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On coarse geometric properties of discrete and locally compact groups

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ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES
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The reduced group C*-algebra is one place where geometric group theory and operator theory overlap. Usually one can expect that a geometric property can be captured by this group algebra. For example a discrete group is amenable if and only if the reduced group C*-algebra is nuclear. In this thesis we investigate the relationship between exactness of the reduced group C*-algebra and amenable actions on compact Hausdorff spaces (amenability at infinity) for locally compact second countable groups. In the discrete case, it is known that a group is amenable at infinity if and only if the reduced group C*-algebra is exact.

Amenability at infinity is known to satisfy strong topological and index type conjectures, such as the Novikov and the coarse Baum–Connes conjecture. The Baum–Connes conjectures serve as a unifying theme throughout this thesis and is part of the motivation to study large-scale (or coarse) invariants of the group. Indeed one coarse invariant we study is whether a group can coarsely embed into a Hilbert space. It was shown by G. Yu [**127**] and G. Skandalis, J. Tu and G. Yu [**108**] that if a group can coarsely embed into a Hilbert space then the assembly maps in the Baum–Connes conjecture are injective. It was shown by G. Yu and N. Higson and J. Roe in [**127, 67**] that if a group is amenable at infinity then it can coarsely embed into a Hilbert space.

Compression was defined to measure how close a coarse embedding is to a quasi-isometric embedding. A lot of research has been done to calculate the precise compression value of a group embedding into a Hilbert space. In this thesis we will study different group constructions that preserve the positivity of the compression.

Chapter 2 is devoted to the study permanence properties of equivariant compression. In particular we give results that control the equivariant compression of a group in terms of properties of open subgroups whose direct limit is the group. We also study the behaviour of equivariant compression under amalgamation of free products where the common subgroup has finite index inside the two larger groups.

Chapter 3 is devoted to showing that coarse embeddability into a Hilbert space is preserved over generalised metric wreath products. We show that positive Hilbert space compression is also preserved by taking generalised metric wreath products.

Chapter 4 is devoted to the study of reduced cross products. When a group G acts on a C*-algebra A, we can form a larger C*-algebra that encodes that action. This is called the reduced cross product which we denote by $A \rtimes_r G$. Indeed the reduced group C*-algebra of G is $*$-isomorphic to $\mathbb{C} \rtimes_r G$. In this chapter we show that a locally compact second countable group is amenable at infinity if and only if its reduced cross product preserves short exact sequences.

Contents

Declaration of Authorship

I, Christopher Cave, declare that the thesis entitled *On coarse geometric properties of discrete and locally compact groups* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:

  [**26**] Chris Cave and Dennis Dreesen. Embeddings of locally compact hyperbolic groups into L_p-spaces. Preprint arXiv:1303.4250.

  [**27**] Chris Cave and Dennis Dreesen. Equivariant compression of certain direct limit groups and amalgamated free products. Preprint arXiv:1309.4636.

  [**28**] Chris Cave and Dennis Dreesen. Embeddability of generalized wreath products. *Bull. Aust. Math. Soc,* **2015**, *91*, 250-263

Signed:...

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The material in Chapters 2 and 3 and Section 4.7 is joint work Dennis Dreesen and can be found in [**27, 28, 26**] respectively. The material in Section 4.6 is joint work with Jacek Brodzki and Kang Li.

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Introduction

The Baum–Connes conjecture serves as a unifying theme throughout this thesis. It is because of this conjecture that several notions in this thesis were first introduced. In essence, the Baum–Connes conjecture is one of the tools we have to analyse the group algebra of an infinite or a topological group. We begin by introducing some some classical representation theory and soon we will see the importance of the group algebra.

Representation theory

When an abstract group comes into existence the group yearns to have a concrete geometric interpretation. This naturally gives rise to the study of representing groups as invertible operators on a vector space. For a discrete group G we can form a natural algebra called the *group algebra* where elements are finite formal sums of group elements with coefficients in the field of complex numbers. There is a nice universal property for finite groups: for any $\mathbb{C}$-algebra A, if there is a group homomorphism $G \to A^{\times}$ then this uniquely lifts to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \to A$. This means that representation theory of the group is encoded in the $\mathbb{C}$-algebra representations of $\mathbb{C}[G]$.

The representation theory of finite groups first began around 100 years ago when in 1896, F.G. Frobenius first extended the definition of characters from finite abelian groups to finite non-abelian groups. Later he proved the very powerful correspondence between linear representations and characters. One of the first applications of representation theory was in the proof of Burnside's theorem: all groups of order $p^a q^b$ (primes p and q) are soluble [**24**]. Nowadays character tables have several applications to chemistry and molecular vibrations [**20, 68**]

However what can one say when the group is no longer finite, i.e. countably infinite or is a topological group? Suppose first the group is countably infinite. We now represent the group as unitary operators on a Hilbert space. For a group G we shall denote the collection of equivalence classes of irreducible unitary representations by $\widehat{G}$. Given a unitary representation $U\colon G \to \mathcal{U}(\mathcal{H})$ of a discrete group we can uniquely extend this to a $*$-algebra representation of $\mathbb{C}[G] \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the space of bounded linear operators on the Hilbert space $\mathcal{H}$.

However the image of $\mathbb{C}[G]$ inside $\mathcal{B}(\mathcal{H})$ is not complete and if we want to use the power of functional analysis and operator algebras then we should really find a way to complete this. There are two completions we have in mind: the reduced completion and the maximal completion. Every group G acts by unitaries on $\ell^2(G)$ by extending the left multiplication action on itself to finitely supported functions. The completion of $\mathbb{C}[G]$ with respect to this $*$-representation is called the *reduced group C^*-algebra*, which we denote by $C^*_r(G)$. The completion of $\mathbb{C}[G]$ with respect to the direct sum of all (cyclic) unitary representations of G is called the *maximal group C^*-algebra*. When G is a toplogical group we can form continuous analogues of $C^*_r(G)$ and $C^*(G)$ by considering completions of $C_c(G)$, the continuous compactly supported functions on G.

When the group is locally compact and abelian then every irreducible unitary representation is one dimensional. This means we can identify $\widehat{G}$ with the group of homomorphisms from the group to the unit circle in $\mathbb{C}$. In particular $\widehat{G}$ is abelian and becomes a locally compact group when equipped with the topology of uniform convergence on compact subsets of G. In fact Pontrjagin's Duality theorem shows that $\hat{\hat{G}}$ and G are homeomorphic and canonically isomorphic as groups.

Plancherel's theorem shows that the Fourier transform $\mathcal{F}\colon L^1(G) \to C_0(\widehat{G})$ extends to a unitary operator $\mathcal{F}\colon L^2(G) \to L^2(\widehat{G})$. As a consequence the left regular representation is unitarily equivalent to the multiplication operator $C_c(G) \to \mathcal{B}(L^2(\widehat{G}))$, $f \mapsto T_{\mathcal{F}f}$ where $(T_{\mathcal{F}f}\xi)(\chi) = \mathcal{F}f(\chi)\xi(\chi)$ for all $\xi \in L^2(\widehat{G})$ and $\chi \in \widehat{G}$. The Stone–Weierstrass theorem gives us that the C*-algebras $C_r^*(G)$ and $C_0(\widehat{G})$ are $*$-isomorphic for all locally compact abelian groups G.

When we are out of the world of locally compact abelian groups we can form the set of irreducible unitary representations of the group and equip it with a topology that coincides with uniform convergence on compact sets when the group is abelian. This space is usually very badly behaved, e.g. the space is usually not Hausdorff. However we do have a better object at our disposal to study: $C_r^*(G)$.

Baum–Connes conjectures

Unfortunately $C_r^*(G)$ is not that much better behaved. For example the C*-algebra is usually simple. Indeed if $C_r^*(G)$ is simple then the amenable radical (the largest normal amenable subgroup) is trivial. One can think of simplicity as the opposite of amenability. In fact it has been an open question for many years whether this is the only obstruction to simplicity [**41**, Question 4]. Recently there has been progress in this area: for a discrete group G, $C_r^*(G)$ is simple if and only if the action of G on its Furstenberg boundary is topologically free [**69**, Theorem 1.5.].

We do have algebraic topological tools to help with this algebra $C_r^*(G)$. Indeed when G is discrete and abelian then we have the nice isomorphism

$$K^j(\widehat{G}) \cong K_j(C(\widehat{G})) \cong K_j(C_r^*(G))$$

where $K^j(\widehat{G})$ is the topological K-theory of the compact space $\widehat{G}$. When G is no longer abelian then we can not take $K^j(\widehat{G})$ but we can still take the K-theory of $C_r^*(G)$. The idea of the following conjecture is to find the correct object that replaces $K^j(\widehat{G})$ when G is not abelian.

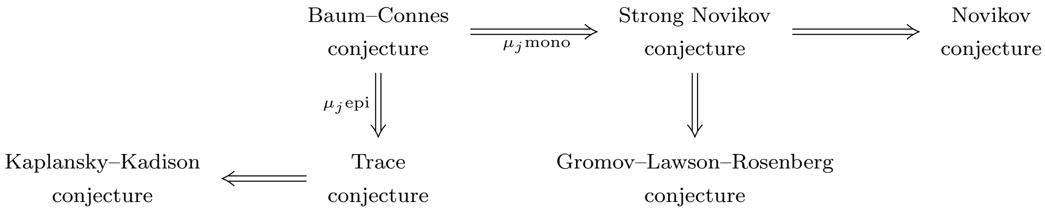
Conjecture (Baum–Connes conjecture)**.** *The following assembly maps*

$$(*) \qquad \mu_j\colon K_j^G(\underline{E}G) \to K_j(C_r^*(G)) \quad (j = 0, 1)$$

are isomorphisms of abelian groups.

The left hand side is the G-equivariant K-homology with compact supports and the right hand side is K-theory of the reduced C*-algebra. The space $\underline{E}G$ is the universal example of proper actions of G [**15**, Definition 1.6.]. In a lot of cases there exists simple models for $\underline{E}G$ [**15**, Section 2.] which makes the left hand side usually more computable than the right hand side. See the books [**87, 119**] for introductory texts to the Baum–Connes conjecture.

The diagram below shows that the Baum–Connes conjecture is a part of several other conjectures from topology and functional analysis.



For a full digram of implications, statements of conjectures and references of implications see [**87**, Section 7]. The current formulation of the conjecture was given in [**15**] but was first set forth in a 1982 preprint of P. Baum and A. Connes [**14**] and was published 18 years after it was first written. The conjecture originates in work of G. Kasparov [**70**] and A. Mishchenko [**86**] in the Novikov conjecture, ideas of A. Connes in foliation theory [**36**] and P. Baum's geometric description of K-homology theory [**16**]. The following is a stronger version of the conjecture.

Conjecture (Baum–Connes conjecture with coefficients)**.** *Assume A is a separable C^*-algebra with an action of a locally compact group G. Then the following assembly maps*

$$(**) \qquad \mu_j : K_j^G(\underline{E}G; A) \to K_j(A \rtimes_r G)$$

are isomorphisms of abelian groups.

In 1997, N. Higson and G. Kasparov proved that a large class of groups satisfy the Baum–Connes conjecture with coefficients: countable groups that have the Haagerup property [**65**]. The Haagerup property first appeared in 1978 when U. Haagerup proved that the reduced group C*-algebra of any non-abelian free group of finite rank has Grothendieck's metric approximation property [**60**]. Indeed a group has the Haagerup property if it can act properly and affinely on a Hilbert space. This includes the class of amenable groups [**19**].

It is known that the assembly map in $(**)$ is an isomorphism for all discrete hyperbolic groups. In 2003 G. Kasparov and G. Skandalsis proved injectivity [**71**] and in 2012, V. Lafforgue proved surjectivity [**78**]. There are no known counterexamples to the Baum–Connes conjecture but there do exist counterexamples to the Baum–Connes conjecture with coefficients [**64**]. One of the ways of attacking the Baum–Connes conjecture is using the coarse Baum–Connes conjecture.

Conjecture (Coarse Baum–Connes conjecture)**.** *Let X be a proper discrete metric space with bounded geometry. Then the following assembly maps are an isomorphism of groups:*

$$(***) \qquad A_\infty : KX_i(X) \to K_i(C^*X) \quad (i = 0, 1).$$

The conjecture was first outlined in 1993 by J. Roe [**98**] and precisely formulated in 1995 by N. Higson and J. Roe [**66**]. The right hand side is the K-theory of the Roe algebra and the left hand side is the coarse K-homology of the space X. The right hand side is a coarse invariant and captures the coarse geometry of the space while the left hand side captures the local topological data of smoothened versions of X of increasing scale. The "moral" of this conjecture is that the study of the coarse structure of the space is equivalent to studying the topology of smoothened copies of X of increasing scale.

If a countable group, when considered as a discrete proper metric space satisfies the coarse Baum–Connes conjecture and admits a finite complex as a classifying space then the assembly maps in $(*)$ are injective [**99**, Theorem 8.4.]. An important class of spaces that satisfy the coarse

Baum–Connes conjecture are those that coarsely embed into a Hilbert space [**127**]. Indeed it was M. Gromov that introduced coarse embeddability into a Hilbert space in [**53**] and hinted at its importance to the Novikov conjecture in [**50**, Problems (4) and (5)]. In 2000, G. Yu showed that if a discrete group coarsely embeds into a Hilbert space and admits a finite complex as a classifying space then it satisfies the Novikov conjecture. This result was strengthened in 2002 by G. Skandalis, J. Tu and G. Yu where they showed the assembly maps (∗∗) are injective for any discrete group that coarsely embeds into a Hilbert space [**108**, Theorem 6.1.]. In 2012, G. Kasparov and G. Yu gave the same result for any group that coarsely embeds into $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$ [**72**].

In [**127**], G. Yu introduced a coarse invariant called property A. If a metric space has property A then the space can coarsely embed into a Hilbert space and so property A gives a criterion for coarse embeddability into a Hilbert space. In [**101**] J. Roe generalised this idea to general metric spaces and in 2014, S. Deprez and K. Li extended the result of G. Skandalis, J. Tu and G. Yu to locally compact second countable groups [**44**].

The following table is the current status of the Baum–Connes conjectures.

	Status	**Class of groups/spaces conjecture is true**	**Counterexamples**	
			Injectivity fails	**Surjectivity fails**
Baum–Connes conjecture with coefficients.	False.	The class of countable groups $\mathbf{LH}\mathcal{ETH}$, which includes all countable Haagerup groups [**65, 87**], discrete hyperbolic groups [**78**].	Open.	$B \rtimes_r \Gamma$ where Γ is the Gromov monster and B is a particular separable Γ-C^*-subalgebra of $\ell^\infty(\mathbb{N}; c_0(\Gamma))$ [**64**]. The assembly map in this example is injective [**124, 125**].
Baum–Connes conjecture.	Open.	One relator groups [**17**], fundamental groups of Haken 3-manifolds [**93, 117**], groups with property (RD) that admit proper, cocompact, isometric action on a strongly bolic metric space [**76**].	Open.	Open.
Coarse Baum–Connes conjecture.	False for metric spaces, open for discrete groups.	Spaces that coarsely embed into a Hilbert space [**127**].	Open.	A sequence of expander graphs [**64**].

Coarse geometry

The right hand side of the coarse Baum–Connes conjecture is the K-theory of the Roe algebra. This is a coarse invariant and so does not depend on the local topological structure but the macroscopic properties of the space. This allows a lot of flexibility and gives a nice link to the study of asymptotic behaviour of groups. The philosophy of studying asymptotic behaviour of groups can be captured in the following quote by M. Gromov in [**53**]:

> “To regain the geometric perspective one has to change his/her position and move the observation point far away from Γ [a finitely generated group]. Then the metric in Γ seen from the distance d becomes the original distance divided by d and for $d \to \infty$ the points

> in Γ coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy."

The idea of asymptotic study of infinite groups begins with viewing a finitely generated group as a metric space by equipping it with a word metric. The metric depends on the generating set so we can not hope for metrics from different generating sets to be isometric. Fortunately the metrics are quasi-isometric. This idea does not end at finitely generated groups. Every locally compact second countable group has a proper left invariant metric that generates the topology and any two such metrics are coarsely equivalent [**59, 109**]. This motivates the study of finding coarse invariants of a group.

The first asymptotic ideas in group theory appeared in the mid-fifties in the papers by V. Efremovic [**47**], E. Følner [**51**], and A. Švarc [**111**]. However the area was revolutionised in 1993 by M. Gromov's paper [**53**]. In the early 50s, V. Efremovic and A. Švarc both observed that the growth rate of the volume of balls in the universal cover of a Riemannian manifold is a topological invariant and only depends on the fundamental group [**47, 111**]. In 1955, A. Švarc applied this idea to show that the fundamental group of a compact n-dimensional manifold can not be an abelian group of rank less than n [**111**]. Similar results on non-positively curved spaces were obtained independently by J. Milnor in 1968 [**85**].

Amenability and amenable actions. In [**51**], E. Følner gave a geometric characterisation of amenability in terms of slow growth of boundaries of finite subsets of the group. This automatically gives that any group with subexponential growth is amenable and that amenability is a quasi-isometric invariant for discrete groups [**38**, Proposition 3.D.32.].

The story of amenability started in 1904 with H. Lebesgue where in [**79**] he gave a list of properties that uniquely specified his integral on $\mathbb{R}$. The only property he listed that differed from the Riemann integral was the monotone convergence theorem. In [**12**], S. Banach considers three questions all of which involve the invariance of finitely additive measures.

Question. Let $\mathcal{M}_b(\mathbb{R})$ be the family of bounded Lebesgue measurable sets on $\mathbb{R}$. If μ is a finitely additive positive translation invariant measure on $\mathcal{M}_b(\mathbb{R})$ such that $\mu([0,1]) = 1$, does $\mu = \lambda$ where λ is the Lebesgue measure on $[0,1]$?

Question. Let $G_n = \mathbb{R}^n \rtimes O(n)$ be the group of isometries on $\mathbb{R}^n$. Does there exists a G_n-invariant, finitely additive, positive measure μ on $\mathcal{P}(\mathbb{R}^n)$, the power set of $\mathbb{R}^n$, such that $\mu([0,1]^n) = 1$?

Question. Does there exist a finitely additive $O(n+1)$-invariant, positive measure μ on $\mathcal{M}_b(S^n)$ such that $\mu(S^n) = 1$ and $\mu \neq \lambda$ where λ is the Lebesgue measure on S^n?

The first question was raised initially by H. Lebesgue in [**79**]. H. Lebesgue asked if the integral was still uniquely specified if the monotone convergence theorem was dropped. S. Banach showed that the answer to the first question is negative. He constructs such a measure on $\mathcal{P}_b(X)$, the family of bounded subsets of $\mathbb{R}$ such that

(1) $\mu(A) < \infty$ for every bounded subset A of $\mathbb{R}$.
(2) $\int_a^b \varphi(x)\,d\mu(x) = \int_a^b \varphi(x)\,dx$ for every Riemann integrable functions φ on $[a,b]$.
(3) There exists a Lebesgue integrable function ψ on an interval $[c,d]$ such that

$$\int_c^d \psi(x)\,d\mu(x) \neq \int_c^d \psi(x)\,d\lambda(x)$$

where λ is the Lebesgue measure on $\mathbb{R}$.

The second question arose out of results by F. Hausdorff in 1914 in his paper "Grundzüge der Mengenlehre" (see the collected works [**61**]). F. Hausdorff showed that no such measure exists for $n \geq 3$. His line of thinking initiated the idea of paradoxical decompositions which is central to the Banach–Tarski paradox in [**13**] and Tarski's theorem on amenability in [**113, 114**]. For the cases $n = 1, 2$, S. Banach showed that such measures in the second question do exist and this is because G_1 and G_2 are amenable.

It was J. von Neumann in [**120**] that realised the cases for $n = 1, 2$ could be generalised to groups that carry a finitely additive, invariant, positive measure of total mass one. It was in [**120**] where amenability was first defined under the German name "messbar".

The third question is known as the Banach–Ruziewicz Problem. S. Banach showed that the answer to this question is positive for S^1 but left the cases $n \geq 2$ unanswered. It was not until the 1980s when the other cases were answered. For $n \geq 2$ the answer is negative. For the cases $n \geq 4$ the problem was solved independently by D. Sullivan [**110**] and G. Margulis [**83**]. For the cases $n = 2, 3$ the problem was solved by V. Drinfel′d in [**46**]. For an extensive survey on this topic see A. Paterson's book on amenability and S. Wagon's book on the Banach–Tarski paradox [**95, 121**]

The notion of amenable ergodic actions was first introduced by R. Zimmer in 1978 and has had great influence in ergodic theory and von Neumann algebras [**128**]. C. Anantharaman-Delaroche generalised the ideas in [**128**] and introduced an amenable group action on a von Neumann algebra [**2, 3**]. In 1987 these ideas were generalised further to incorporate amenable actions on C*-algebras and it was shown that a group acts amenably on a C*-algebra if and only if the reduced cross product was nuclear [**4**]. In 2000, C. Anantharaman-Delaroche and J. Renault extended the definition of amenability to groupoids and this encodes the definition of a topological amenable action on a locally compact space.

In 2000, N. Higson and J. Roe showed that a discrete group acts topologically amenably on a compact Hausdorff space (also known as amenable at infinity) if and only if the group has G. Yu's property A [**67**]. In particular amenability at infinity is a coarse invariant and satisfies the coarse Baum–Connes conjecture and the Novikov conjecture. This result was extended to locally compact second countable groups by S. Deprez and K. Li in 2014 [**44**].

In 1999, E. Kirchberg and S. Wassermann in [**75**] introduced a seemingly separate notion of exact groups. For any locally compact group G, the universal cross product functor from the category of G-C*-algebras to G-C*-algebras preserves short exact sequences. In [**75**] the authors introduced the class of exact groups: the groups of which the *reduced* cross product functor preserves short exact sequences. They asked whether every group is exact and in the same paper they showed that for discrete groups, exactness of the reduced group C*-algebra is equivalent to exactness of the group but left the question open for locally compact groups.

Amenability has a lot of characterisations in different areas of mathematics. In particular in operator theory, a discrete group is amenable if and only if the reduced group C*-algebra is nuclear. An analogue exists for groups that are amenable at infinity. Indeed it was proved independently by N. Ozawa and C. Anantharaman-Delaroche that amenability at infinity for discrete groups is characterised by exactness of the reduced group C*-algebra [**5, 94**]. They left the question of equivalence open for locally compact groups.

C. Anantharaman-Delaroche in [**5**] showed that if a locally compact group is amenable at infinity then the group is exact in the sense of E. Kirchberg and S. Wassermann. However in the same paper she introduced property (W) to provide a partial converse. That is if a locally compact group is exact and has property (W) then the group is amenable at infinity.

Property (W) can be thought of as a weaker version of inner amenability and is satisfied by every discrete countable group. In Chapter 4 we answer one of the questions in [**5**, Question 9.3.] for locally compact second countable groups and show the converse is true without the assumption of property (W).

Theorem (Theorem 4.6.3)**.** *Let G be a locally compact second countable group that does not have property A. Then there exists a non-compact operator in the kernel of the natural surjective map $C_{lu}(G) \rtimes_{L,r} G \to (C_{lu}(G)/C_0(G)) \rtimes_{L,r} G$.*

We have the following consequence of this result that characterises amenability at infinity for locally compact second countable groups.

Corollary (Corollary 4.6.4)**.** *Let G be a locally compact second countable group. Then the following are equivalent.*

(1) *G has property A.*
(2) *G is amenable at infinity.*
(3) *G is exact.*
(4) *The following sequence*

$$0 \longrightarrow C_0(G) \rtimes_{L,r} G \longrightarrow C_{lu}(G) \rtimes_{L,r} G \longrightarrow (C_{lu}(G)/C_0(G)) \rtimes_{\dot{L},r} G \longrightarrow 0$$

is exact.

Coarse embeddings into Hilbert space. We have established that coarse embeddings into a Hilbert space have important consequences in the Baum–Connes and Novikov conjectures. To establish which groups coarsely embed into a Hilbert space, it is interesting to establish permanence properties of this class of groups. In general it is unknown whether coarse embeddings into a Hilbert space are preserved by extensions however if the quotient group has property A and the subgroup is coarsely embeddable then the central group can coarsely embed [**57**].

In [**39**], the authors proved that coarse embeddability into a Hilbert space is preserved by particular extensions. Indeed if G and H coarsely embed into a Hilbert space then $G \wr H = \bigoplus_H G \rtimes H$ also coarsely embeds into a Hilbert space without the mention of property A. In Chapter 3 we extend these results.

Theorem (Theorem 3.3.5)**.** *Assume X is a proper metric space with bounded geometry and coarsely embeds into a Hilbert space. If G and H are groups that coarsely embeds into Hilbert spaces and H acts transitively on X then $G \wr_X H = \bigoplus_X G \rtimes H$ coarsely embeds into a Hilbert space.*

In [**58**], E. Guentner and J. Kaminker introduce compression to measure how close a coarse embedding is to a quasi-isometry in both an equivariant and non-equivariant sense. A lot of research has gone into finding precise values of compression for discrete groups [**8, 9, 11, 21, 40, 80, 90, 89**]. In Chapter 2 we find a lower bound of the behaviour of equivariant compression under direct limits.

Theorem (Theorem 2.1.3)**.** *Let G be a locally compact, second countable group equipped with a proper left invariant metric d that generates the topology of G. Suppose there exists a sequence of open subgroups $(G_i)_{i\in\mathbb{N}}$, each equipped with the restriction of d to G_i, such that $\varinjlim G_i = G$ and $\alpha = \inf\{\alpha_2^{\#}(G_i, d)\} > 0$. If $(G_i)_{i\in\mathbb{N}}$ has (α, l, q)-polynomial property, then we* *have the following two cases:*

$$l \geq q \Rightarrow \alpha_2^{\#}(G,d) \geq \frac{\alpha}{2l+1}$$

or,

$$l \leq q \Rightarrow \alpha_2^{\#}(G,d) \geq \frac{\alpha}{l+q+1}.$$

One useful application of compression is that whenever a finitely generated group has compression strictly greater than 1/2 then the group has property A [**58**]. Likewise when a finitely generated group has equivariant compression strictly greater than 1/2 then the group is amenable. In [**40**], the authors extend the equivariant result to all locally compact, compactly generated groups. In section 4.7 we generalise the non-equivariant result to all locally compact second countable groups.

Theorem (Theorem 4.7.7)**.** *Let G be a locally compact second countable group and let d be a **plig** metric with exponentially controlled growth. If $\alpha_2(G,d) > 1/2$ then G has property A.*

Overview of the thesis

In Chapter 1 we introduce the language of coarse geometry and the metric properties that are central to this thesis: property A, coarse embeddability into a Hilbert space and compression. Near the end of the chapter we give a list of examples of groups that have property A, groups that are coarsely embeddable into a Hilbert space and the compression of some particular groups.

In Chapter 2 we study the behaviour of compression under direct limits. To do this we introduce the (α, l, q)-polynomial property, which measures the growth of Lipschitz constants, and apply it to find a lower bound of the compression in terms of α, l and q.

In Chapter 3 we show that coarse embeddability into a Hilbert space is preserved under wreath products. Indeed we do this in a more general context than groups and we introduce the notion of wreath products of metric spaces. We then show that this metric space construction preserves coarse embeddability into a Hilbert space when it has the coarse path lifting property.

In Chapter 4 we prove that in the class of locally compact second countable groups, amenability at infinity and exactness in the sense of E. Kirchberg and S. Wassermann [**75**] are equivalent. This is done by presenting a sequence of algebras that fails to be exact after taking the reduced cross product functor whenever the group is not amenable at infinity. This is done by using results known in the discrete case about a particular ideal of operators and lifting to the locally compact setting. Then we use a slice map to show that these lifted operators prevents the exactness of a particular sequence. In the last section of this chapter we generalise a result of compression to the locally compact second countable case.

The material in Chapters 2, 3 and Section 4.7 has been submitted as the following papers:

- [**26**] Chris Cave and Dennis Dreesen. Embeddings of locally compact hyperbolic groups into L_p-spaces. Preprint arXiv:1303.4250.
- [**27**] Chris Cave and Dennis Dreesen. Equivariant compression of certain direct limit groups and amalgamated free products. Preprint arXiv:1309.4636.
- [**28**] Chris Cave and Dennis Dreesen. Embeddability of generalized wreath products. *Bull. Aust. Math. Soc,* **2015**, *91*, 250-263

The material in Section 4.6 is currently a preprint in preparation:

- [**22**] Jacek Brodzki, Chris Cave and Kang Li. Exactness of locally compact second countable groups. In preparation.

CHAPTER 1

Background

1.1. Metric Geometry

Definition 1.1.1. Let (X, d) be a metric space. A metric is *proper* if every bounded subset is relatively compact. If $X = G$ is a group then the metric is *left invariant* if $d(gh, gh') = d(h, h')$ for all $g, h, h' \in G$.

