UNIVERSITY OF SOUTHAMPTON

Faculty of Social and Human Sciences

Mathematical Sciences

Gravitational self-force from curvature scalars

by

Cesar Antonio Merlin Gonzalez

Thesis for the degree of Doctor of Philosophy

August 2015

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL AND HUMAN SCIENCES

Mathematical Sciences

Doctor of Philosophy

by Cesar Antonio Merlin Gonzalez

In this work we study the two-body problem in general relativity for the extreme-mass-ratio regime, where the problem is amenable to perturbation theory. The orbital dynamics in this configuration is driven by a back-reaction or self-force, caused by the interaction of a particle with its own gravitational field. In this thesis we develop and implement a new approach for self-force calculations in Kerr spacetime.

We choose to move from the original Lorenz-gauge formulation of the self-force to work in a radiation gauge. In the Lorenz gauge the perturbation is obtained by solving a set of ten coupled differential equations, and in Kerr the equations are not separable. In the radiation gauges the computational cost is reduced by solving the fully separable Teukolsky equation to obtain curvature scalars, and applying certain differential operators to recover the metric perturbations. There are two main challenges in calculating the self-force in these radiations gauges: understanding how to include the "completion" piece that is not recovered in the reconstruction procedure (but it's necessary to satisfy the linearised Einstein field equations); and having a rigorous and well-justified self-force formalism to use these radiation-gauge perturbation.

We identify three types of radiation gauges according to their singular structure: *half-*, *full-* and *no-*string gauges. We obtain modifications to the standard Lorenz-gauge mode-sum formula for the half- and no-string gauges, and explain why the full-string gauges are too pathological to be considered in a numerical implementation. Our method is based on a local analysis of the gauge transformation relating the Lorenz and radiation gauges. This analysis provides the framework to modify the Lorenz-gauge self-force formulation and obtain modifications to the traditional Lorenz-gauge mode-sum formula.

We propose a new method to address the inclusion of the completion piece of the perturbation in Kerr. It is based on imposing smoothness of certain auxiliary gauge-invariant-quantities away from the particle to determine the amplitudes of the mass and angular momentum perturbations that are not accessible through the metric-reconstruction procedure. We obtain the completion piece for Schwarzschild, and for equatorial orbits in Kerr. We discuss how our method could be extended for geodesic non-equatorial orbits around Kerr.

As a first implementation of our formalism, we compute the gravitational self-force in the frequency domain for a particle moving on a circular orbit around a Schwarzschild black-hole. This calculation is carried out using our new version of the mode-sum formula. We obtained numerical solutions to the spin-± 2 Teukolsky equation and apply the reconstruction procedure. We compare our numerical integration with the analytical method of Mano, Susuki and Takasugi. We test the numerical efficiency of our method compared with Lorenz-gauge implementations available in the literature. We find numerical agreement between the results obtained in the outgoing and ingoing radiation-gauges for our particular setup. We show that our results for the self-force agree with the Lorenz-gauge ones at large orbital-radii, and provide an explanation of why this is expected. We discuss the extension of this implementation to more general orbits around Kerr.

Contents

List of Tables

List of Figures

Declaration of Authorship

I, Cesar Antonio Merlin Gonzalez, declare that the thesis entitled:

Gravitational self-force from curvature scalars

and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:

  1. A. Pound, C. Merlin, and L. Barack, "Gravitational self-force from radiation-gauge metric perturbations," *Phys. Rev. D*, vol. 89, p. 024009, 2014
  2. C. Merlin and A. G. Shah, "Self-force from reconstructed metric perturbations: Numerical implementation in schwarzschild spacetime," *Phys. Rev. D*, vol. 91, p. 024005, Jan 2015

Signed:

Date:

Acknowledgements

First and foremost, I want to thank my supervisor Leor Barack. Leor gave me the great opportunity to come and work under his supervision and I can only hope I have made the best efforts to correspond the responsibilities he entrusted me. On top of that, he provided constant support and guidance during my PhD in the form of useful suggestions and feedback to improve my work.

I want to thank my examiners Adrian C. Ottewill and Ian D. Jones, for taking the time for reading this thesis. All the comments and suggestions that they consider pertinent will be taken into account to improve this work.

Among the group of colleagues at Southampton University, I would like to highlight Niels Warburton, Marta Colleoni and Lucy Keer. I shared long hours in the office (even on weekends), dinners, trips, and occasional drinks with them, rendering them irreplaceable companions of my stay in Southampton.

During my stay in Southampton the self-force group has had many members, whose diverse knowledge and expertise have enriched my research through discussions and collaborations. I am deeply grateful to all of them: Sam Dolan, Sarp Akcay, Adam Pound, Maarten van de Meent, Abhay G. Shah, Charalampos Markakis, Jeremy Miller and Paco Giudice.

I also thank all the members of the Capra community. From the diverse talks and discussions presented in the yearly Capra meetings I have managed to get a wider and better picture of the different efforts of the extreme-mass-ratio-inspiral modelling.

I thank my family for all the support and sacrifices they have made to give me the best opportunities within their reach. They have also taught me the value of constant work and encouraged me to fulfill my goals.

Being away from my native country has proven a personal challenge. However a few people have lessen the burden by maintaining constant communication, among them I'd like to mention: Ernesto Navarro, Laura Rosas Patterson, Gabriela Ruelas and Iliana Urzua. I dedicate a special mention to Rodolfo Meza, Ana Avilez, and Georgina Lopez who have also shortened the physical distance when it was possible.

Finally I give thanks to CONACyT (fellowship No. 214690) for providing the necessary financial support throughout my post-graduate degree. It is only fair to say that without this funding it would have been impossible for me to afford my studies outside Mexico.

List of Acronyms

ADM	Arnowitt-Deser-Misner
AM	Angular Momentum
BH	Black Hole
BL	Boyer-Lindquist
CCK	Chrzanowski-Cohen-Kegeles
EFE	Einstein Field Equation
EH	Event Horizon
EM	Electro-Magnetic
EMRI	Extreme-Mass-Ratio Inspiral
GR	General Relativity
GSF	Gravitational Self-Force
IMRI	Intermediate Mass Ratio Inspiral
IRG	Ingoing Radiation Gauge
ISCO	Innermost-Stable Circular Orbit
LG	Lorenz Gauge
LIGO	Laser Interferometer Gravitational-Wave Observatory
LL	Locally-Lorenz
MP	Metric Perturbations
MST	Mano-Suzuki-Takasugi
NR	Numerical Relativity
ORG	Outgoing Radiation Gauge
PN	Post-Newtonian
RG	Radiation Gauge
SF	Self Force

Chapter 1

Introduction

1.1 2-body problem in general relativity and the self-force approach

The gravitational two-body problem in physics consists of an isolated system of two objects in motion due to their gravitational interaction. In the Newtonian limit the motion of a gravitationally bound system of two point masses has two conserved quantities — the energy and the angular momentum (AM) — and the motion is precisely periodic. However, when we take into account the relativistic behaviour of the motion, these quantities will not remain constant due to the emission of gravitational waves, which causes a reduction of the orbital period of the two bodies. This phenomenon was first observed in 1974 by Hulse and Taylor for the PSR 1913+16 binary-pulsar [3]. The two-body problem in General Relativity (GR) is as old as the theory itself. Lorentz and Droste [4] obtained the first relativistic correction to the Newtonian interaction. Einstein himself, with Infeld and Hoffmann [5], formulated a method to approximate the equations of particles moving in a relativistic field, giving birth to post-Newtonian (PN) theory.

In the relativistic context the concept of point particle is not suitable to approach this problem directly, since we can not take advantage of the linearity that the equations of motion exhibit in the Newtonian case: the usual representation of the point-particle as a delta-function becomes inconsistent with the non-linearity of Einstein's field equation (EFE) [6]. The simplest problem we can try to understand in GR is that of a binary black-hole (BH) system, without taking into account any internal properties that make the problem considerably more complicated.

The description of this problem in GR can be treated in different ways depending on the mass-ratio and separation of the orbiting objects. We can identify three different regimes, see Fig. 1.1. The first one corresponds to a sufficiently large separation between the two objects, where the objects are treated as point-like at first approximation, and this regime allows a PN treatment. In this scheme we incorporate GR corrections to the Newtonian dynamics order by order in the separation. However, when the two masses are of the same scale and the separation distance is of the same order of magnitude as the radius of the bodies then the only description possible is given by Numerical Relativity (NR) simulations.

There is a third scenario possible, the so-called extreme-mass-ratio inspiral (EMRI), see Fig. 1.2. In this regime the separation distance is small but the mass-ratio of the bodies is large. The problem is then amenable to a perturbative treatment in which at zeroth-order the motion is geodesic in the background geometry of the large BH. At the next order we take into account the

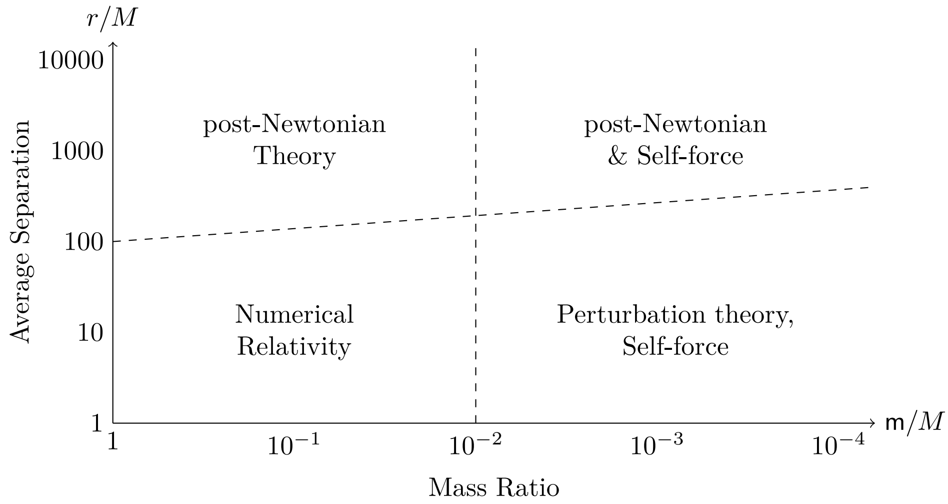

Figure 1.1: Parameter space of the relativistic two-body problem. The mass ratio of the two bodies in log scale is on the x-axis, the average separation of the orbiting bodies in log scale is on the y-axis. The overlap between PN theory and self-force calculations allows to test both frameworks and obtain high-order PN terms. The dotted lines indicate a blur and smooth transition between these regimes.

linear perturbation due to the small but finite mass of the particle. This arrangement gives rise to an effective gravitational-self-force (GSF) which "accelerates" the particle.

We may identify two pieces of the self-force (SF), the conservative and the dissipative. The dissipative piece of the SF removes energy and AM from the orbiting bodies, and radiates them away as gravitational waves. The conservative piece of the SF modifies the positional elements of the orbit; for example, it is responsible for the shift in orbital precession [7, 8].

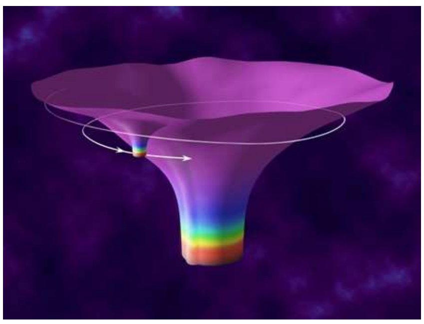

Figure 1.2: Artistic representation of the extreme-mass-ratio-inspiral (EMRI) regime, where a compact object of mass m is embedded in the gravitational field of a central BH of mass $M \gg \mathsf{m}$. The 'small' particle experiences a back-reaction effect or self-force. Credit: NASA.

One of the key sources of gravitational waves for low-frequency gravitational-wave detectors is the inspiral of compact objects into massive BHs in galactic nuclei. Ground and future space-based detectors require accurate models of the inspiral orbits, which must take into account general-relativistic radiation-reaction and other gravitational back-reaction effects of the SF. The European Space Agency plans to launch in the year 2034 the *European New Gravitational Wave Observatory* [9] (based on the *Laser Interferometer Space Antenna*, LISA) which would have its peak sensitivity around 1 mHz. This sensitivity would enable observation of signals from inspirals into Kerr BHs with masses in the range of $\sim 5 \times 10^5 - 5 \times 10^7 M_\odot$. Ground-based detectors, such as the *Laser Interferometer Gravitational-Wave Observatory* (LIGO), have not been able to detect gravitational waves so far, but the ongoing upgrade to *advanced LIGO* will lead to an improvement in the sensitivity [10] by at least a factor of 10 and it may be able to observe intermediate mass-ratio inspirals (IMRIs) with mass-ratios in the range of $\sim 10:1$ to $\sim 100:1$ [11]. Such IMRIs could be modelled combining GSF results and NR simulations. The planned underground *Einstein Telescope* [12] (one of the third generation gravitational-wave detectors), with improved sensitivity at frequencies in the range $\sim 1-10$ Hz, may be able to see from a few to several hundred IMRIs events per year [13]. To make an accurate parameter-extraction and exploit the full scientific value of EMRI signals it is required to have accurate theoretical templates of EMRI waveforms, which requires the knowledge of the SF as prerequisite.

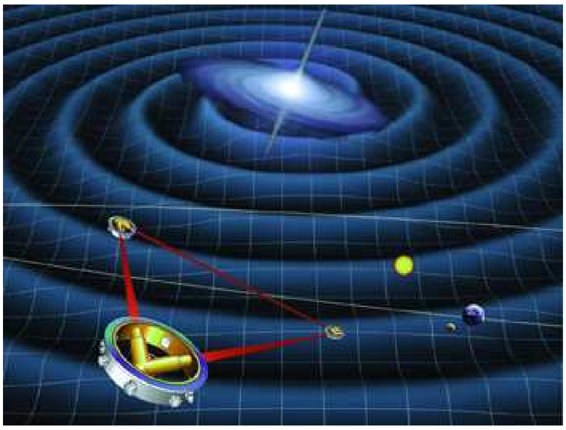

Figure 1.3: Artistic representation of a space-based detector (like the project eLISA) detecting gravitational waves. Credit: NASA.

The modern history of SF calculations began in 1997 with the formulation of the first-order equations of motion for the GSF by Mino, Sasaki and Tanaka [14], and independently by Quinn and Wald [15]. The resulting equation of motion is usually referred to simply as the MiSaTaQuWa equation [see Eq. (2.41)]. Shortly after that, in 1998, the inaugural Capra meeting was held in California and has continuously brought together relativists devoted to the various aspects of SF calculations and its application to the exciting prospect of detecting gravitational waves emitted by EMRIs.

