$ is the natural projection. We see that

$$\psi(c_1)C_{G/L}(\psi(g_1)) \cap \cdots \cap \psi(c_n)C_{G/L}(\psi(g_n)) \subseteq \psi(c_1 C_G(g_1)K) \cap \cdots \cap \psi(c_n C_G(g_n)K) \text{ in } G/L.$$

Suppose that there is some $c \in G$ such that

$$cL \in \psi(c_1 C_G(g_1)K) \cap \cdots \cap \psi(c_n C_G(g_n)K) \text{ in } G/L.$$

This means that

$$\begin{aligned} c \in & \psi^{-1}\left(\psi(c_1 C_G(g_1)K) \cap \cdots \cap \psi(c_n C_G(g_n)K)\right) = \\ & \psi^{-1}\left(\psi(c_1 C_G(g_1)K)\right) \cap \cdots \cap \psi^{-1}\left(\psi(c_n C_G(g_n)K)\right) = \\ & (c_1 C_G(g_1)K)L \cap \cdots \cap (c_n C_G(g_n)K)L = \\ & c_1 C_G(g_1)K \cap \cdots \cap c_n C_G(g_n)K = \emptyset \end{aligned}$$

which is a contradiction. Therefore

$$\psi(c_1)C_{G/L}(\psi(g_1)) \cap \cdots \cap \psi(c_n)C_{G/L}(\psi(g_n)) = \emptyset \text{ in } G/L. \tag{2}$$

It follows that for the induced homomorphism $\tilde{\psi} \colon \operatorname{Aut}(G) \to \operatorname{Aut}(G/L)$ we have $\tilde{\psi}(\phi) \notin \operatorname{Inn}(G/L)$. We see that G is IAS. $\square$

4. Properties of graph products of groups

In this section we will recall some basic theory of graph products that was introduced in [7] by Green and theory of cyclically reduced elements leading to conjugacy criterion for graph products of groups introduced in [10].

Let $G = \Gamma\mathcal{G}$ be a graph product. Every $g \in G$ can be obtained as a product $g = g_1 \ldots g_n$, where $g_i \in G_{v_i}$ for some $v_i \in V\Gamma$. However, this is not given uniquely. We say that a finite sequence $W \equiv (g_1, \ldots, g_n)$ is a *word* in $\Gamma\mathcal{G}$ if $g_i \in G_{v_i}$ for some $v_i \in V\Gamma$ for $i = 1, \ldots, n$. We say that g_i is a *syllable* of W and that the number n is the *length* of W.

We say that the word W *represents* $g \in G$ if $g = g_1 \dots g_n$. We can define the following three types of transformations on the word W:

(T1) remove a syllable g_i if $g_i = 1$,
(T2) remove two consecutive syllables g_i, g_{i+1} belonging to the same vertex group and replace them by a single syllable $g_i g_{i+1}$,
(T3) interchange consecutive syllables $g_i \in G_u$ and $g_{i+1} \in G_v$ if $\{u, v\} \in E\Gamma$.

Transformations of type (T3) are called *syllable shuffling.* Note that the transformations of types (T1) and (T2) reduce the length of W by 1, whereas (T3) preserves it. We say that word W is *reduced* if it is of minimal length, i.e. no sequence of transformations (T1) - (T3) will produce a word of shorter length. Obviously, if we start with a word W representing an element $g \in G$ then by applying finitely many of the above transformations we will rewrite W to a reduced word W' that represents the same element g. The following theorem was proved by Green [7, Theorem 3.9] in her Ph.D. thesis.

Theorem 4.1 (The normal form theorem)**.** *Every element $g \in \Gamma\mathcal{G}$ can be represented by a reduced word. Moreover, if two reduced words represent the same element of the group, then one can be obtained from the other by applying a finite sequence of syllable shuffling. In particular, the length of a reduced word is minimal among all words representing g, and a reduced word represents the identity if and only if it is the empty word.*

Thanks to Theorem 4.1 the following definitions make sense. Let g be an arbitrary element of G and let $W \equiv (g_1, \dots, g_n)$ be a reduced word representing g in G. We use $|g| = n$ to denote the *length* of g and we define the *support* of g to be

$$\operatorname{supp}(g) = \{v \in V\Gamma \mid \exists i \in \{1, \dots, n\} \text{ such that } g_i \in G_v\}.$$

We define $\mathrm{FL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there is a reduced word W that represents the element g and starts with a syllable from G_v. Similarly we define $\mathrm{LL}(g) \subseteq V\Gamma$ as the set of all $v \in V\Gamma$ such that there is a reduced word W that represents the element g and ends with a syllable from G_v. Note that $\mathrm{FL}(g) = \mathrm{LL}(g^{-1})$.

Every subset of vertices $X \subseteq V\Gamma$ induces a full subgraph Γ_X of the graph Γ. Let G_X be the subgroup of G generated by the vertex groups corresponding to the vertices contained in X. Subgroups of G that can be obtained in such way are called *full subgroups* of G; according to standard convention, $G_\emptyset = \{1\}$. Using the normal form theorem one can easily show that G_X is naturally isomorphic to the graph product of the family $\mathcal{G}_X = \{G_v \mid v \in X\}$ with respect to the full subgraph Γ_X. It is also easy to see that there is a canonical retraction $\rho_X \colon G \to G_X$ defined on the standard generators of G as follows:

$$\rho_X(g) = \begin{cases} g & \text{if } g \in G_v \text{ for some } v \in X, \\ 1 & \text{otherwise.} \end{cases}$$

We will often abuse the notation and sometimes consider the retraction ρ_X as a surjective homomorphism $\rho_X \colon G \to G_X$ and sometimes as an endomorphism $\rho_X \colon G \to G$. In that case writing $\rho_X \circ \rho_Y$, where $Y \subseteq V\Gamma$, makes sense.

Let $A, B \subseteq V\Gamma$ be arbitrary. Let $G_A, G_B \leq G$ be the corresponding full subgroups of G and let ρ_A, ρ_B be the corresponding retractions. One can easily check that ρ_A and ρ_B commute: $\rho_A \circ \rho_B = \rho_B \circ \rho_A$. It follows that $G_A \cap G_B = G_{A \cap B}$ and $\rho_A \circ \rho_B = \rho_{A \cap B}$.

For a vertex $v \in V\Gamma$ we will use $\mathrm{link}(v)$ to denote $\{u \in V\Gamma \mid \{u, v\} \in E\Gamma\}$, the set of vertices adjacent to v, and we will use $\mathrm{star}(v)$ to denote $\mathrm{link}(v) \cup \{v\}$. If $S \subseteq V\Gamma$ then $\mathrm{link}(S) = \cap_{v\in S} \mathrm{link}(v)$ and $\mathrm{star}(S) = \cap_{v\in S} \mathrm{star}(v)$.

We will use $N_G(H)$ to denote the normaliser of a subgroup H in a group G. The following remark is a special case of [1, Proposition 3.13].

Remark 4.2. *Let $v \in V\Gamma$. Then $N_G(G_v) = G_{\mathrm{star}(v)} = G_v G_{\mathrm{link}(v)} \simeq G_v \times G_{\mathrm{link}(v)}$.*

For $g \in G$ and $H \leq G$ we will use $C_H(g)$ to denote $\{c \in H \mid cg = gc\}$, the H-centraliser of g in G. The following remark is a special case of [10, Lemma 3.7].

Remark 4.3. *Let $v \in V\Gamma$ and let $a \in G_v \backslash \{1\}$ be arbitrary. Then $C_G(a) = C_{G_v}(a) G_{\mathrm{link}(A)} \simeq C_{G_v}(a) \times G_{\mathrm{link}(A)}$.*

Let $G = \Gamma\mathcal{G}$ be a graph product and let $g_1, \ldots, g_n \in G$ be arbitrary. We say that the element $g = g_1 \ldots g_n$ is a *reduced product* of $g_1, \ldots, g_n$ if $|g| = |g_1| + \cdots + |g_n|$.

Let $g \in G$. We define $\mathrm{S}(g) = \mathrm{supp}(g) \cap \mathrm{star}(\mathrm{supp}(g))$. We also define $\mathrm{P}(g) = \mathrm{supp}(g) \setminus \mathrm{S}(g)$. Obviously g uniquely factorises as a reduced product $g = \mathrm{s}(g)\,\mathrm{p}(g)$ where $\mathrm{supp}(\mathrm{s}(g)) = \mathrm{S}(g)$ and $\mathrm{supp}(\mathrm{p}(g)) = \mathrm{P}(g)$. We call this factorisation the *P-S decomposition of g*.

Let $g \in G$, let $W \equiv (g_1, \ldots, g_n)$ be a reduced expression for g. We say that a sequence $W' = (g_{j+1}, \ldots, g_n, g_1, \ldots, g_j)$, where $j \in \{1, \ldots, n-1\}$, is a *cyclic permutation* of W. We say that the element $g' \in G$ is a *cyclic permutation* of g if g' can be expressed by a cyclic permutation of some reduced expression for g.

Let $W \equiv (g_1, \ldots, g_n)$ be some reduced expression in G. We say that W is *cyclically reduced* if all cyclic permutations of W are reduced. The following lemma was proved in [10, Lemma 3.8].

Lemma 4.4. *Let $g \in G$ be arbitrary and let $W \equiv (g_1, \ldots g_n)$ be some reduced expression for g. If W is cyclically reduced then all reduced expressions representing g are cyclically reduced.*

Let $g \in G$ be arbitrary. We say that g is *cyclically reduced* if either g is trivial or some reduced word representing g is cyclically reduced. The following characterisation of cyclically reduced elements was given in [10, Lemma 3.11]

Lemma 4.5. *Let $g \in G$. Then the following are equivalent:*

(i) *g is cyclically reduced,*
(ii) $(\mathrm{FL}(g) \cap \mathrm{LL}(g)) \setminus \mathrm{S}(g) = \emptyset$,
(iii) $\mathrm{FL}(\mathrm{p}(g)) \cap \mathrm{LL}(\mathrm{p}(g)) = \emptyset$,
(iv) *$\mathrm{p}(g)$ is cyclically reduced.*

One of the consequences of Lemma 4.5 is the fact that for every $g \in G$ there is $g_0 \in G$ such that $g \sim_G g_0$ and g_0 is cyclically reduced. We will use this fact often without mentioning.

Conjugacy criterion for graph products of groups was proved in [10, Lemma 3.12].

Lemma 4.6 (Conjugacy criterion for graph products)**.** *Let x, y be cyclically reduced elements of $G = \Gamma\mathcal{G}$. Then $x \sim_G y$ if and only if the all of the following are true:*

(i) $|x| = |y|$ *and* $\mathrm{supp}(x) = \mathrm{supp}(y)$,

(ii) $\mathrm{p}(x)$ *is a cyclic permutation of* $\mathrm{p}(y)$,
(iii) $\mathrm{s}(y) \in \mathrm{s}(x)^{G_{\mathrm{S}(x)}}$.

5. Poitwise inner automorphisms of graph products

The aim of this section is to prove Theorem 1.1 and Corollary 1.2.

We will need the following technical lemma about conjugators of minimal length.