Definition 1.1.2. A metric space is *discrete* if every point is an open set. A metric space is *uniformly discrete* if there exists $\delta > 0$ such that $B(x, \delta) = \{x\}$ for all $x \in X$.

Unless stated otherwise, all discrete metric spaces are assumed to be countable.

Definition 1.1.3. Let G be a group. A map $l\colon G \to \mathbb{R}^+$ is a *length function* if it satisfies the following conditions

(1) $l(g) = 0 \Leftrightarrow g = 1$.
(2) $l(g) = l(g^{-1})\ \forall g \in G$.
(3) $l(gh) \le l(g) + l(h)\ \forall g, h \in G$.

If l is a length function then $d(x, y) := l(x^{-1}y)$ is a left invariant metric. If d is a left invariant metric then $l(g) := d(e, g)$ is a length function.

Definition 1.1.4 ([**100**, Definition 1.8.])**.** Let X and Y be metric spaces and let $f\colon X \to Y$ be a map.

(1) The map f is *proper* if the pre-image of every bounded subset of Y is a bounded subset of X.
(2) The map is *bornologous* if for every $R > 0$ there is $S > 0$ such that
$$d(x, x') < R \Rightarrow d(f(x), f(x')) < S \quad \forall x, x' \in X.$$
(3) The map f is *coarse* if it is proper and bornologous.

The composition of proper (bornologous or coarse) maps is proper (bornologous or coarse).

Definition 1.1.5. Two maps f, f' from a set X to a metric space Y are *close* if $d(f(x), f'(x))$ is bounded, uniformly in x. We say two metric spaces X and Y are *coarsely equivalent* if there exists coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and X respectively.

The fundamental example is the inclusion $\mathbb{Z} \to \mathbb{R}$ and the ceiling or floor map $\mathbb{R} \to \mathbb{Z}$. It is obvious that coarse equivalence is an equivalence relation among metric spaces.

Definition 1.1.6. Let X and Y be metric spaces. A function $f\colon X \to Y$ is a *coarse embedding* if there exists increasing functions $\rho_\pm\colon [0, \infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} \rho_\pm(t) = \infty$ and
$$\rho_-(d(x, x')) \le d(f(x), f(x')) \le \rho_+(d(x, x')) \quad \forall x, x' \in X.$$
A function $f\colon X \to Y$ is called a *coarse surjection* if there exists $C > 0$ such that for all $y \in Y$ there exists $x \in X$ such that $d(f(x), y)) \le C$.

It is not hard to show that a function is a coarse equivalence if and only if it is a coarse embedding and a coarse surjection. A particular class of metric spaces we shall be interested in are those that can coarsely embed into a Hilbert space.

Definition 1.1.7. A map $f\colon X \to Y$ is *large-scale Lipschitz* if there exists constants A, B such that $d(f(x), f(x')) \leq Ad(x, x') + B$ for all $x, x' \in X$. The map f is called *Lipschitz* if $B = 0$ and called a *quasi-isometry* if

$$\frac{1}{A}d(x,x') - B \leq d(f(x), f(x')) \leq Ad(x,x') + B \quad \forall x, x' \in X.$$

If $B = 0$ then f is called *bi-Lipschitz.* Two metric spaces X and Y are *quasi-isometric* if there exists a coarse surjective quasi-isometry between the two.

Observe that if X is a uniformly discrete metric space then every large-scale Lipschitz map (quasi-isometry) is a Lipschitz (bi-Lipschitz) map.

Definition 1.1.8. A discrete metric space (X, d) is called *quasi-geodesic* if there exists $\delta > 0$ and $\lambda \geq 1$ such that for all $x, y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of X such that

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \leq \lambda d(x,y) \quad \text{and} \quad d(x_i, x_{i+1}) \leq \delta \quad \text{for all } 1 \leq i \leq n.$$

***Proposition* 1.1.9** ([**58**, Proposition 2.9.])**.** *Let X and Y be metric spaces and suppose X is a quasi-geodesic space. If $f\colon X \to Y$ is bornologous then f is large-scale Lipschitz.*

Let G be a locally compact group. A set F generates G if the subgroup generated by F is equal to G. That is every element of G can be written as a word of finitely many elements in F.

Definition 1.1.10. Let G be a group and suppose F generates G. For $g \in G$ the *word length* of g relative to F is

$$|g|_F := \inf\left\{n : g = h_1 h_2 \cdots h_n \text{ for some } h_1, \ldots, h_k \in F\right\}.$$

This forms a length function on G and so defines the *word metric* $d(x, y) := |x^{-1}y|$ relative to F. This is the same metric as the graph metric on the Cayley graph $\mathrm{Cay}(G, F)$. For a subset $S \subset G$, the Cayley graph of G with respect to S has vertices as elements of G and two vertices g, h are connected by an edge if and only if $h = gs$ for some $s \in S \cup S^{-1}$. It follows that $\mathrm{Cay}(G, S)$ is connected if and only if S generates G and the graph metric on $\mathrm{Cay}(G, S)$ is precisely the word metric on G relative to S. Clearly the word metric depends on the generating set however the coarse equivalence class does not.

***Proposition* 1.1.11** ([**100**, Proposition 1.15.][**82**, pp14–16])**.** *Suppose G is compactly generated and suppose d and d' are word metrics associated to compact generating sets. Then (G, d) and (G, d') are quasi-isometric.*

When G is not compactly generated, it is not obvious what metric to use. The word metric with respect to a generating set will no longer be proper. Fortunately with the following result we can equip all locally compact second countable groups with a sensible metric.

***Theorem* 1.1.12** ([**109**], [**38**, Theorem 2.B.4.])**.** *For a locally compact group G the following are equivalent*

(1) *G is second countable.*
(2) *G is σ-compact and first countable.*

(3) *there exists a proper left invariant metric on G that generates the topology.*

We shall call such metrics **plig** metrics.

Proof (Sketch). (1) $\Leftrightarrow$ (2). This is well known.

(3) $\Rightarrow$ (2). Any metrizable space is first countable. As the metric is proper then G is σ-compact. This is because $G = \cup_{n\in\mathbb{N}}\overline{B(1,n)}$ and by properness $\overline{B(1,n)}$ is compact.

(2) $\Rightarrow$ (3). This proof is originally from [**38**, Theorem 2.B.4.]. Let $\mathcal{V} = (V_n)_{n\geq 0}$ be a countable neighbourhood basis of the identity. As G is locally compact, it follows after relabeling that V_0 is relatively compact. As G is σ-compact, $G = \bigcup_{n\geq 0} L_n$ where L_n are symmetric compact subsets that contain the identity for all $n \geq 0$. Set $K_0 = L_0 \cup \overline{V}_0$ and define inductively for $n \geq 1$, $K_{n+1} = L_n \cup (K_n)^3$.

There exists $A_0 \in \mathcal{V}$ such that $(A_0)^3 \subset V_0$. This is because multiplication is continuous on G. Set $K_{-1} = A_0 \cap V_1$. Then $(K_{-1})^3 \subset K_0$ and $K_{-1} \subset V_1$. In particular, K_{-1} is a neighbourhood of the identity so there exists $A_{-1} \in \mathcal{V}$ such that $(A_{-1})^3 \subset K_{-1}$. Set $K_{-2} = A_{-1} \cap V_2$. Then $(K_{-2})^3 \subset K_{-1}$ and $K_{-2} \subset V_2$. We continue this procedure so that we obtain $(K_n)_{n\in\mathbb{Z}}$ so that K_n is symmetric, contain the identity, have non-empty interior and $G = \cup_{n\in\mathbb{Z}} K_n$. Then define the length function to be

$$|g| = \inf\left\{ t \in \mathbb{R} : g = w_{n_1}\cdots w_{n_k} \text{ such that } w_{n_j} \in K_{n_j} \text{ and } t = \sum_{j=1}^{k} 2^{n_j} \right\}$$

By construction $|\cdot|$ is continuous and proper. By using an inductive argument one can show that if $|g| < 2^n$ then $g \in K_n$ for all n. If U is an open set then $x^{-1}U$ is an open neighbourhood of the identity for all $x \in U$. Therefore for all $x \in U$ there exists $n(x) \in \mathbb{N}$ such that $V_{n(x)} \subset x^{-1}U$. As $K_{-n} \subset V_n$ for all $n \in \mathbb{N}$, we have that $B(x, 2^{-n(x)}) \subset U$ for all $x \in U$. Hence this metric generates the topology on G. □

Theorem* 1.1.13** ([**38**, Corollary 4.A.6 (2).][**59**, Theorem 2.8.])**.** *Let G be a locally compact second countable group. Assume d and d' are* ***plig *metrics on G. Then the identity map $(G,d) \to (G,d')$ is a coarse equivalence.*

Proof (Sketch). Each R-ball of d is compact therefore there exists S_R such that $B_d(1,R) \subset B_{d'}(1,S_R)$ for all $R > 0$ as d' is proper. By reversing the roles of d and d' we have that the identity map is a coarse equivalence. □

Observe that word metrics relative to generating sets do not usually generate the topology. For example $[-1,1]$ generates $\mathbb{R}$ but the length function is not continuous with respect to the standard topology. However word metrics with respect to a compact generating set is coarsely equivalent to one (hence all) **plig** metrics [**38**, Corollary 4.A.6(2).]. See [**38**] for an extensive survey on metric geometry of locally compact groups.

Remark 1.1.14. It is important to observe that the coarse equivalence is not necessarily a quasi-isometry. Take $\mathbb{F}_\infty$, the free group on countably many generators. Label the generators by $(x_n)_{n\in\mathbb{N}}$ and define two functions $l_1(x_n^{\pm 1}) = n$ and $l_2(x_n^{\pm 1}) = n^2$. We extend to length functions on all of $\mathbb{F}_\infty$ by setting

$$l_i(g) := l_i(x_{n_1}^{\epsilon_1}) + \cdots + l_i(x_{n_k}^{\epsilon_k}) \quad \text{for } i = 1, 2$$

where $g = x_{n_1}^{\epsilon_1}\cdots x_{n_k}^{\epsilon_k}$ and $\epsilon_i = \pm 1$. The metrics $d_1(x,y) := l_1(x^{-1}y)$ and $d_2(x,y) = l_2(x^{-1}y)$ are proper and left invariant but not quasi-isometric because for all constants $A, B > 0$ there exists $n \in \mathbb{N}$ such that $n^2 \geq An + B$. However these metrics are coarsely equivalent.

Definition 1.1.15. A discrete metric space (X, d) has *bounded geometry* if for all $R > 0$ there exists N_R such that $|B(x, R)| \leq N_R$ for all $x \in X$.

Definition 1.1.16. A general metric space (i.e. not necessarily discrete) has *bounded geometry* if it is coarsely equivalent to a discrete metric space with bounded geometry.

If a metric space is coarsely equivalent to a uniformly discrete metric space with bounded geometry then we call the image of the discrete metric space under the coarse equivalence a *coarse lattice.*

***Proposition* 1.1.17** ([**59**, Lemma 3.3.])**.** *Every locally compact second countable group equipped with a **plig** metric is coarsely equivalent to a uniformly discrete space with bounded geometry.*

Proof (Sketch). This proof is originally from [**59**, Lemma 3.3.]. Choose a maximal family of elements $Z = \{z_i\}_{i\in\mathbb{N}}$ such that $d(z_i, z_j) \geq 1$. Therefore $G = \bigcup_{i\in\mathbb{N}} B(z_i, 1)$ and so the metric space (G, d) is coarsely equivalence to Z when equipped with the subspace metric from G.

Fix $R > 0$ and $z_0 \in Z$. Then $\sum_{z\in B(z_0,R)\cap Z} \mu(B(z, 1/2)) \leq \mu(z, R + 1/2)$, because the sets $B(z, 1/2)$ and $B(z', 1/2)$ are disjoint for any two distinct points. Hence by left invariance of the Haar measure, $|B(z, R) \cap Z| \leq \mu(1, R + 1/2)/\mu(B(1, 1/2))$ for any $z \in Z$. Hence Z has bounded geometry. □

1.2. Some metric geometry properties

1.2.1. Property A. A *kernel* on a set X is a function $k\colon X \times X \to \mathbb{C}$ (or $\mathbb{R}$). Usually one can think of k as an infinite $X \times X$ matrix and $k(x, y)$ is the value at the x-th row and y-th column.

Definition 1.2.1. Let X be a set. A kernel of *positive type* is a function $k\colon X \times X \to \mathbb{C}$ such that for all finite sequences $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} k(x_i, x_j) \geq 0.$$

Definition 1.2.2. Let X be a set. A kernel of *negative type* is a function $k\colon X \times X \to \mathbb{R}$ such that for all finite sequences $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 0$,

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j k(x_i, x_j) \leq 0.$$

We say a kernel is *self-adjoint* (or *symmetric* if real valued) if $k(x, y) = \overline{k(y, x)}$ for all $x, y \in X$. We say a positive (negative) type kernel is *normalised* if $k(x, x) = 1$ ($k(x, x) = 0$) for all $x \in X$. For a group G a function $\varphi\colon G \to \mathbb{C}$ is of *positive type* (or *negative type*) if the kernel $k(g, h) := \varphi(g^{-1}h)$ is of positive type (respectively negative type).

For a locally compact, σ-compact space X (e.g. a space with bounded geometry), denote the space of regular Borel probability measures by $\mathrm{Prob}(X)$. The space $\mathrm{Prob}(X)$ can be identified with the space of all positive linear functionals on $C_0(X)$ with norm 1 [**104**, Theorem 6.19.]. For a fixed positive regular Borel measure μ we can identify $\mathrm{Prob}(X) = \{f \in L^1(X, \mu) : \|f\|_1 = 1 \text{ and } f \geq 0\}$ [**35**, Proposition 7.3.8.]. So $\mathrm{Prob}(X)$ comes with two topologies, the norm and the weak-$*$ topology. When $X = G$ is a locally compact second countable group then we will use a fixed Haar measure μ. Recall the following characterisations of amenability.

Definition and Theorem 1.2.3 ([**95**, Theorem 4.4.], [**18**, Theorem G.3.2.])**.** Let G be a locally compact group. Then the following are equivalent

(1) for all compact subsets $K \subset G$ and $\varepsilon > 0$ there exists $f \in \mathrm{Prob}(G)$ such that

$$\sup_{g \in K} \|g \cdot f - f\|_1 \leq \varepsilon$$

where $g \cdot f(h) = f(g^{-1}h)$ for all $g, h \in G$.

(2) The trivial representation is weakly contained in the left regular representation $\lambda \colon G \to L^2(G)$. That is for all compact subsets $K \subset G$ and $\varepsilon > 0$ there exists unit vectors $\xi \in L^2(G)$ such that

$$\sup_{g \in K} |1 - \langle \lambda_g \xi, \xi \rangle| < \varepsilon.$$

If G has one of the two equivalent properties then we say G is *amenable*.

There are a large amount of applications and characterisations of amenability however we shall only use it to demonstrate the similarity between this definition and the definition of property A.

Definition 1.2.4. [**102**, Definition 2.1.] Let X be a proper metric space with bounded geometry. We say X has *property A* if there exists a sequence of weak-$*$ continuous maps $f_n \colon X \to \mathrm{Prob}(X)$ such that

(1) for each n there is an R such that for each $x \in X$, $\mathrm{Supp}(f_n(x)) \subset B(x, R)$ and
(2) for each $S > 0$, as $n \to \infty$

$$\sup_{d(x,y)<S} \|f_n(x) - f_n(y)\|_1 \to 0.$$

When X is discrete, this coincides with the definition of property A from [**127**].

***Proposition* 1.2.5** ([**123**, Proposition 1.1.3.])**.** *Assume X and Y are discrete metric spaces with bounded geometry and X has property A. If there exists a coarse embedding $\iota \colon Y \to X$ then Y has property A.*

***Proposition* 1.2.6** ([**102**, Lemma 2.2.])**.** *Let X be a (not necessarily discrete) proper metric space with bounded geometry. Then X has property A if and only if some (hence every) coarse lattice in X has property A.*

This means when we consider locally compact second countable groups, we can refer to a group having property A without referring to a **plig** metric. For a locally compact group G and a subset $L \subset G$, the *tube of* L is the set $\mathrm{Tube}(L) := \{(x, y) \in G \times G : x^{-1}y \in L\}$. If L is compact and a kernel is supported on $\mathrm{Tube}(L)$ then we say the kernel has *compact width*. If G is discrete then we will say that the kernel has *finite width*. We have the following useful characterisation of property A and coarse embeddability into a Hilbert space.

***Proposition* 1.2.7** ([**44**, Theorem 2.3.] [**118**, Proposition 3.2.])**.** *A locally compact second countable group G has property A if and only if for every compact subset $K \subset G$ and $\varepsilon > 0$, there exists a compact subset $L \subset G$ and a positive type kernel $k \colon G \times G \to \mathbb{C}$ such that* $\mathrm{Supp}(k) \subset \mathrm{Tube}(L)$ *and*

$$\sup_{(s,t) \in \mathrm{Tube}(K)} |1 - k(s, t)| < \varepsilon.$$

***Theorem* 1.2.8** ([**123**, Theorem 3.2.8.])**.** *Let X be a metric space (not necessarily discrete). Then the following are equivalent*

(1) *X is coarsely embeddable into a Hilbert space.*

(2) *For all* $R, \varepsilon > 0$ *there exists a normalised symmetric kernel* $k\colon X \times X \to \mathbb{R}$ *of positive type such that:*
   (a) $\sup_{d(x,y)\leq R} |1 - k(x,y)| < \varepsilon$ *and*
   (b) $\lim_{S\to\infty} \sup\{|k(x,y)| : d(x,y) \geq S\} = 0$.

***Corollary* 1.2.9.** *If a locally compact second countable group is amenable then it has property A.*

Proof. If G is amenable then for all $R > 0$ and $\varepsilon > 0$ there exists a unit vector $\xi \in L^2(G)$ such that $\sup_{|g|<R} |1 - \langle \lambda_g \xi, \xi\rangle| < \varepsilon/3$. Choose a compactly supported unit vector $\eta \in L^2(G)$ such that $\|\xi - \eta\|_2 < \varepsilon/3$. Hence for all $g \in B(1, R)$

$$|1-\langle\lambda_g\eta,\eta\rangle| \leq |1-\langle\lambda_g\xi,\xi\rangle|+|\langle\lambda_g\xi,\xi\rangle-\langle\lambda_g\eta,\eta\rangle| < \varepsilon/3+|\langle\lambda_g\xi,\xi-\eta\rangle|+|\langle\lambda_g(\xi-\eta),\eta\rangle| \leq \varepsilon.$$

Set $\varphi(g) = \langle\lambda_g\eta,\eta\rangle$, this is a compactly supported positive definite function [**18**, Proposition C.4.3.] and so the positive definite kernel $k(g,h) = \varphi(g^{-1}h)$ suffices. □

Finding a coarse embedding into a Hilbert space is difficult. However the following result shows that property A guarantees a coarse embedding into a Hilbert space.

***Theorem* 1.2.10** ([**127**])**.** *A discrete metric space with property A coarsely embeds into a Hilbert space.*

***Proposition* 1.2.11** ([**44**, Proposition 3.2.])**.** *A locally compact second countable group with property A coarsely embeds into a Hilbert space.*

Let G be a locally compact group and define a convolution operation on $L^1(G)$ where

$$f * g(s) = \int_G f(r)g(r^{-1}s)\, d\mu(r).$$

This turns $L^1(G)$ into a Banach $*$-algebra. More generally for any $f \in L^1(G)$ and $g \in L^p(G)$, $f * g$ belongs to $L^p(G)$ [**63**, Corollary 20.14.]. For every unitary representation $\pi\colon G \to \mathcal{U}(G)$ there is an associated $*$-representation $\pi\colon L^1(G) \to \mathcal{B}(\mathcal{H})$ defined by

$$\pi(f) = \int_G f(g)\pi(g)\, d\mu(g)$$

where this operator is uniquely defined by

$$\langle\pi(f)\xi,\eta\rangle = \int_G f(g)\langle\pi(g)\xi,\eta\rangle\, d\mu(g)$$

for all $\xi, \eta \in \mathcal{H}$. Conversely, any non-degenerate $*$-representation of $L^1(G)$ is of this form. There are two important C*-algebras that we can form from this procedure.

Definition 1.2.12. Let $\lambda\colon G \to L^2(G)$ be the left regular representation. Then for any $f \in L^1(G)$ and $\xi \in L^2(G)$, $\lambda(f) = f * \xi$. Denote $C^*_r(G)$ to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_r = \|\lambda(f)\|_{\mathcal{B}(L^2(G))}.$$

This C*-algebra is called the *reduced group C*-algebra.*

The completion of $\mathbb{C}[G]$ with respect to the norm $\|\cdot\|_{\max}\colon L^1(G) \to \mathbb{R}$,

$$\|f\|_{\max} := \sup\{\|\pi(f)\| : \pi \text{ is a non-degenerate } *\text{-homomorphism}\}$$

is called the *maximal* group C*-algebra of G, which we denote by $C^*(G)$. In particular $\|f\|_r \leq \|f\|_{\text{univ}} \leq \|f\|_1$ for all $f \in L^1(G)$.

Suppose X is a countable uniformly discrete metric space with bounded geometry e.g. a countable discrete group. For an operator $T \in \mathcal{B}(\ell^2(X))$ let $T_{x,y}$ be the matrix entry $\langle T\delta_y, \delta_x \rangle$, where δ_x and δ_y are point masses at x and y respectively. An operator $T \in \mathcal{B}(\ell^2(X))$ has *finite propagation* if there exists $R > 0$ such that $T_{x,y} = 0$ whenever $d(x,y) > R$. This forms a $*$-algebra inside $\mathcal{B}(\ell^2(X))$ and the closure of this $*$-algebra is called the *uniform Roe algebra.* We denote this completion by $C_u^*(X)$. For the next proposition, see Section 4.3 for the definition of cross products and reduced cross products.

***Proposition* 1.2.13** ([**23**, Proposition 5.1.3.])**.** *Let Γ be a discrete countable group and let $\lambda\colon \Gamma \to \mathrm{Aut}(\ell^\infty(\Gamma))$ be the action $\lambda_g f(h) = f(g^{-1}h)$ for all $f \in \ell^\infty(\Gamma)$ and $g, h \in \Gamma$. Then $C_u^*(\Gamma) \cong \ell^\infty(\Gamma) \rtimes_{\lambda,r} \Gamma$.*

The proof of this is similar to the proof of Lemma 4.4.2. From this we can see that $C_r^*(\Gamma)$ is a closed $*$-subalgebra of $C_u^*(\Gamma)$ when Γ is a discrete countable group.

Definition 1.2.14. A linear map φ between C*-algebras A and B is *completely positive* if the map $\varphi_n\colon M_n(A) \to M_n(B)$, defined by $\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]_{i,j}$ is positive for every n.

Definition 1.2.15. An *operator space* is a closed subspace of a C*-algebra. A linear map φ from an operator space $X \subset A$ into an operator space $Y \subset B$ is called *completely bounded* if

$$\|\varphi\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\varphi\colon M_n(X) \to M_n(Y)\| < \infty.$$

We say φ is *completely contractive* if $\|\varphi\|_{\mathrm{cb}} \leq 1$.

Definition 1.2.16. Let A and B be separable C*-algebras. A map $\theta\colon A \to B$ is called *nuclear* if there exist a sequence of completely contractive positive maps $\varphi_n\colon A \to M_{k(n)}(\mathbb{C})$ and $\psi_n\colon M_{k(n)}(\mathbb{C}) \to B$ such that

$$\|\psi_n \circ \varphi_n(a) - \theta(a)\| \to 0 \quad \forall a \in A.$$

Definition 1.2.17. Let A be a separable C*-algebra. Then A is *nuclear* if the identity map $\mathrm{id}_A\colon A \to A$ is a nuclear map. A is *exact* if there exists a faithful representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$ such that π is nuclear.

Indeed every nuclear C*-algebra is exact and every closed $*$-algebra of a nuclear or exact C*-algebra is exact. This is because one can restrict the nuclear map to the $*$-subalgebra and this will still be a nuclear map.

We have characterisations of nuclearity and exactness of C*-algebras. Given two C*-algebra, one can form the algebraic tensor product. Similar to the group algebra, there are two different completions on the algebraic tensor product to make the algebra into a C*-algebra: the maximal and a natural minimal completion. We denote these two completions by $\otimes_{\max}$ and $\otimes_{\min}$.

A C*-algebra A is nuclear if and only if $A \otimes_{\max} B = A \otimes_{\min} B$ for any C*-algebra B [**23**, Theorem 3.8.7.] [**34, 74**]. A is exact if and only if the functor $A \otimes_{\min} -$ preserves short exact sequences of C*-algebras [**23**, Theorem 3.9.1.] [**73**]. When a group G acts trivially on a C*-algebra A, then $A \rtimes_r G \cong A \otimes_{\min} C_r^*(G)$.

***Theorem* 1.2.18.** *Let Γ be a countable discrete group. Then the following are equivalent:*

(1) *Γ has property A.*
(2) *$C_r^*(\Gamma)$ is exact.*
(3) *Γ is exact (for exactness see Definition 4.3.7).*
(4) *$C_u^*(\Gamma)$ is nuclear.*

(5) *$C^*_u(\Gamma)$ is exact.*

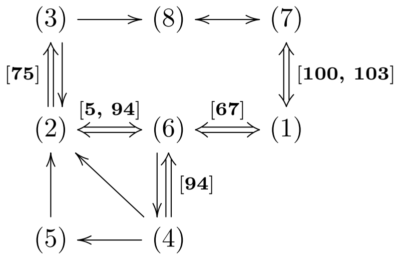
(6) *Γ admits a topological amenable action on its Stone-Čech compactification (for topological amenable action see Definition 4.5.1).*

(7) *Every ghost operator in $C^*_u(\Gamma)$ is compact (for ghost operators see Definition 4.6.1).*

(8) *The following sequence is exact*

$$0 \to c_0(\Gamma) \rtimes_{\lambda,r} G \to \ell^\infty(\Gamma) \rtimes_{\lambda,r} \Gamma \to (\ell^\infty(\Gamma)/c_0(\Gamma)) \rtimes_{\dot{\lambda},r} \Gamma \to 0.$$

We reference where these implications first appeared but the equivalences of (1), (2), (3) and (6) can all be found in [**23**]. The thin arrows indicate when an implication follows easily from definitions or from well known facts which can also be found in [**23**].



We give a sketch of proofs of these equivalence.

Proof (Sketch). (1) ⇔ (6). The definitions of property A and amenable action on a compact Hausdorff space are very similar and it only requires a bit of technical work to show that they are equivalent. If G acts amenably on a compact Hausdorff space then one can use the universal property of the Stone-Čech compactification to put an amenable action on βG. Most of the work in [**67**] was done by showing the original definition of property A from [**127**] is equivalent to the definition we have given here.

(2) ⇔ (3). If Γ is exact then in particular the functor $-\rtimes_{r,\tau}\Gamma$ preserves short exact sequences where τ is the trivial action. Hence $C^*_r(\Gamma)$ is exact. Nothing about discreteness has been used so this direction is true for locally compact groups as well.

For any C*-dynamical system (A, α, Γ) one can define maps $\pi_A \colon A \rtimes_{\alpha,r} \Gamma \hookrightarrow (A \rtimes_\alpha \Gamma) \otimes_{\min} \Gamma$ using Fell's absorption principle [**23**, Proposition 4.1.7.]. Likewise one can also construct $\Phi_A \colon (A \rtimes_\alpha \Gamma) \otimes_{\min} \Gamma \to A \rtimes_{\alpha,r} \Gamma$ such that $\Phi_A \circ \pi_A = \mathrm{id}_{A\rtimes_{\alpha,r}\Gamma}$. If $C^*_r(\Gamma)$ is exact then we have a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \rtimes_{\alpha,r} \Gamma & \longrightarrow & A \rtimes_{\alpha,r} \Gamma & \longrightarrow & (A/I) \rtimes_{\alpha,r} \Gamma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_{A/I}} & & \\
0 & \longrightarrow & I \rtimes_\alpha \Gamma \otimes_{\min} C^*_r(\Gamma) & \longrightarrow & A \rtimes_\alpha \Gamma \otimes_{\min} C^*_r(\Gamma) & \longrightarrow & (A/I) \rtimes_\alpha \otimes_{\min} C^*_r(\Gamma) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_I} & & \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \Phi_{A/I}} & & \\
0 & \longrightarrow & I \rtimes_{\alpha,r} \Gamma & \longrightarrow & A \rtimes_{\alpha,r} \Gamma & \longrightarrow & (A/I) \rtimes_{\alpha,r} \Gamma & \longrightarrow & 0
\end{array}$$

such that the middle row is exact. This is because the maximal cross product preserves short exact sequences. By a diagram chase one can show the other two rows are exact. Interestingly, the construction of the two maps π_A and Φ_A depend on Γ being exact. It is unknown whether this implication is true for locally compact groups, see [**5**, Question 9.3.] and the remarks in [**75**, Section 6.].