1.2 Recent advances in self-force studies

The basic idea behind the MiSaTaQuWa formulation is to identify two length-scales of the problem, one associated with the small orbiting particle and a second one related to the radius of curvature of the background in which the particle is moving. The first scale corresponds to a "near" zone where the geometry is given approximately by the particle's geometry (in the original derivation it was considered Schwarzschild but this restriction was removed by Gralla and Wald [16]) with tidal-type corrections due to the background metric. The "far" zone, where the internal structure of the moving particle becomes less important, is then given by the background spacetime weakly perturbed by the now distant "point-particle". The two asymptotic expansions of the metric are then matched in a "buffer" zone where the two geometrical descriptions are valid. This constrains the motion of the particle (from a far-zone point of view) yielding the expression for the SF in terms of the "*tail* " field. This tail can be interpreted physically as the part of the metric perturbations (MP) arising from the waves being scattered off the background curvature. This is broadly speaking the description of the matched asymptotic expansions method (See Appendix A for a brief review of the method). MiSaTaQuWa equation was later formulated more rigorously by Pound [17] and Gralla-Wald [16]. A full pedagogical derivation can be found in the review by Poisson *et al.* [18].

The SF is a gauge-dependent notion, as we explain in the next Chapter. The behaviour of the SF under a gauge transformation was first studied by Barack and Ori in ref. [19]. The MiSaTaQuWa equation was formulated in the Lorenz gauge (LG), where the field equations become hyperbolic and the representation of the particle's singularity is locally isotropic. These two features (hyperbolicity and local-isotropy) are essential to apply matched asymptotic expansions. This provides a practical way to solve the field equations numerically.

Several schemes have been proposed in order to implement the LG formalism of the SF in practical calculations. The basic challenge is how to subtract the *singular* piece of the MP to obtain the correct tail field, responsible for the SF [see Eq. (2.41) and the discussion below for further details]. Instead of referring to the tail piece of the MP, Detweiler-Whiting [20] proposed that we could decompose the physical MP into *regular* and *singular* pieces [see Eq. (2.47)]. The regular part corresponds to a solution of the vacuum Einstein equation and it is *smooth*, unlike the tail field. The regular field gives rise to the same SF as the one obtained from the tail field and it allows an interesting interpretation of the SF: the particle moves along a geodesic of an effective spacetime with a metric given by the sum of the background metric and the regular field, $g_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$. However, in practical calculations both interpretations involve subtracting divergent quantities, which is not easily done numerically.

Let us present a brief description of three techniques proposed to implement MiSaTaQuWa's formulation in practice. We focus our attention on the one that has been the most successful in practical calculations (see Table 1.1) and it is also the one we will use throughout this work (see Sec. 2.4). This method is referred as 'mode-sum' method.

Worldline/matched expansions. This method[1] involves computing the SF as an integral over the past worldline of the particle. The integrand corresponds to the Green's function for the appropriate wave equation, namely the linearised EFE. This integral is calculated directly by matching together two independent expansions, see Fig. 1.4. As suggested by Anderson [21], in the quasi-local regime the integral is dominated by the recent past, and can be represented using the Hadamard expansion. The analytical form for the Hadamard expansion was obtained to very high accuracy by Ottewill and Wardell [22, 23].

The quasi-local expansion is matched to a second one, which takes into account the distant past along the worldline. It was shown by DeWitt and DeWitt (for the EM case) [24], and by Pfenning and Poisson [25] (for the gravitational case) that this second expansion is relevant for the SF. The signal produced by the particle at a certain time in the past will scatter off the centre of mass of the system and then re-interact with the particle at its current location. The full Green's function was obtained by Casals and Dolan for a static scalar-particle in a Nariai spacetime[2] [26] and for Schwarzschild [27]. Some progress to evaluate the Green's function in the Kerr case was recently reported [28].

Mode-sum method. This method was introduced by Barack and Ori [29–31]. In this approach one calculates the contributions to the tail-field mode by mode in a multipole expansion by subtracting finite quantities, "regularization parameters", for each mode. The values of the regularization parameters are obtained analytically by analysing the singular behaviour of the field near the particle.

[1] Not to be confused with the matched asymptotic expansions method described previously in this section and in Appendix A.

[2] A simple toy model of a BH.

This method is also practical in the sense that it provides a self-testing mechanism: if either the value of the regularization parameters or the value of one of the numerically computed modes of the unregularized "force" are wrong, then the mode-sum formula [see Eq. (2.55) below] may not converge (since at large ℓ the computed modes have to agree with the analytical regularization parameters).

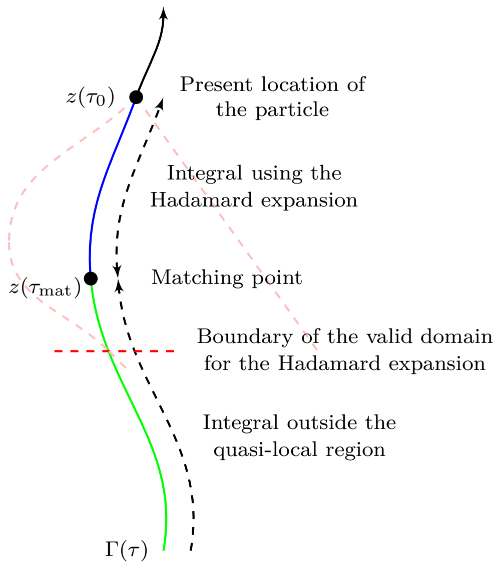

Figure 1.4: Schematic representation of the matched expansions method. Two independent expansions are used to obtain the SF at $z(\tau_0)$, the two of them are matched at $z(\tau_{\rm mat})$ in the past history of the world-line ($\tau_{\rm mat} < \tau_0$). The past light-cone (shown in pink) is bent due to the curvature of the background. The first expansion (blue), in the recent past, is dominated by the Hadamard expansion. The second expansion accounts for the early past of the history (green), and it is computed by an integral over the Green's function along the worldline $\Gamma(\tau)$.

Implementations using the mode-sum formula have been successful so far in a variety of calculations [32–35]. There has been work by Barack and Sago regarding the GSF in a Schwarzschild background for circular orbits [36] and eccentric orbits [37]. Warburton and Barack have computed the scalar SF for a particle orbiting around a Kerr BH for circular-equatorial orbits [38, 39], circular-inclined orbits [40] and for eccentric-equatorial orbits [41]. In the electromagnetic case of a charged particle following a geodesic around a Schwarzschild BH, the SF has been calculated by Haas [42] using mode-sum regularization. Linz *et al.* [43] and independently Zimmerman *et al.* [44] consider the problem of calculating the SF when the gravitational field couples with an electromagnetic field (the work of Zimmerman also considered separately the coupling of the gravitational field with a scalar field). Higher-order regularization parameters have been found in the LG by Heffernan *et al.* [45, 46], and this now allows for a faster rate of convergence of the mode-sum method.

Puncture methods (Effective-Source) [47–52]. This method was proposed for time-domain numerical implementations in $2+1$ or $3+1$ dimensions. It involves splitting the regular part of the MP tensor in terms of an auxiliary *puncture* field and a second *residual* field. The puncture field is given analytically, as an approximation to the singular-field near the particle, so that the residual field will yield the correct SF. Implementations relying on the puncture method have been successful for the scalar SF both in Schwarzschild [50, 51, 53, 54] and Kerr spacetimes [48], with extensions to the GSF in Schwarzschild geometry in 2+1 dimensions [49]. And recent progress has been reported for the Kerr case [55].

1.3 Challenges in self-force calculations and radiation-gauge approach

Traditional calculations of the SF rely on numerical solutions of the linearised EFE in the LG [77]. With the MP as an input one may obtain the value of the SF at the particle's location using the mode-sum method or the puncture method. On Kerr spacetime the tensorial field equations in

Table 1.1: Summary of regularization methods developed for self-force calculations of BH binary inspirals. The references for the scalar SF include a full computation of the SF value, except the *(quasi-local)* entry. The distinction *(quasi-local)* is made to indicate that the full worldline-calculation is not included in the references. The entry labelled with *(approx)* used an approximated expression, accurate to leading-order in M, for the Green's function to calculate the SF. For the EM and gravitational cases, the references for Kerr address only the regularization method and not the full computation of the SF. Similarly the branch cut entry refers to an unpublished attempt to evaluate the branch-cut piece required in the early-time expansion of the worldline method. Taken from [56].

Case		Worldline/matched expansions	Mode-sum	Effective Source
Scalar	Schwarzschild	circular (approx)[21]; generic (quasi-local) [22, 26]; generic[27, 57, 58]; static[59]; accelerated[23];	radial[60]; circular[61–64]; eccentric[32, 45, 51, 65, 66]; static[59];	circular [33, 39, 47, 50, 53, 67]; eccentric[68]; evolving[69];
	Kerr	generic[22]; accelerated[23];	circular[70]; equatorial[38, 46]; inclined circular[40] ; accelerated[43]; static[71, 72];	circular[48]; eccentric[73];
EM	Schwarzschild	static[59];	static[59]; eccentric[42, 45]; static(Schwarzschild- de Sitter)[74]; radial (Reissner- Nordström)[35];	—
	Kerr	—	equatorial[46]; accelerated[43];	—
Gravitational	Schwarzschild	generic (quasi-local)[75]; circular[54]	radial[76]; circular[2, 34, 77–83]; eccentric[1, 45, 84–90]; osculating[91];	circular[49];
	Kerr	circular (quasi-local)[21]; branch cut[28];	equatorial[46]; accelerated[43]; circular[1, 92];	circular[55]; generic[52];

the LG are not separable and one has to deal with partial differential equations. This has been a motivation to work in time-domain implementations [48–53] of MiSaTaQuWa formula with a puncture, but the numerical evolution in this scheme is usually computationally expensive.

The numerical treatment of BH perturbations in Kerr spacetime becomes much simpler in the *radiation* gauges. In these gauges it is possible to use Teukolsky's formalism and the Chrzanowski-Cohen-Kegeles (CCK) procedure to reconstruct the MP from the Weyl curvature scalars [81, 93, 94]. This only involves obtaining the solution of scalar-like wave equations, which admit full separation of variables for each multipole mode. This procedure has been successful calculating gauge-invariant quantities, such as energy fluxes [95, 96] and the red-shift invariant [83, 92] (see Sec. 2.6 for a review of this invariant). However, in this gauge a SF formulation was still unavailable until the present thesis [1], which provides two methods that use the reconstructed perturbations in a radiation gauge (RG).

The perturbation associated with a point-particle in the RGs takes the form of a string-like radial singularity [19] at any given time (this is a gauge artefact of this class of gauges) as we will show in Chapter 3. This string singularity can be removed, but only by paying the price of introducing a discontinuity across a surface intersecting the particle. In short, we can say that while the LG is regular but not practical for SF calculations in Kerr, the RG is practical but generally not regular.

In this sense the methods developed in this thesis are both regular and practical. The basic idea is to work in a gauge where it is relatively easy to obtain the MP numerically, such as the RGs. We will discuss two different classes of gauges that take advantage of the 'simplicity' of BH perturbation theory.

The first of those classes corresponds to a local deformation of the RG to resemble the LG to leading-order. In this class we can directly apply MiSaTaQuWa equation [19] and use the standard LG mode-sum formula. Another advantage is that the interpretation of motion follows the same description as its LG counterpart (see Appendix A for a review of the motion of the centre of mass in the LG). The difficulty of these gauges lies in relating them to the undeformed RG. This idea had been suggested previously [19, 97] but never fully implemented until the present work.

Our second class of gauges corresponds to full RGs without any deformation. In this class the LG mode-sum is not valid any more, and a different regularization method is required. Furthermore the understanding of the LG equation of motion requires modification to accommodate the pathologies of the RG (see Appendix A.2).

The main practical results of these two approaches are modifications to the standard LG mode-sum formula. In other words, two new mode-sum formulae: the first one requires the modes of the unregularised force calculated from either the "inside" or the "outside" limit of the orbit, and certain corrections to the LG regularization parameters; the second one requires both of those one-sided values.

The MP in the RG, recovered using the CCK procedure, do not satisfy the linearised EFE. The full solution — required to obtain the unregularised force — then needs an extra piece, which we will refer to as the *completion* piece. Wald [98] showed that this completion part corresponds to perturbations in the mass and AM parameters, perturbations to other algebraically-special solutions (C-metric and Kerr-NUT), and gauge perturbations. In Schwarzschild, the completion describes solutions to the monopole and dipole parts of the EFE. The situation does not follow directly in the Kerr case where there is an infinite coupling between different harmonic modes. This remained an open problem of BH perturbation theory until we fully addressed it in this thesis [99] for all (equatorial) orbits in Kerr.

We accompany our analytical method with a numerical calculation of the GSF of a particle orbiting a Schwarzschild BH, which serves as a test case for the more complicated problem of Kerr. This test case provides a comparison of the computational cost with respect of a similar computation in the LG. It also serves to anticipate some of the difficulties we will encounter in the Kerr case.

1.4 Layout of this work

In Chapter 2 of this thesis we give a brief review on the general formalism of the SF calculation. In Sec. 2.1 we give review Teukolsky's formalism, which will be useful when we attempt to calculate the MP required to obtain the SF of a particle orbiting a Kerr BH. A practical way to solve the Teukolsky equation numerically comes in the form of the Sasaki-Nakamura equation included in Sec. 2.1.4. The solution of this equation, usually referred as Sasaki-Nakamura field, is related to solutions of the homogeneous Teukolsky equations by a simple transformation. This will be the basis of our numerical implementation, in which we will use the metric-reconstruction procedure first formulated by Chrzanowski [93] and Cohen-Kegeles [94] (CCK reconstruction). This reconstruction starts at the $\ell = 2$ spin-weighted harmonic mode, and it requires the inclusion of the completion piece mentioned above. The treatment of the completion piece requires special considerations and it will be reserved for Chapter 5. Together the reconstructed and completion pieces correspond to what we will refer in Sec. 2.1.6 as a *completed radiation-gauge*. This completed gauge is a solution of the full linearised-EFE. In Sec. 2.2 we give MiSaTaQuWa formula, and the equation of motion in Sec. 2.3. In Sec. 2.4 we will summarize the essence of the mode-sum approach, which is the one we will be using in our implementation. As mentioned above, the SF is not a gauge-invariant quantity. In Sec. 2.5 we consider how gauge transformations from the LG to other regular gauges affect the SF. In Sec. 2.6 we present a gauge-invariant quantity, which is useful for comparisons of SF calculations in different gauges.

In Chapter 3 we give a detailed description of the formalism we obtained for SF calculations. This reformulation will allow us to calculate the SF using the RG reconstructed-perturbation. In Sec. 3.2 we will work in a basis of Fermi-like coordinates, which allows for a straightforward analysis of the Kerr spacetime. We include a detailed calculation of the leading-order term of the gauge-transformation generator that locally relates the singularity of the RG perturbation with the singularity of the LG perturbation. We give a classification of the RGs based on their singular structure. We discuss which of these types of RGs are suitable for numerical implementations, and in particular useful for SF calculations. We develop the approach (proposed in [97]) of locally deforming the RGs to fall in the class of gauges that relate to the LG by a regular gauge-transformation [19], and in which the standard LG mode-sum is still valid. We also discuss the use of the direct (without deformation) computation of the SF using the CCK-reconstructed modes. In the two approaches we just described, using deformed and undeformed RGs, we will require modifications to the Lorenz mode-sum formula. These modifications will be presented in Sec. 3.3 for the locally-deformed gauge, where we also find corrections to the standard LG regularization-parameters. In Sec. 3.4 we find the modifications for the undeformed gauge. The particular expressions for the 'new' regularization parameters will be given explicitly (for a particular extension) in BL coordinates in Appendix D.