Lemma 5.1. *Let $u \in V\Gamma$, $a \in G_u \setminus \{1\}$ and let $\phi \in \mathrm{End}(G)$. Let $v \in V\Gamma$ and $b \in G_v \setminus \{1\}$. Suppose that $\phi(a) \in G_u \setminus \{1\}$, $\phi(b) \in G_v^G \setminus \{1\}$ and $\phi(ab) \sim_G ab$. Pick $b' \in G_v$ and $w \in G$ such that $\phi(b) = wb'w^{-1}$ and $|w|$ is minimal. Then $w \in N_G(G_u) = G_{\mathrm{star}(u)} = G_{\mathrm{link}(u)} G_u$.*

Proof. By assumption $\phi(a) = a'$, for some $a' \in G_u \setminus \{1\}$. Clearly w can be factorised as reduced product $w = xyz$, where $x \in N_G(G_u)$, $z \in N_G(G_v)$ and $\mathrm{LL}(y) \cap \mathrm{star}(v) = \emptyset = \mathrm{FL}(y) \cap \mathrm{star}(u)$. First we show that $z = 1$. Assume $z \in N_G(G_v) \setminus \{1\}$. Then we can set $b'' = zb'z^{-1}$ and $w' = xy$. Clearly, $\phi(b) = w'b''w'^{-1}$ and $|w'| < |w|$ which is a contradiction with our choice of b' and w. We see that $z = 1$.

Now we show that $y = 1$. Obviously

$$ab \sim_G \phi(ab) = a'xyb'y^{-1}x^{-1} \sim_G (x^{-1}a'x)yb'y^{-1}.$$

Denote $s = (x^{-1}a'x)yb'y^{-1}$.

It follows that $x^{-1}a'x \in G_u \setminus \{1\}$ and $b' \in G_v \setminus \{1\}$. Suppose that $y \neq 1$. Since $\mathrm{LL}(y) \cap \mathrm{star}(v) = \emptyset = \mathrm{FL}(y) \cap \mathrm{star}(u)$, we see that s is a reduced product of four non-trivial elements of G and thus $|s| \geq 4$.

Note that $\mathrm{FL}(s) = \{u\}$ and $\mathrm{LL}(s) = \mathrm{LL}(y^{-1}) = \mathrm{FL}(y)$ and thus s is cyclically reduced by Lemma 4.5 as $\mathrm{FL}(s) \cap \mathrm{LL}(s) = \emptyset$. Clearly $|s| > 2$, but also $s \sim_G ab$ and $|ab| \leq 2$ which is a contradiction with Lemma 4.6, thus $y = 1$ and consequently $w \in N_G(G_u)$. By Remark 4.2 we see that $N_G(G_u) = G_{\mathrm{star}(u)} = G_u G_{\mathrm{link}(u)}$. □

Corollary 5.2. *Let $\phi \in \mathrm{End}(G)$ be such that $\phi(g) \sim_G g$ for every $g \in G$ with $|g| = 1$. Suppose that there is $v \in V\Gamma$ and $a \in G_v \setminus \{1\}$ such that $\phi(a) \in G_v$. Then $\phi(G_v) \subseteq G_v$.*

Proof. Take arbitrary $b \in G_v \setminus \{1\}$. We see that $\phi(b) \sim_G b$ by assumption and thus $\phi(b) \neq 1$. Clearly $|ab| \leq 1$ and thus $\phi(ab) \sim_G ab$ as well. Pick $b' \in G_v \setminus \{1\}$ and $w \in G$ such that $\phi(b) = wb'w^{-1}$ and $|w|$ is minimal. By Lemma 5.1 we see that $w \in G_v G_{\mathrm{link}(v)} = N_G(G_v)$ and thus $\phi(b) \in G_v$. We see that $\phi(G_v) \subseteq G_v$. □

Corollary 5.2 tells us that for $v \in V\Gamma$ and $\phi \in \mathrm{End}(G)$, such that $\phi(g) \sim g$ for every $g \in G$ with $|g| = 1$, we have either $G_v \cap \phi(G_v) = \{1\}$ or $\phi(G_v) \subseteq G_v$. Therefore it makes sense to give the following definition. Let $G = \Gamma\mathcal{G}$ be a graph product and let $\phi \in \mathrm{End}(G)$, such that $\phi(g) \sim g$ for every $g \in G$ with $|g| = 1$. We say that a vertex $v \in V\Gamma$ is *stabilised* by ϕ if $\phi(G_v) \subseteq G_v$. We say that a subset $S \subseteq V\Gamma$ is *stabilised* by ϕ if every vertex $v \in S$ is stabilised by ϕ.

Lemma 5.1 together with Corollary 5.2 allows us to formulate the following corollary as an immediate consequence. Recall that for $A, B \subseteq V\Gamma$ we have $G_A \cap G_B = G_{A \cap B}$.

Corollary 5.3. *Let $\phi \in \mathrm{End}(G)$ such that $\phi(g) \sim_G g$ for every $g \in G$ with $|g| \leq 2$. Let $V_0 \subseteq V\Gamma$ be stabilised by ϕ and assume that $V_0 \neq \emptyset$. Let $b \in G_v \setminus \{1\}$ be arbitrary,* *for some* $v \in V\Gamma$. *Pick* $b' \in G_v \setminus \{1\}$ *and* $w \in G$ *such that* $\phi(b) = wb'w^{-1}$ *and* $|w|$ *is minimal. Then*

$$w \in \bigcap_{u \in V_0} N_G(G_u) = \bigcap_{u \in V_0} G_{\text{star}(u)} = G_S,$$

where $S = \cap_{v \in V_0} \text{star}(v)$.

Recall that a vertex $v \in V\Gamma$ is called *central* if $\text{link}(v) = V\Gamma \setminus \{v\}$. Clearly if $v \in V\Gamma$ is a central vertex then $G = G_v \times G_{V\Gamma\setminus\{v\}}$. Note that if $V\Gamma = \{v\}$ then v is central, hence if the graph Γ does not contain a central vertex then necessarily $|V\Gamma| \geq 2$.

Lemma 5.4. *Let* Γ *be a finite graph without central vertices, let* $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ *be a family of non-trivial groups and let* $G = \Gamma\mathcal{G}$ *be the graph product of* $\mathcal{G}$ *with respect to* Γ. *Let* $\phi_0 \in \text{Aut}(G)$ *and assume that* $\phi_0(g) \sim_G g$ *for all* $g \in G$ *such that* $|g| \leq 2$. *Then* $\phi_0 \in \text{Inn}(G)$.

Proof. Pick $\phi \in \text{Inn}(G)\phi_0$ such that the subset of vertices $V_0 \subseteq V\Gamma$ stabilised by ϕ is maximal. Evidently $V_0 \neq \emptyset$. Denote

$$N = \bigcap_{u \in V_0} N_G(G_u) = \bigcap_{u \in V_0} G_{\text{star}(u)} = G_S,$$

where $S = \bigcap_{u \in V_0} \text{star}(u)$. First we show that all vertices of Γ are stabilised by ϕ. Suppose that $V_0 \neq V\Gamma$. Take $v \in V \setminus V_0$ and let $b \in G_v \setminus \{1\}$ be arbitrary. Pick $b' \in G_v \setminus \{1\}$ and $w \in G$ such that $\phi(b) = wb'w^{-1}$ and $w \in G$ and $|w|$ is minimal. Note that $b \sim_G b'$. By Corollary 5.3 we see that $w \in N$. Let ϕ_w be the inner automorphism corresponding to w. Note that $\phi_w^{-1} \circ \phi \in \text{Inn}(G)\phi = \text{Inn}(G)\phi_0$. Clearly $\phi_w(G_u) = G_u$ for all $u \in V_0$ and thus $\left(\phi_w^{-1} \circ \phi\right)(G_u) \subseteq G_u$. Also we see that $\left(\phi_w^{-1} \circ \phi\right)(b) = b' \in G_v$ and thus by Corollary 5.2 we see that $\left(\phi_w^{-1} \circ \phi\right)(G_v) \subseteq G_v$ which is a contradiction as ϕ was chosen so that the set of stabilised vertices is maximal. Hence we see that $V_0 = V\Gamma$.

Note that since Γ does not contain central vertices we have that for every $v \in V\Gamma$ there is $u \in V\Gamma \setminus \{v\}$ such that $\{u, v\} \notin E\Gamma$. Let $v \in V\Gamma$ be arbitrary, take $u \in V \setminus \{v\}$, such that $\{u, v\} \notin E\Gamma$ and let $a \in G_u \setminus \{1\}$, $b \in G_v \setminus \{1\}$ be arbitrary. Clearly $\langle G_u, G_v \rangle = G_{\{u,v\}} \cong G_u * G_v$. Since $G_{\{u,v\}}$ is a retract and $\phi(a)\phi(b) \in G_uG_v \subseteq G_{\{u,v\}}$, we see that $ab \sim_{G_{\{u,v\}}} \phi(a)\phi(b)$. By the conjugacy criterion for free products [12, Theorem 4.2] we get that $\phi(a) = a$ and $\phi(b) = b$ thus $\phi \restriction_{G_v} = \text{id}_{G_v}$.

We see that $\phi \restriction_{G_v} = \text{id}_{G_v}$ for every $v \in V\Gamma$ and thus $\phi = \text{id}_G$. Consequently, $\phi_0 \in \text{Inn}(G)$. $\square$

Note that Lemma 5.4 immediately implies Theorem 1.1.

Proof of Theorem 1.1. Let $\phi \in \text{Aut}_{\text{PI}}(G)$ be arbitrary. Obviously, $\phi(g) \sim g$ for all $g \in G$ and hence $\phi \in \text{Inn}(G)$ by Lemma 5.4. We see that $\text{Aut}_{\text{PI}}(G) \subseteq \text{Inn}(G)$ and thus G has Grossman's property (A). $\square$

We leave the proof of the following lemma as a simple exercise for the reader.

Lemma 5.5. *Let* $G_1, \ldots, G_n$ *be groups. The group* $G = \Pi_{i=1}^n G_i$ *has Grossman's property (A) if and only if the group* G_i *has Grossman's property (A) for each* $i = 1, \ldots, n$.

Now we are ready to prove Corollary 1.2

Proof of Corollary 1.2. Let $C \subseteq V\Gamma$ denote the set of central vertices of the graph Γ. Note that the induced full subgraph $\Gamma_{V\Gamma\setminus C}$ does not contain central vertices, hence the group $G_{V\Gamma\setminus C}$ has Grossman's property (A) by Theorem 1.1. The group G splits as $G = G_{\Gamma V\setminus C} \times \prod_{v\in C} G_v$, a direct product of finitely many groups. By Lemma 5.5 we see that the group G has Grossman's property (A) if and only if G_v has Grossman's property (A) for every $v \in C$. □

Note that Corollary 1.2 does not hold for infinite graphs. Let Γ be a complete graph on countably infinitely many vertices and let $\{G_v \mid v \in V\Gamma\}$ be a family of groups such that $G_v \cong F_2$, where F_2 is the free group on two generators, for every $v \in V\Gamma$. We see that $G = \Gamma\mathcal{G}$ is isomorphic to $\prod_{n\in\mathbb{N}} F_2$. Let $w \in F_2 \setminus \{1\}$ be arbitrary and consider the automorphism $\phi_w \in \mathrm{Aut}(G)$ defined on the coordinates as follows:

$$\phi_w(f_1, f_2, f_3, \dots) = (wf_1w^{-1}, w^2f_2w^{-2}, w^3f_3w^{-3}, \dots).$$

It is obvious that $\phi_w \in \mathrm{Aut_{PI}}(G) \setminus \mathrm{Inn}(G)$, hence G does not have Grossman's property (A), but F_2 has Grossman's property (A) by [8, Lemma 1]. Note that the group G can be actually obtained as a RAAG corresponding to an infinite graph without central vertices.