(2) ⇒ (1). As $C^*_r(G)$ is exact the left regular representation $\lambda \colon C^*_r(\Gamma) \to \mathcal{B}(\ell^2(G))$ is nuclear. So for any finite subset $E \subset \Gamma$ one can find a sequence of map $\theta_n \colon C^*_r(\Gamma) \to \mathcal{B}(\ell^2(G))$ that

factors through a finite dimensional C*-algebra such that $\|\theta_n(\lambda(s)) - \lambda(s)\| \to 0$ uniformly on E. We can define a positive definite kernel $k(s,t) = \langle \theta(\lambda(s^{-1}t)\delta_{t^{-1}}, \delta_s \rangle$. We can ensure k has finite width by carefully choosing the finite dimensional C*-algebra θ factors through. This means that k satisfies the conditions in Proposition 1.2.7.

(4) $\Rightarrow$ (2). $C^*_r(\Gamma)$ is a closed $*$-algebra of $C^*_u(\Gamma) \cong \ell^\infty(\Gamma) \rtimes_{L,r} \Gamma$ because Γ is discrete. Hence if $C^*_u(\Gamma)$ is nuclear then $C^*_r(\Gamma)$ is exact.

(5) $\Rightarrow$ (2). Similarly, exactness passes to closed $*$-subalgebras.

(6) $\Rightarrow$ (4). As Γ acts amenably on $\beta\Gamma$, Γ acts amenably on the C*-algebra $C(\beta\Gamma)$. Hence $C(\beta\Gamma) \rtimes_{L,r} \Gamma = \ell^\infty(\Gamma) \rtimes_{L,r} \Gamma \cong C^*_u(\Gamma)$ is nuclear where L is the left action of Γ on $C(\beta\Gamma)$.

(7) $\Leftrightarrow$ (8). In [**64**, Lemma 9.], there is the following commuting diagram of continuous maps

$$\begin{array}{ccc} A \rtimes_{\alpha,r} \Gamma & \longrightarrow & (A/I) \rtimes_{\alpha,r} \Gamma \\ \downarrow & & \downarrow \\ C_0(\Gamma, A) & \longrightarrow & C_0(\Gamma, A/I). \end{array}$$

When $A = \ell^\infty(\Gamma)$ and $I = c_0(\Gamma)$ then we see that the ghost operators are precisely the kernel of the surjective map in the sequence in (8).

(1) $\Leftrightarrow$ (7). We comment on this equivalence after Theorem 4.6.2 □

Observe the similarities between property A and amenability. A discrete countable group Γ is amenable if and only if $C^*_r(\Gamma)$ is nuclear, if and only if Γ admits a topological amenable action on a point [**23**, Theorem 2.6.8.]. Some of these equivalences can be generalised to uniformly discrete metric spaces with bounded geometry.

***Theorem* 1.2.19** ([**23**, Theorem 5.5.7.] [**103**, Theorem 1.3.]). *Let X be a countable uniformly discrete metric space with bounded geometry. Then the following are equivalent:*

(1) *X has property A.*
(2) *$C^*_u(X)$ is nuclear.*
(3) *Every ghost operator in $C^*_u(X)$ is a compact operator.*

We now give some examples and permanence properties of property A and coarse embeddability into a Hilbert space.

Examples 1.2.20. (1) Every compact metric space has property A.

(2) Every amenable group has property A.

(3) Any discrete metric space with bounded geometry that has finite asymptotic dimension has property A [**67**, Lemma 4.3.]. For a metric space X, a cover $\mathcal{U} = \{U_i\}_{i \in I}$ has *multiplicity* k if each point in X is contained in at most k elements of $\mathcal{U}$. The cover $\mathcal{U}$ has *Lebesgue number* L if any ball of radius at most L is wholly contained in one element of $\mathcal{U}$. We say that a metric space X has *finite asymptotic dimension* if there exists $k > 0$ such that for all $L > 0$ there exists a uniformly bounded cover $\mathcal{U} = \{U_i\}_{i \in I}$ that has Lebesgue number at most L and multiplicity $k+1$. The smallest such k is called the *asymptotic dimension of* X.

In order to show that X has property A we will provide a sequence that satisfies the conditions in Definition 1.2.4 for a space that is coarsely equivalent to X. This proof is original from [**123**, Corollary 2.2.11.]. Fix $R, \varepsilon > 0$ and set $L \geq \frac{R}{35k^2\varepsilon}$. Then there exists a cover $\mathcal{U} = \{U_i\}_{i \in I}$ with Lebesgue number L and multiplicity $k+1$ and there exists N such that $\operatorname{diam}(U_i) < N$ for all $i \in \mathbb{N}$. Define a partition of unity

$$\varphi_i(x) = \frac{d(x, X \setminus U_i)}{\sum_{j\in I} d(x, X \setminus U_j)}.$$

This is well defined as the multiplicity implies there are only at most $k+1$ sums being performed for each $x \in X$. Now define a metric on I, where $d(i,j) = 0$ if $i = j$ and is equal to 1 if $i \neq j$. The space $X \times I$ is coarsely equivalent to X when $X \times I$ is equipped with the metric $d((x,i),(y,j)) = d(x,y) + d(i,j)$. This is because the inclusion $\iota\colon X \to X \times I$, $x \mapsto (x, i_0)$ for some fix $i_0 \in I$ and the projection $p\colon X \times I \to X$ are coarse maps and are uniformly close to the identity map. That is $d(\iota \circ p(x,j),(x,j)) \leq 1$ and $d(p \circ \iota(x), x) = 0$ for all $x \in X$ and $i \in I$.

For each $i \in I$ fix $y_i \in U_i$ and define $\xi_{(x,i)}(y,j) = \varphi_j(x)$ if $y = y_j$ and 0 otherwise. Thus $\|\xi_{(x,i)}\| = 1$ and $\operatorname{Supp}(\xi_{(x,i)})$ is contained in a ball of radius $N+1$ about (x,i). For short hand write $C_x = \sum_{j\in I} d(x, X \setminus U_j)$ for $x \in X$. Observe that $C_x \geq L$ and $d(x, X \setminus U_i)/C_x \leq 1$ and by the triangle inequality and that $\mathcal{U}$ has $k+1$ multiplicity, $|C_x - C_y| \leq (2k+2)d(x,y)$ for all $x, y \in X$. By using the triangle inequality we have that $|\varphi_i(x) - \varphi_i(y)| \leq 1/C_x|d(x, X \setminus U_i) - d(y, X \setminus U_i)| + 1/C_y d(y, X \setminus U_i)|\frac{C_x - C_y}{C_y}| \leq \frac{2k+3}{L} d(x,y)$. Hence by multiplicity again $\sum_{i\in I} |\varphi_i(x) - \varphi_i(y)| \leq \frac{(2k+2)(2k+3)}{L} d(x,y)$. Altogether, if $d(x,y) < R$ then $\|\xi_{(x,i)} - \xi_{(y,j)}\| < \varepsilon$.

(4) Every discrete countable group that is hyperbolic has finite asymptotic dimension and so has property A [**101**].

(5) Every finite dimensional CAT(0) cube complex has compression 1 [**25**, Theorem A.] and so has property A by Theorem 1.3.9. In particular any group acting properly, co-compactly on a finite dimensional CAT(0) cube complex has property A [**25**, Theorem B.]. This includes right-angled Artin groups and Coxeter groups.

(6) Groups that admit a presentation $\langle X|R\rangle$ where R is a single word have property A [**56**, Corollary 2.6.].

(7) Every closed subgroup of a connected Lie group has property A [**5**, Examples 3.2(1).].

(8) Every almost connected group has property A [**5**, Proposition 3.3.].

(9) Any discrete subgroup of $\mathrm{GL}_n(K)$ for a field K has property A [**55**].

(10) Suppose we have the following exact sequence of discrete countable groups

$$1 \to N \to \Gamma \to K \to 1.$$

If K and N have property A then Γ has property A [**123**, Theorem 2.3.6.]. If N has property A and K coarsely embeds into a Hilbert space then Γ also coarsely embeds into a Hilbert space [**37**, Theorem 4.1.].

(11) These results are extended in [**43**] to the locally compact second countable case. In each of the following cases if H has property A (is coarsely embeddable into a Hilbert space) then G has property A (is coarsely embeddable into a Hilbert space).
   (a) $H \subset G$ is a uniform lattice. That is H is discrete and G/H is compact.
   (b) $H \subset G$ is a lattice. That is H is discrete and G/H has finite covolume.
   (c) $H \subset G$ is a closed coamenable subgroup in the sense of Eymard [**49**].
   (d) $H \subset G$ is a closed normal subgroup and G/H has property A.
   (e) $H = G/Q$ where Q is a compact normal subgroup.
   (f) G is a measure equivalence subgroup of H (for measure equivalence see [**43**, Definition 3.5.]).

(12) The class of groups with property A is closed under subgroups, direct limits, amalgamations over a common subgroup and HNN extensions [**123**].

(13) The class of groups that coarsely embed into a Hilbert space is closed under subgroups, direct limits, amalgamation over a common subgroup, HNN extensions [**57**].

(14) The class of groups that coarsely embed into a Hilbert space is also closed under wreath products [**39**, Theorem 5.10.].

(15) Let $X = (V, E)$ be a finite graph. We say X is a C-expander if there exists a $C > 0$ such that for all $f \in \ell^2(V)$,

$$\sum_{x,y \in V} |f(x) - f(y)|^2 \leq C \sum_{(x,y) \in E} |f(x) - f(y)|^2.$$

We call this a *Poincaré inequality*. A sequence of k-regular graphs $(X_n)_{n \in \mathbb{N}}$ is an *expander sequence* if $|V_n| \to \infty$ and there exists a global $C > 0$ such that each X_n is a C-expander. Expanders can be formed by taking the box space of residually finite property (T) groups [**81**, Section 3.3.]. Given a sequence of graphs $(X_n)_{n \in \mathbb{N}}$ we define a metric d on the disjoint union $X = \sqcup_{n \in \mathbb{N}} X_n$ where $d(X_n, X_m) \to \infty$ as $n + m \to \infty$ and d is the graph metric when restricted to each component X_n.

We will show that when $(X_n)_{n \in \mathbb{N}}$ is an expander sequence then the disjoint union X when equipped with the above metric does not coarsely embed into a Hilbert space. This proof is an adaptation of [**92**, Theorem 4.9.]. For a k-regular graph $Y = (V, E)$ and for a fixed distance R there are at most k^R vertices within distance R of a given point. If $R = \log_k(|V|/2)$, then there are at least $|V|/2$ vertices that are distance greater than R away from any given vertex.

Suppose $f\colon X \to \mathcal{H}$ is a coarse embedding and let $\rho_\pm$ be the functions defined in Definition 1.1.6 associated to the coarse embedding. Without loss of generality we can assume that $\mathcal{H} = L^2(0,1)$. By the Poincaré inequality we have that $\sum_{x,y \in V_n} |f_x(t) - f_y(t)|^2 \leq C \sum_{(x,y) \in E_n} |f_x(t) - f_y(t)|^2$ for all $t \in (0,1)$ and all $n \in \mathbb{N}$. By integrating over t we have that $\sum_{x,y \in V_n} \|f_x - f_y\|^2 \leq C \sum_{(x,y) \in E_n} \|f_x - f_y\|^2$ for all $n \in \mathbb{N}$. By substituting $\rho_\pm$ we have that $\sum_{x,y \in V_n} \rho_-(d(x,y))^2 \leq \rho_+(1) C k |V_n|/2$. However because there are at least $|V_n|/2$ vertices that are distance greater than $\log_k(|V_n|/2)$ away from any point we have and that ρ_- is a non-decreasing function we have that $\sum_{x,y \in V_n} \rho_-(d(x,y))^2 \geq \rho_-(\log_k(|V_n|/2)) |V_n|^2/2$. Hence we have a contradiction because ρ_- is unbounded and $|V_n| \to \infty$ as $n \to \infty$ but ρ_- must also satisfy $\rho_-(\log_k(|V_n|/2)) \leq \frac{\rho_+(1)Ck}{2|V_n|}$ for all $n \in N$.

In fact a sequence of graphs $(X_n)_{n \in \mathbb{N}}$ is an expander sequence if and only if it satisfies an ℓ^p-Poincaré inequality for all $1 \leq p < \infty$. So by using a similar argument as above an expander sequence can not coarsely embed into an ℓ^p space for all $1 \leq p < \infty$ [**84**, Proposition 3.], [**100**, Proposition 11.30.].

(16) There exist countable discrete metric spaces with bounded geometry that do not coarsely contain an expander (in fact the metric space does not even weakly contain an expander, see [**10**] for the definition) and does not coarsely embed into any ℓ^p-space for $1 \leq p < \infty$ [**10**].

(17) The first examples of groups that do not coarsely embed into a Hilbert space were constructed by Gromov [**54, 7**]. These groups are finitely generated and weakly contain an expander (see the discussion in [**91**, Subsection 2.4.]), which is enough to prevent a group from coarsely embedding into a Hilbert space.

(18) There exist finitely generated groups that isometrically contain a copy of an expander sequence [**91**] and finitely presented groups that contains a quasi-isometric copy of an expander sequence. Furthermore there exist closed aspherical manifolds of dimension 4 and higher whose fundamental groups contain a quasi-isometric copy of an expander sequence [**91**, Corollary 3.5.][**106**].

(19) There exist finitely generated groups that act properly on a CAT(0) cube complex but do not have property A [**91**, Theorem 6.2.]. In particular these are the first examples of groups that have the Haagerup property but do not have property A [**33**].

1.2.2. The Haagerup property. Let E be a Banach space and let $\mathrm{Isom}(E)$ be the space of bounded linear operators that are isometries. A continuous affine action on a Banach space E consists of a strongly continuous representation $\pi\colon G \to \mathrm{Isom}(E)$ ($v \mapsto \pi_s(v)$ is continuous for all $s \in G$) and a continuous function $b\colon G \to E$ that satisfies the *cocycle* condition

$$b(st) = \pi_s(b(t)) + b(s) \quad \forall s, t \in G.$$

The action is given by $s \cdot v = \pi_s(v) + b(s)$ for all $s \in G$ and $v \in E$. A function that satisfies the cocycle condition is called a *1-cocycle*. Observe that $b(e_G) = 0$ and the image of the 1-cocycle is the orbit of the origin under the action. We say that such an action is *proper* if for all bounded subsets $B, C \subset E$ the set $\{g \in G : \alpha_g(B) \cap C \neq \emptyset\}$ is finite. Notice that α is proper if and only if the 1-cocycle associated to α is a proper map. We have the following useful relation between 1-cocycles on Hilbert spaces and continuous functions of negative type.

***Proposition* 1.2.21** ([**42**, pp 62])**.** *Let $\mathcal{H}$ be a Hilbert space and $b\colon G \to \mathcal{H}$ a 1-cocycle associated to a unitary representation. Then the continuous map $g \mapsto \|b(g)\|^2$ is of negative type.*

***Proposition* 1.2.22** ([**42**, pp 63])**.** *Let $\psi\colon G \to \mathbb{R}$ be a continuous function of negative type. Then there exists an affine isometric action on a Hilbert space $\mathcal{H}$ such that the associated 1-cocycle $b\colon G \to \mathcal{H}$ satisfies, $\psi(g) = \|b(g)\|^2$.*

The following theorem relates the two notions of positive and negative type functions on groups.

***Theorem* 1.2.23** (Schoenberg's theorem [**18**, Theorem C.3.2.,Corollary C.4.19.])**.** *Let G be a topological group and let ψ be a continuous real valued function (kernel) with $\psi(e) = 0$ and $\psi(g) = \psi(g^{-1})$ for all $g \in G$ ($\psi(g, g) = 0$ and $\psi(g, h) = \psi(h, g)$ for all $g, h \in G$). Then the following are equivalent:*

(1) *ψ is of negative type.*
(2) *The function $e^{-t\psi}$ is of positive type for every $t \geq 0$.*

Definition and Theorem 1.2.24 ([**32**, Theorem 2.1.1.])**.** Let G be a locally compact second countable noncompact group. Then the following are equivalent:

(1) G admits a proper, affine, isometric action on a Hilbert space.
(2) There exists a proper, continuous negative type function ψ on G.
(3) There exists a sequence of normalised continuous functions of positive type $(\varphi_n)_{n\in\mathbb{N}}$ such that $\varphi_n \to 1$ uniformly on compact subsets of G.
(4) There exists a C_0-unitary representation $(\pi, \mathcal{H})$, that is the map $g \mapsto \langle \pi(g)\xi, \eta\rangle$ belongs to $C_0(G)$ for all $\xi, \eta \in \mathcal{H}$, which weakly contains the trivial representation. That is for all compact subsets $K \subset G$ and $\varepsilon > 0$ there exists unit vectors $\xi \in L^2(G)$ such

that

$$\sup_{g\in K} |1 - \langle \pi(g)\xi, \xi\rangle| < \varepsilon$$

If G satisfies one of the equivalent properties we say G has the *Haagerup property.*

Proof (Sketch). (1) $\Leftrightarrow$ (2). Every negative type function is of the form $g \mapsto \|b(g)\|^2$ for some 1-cocycle b. This map is proper if and only if the action associated to the 1-cocycle is proper.

(2) $\Leftrightarrow$ (3). By Schoenberg's theorem the function $e^{-n\psi}$ is positive type for all $n \geq 0$. As ψ is proper, the sequence of positive functions $e^{-n\psi}$ converges uniformly to 1 on compact subsets of G.

If $(\varphi_n)_{n\in\mathbb{N}}$ are a sequence of positive definite functions that converge uniformly to 1 on compact sets then define a negative type function by $\psi(g) = \sum_{n\geq 1} \alpha_n(1 - \varphi_n(g))$ for some unbounded, increasing positive sequence α_n. One has to be a bit careful in choosing subsequences on φ_n such that this well defined. As $\varphi_n \to 1$ uniformly on compact sets it follows that ψ is proper.

(3) $\Leftrightarrow$ (4). Every positive type function is of the form $g \mapsto \langle \pi(g)\xi, \xi\rangle$ for some unitary representation $\pi \colon G \to \mathcal{H}$ and unit vectors $\xi \in \mathcal{H}$. So for each n there exists a representation $\pi_n \colon G \to \mathcal{H}_n$ and vectors $\xi_n \in \mathcal{H}_n$ such that $\varphi_n(g) = \langle \pi_n(g)\xi_n, \xi_n\rangle$. Set $\pi = \oplus\pi_n$.

Conversely if π weakly contains the trivial representation then one takes an exhaustive sequence of compact sets K_n such that $G = \cup_n K_n$. Choose a sequence of vectors $\xi_n \in \mathcal{H}$ such that $\lim_{n\to\infty} \sup_{g\in K_n} |1 - \langle \pi(g)\xi_n, \xi_n\rangle| = 0$. Now set $\varphi_n(g) = \langle \pi(g)\xi_n, \xi_n\rangle$ for all $g \in G$ and $n \in \mathbb{N}$.

□

If a group is amenable then the left regular representation weakly contains the trivial representation. It is routine to check that the left regular representation is a C_0-representation and so amenable groups have the Haagerup property. Furthermore, the existence of a sequence of positive type functions that converge uniformly to 1 on compact sets implies that that a group with the Haagerup property admits a coarse embedding into a Hilbert space by Theorem 1.2.8.

1.3. Compression

Let (X, d_X) and (Y, d_Y) be metric spaces and denote $\mathrm{Lip}^{\mathrm{ls}}(X, Y)$ to be the set of large-scale Lipschitz maps from X to Y.

Definition 1.3.1. For $f \in \mathrm{Lip}^{\mathrm{ls}}(X, Y)$ the *compression function of* f, denoted by ρ_f, is

$$\rho_f(r) := \inf_{d_X(x,x')\geq r} d_Y(f(x), f(x')).$$

Definition 1.3.2 ([**58**, Definition 2.2.])**.** Suppose X is a metric space that is unbounded.

(1) For $f \in \mathrm{Lip}^{\mathrm{ls}}(X, Y)$ the *asymptotic compression* R_f is

$$R_f := \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r},$$

where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

(2) The *compression of* X *in* Y is

$$R(X, Y) := \sup\left\{R_f : f \in \mathrm{Lip}^{\mathrm{ls}}(X, Y)\right\}.$$

(3) If $Y = L^p(\Omega)$ for some measure space Ω then

$$\alpha_p(X) := R(X, L^p(\Omega)).$$

Remark 1.3.3. For $f \in \mathrm{Lip}^{\mathrm{ls}}(X,Y)$ one can think of compression R_f as the supremum over all $\alpha \in [0,1]$ such that

$$\frac{1}{A} d_X(x,x')^\alpha - B \leq d_Y(f(x), f(x')) \leq A d_X(x,x') + B \quad \forall x, x' \in X$$

for some constants A, B depending on α. Compression is measuring how close one can quasi-isometrically embed a metric space X into Y.

***Theorem* 1.3.4** ([**58**, Theorem 2.12.]). *Let X_1 and X_2 be metric spaces. If there exists a quasi-isometry $\varphi\colon X_1 \to X_2$ then $R(X_1, Y) \geq R(X_2, Y)$ for every metric space Y.*

***Corollary* 1.3.5** ([**58**, Corollary 2.13.]). *If the metric spaces X_1 and X_2 are quasi-isometric then $R(X_1, Y) = R(X_2, Y)$ for all metric spaces Y.*

This means that compression is a quasi-isometric invariant. In particular when we consider word metrics on groups associated to compact generating sets, the compression of the group does not depend on the choice of the compact generating set. Unfortunately for general locally compact second countable groups G, compression is no longer an invariant and does depend on the **plig** metric used. This is because **plig** metrics are not necessarily quasi-isometric. When we specify the **plig** metric d we shall write $\alpha_p(G,d)$. However we will show in Section 4.7 that it is still useful to consider compression with respect to particular **plig** metrics.

We can incorporate an action of a group to the previous ideas and obtain equivariant compression. Let X be a metric space and fix an isometric action of G on X. For a Banach space E and an affine isometric action α of the group G on E we consider the following space of functions.

$$\mathrm{Lip}^{\mathrm{ls}}_G(X,E,\alpha) = \left\{ f \in \mathrm{Lip}^{\mathrm{ls}}(X,E) \mid f \text{ is } G\text{-equivariant with respect to } \alpha \right\}.$$

We write $\mathrm{Lip}^{\mathrm{ls}}_G(X,E) = \bigcup \left\{ \mathrm{Lip}^{\mathrm{ls}}_G(X,E,\alpha) : \alpha \text{ is an affine isometric action on } E \right\}$. The *$G$-equivariant Banach space compression of X* is defined by

$$R_G(X,E) := \sup \left\{ R_f : f \in \mathrm{Lip}^{\mathrm{ls}}_G(X,E) \right\}.$$

This definition depends on the isometric action of G on X. However most of the time we shall consider G acting on itself by left multiplication.

***Theorem* 1.3.6** ([**58**, Theorem 5.1.]). *Let X and Y be metric spaces where G acts by isometries. If there exists an equivariant quasi-isometry $X \to Y$ then $R_G(X,E) \geq R_G(Y,E)$ for any Banach space E.*

***Corollary* 1.3.7** ([**58**, Corollary 5.2.]). *Let G be a compactly generated group. Then $R_G(G,E)$ is independent of the choice of compact generating set.*

For all $1 \leq p < \infty$ we denote $R_G(G, L^p)$ by $\alpha_p^\#(G)$. Unfortunately in the general setting the equivariant Hilbert space compression depends on the choice of **plig** metric d. When we want to specify we shall write $\alpha_p^\#(G,d)$. Suppose G is generated by a compact symmetric set S and suppose α is an affine isometric action on a Hilbert space $\mathcal{H}$. Let b be the associated 1-cocycle of α and set $M = \max\{\|b(s)\| : s \in S\}$. For $g \in G$ there exists $s_1, \dots, s_n \in S$ such that $g = s_1 \cdots s_n$ and $n = |g|_S$. Using the cocycle relation we have that

$$\|b(g)\| = \|b(s_1 \cdots s_n)\| \leq \sum_{i=1}^n \|b(s_i)\| \leq M|g|_S \quad \text{and} \quad \|b(x) - b(y)\| = \|b(x^{-1}y)\| \leq M|x^{-1}y|_S$$

for all $x, y \in G$. This means that every 1-cocycle is an equivariant Lipschitz map. Conversely suppose $f\colon G \to \mathcal{H}$ is a G-equivariant large-scale Lipschitz map. As f is equivariant it follows that $\|f(x)\| = \|\alpha(x)f(1)\| = \|\pi(x)f(1)+b(x)\|$ where π and b are the orthogonal and translation parts of the action α respectively. Hence

$$\|b(x)\| - \|f(1)\| \leq \|f(x)\| \leq \|b(x)\| + \|f(1)\|$$

for all $x \in G$. This implies that the compression of b is equal to the compression of f. Therefore when we consider equivariant compression of a compactly generated group it is enough to restrict ourselves to the set of all 1-cocycles.

Far less is known about equivariant compression than non-equivariant compression.

Examples 1.3.8. (1) Let (X, d) be a metric space such that $|X| < \infty$ and let $C = \mathrm{diam}(X)$. Let E be a Banach space and $f\colon X \to E$ be the zero map. Then $d(x, y) - C \leq \|f(x) - f(y)\|_E \leq d(x, y)$ for all $x, y \in X$. Hence $R(X, E) = 1$ for any finite metric space and any Banach space E.

(2) Let (X, d) be a metric space such that $|X| < \infty$ and let $C = \mathrm{diam}(X)$. Suppose G acts by isometries on X and let G act trivially on a Banach space E. That is $g \cdot v = v$ for all $v \in E$. Let $x_1, \ldots, x_n \in X$ be representatives of the orbits of the action of G on X. Thus $X = \sqcup_{i=1}^{n} Gx_i$. Let f be a function that is constant on the orbits of the action of G on X. This implies f is G-equivariant. Set $\delta := \min\{d(x, y) : x, y \in X\}$ and $M = \max\{\|f(x_i) - f(x_j)\|_E : 1 \leq i, j \leq n\}$. Therefore $\|f(x) - f(y)\|_E \leq \frac{M}{\delta} d(x, y)$ for all $x, y \in X$ and so f is Lipschitz. It follows that $d(x, y) - C \leq \|f(x) - f(y)\|_E \leq \frac{M}{\delta} d(x, y)$ for all $x, y \in X$. Hence $R_G(X, E) = 1$ for any finite metric space X and for any group G acting isometrically on X and any Banach space E.

(3) $R_G(X, E) \leq R(X, E)$ for any metric space X, Banach space E and any group G.

(4) If Z is a subspace of a metric space X then $R(X, Y) \leq R(Z, Y)$ and $R_G(X, E) \leq R_G(Z, E)$ for any metric space Y, Banach space E and any group G.

(5) For metric spaces X and Y, $\alpha_2(X \times Y) = \min\{\alpha_2(X), \alpha_2(Y)\}$ where $X \times Y$ has the ℓ^1-metric [**58**, Proposition 4.1.].

(6) Let G be a compactly generated group. Then $\alpha_2(G) \leq \alpha_p(G)$ for all $1 \leq p < \infty$ [**26**, Proposition 1.4.][**89**, Lemma 2.3.].

(7) In [**58**, Proposition 4.2.] it was shown that $\alpha_2(\mathbb{F}_2) = 1$. This was generalised so that $\alpha_p(G) = 1$ for all $1 \leq p < \infty$ for any finitely generated word hyperbolic group [**21**] [**115**, Corollary 2.].

(8) In [**115**] it was shown that for all $1 \leq p < \infty$, $\alpha_p(G) = 1$ for any group G in the class of groups denoted by $\mathcal{L}'$ [**115**, Corollary 2.]. This includes polycyclic groups, connected amenable Lie groups, Baumslag–Solitar groups $BS(1, m)$ for any $m \geq 1$, wreath products $F \wr \mathbb{Z}$ for any finite group F, connected Lie groups and their cocompact lattices [**115**] and finitely generated word hyperbolic groups.

(9) $\alpha_2^{\#}(\mathbb{F}_2) = 1/2$ [**58**, pp 15–16]. Let $X = \mathrm{Cay}(\mathbb{F}_2, a, b)$ and consider $\mathcal{H} := \ell^2(E)$ where E is the set of edges in X. Define a 1-cocycle

$$b\colon \mathbb{F}_2 \to \ell^2(E) \quad b(s) = \chi_{[1,s]}$$

where $\chi_{[1,s]}$ is the characteristic function on the unique path from s to the identity. It follows that $\|b(s) - b(t)\| = \sqrt{d(s, t)}$. As $\mathbb{F}_2$ is not amenable it follows that $\alpha_2^{\#}(\mathbb{F}_2) = 1/2$.