In Chapter 4 we present the numerical results of the first implementation of the method described in Chapter 3. We specialise to a particle orbiting a Schwarzschild BH in a circular orbit. The algorithm of our computation appears as Sec. 4.1. This algorithm can be used for the Kerr case with minor modifications. The details of the implementation are given in Sec. 4.2. In Sec. 4.2.1 we discuss the inclusion of the analytical axially-symmetric modes. In Sec. 4.2.2 we discuss the numerical evolution of the Sasaki-Nakamura field. This implementation uses the RG reconstructed modes of the MP for obtaining the retarded force, and we work in both the Ingoing and Outgoing RGs. The final value of the GSF will be obtained using the appropriate mode-sum formula. The completion, which is not obtained in the reconstruction procedure, is included in the LG as described in Sec. 4.2.4. Our numerical implementation will have a cut-off in the number of calculated modes, and the remaining modes are included by performing a fitting to a power series. In Sec. 4.2.5 we describe the details of this fitting for the force. The numerical results appearing in Sec. 4.3 include the convergence plots of the radial and temporal components of the SF, the convergence plot of the gauge-invariant quantity H (see Sec. 2.6). The sources of numerical errors are discussed in Sec. 4.3.1, and in Sec. 4.3.2 we show the convergence of the GSF. In Sec. 4.3.3 we show results for the energy-fluxes and H generated with our code, and demonstrate agreement with the literature. In Sec. 4.3.4 we make an asymptotic comparison of our GSF values with the corresponding LG values, and estimate the gain in efficiency of working in the RG against a LG implementation [36].

The homogeneous pieces that complete the reconstructed MP, namely the completion piece we mentioned before, can be included in the 'Boyer-Lindquist gauge'. This gauge corresponds to variations of the Kerr mass and AM in Boyer-Lindquist (BL) coordinates to other solutions with arbitrary amplitudes. In Chapter 5 we present a rigorous procedure to determine those amplitudes, based on imposing regularity of some gauge-invariant quantities off the particle. In Sec. 5.1 we give a derivation of those auxiliary invariants. We will briefly discuss other approaches to this problem: in Schwarzschild by Price [100], Keidl *et al.* [81] and by Dolan-Barack [49]. We also discuss a numerical method for a similar configuration, in Schwarzschild and Kerr, that was recently presented by Sano-Tagoshi [101, 102]. The analytical implementation of our method for circular orbits (around Schwarzschild and Kerr) will appear in detail in Sec. 5.2 and 5.3 respectively. In Sec. 5.4 we perform the appropriate modifications to our procedure, and consider eccentric-equatorial orbits around Kerr spacetime. This will be done analytically just like the circular-orbit cases. Our method is consistent in the case of circular orbits with the results of Dolan-Barack [49]. While logically this should appear before the formulation of Chapter 3, we address it last to avoid impacting the flow of the discussion.

Notation: Throughout this work we use geometrized units (with $G = c = 1$) and the metric signature $-+++$. For gauge transformations generated by a vector ξ^α, we use the sign convention $x^\alpha \to x^\alpha - \xi^\alpha$. Greek indices $\alpha,\ \beta,\ \gamma, ...$ run from 0 to 4. Lowercase Latin indices refer to spatial coordinates and run from 1 to 3, except briefly in Chapter 5. Uppercase Latin indices refer to the two angular coordinates $\{\theta, \varphi\}$. We denote the metric-compatible covariant derivatives by semicolon, and partial derivatives with a comma. Bold indices correspond to projections with respect to the Kinnersley tetrad $(\ell, n, m, \bar{m})$. We denote complex conjugation of any quantity a by $\bar{a}$ except for the metric perturbation h in Chapter 2, where $\bar{h}$ will denote the trace-reversed perturbation. Metric-compatible connections are calculated using $\Gamma^\alpha_{\ \beta\gamma} = \frac{g^{\alpha\delta}}{2}\left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}\right)$ and components of the Riemann tensor are calculated with $R^\alpha_{\ \beta\gamma\delta} = \Gamma^\alpha_{\ \beta\delta,\gamma} - \Gamma^\alpha_{\ \beta\gamma,\delta} + \Gamma^\alpha_{\ \sigma\gamma}\Gamma^\sigma_{\ \beta\delta} - \Gamma^\alpha_{\ \sigma\delta}\Gamma^\sigma_{\ \beta\gamma}$ according to the notation in [103].

Chapter 2

Self-force Preliminaries

This Chapter will introduce the concepts that are usually the starting point in the description of BH perturbation theory for EMRI modelling. In Sec. 2.1 we include a brief review of BH perturbation theory in the Newman-Penrose formalism, in which Teukolsky equation is formulated. As part of this review, in Sec. 2.1.4 we give a short discussion of the Sasaki-Nakamura transformation. This transformation allows a straightforward numerical evolution of Teukolsky equation. This Sasaki-Nakamura transformation will be relevant in Chapter 4, where we will use it to obtain the homogeneous solutions of Teukolsky equation. We describe in Sec. 2.1.5 the analytical method by Mano-Suzuki-Takasugi [104] which allows to compute highly accurate homogeneous solutions of Teukolsky equation. Even though we do not use this method in practice, we include it in this Chapter for completeness.

In Sec. 2.2, Eq. (2.41) corresponds to MiSaTaQuWa equation as it was derived using the method of matched asymptotic expansions of Appendix A. We will deviate from the original interpretation of the SF arising from a 'tail' field and rather adopt the more intuitive (effective) interpretation by Detweiler-Whiting [20] where the SF arises from certain regular piece of the MP.

In Sec. 2.3 we discuss the motion of a particle in the perturbed spacetime. This will be presented as a correction to the geodesic equation of the background spacetime. This deviation arises from the curvature of the background where the particle is embedded, and from a SF term.

Still in the context of the LG, we provide in Sec. 2.4 a brief review of the mode-sum formula. The LG mode-sum is (to date) the most successful regularization method to obtain the SF (see Table 1.1 in the previous Chapter). The explicit expressions for the regularization parameters in the LG are given in Appendix B.

In Sec. 2.5 we describe how the SF (and the motion driven by it) is a gauge-dependent quantity. We consider how moving away form the LG requires careful considerations in terms of the equation of motion. We consider three general classes of gauges: gauges related to the LG by a continuous gauge transformation; gauges related to LG by a discontinuous (but bounded at the particle's limit) transformation; and "parity-regular" gauges where the SF can be obtained using averaging procedures around the particle.

2.1 Gravitational perturbations of a Kerr Black-Hole

The study of gravitational perturbations around a Kerr BH can be done using Newman-Penrose formalism [105]. In particular we are interested in reconstructing the components of the MP in a RG h^{Rad} (See Eq. (2.18) in Sec. 2.1.3 below). Full derivations and detailed explanations can be found in Chandrasekhar's book [106].

In BL coordinates the line-element for the Kerr geometry is given by [103],

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\varphi + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{2.1}$$

where we have used

$$\Sigma \equiv r^2 + a^2\cos^2\theta, \quad \text{and} \quad \Delta \equiv r^2 + a^2 - 2Mr. \tag{2.2}$$

The event-horizon (EH) of the Kerr BH in these coordinates is at $r = r_+ \equiv M + \sqrt{M^2 - a^2}$ and the inner horizon is at $r = r_- \equiv M - \sqrt{M^2 - a^2}$.

The Kerr metric has two Killing-vector fields $\xi^\alpha_{(t)} = (\partial/\partial t)^\alpha$ and $\xi^\alpha_{(\varphi)} = (\partial/\partial\varphi)^\alpha$ and corresponding conserved quantities: the specific energy $\mathcal{E} = -\xi^\alpha_{(t)}u_\alpha = -u_t$ and specific azimuthal component of the AM $\mathcal{L} = \xi^\alpha_{(\varphi)}u_\alpha = u_\varphi$. Namely the Kerr BH is stationary and axially symmetric. The Kerr metric also admits a Killing tensor $Q^{\alpha\beta}$ with the Carter constant $\mathcal{Q} = Q^{\alpha\beta}u_\alpha u_\beta$ as associated conserved-quantity. Any orbit described by a test particle around a Kerr BH is fully specified by these three parameters up to initial phases. The quantity $\mathcal{Q}$ is related to $\mathcal{E}$ and $\mathcal{L}$ according to [107]

$$\mathcal{Q} = u_\theta^2 + \cos^2\theta\left[a^2(1 - \mathcal{E}^2 + \csc^2\theta\mathcal{L}^2)\right], \tag{2.3}$$

where θ is evaluated at the test particle. Eq. (2.3) vanishes for equatorial orbits where $\theta = \pi/2$ and $u_\theta = 0$.

2.1.1 Geodesic equations

A time-like test body of mass m in any spacetime will follow the geodesic equation $\mathsf{m}u^\alpha\nabla_\alpha u^\beta = 0$, see Fig. 2.1. In BL coordinates the components of the geodesic equation in Kerr are [107]

$$\Sigma^2\left(\frac{dr}{d\tau}\right)^2 = \left[\mathcal{E}(r^2+a^2) - a\mathcal{L}\right]^2 - \Delta\left[r^2 + (\mathcal{L} - a\mathcal{E})^2 + \mathcal{Q}\right] \equiv V_r, \tag{2.4a}$$

$$\Sigma^2\left(\frac{d\theta}{d\tau}\right)^2 = \mathcal{Q} - \mathcal{L}\cot^2\theta - a^2\cos^2\theta(1-\mathcal{E})^2 \equiv V_\theta, \tag{2.4b}$$

$$\Sigma\left(\frac{d\varphi}{d\tau}\right)^2 = \mathcal{L}\csc^2\theta + a\mathcal{E}\left(\frac{r^2+a^2}{\Delta} - 1\right) - \frac{a^2\mathcal{L}}{\Delta}, \tag{2.4c}$$

$$\Sigma\left(\frac{dt}{d\tau}\right)^2 = \mathcal{E}\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right] + a\mathcal{L}\left(1 - \frac{r^2+a^2}{\Delta}\right). \tag{2.4d}$$

where the roots of $V_r = 0$ and $V_\theta = 0$ give the turning points of the orbit. In Chapter 5 we use a more practical parametrization for eccentric-equatorial orbits.

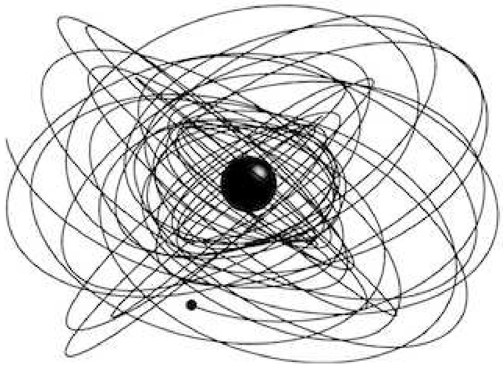

Figure 2.1: Example of an approximated geodesic orbit. A small BH follows the orbit as it falls into a supermassive BH. Credit: Drasco and Cutler [108]

2.1.2 Newman-Penrose formalism

Let us now summarise some useful results from the method developed by Newman and Penrose [105]. We will use this method as a starting point for the MP reconstruction.

The principal null-vectors are given in BL coordinates by

$$\ell^\alpha = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\alpha = \frac{1}{2\Sigma}\left(r^2+a^2, -a, 0, a\right), \tag{2.5a}$$

which are normalized as $\ell^\alpha n_\alpha = -1 = \ell_\alpha n^\alpha$ according to Kinnersley's choice [103]. The two remaining null-vectors of the tetrad are

$$m^\alpha = \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right) \qquad \text{and} \tag{2.6a}$$

$$\bar{m}^\alpha = \frac{-1}{\sqrt{2}(r-ia\cos\theta)}\left(ia\sin\theta, 0, -1, \frac{i}{\sin\theta}\right), \tag{2.6b}$$

with $m^\alpha \bar{m}_\alpha = 1 = m_\alpha \bar{m}^\alpha$. The null-vectors in Eqs. (2.5) and (2.6) also satisfy orthogonality relations: $\ell_\alpha m^\alpha = \ell_\alpha \bar{m}^\alpha = n_\alpha m^\alpha = n_\alpha \bar{m}^\alpha = 0$ and $\ell^\alpha \ell_\alpha = n^\alpha n_\alpha = m^\alpha m_\alpha = 0$. The corresponding directional derivatives are $\boldsymbol{D} = \ell^\mu \partial_\mu$, $\boldsymbol{\Delta} = n^\mu \partial_\mu$, $\boldsymbol{\delta} = m^\mu \partial_\mu$ and $\bar{\boldsymbol{\delta}} = \bar{m}^\mu \partial_\mu$. The non-vanishing spin-coefficients are

$$\varrho = -\frac{1}{r-ia\cos\theta}, \quad \beta = -\bar{\varrho}\frac{\cot\theta}{2\sqrt{2}}, \quad \pi = \frac{i}{\sqrt{2}}a\varrho^2\sin\theta, \quad \tau = -\frac{i}{\sqrt{2}}a\varrho\bar{\varrho}\sin\theta,$$
$$\mu = \frac{1}{2}\varrho^2\bar{\varrho}\Delta, \quad \gamma = \mu + \frac{1}{2}\varrho\bar{\varrho}(r-M), \quad \alpha = \pi - \bar{\beta}, \tag{2.7}$$

where $\Sigma = (\varrho\bar{\varrho})^{-1}$.

The ten independent components of the traceless part of the Riemann tensor are encoded in five complex curvature-scalars or Weyl scalars. These scalars are given in terms of the components of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ as [103]

$$
\begin{aligned}
\psi_0 &= -\, C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \ell^\gamma m^\delta, &(2.8a)\\
\psi_1 &= -\, C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \ell^\gamma n^\delta, &(2.8b)\\
\psi_2 &= -\, C_{\alpha\beta\gamma\delta}\, \ell^\alpha m^\beta \bar{m}^\gamma n^\delta, &(2.8c)\\
\psi_3 &= -\, C_{\alpha\beta\gamma\delta}\, \ell^\alpha n^\beta \bar{m}^\gamma n^\delta, &(2.8d)\\
\psi_4 &= -\, C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, &(2.8e)
\end{aligned}
$$

and asymptotically behave as $\psi_i = O(r^{-5+i})$ with $i = 0, 1, ..., 4$ for outgoing waves [105]. For a Kerr BH the curvature scalars have values $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$, and $\psi_2 = M\varrho^3$.

The presence of the orbiting particle will produce a perturbation $\delta\psi_i$, for $i = 0, 1, ..., 4$ as before. In general we write $\psi_i = \psi_i^{(0)} + \delta\psi_i$. In the full EMRI system ψ_0, and ψ_4 correspond to the perturbations themselves. The two scalars ψ_1 and ψ_3 are gauge dependent, and one can always use the gauge freedom to also set them to zero. This leaves only $\psi_2^{(0)} = M\varrho^3$ in the background, and the perturbation is denoted by $\delta\psi_2$.