In the rest of the section we prove three technical results about conjugacy in graph products of groups that will be useful in Section 6

Lemma 5.6. *Let $u, v \in V\Gamma$ be such that $\{u, v\} \in E\Gamma$ and let $a \in G_u \setminus \{1\}$, $b \in G_v \setminus \{1\}$ be arbitrary. Let $\phi_0 \in \mathrm{End}(G)$ and assume that $\phi_0(a) \sim_G a$ and $\phi_0(b) \sim_G b$. Then $\phi_0(ab) \sim_G ab$.*

Proof. Pick $\phi \in \mathrm{Inn}(G)\phi_0$ such that $\phi(a) = a$. Then $\phi(b) = cbc^{-1}$ for some $c \in G$. We see that $\phi(a)\phi(b) = \phi(b)\phi(a)$ and thus $cbc^{-1} \in C_G(a)$. By Lemma 4.3 we see that $C_G(a) = C_{G_u}(a)G_{\mathrm{link}(u)} \leq G_uG_{\mathrm{link}(u)}$. Note that $G_uG_{\mathrm{link}(u)} = G_{\{u\}\cup\mathrm{link}(u)}$ is a retract of G, let $\rho\colon G \to G_uG_{\mathrm{link}(u)}$ be the corresponding retraction. Since $v \in \mathrm{link}(u)$ we see that $\rho(b) = b$. We see that $cbc^{-1} = \rho(cbc^{-1}) = \rho(c)b\rho(c)^{-1}$. Set $c_1 = \rho(c)$. We see that $c_1^{-1}\phi(ab)c_1 = c_1^{-1}ac_1b$. Since $c_1 \in G_uG_{\mathrm{link}(u)}$ there is $c_2 \in G_u$ such that $c_1^{-1}ac_1 = c_2^{-1}ac_2$. As $c_2 \in G_u$ and $\{u, v\} \in E\Gamma$ we see that

$$c_1^{-1}\phi(ab)c_1 = c_1^{-1}ac_1b = c_2^{-1}ac_2b = c_2^{-1}abc_2.$$

It follows that $\phi_0(ab) \sim_G ab$. □

Corollary 5.7. *Let $\phi \in \mathrm{End}(G)$. Assume that $\phi(g) \sim_G g$ for every $g \in G$ such that g is cyclically reduced and $\mathrm{S}(g) = \emptyset$. Furthermore, suppose that $\phi(g) \sim_G g$ for all $g \in G$ such that $|g| = 1$ as well. Then $\phi(g) \sim_G g$ for all $g \in G$ such that $|g| \leq 2$.*

Proof. Let $g \in G$ be arbitrary such that $|g| = 2$. Clearly, $\mathrm{supp}(g) = \{u, v\}$ for some $u, v \in V\Gamma$ such that $u \neq v$. One can easily check that g is cyclically reduced using Lemma 4.5. Suppose that $\{u, v\} \notin E\Gamma$. Then $\mathrm{S}(g) = \emptyset$ and $\phi(g) \sim_G g$ by assumption.

Now suppose that $\{u, v\} \in E\Gamma$. Then $g = ab$ for some $a \in G_u \setminus \{1\}$, $b \in G_v \setminus \{1\}$. By assumption, $\phi(a) \sim_G a$ and $\phi(b) \sim_G b$ as $|a| = |b| = 1$. Then $\phi(ab) \sim_G ab$ by the previous lemma and we are done. □

Lemma 5.8. *Let $\phi_0 \in \mathrm{End}(G)$ and let $u, v \in V\Gamma$ be such that $\{u, v\} \notin E\Gamma$ and $u \neq v$. Let $a \in G_u \setminus \{1\}$ and $b \in G_v \setminus \{1\}$ be arbitrary and assume that $\phi_0(ab) \sim_G ab$, $\phi_0(a) \in G_u^G \setminus \{1\}$ and $\phi_0(b) \in G_v^G \setminus \{1\}$. Then $\phi_0(a) \sim_G a$ and $\phi_0(b) \sim_G b$.*

Proof. By assumption $\phi_0(a) = w_a a' w_a^{-1}$ for some $a' \in G_u \setminus \{1\}$. Set $\phi = \phi_{w_a}^{-1} \circ \phi_0$, where $\phi_{w_a} \in \mathrm{Inn}(G)$ is the inner automorphism of G corresponding to w_a. Clearly $\phi(ab) \sim_G ab$, $\phi(a) = a' \in G_u \setminus \{1\}$ and $\phi(b) \in G_v^G \setminus \{1\}$. Pick $b' \in G_v \setminus \{1\}$ and $w \in G$ such that $\phi(b) = wb'w^{-1}$ and $|w|$ is minimal. By Lemma 5.1 we see that $w \in G_{\mathrm{link}(u)} G_u = N_G(G_u)$. We have $ab \sim_G \phi(ab) = a'wb'w^{-1}$ and consequently $ab \sim_G w^{-1}a'wb'$. Note that $w^{-1}a'w \in G_u$ since $w \in N_G(G_u)$. Denote $a'' = w^{-1}a'w$. Let $\rho\colon G \to G_{\{u,v\}}$ be the canonical retraction corresponding to the set of vertices $\{u, v\}$. Clearly $\rho(ab) = ab$ and $\rho(a''b') = a''b'$ and $ab \sim_{G_{\{u,v\}}} a''b'$. Note that $G_{\{u,v\}} \cong G_u * G_v$ and thus by the conjugacy criterion for free products of groups (see [12, Theorem 4.2]) we see that $a'' = a$ and $b' = b$. It follows that $\phi_0(a) \sim_G a$ and $\phi_0(b) \sim b$. ☐

6. Conjugacy distinguishable pairs in graph products

We say that a class $\mathcal{C}$ is an *extension closed variety of finite groups* if the class $\mathcal{C}$ of finite groups is closed under taking subgroups, finite direct products, quotients and extensions. Obvious examples of extension closed varieties of finite groups are the following:

- the class of all finite groups;
- the class of all finite p-groups, where p is a prime number;
- the class of all finite solvable groups.

Unless stated otherwise (see Lemma 6.5 and Lemma 6.6), in this section we will assume that the class $\mathcal{C}$ is an extension closed variety of finite groups. This will allow us to use the following lemma which is a direct consequence of [10, Theorem 1.2]

Lemma 6.1. *Let $\mathcal{C}$ be an extension closed variety of finite groups and let $G = \Gamma\mathcal{G}$ be a graph product of $\mathcal{C}$-groups. Then the group G is $\mathcal{C}$-CS.*

The main result of this section is the following proposition.

Proposition 6.2. *Let Γ be a finite simplicial graph without central vertices and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually-$\mathcal{C}$ groups. Then the group $\Gamma\mathcal{G}$ is $\mathcal{C}$-IAS.*

To prove Proposition 6.2 we will give sufficient conditions for the pair (f, g) to be $\mathcal{C}$-CD in the graph product (see Lemma 6.7) and then use this description to show that if we have and automorphism ϕ such that $(g, \phi(g))$ is not $\mathcal{C}$-CD for all $g \in G$ then necessarily ϕ must be inner.

We will use the fact that graph products have functorial property.

Remark 6.3. *Let Γ be a simplicial graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$, $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ be two families of groups such that for every $v \in V\Gamma$ there is a homomorphism $\phi_v\colon G_v \to F_v$. Then there is unique group homomorphism $\phi\colon \Gamma\mathcal{G} \to \Gamma\mathcal{F}$ such that $\phi \restriction_{G_v} = \phi_v$ for every $v \in V\Gamma$.*

The following lemma is an easy consequence of [10, Lemma 7.2].

Lemma 6.4. *Let $\Gamma\mathcal{G}$ be a graph product of residually-$\mathcal{C}$ groups. Let $f, g \in G$ be cyclically reduced in G and assume that $f \neq g$. Then there is $\mathcal{F} = \{F_v | v \in V\Gamma\}$, a family of $\mathcal{C}$-groups indexed by $V\Gamma$, and a homomorphism $\phi_v : G_v \to F_v$, for every $v \in V\Gamma$, such that for the corresponding extension $\phi : G \to F$, where $F = \Gamma\mathcal{F}$, all of the following are true:*

(i) $|g| = |\phi(g)|$ *and* $\operatorname{supp}(g) = \operatorname{supp}(\phi(g))$,
(ii) $|f| = |\phi(f)|$ *and* $\operatorname{supp}(f) = \operatorname{supp}(\phi(f))$,
(iii) $\phi(f), \phi(g)$ *are cyclically reduced in* F,
(iv) $\phi(f) \neq \phi(g)$ *in* F.

We utilise Lemma 6.4 to show that conjugacy classes of certain pairs of elements of graph products of residually-$\mathcal{C}$ groups can be separated in a graph product of $\mathcal{C}$-groups.

Lemma 6.5. *Suppose that $\mathcal{C}$ is a class of finite groups closed under taking direct products and subgroups. Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of residually-$\mathcal{C}$ groups. Let $G = \Gamma\mathcal{G}$ and suppose that $f, g \in G$ are cyclically reduced elements of G such that $f \not\sim_G g$ and either* $\operatorname{supp}(f) \neq \operatorname{supp}(g)$ *or* $|f| \neq |g|$. *Then there is a family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ and a homomorphism $\phi : G \to F$, where $F = \Gamma\mathcal{F}$, such that $\phi(f) \not\sim_F \phi(g)$.*

Proof. By Lemma 6.4 there is a family $\mathcal{C}$-groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ such that for every $v \in V\Gamma$ there is a homomorphism $\phi_v : G_v \to F_v$, such that for the corresponding extension $\phi : G \to F = \Gamma\mathcal{F}$ we have $|g| = |\phi(g)|$, $\operatorname{supp}(g) = \operatorname{supp}(\phi(f))$, $|f| = |\phi(f)|$, $\operatorname{supp}(f) = \operatorname{supp}(\phi(f))$ and both $\phi(f)$, $\phi(g)$ are cyclically reduced. By Lemma 4.6 we see that $\phi(f) \not\sim_F \phi(g)$. $\square$

As it turns out, conjugacy classes of cyclically reduced elements with specific P-S decomposition can be always separated.

Lemma 6.6. *Suppose that $\mathcal{C}$ is a class of finite groups closed under taking direct products and subgroups. Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of residually-$\mathcal{C}$ groups. Let $G = \Gamma\mathcal{G}$ be a graph product of $\mathcal{G}$ with respect to Γ. Let $f \in G$ be cyclically reduced element of G such that* $\mathrm{S}(f) = \emptyset$. *Then for every $g \in G \setminus f^G$ there is a family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ and a homomorphism $\phi : G \to F$, where $F = \Gamma\mathcal{F}$, such that $\phi(f) \not\sim_F \phi(g)$.*

Proof. Let $g \in G \setminus f^G$ be arbitrary. Pick $g_0 \in G$ such that $g_0 \sim_G g$ and g_0 is cyclically reduced. Since $\mathrm{S}(f) = \emptyset$ we see that $\mathrm{p}(f) = f$. Combining Lemma 4.6 with the fact that $\mathrm{S}(f) = \emptyset$ we see that there are two possibilities to consider:

(i) $\operatorname{supp}(f) \neq \operatorname{supp}(g_0)$ or $|f| \neq |g_0|$,
(ii) $\mathrm{p}(g_0)$ is not a cyclic permutation of f.

We can use Lemma 6.5 do deal with case (i).