(10) For any locally compact second countable amenable group G, $\alpha_2(G,d) = \alpha_2^{\#}(G,d)$ where d is a proper left invariant metric that generates the topology on G [**40**, Proposition 4.4.].

(11) There exists a finitely generated amenable group G such that $\alpha_2(G) = 0$ [**11**].

(12) For any $\alpha \in [0,1]$ there exists a finitely generated group G_α with asymptotic dimension at most 2 and with $\alpha_2(G_\alpha) = \alpha$ [**9**, Theorem 1.5.]. We shall use this example in section 2.3.

The construction is as follows: we take a carefully chosen decreasing chain of finite index normal subgroups of a discrete lattice Γ in $\mathrm{SL}_3(F)$ for a local field F where for some $m \in \mathbb{N}$, Γ is generated by m involutions. Let $(M_k)_{k\in\mathbb{N}}$ be the family of quotients and denote $\sigma_1(k), \ldots, \sigma_m(k)$ to be the image of the involutions under the quotient map.

There exists a natural metric on the finite quotients M_k and we denote the family of metric spaces by $(\Pi_k)_{k\in\mathbb{N}}$. We equip the disjoint union $\sqcup\Pi_k$ such that the restriction to each quotient is the natural metric and that $d(\Pi_k, \Pi_j) \geq \mathrm{diam}(\Pi_k) + \mathrm{diam}(\Pi_j)$ for all $k \neq j$. It is shown in [**77**] that such $\sqcup_k \Pi_k$ does not embed into any uniformly convex Banach space.

For each $\alpha \in [0,1]$ there exists a sequence of constants $(\lambda_k)_{k\in\mathbb{N}}$ such that the rescaled family of metric spaces $(\lambda_k\Pi_k)_{k\in\mathbb{N}}$ has compression $\alpha_2(\sqcup\lambda_k\Pi_k) = \alpha$ [**9**, Proposition 1.4.]. For all $k \in \mathbb{N}$ define m_k to be the integer part of $\frac{\lambda_k - 1}{2}$.

The group is constructed as a graph of groups. Let F be the free product $*_{k\in\mathbb{N}}M_k$ and for every $1 \leq i \leq m$ let H_i be the free product $\mathbb{Z}/2 * \mathbb{Z}$ where the $\mathbb{Z}/2$-factor is denoted by σ_i and the $\mathbb{Z}$-factor is denoted by t_i. For every $k \in \mathbb{Z}$ we denote the element $t_i^k \sigma_i t_i^{-k}$ by $\sigma_i^{(k)}$. The vertex groups are F and $H_1, \ldots, H_m$ and the only edges are (F, H_i). The edge group of (F, H_i) are free products $*_{k\in\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$, where the k-factor $\mathbb{Z}/2\mathbb{Z}$ is identified with $\sigma_i(k) \in M_k$ in F and with $\sigma_i^{(m_k)}$ in H_i.

Now G_α is taken to be the fundamental group of this graph of groups and so is generated by the set $\{\sigma_1(1), \ldots, \sigma_m(1), t_1, \ldots, t_m\}$. In particular the word metric is proper and left-invariant. The free product naturally embeds into the fundamental group G_α [**107**, Chapter I, Section 5]. We equip $*_kM_k$ with the subspace metric from G_α and so this gives a proper left-invariant metric on $*_kM_k$. In [**9**, Lemma 5.7.] it is shown that

$$d_{G_\alpha}(g,h) = (2m_k + 1)d_{M_k}(g,h)$$

for all $g, h \in M_k$. Thus for all $k \in \mathbb{N}$, the metric space $\lambda_k\Pi_k$ is uniformly quasi-isometric to (M_k, d_{G_α}) and so $(*_kM_k, d_{G_\alpha})$ contains a quasi-isometric copy of the metric space $\sqcup_k\lambda_k\Pi_k$. Thus α is an upper bound of the compression of G_α and careful analysis of the word metric shows this bound is realised [**9**, Theorem 5.5.].

In the example in Section 2.3 we will chose α to be zero. Hence $*_kM_k$ will have compression 0 when equipped with the word metric from G_α.

(13) Define recursively $\mathbb{Z}_{(1)} = \mathbb{Z}$ and $\mathbb{Z}_{(k+1)} = \mathbb{Z}_{(k)} \wr \mathbb{Z}$. Then $\alpha_2(\mathbb{Z}_{(k)}) = \alpha_2^{\#}(\mathbb{Z}_{(k)}) = \frac{1}{2-2^{1-k}}$ [**89**, Corollary 1.3.]. In the same article it is shown that $\alpha_2^{\#}(\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}) = 1/2$ [**89**, Corollary 1.3.].

It was first shown that $\alpha_2^{\#}(\mathbb{Z}\wr\mathbb{Z}) \in [1/2, 3/4]$ [**8**, Theorem 3.10.]. The upper bound is obtained by showing that $\alpha_2^{\#}(\bigoplus\mathbb{Z}, d_{\mathbb{Z}\wr\mathbb{Z}}) = \alpha_2(\bigoplus\mathbb{Z}, d_{\mathbb{Z}\wr\mathbb{Z}}) \leq 3/4$ [**8**, Theorem 3.9.] and we shall use this compression bound in Example 2.1.1.

(14) In the same article it is shown that $\alpha_2^{\#}(F) = \alpha_2(F) = 1/2$ for Thompson's group F [**8**, Theorem 1.3.].

(15) There exists a general lower bound for wreath products. Indeed for any finitely generated groups G and H

$$\alpha_p(G \wr H) \geq \max\{1/p, 1/2\} \min\left\{\alpha_1(G), \frac{\alpha_1(H)}{\alpha_1(H)+1}\right\}$$

for any $1 \leq p < \infty$ [**80**, Theorem 1.1.].

(16) Let G and H be finitely generated groups and let G be the free product $G = G_1 * G_2$. Then

$$\min\{\alpha_p(G_1), \alpha_p(G_2), 1/p\} \leq \alpha_p(G) \leq \min\{\alpha_p(G_1), \alpha_p(G_2)\}$$

for all $1 \leq p < \infty$ [**45**, Corollary 2.5.].

We have the following useful application of compression.

***Theorem* 1.3.9** ([**58**, Theorem 3.2., Theorem 5.3.]). *Let Γ be a finitely generated group. If $\alpha_2(\Gamma) > 1/2$ then Γ has property A. If $\alpha_2^{\#}(\Gamma) > 1/2$ then Γ is amenable.*

This has been partially extended extended to locally compact, compactly generated groups.

***Theorem* 1.3.10** ([**40**, Theorem 4.1.]). *Let G be a locally compact, compactly generated group. If $\alpha_2^{\#}(G) > 1/2$ then G is amenable.*

In section 4.7 we shall extend the non-equivariant part of Theorem 1.3.9 to all locally compact second countable groups.

CHAPTER 2

Compression of direct limits of groups and amalgamated free products

The results in this Chapter were done in joint work with Dennis Dreesen and can be found in [**27**].

2.1. (α, l, q)-polynomial property

In this section of results we compute the equivariant Hilbert space compression of certain direct limits of groups. Specifically we assume that a given group G, equipped with a proper length function, can be viewed as a direct limit of open (hence closed) subgroups $G_1 \subset G_2 \subset G_3 \subset \ldots \subset G$. We equip each G_i with the subspace metric from G. Our main objective will be to find bounds on $\alpha_2^{\#}(G)$ in terms of properties of the G_i. Observe that, as each G_i is a metric subspace of G, we have $\alpha_2^{\#}(G) \leq \inf_{i\in\mathbb{N}} \alpha_2^{\#}(G_i)$. The main challenge is to find a sensible lower bound on $\alpha_2^{\#}(G)$. The next example will show that it is not enough to only consider $\alpha_2^{\#}(G_i)$.

Example 2.1.1. Consider the wreath product $\mathbb{Z} \wr \mathbb{Z}$ equipped with the standard word metric relative to $\{(\delta_1, 0), (0, 1)\}$, where δ_1 is the characteristic function of $\{0\}$. Let $\mathbb{Z}^{(\mathbb{Z})} = \{f\colon \mathbb{Z} \to \mathbb{Z} : \ f \text{ is has finite support}\}$ be equipped with the subspace metric from $\mathbb{Z}\wr\mathbb{Z}$. Consider the direct limit of groups

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \hookrightarrow \mathbb{Z}^5 \cdots \hookrightarrow \mathbb{Z}^{(\mathbb{Z})}$$

where $\mathbb{Z}^{2n+1}$ has the subspace metric from $\mathbb{Z}^{(\mathbb{Z})}$. This metric is quasi-isometric to the standard word metric on $\mathbb{Z}^{2n+1}$ and so each term has equivariant compression 1. So $\mathbb{Z}^{(\mathbb{Z})}$ is a direct limit of groups with equivariant compression 1 but by [**8**, Theorem 3.9.], $\mathbb{Z}^{(\mathbb{Z})}$ has equivariant compression less than 3/4. On the other hand the sequence

$$\mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z}$$

is a sequence of groups with equivariant compression 1 and the equivariant compression of the direct limit is 1.

This example shows that in order to predict the equivariant compression of the direct limit it will be necessary to incorporate more information than only the compression exponent of 1-cocycles.

The key property that we introduce is the (α, l, q)-polynomial property. We assume that the sequence $(G_i)_{i\in\mathbb{N}}$ is *normalized*, i.e. each open ball $B(1, i) \subset G$ is contained in G_i. Up to taking a subsequence, one can make this assumption without loss of generality.

Definition 2.1.2. Let G be a topological group equipped with a proper length function $|\cdot|$ and suppose that $(G_i)_{i\in\mathbb{N}}$ is a normalized nested sequence of open subgroups such that $\varinjlim G_i = G$. Assume that $\alpha := \inf_{i\in\mathbb{N}} \alpha_2^{\#}(G_i) \in (0, 1]$. For $l, q \in \mathbb{N}$ the sequence $(G_i)_{i\in\mathbb{N}}$ has the (α, l, q)-*polynomial property* ((α, l, q)-PP) if there exists:

(1) a sequence $(\eta_i)_{i\in\mathbb{N}} \subset \mathbb{R}^+$ converging to 0 such that $\eta_i < \alpha$ for each $i \in \mathbb{N}$,

(2) $(A_i, B_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{>0} \times \mathbb{R}^{\geq 0}$,

(3) a sequence of 1-cocycles $(b_i\colon G_i \to \mathcal{H}_i)_{i\in\mathbb{N}}$, where each b_i is associated to a unitary action π_i of G_i on a Hilbert space $\mathcal{H}_i$

such that

$$\frac{1}{A_i}|g|^{2\alpha-\eta_i} - B_i \leq \|b_i(g)\|^2 \leq A_i|g|^2 + B_i \quad \forall g \in G_i, \forall i \in \mathbb{N}$$

and there is $C, D > 0$ such that $A_i \leq Ci^l$, $B_i \leq Di^q$ for all $i \in \mathbb{N}$.

Observe that the only real restrictions are the inequalities $A_i \leq Ci^l$, $B_i \leq Di^q$; we exclude sequences A_i, B_i that are superpolynomial. The intuition is that equivariant compression is a polynomial property, so sequences growing faster than every polynomial would force the compression of the limit group to be 0. On the other hand if the sequences grow polynomially, then one can use compression to compensate for this growth. One then obtains a strictly positive lower bound on $\alpha_2^{\#}(G)$ which may decrease depending on how fast the sequences grow.

Every locally compact second countable group G has a proper left invariant metric d that generates the topology on G, see Theorem 1.1.12. Define a metric d' such that for any $x \neq y$,

$$d'(x,y) = \begin{cases} 1 & \text{if } d(x,y) \leq 1 \\ d(x,y) & \text{otherwise.} \end{cases}$$

Then (G, d') is quasi-isometric to (G, d) and for any $x \in G\backslash\{e\}$, $|x| = d'(x, e) \geq 1$. In particular compression does not change. Without loss of generality we assume that the metric on the group is 1-uniformly discrete. That is $|g| \geq 1$ for all $g \in G \setminus \{e\}$.

***Theorem* 2.1.3.** *Let G be a locally compact, second countable group equipped with a proper, 1-uniformly discrete metric d. Suppose there exists a sequence of open subgroups $(G_i)_{i\in\mathbb{N}}$, each equipped with the restriction of d to G_i, such that $\varinjlim G_i = G$ and $\alpha = \inf\{\alpha_2^{\#}(G_i, d)\} > 0$. If $(G_i)_{i\in\mathbb{N}}$ has (α, l, q)-PP, then there are the following two cases:*

$$l \geq q \Rightarrow \alpha_2^{\#}(G,d) \geq \frac{\alpha}{2l+1}$$

or,

$$l \leq q \Rightarrow \alpha_2^{\#}(G,d) \geq \frac{\alpha}{l+q+1}.$$

We have the following useful characterisation of (α, l, q)-polynomial property.

***Lemma* 2.1.4.** *Let G be a topological group equipped with a proper length function $|\cdot|$ and suppose there exists a sequence of open subgroups $(G_i)_{i\in\mathbb{N}}$ such that $\varinjlim G_i = G$. Then $(G_i)_{i\in\mathbb{N}}$ has the (α, l, q)-polynomial property if and only if there exists $C, D > 0$ such that for all $\varepsilon > 0$ there exists*

(1) a sequence $(A_i, B_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{>0} \times \mathbb{R}^{\geq 0}$ such that $A_i \leq Ci^l$ and $B_i \leq Di^q$;

(2) a sequence of 1-cocycles $(b_i\colon G_i \to \mathcal{H}_i)_{i\in\mathbb{N}}$

such that

$$\frac{1}{A_i}|g|^{2\alpha-\varepsilon} - B_i \leq \|b_i(g)\|^2 \leq A_i|g|^2 + B_i \quad \forall g \in G_i, \forall i \in \mathbb{N}.$$

Proof. Suppose we satisfy the conditions of the lemma. Then take $\varepsilon_n = \alpha/2n$ to obtain a sequence of 1-cocycles that satisfy the conditions in Definition 2.1.2.

Suppose $(G_i)_{i\in\mathbb{N}}$ has the (α, l, q)-PP with respect to a sequence η_i converging to 0. Choose k_ε large enough so that $\eta_k < \varepsilon$ whenever $k \geq k_\varepsilon$. The 1-cocycles b_k associated to (α, l, q)-PP will satisfy the conditions in the lemma for $k \geq k_\varepsilon$. For $k \leq k_\varepsilon$ restrict b_{k_ε} to G_k so the sequence of 1-cocycles satisfy the conditions for every group G_i. □

***Proposition* 2.1.5.** *Let G be a locally compact second countable group equipped with a proper length function $|\cdot|$ and suppose there exists a sequence of open subgroups $(G_i)_{i\in\mathbb{N}}$ such that $\varinjlim G_i = G$. If $\alpha := \alpha_2^\#(G) > 0$ then $(G_i)_{i\in\mathbb{N}}$ has $(\alpha, 0, 0)$-polynomial property.*

Proof. For all $0 < \varepsilon < \alpha$ there exists a 1-cocycle b such that

$$\frac{1}{A}|g|^{\alpha-\varepsilon} - B \leq \|b(g)\| \quad \forall g \in G.$$

The restriction of b to each G_i is a 1-cocycle and gives $(G_i)_{i\in\mathbb{N}}$ the $(\alpha, 0, 0)$-polynomial property. □

Combining this result with Theorem 2.1.3 we can confirm our intuition that if the sequences of Lipschitz constants grow superpolynomially then the compression of the direct limit group is forced to be 0.

***Corollary* 2.1.6.** *Suppose G is a locally compact second countable group with a **plig** metric d. Then $(G_i)_{i\in\mathbb{N}}$ has the (α, l, q)-polynomial property for some $\alpha \in (0, 1]$ and $l, q \geq 0$ if and only if $\alpha_2^\#(G, d) > 0$*

2.2. The proof of Theorem 2.1.3

Proof of Theorem 2.1.3. Take sequences $(\psi_i\colon G_i \to \mathbb{R})_{i\in\mathbb{N}}$, $(\eta_i)_i$ and $(A, B) = (A_i, B_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{>0} \times \mathbb{R}^{\geq 0}$ satisfying the conditions of (α, l, q)-PP. We assume here, without loss of generality, that the sequences $(A_i)_i, (B_i)_i$ are non-decreasing.

For each $i \in \mathbb{N}$, define a sequence of continuous maps $(\varphi_k^i\colon G \to \mathbb{R})_{k\in\mathbb{N}}$ by

$$\varphi_k^i(g) = \begin{cases} \exp\left(\frac{-\psi_i(g)}{k}\right) & \text{if } g \in G_i \\ 0 & \text{otherwise.} \end{cases}$$

Each φ_k^i is continuous as G_i is open and also closed, being the complement of $\cup_{g\notin G_i} gG_i$. Observe that for all $i, k \in \mathbb{N}$, $\varphi_k^i(e) = 1$. By (α, l, q)-PP, for all $i, k \in \mathbb{N}$, we have

$$\exp\left(\frac{-A_i|g|^2 - B_i}{k}\right) \leq \varphi_k^i(g) \quad \forall g \in G_i, \text{ and}$$

$$\varphi_k^i(g) \leq \exp\left(\frac{-|g|^{2\alpha-\eta_i} + A_iB_i}{A_ik}\right) \quad \forall g \in G.$$

Fix some $p > 0$, set $J(i) = (A_i + B_i)i^{1+p}$ and define $\overline{\psi}\colon G \to \mathbb{R}$ by

$$\overline{\psi}(g) = \sum_{i\in\mathbb{N}} 1 - \Phi_i(g),$$

where $\Phi_i(g) := \varphi_{J(i)}^i(g)$. To check that $\overline{\psi}$ is well defined, choose any $g \in G$ and note that for $i > |g|$, we have $g \in G_i$ and so $\varphi_k^i(g) \geq \exp(\frac{-A_i|g|^2 - B_i}{k})$. Hence

$$\sum_{i>|g|} 1 - \Phi_i(g) \leq \sum_{i>|g|} 1 - \exp\left(\frac{-A_i|g|^2 - B_i}{(A_i + B_i)i^{1+p}}\right) \leq \sum_{i>|g|} 1 - \exp\left(\frac{-|g|^2}{i^{1+p}}\right) \leq \sum_{i>|g|} \frac{|g|^2}{i^{1+p}}$$
$$= |g|^2 \sum_{i>|g|} \frac{1}{i^{1+p}}.$$

As $\overline{\psi}(g) = \sum_{i=1}^{|g|} 1 - \Phi_i(g) + \sum_{i>|g|} 1 - \Phi_i(g)$, we see that $\overline{\psi}$ is well defined and that it can be written as a limit of continuous functions converging uniformly on compact sets. Consequently, it is itself continuous. By Schoenberg's theorem [**42**, Theorem 5.16.], all of the maps φ_k^i are positive definite on G [**62**, Section 32.43(a)]. Hence, $\overline{\psi}$ is a conditionally negative definite map [**18**, Proposition C.2.4(i),(iii).]. Moreover, using that $|\cdot|$ is 1-uniformly discrete, we can find a constant $E > 0$ such that

$$\overline{\psi}(g) \leq |g| + |g|^2 \sum_{i>|g|} \frac{1}{i^{1+p}} \leq E|g|^2 \tag{1}$$

so the 1-cocycle associated to $\overline{\psi}$ via Proposition 1.2.22 is large-scale Lipschitz.

We now find a lower bound to the compression of this 1-cocycle. Set $VI\colon \mathbb{N} \to \mathbb{R}$ to be the function

$$VI(i) = (A_i J(i)\ln(2) + A_i B_i)^{\frac{1}{2\alpha-\eta_i}}.$$

One checks easily that

$$|g| \geq VI(i) \Rightarrow \Phi_i(g) = \varphi^i_{J(i)}(g) \leq \frac{1}{2}. \tag{2}$$

To make the function VI more concrete, let us look at the values of A_i, B_i and $J(i)$. Recall that by assumption, we have $A_i \leq Ci^l, B_i \leq Di^q$. Hence for i sufficiently large, we have $J(i) \leq (Ci^l + Di^q)i^{1+p} \leq Fi^X$ where F is some constant and $X = 1 + 2p + \max(l, q)$. We thus obtain that there is a constant $K > 0$ such that for i sufficiently large,

$$VI(i) \leq Ki^{Y/(2\alpha-\eta_i)},$$

where

$$Y = \max(X + l, l + q) = \max(1 + 2p + 2l, 1 + 2p + l + q).$$

The sequences η_i converges to 0 so for all $\varepsilon > 0$ there exists I_ε such that $\eta_i < \varepsilon$ any $i > I_\varepsilon$. Hence for all $i > I_\varepsilon$,

$$VI(i) \leq Ki^{Y/(2\alpha-\varepsilon)}.$$

Together with Equation (2), this implies that for $i > I$,

$$|g| \geq Ki^{Y/(2\alpha-\varepsilon)} \Rightarrow \Phi_i(g) = \varphi^i_{J(i)}(g) \leq \frac{1}{2}. \tag{3}$$

For every $g \in G$, set

$$c(g)_{p,\varepsilon} = \sup\left\{i \in \mathbb{N} \mid Ki^{Y/(2\alpha-\varepsilon)} \leq |g|\right\}.$$

We then have for every $g \in G$ with $|g|$ large enough, that

$$\overline{\psi}(g) \geq \sum_{i=1}^{c(g)_{p,\varepsilon}} 1 - \varphi^i_{J(i)}(g) \geq \sum_{i=I_\varepsilon+1}^{c(g)_{p,\varepsilon}} 1/2 = \frac{c(g)_{p,\varepsilon} - I}{2}.$$

As $c(g)_{p,\varepsilon} \geq \left(\frac{|g|}{K}\right)^{(2\alpha-\varepsilon)/Y} - 1$, we conclude that

$$R(b) \geq \frac{2\alpha - \varepsilon}{2\max(1 + 2p + 2l, 1 + 2p + l + q)},$$

for all $\varepsilon > 0$. By taking the limit as $\varepsilon, p \to 0$ we that $\alpha_2^{\#}(G) \geq \frac{\alpha}{\max(1+2l,1+l+q)}$. Hence, we have the following two cases:

$$l \geq q \Rightarrow \alpha_2^{\#}(G) \geq \frac{\alpha}{1 + 2l}$$

or,

$$l \leq q \Rightarrow \alpha_2^{\#}(G) \geq \frac{\alpha}{l + q + 1}.$$

□

2.3. Examples

***Theorem* 2.3.1.** *Let G and H be finitely generated groups where H has polynomial growth of degree $d > 1$. Further assume that $0 < \alpha_2^{\#}(G) < \frac{1}{2(1+d)}$. Then*

$$\alpha_2^{\#}\left(\bigoplus_H G\right) \geq \frac{\alpha_2^{\#}(G)}{1 + 2\alpha_2^{\#}(G)(1+d)}.$$

Remark 2.3.2. At the time of writing, these assumptions are empty because the values of $\alpha_2^{\#}(G)$ are not as well understood as the non-equivariant counterpart. The only known values for $\alpha_2^{\#}(G)$ are 1, 1/2, 0 and $\frac{1}{2-2^{1-k}}$ for $k \in \mathbb{N}$ [**8, 89, 11**] but in the non-equivariant case any value for compression can be achieved [**9**]. It is likely there exists values for $\alpha_2^{\#}$ between 0 and 1/2. For groups where equivariant compression is known, [**80**, Theorem 1.1.] provides a lower bound of $\alpha_2^{\#}(G)/2$. However whenever $0 < \alpha_2^{\#}(G) < \frac{1}{2(1+d)}$ then the above theorem provides a larger lower bound than $\alpha_2^{\#}(G)/2$.

Proof. We consider $\bigoplus_H G$ to be the group of functions $\mathbf{f}\colon H \to G$ that have finite support. Let $\mathbf{f} \in \bigoplus_H G$ and let $\mathrm{Supp}(\mathbf{f}) = \{h_1, \dots, h_n\} \subset H$. Set the length of $\mathbf{f}$ as follows

$$|\mathbf{f}|_{G \wr H} = \inf_{\sigma \in S_n}\left(d_H(1, h_{\sigma(1)}) + \sum_{i=1}^{n} d_H(h_{\sigma(i)}, h_{\sigma(i+1)}) + d_H(h_{\sigma(n)}, 1)\right) + \sum_{h \in H} |\mathbf{f}(h)|_G.$$

This is the induced length metric from $G \wr H$ and so this is a proper length function on $\bigoplus_H G$. Consider the following group

$$G_i = \{\mathbf{f}\colon H \to G \mid \mathrm{Supp}(\mathbf{f}) \subset B(1,i)\}$$

and set $n_i = |B(1,i)|$. It follows from the definition of the metric on $G \wr H$ that

$$|(g_1, \dots, g_{n_i})|_{G\wr H} - 2i|B(1,i)| \leq \sum_{j=1}^{n_i} |g_j|_G \leq |(g_1, \dots, g_{n_i})|_{G \wr H}$$

for all $(g_1, \dots, g_{n_i}) \in G_i$. Hence the metric on G_i induced from $G \wr H$ is quasi-isometric to the metric on G_i induced from the word metric on G. Hence $\alpha_2^{\#}(G_i) = \alpha_2^{\#}(G)$ for all $i \in \mathbb{N}$ [**58**, Proposition 4.1. and Corollary 2.13.]. Whenever $|(g_1, \dots, g_{n_i})| > 4i|B(1,i)|$ then it follows that

$$|(g_1, \dots, g_n)|_{G \wr H} - 2i|B(1,i)| > \frac{1}{2}|(g_1, \dots, g_n)|_{G \wr H}.$$

Set $0 < \alpha < \alpha_2^{\#}(G)$ and consider a 1-cocycle $b\colon G \to \mathcal{H}$ such that

$$\frac{1}{C}|g|_G^{2\alpha} \leq \|b(g)\|^2 \leq C|g|_G^2.$$

Define a 1-cocycle $b_i\colon G_i \to \mathcal{H}^{n_i}$, where $b_i(g_1, \dots, g_{n_i}) = (b(g_1), \dots, b(g_{n_i}))$. First suppose $|(g_1, \dots, g_{n_i})|_{G \wr H} > 4i|B(1,i)|$. It follows that because

$$(a+b)^x \leq a^x + b^x \quad \text{for all } a, b \geq 1 \text{ and } x \in [0,1]$$

then

$$\|b_i(g_1, \dots, g_{n_i})\|^2 = \sum_{j=1}^{n_i} \|b(g_j)\|^2 \geq \frac{1}{C}\sum_{j=1}^{n_i} |g_j|_G^{2\alpha} \geq \frac{1}{C}\left(\sum_{j=1}^{n_i} |g_j|\right)^{2\alpha}$$
$$\geq \frac{1}{C}\left(|(g_1, \dots, g_{n_i})|_{G \wr H} - 2i|B(1,i)|\right)^{2\alpha} \geq \frac{1}{4^\alpha C}|(g_1, \dots, g_{n_i})|_{G \wr H}^{2\alpha}.$$

If $|(g_1,\dots,g_{n_i})|_{G\wr H} < 4i|B(1,i)|$ then $|(g_1,\dots,g_{n_i})|^{2\alpha}_{G\wr H} - 8i^{2\alpha}|B(1,i)|^{2\alpha} \leq 0$. Putting this together we have that for all $(g_1,\dots,g_{n_i}) \in G_i$ it follows that

$$\frac{1}{2C}|(g_1,\dots,g_{n_i})|^{2\alpha}_{G\wr H} - \frac{8i^{2\alpha}}{4^\alpha C}|B(1,i)|^{2\alpha} \leq \|b_i(g_1,\dots,g_{n_i})\|^2.$$

Hence $(G_i)_{i\in\mathbb{N}}$ has the $(\alpha, 0, 2\alpha(1+d))$ polynomial property for all $\alpha < \alpha^{\#}(G)$. Hence

$$\alpha_2^{\#}\left(\bigoplus_H G\right) \geq \frac{\alpha}{1+2\alpha(1+d)} \qquad \square$$

Example 2.3.3. Our result also allows to consider spaces $\bigoplus_H G_h$ where G_h actually depends on the parameter $h \in H$. For example, we could take a collection of finite groups F_i with $F_0 = \{0\}$ and look at $G = \bigoplus_{i\in\mathbb{N}} F_i$ equipped with a proper length function $|\cdot|$ as follows:

$$|g| := \min(n \in \mathbb{N} \mid g \in \oplus_{i=0}^n F_i) \quad \forall g \in G$$

Set $G_i = \bigoplus_{j=0}^i F_j$ and note that $\alpha_2^{\#}(G_i) = 1$ as G_i is finite. Moreover, it is easy to see that the sequence $(G_i)_i$ is normalized. Define $f_i : G_i \to \mathbb{R}$ to be the 0-map. This is a 1-cocycle of G_i relative to any unitary representation of G_i. The associated conditionally negative definite map satisfies

$$\forall g \in G_i: \ |g|^2 - i^2 \leq \psi_i(g) \leq |g|^2 + i^2.$$

We obtain the lower bound $\alpha_2^{\#}(G) \geq 1/3$ by Theorem 2.1.3. This is the first available lower bound on the equivariant compression of G.