The first-order perturbations ψ_0 and ψ_4 can be obtained by solving the inhomogeneous Teukolsky equation [109] sourced by T_s [explicitly given below in Eq. (2.16)]. For a particular spin-value s the equation is given by

$$
\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi_s}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi_s}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi_s}{\partial\varphi^2}\\
&-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\psi_s}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_s}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi_s}{\partial\varphi}\\
&-2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi_s}{\partial t} + (s^2\cot^2\theta - s)\psi_s = 4\pi\Sigma T_s, \qquad (2.9)
\end{aligned}
$$

which was shown to be separable (also by Teukolsky [109]), if we write a decomposition of the form

$$
\psi_s = \sum_{\ell m} e^{i(m\varphi-\omega t)} R_{s\ell m}(r)\, {}_sS_{\ell m}(\theta). \qquad (2.10)
$$

The relevant gravitational perturbations are given by $\psi_{s=-2}(r) \equiv \varrho^{-4}\psi_4(r)$ and $\psi_{s=2}(r) \equiv \psi_0(r)$. In the circular-equatorial orbits case (that will concern us in Chapter 4) we have $\omega \equiv m\Omega$, where Ω corresponds to the angular frequency. This frequency is calculated in terms of components of the four-velocity by

$$
\Omega \equiv \frac{u^\varphi}{u^t} = \frac{M^{1/2}}{r_0^{3/2} + aM^{1/2}}. \qquad (2.11)
$$

In vacuum ($T_s = 0$) Eq. (2.9) separates into

$$
\begin{aligned}
\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR_s}{dr}\right) &+ \big\{\big[(r^2+a^2)^2\omega^2 - 4aMr\omega m + a^2m^2 + 2ia(r-M)ms\\
&-2iM(r^2-a^2)\omega s\big]\Delta^{-1} + 2ir\omega s - \Lambda_{\ell m} - a^2\omega^2\big\}R_s = 0, \qquad (2.12a)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) &+ \bigg(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta}\\
&-s^2\cot^2\theta + \Lambda_{\ell m} - s^2\bigg)S = 0. \qquad (2.12b)
\end{aligned}
$$

We have omitted the arguments and harmonic indices ℓm of $R_s(r)$ and $S(\theta)$ in Eq. (2.12) for simplicity. When $a \neq 0$ the eigenvalues of the spheroidal harmonics are given by $\Lambda_{\ell m} = \lambda_s + 2am\omega - a^2\omega^2 + s + s^2$, with $\lambda_s = (\ell - s)(\ell + s + 1)$. When $a = 0$, $\Lambda_{\ell m}$ reduces to $\ell(\ell+1)$ and the eigenfunctions correspond to the spin-weighted spherical harmonics ${}_sS_{\ell m}(\theta)e^{im\varphi} = {}_sY_{\ell m}(\theta, \varphi)$ (see Appendix E for further details).

Explicitly for $s = 2, -2$ we get equations for $R_{s=2}(r) \equiv R_0(r)$ and $R_{s=-2}(r) \equiv R_4(r)$:

$$\Delta^{-2}\frac{d}{dr}\left(\Delta^3\frac{dR_0}{dr}\right) + V_2(r)R_0 = -\,4\pi T_2, \tag{2.13a}$$

$$\Delta^{2}\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_4}{dr}\right) + V_{-2}(r)R_4 = -\,4\pi T_{-2}, \tag{2.13b}$$

where the potential $V_s(r)$ is read from Eq. (2.12a):

$$V_s(r) = \frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda_s, \qquad \text{with,} \tag{2.14a}$$

$$K(r) = (r^2 + a^2)\omega - ma. \tag{2.14b}$$

The stress-energy tensor for the orbiting particle is modelled as a δ-function distribution along the worldline. It can be explicitly written as

$$\begin{aligned} T^{\alpha\beta} =& \mathsf{m}\int u^\alpha u^\beta \delta^{(4)}(x^\alpha - x_0^\alpha(\tau))(-g)^{-1/2}d\tau \\ =& \mathsf{m}\int \frac{u^\alpha u^\beta}{\Sigma\sin\theta}\delta(r - r_0(\tau))\delta(\cos\theta - \cos\theta_0(\tau))\delta(\varphi - \varphi_0(\tau))\delta(t - t_0(\tau))d\tau \\ =& \frac{\mathsf{m}}{u^t r_0^2}u^\alpha u^\beta \delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\varphi - \varphi_0), \end{aligned} \tag{2.15}$$

where m is located at the point x_0 with coordinates $(r_0, t_0, \theta_0, \varphi_0)$, and $(-g)^{1/2} = \Sigma\sin\theta$. The third line is obtained by changing the integration variable from τ to t and integrating in a time-t interval containing t_0. The source T_s can be obtained in terms of the components of $T^{\alpha\beta}$ projected along the Newman-Penrose tetrad. Explicitly for $s = -2, 2$ (which are the values of s relevant to our work) we have [106]

$$\begin{aligned} T_{-2} =& 2\varrho^{-4}\left\{(\boldsymbol{\Delta} + 3\gamma - \bar\gamma + 4\mu + \bar\mu)\left[(\bar{\boldsymbol{\delta}} - 2\bar\tau + 2\alpha)T_{\boldsymbol{24}} - (\boldsymbol{\Delta} + 2\gamma - 2\bar\gamma + \bar\mu)T_{\boldsymbol{44}}\right]\right. \\ &\left. + (\bar{\boldsymbol{\delta}} - \bar\tau + \bar\beta + 3\alpha + 4\pi)\left[(\boldsymbol{\Delta} + 2\gamma + 2\bar\mu)T_{\boldsymbol{24}} - (\bar{\boldsymbol{\delta}} - \bar\tau + 2\bar\beta + 2\alpha)T_{\boldsymbol{22}}\right]\right\}, \end{aligned} \tag{2.16a}$$

$$\begin{aligned} T_2 =& 2\left\{(\boldsymbol{\delta} + \bar\pi - \bar\alpha - 3\beta - 4\tau)\left[(\boldsymbol{D} - 2\epsilon - 2\bar\varrho)T_{\boldsymbol{13}} - (\boldsymbol{\delta} + \bar\pi - 2\bar\alpha - 2\beta)T_{\boldsymbol{11}}\right]\right. \\ &\left. + (\boldsymbol{D} - 3\epsilon + \bar\epsilon - 4\varrho - \bar\varrho)\left[(\boldsymbol{\delta} + 2\bar\pi - 2\beta)T_{\boldsymbol{13}} - (\boldsymbol{D} - 2\epsilon + 2\bar\epsilon - \bar\varrho)T_{\boldsymbol{33}}\right]\right\}, \end{aligned} \tag{2.16b}$$

where the projections of the stress-energy tensor are given by:

$$T_{\boldsymbol{ab}} \equiv e^\alpha_{\boldsymbol{a}} e^\beta_{\boldsymbol{b}} T_{\alpha\beta}, \tag{2.17}$$

with $e^\alpha_{\boldsymbol{a}} = \{\ell^\alpha, n^\alpha, m^\alpha, \bar m^\alpha\}$.

2.1.3 CCK metric reconstruction procedure

The RGs are given by the conditions

$$\ell^\beta h^{\rm IRG}_{\alpha\beta} = 0 = g^{\alpha\beta} h^{\rm IRG}_{\alpha\beta}, \qquad \text{for the ingoing radiation-gauge, and} \tag{2.18a}$$

$$n^\beta h^{\rm ORG}_{\alpha\beta} = 0 = g^{\alpha\beta} h^{\rm ORG}_{\alpha\beta}, \qquad \text{for the outgoing radiation-gauge,} \tag{2.18b}$$

where $g_{\alpha\beta}$ is the metric of the background spacetime and $h_{\alpha\beta}$ corresponds to the MP. The second equality corresponds to the extra requirements of the RGs to be trace-free.

The procedure to obtain the *vacuum* perturbation in the RG starting from the curvature scalars ψ_0 and ψ_4 was first proposed by Cohen and Kegeles for the electromagnetic case [110], and soon after generalized to the gravitational case by Chrzanowski [93] and independently by Cohen-Kegeles [111]. The CCK reconstruction (after the names of the authors) can be computed from the expressions

$$\begin{aligned} h^{\rm ORG}_{\alpha\beta} = & -\varrho^{-4}\left\{n_\alpha n_\beta\left(\bar{\boldsymbol{\delta}} - 3\alpha - \bar\beta + 5\pi\right)\left(\bar{\boldsymbol{\delta}} - 4\alpha + \pi\right) + \bar m_\alpha \bar m_\beta\left(\boldsymbol{\Delta} + 5\mu - 3\gamma + \bar\gamma\right)\left(\Delta + \mu - 4\gamma\right)\right. \\ & -n_{(\alpha}\bar m_{\beta)}\left[\left(\bar{\boldsymbol{\delta}} - 3\alpha + \bar\beta + 5\pi + \bar\tau\right)\left(\boldsymbol{\Delta} + \mu - 4\gamma\right)\right. \\ & \left.\left. + \left(\boldsymbol{\Delta} + 5\mu - \bar\mu - 3\gamma - \bar\gamma\right)\left(\bar{\boldsymbol{\delta}} - 4\alpha + \pi\right)\right]\right\}\Psi^{\rm ORG} + c.c., \end{aligned} \tag{2.19a}$$

$$\begin{aligned} h^{\rm IRG}_{\alpha\beta} = & -\left\{\ell_\alpha \ell_\beta\left(\bar{\boldsymbol{\delta}} + \alpha + 3\bar\beta - \bar\tau\right)\left(\bar{\boldsymbol{\delta}} + 4\bar\beta + 3\bar\tau\right) + \bar m_\alpha \bar m_\beta\left(\boldsymbol{D} - \bar\varrho\right)\left(\Delta + 3\bar\varrho\right)\right. \\ & \left. -\ell_{(\alpha}\bar m_{\beta)}\left[\left(\bar{\boldsymbol{\delta}} - \alpha + 3\bar\beta - \pi - \bar\tau\right)\left(\boldsymbol{D} + 3\bar\varrho\right) + \left(\boldsymbol{D} + \varrho - \bar\varrho\right)\left(\bar{\boldsymbol{\delta}} + 4\bar\beta + 3\bar\tau\right)\right]\right\}\bar\Psi^{\rm IRG} + c.c., \end{aligned} \tag{2.19b}$$

where Ψ is the appropriate 'Hertz potential'. The Hertz potential can be obtained from the Weyl scalars according to [112]

$$8\psi_0 = \mathfrak{L}^4\bar\Psi^{\rm ORG} + 12\varrho^{-3}\psi_2^{(0)}\partial_t\Psi^{\rm ORG}, \qquad 32\varrho^{-4}\psi_4 = \boldsymbol{\Delta}^2\tilde{\boldsymbol{D}}^4\boldsymbol{\Delta}^2\bar\Psi^{\rm ORG}, \tag{2.20a}$$

$$8\varrho^{-4}\psi_4 = \mathfrak{L}^4\bar\Psi^{\rm IRG} - 12\varrho^{-3}\psi_2^{(0)}\partial_t\Psi^{\rm IRG}, \qquad 2\psi_0 = \boldsymbol{D}^4\bar\Psi^{\rm IRG}, \tag{2.20b}$$

with $\mathfrak{L} = -\left[\partial_\theta - s\cot\theta + i\csc\theta\partial_\varphi\right] - ia\sin\theta\partial_t$ and

$$\tilde D \equiv -\frac{2\Delta}{\Sigma}\boldsymbol{\Delta} = -\frac{r^2+a^2}{\Delta}\partial_t + \partial_t - \frac{a}{\Delta}\partial_\varphi. \tag{2.21}$$

In the vacuum region, Ψ satisfies the homogeneous Teukolsky equation with the opposite spin of that of the Weyl scalar from which it is obtained. This implies that Ψ can be also decomposed into harmonics as

$$\Psi_s = \sum_{\ell m}^{\infty} \tilde R_{s\,\ell m}(r)\,{}_sS_{\ell m}(\theta)e^{i(m\varphi - \omega t)}, \tag{2.22}$$

where $\tilde R_{s\,\ell m}(r)$ and ${}_sS_{\ell m}(\theta)$ are solutions of Eq. (2.12) as before. The function $\tilde R_{s\,\ell m}(r)$ can be obtained by inverting Eq. (2.20) mode by mode. We also require the Teukolsky–Starobinsky identity[106]:

$$\mathfrak{L}^4\,{}_2S_{\ell m} = D\,{}_{-2}S_{\ell m}, \tag{2.23}$$

where $D^2 = \lambda_{Ch}^2(\lambda_{Ch}+2)^2 + 8a\omega(m-a\omega)\lambda_{Ch}(5\lambda_{Ch}+6) + 48a^2\omega^2[2\lambda_{Ch}+3(m-a\omega)^2]$. The eigenvalue λ_{Ch} used by Chandrasekhar in [106] is related to the one we used in Eq. (2.12) by $\lambda_{Ch} = \lambda_s + s + 2$.

2.1.4 Sasaki–Nakamura transformation

In principle one should be able to integrate the homogeneous Teukolsky equation from the horizon (and from infinity). This approach does not work well in numerical implementations. The fundamental reason for this difficulty is that the Teukolsky potential is long-ranged, and the asymptotic form of its homogeneous solutions are ill-behaved. This long-range potential makes it difficult to properly set the phases of the asymptotic solutions [113]. The outgoing piece of the solution grows $\sim r^4$ times the ingoing piece as $r \to \infty$ in BL coordinates. The latter is easily lost in numerical calculations.