Assume that $\operatorname{supp}(f) = \operatorname{supp}(g_0)$, $|f| = |g_0|$ and that $\mathrm{p}(g_0)$ is not a cyclic permutation of f. Let $\{f_1, \dots, f_m\} \subset G$ be the set of all cyclic permutations of f (including f). We use Lemma 6.4 for each pair f_i, g_0, where $1 \leq i \leq m$, to obtain a family $\mathcal{C}$-groups $\mathcal{F}_i = \{F_v^i | v \in V\Gamma\}$ with homomorphisms $\phi_v^i : G_v \to F_v^i$ for all $v \in V\Gamma$. For every $v \in V\Gamma$ set $K_v = \bigcap_{i=1}^m \ker(\phi_v^i)$ and denote $F_v = G_v/K_v$. Note that as the class $\mathcal{C}$ is closed under taking subgroups and direct products the set $\mathcal{N}_{\mathcal{C}}(G_v)$ is closed under intersection for every $v \in V\Gamma$ (see [10, Lemma 2.1]) and thus $F_v \in \mathcal{C}$ for every $v \in V\Gamma$. Set

$\mathcal{F} = \{F_v | v \in V\Gamma\}$ and let $\phi_v \colon G_v \to F_v$ be the natural projection corresponding to v. Let $\phi \colon G \to \Gamma\mathcal{F}$ be the natural extension. Note that $\mathrm{p}(\phi(g_0)) = \phi(\mathrm{p}(g_0))$, $\mathrm{p}(\phi(f)) = \phi(\mathrm{p}(f))$ and $\phi(f)$, $\phi(g_0)$ are cyclically reduced in $\Gamma\mathcal{F}$. Clearly the set $C = \{\phi(f_1), \dots, \phi(f_m)\}$ is the set of all cyclic permutations of $\mathrm{p}(\phi(f))$ and we see that $\mathrm{p}(\phi(g_0)) \notin C$, hence $\mathrm{p}(\phi(g_0))$ is not a cyclic permutation of $\phi(f)$. By Lemma 4.6 we see that $\phi(g_0) \not\sim_{\Gamma\mathcal{F}} \phi(f)$ and thus $\phi(g) \not\sim_{\Gamma\mathcal{F}} \phi(f)$. $\square$

Combining Lemma 6.5 and Lemma 6.6 together with Lemma 6.1 we get the following description of $\mathcal{C}$-CD pairs in graph products.

Lemma 6.7. *Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of residually-$\mathcal{C}$ groups. Let $G = \Gamma\mathcal{G}$ be a graph product of $\mathcal{G}$ with respect to Γ. Let $g_1, g_2 \in G$ be cyclically reduced elements of G such that $g_1 \not\sim_G g_2$ and either $\mathrm{supp}(g_1) \neq \mathrm{supp}(g_2)$ or $|g_1| \neq |g_2|$. Then the pair (g_1, g_2) is $\mathcal{C}$-CD in G. Furthermore, if $f \in G$ is cyclically reduced with $\mathrm{S}(f) = \emptyset$ then the pair (f, g) is $\mathcal{C}$-CD for every $g \in G \setminus f^G$.*

Proof. If $g_1, g_2 \in G$ are cyclically reduced and either $\mathrm{supp}(g_1) \neq \mathrm{supp}(g_2)$ or $|g_1| \neq |g_2|$ then by Lemma 6.5 we see that there is a family of $\mathcal{C}$-groups $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ and a homomorphism $\gamma \colon G \to F = \Gamma\mathcal{F}$ such that $\gamma(g_1) \not\sim_F \gamma(g_2)$. By Lemma 6.1 we see that the group F is $\mathcal{C}$-CS and thus the pair $(\gamma(g_1), \gamma(g_2))$is $\mathcal{C}$-CD in F. Using Remark 3.1 we see that the pair (g_1, g_2) is $\mathcal{C}$-CD in G.

Similarly, if $f \in G$ is cyclically reduced with $\mathrm{S}(f) = \emptyset$ then by Lemma 6.6 there is a family $\mathcal{F} = \{F_v \mid v \in V\Gamma\}$ and a homomorphism $\gamma \colon G \to F = \Gamma\mathcal{F}$ such that $\gamma(f) \not\sim_F \gamma(g)$. Again, by Lemma 6.1 we see that the group F is $\mathcal{C}$-CS and thus the pair $(\gamma(f), \gamma(g))$ is $\mathcal{C}$-CD in F. Using Remark 3.1 we see that the pair (f, g) is $\mathcal{C}$-CD in G. $\square$

Now we are ready to prove Proposition 6.2

Proof of proposition 6.2. Our aim is to use Lemma 3.2, hence we want to show that for every $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(g)$ there is an element $g \in G$ such that $\phi(g) \not\sim_G g$ and the pair $(\phi(g), g)$ is $\mathcal{C}$-CD.

Let $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ be arbitrary and assume that for every $g \in G$ the pair $(\phi(g), g)$ is not $\mathcal{C}$-CD in G.

If $g \in G$ is cyclically reduced with $\mathrm{S}(g) = \emptyset$ then the pair (f, g) is $\mathcal{C}$-CD for every $f \in G \setminus g^G$ by Lemma 6.7. We see that we may assume that $\phi(g) \sim_G g$ for every cyclically reduced element $g \in G$ such that $\mathrm{S}(g) = \emptyset$. In particular $\phi(ab) \sim_G ab$, whenever $a \in G_u \setminus \{1\}$ and $b \in G_v \setminus \{1\}$ for some $u, v \in V\Gamma$ such that $\{u, v\} \notin E\Gamma$.

Let us analyse what happens to $g \in G$ with $|g| = 1$. Let $u \in V\Gamma$ and $a \in G_u \setminus \{1\}$ be arbitrary. Pick $h \in G$ such that $h \sim_G \phi(a)$ and h is cyclically reduced. There are three cases to consider:

(i) $1 < |h|$,
(ii) $|h| = 1$ and $\mathrm{supp}(h) \neq \{u\} = \mathrm{supp}(a)$,
(iii) $|h| = 1$ and $\mathrm{supp}(h) = \{u\}$.

Using Lemma 6.7 we see that if (i) or (ii) is the case then the pair (a, h) is $\mathcal{C}$-CD. This means that there is a group $C \in \mathcal{C}$ and a homomorphism $\gamma \colon G \to C$ such that $\gamma(a) \not\sim_C \gamma(h)$. Consequently $\gamma(\phi(a)) \not\sim_C \gamma(a)$ and the pair $(\phi(a), a)$ is $\mathcal{C}$-CD in G. We see that without loss of generality we may assume that $\phi(g) \in G_v^G$, whenever $g \in G_v \setminus \{1\}$

for some $v \in V\Gamma$, because otherwise the pair $(\phi(g), g)$ would be conjugacy distinguishable as we just demonstrated.

As Γ does not contain central vertices we know that for every $u \in V\Gamma$ there is $v \in V\Gamma \setminus \{u\}$ such that $\{u, v\} \notin E\Gamma$. Let $b \in G_v \setminus \{1\}$ be arbitrary. We see that $\phi(a) \in G_u^G$ and $\phi(b) \in G_v^G$. Clearly the element ab is cyclically reduced and $\mathrm{S}(ab) = \emptyset$, hence $\phi(ab) \sim_G ab$ by assumption. Then by Lemma 5.8 we see that $\phi(a) \sim_G a$ and $\phi(b) \sim_G b$. This means that we may assume that $\phi(g) \sim g$ for all $g \in G$ such that $|g| = 1$. Consequently, by Corollary 5.7 we see that $\phi(g) \sim_G g$ for all $g \in G$ such that $|g| \leq 2$. However, using Lemma 5.4 we see that $\phi \in \mathrm{Inn}(G)$, which is a contradiction with our original assumption that $\phi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$.

We see that our original assumption cannot be true, i.e. there must be an element $g \in G$ such that $\phi(g) \not\sim_G g$ and the pair $(\phi(g), g)$ is $\mathcal{C}$-CD in G. As G is a finite graph product of finitely generated groups it is finitely generated and thus by Lemma 3.2 we see that the group G is $\mathcal{C}$-IAS. □

Corollary 6.8. *Let Γ be a finite graph and let $\mathcal{G} = \{G_v \mid v \in V\Gamma\}$ be a family of non-trivial finitely generated residually-$\mathcal{C}$ groups such that the group G_v is $\mathcal{C}$-IAS whenever the vertex v is central in Γ. Then the group $G = \Gamma\mathcal{G}$ is $\mathcal{C}$-IAS.*

Proof. Let $C \subseteq V\Gamma$ denote the set of central vertices of graph Γ. Note that the induced full subgraph $\Gamma_{V\Gamma\setminus C}$ does not contain central vertices, hence the group $G_{V\Gamma\setminus C}$ is $\mathcal{C}$-IAS by Lemma 6.2. The group $G_{V\Gamma\setminus C}$ is residually-$\mathcal{C}$ by [10, Lemma 6.6]. The group G splits as $G = G_{\Gamma V\setminus C} \times \prod_{v\in C} G_v$, a direct product of finitely many finitely generated $\mathcal{C}$-IAS residually-$\mathcal{C}$ groups, and thus G is $\mathcal{C}$-IAS by Proposition 2.1. □

Applying Proposition 6.2 and Corollary 6.8 to the class of all finite groups we immediately obtain Theorem 1.3 and Corollary 1.4.

7. Graph products of residually-p groups

Let G be a group and let p be a prime number. Set $K_p = [G, G]G_p \leq G$, where G_p is the subgroup of G generated by all elements of the form g^p for $g \in G$. Note that K_p is characteristic in G and thus the natural projection $\pi \colon G \to G/K_p$ induces a homomorphism $\tilde{\pi} \colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/K_p)$ given by $\tilde{\pi}(\phi)(gK_p) = \phi(g)K_p$ for every $\phi \in \mathrm{Aut}(G)$. We will use $\mathrm{Aut}_p(G)$ to denote $\ker(\tilde{\pi})$, i.e. the automorphisms that act trivially on the first mod-p homology of G. Note that if G is finitely generated then G/K_p is actually the direct product of copies of C_p, the cyclic group of order p, and we see that G/K_p is a finite p-group and thus K_p is of finite index in G. Consequently, if G is finitely generated then $\mathrm{Aut}_p(G)$ is of finite index in $\mathrm{Aut}(G)$. Also since G/K_p is abelian we see that $\mathrm{Inn}(G) \leq \mathrm{Aut}_p(G)$ and thus $\mathrm{Out}_p(G) = \mathrm{Aut}_p(G)/\mathrm{Inn}(G) \leq \mathrm{Out}(G)$. Again, if G is finitely generated then $\mathrm{Out}_p(G)$ is actually of finite index in $\mathrm{Out}(G)$.

The following is a classical result of P. Hall (see [17, 5.3.2, 5.3.3]).

Lemma 7.1. *If G is a finite p-group, then $\mathrm{Aut}_p(G)$ is also a finite p-group.*

Recall that if $\mathcal{C}$ is the class of all finite p-groups, then the corresponding pro-$\mathcal{C}$ topology on a group G is referred to as the pro-p topology on G. We say that a subset $X \subseteq G$ is p-closed in G if it is closed in pro-$p(G)$. If group G is $\mathcal{C}$-IAS then we say that G is p-IAS, similarly for p-CS and p-Grossmanian groups.