Example 2.3.4. We will use the construction in [**9**] to provide an example of a sequence that does not have (α, l, q)-polynomial property for any $\alpha \in (0,1]$ and $l, q > 0$. Let Π_k, $k \geq 1$ be a sequence of Lafforgue expanders that do not embed into any uniformly convex Banach space [**77**]. These are finite factor groups M_k of a discrete lattice Γ of $\mathrm{SL}_3(F)$ for a local field F.

For every $\alpha \in [0,1]$ there exists a finitely generated group G and a sequence of scaling constants λ_k such that $\lambda_k\Pi_k$ has compression α and G is quasi-isometric to $\lambda_k\Pi_k$. Furthermore G contains the free product $*_k M_k$ as a subgroup. Let $\alpha = 0$ and let G and the scaling constants λ_k be such that G has compression 0. We can equip $*_k M_k$ with a proper left invariant metric coming from G. Hence we have a sequence

$$M_1 \hookrightarrow M_1 * M_2 \hookrightarrow \cdots \hookrightarrow *_{k=1}^n M_k \hookrightarrow \cdots \hookrightarrow *_k M_k.$$

For each $n > 0$, $*_{k=1}^n M_k$ has equivariant compression 1/2 [**45**, Theorem 1.4.] however the limit group $*_k M_k$ contains a quasi-isometric copy of $\lambda_k\Pi_k$ and so has compression 0. Thus this sequence can not have the (α, l, q)-polynomial property for any $\alpha \in (0,1]$ and $l, q > 0$.

2.4. The behaviour of compression under free products amalgamated over finite index subgroups

In [**52**], S.R. Gal proves the following result.

***Theorem* 2.4.1** ([**52**, Corollary 5.3.])**.** *Let G_1 and G_2 be finitely generated groups with the Haagerup property and have a common finite index subgroup H. For each $i = 1, 2$, let β_i be a proper affine isometric action of G_i on a Hilbert space $V_i (= l^2(\mathbb{Z}))$. Assume that $W < V_1 \cap V_2$ is invariant under the actions $(\beta_i|_H)$ and moreover that both these (restricted) actions coincide on W. Then $G_1 *_H G_2$ is Haagerup.*

Under the same conditions as above, we want to give estimates on $\alpha_2^{\#}(G_1 *_H G_2)$ in terms of the equivariant Hilbert space compressions of G_1, G_2 (see Theorem 2.4.3 below). Note that the following lemma shows that $\alpha_2^{\#}(G_1) = \alpha_2^{\#}(H) = \alpha_2^{\#}(G_2)$ when H is of finite index in both G_1 and G_2. We are indebted to Alain Valette for this lemma and its proof.

***Lemma* 2.4.2.** *Let G be a compactly generated, locally compact group, and let H be an open, finite-index subgroup of G. Then $\alpha_p^{\#}(H) = \alpha_p^{\#}(G)$ for all $1 \leq p < \infty$.*

Proof. As H is embedded H-equivariantly, quasi-isometrically in G, we have $\alpha_p^{\#}(H) \geq \alpha_p^{\#}(G)$. To prove the converse inequality, we may assume that $\alpha_p^{\#}(H) > 0$. Let S be a compact generating subset of H. Let $A(h)v = \pi(h)v + b(h)$ be an affine isometric action of H on L^p, such that for some $\alpha < \alpha_p^{\#}(H)$ we have $\|b(h)\|_p \geq C|h|_S^{\alpha}$, for every $h \in H$. Now we induce up the action A from H to G, as on p.98 in Section 2.5. of [**18**][1]. The affine space of the induced action is

$$E := \{f : G \to L^p : f(gh) = A(h)^{-1}f(g),\ \forall h \in H \text{ and almost every } g \in G\},$$

with distance given by $\|f_1 - f_2\|_p^p = \sum_{x\in G/H} \|f_1(x) - f_2(x)\|_p^p$. The induced affine isometric action $\tilde{A}$ of G on E is then given by $(\tilde{A}(g))f(g') = f(g^{-1}g')$, for $f \in E$, $g, g' \in G$.

A function $\xi_0 \in E$ is then defined as follows. Let $s_1 = e, s_2, , ..., s_n$ be a set of representatives for the left cosets of H in G. Set $\xi_0(s_ih) = b(h^{-1})$, for $h \in H$, $i = 1, ..., n$. Define the 1-cocycle $\tilde{b}$ on G by $\tilde{b}(g) = \tilde{A}(g)\xi_0 - \xi_0$, for $g \in G$. For an $h \in H$, we then have:

$$\|\tilde{b}(h)\|_p^p = \sum_{i=1}^{n} \|\xi_0(h^{-1}s_i) - \xi_0(s_i)\|_p^p = \sum_{i=1}^{n} \|\xi_0(h^{-1}s_i)\|_p^p \geq \|\xi_0(h^{-1})\|_p^p = \|b(h)\|_p^p.$$

Set $K = \max_{1\leq i\leq n} \|\tilde{b}(s_i)\|_p$. Take $T = S \cup \{s_1, ..., s_n\}$ as a compact generating set of G. For $g \in G$, write $g = s_ih$ for $1 \leq i \leq n$, $h \in H$. Then

$$\begin{aligned}\|\tilde{b}(g)\|_p \geq \|\tilde{b}(h)\|_p - K \geq \|b(h)\|_p - K \geq C|h|_S^{\alpha} - K \geq C|h|_T^{\alpha} - K\\ \geq C(|g|_T - 1)^{\alpha} - K \geq C'|g|_T^{\alpha} - K'.\end{aligned}$$

So the compression of the 1-cocycle $\tilde{b}$ is at least α, hence $\alpha_p^{\#}(G) \geq \alpha_p^{\#}(H)$. □

The following proof uses a construction by S.R. Gal, see page 4 of [**52**].

***Theorem* 2.4.3.** *Let H be a finite index subgroup of G_1 and G_2 and assume there is a proper affine isometric action β_i (with compression α_i) of each G_i on a Hilbert space V_i. Assume that $W < V_1 \cap V_2$ is invariant under the actions $(\beta_i|_H)$ and moreover that both these (restricted) actions coincide on W. Then $\alpha_2^{\#}(G_1 *_H G_2) \geq \frac{\min(\alpha_1,\alpha_2)}{2}$. In particular, $\alpha_2^{\#}(G_1 *_H G_2) \geq \frac{\alpha_2^{\#}(H)}{2}$.*

Proof. Following [**52**], let us build a Hilbert space W_Γ on which $\Gamma = G_1 *_H G_2$ acts affinely and isometrically. Let ω be a finite alternating sequence of 1's and 2's and suppose π is a linear action of H on some Hilbert space denoted $\mathcal{H}_\omega$. One can induce up the linear action from H to G_i, obtaining a Hilbert space

$$V := \{f\colon G_i \to \mathcal{H}_\omega \mid \forall h \in H,\ f(gh) = \pi(h^{-1})f(g)\}$$

and an orthogonal action $\pi_i\colon G_i \to \mathcal{O}(V)$ defined by $\pi_i(g)f(g') = f(g^{-1}g')$. The subspace

$$\{f\colon G_i \to \mathcal{H}_\omega \mid \forall h \in H,\ f(h) = \pi(h^{-1})f(1), f|_{G_i\setminus H} = 0\}$$

[1]We seize this opportunity to correct a misprint in the definition of the vector ξ_0 in that construction on p.98 of [**18**].

can be identified with $\mathcal{H}_\omega$ by letting an element f correspond to $f(1)$. It is clear that the action π_i restricted to H coincides with the original linear action π via this identification.

So, starting from any linear H-action on a Hilbert space $\mathcal{H}_\omega$, we can obtain a linear action of say G_1 on a Hilbert space that can be written as $\mathcal{H}_\omega \oplus \mathcal{H}_{1\omega}$ for some $\mathcal{H}_{1\omega}$. We can restrict this action to a linear H-action on $\mathcal{H}_{1\omega}$ and we can lift this to an action of G_2 on a space $\mathcal{H}_{1\omega} \oplus \mathcal{H}_{21\omega}$ and so on, repeating the process indefinitely. Here, we will execute this infinite process twice.

The first linear H-action on which we apply the process is obtained as follows. As $\beta_i(H)(W) = W$ for each $i = 1, 2$, the restriction to H of β_1, gives naturally a linear H-action on $\mathcal{H}_1 := V_1/W$. The second linear H-action is obtained by similarly noting that the restriction to H of β_2 gives a linear H-action on $\mathcal{H}_2 := V_2/W$. We then apply the above process indefinitely.

$$\mathcal{H}_1^\bullet := \overbrace{\mathcal{H}_1 \oplus \underbrace{\mathcal{H}_{21}}^{G_2\curvearrowright} \oplus \overbrace{\mathcal{H}_{121}}_{G_1\curvearrowright} \oplus \underbrace{\mathcal{H}_{2121}}^{G_2\curvearrowright} \oplus \cdots}, \quad \mathcal{H}_2^\bullet := \overbrace{\mathcal{H}_2 \oplus \underbrace{\mathcal{H}_{12}}^{G_1\curvearrowright} \oplus \overbrace{\mathcal{H}_{212}}_{G_2\curvearrowright} \oplus \underbrace{\mathcal{H}_{1212}}^{G_1\curvearrowright} \oplus \cdots},$$

where for ω a sequence of alternating 1's and 2's, G_i acts on $\mathcal{H}_\omega \oplus \mathcal{H}_{i\omega}$. Note that there are two H-actions on $\mathcal{H}_1^\bullet$ as H acts on the first term $\mathcal{H}_1$. One can verify that both H-actions coincide (this fact is also mentioned in [**52**],page 4). The same is true for $\mathcal{H}_2^\bullet$.

Denote $\mathcal{H}_1^\circ = \mathcal{H}_1^\bullet \ominus \mathcal{H}_1 = \mathcal{H}_{21} \oplus \mathcal{H}_{121} \oplus \mathcal{H}_{2121} \oplus \cdots$, and similarly, set $\mathcal{H}_2^\circ = \mathcal{H}_2^\bullet \ominus \mathcal{H}_2 = \mathcal{H}_{12} \oplus \mathcal{H}_{212} \oplus \mathcal{H}_{1212} \oplus \cdots$. We denote

$$W_\Gamma = W \oplus \mathcal{H}_1^\bullet \oplus \mathcal{H}_2^\bullet = V_1 \oplus \mathcal{H}_1^\circ \oplus \mathcal{H}_2^\bullet = V_2 \oplus \mathcal{H}_2^\circ \oplus \mathcal{H}_1^\bullet.$$

The above formula, which decomposes W as a direct sum in three distinct ways, shows that both G_1 and G_2 act on W_Γ. As mentioned before, the actions coincide on H and so we obtain an affine isometric action of Γ on W_Γ. Note that the corresponding 1-cocycle, when restricted to G_1 (or G_2), coincides with the 1-cocycle of β_1(or β_2).

We inductively define a length function $\psi_T\colon \Gamma \to \mathbb{N}$ by $\psi_T(h) = 0$ for all $h \in H$ and $\psi_T(\gamma) = \min\{\psi_T(\eta) + 1 \mid \gamma = \eta g, \text{ where } g \in G_1 \cup G_2\}$. By [**96**, Theorem 1.] we see that this map is conditionally negative definite and thus the normed square of a 1-cocycle associated to an affine isometric action of Γ on a Hilbert space.

Let ψ_Γ be the conditionally negative definite function associated to the action of Γ on W_Γ. We now find the compression of the conditionally negative definite map $\psi = \psi_\Gamma + \psi_T$. First set

$$M = \max\left\{|t_j^i|_{G_i} : i = 1, 2 \text{ and } 1 \le j \le [G_i : H]\right\},$$

where t_j^i are right coset representatives of H in G_i such that $t_1^i = 1_{G_i}$ for $i = 1, 2$.

Denote $\alpha = \min(\alpha_1, \alpha_2)$ and fix some $\varepsilon > 0$ arbitrarily small. Let $\gamma \in \Gamma$ and suppose in normal form $\gamma = g t_{j_1}^{i_1} \cdots t_{j_k}^{i_k}$, where $g \in G_i$ for some $i = 1, 2$. Assume first that $\psi_T(\gamma) \ge \frac{|\gamma|^{\alpha-\varepsilon}}{M}$. In that case, $\psi(\gamma) \ge \frac{|\gamma|^{\alpha-\varepsilon}}{M}$. Else, we have that $\psi_T(\gamma) < \frac{|\gamma|^{\alpha-\varepsilon}}{M}$ and so for all $\gamma \in \Gamma$ such that $|\gamma|$ is sufficiently large, we have

$$\begin{aligned}\psi(\gamma) \ge \psi_\Gamma(\gamma) = \|\gamma \cdot 0\|^2 \ge (\|g \cdot 0\| - \psi_T(\gamma)M)^2 &\gtrsim ((|\gamma| - \psi_T(\gamma)M)^{\alpha-\varepsilon/2} - \psi_T(\gamma)M)^2 \\ &\ge ((|\gamma| - |\gamma|^{\alpha-\varepsilon})^{\alpha-\varepsilon/2} - |\gamma|^{\alpha-\varepsilon})^2 \gtrsim |\gamma|^{2\alpha-\varepsilon},\end{aligned}$$

where $\gtrsim$ represents inequality up to a multiplicative constant; we use here that one can always assume, without loss of generality, that the 1-cocycles associated to β_1 and β_2 satisfy $\|b_i(g_i)\| \gtrsim |g_i|^{\alpha-\varepsilon}$ (see Lemma 3.4 in [**6**]).

So now, by the first case, $\psi(\gamma) \geq |\gamma|^{\alpha-\varepsilon}$ for all $\gamma \in \Gamma$ that are sufficiently large. Hence, we obtain the lower bound $\alpha_2^{\#}(\Gamma) \geq \alpha_2^{\#}(H)/2$. □

CHAPTER 3

Coarse embeddability of generalised wreath products

The results in this Chapter were done in joint work with Dennis Dreesen and can be found in [**28**].

3.1. Generalised wreath products

Throughout this chapter groups will be finitely generated and metric spaces will be countable and discrete. Given two finitely generated groups G and H, the wreath product, written as $G \wr H$ is the set of pairs $(\mathbf{f}, h)$ where $h \in H$ and $\mathbf{f} \colon H \to G$ is a finitely supported function (i.e. $\mathbf{f}(h) = e_G$ for all but finitely many $h \in H$) together with a group operation

$$(\mathbf{f}, h) \cdot (\mathbf{g}, h') = (\mathbf{f} \cdot (h\mathbf{g}), hh')$$

where $(h\mathbf{g})(z) = \mathbf{g}(h^{-1}z)$ for all $h, z \in G$. One can think of $G \wr H$ as being the semi-direct product $\bigoplus_H G \rtimes H$ where H acts on $\bigoplus_H G$ by permuting the indices. If finite sets S and T generate G and H respectively then $G \wr H$ is generated by the finite set

$$\{(\mathbf{e}, t) : t \in T\} \cup \{(\delta_s, e_H) : s \in S\}$$

where $\mathbf{e}(h) = e_G$ for all $h \in H$ and

$$\delta_s(h) = \begin{cases} s & \text{if } h = e_H \\ e_G & \text{otherwise.} \end{cases}$$

The word metric on $G \wr H$ coming from this generating set can be thought of as follows. Given two elements $(\mathbf{f}, x)$ and $(\mathbf{g}, y)$, take the shortest path in the Cayley graph $\mathrm{Cay}(H, T)$ going from x to y that passes through the points in $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = \{h_1, \dots, h_n\}$. At each point $h_i \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ travel from $\mathbf{f}(h_i)$ to $\mathbf{g}(h_i)$ in G. Explicitly for $(\mathbf{f}, x), (\mathbf{g}, y) \in \bigoplus_{g \in H} G \rtimes H$ and $\mathrm{Supp}\ (\mathbf{f}^{-1}\mathbf{g}) = \{h_1, \dots, h_n\}$ define

$$p_{(x,y)}(\mathbf{f}, \mathbf{g}) = \inf_{\sigma \in S_n} \left(d_H(x, h_{\sigma(1)}) + \sum_{i=1}^{n} d_H(h_{\sigma(i)}, h_{\sigma(i+1)}) + d_H(h_{\sigma(n)}, y) \right)$$

where the infimum is taken over all permutations in S_n. The number $p_{(x,y)}(\mathbf{f}, \mathbf{g})$ corresponds to the shortest path between x and y in H going through each element in $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$. Hence the distance between $(\mathbf{f}, x)$ and $(\mathbf{g}, y)$ is

$$d_{G \wr H}((\mathbf{f}, x), (\mathbf{g}, y)) = p_{(x,y)}(\mathbf{f}, \mathbf{g}) + \sum_{h \in H} d_G(\mathbf{f}(h), \mathbf{g}(h)).$$

Suppose G, H are groups and H acts transitively on a set X. Fix a base point $x_0 \in X$ and define the *permutational wreath product* to be the group $G \wr_X H := \bigoplus_X G \rtimes H$ where

$$\bigoplus_X G = \{\mathbf{f} \colon X \to G : \mathbf{f}(x) = e_G \text{ for all but finitely many } x \in X\}$$

and H acts on $\bigoplus_X G$ by permuting the indices. If S and T generate G and H respectively then $G \wr_X H$ is generated by

$$\{(\mathbf{e}, t) : t \in T\} \cup \{(\delta_s, e_H) : s \in S\}$$

where $\mathbf{e}(x) = e_G$ for all $x \in X$ and

$$\delta_s(x) = \begin{cases} s & \text{if } x = x_0 \\ e_G & \text{otherwise.} \end{cases}$$

The metric on $G \wr_X H$ from the generating set can be thought of as follows. Given two elements $(\mathbf{f}, x)$ and $(\mathbf{g}, y)$ take the shortest path going from x to y in $\mathrm{Cay}(H, T)$ that passes through points $\{h_1, \dots, h_n\}$ such that $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = \{h_1 x_0, \dots, h_n x_0\}$. At each element $h_i \in \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ travel from $\mathbf{f}(h_i x_0)$ to $\mathbf{g}(h_i x_0)$ in G. In general the shortest path is not necessarily unique.

Explicitly for $(\mathbf{f}, x), (\mathbf{g}, y) \in \bigoplus_{x \in X} G \rtimes H$, let $I = \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ and let $n = |\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|$. Define $\mathcal{P}_I$ to be the set

$$\mathcal{P}_I := \{(h_1, \dots, h_n) \subset H^n : \{h_1 x_0, \dots, h_n x_0\} = I\}.$$

In particular if $(h_1, \dots, h_n) \in \mathcal{P}_I$ then any permutation of $(h_1, \dots, h_n)$ is also in $\mathcal{P}_I$. Hence the length of the shortest path between x and y in H passing though the points that project onto $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ is precisely

$$\rho_{(x,y)}(\mathbf{f}, \mathbf{g}) := \inf_{(h_1, \dots, h_n) \in \mathcal{P}_I} \left(d(x, h_1) + \sum_{i=1}^{n-1} d(h_i, h_{i+1}) + d(h_n, y) \right).$$

Hence the distance between $(\mathbf{f}, x)$ and $(\mathbf{g}, y)$ is

$$d_{G \wr_X H}((\mathbf{f}, x), (\mathbf{g}, y)) = \rho_{(x,y)}(\mathbf{f}, \mathbf{g}) + \sum_{z \in X} d_G(\mathbf{f}(z), \mathbf{g}(z)).$$

One can ask whether we can generalise this construction. Suppose Y and Z are metric spaces and $p \colon Y \to Z$ is a C-dense map, i.e. $B_Z(p(Y), C) = Z$. Given two points $y, y' \in Y$ and a finite sequence of points $I = \{z_1, \dots, z_n\}$ in Z, we define $\mathcal{P}_I$ to be the set

$$\mathcal{P}_I := \left\{(y_1, \dots, y_n) \subset Y^n : \exists \sigma \in S_n \text{ such that } \forall i,\ p(y_i) \in B(z_{\sigma(i)}, C)\right\}.$$

In particular, if $(y_1, \dots, y_n) \in \mathcal{P}_I$ then any permutation of $(y_1, \dots, y_n)$ also lies in $\mathcal{P}_I$. We now define the *length of the path from y to y' going through I* by

$$\mathrm{path}_I(y, y') = \inf_{(y_1, \dots, y_n) \in \mathcal{P}_I} \left(d_Y(y, y_1) + \sum_{i=1}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y') \right).$$

Let X be another metric space and fix a base point $x_0 \in X$. Define $\bigoplus_Z X$ to be the set

$$\bigoplus_Z X = \{\mathbf{f} \colon Z \to X : \mathbf{f}(z) = x_0 \text{ for all but finitely many } z \in Z\}.$$

For $\mathbf{f}, \mathbf{g} \in \bigoplus_Z X$ define $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g}) = (\mathrm{Supp}(\mathbf{f}) \cup \mathrm{Supp}(\mathbf{g})) \setminus \{z \in Z : \mathbf{f}(z) = \mathbf{g}(z)\}$. Let $(\mathbf{f}, y), (\mathbf{g}, y') \in \bigoplus_Z X \times Y$ and let $I = \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$. Define a metric on the set $\bigoplus_Z X \times Y$ by

$$d((\mathbf{f}, y), (\mathbf{g}, y')) = \mathrm{path}_I(y, y') + \sum_{z \in Z} d_X(\mathbf{f}(z), \mathbf{g}(z)).$$

We obtain a metric space $(\bigoplus_Z X \times Y, d)$, which we denote by $X \wr_Z^C Y$. Here the C refers to the C-dense map $p \colon Y \to Z$. When X and Y are groups and Y acts transitively on Z then the map $p \colon Y \to Z$ is surjective thus $C = 0$. When there is no risk for confusion, we will omit

C from this notation. When X and Y are graphs, then the metric wreath product $X \wr_Y Y$ coincides with the wreath product of graphs [**48**, Definition 2.1.].

3.2. Measured wall structures

Let X be a set and 2^X the power set of X. We endow 2^X with the product topology. For $x \in X$, denote $\mathcal{A}_x = \{A \subset X : x \in A\}$. This is a clopen subset in 2^X. For two elements $x, y \in X$ we say a set $A \subset X$ *cuts* x and y, denoted $A \vdash \{x, y\}$ if $x \in A$ and $y \in A^c$ or $x \in A^c$ and $y \in A$. Likewise we say that A cuts another set Y if neither $Y \subset A$ nor $Y \subset A^c$.

Definition 3.2.1. A *measured wall structure* on a set X is a Borel measure μ on 2^X such that for every $x, y \in X$,

$$d_\mu(x, y) := \mu\left(\left\{A \in 2^X : A \vdash \{x, y\}\right\}\right) < \infty.$$

Since $\left\{A \in 2^X : A \vdash \{x, y\}\right\} = \mathcal{A}_x \triangle \mathcal{A}_y$, the set is measurable. It follows that d_μ is well defined and is a pseudometric on X, called the *wall metric associated to μ*.

If $f\colon X \to Y$ is a map between sets and (Y, μ) is a measured wall structure, then we can push forward the measure μ via the inverse image map $f^{-1}\colon 2^Y \to 2^X$ and obtain a measured wall structure $(X, f^*\mu)$, where for $A \subset 2^X$, $f^*\mu(A) = \mu(\{f(B) \mid B \in A, B = f^{-1}(f(B))\})$. It follows that $d_{f^*\mu}(x, x') = d_\mu(f(x), f(x'))$ [**39**, Section 2.].

Given a family of spaces X_i with measured wall structures μ_i and the natural projection maps $p_i\colon \bigoplus X_j \to X_i$, then the measure $\mu = \sum_I p_i^*\mu_i$ defines a measured wall structure on $\bigoplus_i X_i$. The associated wall metric is $d_\mu((x_i), (y_i)) = \sum_i d_{\mu_i}(x_i, y_i)$.

***Proposition* 3.2.2** ([**29**, Proposition 6.16.], [**39**, Proposition 2.6.])**.** *Let X be a set and $k\colon X \times X \to \mathbb{R}_+$ a kernel. Then the following are equivalent:*

(1) *There exists $f\colon X \to L^1(X)$ such that $k(x, y) = \|f(x) - f(y)\|_1$ for all $x, y \in X$.*
(2) *For every $p \geq 1$, there exists $f\colon X \to L^p(X)$ such that $(k(x, y))^{1/p} = \|f(x) - f(y)\|_p$ for all $x, y \in X$.*
(3) *$k = d_\mu$ for some measured wall structure (X, μ).*

In order to prove our main result we make use of a method of lifting measured wall structures. First we require some technical definitions. Let W, X be sets and $\mathcal{A} = 2^{(X)}$, the set of finite subsets of X.

Definition 3.2.3 ([**39**, Definition 3.1.])**.** An $\mathcal{A}$-gauge on W is a function $\phi\colon W \times W \to \mathcal{A}$ such that:

$$\phi(w, w') = \phi(w', w) \quad \forall w, w' \in W$$
$$\phi(w, w'') \subset \phi(w, w') \cup \phi(w', w'') \quad \forall w, w', w'' \in W.$$

If W is a group then ϕ is called *left invariant* if $\phi(ww', ww'') = \phi(w', w'')$ for all $w, w', w'' \in W$.

***Theorem* 3.2.4** ([**39**, Theorem 4.2])**.** *Let X, W be sets, $\mathcal{A} = 2^{(X)}$. Let ϕ be an $\mathcal{A}$-gauge on W and assume that $\phi(w, w) = \emptyset$ for all $w \in W$. Let (X, μ) be a measured wall structure. Then there is a naturally defined measure $\widetilde{\mu}$ on $2^{W \times X}$ such that $(W \times X, \widetilde{\mu})$ is a measured wall structure with corresponding pseudometric*

$$d_{\widetilde{\mu}}(w_1 x_1, w_2 x_2) = \mu\left(\{A \in \mathcal{A} : A \vdash \phi(w_1, w_2) \cup \{x_1, x_2\}\}\right).$$

A consequence of this theorem is that if X, Y, Z are metric spaces where X has a fixed point $x_0 \in X$ then $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})$ is an $\mathcal{A}$-gauge on $\bigoplus_Z X$, where $\mathcal{A} = 2^{(Z)}$. Hence if Z has a measured wall structure there exists a lifted measured wall structure on $\bigoplus_Z X \times Z$.

3.3. Coarse embeddings of wreath products

Definition 3.3.1. We say that a metric space X has C-bounded geometry for some $C > 0$, if there exists a constant $N(C) > 0$ such that $|B(x, C)| \leq N(C)$ for all $x \in X$. A metric space has *bounded geometry* if it has C-bounded geometry for every $C > 0$.

Example 3.3.2. Note that C-bounded geometry for some C does not in general imply bounded geometry. As an easy example, one can consider an infinite metric space equipped with the discrete metric, i.e. $d(x, y) = 1$ for every $x, y \in X$ distinct.

Definition 3.3.3. [**122**, Definition 1.2.] Let Y and Z be metric spaces. A map $p\colon Y \to Z$ has the *C-coarse path lifting property* if there exists $C > 0$ and a non-decreasing function $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$, such that for any $z, z' \in Z$ and $y \in Y$ with $d_Z(p(y), z) \leq C$ there exists $y' \in Y$ such that $d_Z(p(y'), z') \leq C$ and $d_Y(y, y') \leq \theta(d_Z(z, z'))$.

Example 3.3.4. The path lifting property occurs naturally in the setting of groups. Let $Y = H$ be a group and let $N \triangleleft H$ be a normal subgroup. The most natural way of defining a distance function on $Z := H/N$ is by setting $d(hN, h'N)$ to be the infimum of $d(hn, h'n')$ over all $n, n' \in N$. The projection map $p : H \to H/N$ is a bornologous map and one checks easily that it satisfies the coarse path lifting property. Actually, one only needs the fact that N is "almost normal" in H, i.e. that for every finite subset F of H, there exists a finite subset $F' \subset H$ with $NF \subset F'N$.

For each $R > 0$ there exists a finite subset F_R of H such that $NB(1, R) \subset F_R N$. Set $\theta\colon R \mapsto \max\{d(e, f) : f \in F_R\}$. Then $d(xN, yN) \leq R$ if and only if $x^{-1}y \in NB(1, R)N$ and so as N is almost normal it follows that $x^{-1}y \in F_R$. Hence the quotient map has the coarse path lifting property.

Another example can be obtained by taking Z to be the set of right N-cosets of H, where N is any (not necessarily normal) subgroup of H. In this case, the projection map $p : H \to N\backslash H, g \mapsto Ng$ is a bornologous map that has the coarse path lifting property.