To overcome this difficulty, Sasaki and Nakamura [114] introduced a new variable to obtain Teukolsky's radial function $R_4(r)$. The so-called Sasaki-Nakamura function $X(r)$ is governed by an equation with a short-ranged potential. The physical solutions of the Sasaki-Nakamura equation have desired asymptotic behaviours at horizon and infinity [106]. The Sasaki-Nakamura equation in Kerr is

$$\frac{d^2X}{dr_*^2} - F(r)\frac{dX}{dr_*} - U(r)X = 0. \tag{2.24}$$

The tortoise coordinate is defined as

$$\frac{dr_*}{dr} \equiv \frac{r^2+a^2}{\Delta}, \tag{2.25}$$

and it is explicitly given in Kerr by

$$r_*(r) = r + \frac{2Mr_+}{r_+ - r_-}\ln\frac{r-r_+}{2M} - \frac{2Mr_-}{r_+ - r_-}\ln\frac{r-r_-}{2M}. \tag{2.26}$$

The functions $F(r)$ and $U(r)$ of Eq. (2.24) are given [114] by

$$F(r) = \frac{1}{\eta}\frac{d\eta}{dr}\frac{\Delta}{r^2+a^2}, \tag{2.27a}$$

$$U(r) = \frac{\Delta U_1(r)}{(r^2+a^2)^2} + G(r)^2 + \frac{\Delta}{r^2+a^2}\frac{dG}{dr} - F(r)G(r), \tag{2.27b}$$

with

$$\eta(r) = c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} + \frac{c_4}{r^4}, \tag{2.28}$$

and

$$G(r) = -\frac{2(r-M)}{r^2+a^2} + \frac{r\Delta}{(r^2+a^2)^2}, \tag{2.29a}$$

$$U_1(r) = V_{-2}(r) + \frac{\Delta^2}{\beta}\left[\frac{d}{dr}\left(2\alpha + \frac{1}{\Delta}\frac{d\beta}{dr}\right) - \frac{1}{\eta}\frac{d\eta}{dr}\left(\alpha + \frac{1}{\Delta}\frac{d\beta}{dr}\right)\right], \tag{2.29b}$$

$$\alpha = -iK\frac{\beta}{\Delta^2} + 3i\frac{dK}{dr} + \lambda_s + 6\frac{\Delta}{r^2}, \tag{2.29c}$$

$$\beta = 2\Delta\left[-iK + r - M - 2\frac{\Delta}{r}\right], \tag{2.29d}$$

where the functions $K(r)$ and $V_{-2}(r)$ are taken from the Teukolsky equation (2.13a) given above, and λ_s is the same as that in Eq. (2.12b). The coefficients of η are

$$c_0 = -12i\omega M + \lambda_s(\lambda_s+2) - 12a\omega(a\omega - m), \tag{2.30a}$$

$$c_1 = 8ia[3a\omega - \lambda_s(a\omega - m)], \tag{2.30b}$$

$$c_2 = -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \tag{2.30c}$$

$$c_3 = 24ia^3(a\omega - m) - 24Ma^2, \tag{2.30d}$$

$$c_4 = 12a^4. \tag{2.30e}$$

The explicit transformation that allows to calculate the radial function $R_4(r)$ that satisfies Teukolsky equation in terms of the Sasaki-Nakamura field $X(r)$ is

$$R_4(r) = \frac{1}{\eta}\left[\left(\alpha + \frac{\beta'}{\Delta}\right)\chi - \frac{\beta}{\Delta}\chi'\right], \tag{2.31}$$

where the prime denotes derivative with respect of r, and $\chi \equiv X(r)\Delta/\sqrt{r^2+a^2}$.

2.1.5 MST (Mano-Suzuki-Takasugi) method

Analytical solutions to the radial part of the homogeneous Teukolsky equation were given by Mano, Suzuki and Tagoshi, usually referred simply as *MST method* [104, 115]. These solutions are written as an infinite series of known hypergeometric functions: the ingoing solution R_H (which is regular at the EH) is written as a sum over hypergeometric functions ${}_2F_1$ and the outgoing solution R_∞ (regular at infinity) is written as a sum over (Tricomi's) confluent-hypergeometric functions $\mathcal{U}$,

$$\begin{aligned} R_H =& A_s e^{i\epsilon\kappa x}(-x)^{-2-i\epsilon_+}(1-x)^{i\epsilon_-}\sum_{n=-\infty}^{n=\infty} a_n^\nu(s)\,{}_2F_1\left(n+\nu+1-i\tau, -n-\nu-i\tau; 1-s-2i\epsilon_+; x\right), \\ R_\infty =& 2^\nu e^{-\pi\epsilon}e^{-i\pi(\nu+1+s)}e^{iz}z^{\nu+i\epsilon_+}(z-\epsilon\kappa)^{-s-i\epsilon_+}\times \\ & \sum_{n=-\infty}^{n=\infty} i^n\frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n}(2z)^n a_n^\nu(s)\mathcal{U}(n+\nu+1+s-i\epsilon, 2n+2\nu+2; -2iz), \end{aligned} \tag{2.32}$$

where the different variables are defined as

$$\begin{aligned} x = -\frac{\omega}{\epsilon\kappa}(r-r_+), \quad \epsilon = 2M\omega, \quad z =& \epsilon\kappa(1-x) = \epsilon\kappa\tilde{x}, \quad \epsilon_\pm = \frac{\epsilon\pm\tau}{2}, \\ \kappa = \sqrt{1-\tilde{a}^2}, \quad \tau =& \frac{\epsilon - m\tilde{a}}{\kappa}, \quad \tilde{a} = \frac{a}{M}. \end{aligned} \tag{2.33}$$

The parameter ν (*renormalized AM*) has the low frequency limit $\nu \to \ell$ as $\epsilon \to 0$. In general ν is determined by solving the condition

$$R_n(\nu)L_{n-1}(\nu) = 1, \tag{2.34}$$

where $R_n(\nu)$ and $L_n(\nu)$ are the continued fractions defined by

$$\begin{aligned} R_n(\nu) =& \frac{a_n^\nu(s)}{a_{n-1}^\nu(s)} = -\frac{\gamma_n^\nu(s)}{\beta_n^\nu(s) + \alpha_n^\nu(s)R_{n+1}(\nu)}, \\ L_n(\nu) =& \frac{a_n^\nu(s)}{a_{n+1}^\nu(s)} = -\frac{\alpha_n^\nu(s)}{\beta_n^\nu(s) + \gamma_n^\nu(s)L_{n-1}(\nu)}. \end{aligned} \tag{2.35}$$

The coefficients α_ν^n, β_ν^n and γ_ν^n are given [116] by

$$\begin{aligned} \alpha_\nu^n =& \frac{i\epsilon\kappa(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)}, \\ \beta_\nu^n =& -\lambda_s - s(s+1) + (n+\nu)(n+\nu+1) + \epsilon^2 + \epsilon(\epsilon - m\tilde{a}) + \frac{\epsilon(\epsilon-m\tilde{a})(s^2+\epsilon^2)}{(n+\nu)(n+\nu+1)}, \end{aligned}$$

$$\gamma_\nu^n = -\frac{i\epsilon\kappa(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}. \tag{2.36}$$

The normalization coefficients A_s in Eq. (2.32) are

$$A_{-s} = 1, \quad A_s = \bar{C}_s \left(\frac{\omega}{\epsilon\kappa}\right)^{2s} \frac{\Gamma(1+s-2i\epsilon_+)}{\Gamma(1-s-2i\epsilon_+)} \left|\frac{\Gamma(\nu+1-s-i\epsilon)}{\Gamma(\nu+1-s+i\epsilon)}\right|^2, \tag{2.37}$$

with the relevant Starobinsky constants C_s (with $s=2$):

$$\begin{aligned} |C_2| =& (Q_2^2 + 4a\omega m - 4a^2\omega^2)\left[(Q_2-2)^2 + 36a\omega m - 36a^2\omega\right] \\ &+ (2Q_2-1)(96a^2\omega^2 - 48a\omega m) + 144\omega^2(M^2-a^2), \quad \text{with} \quad Q_s = \Lambda_{\ell m} + a^2\omega^2 - 2a\omega m. \end{aligned} \tag{2.38}$$

2.1.6 Completed Radiation Gauges

The CCK-reconstruction procedure described in Sec. 2.1.3 starts at the $\ell = 2$ spin-weighted harmonic mode, since the spin-weighted spherical harmonics are not defined when $\ell < |s|$. The Weyl scalars (ψ_0 or ψ_4) required in this procedure give the full gauge-invariant information about the radiative content of the solution. For vacuum perturbations Wald showed [98] that the remaining contributions correspond to perturbations of the mass and AM, gauge perturbations and perturbations of the Kerr metric to other solutions, namely to C-metric and Kerr-NUT solutions. The mass and AM perturbations of Schwarzschild were studied (in the LG) by Detweiler and Poisson [117], who showed the importance of these contributions in the context of BH perturbation theory and SF calculations.

Thus the full MP in a RG has two pieces: a perturbation constructed using the CCK-reconstruction procedure together with a 'completion' piece. We write this *completed radiation-gauge* as

$$h_{\alpha\beta}^{\mathrm{Rad}} = h_{\alpha\beta}^{(\mathrm{rec})} + h_{\alpha\beta}^{(\mathrm{comp})}, \tag{2.39}$$

where $h_{\alpha\beta}^{(\mathrm{rec})}$ is the CCK-reconstructed piece (given in the IRG or ORG) and $h_{\alpha\beta}^{(\mathrm{comp})}$ is whatever is required so that $h_{\alpha\beta}^{\mathrm{Rad}}$ satisfies the linearised EFE. The completion piece in Schwarzschild can be obtained by solving the $\ell = 0, 1$ modes of the EFE [83]. The non-separability of the EFE in Kerr makes the problem much more difficult, which we address in Chapter 5.

2.2 The MiSaTaQuWa formula and Detweiler-Whiting reformulation

Consider a point-like particle of mass m moving on the geometry of a Kerr BH with metric $g_{\alpha\beta}$. We recall that in general the concept of point-particle is not suitable in the context of GR. However through the method of matched asymptotic expansions — as discussed in Appendix A— it has been shown [14, 15] that the particle is described by the usual delta-function distribution within linear perturbation-theory. The Kerr BH is characterized by its mass M and spin parameter a, and we consider $\mathsf{m} \ll M$. Due to the finite mass of the particle, the motion will not be geodesic in g. Let $h_{\alpha\beta}^{\mathrm{Lor}}$ represent the MP due to m in the LG. $h_{\alpha\beta}^{\mathrm{Lor}}$ satisfies the gauge condition

$$g^{\beta\gamma} h_{\alpha\beta;\gamma}^{\mathrm{Lor}} = \frac{1}{2} g^{\beta\gamma} h_{\beta\gamma;\alpha}^{\mathrm{Lor}}. \tag{2.40}$$

In what follows, for simplicity, we will refer to the LG perturbation as just $h_{\alpha\beta}$. The distinction from other gauges will only be required later in Sec. 2.5.

Assume that the trajectory of the particle in $\mathsf{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$ is given by $x_0^\alpha(\lambda)$, where λ is an arbitrary parameter. Given a choice of a coordinate system in g we can project the worldline $x_0^\alpha(\lambda)$ onto the background g on the basis of "same coordinates values" (we assume that the coordinates in the two spacetimes would be the same in the limit $\mathsf{m} \to 0$ where the motion is geodesic). This projection defines an accelerated worldline on the background, and we interpret such acceleration as being caused by a GSF $F^\alpha_{\rm self}$. We denote by τ the proper-time along this worldline and the four-velocity of the particle is given by $u^\alpha \equiv dx_0^\alpha/d\tau$.

According to the MiSaTaQuWa formula [14, 15] the GSF at a given point x_0^α can be calculated from the tail field $h^{\rm tail}_{\alpha\beta}$. This tail arises from waves being scattered due to the spacetime-curvature and interacting with the field in later times. The tail field is continuous and differentiable everywhere, even on the worldline, but it is not a smooth function on the worldline: generally it is not twice differentiable. The MiSaTaQuWa equation, as derived using the method of matched asymptotic expansions of Appendix A, takes the form

$$F^\alpha_{\rm self}(x_0) = \lim_{x \to x_0} \mathsf{m} \nabla^{\alpha\beta\gamma} \bar{h}^{\rm tail}_{\beta\gamma}(x). \tag{2.41}$$

Here $\bar{h}_{\alpha\beta}$ is the trace-reversed MP [1] given by

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} h_{\mu\nu}. \tag{2.42}$$

The operator $\nabla^{\alpha\beta\gamma}$ in Eq. (2.41) is the "force" operator [97]. The explicit form of the force operator arises from considering the difference between the trajectory in a perturbed spacetime and the one in the background spacetime, where the particle experiences an external force perpendicular to its velocity. The perturbation can be *any* smooth external weak gravitational-perturbation, and produces a fictitious 'gravitational' force $F^\alpha_{\rm grav}$. When the perturbation h is produced by the test particle, $F^\alpha_{\rm grav}$ corresponds to the SF. This difference between the accelerations in the two spacetimes can be expressed according to [19]

$$F^\alpha_{\rm grav} \equiv -\mathsf{m}(\delta^\alpha_\lambda + u^\alpha u_\lambda) \Delta\Gamma^\lambda_{\mu\nu} u^\mu u^\nu, \tag{2.43}$$

where $\Delta\Gamma^\lambda_{\mu\nu} \equiv \Gamma'^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}$ is the difference of the connections compatible with the perturbed metric (Γ') and the background metric (Γ). $\Delta\Gamma^\lambda_{\mu\nu}$ can be written in terms of h as

$$\Delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(h_{\alpha\mu;\nu} + h_{\alpha\nu;\mu} - h_{\mu\nu;\alpha} \right). \tag{2.44}$$

Explicitly, the force operator is given in terms of the metric tensor $g_{\alpha\beta}$, the metric-compatible covariant derivative ∇_α and u^α by

$$\nabla^{\alpha\beta\gamma} = \frac{1}{4} \left(2g^{\alpha\delta} u^\beta u^\gamma - 4g^{\alpha\beta} u^\gamma u^\delta - 2u^\alpha u^\beta u^\gamma u^\delta + u^\alpha g^{\beta\gamma} u^\delta + g^{\alpha\delta} g^{\beta\gamma} \right) \nabla_\delta, \tag{2.45}$$

where u^α is a smooth extension of the four velocity in the neighbourhood of the worldline. A useful way to calculate $F_{\rm grav}$ (which we will use in Chapters 3 and 4) is to substitute Eqs. (2.45) and (2.42)

[1]Here and for any MP $\bar{h}$ refers to trace-reversed fields, *not to complex conjugation.*

in Eq. (2.41) to obtain

$$\begin{aligned} F^{\alpha}_{\text{grav}}(x) &= -\frac{\mathsf{m}}{2}(g^{\alpha\beta} + u^{\alpha}u^{\beta})\left[\nabla_{\mu}h_{\nu\beta}(x) + \nabla_{\nu}h_{\mu\beta}(x) - \nabla_{\beta}h_{\mu\nu}(x)\right]u^{\mu}u^{\nu} \\ &= -\,\mathsf{m}P^{\alpha\beta}\left[\nabla_{\mu}h_{\nu\beta}(x) - \frac{1}{2}\nabla_{\beta}h_{\mu\nu}(x)\right]u^{\mu}u^{\nu}. \end{aligned} \tag{2.46}$$

where $P^{\alpha\beta} \equiv g^{\alpha\beta} + u^{\alpha}u^{\beta}$. When the $h_{\alpha\beta}$ in Eq. 2.46 is replaced by the tail part of the LG perturbation F^{α}_{grav} gives the SF.

Let us consider the retarded perturbation $h^{(\text{ret})}_{\alpha\beta}$, which satisfies the inhomogeneous EFE. Detweiler and Whiting showed that the same physical SF— as the one obtained from the tail perturbation— can be obtained in terms of a *regular* (smooth) field $\bar{h}^{\text{R}}_{\alpha\beta}$, which is a solution of the vacuum Einstein equation [118]. This regular field is related to the retarded perturbation as

$$\bar{h}^{(\text{ret})}_{\alpha\beta} = \bar{h}^{\text{S}}_{\alpha\beta} + \bar{h}^{\text{R}}_{\alpha\beta}, \tag{2.47}$$

where $\bar{h}^{\text{S}}_{\alpha\beta}$ is certain locally-defined singular piece of the retarded field near the location of the particle [the leading-order expression for the S part of the MP in the Lorenz gauge will be given in Fermi-like coordinates explicitly in Eq. (3.8)]. The Detweiler-Whiting singular field at a point x^{α}, off the worldline, depends only on the points of the worldline that are space- and null-like separated. The regular field depends an all the points on the worldline up to the advanced time v, as shown in Fig. 2.2.