Lemma 7.2. *Let G be a finitely generated p-IAS group. Then the group* $\mathrm{Out}_p(G)$ *is residually p-finite and, consequently, the group* $\mathrm{Out}(G)$ *is virtually residually p-finite.*

Proof. Let $\phi \in \mathrm{Aut}_p(G) \setminus \mathrm{Inn}(G)$ be arbitrary. By definition there is $N \in \mathcal{N}_p(G)$ characteristic in G such that the natural projection $\pi\colon G \to G/N$ induces a homomorphism $\tilde{\pi}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$ such that $\tilde{\pi}(\phi) \notin \mathrm{Inn}(G/N)$. By Lemma 7.1 we see that $\mathrm{Aut}_p(G/N)$ is a finite p-group. Note that $\tilde{\pi}(\mathrm{Aut}_p(G)) \leq \mathrm{Aut}_p(G/N)$ and thus $\tilde{\pi}(\phi) \in \mathrm{Aut}_p(G/N) \setminus \mathrm{Inn}(G/N)$, therefore $\mathrm{Inn}(G)$ is p-closed in $\mathrm{Aut}_p(G)$ and consequently $\mathrm{Out}_p(G)$ is residually p-finite. As G is finitely generated, $\mathrm{Out}_p(G)$ is of finite index in $\mathrm{Out}(G)$ and we see that $\mathrm{Out}(G)$ is virtually residually p-finite. □

This gives us everything we need to prove Theorem 1.6.

Proof of Theorem 1.6. Using Theorem 6.8 in the context of the class of all p-finite groups we see that the group $\Gamma\mathcal{G}$ is p-IAS. The rest follows by Lemma 7.2. □

Applying Proposition 2.1 to the class of all p-finite groups we get the following p-analogue of Corollary 2.5.

Lemma 7.3. *Let A, B be finitely generated residually p-finite p-IAS groups. Then $A \times B$ is p-IAS and, consequently,* $\mathrm{Out}_p(G)$ *is residually p-finite and* $\mathrm{Out}(G)$ *is virtually residually p-finite.*

Proof. Applying Proposition 2.1 to the case when $\mathcal{C}$ is the class of all finite p-groups we see that $A \times B$ is p-IAS. The rest follows by Lemma 7.2. □

Proof of Corollary 1.7. Denote $G = \Gamma\mathcal{G}$. Let $C \subseteq V\Gamma$ be set of central vertices of Γ. Note that the induced subgraph $\Gamma_{V\Gamma\setminus C}$ does not contain central vertices and thus by Theorem 1.6 we see that the full subgroup $G_{V\Gamma\setminus C}$ is p-IAS. We see that G splits as $G = G_{V\setminus C} \times \prod_{v\in C} G_V$, a direct product of p-IAS groups. The rest follows by Lemma 7.3 □

Let G be a group. Consider the natural homomorphism $\pi\colon G \to G/[G,G]$. Clearly $[G,G]$ is a characteristic subgroup of G and thus π induces a homomorphism

$$\tilde{\pi}\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/[G,G]).$$

Note that $\mathrm{Inn}(G) \leq \ker(\tilde{\pi})$, hence $\tilde{\pi}$ induces a homomorphism

$$\pi^*\colon \mathrm{Out}(G) \to \mathrm{Out}(G/[G,G]).$$

The kernel of this homomorphism is the *Torelli group* of G ($\mathrm{Tor}(G)$), i.e. $\phi \in \mathrm{Out}(G)$ belongs to $\mathrm{Tor}(G)$ if and only if ϕ acts trivially on the abelianisation of G. Note that $\mathrm{Tor}(G) \subseteq \mathrm{Out}_p(G)$ for every prime number p.

We say that a group G is bi-orderable if there exist a total ordering $\preceq$ of G such that if $f \preceq g$ then $cf \preceq cg$ and $fc \preceq gc$ for all $c, f, g \in G$.

Proof of Theorem 1.8. Every residually torsion free nilpotent group is residually p-finite for every prime p by [9, Theorem 2.1] and thus we see that $\mathrm{Out}_p(G)$ is residually p-finite by Theorem 1.6. As discussed earlier, $\mathrm{Tor}(G) \leq \cap_{p\in\mathbb{P}} \mathrm{Out}_p(G)$, where $\mathbb{P}$ denotes the set of all prime numbers. Being residually p-finite is a hereditary property and thus $\mathrm{Tor}(G)$ is residually p-finite for every prime number p. Consequently, by [18] we see that $\mathrm{Tor}(G)$ is bi-orderable. □

8. Open question

Let A, B be finitely generated RF groups such that $\mathrm{Out}(A)$ and $\mathrm{Out}(B)$ are RF as well. As follows from Corollary 2.5, if we assume that groups A, B are IAS then $\mathrm{Out}(A \times B)$ is RF as well. However, what if we drop this assumption? What can be said about residual finiteness of $\mathrm{Out}(A \times B)$?

Question 8.1. *Let A, B be finitely generated RF groups such that $\mathrm{Out}(A), \mathrm{Out}(B)$ are RF. Is $\mathrm{Out}(A \times B)$ RF?*

Clearly, if every finitely generated RF group with $\mathrm{Out}(G)$ RF was IAS then the class of finitely generated groups with RF outer automorphism would be closed under taking direct products by Proposition 2.1. This naturally leads to another question.

Question 8.2. *Is there a finitely generated RF group G such that $\mathrm{Out}(G)$ is an infinite RF group but G is not IAS?*

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RESIDUAL PROPERTIES OF GRAPH PRODUCTS OF GROUPS

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Abstract. We prove that the class of residually $\mathcal{C}$ groups is closed under taking graph products, provided that $\mathcal{C}$ is closed under taking subgroups, finite direct products and that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$. As a consequence, we show that local embeddability into various classes of groups is stable under graph products. In particular, we prove that graph products of residually amenable groups are residually amenable, and that locally embeddable into amenable groups are closed under taking graph products.

1. Introduction and motivation

Graph products were introduced by Green [8], and are a common generalisation of direct and free products. When all the groups involved are infinite cyclic, the graph products are known as right-angled Artin groups (RAAGs). In this sense, graph products generalise direct and free products in the same way as RAAGs generalise free abelian and free groups.

Let $\Gamma = (V, E)$ be a simplicial graph, i.e. V a set and $E \subseteq \binom{V}{2}$ a graph with no loops and no multiple edges, and let $\mathcal{G} = \{G_v \mid v \in V\}$ be a family of groups indexed by the vertex set V. Note that the set V can be of arbitrary cardinality. The *graph product* $\Gamma\mathcal{G}$ of the groups $\mathcal{G}$ with respect to the graph Γ is defined as the quotient of the free product $*_{v\in V} G_v$ obtained by adding all the relations of the form

$$g_u g_v = g_v g_u \quad \forall g_u \in G_u, g_v \in G_v, \{u, v\} \in E.$$

The groups $G_v \in \mathcal{G}$ are called the *vertex groups* of $\Gamma\mathcal{G}$.

Properties that are stable under direct and free products are often inherited by graph products too. Green originally proved that a graph product of residually finite (resp.: p-finite) groups is again residually finite (resp.: p-finite) [8]. More recently, the second named author proved that graph products of residually finite solvable groups are again residually finite solvable [7, Lemma 6.8]. Other examples are soficity [5], (hereditary) conjugacy separability [7], Tits alternatives [2], the Haagerup property or finiteness of asymptotic dimension [1].

In this work we study residual properties of graph products. We adopt an approach that unifies and recovers the known facts concerning residual properties. Moreover, it allows us to prove new results in this direction.

If $\mathcal{C}$ is a class of groups, then we say that a group G is *residually* $\mathcal{C}$ if for every non-trivial element $g \in G$ there is a group $C \in \mathcal{C}$ and a surjective homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $\varphi(g)$ is non-trivial in C.

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Theorem A. Let $\mathcal{C}$ be a class of groups closed under taking subgroups and finite direct products. Assume that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$, then the class of residually $\mathcal{C}$ groups is closed under taking graph products.

In [3] the first named author considered the class of residually amenable groups. Among other things, it is proved there that such class is closed under taking free products [3, Corollary 1.2]. As a consequence of Theorem A, we deduce that the class of residually amenable groups is also closed under graph products.

Corollary A. The class of residually amenable groups is closed under taking graph products. The same is true for residually elementary amenable groups.

Moreover, a result in the same spirit is true for a weaker form of approximation, the one of local embeddability into a class $\mathcal{C}$ (LE-$\mathcal{C}$ for short). See the beginning of Section 5 for the precise definitions.

The class of LEA (locally embeddable into amenable) groups is of great interest because of its relation with sofic groups [10]. While every LEA group is sofic, Gromov's question whether the other implication holds as well turned out to have a negative answer. The first example of a sofic non-LEA group is due to de Cornulier [6], and recently others were presented [12].

Even though sofic groups are quite obscure (it is not known whether there exists a non-sofic group), LEA groups are easier to understand. Moreover, since evey LEA group is sofic, they satisfy deep conjectures that are known to hold for sofic groups, such as Gottschalk's surjunctivity conjecture, Connes' embedding conjecture or Kaplansky's direct and stable finiteness conjectures.

Theorem B. Let $\mathcal{C}$ be a class of groups, suppose that $\mathcal{C}$ is closed under taking subgroups, finite direct products and that graph products of residually $\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of LE-$\mathcal{C}$ groups is closed under graph products.

This general result allows us to establish that the property of being LE-$\mathcal{C}$ is stable under graph products for certain classes of groups.

Corollary B. Let $\mathcal{C}$ be one of the following classes:

(1) finite groups,
(2) finite p-groups,
(3) solvable groups,
(4) finite solvable groups,
(5) elementary amenable groups,
(6) amenable groups.

Then the class of LE-$\mathcal{C}$ groups is closed under graph products.

2. Preliminaries

2.1. **Notations.** Throughout this work, all graphs considered are simplicial graphs, even if not explicitly stated.

The identity element of a group G is denoted by e_G, or simply by e if the group G is clear from the context. Given two elements $h, k \in G$, the commutator $hkh^{-1}k^{-1}$ is denoted by $[h, k]$. If $H, K \leq G$ are two subgroups, $[H, K]$ denotes the subgroup of G

generated by the elements $[h, k]$, where $h \in H$ and $k \in K$. A surjective homomorphism is usually indicated by $G \twoheadrightarrow H$.

We use the standard notation of an $\mathcal{A}$-by-$\mathcal{B}$ group to denote a group G with a normal subgroup $N \trianglelefteq G$ such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are given classes of groups (e.g. $\mathcal{A}$ being the free groups and $\mathcal{B}$ being the amenable groups, in the case of a free-by-amenable group).

For a residually finite solvable group we mean a residually (finite solvable) group, not a group which is solvable and, at the same time, residually finite.

2.2. **Graph products.** We recall here some terminology and facts about graph products that will be used in this paper. Let $G = \Gamma\mathcal{G}$ be a graph product. Every element $g \in G$ can be obtained as a product of a sequence $W \equiv (g_1, g_2, \dots, g_n)$, where each g_i belongs to some $G_{v_i} \in \mathcal{G}$. We say that W is a *word* in G and that the elements g_i are its *syllables*. The *length* of a word is the number of its syllables, and it is denoted by $|W|$.

Transformations of the three following types can be defined on words in graph products:

(T1) remove the syllable g_i if $g_i = e_{G_v}$, where $v \in V$ and $g_i \in G_v$,
(T2) remove two consecutive syllables g_i, g_{i+1} belonging to the same vertex group G_v and replace them by the single syllable $g_i g_{i+1} \in G_v$,
(T3) interchange two consecutive syllables $g_i \in G_u$ and $g_{i+1} \in G_v$ if $\{u, v\} \in E$.