***Theorem* 3.3.5.** *Let X, Y, Z be metric spaces and $p\colon Y \to Z$ be a C-dense bornologous map with the C'-coarse path lifting property where $C \leq C'$. Let $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function satisfying the properties in Definition 3.3.3. Assume that Y is uniformly discrete and that Z has C-bounded geometry. If X, Y, Z are coarsely embeddable into an L^1-space, then so is $X \wr_Z^C Y$.*

Remark 3.3.6. Observe that, by Proposition 3.2.2, the conclusion of the theorem also implies L^p-embeddability of $X \wr_Z Y$ for any $p \geq 1$. On the other hand, it is known that L^p embeds isometrically into L^1 for $1 \leq p \leq 2$. Hence in the formulation of Theorem 3.3.5, we can just as well replace L^1-embeddability by L^p-embeddability for $1 \leq p \leq 2$.

Proof. By Proposition 3.2.2, there exists measured wall structures (X,σ), (Y,ν), (Z,μ) and functions $\rho_X,\rho_Y,\rho_Z,\eta_X,\eta_Y,\eta_Z\colon \mathbb{R}^+\to\mathbb{R}^+$ increasing to infinity, such that

$$\rho_X(d_X(x_1,x_2))\le d_\sigma(x_1,x_2)\le \eta_X(d_X(x_1,x_2))\quad \forall x_1,x_2\in X \tag{4}$$

$$\rho_Y(d_Y(y_1,y_2))\le d_\nu(y_1,y_2)\le \eta_Y(d_Y(y_1,y_2))\quad \forall y_1,y_2\in Y \tag{5}$$

$$\rho_Z(d_Z(z_1,z_2))\le d_\mu(z_1,z_2)\le \eta_Z(d_Z(z_1,z_2))\quad \forall z_1,z_2\in Z. \tag{6}$$

By Theorem 3.2.4, there exists a measured wall structure $\widetilde{\mu}$ on $\bigoplus_Z X\times Z$ where for $(\mathbf{f},z),(\mathbf{g},z')\in\bigoplus_Z X\times Z$

$$d_{\widetilde{\mu}}((\mathbf{f},z),(\mathbf{g},z'))=\mu(\{A: A\vdash \mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})\cup\{z,z'\}\}).$$

We have a projection map $p\colon \bigoplus_Z X\times Y\to\bigoplus_Z X\times Z$ where $(\mathbf{f},y)\mapsto(\mathbf{f},p(y))$. Using this we can pullback a measured wall structure on $\bigoplus_Z X\times Y$ where

$$d_{p\widetilde{\mu}}((\mathbf{f},y),(\mathbf{g},y'))=d_{\widetilde{\mu}}((\mathbf{f},p(y)),(\mathbf{g},p(y')))$$

We define three other wall structures, $\widetilde{\sigma},\widetilde{\nu}$ and $\widetilde{\omega}$, on $X\wr_Z Y$ where

$$\begin{aligned} d_{\widetilde{\sigma}}((\mathbf{f},y),(\mathbf{g},y')) &=\sum_{z\in Z} d_\sigma(\mathbf{f}(z),\mathbf{g}(z)),\\ d_{\widetilde{\nu}}((\mathbf{f},y),(\mathbf{g},y')) &= d_\nu(y,y'),\\ d_{\widetilde{\omega}}((\mathbf{f},y),(\mathbf{g},y')) &=|\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|.\end{aligned}$$

It is clear from our comments in Section 3.2 that $\widetilde{\sigma}$ and $\widetilde{\nu}$ are indeed wall space structures. Observe that $d_{\widetilde{\omega}}$ is associated to the map $\Lambda:\bigoplus_Z X\times Y\to\ell^1(X\times Z)$, where

$$\Lambda(\mathbf{f},y):(x,z)\mapsto\begin{cases}1/2 & \text{if } f(z)=x\\ 0 & \text{if otherwise}\end{cases}$$

and $\|\Lambda(\mathbf{f},y)-\Lambda(\mathbf{g},y')\|_1=|\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|=d_{\widetilde{\omega}}((\mathbf{f},y),(\mathbf{g},y'))$. We now aim to show that we can coarsely embed $X\wr_Z Y$ into an L^1-space. Define $\lambda=p\widetilde{\mu}+\widetilde{\sigma}+\widetilde{\nu}+\widetilde{\omega}$ to be a measured wall space structure on $X\wr_Z Y$. By Proposition 3.2.2, it suffices to show that for every $R>0$ if $d_\lambda((\mathbf{f},y),(\mathbf{g},y'))\le R$ then $d_{X\wr_Z Y}((\mathbf{f},y),(\mathbf{g},y'))\le C_1(R)$ and if $d_{X\wr_Z Y}((\mathbf{f},y),(\mathbf{g},y'))\le R$ then $d_\lambda((\mathbf{f},y),(\mathbf{g},y'))\le C_2(R)$ where C_1,C_2 are constants depending only on R. Fix $R>0$ and suppose $d_\lambda((\mathbf{f},y),(\mathbf{g},y'))\le R$ for some $(\mathbf{f},y),(\mathbf{g},y')\in X\wr_Z Y$. In particular

$$d_{\widetilde{\mu}}((\mathbf{f},p(y)),(\mathbf{g},p(y')))\le R, \tag{7}$$

$$\sum_{z\in Z} d_\sigma((\mathbf{f}(z),\mathbf{g}(z))\le R, \tag{8}$$

$$d_\nu(y,y')\le R, \tag{9}$$

$$|\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})|\le R. \tag{10}$$

Set $z_0:=p(y)$ and write $\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})=\{z_1,z_2,\dots,z_n\}$ for some $n\le R$. By (7) it follows that $\mu(A: A\vdash\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})\cup\{p(y),p(y')\})\le R$. In particular $d_\mu(a,b)\le R$ for all $a,b\in\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})\cup\{p(y),p(y')\}$. By Equation (6), this implies that $d_Z(a,b)\le\rho_Z^{-1}(R)$ for all $a,b\in\mathrm{Supp}(\mathbf{f}^{-1}\mathbf{g})\cup\{p(y),p(y')\}$. Starting from $y_0=y$, by the path lifting property, we can find y_1 such that $d_Z(p(y_1),z_1)\le C$ and $d_Y(y,y_1)\le\theta(\rho_Z^{-1}(R))$. We can then find y_2 with $d_Z(p(y_2),z_2)\le C$ and $d_Y(y_1,y_2)\le\theta(\rho_Z^{-1}(R))$. Continuing inductively and by the triangle inequality, we obtain

$$\sum_{i=0}^{n-1} d_Y(y_i,y_{i+1})+d_Y(y_n,y_0)\le 2\sum_{i=0}^{n-1}\theta(\rho_Z^{-1}(R))\le 2R\theta(\rho_Z^{-1}(R)).$$

Using Equation (9) and denoting $y_0 = y$, we thus have that

$$\text{path}_I(y, y') \leq \sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y_0) + d_Y(y, y') \leq 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R). \tag{11}$$

Now we can deduce that

$$\begin{aligned}
&\text{path}_I(y, y') + \sum_{z \in Z} d_X((\mathbf{f}(z), \mathbf{g}(z)) \\
&\quad \leq 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R) + \sum_{z \in \text{Supp}(\mathbf{f}^{-1}\mathbf{g})} \rho_X^{-1}(R) \qquad \text{by (4), (8) and (11)} \\
&\quad \leq 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R) + R\rho_X^{-1}(R) \qquad \text{by (10).}
\end{aligned}$$

It suffices to set $C_1(R) = 2R\theta(\rho_Z^{-1}(R)) + \rho_Y^{-1}(R) + R\rho_X^{-1}(R)$. Now suppose conversely that $d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) \leq R$. In particular

$$\text{path}_I(y, y') \leq R \tag{12}$$

$$\sum_{z \in Z} d_X(f(z), g(z)) \leq R. \tag{13}$$

Let $(y_1, \dots, y_n) \in \mathcal{P}_I$ such that

$$d_Y(y, y_1) + \sum_{i=1}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y') \leq R + 1. \tag{14}$$

As Y is uniformly discrete, we have $\delta_Y := \inf(d(a, b) \mid a, b \in Y) > 0$. This implies that, although some of the y_i may be equal, the number of distinct y_i is bounded by $\frac{R+1}{\delta_Y}$. Any point in the support of $\mathbf{f^{-1}g}$ lies, by definition, in a C-neighbourhood of some $p(y_i)$. As such neighbourhoods contain at most $N(C)$ elements, we can conclude that

$$n = |\text{Supp}(\mathbf{f^{-1}g})| \leq E(R) := N(C)\frac{R+1}{\delta_Y}. \tag{15}$$

From Equation (14) and the triangle inequality, it follows that

$$d_Y(a, b) \leq R + 1 \qquad \forall a, b \in \{y, y', y_1, \dots, y_n\}. \tag{16}$$

As p is bornologous, there exists $S = S(R+1)$ such that for all $z, z' \in \{p(y), p(y'), p(y_1), \dots, p(y_n)\}$, we have $d_Z(z, z') \leq S$. By definition of $(y_1, \dots, y_n)$ and the triangle inequality it follows that $d_Z(z, z') \leq S + 2C$ for every $z, z' \in \text{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}$. By (6) it follows that

$$d_\mu(z, z') \leq \eta_Z(S + 2C) \qquad \forall z, z' \in \text{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}. \tag{17}$$

Let us enumerate $\text{Supp}(\mathbf{f}^{-1}\mathbf{g}) \sqcup \{p(y), p(y')\} = \{p(y) = z_0, z_1, \dots, z_{n+1} = p(y')\}$. Note that, if A cuts $\text{Supp}(\mathbf{f}^{-1}\mathbf{g}) \cup \{p(y), p(y')\}$, then A must cut $\{z_i, z_{i+1}\}$ for some $i \in \{0, 1, \dots, m-1\}$. Hence $d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) \leq \sum_{i=0}^{n} d_\mu(z_i, z_{i+1})$. It now follows that

$$\begin{aligned}
d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) &= (d_{\widetilde{\nu}} + d_{p\widetilde{\mu}} + d_{\widetilde{\sigma}} + d_{\widetilde{\omega}})((\mathbf{f}, y), (\mathbf{g}, y')) \\
&\leq d_\nu(y, y') + \sum_{i=0}^{n} d_\mu(z_i, z_{i+1}) + \sum_{z \in Z} d_\sigma(\mathbf{f}(z), \mathbf{g}(z)) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) \\
&\leq \eta_Y(R) + \sum_{i=0}^{n} \eta_Z(S + 2C) + \sum_{z \in Z} \eta_X(R) + E(R) \qquad \text{by (5), (13), (17), (15)} \\
&\leq \eta_Y(R) + E(R)\eta_Z(S + 2C) + E(R)\eta_X(R) + E(R).
\end{aligned}$$

Hence, it suffices to set $C_2(R) := \eta_Y(R) + E(R)(\eta_Z(S + 2C) + \eta_X(R) + 1)$. This shows by Proposition 3.2.2 that $X \wr_Z Y$ embeds coarsely into an L^p-space. □

Remark 3.3.7. The only time that we used the conditions Y is uniformly discrete was to show that Equations (12) and (13) imply that $|\mathrm{Supp}(\mathbf{f^{-1}g})|$ is bounded by some function of R. One checks easily that this condition can be replaced by Y has bounded geometry. Alternatively, it would also be sufficient to require nothing on Y and Z but to ask that X is a uniformly discrete metric space.

3.4. The compression of $X \wr_Z Y$

We can modify the previous proof to give information on the L^1-compression of $X \wr_Z Y$ in terms of the growth behaviour of θ and the L^1-compression of X, Y and Z.

Definition 3.4.1. Let Y and Z be metric spaces and let $p\colon Y \to Z$ be a C-dense map with the coarse path lifting property with respect to a non-decreasing function $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$. If $\delta > 0$ is such that $\theta(r) \lesssim r^\delta + 1$ for every $r \in \mathbb{R}^+$, then we say that p has the *δ-polynomial path lifting property.*

Here, $\lesssim$ denotes inequality up to a multiplicative constant.

***Theorem* 3.4.2.** *Let X, Y, Z be metric spaces as in Theorem 3.3.5 and $p\colon Y \to Z$ a C-dense large-scale Lipschitz map. If p has the δ-polynomial path lifting property for some $\delta > 0$ then*

$$\alpha_1(X \wr_Z^C Y) \geq \min\left(\alpha_1(X), \alpha_1(Y), \frac{\alpha_1(Z)}{\alpha_1(Z) + \delta}\right)$$

Remark 3.4.3. Our bound generalizes the bound of Theorem 1.1 in [**80**], which covers the cases when X and Y are finitely generated groups and $Z = Y$. Observe that, as both X and Y can be considered as metric subspaces of $X \wr_Z Y$, one also has an upper bound, namely $\min(\alpha, \beta)$, for the compression of $X \wr_Z Y$.

Proof. Assume that there are constants $a, b > 0$ such that $d_Z(p(y), p(y')) \leq a d_Y(y, y') + b$ for every $y, y' \in Y$.The starting point for this proof is the proof of Theorem 3.3.5 and we will often refer to inequalities stated there. For now, assume that α, β, γ are real numbers and that $f_1\colon X \to L^1$, $f_2\colon X \to L^1$ and $f_3\colon Z \to L^1$ are large scale Lipschitz functions into L^1-spaces such that

$$d_X(x, x')^\alpha \lesssim \|f_1(x) - f_1(x')\|_1$$
$$d_Y(y, y')^\beta \lesssim \|f_2(y) - f_2(y')\|_1$$
$$d_Z(z, z')^\gamma \lesssim \|f_3(z) - f_3(z')\|_1.$$

Let d_σ, d_ν, and d_μ be the measured wall space structures associated to the functions f_1, f_2, f_3 by Proposition 3.2.2. Define the measured wall $d_{p\widetilde{\mu}}, d_{\widetilde{\mu}}, d_{\widetilde{\sigma}}, d_{\widetilde{\omega}}$ on $X \wr_Z Y$ as in Theorem 3.3.5. As a first step, we are going to show that the function associated to the measured wall $d_\lambda = d_{p\widetilde{\mu}} + d_{\widetilde{\mu}} + d_{\widetilde{\sigma}} + d_{\widetilde{\omega}}$ is Lipschitz. That is, there is a constant $\widetilde{C} \in \mathbb{R}$ such that for every $(\mathbf{f}, y), (\mathbf{g}, y') \in X \wr_Z Y$,

$$d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) \leq \widetilde{C} d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')).$$

By Equation (15), it follows that $d_{\widetilde{\omega}}$ corresponds to a large-scale Lipschitz function if Y is uniformly discrete and Z has C-bounded geometry. Starting from Equation (12) and (13), one can easily show the same fact using only uniform discreteness of X.

As d_ν and d_σ both correspond to large scale Lipschitz functions, this implies that so does $d_{\widetilde{\nu}} + d_{\widetilde{\sigma}}$:

$$d_{\widetilde{\nu}}((\mathbf{f}, y), (\mathbf{g}, y')) + d_{\widetilde{\sigma}}((\mathbf{f}, y), (\mathbf{g}, y')) = d_\nu(y, y') + \sum_{z \in Z} d_\sigma(\mathbf{f}(z), \mathbf{g}(z))$$
$$\lesssim d_Y(y, y') + 1 + \sum_{z \in Z} d_X(\mathbf{f}(z), \mathbf{g}(z)) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) \lesssim d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) + 1.$$

It thus remains to show that $d_{p\widetilde{\mu}}$ corresponds to a Lipschitz function. Denote $y_0 = y, y_{n+1} = y'$ and choose $(y_1, \dots, y_n) \in \mathcal{P}_I$ such that

$$\text{path}_I(y, y') \leq \sum_{i=0}^{n} d_Y(y_i, y_{i+1}) \leq \text{path}_I(y, y') + 1.$$

Write $z_0 = p(y),\ z_{n+1} = p(y')$ and enumerate the elements of $\text{Supp}(\mathbf{f}^{-1}\mathbf{g})$ as $\{z_1, z_2, \dots, z_n\}$ where each z_i lies in a C-ball around $p(y_i)$. As p is bornologous, we have that $d_Z(z_i, z_{i+1}) \leq 2C + ad(y_i, y_{i+1}) + b$ for each i. Hence,

$$d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) \leq \sum_{i=0}^{n} d_\mu(z_i, z_{i+1}) \lesssim \sum_{i=0}^{n} d_Z(z_i, z_{i+1}) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y'))$$
$$\leq n(2C+b) + a\sum_{i=0}^{n} d_Y(y_i, y_{i+1}) + d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y')) = d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y'))(2C+b+1) + a\sum_{i=0}^{n} d_Y(y_i, y_{i+1})$$
$$\leq d_{\widetilde{\omega}}((\mathbf{f}, y), (\mathbf{g}, y'))(2C + b + 1) + a + a\ \text{path}_I(y, y') \lesssim d_{X \wr_Z Y}((\mathbf{f}, y), (\mathbf{g}, y')) + 1,$$

where we use that $d_{\widetilde{\omega}}$ corresponds to a large-scale Lipschitz function. We conclude that d_λ is associated to a large scale Lipschitz map of $X \wr_Z Y$ into an L^1-space.

As a second step, we calculate the compression of d_λ. Assume first that $d_\lambda((\mathbf{f}, y), (\mathbf{g}, y')) \leq R$ for some $R > 0$ such that Equations (7), (8), (9) and (10) are valid. Enumerate the elements of $\text{Supp}(\mathbf{f}^{-1}\mathbf{g})$, say $z_1, z_2, \dots, z_n$. Set $z_0 = p(y)$. Denote $y_0 = y$, then use the path lifting property to take y_1 such that $d_Z(p(y_1), z_1) < C$ and $d(y_0, y_1) \leq ad(z_0, z_1)^\delta + b$. Next, take y_2 such that $d_Z(p(y_2), z_2) < C$ and such that $d(y_1, y_2) \leq ad_Z(z_1, z_2)^\delta + b$ and so on. By definition, we have

$$\text{path}_I(y, y') \leq (\sum_{i=0}^{n-1} d_Y(y_i, y_{i+1})) + d_Y(y_n, y').$$

We now obtain

$$\text{path}_I(y, y') \leq \sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y_n, y') \lesssim \sum_{i=0}^{n-1} d_Y(y_i, y_{i+1}) + d_Y(y, y')$$
$$\lesssim \sum_{i=0}^{n-1} (d_Z(z_i, z_{i+1})^\delta + 1) + d_Y(y, y') \lesssim R + \sum_{i=0}^{n-1} d_Z(z_i, z_{i+1})^\delta + d_\nu(y, y')^{1/\beta}$$
$$\leq R + \sum_{i=0}^{n-1} d_Z(z_i, z_{i+1})^\delta + R^{1/\beta} \lesssim R + \sum_{i=0}^{n-1} d_\mu(z_i, z_{i+1})^{\delta/\gamma} + R^{1/\beta} \lesssim R + RR^{\delta/\gamma} + R^{1/\beta},$$

where the last inequality follows from the fact that

$$d_\mu(z_i, z_{i+1}) \leq d_{p\widetilde{\mu}}((\mathbf{f}, y), (\mathbf{g}, y')) \leq R.$$

Consequently, we obtain

$$d_{X\wr_Z Y}((\mathbf{f},y),(\mathbf{g},y')) = \text{path}_I(y,y') + \sum_{z\in Z} d_X(f(z),g(z)) \lesssim R^{\frac{\delta}{\gamma}+1} + R^{1/\beta} + \sum_{z\in Z} d_\sigma(f(z),g(z))^{1/\alpha}$$
$$\lesssim R^{\frac{\delta+\gamma}{\gamma}} + R^{1/\beta} + (\sum_{z\in Z} d_\sigma(f(z),g(z)))^{1/\alpha} \lesssim R^X,$$

where $X = \max(\frac{\delta+\gamma}{\gamma}, \frac{1}{\alpha}, \frac{1}{\beta})$. Consequently, the compression of d_λ, and hence of $X \wr_Z Y$ is bounded from below by

$$\min\left(\alpha, \beta, \frac{\gamma}{\delta+\gamma}\right).$$ □

Remark 3.4.4. At the end of Section 2 in [**80**], the author shows that the L^p-compression $\alpha_p^*(X)$ of a metric space X is always greater than $\max(\frac{1}{2}, \frac{1}{p})\alpha_1^*(X)$. Moreover, L^p embeds isometrically into L^1 for any $p \in [1,2]$. So, for $p \in [1,2]$, we deduce that the positivity of the L^p-compression is preserved under generalized wreath products with the polynomial path lifting property.

CHAPTER 4

Exactness of locally compact groups

In this chapter we show that exactness of a locally compact second countable group is equivalent to amenability at infinity. First we need revise some preliminaries.

Given a Hilbert space $\mathcal{H}$, the space of unitaries $\mathcal{U}(\mathcal{H})$ with the operator norm topology forms a topological group. However this topology is usually too strong, for example the left regular representation $\lambda\colon G \to \mathcal{U}(\mathcal{H})$, where $\lambda_g(\xi)(h) = \xi(g^{-1}h)$ is only a continuous function when when G is discrete. Fortunately the representation is continuous in the strong operator topology. That is a net $(U_\lambda)_{\lambda\in\Lambda} \subset \mathcal{U}(\mathcal{H})$ converges to U in $\mathcal{U}(\mathcal{H})$ if and only if $U_\lambda(\xi) \to U_\lambda(\xi)$ for all $\xi \in \mathcal{H}$. When we consider unitary representations of groups we will assume they are continuous with respect to the strong operator topology.

Suppose G is a locally compact second countable group with a fixed Haar measure μ and let A be a C*-algebra where $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a non-degenerate representation. Suppose $f \in C_c(G, A)$ and $U\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G. We want to make sense out of the integrals

$$(*) \qquad \int_G \pi(f(s))U_s \, d\mu(s) \in \mathcal{B}(\mathcal{H}).$$

4.1. Von Neumann algebras

There is an enormous amount of research into von Neumann algebras however in this section we shall only introduce the results we shall be using in Section 4.6. We refer the reader to [**112**, Chapter IV] and [**88**, Chapter 4] for introductory texts.

For a Banach space E, the *dual space* of E is the space of all continuous linear functionals on E. We denote this space by E^*. It is a well known fact that E can isometrically embed into E^{**}, the double dual of E.

Definition 4.1.1. Let $\mathcal{H}$ be a Hilbert space and let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. If A is strongly closed then we call A a *von Neumann* algebra.

Since the strong topology is weaker than the norm topology it follows that every von Neumann algebra is a C*-algebra. One example of a von Neumann algebra is the space of bounded linear operators on a Hilbert space. The intersection of a family of von Neumann algebras is a von Neumann algebra [**88**, pp 117]. Thus for any set $C \subset \mathcal{B}(\mathcal{H})$ there is the smallest von Neumann algebra that contains C. We call this the von Neumann algebra *generated* by C.

***Theorem* 4.1.2.** [**88**, Theorem 4.2.9.] *Let A be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then there exists a Banach space A_* such that $(A_*)^*$ is linearly isometrically isomorphic to A.*

We shall call A_* the *pre-dual* of A. One useful construction we shall be using is the tensor product of von Neumann algebras. Given two vector spaces X and Y we denote their algebraic tensor product by $X \odot Y$.

Definition 4.1.3. Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. The *tensor product of* $\mathcal{H}$ *and* $\mathcal{K}$ is the completion of $\mathcal{H} \odot \mathcal{K}$ with respect to the inner product

$$\langle \sum_i h_i \otimes k_i, \sum_j h'_j \otimes k'_j \rangle = \sum_{i,j} \langle h_i, h'_j \rangle \langle k_i, k'_j \rangle.$$

We denote this completion by $\mathcal{H} \otimes \mathcal{K}$.

***Proposition* 4.1.4.** [**23**, Proposition 3.2.3.] *If $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{K})$ then there exists a unique linear operator $S \otimes T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that*

$$S \otimes T(v \otimes w) = Sv \otimes Tw$$

for all $v \in \mathcal{H}$, $w \in \mathcal{K}$. Moreover $\|S \otimes T\| = \|S\|\|T\|$.

As a consequence there is a natural injective $*$-homomorphism $\mathcal{B}(\mathcal{H}) \odot \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Indeed we can identify $\mathcal{B}(\mathcal{H})$ with $\mathcal{B}(\mathcal{H}) \odot \mathbb{C}1 \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Definition 4.1.5. Let A and B be von Neumann algebras on $\mathcal{H}$ and $\mathcal{K}$ respectively. The von Neumann algebra on $\mathcal{H} \otimes \mathcal{K}$ generated by $a \otimes b$, $a \in A$, $b \in B$ is called the *von Neumann tensor product* of A and B. This is denoted by $A \bar{\otimes} B$.

***Proposition* 4.1.6** ([**112**, Chapter 4, Proposition 1.6.])**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{B}(\mathcal{K}) = \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$*

4.2. Bochner integral

Let G be a locally compact group and E a Banach space. Let $C_c(G,)$ be the space of continuous, compactly supported functions on G with values in E. We now introduce the inductive limit topology on $C_c(G, E)$. For each compact subset $K \subset G$, let $C_K(G, E)$ denote the space of continuous functions with support contained in K. Indeed $C_c(G, E) = \bigcup \{C_K(G, E) : K \subset G \text{ is compact}\}$

***Proposition* 4.2.1** ([**126**, Proposition D.7.])**.** *There exists a topology on $C_c(G, E)$ such that*

(1) *$C_c(G, E)$ is a locally convex topological vector space.*
(2) *For all compact subsets $K \subset G$, the inclusions $\iota_K : C_K(G, E) \hookrightarrow C_c(G, E)$ are homeomorphisms onto their images, when $C_K(G, E)$ is equipped with the supremum norm.*
(3) *For any locally convex topological vector space M and any linear map $F : C_c(G, E) \to M$, F is continuous if and only if $F \circ \iota_K : C_K(G, E) \to M$ is continuous for all compact subsets $K \subset G$.*

Proof (Sketch). For each $K \subset G$, let $\mathcal{T}_K(0)$ be a neighbourhood basis of the identity of $C_K(G, E)$. We define a neighbourhood basis of the identity $\mathcal{T}(0)$ of $C_c(G, E)$ where

$$\mathcal{T}(0) = \{X \subset C_c(G, E) : X \text{ is convex and } X \cap C_K(G, E) \in \mathcal{T}_K(0) \text{ for all compact } K \subset G\}.$$

□

We call this topology the *inductive limit topology.* We say that a net $(f_\lambda)_{\lambda \in \Lambda}$ is *eventually compactly supported* if there exists a compact set K_0 and an index $\lambda_0 \in \Lambda$ such that $\mathrm{Supp}(f_\lambda) \subset K_0$ for all $\lambda \geq \lambda_0$. This means that a net $(f_\lambda)_{\lambda \in \Lambda}$ of eventually compactly supported functions converges to f in the inductive limit topology if and only if the net converges to f uniformly. This is because the inclusion functions are homeomorphisms onto their image and there exists a compact $K \subset G$ such that for all $\lambda \geq \lambda_0$, $f_\lambda \in C_K(G, E)$.

***Proposition* 4.2.2.** *Let M be a locally convex topological vector space and let $F\colon C_c(G) \to M$ be a linear map. Then F is continuous if and only if F maps eventually compactly supported convergent nets to convergent nets in M.*

Proof (Sketch). Suppose $f_\lambda \to f$ in $C_K(G,E)$. Indeed $\iota_K(f_\lambda) \to \iota_K(f)$ in the inductive limit topology and is eventually compactly supported. Hence $F(\iota_K(f_\lambda)) \to F(\iota_K(f))$ and so $F \circ \iota_K$ is continuous for all compact subsets $K \subset G$. By the previous proposition this is enough. □

There is a natural inclusion of $C_c(G) \odot E$ into $C_c(G,E)$ where $z \otimes a$ maps to the function $s \mapsto z(s)a$.

***Lemma* 4.2.3** ([**126**, Lemma 1.87.])**.** *Suppose E_0 is a dense subset of a Banach space E. Then the span of elements $z \otimes a \in C_c(G) \odot E$ where $z \in C_c(G)$ and $a \in E_0$ is dense in $C_c(G,E)$ with the inductive limit topology.*

Proof (Sketch). In fact we show a stronger result: for each $f \in C_c(G,E)$ there exists a compact $K \subset G$ such that for all small enough $\varepsilon > 0$ there exist $g \in C_K(G) \odot E_0$ such that $\|f - g\|_\infty < \varepsilon$.