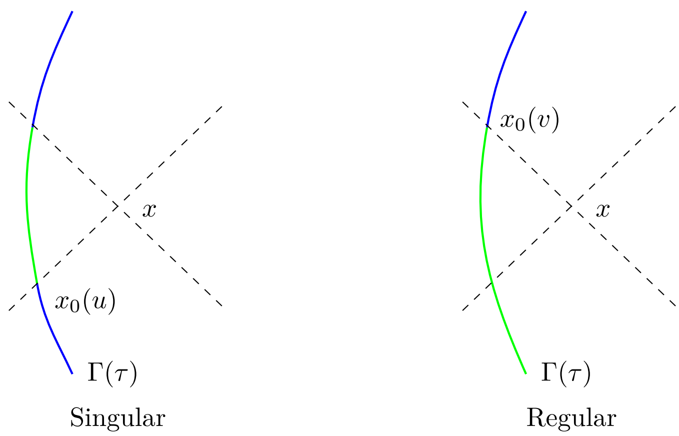

Figure 2.2: In green we show the region of the worldline which supports the Green's function of a point x. The worldline $\Gamma(\tau)$ appears in blue. In curved spacetime the singular field depends on the history of the particle in the interval $u \le \tau \le v$; the regular field depends only on the interval $-\infty < \tau < v$. u and v are the retarded and advanced times respectively. The regular field is only causal at coincidence.

The singular part of the MP does not contribute to the value of the SF [118]. We can then calculate the SF as

$$F^{\alpha}_{\text{self}}(x_0) = \mathsf{m}\nabla^{\alpha\beta\gamma}\bar{h}^{\text{R}}_{\beta\gamma}(x_0). \tag{2.48}$$

We comment that the fact that the particle will move along a geodesic of the spacetime $\mathsf{g}_{\alpha\beta} = g_{\alpha\beta} + h^{\text{R}}_{\alpha\beta}$ is not enough to give any physical "substance" to the R-field of the particle, and it should be understood as an *effective* field; the physical perturbation is still given by $h^{(\text{ret})}_{\alpha\beta}$.

2.3 Equation of motion in the Lorenz-gauge

Defining the position of the particle requires more than just giving the value of the SF. The SF is gauge dependent, and any expression of it must be accompanied by the information about the gauge to which it corresponds.

Let us consider that the point-mass m produces a perturbation $\varepsilon h_{\mu\nu} + O(\varepsilon^2)$, where $\varepsilon \equiv 1$ is used to count powers of m, in the background spacetime $g_{\mu\nu}$. We write the object's worldline as the perturbative expansion[2]

$$z^\mu(\tau, \varepsilon) = z_0^\mu(\tau) + \varepsilon z_1^\mu(\tau) + O(\varepsilon^2). \tag{2.49}$$

The leading term $z_0^\mu(\tau)$ is the coordinate description of a geodesic Γ of the background spacetime. The term, z_1^μ, is a vector field defined on Γ and describes the first-order deviation of the object's centre of mass from the worldline, where the centre of mass is defined by the object's mass dipole moment in a locally inertial frame centred on Γ, see Fig. 2.3. This first-order correction is [16]

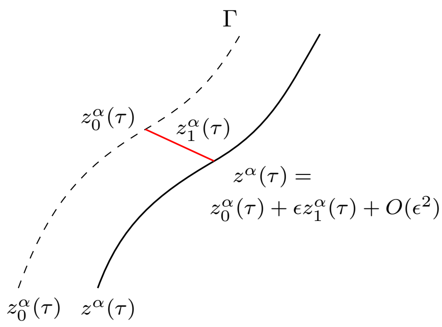

Figure 2.3: Perturbative treatment of the worldline Γ. The vector field $z_1^\alpha(\tau)$ describes the first order deviation of the object's centre of mass. We neglect the contributions of terms $O(\epsilon^2)$.

$$\mathsf{m}\frac{D^2 z^\alpha_{1\mathrm{Lor}}}{d\tau^2} = -\mathsf{m} R^\alpha{}_{\mu\beta\nu} u^\mu z^\beta_{1\mathrm{Lor}} u^\nu + F^\alpha_{\mathrm{self}}, \tag{2.50}$$

where $F^\alpha_{\mathrm{self}} \propto \mathsf{m}^2$ is the SF in the LG produced by the MP of the point-mass moving along Γ. In addition to the SF, the equation of motion contains the term $-R^\alpha{}_{\mu\beta\nu} u^\mu z^\beta_{1\mathrm{Lor}} u^\nu$, which is purely a background effect. This term describes the fact that if F^α_{self} forces the small object slightly off Γ, the object continues to move relative to Γ due to the background curvature.

The SF in the LG can be written in several (equivalent) forms: an alternative form to Eqs. (2.41) and (2.48) is that by Gralla. Gralla, based on the work by Quinn and Wald [15], showed that the tail formula of Eq. (2.41) can be written in terms of a spatial average [123], which we shall refer as the Quinn-Wald-Gralla form

$$F^\alpha_{\mathrm{self}} = \lim_{s\to 0} \frac{1}{4\pi s^2} \int F^{\alpha\,\mathrm{Lor}}_{(\mathrm{ret})} dS. \tag{2.51}$$

The integral in Eq. (2.51) is taken over a small two-sphere centred on the worldline Γ with a constant geodesic radius s perpendicular to Γ, $dS = s^2 d\Omega$ is the surface element on that sphere and $d\Omega \equiv \sin\theta d\theta d\phi$ corresponds to the surface element on a unit sphere. The angles (θ, ϕ) on the unitary sphere around the particle are defined in the usual way from $x^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The integration is performed for each component in a local coordinate frame centred on the worldline[3]. The choice of local coordinates warranties asymptotic flatness to leading-order [123]. The quantity $F^{\alpha\,\mathrm{Lor}}_{(\mathrm{ret})}$ is the retarded force, calculated from $h^{\mathrm{ret}}_{\alpha\beta}$ using Eq. (2.46). $F^{\alpha\,\mathrm{Lor}}_{(\mathrm{ret})}$ diverges at the particle, and it is only defined as a field off the particle by taking a smooth extension of the four velocity

[2]An alternative "self-consistent" description of the motion, used often in the literature and put on a systematic basis in Refs. [119–122], instead describes the trajectory in its unexpanded form $z^\mu(\tau, \varepsilon)$. We prefer to use a perturbative expansion of the worldline, as presented by Gralla and Wald [16].

[3]Gralla showed that Eq. (2.51) can be expressed alternatively as an average over a circle or over two antipodal points [123].

u^α off Γ, which we denote as $\tilde{u}^\alpha$. This extension is defined in [123] by parallel propagation along geodesics perpendicular to Γ.

2.4 Mode-Sum regularization

A practical way to implement MiSaTaQuWa formula and obtain the SF is given by the mode-sum regularization procedure. As we mentioned in previous sections, the SF (at the particle's location $x^\alpha = x_0^\alpha$) can be obtained by subtracting the singular part of the force from the retarded value

$$F^\alpha_{\rm self}(x_0) = \lim_{x\to x_0}\left[F^\alpha_{\rm (ret)}(x) - F^\alpha_{\rm S}(x)\right], \tag{2.52}$$

where the fields $F^\alpha_{\rm (ret)}(x) = \mathsf{m}\nabla^{\alpha\beta\gamma}\bar{h}^{\rm (ret)}_{\beta\gamma}(x)$ and also $F^\alpha_{\rm S}(x) = \mathsf{m}\nabla^{\alpha\beta\gamma}\bar{h}^{\rm S}_{\beta\gamma}(x)$.

Let us expand $F^\alpha_{\rm (ret)}(x)$ and $F^\alpha_{\rm S}(x)$ in spherical harmonics on the surface $t, r = const.$ (ignoring the vectorial nature of the SF and treating each of their components as a scalar function; see [66] for a more sophisticated covariant approach). The SF can be written as a sum over finite ℓ-modes (obtained by summing over the m dependence of the harmonic modes for a given ℓ) [124]:

$$F^\alpha_{\rm self}(x_0) = \lim_{x\to x_0}\sum_{\ell=0}^{\infty}\left[F^{\alpha\ell}_{\rm (ret)}(x) - F^{\alpha\ell}_{\rm S}(x)\right]. \tag{2.53}$$

The quantities $F^\alpha_{\rm (ret)}(x)$ and $F^\alpha_{\rm S}(x)$ are divergent at the particle since the retarded and singular perturbations diverge there. However, each of the individual ℓ-modes $F^{\alpha\ell}_{\rm (ret)}(x)$ and $F^{\alpha\ell}_{\rm S}(x)$ are finite, even at the particle.

It is known that $F^{\alpha\ell}_{\rm S}$ has the large-ℓ expansion $F^{\alpha\ell}_{\rm S} = A^\alpha L + B^\alpha + C^\alpha/L + ...$ [124], with $L \equiv \ell + 1/2$. Since the mode-sum in Eq. (2.53) converges faster than any power of $1/\ell$ (recall $F_{\rm (ret)}(x) - F_{\rm S}(x)$ is smooth), we expect that both the retarded and singular pieces share the same large-ℓ power expansion with the same coefficients. We can then express Eq. (2.53) as a difference of two convergent sums, in the form

$$F^\alpha_{\rm self}(x_0) = \sum_{\ell=0}^{\infty}\left[F^{\alpha\ell}_{\rm (ret)\pm}(x_0) \mp A^\alpha L - B^\alpha - C^\alpha/L\right] - \sum_{\ell=0}^{\infty}\left[F^{\alpha\ell}_{\rm S\pm}(x_0) \mp A^\alpha L - B^\alpha - C^\alpha/L\right], \tag{2.54}$$

where the sign $\pm$ depends on the side we approach the value of r_0, (the quantity $F^{\alpha\ell}_{\rm (ret)\pm} \mp A^\alpha L$ turns out to be direction independent). We expect that both sums converge at least as $\sim 1/\ell$. We arrive at

$$F^\alpha_{\rm self}(x_0) = \sum_{\ell=0}^{\infty}\left(F^{\alpha\ell}_{\rm (ret)\pm}(x_0) \mp A^\alpha L - B^\alpha - C^\alpha/L\right) - D^\alpha, \tag{2.55}$$

with

$$D^\alpha \equiv \sum_{\ell=0}^{\infty}\left(F^{\alpha\ell}_{\rm S\pm}(x_0) \mp A^\alpha L - B^\alpha - C^\alpha/L\right). \tag{2.56}$$

Equation (2.55) is the mode-sum formula to calculate the SF in the LG. The coefficients A^α, B^α, C^α and D^α are the ℓ-independent regularization parameters for each component of the SF. The LG regularization parameters appear explicitly for eccentric orbits of Kerr in Appendix B of this thesis (see [125] for a full derivation). The values of the regularization parameters remain invariant under gauge transformations from LG that are sufficiently regular [19]. As in Eq. (2.51), $F_{(\mathrm{ret})}(x)$ is only defined as a field off the particle by choosing an extension $\tilde{u}^\alpha$. The choice is arbitrary in Eq. (2.55) as long as the regularization parameter A^α, B^α, C^α, and D^α are calculated accordingly.

2.5 Gauge and motion

Let us now consider the effect on the SF induced by a gauge transformation. First we look at the class of gauges studied by Barack and Ori [19], namely those related to LG by a continuous gauge transformation. Calculations of the SF in a different gauge correspond simply to determining how Eq. (2.50) transforms under the gauge transformation that relates the new gauge with LG, $x^\alpha \to x'^\alpha = x^\alpha - \xi^\alpha$.

Let us prescribe a foliation of spacetime near Γ with 3-dimensional spatial hypersurfaces Σ intersecting Γ orthogonally. Let x^a be coordinates on each Σ, with $x^a = 0$ at Γ. We can arrange for z_1^α to be orthogonal to Γ and then focus on the spatial component z_1^a. Due to our foliation of spacetime, the ξ^a component is tangent to the spatial hypersurfaces, and the parallel component does not contribute to the SF. We shall require ξ_a to be bounded in the limit to the worldline. The remaining Σ-perpendicular component can diverge as we take the limit to Γ, but no more strongly than $\ln s$. This divergence must also be spherically symmetric. Furthermore, these statements must be valid on each Σ, eliminating pathological changes in the singular structure as we move forward in time. Among other things, these conditions imply that: (a) the divergence of the first-order MP in the new gauge, $h_{\alpha\beta} = h^{\mathrm{Lor}}_{\alpha\beta} + 2\xi_{(\alpha;\beta)}$, is no stronger than in the LG, behaving as $1/s$ near the particle; and (b) the leading-order singularity is constant in time. If ξ_α satisfies these conditions, which also imply that Eq. (2.59) together with its proper-time derivatives along Γ evaluate to a finite result, we say the gauge is *sufficiently regular* to define the SF [1].

The gauge perturbation $\delta h_{\alpha\beta} \equiv 2\xi_{(\alpha;\beta)}$ induces a change in the SF $\delta F^\alpha_{\mathrm{Lor}}$ which can be calculated using Eq. (2.46):

$$\delta F^\alpha_{\mathrm{Lor}} = -\frac{1}{2}\mathsf{m} P^{\alpha\lambda}\left(\delta h_{\alpha\mu;\nu} + \delta h_{\alpha\nu;\mu} - \delta h_{\mu\nu;\alpha}\right) u^\mu u^\nu. \tag{2.57}$$

Substituting δh and using the Ricci identity $\xi_{\mu;\lambda\nu} - \xi_{\mu;\nu\lambda} = \xi_\rho R^\rho{}_{\mu\lambda\nu}$, we obtain [19]

$$\begin{aligned} \delta F^\alpha_{\mathrm{Lor}} &= -\,\mathsf{m} P^{\alpha\lambda}\left(\xi_{\lambda;\mu\nu} + \xi_\rho R^\rho{}_{\mu\lambda\nu}\right) u^\mu u^\nu \\ &= -\,\mathsf{m}\left[P^{\alpha\lambda}\frac{D^2\xi_\lambda}{D\tau^2} + R^\alpha{}_{\mu\lambda\nu} u^\mu \xi^\lambda u^\nu\right], \end{aligned} \tag{2.58}$$

where $D\xi_\lambda/D\tau \equiv \xi_{\lambda;\mu}u^\mu$ stands for the covariant derivative with respect of the proper-time along Γ.

In the Barack-Ori class of gauges ξ_λ is continuous, and Eq. (2.58) has a definite value. The equality used to get the second line of Eq. (2.58) holds only for geodesics. When u^μ is not geodesic we will have an extra term $\sim a^\mu \xi_{\lambda;\mu}$, with a being the acceleration with respect to the geodesic.