The last transformation is also called *syllable shuffling.* Note that transformations (T1) and (T2) decrease the length of a word, whereas transformations (T3) preserve it. Thus, applying finitely many of these transformations to a word W, we obtain a word W' which is of minimal length and that represents the same element in G.

For $1 \le i < j \le n$, we say that syllables g_i, g_j can be *joined together* if they belong to the same vertex group and 'everything in between commutes with them'. More formally: $g_i, g_j \in G_v$ for some $v \in V$ and for all $i < k < j$ we have that $g_k \in G_{v_k}$ for some $v_k \in \operatorname{link}(v) := \{u \in V \mid \{u, v\} \in E\}$. In this case the words

$$W \equiv (g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_{j-1}, g_j, g_{j+1}, \dots, g_n)$$

and

$$W' \equiv (g_1, \dots, g_{i-1}, g_i g_j, g_{i+1}, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$$

represent the same group element in G, and the length of the word W' is strictly shorter than W.

We say that a word $W \equiv (g_1, g_2, \dots, g_n)$ is *reduced* if it is either the empty word, or if $g_i \neq e$ for all i and no two distinct syllables can be joined together. As it turns out, the notion of being reduced and the notion of being of minimal length coincide, as it was proved by Green [8, Theorem 3.9]:

Theorem 2.1 (Normal Form Theorem)**.** *Every element g of a graph product G can be represented by a reduced word. Moreover, if two reduced words W, W' represent the same element in the group G, then W can be obtained from W' by a finite sequence of syllable shufflings. In particular, the length of a reduced word is minimal among all words representing g, and a reduced word represents the trivial element if and only if it is the empty word.*

Thanks to Theorem 2.1 the following are well defined. Let $g \in G$ and let $W \equiv (g_1, \dots, g_n)$ be a reduced word representing g. We define the *length* of g in G to be $|g| = n$ and the *support* of g in G to be

$$\operatorname{supp}(g) = \{v \in V \mid \exists i \in \{1, \dots, n\} \text{ such that } g_i \in G_v \setminus \{e\}\}.$$

Let $x, y \in G$ and let $W_x \equiv (x_1, \dots, x_n), W_y \equiv (y_1 \dots, y_m)$ be reduced expressions for x and y, respectively. We say that the product xy is a reduced product if the word $(x_1, \dots, x_n, y_1, \dots, y_m)$ is reduced. Obviously, xy is a reduced product if and only if $|xy| = |x| + |y|$. We can naturally extend this definition: for $g_1, \dots, g_n \in G$ we say that the product $g_1 \dots g_n$ is reduced if $|g_1 \dots g_n| = |g_1| + \dots + |g_n|$.

A subset $X \subseteq V$ induces the full subgraph Γ_X of Γ. Let G_X be the subgroup of $G = \Gamma\mathcal{G}$ generated by the vertex groups corresponding to X and, by convention, let $G_\emptyset$ be the trivial subgroup. It follows from Theorem 2.1 that G_X is isomorphic to the graph product of the family $\mathcal{G}_X = \{G_v \in \mathcal{G} \mid v \in X\}$ with respect to the full subgraph Γ_X. Subgroups of G that can be obtained in this way are called *full subgroups*. For such subgroups, there is a canonical retraction $\rho_X \colon G \twoheadrightarrow G_X$ defined on the vertex groups as

$$\rho_X(g) = \begin{cases} g & \text{if } g \in G_v \text{ and } v \in X, \\ e & \text{otherwise.} \end{cases}$$

Thus, G splits as the semidirect product $G \cong \ker(\rho_X) \rtimes G_X$, and full subgroups are retracts of G.

2.3. **Special amalgams.** Let $B \leq A$ and C be groups, we define $A \star_B C$, the *special amalgam* of A and C over B, to be the free product with amalgamation

$$A \star_B C := A *_B (B \times C) = \langle A, C \parallel [b, c] = e \ \forall b \in B, \forall c \in C \rangle.$$

Special amalgams generalise the notion of special HNN extensions: every special HNN extension

$$A *_{\mathrm{id}_B} = \langle A, t \parallel tbt^{-1} = b \ \forall b \in B \rangle$$

is isomorphic to $A \star_B \langle t \rangle = A *_B (B \times \mathbb{Z})$.

Graph products can be seen in a natural way as special amalgams of their full subgroups:

Fact 2.2. *Let $G = \Gamma\mathcal{G}$ be a graph product. Then, for every $v \in V$, we have that $G \cong G_A \star_{G_B} G_C$, where $A = V \setminus \{v\}$, $B = \operatorname{link}(v)$ and $C = \{v\}$.*

Consider $G = A \star_B C$, then every element $g \in G$ can be represented as a product $a_0 c_1 a_1 \dots c_n a_n$, where $a_i \in A$ for $i = 0, 1, \dots, n$ and $c_j \in C$ for $j = 1, \dots, n$. We say that $g = a_0 c_1 a_1 \dots c_n a_n$ is in a *reduced form* if $a_i \notin B$ for $i = 1, \dots, n-1$ and $c_j \neq e$ for $j = 1, \dots, n$. The following lemma was proved in [7, Lemma 5.3].

Lemma 2.3. *Let $B \leq A$ and C be groups, and consider $G = A \star_B C$. Suppose that $g = a_0 c_1 a_1 \dots c_n a_n$ is in reduced form, where $a_i \in A$, $c_j \in C$ and $n \geq 1$. Then g is not the trivial element of G.*

Moreover, suppose that $f = x_0 y_1 x_1 \dots y_m x_m$ is in reduced form, where $x_i \in A$, $y_j \in C$. If $f = g$ then $m = n$ and $c_i = y_i$ for all $i = 1, \dots, n$.

Special amalgams satisfy a functorial property.

Fact 2.4. *Let $B \leq A, C, P, Q$ be groups and let $\psi_A \colon A \to P$, $\psi_C \colon C \to Q$ be homomorphisms. By the universal property of amalgamated free products the homomorphisms ψ_A and ψ_C uniquely extend to the homomorphism $\psi \colon G \to H$, defined on the generators by*

$$\psi(g) = \begin{cases} \psi_A(a) & \text{if } g = a \text{ for some } a \in A, \\ \psi_C(c) & \text{if } g = c \text{ for some } c \in C, \end{cases}$$

where $G := A \star_B C$ and $H := P \star_{\psi_A(B)} Q$.

3. Pro-$\mathcal{C}$ topologies on groups

Let $\mathcal{C}$ be a class of groups and let G be a group. We say that a normal subgroup $N \trianglelefteq G$ is a *co-$\mathcal{C}$ subgroup* of G if $G/N \in \mathcal{C}$, and we denote by $\mathcal{N}_{\mathcal{C}}(G)$ the set of co-$\mathcal{C}$ subgroups of G.

Consider the following closure properties for a class of groups $\mathcal{C}$:

(c0) $\mathcal{C}$ is closed under taking finite subdirect[1] products,
(c1) $\mathcal{C}$ is closed under taking subgroups,
(c2) $\mathcal{C}$ is closed under taking finite direct products.

Note that

$$(c0) \Rightarrow (c2) \qquad \text{and} \qquad (c1) + (c2) \Rightarrow (c0).$$

If the class $\mathcal{C}$ satisfies (c0) then, for every group G, the set $\mathcal{N}_{\mathcal{C}}(G)$ is closed under intersections, that is to say, if $N_1, N_2 \in \mathcal{N}_{\mathcal{C}}(G)$ then also $N_1 \cap N_2 \in \mathcal{N}_{\mathcal{C}}(G)$. This implies that $\mathcal{N}_{\mathcal{C}}(G)$ is a base at e_G for a topology on G.

Hence the group G can be equipped with a group topology, where the base of open sets is given by

$$\{gN \mid g \in G, N \in \mathcal{N}_{\mathcal{C}}(G)\}.$$

This topology, denoted by pro-$\mathcal{C}(G)$, is called the *pro-$\mathcal{C}$ topology* on G.

If the class $\mathcal{C}$ satisfies (c1) and (c2), or equivalently, (c0) and (c1), then one can easily see that equipping a group G with its pro-$\mathcal{C}$ topology is a faithful functor from the category of groups to the category of topological groups. That is to say, for groups G, H every homomorphism $\varphi \colon G \to H$ is a continuous map with respect to the corresponding pro-$\mathcal{C}$ topologies.

A set $X \subseteq G$ is *$\mathcal{C}$-closed* in G if X is closed in pro-$\mathcal{C}(G)$: for every $g \notin X$ there exists $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that the open set gN does not intersect X, that is, $gN \cap X = \emptyset$. This is equivalent to $gN \cap XN = \emptyset$, and hence $\varphi(g) \notin \varphi(X)$ in G/N, where $\varphi \colon G \twoheadrightarrow G/N$ is the canonical projection onto the quotient G/N. Accordingly, a set is *$\mathcal{C}$-open* in G if it is open in pro-$\mathcal{C}(G)$.

The following lemma was proved by Hall [11, Theorem 3.1].

Lemma 3.1. *Let $\mathcal{C}$ be a class of groups closed under subgroups and finite direct products, and let G be a group. A subgroup $H \leq G$ is $\mathcal{C}$-open in G if and only if there is $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $N \leq H$. Moreover, every $\mathcal{C}$-open subgroup of G is $\mathcal{C}$-closed in G.*

The following lemma is crucial for the proofs of the next section.

[1]A subdirect product of the family $\{G_i\}_{i \in I}$ is a subgroup $H \leq \prod_{i \in I} G_i$ such that the projections $H \to G_i$ are surjective for all $i \in I$.

Lemma 3.2. *Let $\mathcal{C}$ be a class of groups closed under subgroups and finite direct products, and let G be a residually $\mathcal{C}$ group. Then a retract R of G is $\mathcal{C}$-closed in G.*

Proof. Let $\rho\colon G \twoheadrightarrow R$ be the retraction corresponding to R and let $g \in G \setminus R$ be arbitrary. Note that $\rho(g) \neq g$ as $g \notin R$. By assumption, there is $C \in \mathcal{C}$ and a homomorphism $\varphi\colon G \twoheadrightarrow C$ such that $\varphi(\rho(g)) \neq \varphi(g)$ in C.

Let $\psi\colon G \to C \times C$ be the homomorphism defined by $\psi(f) = (\varphi(f), \varphi(\rho(f)))$ for all $f \in G$. Let $D = \{(c, c) \in C \times C \mid c \in C\} \leq C \times C$ be the diagonal subgroup of $C \times C$. Clearly $\psi(R) \leq D$ and $\psi(g) \notin D$, thus $\psi(g) \notin \psi(R)$. As the class $\mathcal{C}$ is closed under taking subgroups and direct products we see that $\psi(G) \in \mathcal{C}$ and so $\ker(\psi) \in \mathcal{N}_{\mathcal{C}}(G)$ and $g \ker(\psi) \cap R = \emptyset$. Hence R is $\mathcal{C}$-closed in G. □

When the class $\mathcal{C}$ contains only finite groups this statement has been proved in [13, Lemma 3.1.5].

4. Graph products of residually $\mathcal{C}$ groups

The following was proved in [7, Lemma 6.6].