Let W be a fixed compact symmetric neighbourhood of the identity. If $\varepsilon > 0$ then choose a symmetric open neighbourhood of the identity $V \subset W$ such that $\|f(s) - f(r)\| < \varepsilon$ whenever $s^{-1}r \in V$. Choose $s_1, \ldots, s_n \in \operatorname{Supp}(f)$ such that $\operatorname{Supp}(f) \subset \cup_{i=1}^n s_i V$. This means that $\operatorname{Supp}(f)^c \cup s_1 V \cup \cdots \cup s_n V$ is an open cover of G and so choose a partition of unity $\{z_i\}_{i=0}^n$ such that $\operatorname{Supp}(z_0) \subset \operatorname{Supp}(f)^c$ and $\operatorname{Supp}(z_i) \subset s_i V$ for all $i = 1, \ldots, n$. For each s_i, choose $x_i \in E_0$ such that $\|f(s_i) - x_i\| < \varepsilon/2$. Now set $g = \sum_{i=1}^n z_i \otimes x_i$. It follows that g is compactly supported as $\operatorname{Supp}(g) \subset KW$, and for all $r \in G$,

$$\|f(r) - g(r)\| \le \sum_{i=1}^n z_i(r)\|f(r) - x_i\| \le \sum_{i=1}^n z_i(r)\|f(r) - f(s_i)\| + \sum_{i=1}^n z_i(r)\|f(s_i) - x_i\| \le \varepsilon. \ \square$$

Let $\mathcal{H}$ be a Hilbert space and set $L^2(G,\mathcal{H}) = \left\{f\colon G \to \mathcal{H} \mid \int_G \|f(s)\|_{\mathcal{H}}^2 < \infty\right\}$. For $f \in L^2(G,\mathcal{H})$ we will usually use the notation f_g for $f(g) \in \mathcal{H}$. This is a Hilbert space under the inner product $\langle f, g\rangle = \int_G \langle f(h), g(h)\rangle_{\mathcal{H}}\, d\mu(h)$ for all $f, g \in L^2(G,\mathcal{H})$. We can also form $L^2(G) \otimes \mathcal{H}$, which is a Hilbert space under the inner product $\langle \xi \otimes v, \eta \otimes v'\rangle = \langle \xi, \eta\rangle\langle v, v'\rangle$ for all $\xi, \eta \in L^2(G)$ and $v, v' \in \mathcal{H}$. Observe that $\|z \otimes a\|$ is equal to the norm of the function $s \mapsto z(s)a$.

***Corollary* 4.2.4.** *The inclusion of $C_c(G) \odot \mathcal{H}$ into $C_c(G,\mathcal{H})$ extends to an isometric linear isomorphism from $L^2(G) \otimes \mathcal{H}$ to $L^2(G,\mathcal{H})$.*

Proof. It is clear that $C_c(G) \odot \mathcal{H}$ is dense inside $L^2(G) \otimes \mathcal{H}$ and $C_c(G,\mathcal{H})$ is dense inside $L^2(G,\mathcal{H})$. It suffices to show that $C_c(G) \odot \mathcal{H}$ is dense inside $C_c(G,\mathcal{H})$ with respect to the inner product on $L^2(G,\mathcal{H})$. If $f \in C_c(G,\mathcal{H})$ then by the previous proposition there exists an eventually compactly supported net $(f_\lambda)_{\lambda\in\Lambda}$ such that $f_\lambda \to f$ uniformly. Hence there exists $K \subset G$ and $\lambda_0 \in \Lambda$ such that $\operatorname{Supp}(f_\lambda) \cup \operatorname{Supp}(f) \subset K$ for all $\lambda \ge \lambda_0$. If $\varepsilon > 0$ then choose $\lambda \ge \lambda_0$ large enough so that $\|f_\lambda - f\|_\infty < \varepsilon$. Hence $\int_G \|f_\lambda(s) - f(s)\|_{\mathcal{H}}^2\, d\mu(s) \le \mu(K)\varepsilon^2$. Therefore $C_c(G) \odot \mathcal{H}$ is dense inside $C_c(G,\mathcal{H})$ and so the operator extends to an isometric linear isomorphism. □

For $f \in C_c(G,E)$, the function $s \mapsto \|f(s)\|_E$ belongs to $C_c(G)$. Define the L^1-norm as

$$\|f\|_1 := \int_G \|f(s)\|_E\, d\mu(s).$$

***Proposition* 4.2.5** ([**126**, Lemma 1.91.]). *Suppose E is a Banach space and G is a locally compact group with left Haar measure μ. Then there is a unique linear map $I_E\colon C_c(G,E)\to E$ such that*

(1) *$I(z\otimes a)=a\int_G z(s)\,d\mu(s)$.*
(2) *$\|I(f)\|_E\le\|f\|_1$.*
(3) *$\varphi(I(f))=\int_G\varphi(f(s))\,d\mu(s)$ for all $\varphi\in E^*$.*
(4) *If $L\colon E\to F$ is a bounded linear operator then $L(I_E(f))=I_F(L(f))$, where $L(f)$ is the function $s\mapsto L(f(s))$.*

For $f\in C_c(G,E)$ we shall write $I(f)$ as $\int_G f(s)\,d\mu(s)$.

***Proposition* 4.2.6** ([**126**, Proposition 1.105.]). *Let E be a Banach space and G a locally compact group with a left Haar measure μ. Suppose that $F\in C_c(G\times G,E)$. Then the maps $s\mapsto\int_G F(s,r)\,d\mu(r)$ and $r\mapsto\int_G F(s,r)\,d\mu(s)$ belong to $C_c(G,E)$ and the iterated integrals $\int_G\int_G F(s,r)\,d\mu(r)d\mu(s)$, $\int_G\int_G F(s,r)\,d\mu(s)d\mu(r)$ are well defined and have common value.*

Definition 4.2.7. A net T_i converges strictly to T if and only if $\|T_iK-TK\|$ and $\|KT_i-KT\|$ tend to 0 for all compact operators $K\in\mathcal{K}(\mathcal{H})$. We denote $\mathcal{B}_s(\mathcal{H})$ to be the space of bounded linear operators with the strict topology.

Definition 4.2.8. A net T_i converges $*$-strongly to T if and only if T_i and T_i^* converge strongly to T and T^* respectively.

***Proposition* 4.2.9.** [**97**, Proposition C.7.] *On $\mathcal{B}(\mathcal{H})$, strict convergence implies $*$-strong convergence. On norm bounded subsets of $\mathcal{B}(\mathcal{H})$ the strict and $*$-strong topologies coincide.*

***Proposition* 4.2.10.** [**97**, Corollary C.8.] *Suppose $U\colon G\to\mathcal{U}(\mathcal{H})$ is a unitary representation of G. Then U is strictly continuous.*

Suppose $\pi\colon A\to\mathcal{B}(\mathcal{H})$ is a non-degenerate representation of a C*-algebra A and let $U\colon G\to\mathcal{B}(\mathcal{H})$ be a unitary representation of a locally compact group G. For any function $f\in C_c(G,A)$, the function $G\to\mathcal{B}(\mathcal{H})$, $s\mapsto\pi(f(s))U_s$ is not necessarily continuous under the operator norm and so we can not apply Proposition 4.2.5 directly. However this does not stop us defining the integral.

***Proposition* 4.2.11** ([**126**, Lemma 1.101]). *There is a unique linear map $I\colon C_c(G,\mathcal{B}_s(\mathcal{H}))\to\mathcal{B}(\mathcal{H})$ such that*

$$\langle I(f)v,w\rangle=\int_G\langle f(s)v,w\rangle\,d\mu(s)\quad\forall f\in C_c(G,\mathcal{B}_s(\mathcal{H}))\text{ and }\forall v,w\in\mathcal{H}.$$

We write $I(f)$ as $\int_G f(s)\,d\mu(s)$. If $L\colon\mathcal{H}\to\mathcal{K}$ is a linear map then $L(\int_G f(s)\,d\mu(s))=\int_G L(f(s))\,d\mu(s)$.

This uniquely defines the operator in $(*)$.

4.3. Cross products

Definition 4.3.1. Let (A,G,α) be a triple consisting of C*-algebra A, a locally compact group G, and a group homomorphism $\alpha\colon G\to\operatorname{Aut}(A)$, $g\mapsto\alpha_g$. The triple (A,G,α) is called a *C*-dynamical system* if for every $a\in A$ the map $G\to A$, $g\mapsto\alpha_g(a)$ is continuous.

Given a C*-dynamical system (A, G, α) we can construct a Banach $*$-algebra encoding this action. Define a convolution product and adjoint on $C_c(G, A)$ by

$$f *_\alpha g(s) := \int_G f(r)\alpha_r(g(s))\, d\mu(r), \quad f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*).$$

One can show that $C_c(G, A)$ becomes a $*$-algebra under convolution, involution and pointwise addition [**126**, Section 2.3.]. We denote the completion of $C_c(G, A)$ with respect to the L^1-norm, $\|\cdot\|_1$, by $L^1(G, A)$. This is a Banach $*$-algebra.

Definition 4.3.2. Let (A, G, α) be a C*-dynamical system. A *covariant representation* of (A, G, α) is a pair (π, U) where $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism and $U\colon G \to \mathcal{B}(\mathcal{H})$ is a unitary representation of G such that $U_g\pi(a)U_g^* = \pi(\alpha_g(a))$ for all $g \in G$ and $a \in A$.

Definition 4.3.3. Let A be a $*$-algebra. We say that a $*$-homomorphism $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is *non-degenerate* if the set

$$\{\pi(a)h : a \in A \text{ and } h \in \mathcal{H}\}$$

spans a dense subset of $\mathcal{H}$.

***Proposition* 4.3.4** ([**126**, Proposition 2.23.])**.** *Suppose that (π, U) is a covariant representation of a C*-dynamical system (A, G, α) on $\mathcal{H}$. Then*

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s\, d\mu(s)$$

defines a $*$-homomorphism of $C_c(G, A)$ to $\mathcal{B}(\mathcal{H})$. Furthermore $\|\pi \rtimes U(f)\|_{\mathcal{B}(\mathcal{H})} \leq \|f\|_1$ and $\pi \rtimes U$ is non-degenerate if π is non-degenerate. Conversely every non-degenerate $*$-homomorphism of $C_c(G, A)$ arises from some covariant representation.

We can complete $C_c(G, A)$ with respect to the norm $\|f\| := \|\pi \rtimes U(f)\|_{\mathcal{B}(L^2(G))}$ and we denote the completion by $A \rtimes_{\pi,U,\alpha} G$. On $C_c(G, A)$ we can define a norm

$$\|f\|_{\max} := \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\} \quad \forall f \in C_c(G, A).$$

This is called the *maximal norm* and the completion of $C_c(G, A)$ with respect to this norm is called *maximal cross product*, which is denoted by $A \rtimes_\alpha G$. For every C*-dynamical system (A, G, α) there exists a naturally associated cross product. Let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a faithful representation of A. Define $\tilde{\pi}\colon A \to \mathcal{B}(L^2(G, \mathcal{H}))$ by

$$(\tilde{\pi}(a)f)_s := \pi(\alpha_{s^{-1}}(a))(f_s)$$

for all $f \in L^2(G, \mathcal{H})$, $s \in G$ and $a \in A$. Define a unitary representation $\Lambda\colon G \to \mathcal{U}(L^2(G, \mathcal{H}))$, $(\Lambda_t f)_s = f_{t^{-1}s}$ for all $s, t \in G$. It is routine to show that $(\tilde{\pi}, \Lambda)$ is a covariant representation of (A, G, α).

***Lemma* 4.3.5** ([**126**, Lemma 2.26.])**.** *Let (A, G, α) be a C*-dynamical system. If $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a faithful representation then the representation $\tilde{\pi} \rtimes \Lambda\colon C_c(G, A) \to \mathcal{B}(L^2(G, \mathcal{H}))$ is also a faithful.*

***Proposition* 4.3.6** ([**126**, Lemma 7.8.])**.** *Suppose π_1 and π_2 are two faithful representations of A. Then $A \rtimes_{\tilde{\pi}_1,\Lambda,\alpha} G$ and $A \rtimes_{\tilde{\pi}_2,\Lambda,\alpha} G$ are $*$-isomorphic.*

The consequence of this is that the completion only depends on the C*-dynamical system. We call this completion the *reduced cross product* of (A, G, α) and denote the C*-algebra by $A \rtimes_{\alpha,r} G$. If (A, α, G) and (B, β, G) are C*-dynamical systems and $\theta\colon A \to B$ is an equivariant

$*$-homomorphism, then there is a canonical $*$-homomorphism $\theta_r\colon A \rtimes_{\alpha,r} G \to B \rtimes_{\beta,r} G$ such that for all $f \in C_c(G, A)$, $\theta_r(f)(g) = \theta(f(g))$ for all $g \in G$.

Let (A, G, α) be a C*-dynamical system and $J \subset A$ a closed two sided ideal of A such that for all $\alpha_g(J) \subset J$ for all $g \in G$. In this situation α restricts to an action on J and α is well defined on the quotient A/J which we denote by $\dot\alpha$. Hence $(J, G, \alpha|J)$ and $(A/J, G, \dot\alpha)$ are also C*-dynamical systems.

Definition 4.3.7 ([**75**])**.** A locally compact group G is *exact* if for every C*-dynamical system (A, G, α) and every α-invariant closed two sided ideal J of A the sequence

$$0 \to J \rtimes_{\alpha|J,r} G \to A \rtimes_{\alpha,r} G \to (A/J) \rtimes_{\dot\alpha,r} G \to 0$$

is exact.

Let (A, α, G) be a C*-dynamical system and suppose $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is a faithful representation. For $\varphi \in \mathcal{B}(L^2(G))_*$ we can consider φ as a linear functional on $\mathcal{B}(L^2(G))$ because φ also belongs to the double dual $(\mathcal{B}(L^2(G))_*)^{**} = \mathcal{B}(L^2(G))^*$. In particular $\mathrm{im}(\varphi) \subset \mathbb{C}$. We denote by S^A_φ the restriction to $A \rtimes_{\alpha,r} G$ of the slice map $\varphi\bar\otimes\mathrm{id}_{\mathcal{B}(\mathcal{H})}\colon \mathcal{B}(L^2(G))\bar\otimes\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. The map $\varphi\bar\otimes\mathrm{id}_{\mathcal{B}(\mathcal{H})}$ is continuous when $\mathcal{B}(\mathcal{H})$ has the operator norm topology and $\mathcal{B}(L^2(G))\bar\otimes\mathcal{B}(\mathcal{H})$ has the strong operator topology, indeed $\|\varphi\bar\otimes\mathrm{id}\| = \|\varphi\|$ [**116**, Proposition 12.4.4.]. We shall refer to S^A_φ as the *slice map* corresponding to φ.

***Lemma* 4.3.8** ([**75**, Lemma 2.1.])**.** (1) *If $x \in A \rtimes_{\alpha,r} G$ then $S^A_\varphi(x) \in A$.*

(2) *Let (A, α, G) and (B, β, G) be C*-dynamical systems and let $\theta\colon A \to B$ be a G-equivariant $*$-homomorphism. Then the following diagram commutes*

$$\begin{array}{ccc} A \rtimes_{\alpha,r} G & \xrightarrow{\theta_r} & B \rtimes_{\beta,r} G \\ \downarrow S^A_\varphi & & \downarrow S^B_\varphi \\ A & \xrightarrow[\theta]{} & B. \end{array}$$

for all $\varphi \in \mathcal{B}(L^2(G))_*$.

Proof (Sketch). Fix $\xi, \eta \in C_c(G)$ and define $\varphi \in \mathcal{B}(L^2(G))_*$ by $\varphi(S) = \langle S\xi, \eta\rangle$ for all $S \in \mathcal{B}(L^2(G))$. For $T \in \mathcal{B}(L^2(G))\bar\otimes\mathcal{B}(\mathcal{H})$, $\varphi(T)\bar\otimes\mathrm{id} \in \mathcal{B}(\mathcal{H})$ is defined to be the unique operator that satisfies $\langle \varphi(T)\bar\otimes\mathrm{id}\, v, v'\rangle = \sigma_T(v, v')$, where $\sigma_T(v, v') = \langle T(\xi \otimes v), \eta \otimes v'\rangle$ for all $v, v' \in \mathcal{H}$ [**116**, Proposition 12.4.4.].

For $f \in C_c(G, A)$, write T_f to be the operator $(\tilde\pi \rtimes \Lambda)(f) \in \mathcal{B}(L^2(G))\bar\otimes\mathcal{B}(\mathcal{H})$. Hence for $h \in G$, $T_f(\xi \otimes v)(h) = \int_G \xi(g^{-1}h)\pi(\alpha_{h^{-1}}(f_g))(v)\, d\mu(g)$. Therefore

$$S^A_\varphi(T_f) = \int_G \int_G \xi(g^{-1}h)\overline{\eta(h)}\alpha_{h^{-1}}(f_g)\, d\mu(g)d\mu(h) \quad \forall f \in C_c(G, A). \tag{18}$$

For short hand write $F(g, h) = \xi(g^{-1}h)\overline{\eta(h)}\alpha_{h^{-1}}(f_g)$. By Proposition 4.2.6, the iterated integral $\int_G \int_G F(g, h)\, d\mu(g)d\mu(h)$ belongs to A. As $C_c(G, A)$ is norm dense in $A \rtimes_{\alpha,r} G$ and S^A_φ is continuous then $S^A_\varphi(T) \in A$ for any $T \in A \rtimes_{\alpha,r} G$.

The span of the linear functionals of the form $T \mapsto \langle T\xi, \eta\rangle$ for some $\xi, \eta \in C_c(G)$ is dense in $\mathcal{B}(L^2(G))_*$ [**112**, Theorem 1.6.]. So for $\varphi \in \mathcal{B}(L^2(G))_*$ and $\varepsilon > 0$, there exists a sequence $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in C_c(G)$ such that $\|\varphi(T) - \sum_{i=1}^n \langle T\xi_n, \eta_n\rangle\| < \varepsilon$ for all $T \in \mathcal{B}(L^2(G))$ with $\|T\| \leq 1$. For short hand write the linear functional $T \mapsto \langle T\xi, \eta\rangle$ as $\omega_{\xi,\eta}$. Hence for any $T \in A \rtimes_{\alpha,r} G$, $\|S^A_\varphi(T) - \sum_{i=1}^n S^\varphi_{\omega_{\xi_i,\eta_i}}(T)\| \leq \|\varphi - \sum_{i=1}^n \omega_{\xi_i,\eta_i}\|\|T\| \leq \varepsilon\|T\|$. Thus $S^A_\varphi(T) \in A$ for all $\varphi \in \mathcal{B}(L^2(G))$ and $T \in A \rtimes_{\alpha,r} G$.

(2) follows because the diagram commutes on the dense subsets $C_c(G, A)$ and $C_c(G, B)$ and the slice maps S^A_φ, S^B_φ are continuous. □

***Proposition* 4.3.9** ([**75**, Proposition 2.2.]). *Let (A, α, G) be a C^*-dynamical system and let J be a two-sided ideal in A such that $\alpha(G)J \subset J$. Denote the surjection of A onto A/J by θ and the canonical $*$-homomorphism $\theta_r \colon A \rtimes_{\alpha,r} G \to (A/J) \rtimes_{\dot{\alpha},r} G$. Then for $T \in A \rtimes_{\alpha,r} G$ the following are equivalent:*

(1) $T \in \ker(\theta_r)$.
(2) $S^A_\varphi(T) \in J$ *for all* $\varphi \in \mathcal{B}(L^2(G))_*$.
(3) $S^A_{\omega_{\xi,\eta}}(T) \in J$ *for all* $\xi, \eta \in C_c(G)$.

4.4. Left and right uniformly continuous functions on G

Let $C_b(G)$ be the space of bounded continuous complex valued functions on G equipped with the supremum norm $\|\cdot\|_\infty$. Let π be the faithful representation $\pi \colon C_b(G) \to \mathcal{B}(L^2(G))$ where $(\pi(f)\xi)(x) = f(x)\xi(x)$ for $f \in C_b(G)$, $\xi \in L^2(G)$ and $x \in G$. Let L and R be homomorphisms from G to $\mathrm{Aut}(C_b(G))$ such that $(L_g f)(x) = f(g^{-1}x)$ and $R_g f(x) = f(xg)$ for all $f \in C_b(G)$ and $x, g \in G$. A bounded continuous function $f \colon G \to \mathbb{C}$ is *left uniformly continuous* if $\|L_g f - f\|_\infty \to 0$ as $g \to 1$ and *right uniformly continuous* if $\|R_g f - f\|_\infty \to 0$ as $g \to 1$. We denote the space of bounded left (right) uniformly continuous functions by $C_{lu}(G)$ (respectively $C_{ru}(G)$). We denote the restriction of π to $C_{lu}(G)$ ($C_{ru}(G)$) by π^L (respectively π^R).

We have the left and right regular unitary representations $\lambda, \rho \colon G \to \mathcal{U}(L^2(G))$ where for $\xi \in L^2(G)$ and $g, x \in G$, $(\lambda_g \xi)(x) = \xi(g^{-1}x)$ and $(\rho_g \xi)(x) = \xi(xg)\Delta(g)^{1/2}$. It is not hard to show that (π^L, λ) and (π^R, ρ) are covariant representations of the C*-dynamical systems $(C_{lu}(G), G, L)$ and $(C_{ru}(G), G, R)$ respectively. The aim of this section is to show that the following $*$-isomorphisms hold

$$C_{ru}(G) \rtimes_{\pi^R,\rho,R} G \cong C_{lu}(G) \rtimes_{\pi^L,\lambda,L} G \cong C_{lu}(G) \rtimes_{L,r} G$$

***Proposition* 4.4.1.** *There exists a unitary $U_3 \in \mathcal{B}(L^2(G))$ such that*

$$U_3(C_{ru}(G) \rtimes_{\pi^R,\rho,R} G)U_3^* = C_{lu}(G) \rtimes_{\pi^L,\lambda,L} G$$

Proof. Define a G-equivariant $*$-isomorphism $\psi \colon C_{ru}(G) \to C_{lu}(G)$ such that $\psi(f)(x) = f(x^{-1})$ for all $f \in C_{ru}(G)$ and $x \in G$. Define a unitary $U_3 \colon L^2(G) \to L^2(G)$ such that $(U_3\xi)(x) = \xi(x^{-1})\Delta(x^{-1})^{1/2}$ for all $\xi \in L^2(G)$ and $x \in G$. Then $U_3\rho_g = \lambda_g U_3$ and $U_3\pi^R(f) = \pi^L(\psi(f))U_3$ for all $g \in G$ and $f \in C_{ru}(G)$. Hence this extends to a $*$-isomorphism between $C_{ru}(G) \rtimes_{\pi^R,\rho,R} G$ and $C_{lu}(G) \rtimes_{\pi^L,\lambda,L} G$ □

Let $1 \otimes \pi^L \colon C_{lu}(G) \to \mathcal{B}(L^2(G, L^2(G)))$ where $[(1 \otimes \pi^L)(f)(\xi)]_g = f(g)\xi_g$ and $\Lambda \colon G \to \mathcal{B}(L^2(G, L^2(G)))$ where $[(\Lambda_h\xi)]_g = \xi_{h^{-1}g}$ for all $f \in C_{lu}(G)$, $\xi \in L^2(G, L^2(G))$ and $g, h \in G$. It is clear that $(1 \otimes \pi^L, \Lambda)$ is a covariant representation of the C*-dynamical system $(C_{lu}(G), G, L)$. Furthermore it is also clear that there exists a unitary $U_2 \colon L^2(G) \to L^2(G, L^2(G))$ such that

$$U_2(C_{lu}(G) \rtimes_{\pi^L,\lambda,L} G)U_2^* = C_{lu}(G) \rtimes_{1\otimes\pi^L,\Lambda,L} G.$$

Indeed fix $\eta_0 \in L^2(G)$ such that $\|\eta_0\|_2 = 1$ and for any $\xi \in L^2(G)$ define $U_2 \colon L^2(G) \to L^2(G, L^2(G))$, where $(U_2\xi)_g(x) = \xi(g)\eta_0(x)$. Thus $\|U_2\xi\|_2 = \|\xi\|_2$, $U_2\pi^L(f) = 1 \otimes \pi^L(f)U_2$ and $U_2\lambda_g = \Lambda_g U_2$ for all $\xi \in L^2(G)$, $f \in C_{lu}(G)$ and $g \in G$.

***Lemma* 4.4.2.** *Let $U_1^*\colon L^2(G, L^2(G)) \to L^2(G, L^2(G))$ be a linear map such that $(U_1^*\xi)_g(x) = \xi_{gx^{-1}}(x)\Delta(x^{-1})^{1/2}$. Then U_1^* is a unitary map and $U_1^*(C_{lu}(G) \rtimes_{L,r} G)U_1 = (C_{lu}(G) \rtimes_{1\otimes\pi^L,\Lambda,L} G)$, where U_1 is the adjoint of U_1^*.*

Proof. This is an adaptation of the map in [**23**, Proposition 5.1.3.]. By changing variables $x \mapsto x^{-1}g$, and the property of the modular function we have

$$\int_G |\xi_{gx^{-1}}(x)|^2\Delta(x^{-1})\, d\mu(x) = \int_G |\xi_x(x^{-1}g)|^2\, d\mu(x).$$

Thus by Fubini's theorem and left invariance of the Haar integral $\|U_1^*\xi\|_2^2 = \|\xi\|_2^2$.

It follows that $U_1^*\Lambda_g = \Lambda_g U_1^*$ and $U_1^*\tilde{\pi}^L(f) = 1 \otimes \pi^L(f)U_1^*$ for all $g \in G$ and $f \in C_{lu}(G)$, where $\tilde{\pi}^L$ is defined after Proposition 4.3.4. □

We denote the composition $U_1 \circ U_2 \circ U_3\colon L^2(G) \to L^2(G, L^2(G))$ and the adjoint $U_3^* \circ U_2^* \circ U_1^*\colon L^2(G, L^2(G)) \to L^2(G)$ by $U_{1,2,3}$ and $U_{1,2,3}^*$ respectively. Putting this all together we obtain the following isomorphisms.

***Corollary* 4.4.3.**

$$U_{1,2,3}(C_{ru}(G) \rtimes_{\pi^R,\rho,R} G)U_{1,2,3}^* = C_{lu}(G) \rtimes_{L,r} G$$

4.5. Amenable actions

We will assume that all topological spaces in this section are Hausdorff. A group G acts on a locally compact space X if there exists a homomorphism $\alpha\colon G \to \mathrm{Homeo}(X)$ such that the map $X \times G \to X$, $(x, s) \mapsto \alpha_s(x)$ is continuous.

Definition 4.5.1. A locally compact group G admits a *topologically amenable* action on a locally compact space X if there exists a net $(m_i)_{i\in I}$ of weak-$*$ continuous maps $m_i\colon X \to \mathrm{Prob}(G)$ such that for all $\varepsilon > 0$ and compact sets $K \times K' \subset X \times G$ there exists $J \in I$ such that for all $i \geq J$,

$$\sup_{(x,s)\in K\times K'} \|sm_i^x - m_i^{sx}\|_1 < \varepsilon,$$

where $\mathrm{Prob}(G)$ is the space of probability measures on G and $sm_i^x(g) = m_i^x(s^{-1}g)$ for all $s, g \in G$ and $x \in X$.

We say that a locally compact group is *amenable at infinity* if it admits an amenable action on a compact space. The spectrum of a C*-algebra is the set of unitary equivalence classes of irreducible representations of the C* algebra endowed with the hull kernel topology [**88**, Section 5.4.]. The spectrum of $C_{lu}(G)$ is compact because $C_{lu}(G)$ is unital [**88**, Theorem 5.4.8]. We shall denote the spectrum by $\beta^u G$ and when G is discrete then $\beta^u G$ is the Stone–Čech compactification of G.

***Proposition* 4.5.2** ([**5**, Proposition 3.4.])**.** *Let G be a locally compact group. Then G is amenable at infinity if and only if the action of G on $\beta^u G$ is amenable.*

Examples 4.5.3.
1. An amenable group acts amenably on a point.
2. Every discrete hyperbolic group acts amenably on its Gromov boundary [**1**], [**23**, Theorem 5.3.15.].
3. Every almost connected group is amenable at infinity [**5**, Proposition 3.3.].

Recall the definition of property A for general metric space.

Definition 4.5.4 ([**102**, Definition 2.1.])**.** Let X be a bounded geometry proper metric space. We say that X has *property A* if there exists a sequence of weak-$*$ continuous maps $f_n \colon X \to \mathrm{Prob}(X)$ such that

- for all n there exists an $R > 0$ such that $\mathrm{Supp}(f_n(x)) \subset B(x, R)$,
- for each $S > 0$, as $n \to \infty$,

$$\sup_{d(x,y)<S} \|f_n(x) - f_n(y)\|_1 \to 0.$$

***Theorem* 4.5.5** ([**5**, Proposition 3.5.] [**44**, Theorem 2.3.])**.** *A locally compact second countable group has property A if and only if it acts amenably on a compact Hausdorff space.*

***Theorem* 4.5.6** ([**5**, Theorem 7.2.])**.** *If a locally compact second countable group is amenable at infinity then it is exact.*

4.6. Exactness of locally compact second countable groups

In this section we show that exactness of a locally compact second countable group is equivalent to admitting an amenable action on a compact Hausdorff space. The material here is joint work with Jacek Brodzki and Kang Li.