Let us consider the LG equation of motion, Eq. (2.50). Under a gauge transformation, z_1^a transforms as $z_1^a \to z_1^a + \Delta z_1^a$, with

$$\Delta z_1^a = -\lim_{s\to 0}\frac{3}{4\pi}\int n^a n^b \xi_b d\Omega, \tag{2.59}$$

where n^a is the unit vector normal to the two-sphere centred on the worldline and containing the particle, as before. A derivation of Eq. (2.59) appears in Appendix A.2.

The vector ξ_α, that transform from the LG, can be written as $\xi_\alpha(x^a) = \lim_{x^a\to 0}\xi_\alpha(x^a) + o(1)$ for a gauge within the Barack-Ori class. Considering the identity $\int n^a n^b d\Omega = 4\pi/3$, we can evaluate Eq. (2.59) to find

$$\Delta z_1^a = -\xi^a|_\Gamma. \tag{2.60}$$

In words, the gauge contribution to the deviation term from moving away from the LG is just the transformation $x^a \to x^a - \xi^a$ evaluated at the worldline. This means that any gauge transformation within the Barack-Ori class are just translations of the centre of mass.

Barack and Ori [19] showed that the regularization parameters are gauge-independent under a continuous gauge-transformation from the LG. This continuity condition can be relaxed as long as the gauge vector has a well-defined limit at the particle's location.

We want to extend the class of gauges where Eqs. (2.50) and (2.58) may still be used, in particular we want to include discontinuous gauges. This requires investigating how $z^\alpha_{1\mathrm{Lor}}$, in Eq. (2.50), is affected by a discontinuous gauge-transformation.

Gralla and Wald [16, 126] showed how the SF can be obtained in gauges related to LG by a transformation whose generator may have a direction dependence at the particle (but is bounded there, and smooth elsewhere). For a subset of the Gralla-Wald class satisfying a certain parity condition near the particle, Gralla eliminated the preferred role of the LG [123], showing that the SF in this "parity-regular" class can be obtained by averaging the retarded force over a small sphere around the particle, using Eq. (2.51). Gralla also showed that the LG mode-sum formula applies within this class. Let us consider a gauge in this class: this gauge is related to the LG by a gauge vector ξ_α that is smooth off Γ, but is allowed a certain type of ill-defined limit to Γ. The vector must be bounded at Γ and its spatial components must have the local form $\xi_a(x^b) = Z_a(0) + K_a(n^b) + O(s)$ with K_a having odd parity, $K_a(-n^b) = -K_a(n^b)$, under the parity transformation $n^a \to -n^a$. We say that any ξ_α is parity-regular if its spatial components have the leading-order form $Z_a(0)+K_a(n^b)$ with odd K_a. Note that the integral of $n^a n^b K_b(n^c)$ vanishes because K_b is odd and $n^a n^b$ is even. For such a gauge vector we can reduce Eq. (2.59) to the simple average

$$\Delta z_1^a = -\frac{1}{4\pi}\lim_{s\to 0}\int \xi^a d\Omega, \tag{2.61}$$

which gives $\Delta z_1^a = -Z_a(0)$. This type of transformation of the object's position are as reasonable as the result $\Delta z_1^a = -\xi^a|_\Gamma$: if the shift in position of a point depends on the direction one approaches it from, then the average over all directions yields the net shift. Gralla also showed that for any MP in his class, the GSF is given by the same simple spherical average of Eq. (2.51) as in the LG. This form was originally taken as an axiom by Quinn and Wald in their derivation of the GSF in the LG [15]. Gralla's work shows, without assuming it as an axiom, that it holds true in a large class of gauges; hence the name Quinn-Wald-Gralla we have given it. Additionally, Gralla showed, based on this result, that in his class of gauges the GSF can be written in the standard mode-sum form of Eq. (2.55), with the standard LG parameter values, lending great utility to these gauges.

Last, a gauge in the Gralla-Wald class is related to LG by a gauge vector ξ^α. ξ^α is smooth off Γ but is allowed an arbitrary (bounded) direction-dependent limit to Γ, as before the spatial component has the form $\xi_a(x^b) = Z_a(0) + K_a(n^b) + O(s)$ but now $K_a(n^b)$ is allowed any smooth dependence on n^a. This means that Eq. (2.51) does not generically holds true, since any piece of $K_a(n^b)$ that is not parity regular will contribute to the integral in a finite amount. $K_a(n^b)$ is referred to as a supertranslation. A parity-irregular MP is related to a parity-regular one by a parity-irregular transformation [1].

2.6 Conservative effects of the GSF and the red-shift invariant

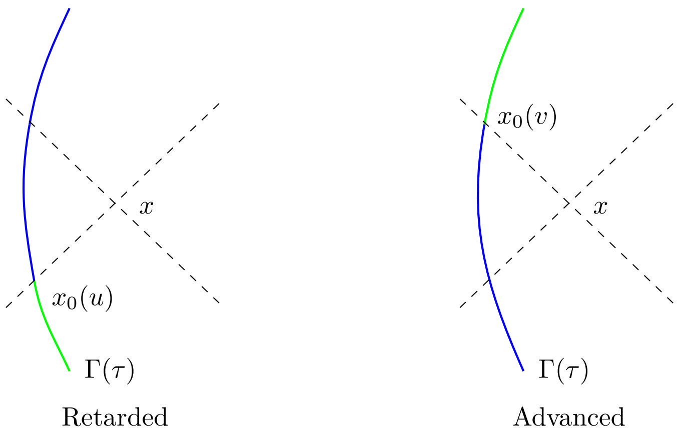

Figure 2.4: In green we show the region of the worldline where the corresponding fields have support. The worldline Γ appears in blue. In curved spacetime the retarded field depends on the past history of the particle where $-\infty < \tau \leq u$; the advanced field depends on the future history where $v \leq \tau < \infty$. u and v are the retarded and advanced times respectively.

To understand the physical consequences of the SF it is useful to distinguish between "conservative" and "dissipative" effects. The physical SF is a sum of two pieces: $F^{\alpha}_{\rm self} = F^{\alpha}_{\rm cons} + F^{\alpha}_{\rm diss}$, where the conservative and dissipative pieces of the SF are defined in terms of the retarded and advanced perturbation, see Fig. 2.4. The advanced perturbation has support starting from the intersection of the future light-cone with Γ. The conservative and dissipative pieces are defined by [97, 127]

$$F^{\alpha}_{\rm cons}(\tau) = \frac{1}{2}\left[F^{\alpha}_{\rm self(ret)}(\tau) + F^{\alpha}_{\rm self(adv)}(\tau)\right], \quad F^{\alpha}_{\rm diss}(\tau) = \frac{1}{2}\left[F^{\alpha}_{\rm self(ret)}(\tau) - F^{\alpha}_{\rm self(adv)}(\tau)\right], \tag{2.62}$$

where $F^{\alpha}_{\rm self(ret)}$ is the retarded SF and $F^{\alpha}_{\rm self(adv)}$ is the advanced one. Both $F^{\alpha}_{\rm self(ret),(adv)}$ satisfy Eq. (2.41) with $\bar{h}^{\rm tail}_{\alpha\beta}(x) \to \bar{h}^{\rm (ret),(adv)}_{\alpha\beta}(x)$ in turn. In Schwarzschild, the components of the retarded and advanced SF are related in the simple way [97]

$$F^{\alpha}_{\rm self(ret/adv)}(\tau) = \epsilon_{(\alpha)} F^{\alpha}_{\rm self(adv/ret)}(-\tau), \tag{2.63}$$

with $\epsilon_{(\alpha)} = (-1, 1, 1, -1)$, and choosing $\tau = 0$ at a radial turning-point of the orbit. For circular orbits $F^{t}_{\rm cons} = F^{\varphi}_{\rm cons} = F^{r}_{\rm diss} = 0$. Hence the t and φ components are just dissipative, and the r component contributes only to the conservative effects.

Suppose that we carry out two independent calculations of the SF in two different gauges and we want to test our results by comparing the two sets of results. As we showed in the previous section, comparing the value of the SF itself would require knowledge of the gauge-generator that relates the two gauges. Detweiler showed, for circular orbits in Schwarzschild [118], that there are two invariant quantities that carry out non-trivial information about the conservative SF dynamics: the orbital frequency $\Omega \equiv u^{\varphi}/u^{t}$ and the contravariant t component of the four velocity $u^{t} \equiv U$. The gauge invariance of Ω and U is restricted to gauge transformations (generated by the vector

ξ^α) that respect the helical symmetry of the perturbed spacetime. In other words, ξ^α satisfies

$$(\partial_t + \Omega\partial_\varphi)\xi^\alpha = 0. \tag{2.64}$$

The physical interpretation of the gauge invariant U is less obvious than that of the orbital frequency Ω. Two physical interpretations of U were discussed by Detweiler [118]. First, U is a measure of the gravitational red-shift experienced by photons emitted by the orbiting particle. The photons are observed at a large distance on the orbital axis in the effective metric $g_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$ rather than the true physical metric $g_{\alpha\beta} + h_{\alpha\beta}$. The second interpretation is related to the helical Killing-vector $k^\alpha \equiv \{1, 0, 0, \Omega\}$ of the perturbed spacetime: the gauge independence of U implies that the constant of motion $\mathcal{E} - \Omega\mathcal{L} \equiv 1/u^t$ is also gauge-independent, while $\mathcal{E} \equiv -u_t$ (energy) and $\mathcal{L} \equiv u_\phi$ (AM) are not.

Explicit expressions for Ω and U, including SF terms, are obtained from the conservative r component of the equation of motion $\mathsf{m}u^\beta\nabla_\beta u^\alpha = F^\alpha_{\rm cons}$: the dissipative piece of the SF is ignored for this analysis. For circular orbits ($u^r = 0$) to linear order in m we get [128]

$$\Omega = \Omega_0\left[1 - \frac{r_0(r_0 - 3M)}{2\mathsf{m}M}F_{r\,\rm cons}\right] \qquad \text{and} \qquad U = U_0\left(1 - \frac{r_0}{2\mathsf{m}}F_{r\,\rm cons}\right), \tag{2.65}$$

where r_0 is the orbital radius (Schwarzschild r coordinate), $\Omega_0 = (M/r_0^3)^{1/2}$ and $U_0 \equiv (1 - 3M/r_0)^{-1/2}$ are the geodesic values of Ω and U, respectively. The expression in Eq. (2.65) tells us that the effect of the conservative SF is to "shift" the values of Ω and U from their non-perturbed values Ω_0 and U_0 at constant r_0.

Despite the formal gauge-invariance of Ω and U, the shift $\Delta\Omega(r_0) \equiv \Omega - \Omega_0$ is in fact gauge-dependent, because the radius r_0 is itself gauge dependent. In other words, two calculations of the SF in different gauges with the same value of r_0 will correspond to two physically distinct orbits. To overcome this problem we can express one of the gauge-invariant quantities in terms of the other.

Let $\tilde\tau$ be proper time along the geodesic of the effective metric $\mathsf{g} = g + h^{\rm R}$. For a given event along the orbit we will have two proper times: $\tilde\tau$ along g and τ along the projection on g. We choose that $\tilde\tau = \tau$ at the initial time, and in general they will be different everywhere else. We associate each point along the trajectory given by g with a point with the same coordinates along the trajectory given by g. Then to $O(\mathsf{m})$ at the worldline of the particle it is easy to show that [82]

$$\frac{d\tau}{d\tilde\tau} = 1 + H^R, \qquad \text{with,} \qquad H^R \equiv \frac{1}{2}h^{\rm R}_{\alpha\beta}u^\alpha u^\beta. \tag{2.66}$$

In terms of the four-velocity $\tilde u^\alpha \equiv dx^\alpha/d\tilde\tau$ we have

$$\tilde\Omega \equiv \tilde u^\varphi/\tilde u^t = u^\varphi/u^t = \Omega, \qquad\qquad \tilde U \equiv \tilde u^t = U(1 + H^R). \tag{2.67}$$

Expressing $\tilde U$ in terms of the gauge-invariant radius $R = \tilde R \equiv (M/\Omega^2)^{1/3}$, we obtain the SF-induced difference

$$\Delta\tilde U(R) \equiv \tilde U(R) - (1 - 3M/R)^{-1/2} = (1 - 3M/R)^{-1/2}H^R. \tag{2.68}$$

Comparing the function $\Delta U(R)$ obtained in different gauges provides a non-trivial test of the calculation of $h^{\rm R}_{\alpha\beta}$, and to some extent of the SF itself.

The equivalence of calculating the SF using different gauges was demonstrated by Sago *et al.* [82] with the explicit calculation of $\Delta U(R)$. Two implementations (one in the LG [36] and the other one in the Regge-Wheeler gauge [118]) showed an agreement for $\Delta\tilde U(R)$ within the computational error ($\sim 10^{-5}$ in fractional terms).

The *regular* part of $H^{(\mathrm{ret})} \equiv \frac{1}{2} h^{(\mathrm{ret})}_{\alpha\beta} u^\alpha u^\beta$ (denoted by H^{R}) is obtained using the mode-sum formula [128]

$$H^{\mathrm{R}} = \sum_{\ell=0}^{\infty} \left[H_\ell^{(\mathrm{ret})}(x_0) - B_H - C_H/\ell \right] - D_H, \tag{2.69}$$

with

$$D_H = \sum_{\ell=0}^{\infty} \left[H_\ell^{\mathrm{S}}(x_0) - B_H - C_H/\ell \right], \tag{2.70}$$

where $H_\ell^{(\mathrm{ret})}$ are the modes computed from the retarded MP $h^{(\mathrm{ret})}_{\alpha\beta}$ and H^{S} is the singular piece of $H^{(\mathrm{ret})}$. The regularization parameters in Eq. (2.69) are explicitly [128]

$$B_H = \frac{2\mathsf{m}}{\pi\sqrt{r_0^2 + \mathcal{L}^2}} \hat{K}\left(\frac{\mathcal{L}^2}{r_0^2 + \mathcal{L}^2} \right), \qquad C_H = D_H = 0, \tag{2.71}$$

where $\hat{K}$ is the complete-integral of first kind as defined in Appendix B.

The quantity H^R is useful for validating different implementations [48, 82, 83, 92] and for the extraction of PN parameters (coefficients in the large-distance expansion). A generalization for eccentric orbits around Schwarzschild was recently presented by Akcay *et al.* [129].

Other gauge-invariant effects of the GSF can be studied. Among those effect we find the shift of the innermost stable circular-orbit [130–132] (ISCO shift) and the periastron advance [128, 133]. Moreover, other gauge-invariant quantities have been identified [134]: the spin precession, and four independent tidal degrees of freedom (which correspond to three eigenvalues —two electric and one magnetic—, and the angle from a scalar product between the electric and magnetic eigenvectors). These invariants quantities have been recently studied [78, 135] and successfully calculated [79] for quasi-circular orbits around Schwarzschild. Recently the invariants in the *octopolar* sector (three derivatives of the metric) have also been computed successfully by Nolan *et al.* [136]. These invariants may be useful to compare between perturbation theory and PN theory.