Lemma 4.1. *Let $\mathcal{C}$ be a class of groups closed under finite direct products, let $A, C \in \mathcal{C}$ and suppose that $B \leq A$. Then the special amalgam $G = A \star_B C$ is free-by-$\mathcal{C}$.*

In the following proposition we characterise precisely which special amalgams are residually $\mathcal{C}$. This should be compared with [3, Theorem 1.9], where a similar statement can be found.

Proposition 4.2. *Let $B \leq A, C$ be groups and suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, finite direct products and that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$.*

The group $G = A \star_B C$ is residually $\mathcal{C}$ if and only if A, C are residually $\mathcal{C}$ and B is $\mathcal{C}$-closed in A.

Proof. Suppose that A, B are residually $\mathcal{C}$ and that the subgroup B is $\mathcal{C}$-closed in A. We need to prove that the group G is residually $\mathcal{C}$.

Let $g \in G \setminus \{e\}$ be arbitrary and let $g = a_0 c_1 a_1 \dots c_n a_n$, where $a_i \in A$ for $i = 0, \dots, n$, $c_j \in C$ for $j = 1, \dots, n$, be a reduced expression.

There are two cases to consider. If $n = 0$, then $g = a_0 \in A \setminus \{e\}$. Note that A is a retract of G and thus for the canonical retraction $\rho_A\colon G \twoheadrightarrow A$ we have $\rho_A(a_0) = a_0 \neq e$ in A. The group A is residually $\mathcal{C}$, so there is a group $H \in \mathcal{C}$ and a surjective homomorphism $\varphi\colon A \twoheadrightarrow H$ such that $\varphi(a_0) \neq e_H$. We see that $(\varphi \circ \rho_A)(g) \neq e_H$.

Suppose now that $n \geq 1$. As B is $\mathcal{C}$-closed in A, there is a group $Q \in \mathcal{C}$ and a surjective homomorphism $\alpha\colon A \twoheadrightarrow Q$ such that $\alpha(a_i) \notin \alpha(B)$ for all $i = 1, \dots, n-1$. Moreover, C is residually $\mathcal{C}$, so there exists a group $S \in \mathcal{C}$ and a surjective homomorphism $\gamma\colon C \twoheadrightarrow S$ such that $\gamma(c_i) \neq e_S$ for all $i = 1, \dots, n$. Let $\psi\colon G \to P$, where $P = Q \star_{\alpha(B)} S$, be the canonical extension of α and γ given by Fact 2.4. It follows that

$$\psi(g) = \alpha(a_0)\gamma(c_1)\alpha(a_1) \dots \gamma(c_n)\alpha(a_n)$$

is a reduced expression for $\psi(g)$ in P. Hence $\psi(g) \neq e_P$ by Lemma 2.3.

The group P is free-by-$\mathcal{C}$ by Lemma 4.1, and thus residually $\mathcal{C}$ by assumption. Hence, G is residually $\mathcal{C}$.

It remains to prove the other implication. So, suppose that G is residually $\mathcal{C}$. As $A, C \leq G$ and $\mathcal{C}$ is closed under subgroups, it follows that the groups A and C are residually $\mathcal{C}$.

Looking for a contradiction, suppose that B is not $\mathcal{C}$-closed in A. Then there exists an element $a \in A \setminus B$ such that $\varphi(a) \in \varphi(B)$ for all surjective homomorphisms $\varphi\colon A \twoheadrightarrow Q$, with $Q \in \mathcal{C}$.

Let $c \in C$ be a non-trivial element, then the element $g := [a, c] \in G$ is not trivial, as $a \notin B$ and C only commutes with B.

The group G is residually $\mathcal{C}$, hence there exist a group $Q \in \mathcal{C}$ and a surjective homomorphism $\varphi\colon G \twoheadrightarrow Q$ such that $\varphi(g) \neq e_Q$.

By the choice of the element a, it follows that $\varphi(a) \in \varphi(B)$. Moreover, B and C commute elementwise in G, so $\varphi(B)$ and $\varphi(C)$ commute elementwise in Q. This implies that

$$\varphi(g) = [\varphi(a), \varphi(c)] \in [\varphi(B), \varphi(C)] = \{e_Q\},$$

contradicting the assumption $\varphi(g) \neq e_Q$. Hence, B is $\mathcal{C}$-closed in A. □

Theorem A. Let $\mathcal{C}$ be a class of groups closed under taking subgroups and finite direct products. Assume that free-by-$\mathcal{C}$ groups are residually $\mathcal{C}$, then the class of residually $\mathcal{C}$ groups is closed under taking graph products.

Proof. Let Γ be a graph and let $\mathcal{G} = \{G_v \mid v \in V\}$ be a family of residually $\mathcal{C}$ groups. We want to prove that the graph product $G := \Gamma\mathcal{G}$ is residually $\mathcal{C}$.

Let $g \in G$ be a non-trivial element and set $S = \operatorname{supp}(g)$. Consider the canonical projection $\rho_S\colon G \twoheadrightarrow G_S$ onto the graph product associated to the finite graph Γ_S. As $\rho_S(g) = g \neq e$, without loss of generality we can assume that the graph Γ is itself finite.

We proceed by induction on $|V|$. If $|V| = 1$ then $G = G_v$ is residually $\mathcal{C}$ by assumption.

Suppose now that $|V| = r > 1$ and that the statement holds for all graph products on graphs with at most $r - 1$ vertices.

Fix a vertex $v \in V$ and let

$$A := V \setminus \{v\}, \qquad B := \operatorname{link}(v), \qquad C := \{v\}.$$

From Fact 2.2 it follows that $G = G_A \star_{G_B} G_C$. Moreover, G_A is a graph product of residually $\mathcal{C}$ groups with respect to a graph with $r - 1$ vertices, hence G_A is residually $\mathcal{C}$ by the induction hypothesis. Note that G_C is a vertex group, thus it is residually $\mathcal{C}$ by assumption. Finally, G_B is a retract of G_A and thus G_B is $\mathcal{C}$-closed in G_A by Lemma 3.2.

Hence, applying Proposition 4.2, we see that G is residually $\mathcal{C}$. □

To give some examples of classes satisfying these properties, we recall here the notion of a root class. We say that a class $\mathcal{C}$ is *non-trivial* if there is $G \in \mathcal{C}$ such that $G \neq \{e\}$. A non-trivial class of groups is called a *root class* if it is closed under taking subgroups, and for every group G and every subnormal series $K \trianglelefteq H \trianglelefteq G$ such that $G/H, H/K \in \mathcal{R}$, there exists $L \trianglelefteq G$ such that $L \subseteq K$ and $G/L \in \mathcal{R}$. In particular, for the choice $K = \{e\}$ one sees that $\mathcal{R}$ is closed under taking extensions.

Finite groups, finite p-groups, and (finite) solvable groups are examples of root classes [9]. This notion was introduced by Gruenberg [9] who proved that, when $\mathcal{R}$ is a root class, a free product of residually $\mathcal{R}$ groups is residually $\mathcal{R}$. In [3, Theorem 1.1, Lemma 3.3], with the aim to generalise this result, the first named author proved the following.

Lemma 4.3. *Let $\mathcal{C}$ be a non-trivial class of groups containing a root class $\mathcal{R}$, and such that*

1. *$\mathcal{C}$ is closed under taking finite direct products,*
2. *every $\mathcal{R}$-by-$\mathcal{C}$ group sits in $\mathcal{C}$,*
3. *for every group in $\mathcal{C}$ there exists a group in $\mathcal{R}$ of the same cardinality.*

Then a free-by-$\mathcal{C}$ group is residually $\mathcal{C}$ and a free product of residually $\mathcal{C}$ groups is again residually $\mathcal{C}$.

Root classes satisfy the assumptions of this lemma. Using Theorem A we extend these results to graph products.

Corollary 4.4. *Let $\mathcal{R}$ be a root class. Then the class of residually $\mathcal{R}$ groups is closed under taking graph products.*

Proof. Root classes are closed under taking subgroups, finite direct products and free-by-$\mathcal{R}$ groups are residually $\mathcal{R}$. Hence we can apply Theorem A. □

Using this, we recover Green's result that residually finite and residually p-finite groups are closed under taking graph products [8, Corollary 5.4, Theorem 5.6]. Corollary 4.4 also covers [7, Lemma 6.8] for the class of residually finite solvable groups. Moreover, it yields the same statement for residually solvable groups:

Corollary 4.5. *Graph products of residually finite groups are residually finite. The same holds for the classes of residually p-finite groups, residually finite solvable groups and residually solvable groups.*

On the other hand, the class of amenable groups and the class of elementary amenable groups are not root classes, yet they satisfy the assumptions of Lemma 4.3. Hence, [3, Corollary 3.4] reads as follows.

Fact 4.6. *If G is free-by-amenable, then it is residually amenable. Moreover, if G is free-by-(elementary amenable), then it is residually elementary amenable.*

As a consequence, these classes are closed under taking free products [3, Corollary 1.2].

The class of amenable groups is closed under taking subgroups and finite direct products. Moreover, free-by-amenable groups are residually amenable by Fact 4.6. Thus we can apply Theorem A to show that the class of residually amenable groups is closed under taking graph products. In the elementary amenable case we can use the same argument. Hence Corollary A.

5. Graph products of LE-$\mathcal{C}$ groups

We recall here the definition of local embeddability. Let G, C be two groups and $K \subseteq G$ be a finite subset. A map $\varphi\colon G \to C$ is called a *K-almost-homomorphism* if

1. $\varphi(k_1k_2) = \varphi(k_1)\varphi(k_2)$ for all k_1, $k_2 \in K$,
2. $\varphi\restriction_K$ is injective.

A group G is *locally embeddable into $\mathcal{C}$* (LE-$\mathcal{C}$ for short) if for all finite $K \subseteq G$ there exist a group $C \in \mathcal{C}$ and a K-almost-homomorphism $\varphi\colon G \to C$.

For classes $\mathcal{C}$ closed under finite direct products this definition yields a generalisation of being residually $\mathcal{C}$ [4, Corollary 7.1.14]. To give an example of finitely generated, non residually finite, LEF group [4, Proposition 7.3.9], consider the subgroup of $\mathrm{Sym}(\mathbb{Z})$ generated by the transposition $(0\ 1)$ and the translation $n \to n+1$. This group cannot be finitely presented.

Theorem A has an analogue for graph products of LE-$\mathcal{C}$ groups.

Theorem B. Let $\mathcal{C}$ be a class of groups, suppose that $\mathcal{C}$ is closed under taking subgroups, finite direct products and that graph products of residually $\mathcal{C}$ groups are residually $\mathcal{C}$. Then the class of LE-$\mathcal{C}$ groups is closed under graph products.

Proof. Let $\Gamma = (V, E)$ be a graph, $\mathcal{G} = \{G_v \mid v \in V\}$ be a family of LE-$\mathcal{C}$ groups and let $G := \Gamma\mathcal{G}$ be the graph product of $\mathcal{G}$ with respect to Γ. Let $K \subseteq G$ be a finite subset of G. The set $\cup_{k\in K}\mathrm{supp}(k)$ is a finite subset of V, thus without loss of generality we can suppose that V itself is finite. Set

$$K' = K^{-1}K = \{k^{-1}k' \mid k, k' \in K \cup \{e_G\}\}$$

and suppose that $K' = \{g_1, \ldots, g_r\}$. Let $W_1, \ldots, W_r$ be reduced words in G representing the elements $g_1, \ldots, g_r$.