4.6.1. Ghost operators. We introduce an important ideal inside the uniform Roe algebra.

Definition 4.6.1. Let Z be a discrete metric space with bounded geometry. An operator $T \in C^*_u(Z)$ is called a *ghost* if $T_{x,y} \to 0$ as $x \to \infty$ and $y \to \infty$. The space of all ghost operators in $C^*_u(Z)$ forms a closed two sided ideal and is called the *ghost ideal*.

Clearly every finite rank operator is a ghost operator so every compact operator is a ghost. So when do these ideals coincide?

***Theorem* 4.6.2** ([**103**, Theorem 1.3.])**.** *Let Z be a discrete metric space with bounded geometry. Then Z has property A if and only if all ghost operators are compact.*

If Z has property A then every ghost operator is compact is an easy result [**100**, Proposition 11.43.]. We can show this another way. Let I be a two-side ideal in $C^*_u(Z)$ and set $\mathrm{Fin}(I)$ to be the intersection of the ideal I with the space of all finite propagation operators. If G is a ghost then $\mathrm{Fin}(\langle G\rangle) \subset \mathcal{K}(\ell^2(Z))$ [**31**, Theorem 3.1.]. It follows that G is compact if Z has property A because [**31**, Theorem 4.4.] states that if Z has property A then $\mathrm{Fin}(I)$ is dense inside I. Hence every ghost is the limit of compact operators and so is compact.

The other direction uses another characterisation of property A called operator norm localisation [**30**] [**105**, Theorem 4.1.] to build an operator that asymptotically has a spectral gap. This is enough to provide a non-compact ghost operator that lives in the uniform Roe algebra.

4.6.2. Plan of proof. We plan to show that if a locally compact second countable group G is not amenable at infinity then the following sequence

$$0 \longrightarrow C_0(G) \rtimes_{L,r} G \xrightarrow{\iota} C_{lu}(G) \rtimes_{L,r} G \xrightarrow{\theta_r} (C_{lu}(G)/C_0(G)) \rtimes_{\dot{L},r} G \longrightarrow 0$$

that is not exact. The maps ι and θ_r are injective and surjective respectively so we will show the sequence is not exact in the middle, i.e $\mathrm{im}\,\iota \subsetneq \ker \theta_r$.

As G is not amenable at infinity, it does not have property A by Theorem 4.5.5. This means that some coarse lattice Z does not have property A by Theorem 1.2.6. By Theorem 4.6.2 there exists a non-compact ghost operator $T \in C^*_u(Z)$. We will isometrically lift T to

$C_{ru}(G) \rtimes_{\pi^R,R,\rho} G$ and map this to $C_{lu}(G) \rtimes_{L,r} G$ via the unitaries in Corollary 4.4.3. By using the slice map in Proposition 4.3.9 we shall show that this operator belongs to $\ker\theta_r$. The algebra $C_0(G) \rtimes_{L,r} G$ is $*$-isomorphic to the compact operators on $L^2(G, L^2(G))$ and since T is non-compact, it follows that the image of the lift in $C_{lu}(G) \rtimes_{L,r} G$ is also non-compact. Hence this operator belongs to $\ker\theta_r \setminus \operatorname{im}\iota$.

In order to apply the slice map we need to make the distinction between the functions in $C_{lu}(G) \rtimes_{L,r} G$ that are in the dense subset $C_c(G, C_{lu}(G))$ and the operators these functions represent. Given a finite propagation operator in $C_u^*(Z)$ we will need to find a function $\widehat{T} \in C_c(G, C_{ru}(G))$ such that the image of $\widehat{T}$ under the representation $\pi^R \rtimes \rho\colon C_c(G, C_{ru}(G)) \to \mathcal{B}(L^2(G))$ is the lift of the operator T belonging to $C_{ru}(G) \rtimes_{\pi^R,R,\rho} G \subset \mathcal{B}(L^2(G))$. The reason we have gone through $C_{ru}(G) \rtimes_{\pi^R,R,\rho} G$ instead of directly to $C_{lu}(G) \rtimes_{L,r} G$ is because it is easier to find this function $\widehat{T}$ in $C_c(G, C_{ru}(G))$

4.6.3. Lifting operators from a coarse lattice. Let G be a locally compact second countable group and Z a coarse lattice inside G. Z is uniformly discrete so fix $\delta > 0$ such that for all $z, z' \in Z$, $d(z, z') > \delta$, whenever $z \neq z'$. Let φ be a continuous positive valued function on G such that $\operatorname{Supp}(\varphi) \subset B(1, \delta/2)$ and

$$\|\varphi\|_2^2 = \int_G |\varphi(g)|^2 \, d\mu(g) = 1.$$

For $z \in Z$, set φ_z to be the function $g \mapsto \varphi(z^{-1}g)$ for $g \in G$. This is a bump function supported on a $\delta/2$-neighbourhood around z. As Z is δ-uniformly discrete, each bump function φ_z has disjoint support. In particular $\{\varphi_z : z \in Z\}$ forms an orthonormal set in $L^2(G)$.

Define an operator $W\colon \ell^2(Z) \to L^2(G)$, $\delta_z \mapsto \varphi_z$ and extend linearly. Hence for $\eta \in \ell^2(Z)$, $(W\eta)(x) = \sum_{z\in Z} \eta(z)\varphi_z(x)$ for all $x \in G$. For $\xi \in L^2(G)$, $W^*\xi(z) = \int \xi(y)\varphi_z(y)\, d\mu(y)$. It is clear that W is an isometry as it sends an orthonormal basis to an orthonormal set.

Let $T \in C_u^*(Z)$ be a finite propagation operator and denote $\langle T\delta_w, \delta_z\rangle$ by $T_{z,w}$. By left invariance of the Haar integral we have that for $x \in G$ and $\xi \in L^2(G)$ we have that

$$(WTW^*)(\xi)(x) = \int_G \sum_{z,w\in Z} \varphi_z(x)\varphi_w(xy)T_{z,w}\xi(xy)\, d\mu(y).$$

As T has finite propagation we are only performing finitely many sums. This means we are able to exchange the order of summation and integration without worry. For all $x, y \in G$ define a continuous function $\widehat{T}_y\colon G \to \mathbb{C}$ such that

$$\widehat{T}_y(x) = \sum_{z,w\in Z} \varphi_z(x)\varphi_w(xy)T_{z,w}\Delta(y)^{-1/2}.$$

The supports of φ_z are pairwise disjoint so for all $x, y \in G$, either $\widehat{T}_y(x) = 0$ or there exists exactly one pair $z, w \in Z$ such that $\varphi_z(x)$ and $\varphi_w(xy)$ are non-zero. Observe $z \in B(x, \delta/2)$ and $w \in B(xy, \delta/2)$ as the support of φ is contained in a ball of radius δ. The map $\widehat{T}_y$ is right uniformly continuous and the function $\widehat{T}\colon G \to C_{ru}(G)$, $y \mapsto \widehat{T}_y$ is compactly supported. This is because if there exists an R such that $T_{z,w} = 0$ whenever $d(z, w) > R$ then the function $y \mapsto \widehat{T}_y$ is supported on a ball of radius $R + \delta$. Therefore the function $\widehat{T}$ belongs to $C_c(G, C_{ru}(G))$.

By construction, the function $\widehat{T}$ is represented by the operator WTW^* on $\mathcal{B}(L^2(G))$. In other words the operator WTW^* is the image of $\widehat{T}$ under the $*$-representation $\pi^R \rtimes \rho\colon C_c(G, C_{ru}(G)) \to \mathcal{B}(L^2(G))$ defined in Proposition 4.3.4 for the covariant representation (π^R, ρ). Indeed for all

$\xi \in L^2(G)$ and $x \in G$

$$(\pi^R \rtimes \rho)(\widehat{T})(\xi)(x) = \int \widehat{T}_y(x)\xi(xy)\Delta(y)^{1/2}\, d\mu(y) = (WTW^*)(\xi)(x).$$

Let $\psi\colon C_{ru}(G) \to C_{lu}(G)$ be the function $\psi(f)(g) = f(g^{-1})$. We can extend this to the function $\psi\colon C_c(G, C_{ru}(G)) \to C_c(G, C_{lu}(G))$ where $\psi(f)_g = \psi(f_g)$. By the $*$-isomorphism in Corollary 4.4.3 it follows that $\psi(\widehat{T})$ is represented by the operator $U_{1,2,3}WTW^*U_{1,2,3}^*$ on $\mathcal{B}(L^2(G, L^2(G)))$. Indeed

$$U_{1,2,3}WTW^*U_{1,2,3}^* = U_{1,2,3}\left(\int_G \widehat{T}_y\rho_y\, d\mu(y)\right)U_{1,2,3}^* = (\tilde{\pi}^L \rtimes \Lambda)(\psi(\widehat{T}))$$

where $\tilde{\pi}^L \rtimes \Lambda$ is the representation associate the reduced cross product in Proposition 4.3.6.

4.6.4. Main theorem.

***Theorem* 4.6.3.** *Let G be a locally compact second countable group that does not have property A. Then there exists an operator in the kernel of $C_{lu}(G) \rtimes_{L,r} G \to (C_{lu}(G)/C_0(G)) \rtimes_{L,r} G$ that is non-compact.*

Proof. For any operator $T \in C_u^*(Z)$ we shall write $\widetilde{T}$ for the operator $U_{1,2,3}WTW^*U_{1,2,3}^* \in C_{lu}(G) \rtimes_{L,r} G$. We shall write $S_{\xi,\eta}$ for the slice map defined in (18) for any $\xi, \eta \in C_c(G)$.

Let Z be a coarse lattice inside G. As G does not have property A then Z does not have property A by Proposition 1.2.6. Hence $C_u^*(Z)$ contains a non-compact ghost operator and call it T. Since T is non-compact, it follows that $\widetilde{T}$ is also non-compact. This is because the lifting map $W\colon \ell^2(Z) \to L^2(G)$ is an isometry and $U_{1,2,3}$ is a unitary operator and so the composition of these operators preserves the non-compactness of T. In order to prove the statement of the theorem it is enough to show that $S_{\xi,\eta}(\widetilde{T})$ belongs to $C_0(G)$ for all $\xi, \eta \in C_c(G)$ by Proposition 4.3.9. We shall show that for any $\varepsilon > 0$ there exists $C > 0$ such that whenever $|x| > C$ then $|S_{\xi,\eta}(\tilde{T})(x)| < \varepsilon$.

The kernel $k(g,h) := \xi(g^{-1}h)\overline{\eta}(h)$ is bounded above by $\|\xi\|_\infty\|\eta\|_\infty$ and is compactly supported, say on $K_1 \times K_2$, as ξ, η are compactly supported. The modular function, the distance function and the length function are all continuous so there exists a $D > 0$ such that $d(g,h) + |h| + \Delta(g)^{-1/2} \le D$ for all $(g,h) \in K_1 \times K_2$.

Set ε' to be

$$\varepsilon' := \frac{\varepsilon}{3D\|\xi\|_\infty\|\eta\|_\infty\mu(K_1)\mu(K_2)}.$$

Choose $M > 0$ such that if $|z|, |w| > M$ then $|T_{z,w}| \le \varepsilon'$. Let $T^{(n)}$ be an operator of finite propagation such that $\|T^{(n)} - T\| \to 0$ as $n \to \infty$. The function $S_{\xi,\eta}\colon A \rtimes_{L,r} G \to A$ is continuous so choose n large enough so that $\|T - T^{(n)}\|_{\mathcal{B}(\mathcal{H})} + \|S_{\xi,\eta}(\widetilde{T}) - S_{\xi,\eta}(\widetilde{T}^{(n)})\|_\infty \le \min(\varepsilon/3, \varepsilon')$. In particular $|T_{z,w} - T_{z,w}^{(n)}| \le \varepsilon'$ for all $z, w \in Z$.

Set $C = M + D + \delta/2$. Hence whenever $|x| > C$ and $(g,h) \in K_1 \times K_2$ it follows that

$$|\widehat{T}_g^{(n)}(x^{-1}h^{-1})| = \varphi_{z_0}(x^{-1}h^{-1})\varphi_{w_0}(x^{-1}h^{-1}g)|T_{z_0,w_0}^{(n)}|\Delta(g)^{-1/2} \le 2\varepsilon' D$$

for some $(z_0, w_0) \in B(x^{-1}h^{-1}, \delta/2) \times B(x^{-1}h^{-1}g, \delta/2)$. This is because $|T_{z_0,w_0}^{(n)}| \le |T_{z_0,w_0}| + \varepsilon'$ and if $|x| > C$ then $|z_0|, |w_0| > M$. Now putting all of this together, whenever $|x| > C$ it follows that

$$|S_{\xi,\eta}(\widetilde{T})(x)| \le |S_{\xi,\eta}(\widetilde{T}^{(n)})(x)| + \varepsilon/3 \le \int_{K_1\times K_2} |k(g,h)\widehat{T}_g^{(n)}(x^{-1}h^{-1})|\, d\mu(g)d\mu(h) + \varepsilon/3 \le \varepsilon \quad \square$$

***Corollary* 4.6.4.** *Let G be a locally compact second countable group. Then the following are equivalent.*

(1) *G has property A.*
(2) *G is amenable at infinity.*
(3) *G is exact.*
(4) *The following sequence*

$$0 \longrightarrow C_0(G) \rtimes_{L,r} G \longrightarrow C_{lu}(G) \rtimes_{L,r} G \longrightarrow (C_{lu}(G)/C_0(G)) \rtimes_{\dot{L},r} G \longrightarrow 0$$

is exact.

Proof. The implication (3) ⇒ (4) is trivial, the equivalence (1) ⇔ (2) was established in [**44**] and the implication (2) ⇒ (3) is done in [**5**]. The above theorem is the implication (4) ⇒ (1) as $C_0(G) \rtimes_{L,r} G$ is isomorphic to the compact operators on $L^2(G, L^2(G))$. □

4.7. Compression of locally compact second countable groups

In this section we show that when compression of a locally compact second countable group is strictly larger than 1/2 then it has property A. We shall use the following characterisation of property A for locally compact second countable groups.

***Theorem* 4.7.1** ([**44**, Theorem 2.3.])**.** *Let G be a locally compact second countable group. Then G has property A if and only if for any compact subset $K \subset G$ there exists a sequence of positive kernels with compact width $u_k \colon G \times G \to \mathbb{C}$ such that*

$$\sup_{(s,t)\in \mathrm{Tube}(K)} |1 - u_k(s,t)| \to 0 \quad \text{as } k \to \infty.$$

Definition 4.7.2. Let (G, d) be a locally compact second countable group with a **plig** metric, and let μ denote the Haar measure on G. Then we say that the metric d has *exponentially controlled growth* if there exists constants $\alpha, \beta > 0$ such that

$$\mu(B_d(e,n)) \leq \beta e^{\alpha n} \quad \forall n \in \mathbb{N}.$$

***Theorem* 4.7.3.** [**59**, Theorem 5.3.] *Every locally compact second countable group G has a **plig** metric d such that the metric has exponentially controlled growth.*

***Proposition* 4.7.4.** [**44**, Proposition 3.3.] *Let G be a locally compact, second countable group. Then the following are equivalent:*

(1) *G admits a coarse embedding into a Hilbert space;*
(2) *G admits a continuous coarse embedding into a Hilbert space.*

***Lemma* 4.7.5.** *Let G be a locally compact, second countable group and d a **plig** metric. Let $f \colon G \to \mathcal{H}$ be a large scale Lipschitz map. Then there exists a continuous large-scale Lipschitz map $\widehat{f}$ such that*

$$R_f = R_{\widehat{f}}.$$

Proof. In the proof of [**44**, Proposition 3.3.] there exists a continuous function $\widehat{f} \colon G \to \mathcal{H}$ and $R > 0$ such that

$$\|f(x) - \widehat{f}(x)\|_{\mathcal{H}} \leq R$$

for all $x \in G$. Hence $\widehat{f}$ is large-scale Lipschitz and has the same compression as f. □

Let G be a locally compact, second countable group. Given a measurable complex-valued kernel $k\colon G \times G \to \mathbb{C}$ define an operator $\mathrm{Op}(k)\colon L^2(G) \to L^2(G)$ by convolution

$$\mathrm{Op}(k)\xi(x) = \int_G k(x,y)\xi(y)\, d\mu(y).$$

***Proposition* 4.7.6.** *Under the following conditions* $\mathrm{Op}(k)$ *is a bounded operator.*

(1) *If k is bounded and has compact width then* $\mathrm{Op}(k)$ *is bounded.*

(2) *Let k be a non-negative and real-valued kernel with the property that there exists $C > 0$ such that*

$$\int_G k(s,t)\, d\mu(s) \le C, \quad \text{for all } t \in G, \text{ and}$$
$$\int_G k(s,t)\, d\mu(t) \le C, \quad \text{for all } s \in G.$$

Then $\mathrm{Op}(k)$ *is bounded and* $\|\mathrm{Op}(k)\| \le C$.

Proof. We shall only prove (1) as (2) is known as the Schur Test. We aim to show that for all $f \in L^2(G)$ there exists a constant $M > 0$ such that $\|\mathrm{Op}(k)f\| \le M\|f\|_{L^2(G)}$.

Suppose $\mathrm{Supp}(k) \subset \mathrm{Tube}(L)$ where L is compact subset of G. If $x \in G$ is fixed and $k(x,y) \neq 0$ then $y \in xL$. Hence $\|\mathrm{Op}(k)f\|_2^2 = \int_G \left(\int_{xL} k(x,y)f(y)\, d\mu(y)\right)^2 d\mu(x)$. Choose $K > 0$ such that $k(x,y) \le K$ for all $x, y \in G$. Then by the Cauchy–Schwarz inequality it follows that $\int_G \left(\int_{xL} k(x,y)f(y)\, d\mu(y)\right)^2 d\mu(x) \le \mu(L)K^2 \int_G \int_{xL} |f(y)|^2\, d\mu(y)d\mu(x)$. By using left invariance of the Haar integral and Fubini's theorem we have that $\int_G \int_{xL} |f(y)|^2\, d\mu(y)d\mu(x) = \int_L \|f\|_2^2\, d\mu(y)$. Hence putting this together we have that $\|\mathrm{Op}(k)f\|_2^2 \le \mu(L)^2K^2\|f\|_2^2$. □

***Theorem* 4.7.7.** *Let G be a locally compact second countable group and d a **plig** metric with exponentially controlled growth. If $\alpha_2(G,d) > 1/2$ then G has property A.*

To prove this we will first have to prove a technical lemma. Let A be the C*-algebra of bounded operators on $L^2(G)$ which is the norm closure of the subalgebra of operators generated by $\mathrm{Op}(k)$, where k is a bounded compact width kernel.

Let f be a large-scale Lipschitz function from a locally compact second countable group G to a Hilbert space $\mathcal{H}$ such that for some $0 < \varepsilon < 1/2$, $d(s,t)^{\frac{1+\varepsilon}{2}} \le \|f(s) - f(t)\|_{\mathcal{H}}$ for all $x, y \in G$. For all $k \ge 1$, Define a kernel $u_k\colon G \times G \to \mathbb{R}$ by $u_k(s,t) = e^{\|f(s)-f(t)\|k^{-1}}$. By Schoenberg's theorem in Theorem 1.2.23, this kernel is positive definite. In fact this kernel satisfies the conditions in Theorem 1.2.8. As the function f is large-scale Lipschitz we have that for some constant D, $u_k(s,t) \ge e^{Dk^{-1}d(s,t)^2}$ for all $s, t \in G$. In particular for all $R, \varepsilon > 0$ we can chose k large enough so that $|1 - e^{k^{-1}d(s,t)^2}| < \varepsilon$ whenever $d(s,t) < R$. However the kernels u_k do not have compact width so the plan of this theorem is to approximate the kernels u_k by positive ones with compact width. Then we would satisfy the conditions in Theorem 4.7.1 and so G would have property A.

***Lemma* 4.7.8.** *The operators* $\mathrm{Op}(u_k)$ *belong to A for all $k \ge 1$.*

Proof. The metric has exponentially controlled growth so let $\alpha, \beta > 0$ be constants such that $\mu(B(e,n)) \le \beta e^{\alpha n}$ for all $n \in \mathbb{N}$. Fix $\kappa > 0$. We will show that the kernel $u\colon G \times G \to \mathbb{C}$ defined by $u(s,t) = e^{-\|f(s)-f(t)\|^2\kappa}$ defines an element in A.

Define for $n \in \mathbb{N}$,

$$k_n(s,t) = \begin{cases} u(s,t), & \text{if } d(s,t) > n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $u - k_n$ is bounded compact width kernel so $\mathrm{Op}(u-k_n)$ belongs to A. We will show that $\|\mathrm{Op}(u) - \mathrm{Op}(u-k_n)\|_2 \to 0$ as $n \to \infty$. Since $\mathrm{Op}(u) - \mathrm{Op}(u-k_n) = \mathrm{Op}(k_n)$ on compactly supported elements of $L^2(G)$, it is enough to show that $\|\mathrm{Op}(k_n)\| \to 0$ as $n \to \infty$. To do this we will provide a sequence C_n that converges to 0 such that $\int_G k_n(s,t)\,d\mu(t) \le C_n$ for all $s \in S$. Then by the Schur test, $\|\mathrm{Op}(k_n)\|$ is bounded above by C_n for each n and so converges to 0.

Fix $s \in G$ and set $F_m = B(s, m+1) \setminus B(s,m)$. Now for all $m \ge 0$, $\int_{F_m} u(s,t)\,d\mu(t) \le \mu(F_m)(e^{-\kappa m^{1+\varepsilon}})$ because $u(s,t) \le e^{-\kappa m^{1+\varepsilon}}$ for all $t \in F_m$. Hence for all $n \in \mathbb{N}$,

$$\int_{B(s,n)^c} u(s,t)\,d\mu(t) \le (\beta e^{\alpha} - 1) \sum_{m \ge n} \left(\frac{\beta e^{\alpha}}{e^{\kappa m^{\varepsilon}}} \right)^m . \tag{19}$$

This is because $B(s,n)^c = \cup_{m \ge n} F_m$ and $\mu(F_m) \le \beta e^{\alpha m}(\beta e^{\alpha} - 1)$ for $m \in \mathbb{N}$. There exists n_0 such that for all $m \ge n_0$, $\beta e^{\alpha} \le e^{\kappa m^{\varepsilon}}$. This means that the sum on the right hand side converges to 0 as $n \to \infty$. Now we are done because $\int_G k_n(s,t)\,d\mu(t) = \int_{B(s,n)^c} u(s,t)\,d\mu(t)$ for all $n \in \mathbb{N}$ and if we set C_n to equal the right hand side of (19) for all $n \ge n_0$ then by the Schur test, $\|\mathrm{Op}(k_n)\| \le C_n$. The sequence C_n converges to 0 so $\mathrm{Op}(u_k) \in A$ for all $k \ge 1$. □

In fact what we can deduce from this proof is that there exists C such that $\|\mathrm{Op}(u_k)\| \le C$ for all $k \ge 1$. Indeed for each k there exists m_k such that $2\beta e^{\alpha} \le e^{k m^{\varepsilon}}$ for all $m \ge m_k$. As k increases, m_k decreases and so $\int_G u_k(s,t)\,d\mu(t) \le \mu(B(1,m_k)) + \int_{B(s,m_k)^c} u_k(s,t)\,d\mu(t)$. From our choice of m_k and from the proof above we can deduced that $\int_{B(s,m_k)^c} u_k(s,t)\,d\mu(t) \le \beta e^{\alpha} - 1$. By the Schur test and the fact that $m_1 \ge m_k$ for all $k \ge 1$, we have that $\|\mathrm{Op}(u_k)\| \le \mu(B(1,m_1)) + \beta e^{\alpha} - 1$ for all $k \ge 1$.

Proof of Theorem 4.7.7. As we did in Section 2.1 we can assume the metric is uniformly discrete. Up to a multiplicative constant we can assume there exists a function $f\colon G \to \mathcal{H}$ such that $d(s,t)^{\frac{1+\varepsilon}{2}} \le \|f(s) - f(t)\|_{\mathcal{H}}$ for some $\varepsilon > 0$. As the kernels u_k are positive definite it follows that the operators $\mathrm{Op}(u_k)$ are positive. Let V_k be the positive square root of $\mathrm{Op}(u_k)$, see [**88**, Theorem 2.2.1.]. By the C*-identity and because $\sup_{k \ge 1} \|\mathrm{Op}(u_k)\| < \infty$ we can deduce that $\sup_{k\ge1} \|V_k\| < \infty$. Therefore by the previous lemma we can choose a sequence of compact width kernels w_k such that such that $\|V_k - \mathrm{Op}(w_k)\| \|V_k\| \to 0$. For short hand we will write $W_k = \mathrm{Op}(w_k)$. Define kernels $\widehat{u}_k$ by

$$\widehat{u}_k(s,t) = \frac{1}{\mu(B(e,1/k))^2} \langle W_k \chi_{B(s,1/k)}, W_k \chi_{B(t,1/k)} \rangle_{L^2(G)} \quad \forall s,t \in G.$$

We shall show that $\widehat{u}_k$ are positive, have compact width and approximate u_k. For short hand we will write B_k for $\mu(B(e,1/k))^2$ and $\chi_{s,k}$ for the function $\chi_{B(s,1/k)}$. It is a fact that T^*T is a positive operator for any operator T on a Hilbert space $\mathcal{H}$. This means that $\langle T^*Tv, v\rangle \ge 0$ for all $v \in \mathcal{H}$. So for any $s_1, \ldots, s_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ it follows that $\langle W^*W(\sum_{i=1}^n \lambda_i \chi_{s_i,k}), \sum_{i=1}^n \lambda_i \chi_{s_i,k}\rangle \ge 0$. In particular $\sum_{i,j}^n \lambda_i \overline{\lambda_j} \widehat{u}_k(s_i,s_j) \ge 0$ and so it is a positive kernel.

We know that each w_k have compact width so suppose for each k there exists $L_k > 0$ such that if $d(s,t) > L_k$ then $w_k(s,t) = 0$. Then for all $s, x \in G$ and $k \in \mathbb{N}$, if $d(x,s) > L_k + 1/2k$ then $\int_{B(s,1/k)} w_k(x,y)\,d\mu(y) = 0$. In particular, if $d(s,t) > L_k + 1/k$ then $\int_{B(s,1/k)} w_k(x,y)\,d\mu(y) \cdot \int_{B(t,1/k)} w_k(x,y)\,d\mu(y) = 0$ for all $x \in G$. Therefore $\langle W_k\chi_{s,k}, W_k\chi_{t,k}\rangle = 0$ whenever $d(s,t) > L_k + 1/k$ and so for all $k \ge 1$, the positive kernels $\widehat{u}_k$ have compact width.

Now we want to show that $\widehat{u}_k$ can approximate u_k uniformly on compact tubes. As an intermediate step define the kernels

$$v_k(s,t) = \frac{1}{B_k}\langle V_k\chi_{s,k}, V_k\chi_{t,k}\rangle_{L^2(G)} \quad \forall s,t \in G.$$

By the Cauchy–Schwarz inequality and that $\|\chi_{s,k}\|_2 = B_k^{1/2}$ it follows that $|\widehat{u}_k(s,t)-v_k(s,t)| \leq \|V_k^*V_k - W_k^*W_k\|$. Hence by the triangle inequality and that V_k is self adjoint, $\|V_kV_k - W_k^*W_k\| \leq \|V_k - W_k\|(\|V_k\|+\|W_k\|) \leq \|V_k - W_k\|(2\|V_k\|+\|V_k - W_k\|)$. Hence $|\widehat{u}_k(s,t)-v_k(s,t)|$ converges to 0 uniformly as $k \to \infty$.

To prove the statement of the theorem it is enough to show that for all $R, \varepsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $|u_k(s,t) - v_k(s,t)| < \varepsilon$ whenever $d(s,t) \leq R$. This is because $\widehat{u}_k$ converges uniformly to v_k.

As the exponential function is a convex function it follows that for all $a, b > 0$, $|e^{-a} - e^{-b}| \leq |a+b|$. In particular, as f is a large-scale Lipschitz function, $|u_k(s,t) - u_k(x,y)| \leq D(d(s,t) + d(x,y))k^{-1}$ for any $s,t,x,y \in G$, where D is the Lipschitz constant for f. Fix $R, \varepsilon > 0$ and choose k large enough so that $D(2R+1)k^{-1} < \varepsilon$. Then $|u_k(s,t) - v_k(s,t)| < \varepsilon$ whenever $d(s,t) < R$. This is because

$$|u_k(s,t) - v_k(s,t)| \leq \frac{1}{\mu(B(e,1/k))^2}\int_{B(s,1/k)}\int_{B(t,1/k)} |u_k(s,t) - u_k(x,y)|\, d\mu(x)d\mu(y)$$

for all $s,t \in G$. Hence when $d(s,t) \leq R$ and x and y are within a ball of radius $1/k$ of s and t respectively then $|u_k(s,t) - u_k(x,y)| \leq D(2R+1)k^{-1} < \varepsilon$. □

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