For every $v \in V$ consider the finite subset

$$K_v := \{e_{G_v}\} \cup \{g \in G_v \mid g \text{ is a syllable of some } W_i\} \subseteq G_v.$$

By assumption, the vertex group G_v is LE-$\mathcal{C}$ for every $v \in V$. Hence, there exist a family of groups $\mathcal{F} = \{F_v \in \mathcal{C} \mid v \in V\}$ and a family of K_v-almost-homomorphisms $\{\varphi_v \colon G_v \to F_v \mid v \in V\}$.

As $e_{G_v} \in K_v$ it follows $\varphi_v(e_{G_v}) = e_{F_v}$. This implies that

$$\varphi_v(g) \neq e_{F_v} \quad \forall g \in K_v \setminus \{e_{G_v}\}.$$

Let $F := \Gamma\mathcal{F}$ be the graph product of $\mathcal{F}$ with respect to Γ.

Let $\mathcal{W}_G$ and $\mathcal{W}_F$ denote the set of all the words in G and F respectively. We define the function $\tilde{\varphi} \colon \mathcal{W}_G \to \mathcal{W}_F$ in the following manner: for $W \equiv (g_1, \ldots, g_n)$, where $g_i \in G_{v_i}$ for some $v_i \in V$ for $i = 1, \ldots, n$, we set

$$\tilde{\varphi}(W) \equiv (\varphi_{v_1}(g_i), \ldots, \varphi_{v_n}(g_n)).$$

By definition, if W is the empty word in G then $\tilde{\varphi}(W)$ is the empty word in F. Let us note that the map $\tilde{\varphi}$ is compatible with concatenation: for all $U, V \in \mathcal{W}_G$ we have $\tilde{\varphi}(UV) = \tilde{\varphi}(U)\tilde{\varphi}(V)$.

Let $g \in G$ be an arbitrary element and let $W \equiv (g_1, \ldots g_n)$, $W' \equiv (g'_1, \ldots, g'_m)$ be two reduced words representing g in G. By Theorem 2.1 we see that $m = n$ and that the word W can be transformed to W' by finite sequence of syllable shufflings. Since the groups G, F are graph products with respect to the same graph, it can be easily seen that the word $\tilde{\varphi}(W)$ can be transformed to $\tilde{\varphi}(W')$ (using the same sequence of syllable shufflings). Hence the words $\tilde{\varphi}(W)$ and $\tilde{\varphi}(W')$ represent the same element in F.

We see that the map $\tilde{\varphi}$ induces a well defined map $\varphi \colon G \to F$ given by

$$\varphi(g) = \varphi_{v_1}(g_1) \ldots \varphi_{v_n}(g_n).$$

Clearly, $\varphi \restriction_{G_v} = \varphi_v$ for every $v \in V$ and thus it makes sense to omit the subscripts and write

$$\varphi(g) = \varphi(g_1) \dots \varphi(g_n).$$

We claim that φ is a K-almost-homomorphism, that is, $\varphi \restriction_K$ is an injective map and $\varphi(kk') = \varphi(k)\varphi(k')$ for all $k, k' \in K$.

First of all, let us show that if the reduced word $W_k \equiv (f_1, \dots, f_n)$ represents $k \in K'$ in the group G, then the word $\tilde{\varphi}(W_k) \equiv (\varphi(f_1), \dots, \varphi(f_n))$, which represents $\varphi(k)$ in F, is a reduced word in F.

As the maps φ_v are K_v-almost-homomorphisms for every $v \in V$, it follows that $\varphi(f_i) \neq e$ in F for $i = 1, \dots, n$, so no syllable of $\tilde{\varphi}(W_k)$ is trivial. Suppose that $\tilde{\varphi}(W_k)$ is not reduced in F. This means that there exist $i < j \in \{1, \dots, n\}$ such that the syllables $\varphi(f_i)$ and $\varphi(f_j)$ can be joined together. However, this implies that the syllables f_i and f_j can be joined in the word W_k, which contradicts the fact that W_k is reduced. Hence $\tilde{\varphi}(W_k)$ is reduced.

Now, let us prove that $\varphi(kk') = \varphi(k)\varphi(k')$ for all $k, k' \in K'$. Let $k, k' \in K$ be arbitrary and let W, W' be reduced words representing k and k' respectively. We want to show that the word $\tilde{\varphi}(WW') \equiv \tilde{\varphi}(W)\tilde{\varphi}(W')$ represents the element $\varphi(kk')$.

Suppose that the product kk' is reduced, i.e. the concatenation WW', which is a word representing kk' in G, is reduced. Using a similar argument as above we see that the word $\tilde{\varphi}(WW') \equiv \tilde{\varphi}(W)\tilde{\varphi}(W')$ is reduced. The word $\tilde{\varphi}(W)\tilde{\varphi}(W)$ represents $\varphi(k)\varphi(k')$ in F by definition, but at the same time we see that the word $\tilde{\varphi}(WW')$ represents $\varphi(kk')$ in F, and thus $\varphi(kk') = \varphi(k)\varphi(k')$.

Now, suppose that the product kk' is not reduced. Let $c, f, g \in G$ be such that k factorises as a reduced product $k = fc$, k' factorises as a reduced product $k' = c^{-1}g$ and $|c|$ is maximal. Clearly, $kk' = fg$. Without loss of generality we may assume that $W \equiv (f_1, \dots, f_n, c_1, \dots c_l)$ and $W' \equiv (c_l^{-1}, \dots, c_1^{-1}, g_1, \dots, g_m)$, where $c = c_1 \dots c_l$, $f = f_1 \dots f_n$ and $g = g_1 \dots g_m$.

Consider the word $X \in \mathcal{W}_F$, where X is

$$(\varphi(f_1), \dots, \varphi(f_n), \varphi(c_1), \dots, \varphi(c_l), \varphi(c_l^{-1}), \dots, \varphi(c_1^{-1}), \varphi(g_1), \dots, \varphi(g_m)).$$

Note that $X = \tilde{\varphi}(WW') = \tilde{\varphi}(W)\tilde{\varphi}(W')$. The syllable $\varphi(c_l)$ can be joined with syllable $\varphi(c_l^{-1})$. Obviously, $c_l \in G_u$ for some $u \in V$. As $\varphi \restriction_{G_u}$ is a K_u-almost-homomorphism and $c_l, c_l^{-1} \in K_u$ we see that $\varphi(c_l)\varphi(c_l^{-1}) = \varphi(c_l c_l^{-1}) = \varphi(e_{G_u})$. As stated before, $\varphi(e_{G_v}) = e_{F_v}$ for every $v \in V$ and thus we can remove the trivial syllable. Note that this transformation is compatible with the function $\tilde{\varphi}$:

$$\varphi(k)\varphi(k') = \varphi(f_1 \dots f_n c_1 \dots c_{l-1})\varphi(c_{l-1}^{-1} \dots c_1^{-1} g_1 \dots g_m).$$

Repeating these two steps $l - 1$ more times, the word X can be rewritten to

$$X' \equiv (\varphi(f_1), \dots, \varphi(f_n), \varphi(g_1), \dots, \varphi(g_m))$$

and thus we see that $\varphi(k)\varphi(k') = \varphi(f)\varphi(g)$.

Note that the word X' is reduced in F if and only if the word

$$(f_1, \dots, f_n, g_1, \dots, g_m)$$

is reduced in G. Suppose that the word X' is reduced. Then clearly

$$\varphi(k)\varphi(k') = \varphi(f)\varphi(g) = \varphi(fg) = \varphi(kk')$$

and we are done.

Suppose that the word X' is not reduced. As all the syllables of X' are nontrivial we see that two syllables of the word X' can be joined together. The word $(\varphi(f_1),\dots,\varphi(f_n))$ is a subword of $\tilde{\varphi}(W)$, which is a reduced word, and thus it is reduced, hence no two syllables of $(\varphi(f_1),\dots,\varphi(f_n))$ can be joined together. The same argument applies to $(\varphi(g_1),\dots,\varphi(g_m))$. Hence, we see that there exist $1 \leq i \leq n$ and $1 \leq j \leq m$ such that the syllables $\varphi(f_i)$ and $\varphi(g_j)$ can be joined together in X'. Again, $f_i, g_j \in G_u$ for some $u \in V$ and thus $\varphi(f_i)\varphi(g_j) = \varphi(f_i g_j)$ as $f_i, g_j \in K_u$. By the assumptions (as $\varphi \restriction_{K_u}$ is injective), $\varphi(f_i g_j) = e_{F_u}$ if and only if $f_i g_j = e_{G_u}$. However, $f_i = g_j^{-1}$ would be a contradiction with the maximality of $|c|$, hence $\varphi(f_i)\varphi(g_j) \neq e_{F_u}$. As $\varphi \restriction_{G_u}$ is a K_u-almost-homomorphism we see that joining the syllable $\varphi(f_i)$ with the syllable $\varphi(g_j)$ is compatible with the map $\tilde{\varphi}$.

Suppose that the syllable $\varphi(f_i g_j)$ can be joined with some $\varphi(f_k)$. By definition, this means that $\varphi(f_k)$ and $\varphi(f_i)$ could have been joined in $\tilde{\varphi}(W)$. This contradicts the fact that $\tilde{\varphi}(W)$ is reduced. By an analogous argument, the syllable $\varphi(f_i g_j)$ cannot be joined with any syllable $\varphi(g_p)$.

By iterating the previous step at most $\min\{n, m\}$ times, we obtain a sequence of transformations compatible with the map $\tilde{\varphi}$. All together, we have shown that the word $\tilde{\varphi}(W)\tilde{\varphi}(W')$ can be rewritten to a reduced word X'', that represents the element $\varphi(k)\varphi(k')$ in F, and each rewriting step is compatible with the map $\tilde{\varphi}$: if we applied the analogous transformations to the word WW' we would obtain a reduced word U, that represents the element kk' in G, such that $\tilde{\varphi}(U) \equiv X''$. It follows that $\varphi(kk') = \varphi(k)\varphi(k')$.

To finish, we need to prove that $\varphi \restriction_K$ is an injective map. Let $k, k' \in K \subseteq K'$ be arbitrary such that $k \neq k'$, or equivalently $k'k^{-1} \neq e_G$. We have already shown that $\varphi(k'k^{-1}) = \varphi(k')\varphi(k^{-1})$. Consider a reduced word $W_{k'k^{-1}}$ representing the element $k'k^{-1} \in K'$. Note that by the construction of the function φ it follows that $\varphi(k) = \varphi(k)^{-1}$ for all $k \in K$. By the previous argumentation, the word $\tilde{\varphi}(W_{k'k^{-1}})$ is reduced in F and thus by Theorem 2.1 we see that $\varphi(k'k^{-1}) = \varphi(k')\varphi(k)^{-1} \neq e_F$. It follows that $\varphi(k) \neq \varphi(k')$.

Thus, we proved that φ is a K-almost-homomorphism.

The graph product $F = \Gamma\mathcal{F}$ is residually $\mathcal{C}$ by assumption. Hence, there exists a surjective homomorphism $\psi\colon F \twoheadrightarrow D \in \mathcal{C}$ which is injective on the finite subset $\varphi(K) \subseteq F$. Thus, the composition $\psi \circ \varphi\colon G \to D$ is a K-almost-homomorphism, and G is LE-$\mathcal{C}$. □

As an immediate corollary we get the following.

Corollary 5.1. *Let $\mathcal{R}$ be a root class. Then the class of LE-$\mathcal{R}$ groups is closed under graph products.*

Note that the first four cases of Corollary B follow from Corollary 5.1. In the remaining cases the assumptions of Theorem B are met by Corollary A, hence Corollary B.

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