Chapter 3

Gravitational self-force from curvature scalars

The previous Chapter provided an outline of the 'traditional' theory behind SF calculations in the LG. We focused our review on the mode-sum regularization method. We stressed the importance of a careful analysis regarding gauge transformations of the SF, in particular those transformations that are not related to LG by a continuous gauge transformation. We also introduced some of the main tools required to apply BH perturbation theory in a RG, namely to recover the MP by solving the separable Teukolsky equation. The formalism to obtain the GSF taking advantage of the RGs requires careful considerations.

The preliminary analysis of Barack-Ori [19] identified that in general the RGs have a string-like singularity, namely a singularity that is not confined to the location of the particle (like the LG singularity), but rather extends from the particle to infinity (or to the EH) along a radial-null direction. These string-like singularities would render the RGs not suitable to directly implement the LG mode-sum formula.

In this Chapter we present a detailed explanation of how to derive a mode-sum formula for the RG. This is a non-trivial task since these gauges fall outside the class of gauges related to the LG by a regular and continuous gauge transformation, for which the usual description of the motion in terms of MiSaTaQuWa equation was first derived. However, some of the RGs will fall within the Quinn-Wald-Gralla class where the net shift in the position is obtained by averaging over the two-sphere containing the particle, as discussed in Sec. 2.5.

The structure of this Chapter is as follows. We start in Sec. 3.1 by defining a set of useful Fermi-like coordinates around the particle's worldline. In Sec. 3.2 we will look at the singular structure of the RGs near a point-particle; this will be done by obtaining the leading-order gauge transformation between the LG and RG perturbations. According to the singular structure of this gauge transformation we will identify three types of RGs: *full*-, *half*- and *no*-string RGs. Our considerations apply to either the ingoing and outgoing RGs. In Sec. 3.2.7 we describe how to change our Fermi-like coordinates results to any other choice of coordinates.

Equipped with the gauge transformation we will allow for slight modifications of the RG to define a different gauge, as it was proposed originally proposed by Barack in [97]. This class of 'Locally Lorenz' (LL) gauges will fall within the class of regular gauges described by Barack-Ori in [19], where the motion driven by the mode-sum SF has a well understood description using matched asymptotic expansions. In Sec. 3.3 we provide the prescription to implement the mode-sum formula for these LL gauges, and we derive corrections to the standard LG mode-sum formula. The expressions for the corrections to the LG regularization parameters in BL coordinates will be relegated to Appendix D.

In Sec. 3.4 we tackle the description in terms of the original undeformed RGs and provide the relevant modification to the mode-sum formula. All the results presented in this Chapter were published in [1]. The use of Fermi-like coordinates was proposed by Adam Pound. These coordinates allowed us to independently check our preliminary results obtained in Eddington-Finkelstein coordinates [137], and the generalization of the formalism to Kerr.

3.1 Fermi-like coordinates

Let Γ denote the zeroth-order geodesic orbit of a particle of mass m, in some arbitrary coordinates $x^\alpha = x_0^\alpha(\tau)$, where τ is the proper time as before. The four velocity of the particle $u^\alpha \equiv dx_0^\alpha/d\tau$ satisfies $u^\alpha u_\alpha = -1$. Let us use Fermi-like coordinates (τ, x^a) centred on Γ, as shown in Fig. 3.1. The usual Fermi normal coordinates are used for convenient calculations near a worldline. We modify them to accommodate the preferred direction given by the principal null-vector ℓ^α (in our analysis this null vector will be either ℓ^α or n^α).

The usual Fermi normal coordinates are constructed by first erecting an orthonormal basis (u^α, e_a^α), with $a = 1,2,3$, that is parallelly propagated along Γ. In a neighbourhood of Γ, a foliation of spatial hypersurfaces Σ_τ is prescribed to spacetime. Each Σ_τ is generated by spatial geodesics orthogonally intersecting Γ at a point $x_0(\tau)$. On each hypersurface, a Cartesian coordinates system is established, with coordinates defined as $x^a \equiv -e^a_{\bar\alpha}\nabla^{\bar\alpha}\sigma(\bar x, x)$. The barred indices correspond to the location of the particle $\bar x \equiv x_0(\tau)$, and $\sigma(\bar x, x)$ is Synge's world function [18], equal to one half the squared geodesic distance from $\bar x$ to x. With this definition, x^a has a magnitude

$$s \equiv \sqrt{\delta_{ab}x^a x^b} \tag{3.1}$$

equal to the geodesic distance to x, which has a direction along the triad leg e_a^α. On the worldline, we have $x^a = 0$. By labelling each point on Σ_τ with the time τ, one arrives at a 4D coordinate system (τ, x^a). In these coordinates the metric $g_{\mu\nu}$ takes the locally flat form $\eta_{\mu\nu} + O(s^2)$, where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, with Christoffel symbols $\Gamma^\alpha_{\beta\gamma} = O(s)$.

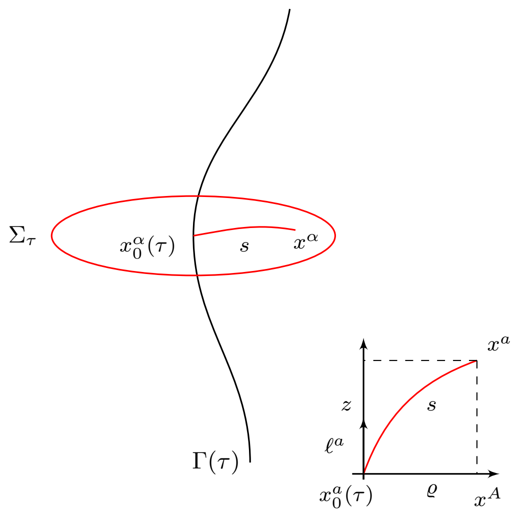

Figure 3.1: Fermi-like coordinates centred on Γ. A set of Cartesian coordinates is established on each spatial hypersurface Σ_τ. The orientation of the coordinates is fixed by choosing the z positive direction to lie along the spatial projection of the null vector ℓ^α.

Let us now define our choice of Fermi-like coordinates. The RG condition [explicitly given for the ingoing and outgoing RGs in Eq. (2.18)], provides a natural choice of singling out the direction along the principal null vector ℓ^α on each Σ_τ. Let $x^a = (x^A, z)$, with $A = 1,2$, and keep the spatial projection of ℓ^α fixed in the positive z direction at $s = 0$, such that

$$\ell^a = \hat\ell\delta^a_z + O(s), \qquad \hat\ell > 0. \tag{3.2}$$

Since ℓ^α is null, we have $\ell^\tau = \hat{\ell} + O(s)$. By keeping the orientation of our coordinates fixed relative to ℓ^α in this way, we cease to parallel propagate the spatial triad e^α_a along Γ, unlike the usual Fermi-normal coordinates. Instead, we allow it to rotate along the worldline, according to

$$\frac{De^\alpha_a}{d\tau} = \omega_a{}^b e^\alpha_b, \tag{3.3}$$

where $\omega_a{}^b$ is a time-dependent rotation matrix. More specifically, we have chosen one of our triad legs to be

$$e^\alpha_3 = \frac{P^\alpha{}_\beta \ell^\beta}{\sqrt{P_{\mu\nu}\ell^\mu \ell^\nu}}, \tag{3.4}$$

where

$$P_{\alpha\beta} \equiv g_{\alpha\beta}(x_0) + u_\alpha u_\beta \tag{3.5}$$

is the operator (defined along Γ) that projects a vector onto Σ_τ. This way we have forced an adaptive rotation of the triad. Despite this rotation, the rest of the coordinate construction is identical to the Fermi construction, with the exception that due to the non-inertial rotation, we now have $g_{\mu\nu} = \eta_{\mu\nu} + O(s)$ and $\Gamma^\alpha_{\beta\gamma} = O(1)$.

We denote the geodesic distance in the direction orthogonal to both u^α and ℓ^a by

$$\varrho \equiv \sqrt{\delta_{AB} x^A x^B}, \tag{3.6}$$

and we also introduce the unit vectors

$$n^a \equiv x^a/s, \qquad \text{and} \qquad N^A \equiv x^A/\varrho, \tag{3.7}$$

which satisfy $\delta_{ab} n^a n^b = 1$ and $\delta_{AB} N^A N^B = 1$. We also note the useful rules $\partial_a s = n_a$ and $\partial_A \varrho = x_A/\varrho$.

3.2 Local singularity structure in radiation gauges

3.2.1 Local gauge transformation

Let us consider the LG perturbation $h^{\rm Lor}_{\alpha\beta}$ [satisfying Eq. (2.40)]. In our Fermi-like coordinates, $h^{\rm Lor}_{\alpha\beta}$ has the leading-order singular form [18]

$$h^{\rm Lor}_{\alpha\beta} = \frac{2\mathsf{m}}{s}\delta_{\alpha\beta} + o(s^{-1}). \tag{3.8}$$

We wish to make a local gauge-transformation to a completed RG perturbation[1] starting from $h^{\rm Lor}_{\alpha\beta}$. For the time being we will assume that the completion piece $h^{\rm (comp)}_{\alpha\beta}$ is given in a gauge regular-enough so that it has no contribution to the leading-order singular structure of the RG perturbation. This will be later obtained explicitly for a Kerr spacetime in Chapter 5. The reconstructed piece $h^{\rm (rec)}_{\alpha\beta}$ satisfies the RG and trace-free conditions [here $h^{\rm (rec)}_{\alpha\beta}$ stands for either $h^{\rm IRG}_{\alpha\beta}$ or $h^{\rm ORG}_{\alpha\beta}$ satisfying Eq. (2.18)] as before.

[1] Let us recall Eq. (2.39) in Sec. 2.1.6, where the completed RG perturbation was defined as $h^{\rm Rad}_{\alpha\beta} \equiv h^{\rm (rec)}_{\alpha\beta} + h^{\rm (comp)}_{\alpha\beta}$. $h^{\rm (rec)}_{\alpha\beta}$ is the piece obtained via the CCK reconstruction procedure, and $h^{\rm (comp)}_{\alpha\beta}$ is the extra price required to satisfy the full linearised EFE.

Let us consider the $O(\mathsf{m})$ gauge transformation[2] $\xi_\alpha = \xi_\alpha^{\text{Rad}\to\text{Lor}}$ which takes $h_{\alpha\beta}^{\text{Rad}}$ to $h_{\alpha\beta}^{\text{Lor}}$:

$$h_{\alpha\beta}^{\text{Lor}} = h_{\alpha\beta}^{\text{Rad}} + \xi_{\alpha;\beta} + \xi_{\beta;\alpha} + o(s^{-1}). \tag{3.9}$$

Here the $o(s^{-1})$ terms account for the contribution from the completion piece. Contracting both sides with ℓ^β and using the gauge conditions leads to

$$\ell^\beta(\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) = \ell^\beta h_{\alpha\beta}^{\text{Lor}} + o(s^{-1}), \tag{3.10}$$

where the covariant derivatives are replaced by partial derivatives, assuming that the singularity in $\xi_{\alpha,\beta}$ is stronger than the singularity in ξ_α, which makes the connection terms sub-dominant. We seek a solution for ξ_α that is well behaved as a function of time τ, i.e., whose τ derivatives do not change the degree of singularity; more precisely, we assume $\partial_\tau \xi_\alpha \sim o(s^{-1})$, such that time derivatives can be neglected.

We recall that in our choice of coordinates we have $\ell^\alpha = \hat{\ell}(\delta_\tau^\alpha + \delta_z^\alpha)$, which allows us to obtain the four components of Eq. (3.10):

$$\partial_z \xi_\tau = \frac{2\mathsf{m}}{\sqrt{\varrho^2 + z^2}} + o(s^{-1}), \tag{3.11a}$$

$$2\partial_z \xi_z + \partial_z \xi_\tau = \frac{2\mathsf{m}}{\sqrt{\varrho^2 + z^2}} + o(s^{-1}), \tag{3.11b}$$

$$\partial_z \xi_A + \partial_A \xi_\tau + \partial_A \xi_z = o(s^{-1}), \tag{3.11c}$$

where we have divided out the common factor $\hat{\ell}$ and used Eq. (3.1) to replace s. The trace-free condition constrains ξ_α to satisfy

$$\begin{aligned} 2\xi^\alpha{}_{,\alpha} =& g^{\alpha\beta} h_{\alpha\beta}^{\text{Lor}} + o(s^{-1}), \quad \text{or} \\ \partial_a \xi^a =& \frac{2\mathsf{m}}{s} + o(s^{-1}) \quad \text{in Fermi-like coordinates.} \end{aligned} \tag{3.12}$$

We can now solve Eq. (3.11) together with Eq. (3.12).

3.2.2 General solutions

One can see by inspection that $\xi_\tau^\pm = \pm 2\mathsf{m} \ln(s \pm z)$ are both solutions to Eq. (3.11a). The most general solution can include arbitrary functions of τ and x^A:

$$\xi_\tau^\pm = \pm 2\mathsf{m} \ln(s \pm z) + \zeta_\tau^\pm(\tau, x^A) + o(1). \tag{3.13}$$

According to condition 1 in Appendix A.2 we could allow $o(\ln s)$ sub-leading terms in the gauge transformation without affecting its regularity. These type of contributions would not correspond to the required form of a LG solution[3], and so we just keep $o(1)$ sub-leading terms. Inspection of

[2] Logically, we should be considering here the opposite transformation, $\xi_\alpha^{\text{Lor}\to\text{Rad}} = -\xi_\alpha$. We instead choose to work with $\xi_\alpha^{\text{Rad}\to\text{Lor}}$ for later convenience.

[3] Recall that the first-order LG perturbation has the form

$$h_{\alpha\beta} = s^{-1} h_{\alpha\beta}^{(1,-1)} + h_{\alpha\beta}^{1,0} + s h_{\alpha\beta}^{(1,1)} + O(s^2), \tag{3.14}$$

where $h_{\alpha\beta}^{(1,-1)}$, $h_{\alpha\beta}^{(1,0)}$, and $h_{\alpha\beta}^{(1,1)}$ are s-independent [138].

Eqs. (3.11b) and (3.11c) similarly yields the general solutions

$$\xi_z^\pm = \zeta_z^\pm(\tau, x^A) + o(1), \tag{3.15a}$$

$$\xi_A^\pm = \frac{2\mathsf{m}x^A}{s \pm z} - z\partial_A \left[\zeta_\tau^\pm(\tau, x^A) + \zeta_z(\tau, x^A)\right] + \zeta_A^\pm(\tau, x^A) + o(1), \tag{3.15b}$$

where $\zeta_\alpha^\pm$ are all arbitrary functions of τ and x^A.

The trace-free condition constrains the arbitrary functions, yielding $\zeta^\pm(\tau, x^A) \equiv 0$. Substituting the general solutions (3.13) and (3.15) into the trace-free condition gives $\partial^A\xi_A^\pm = \frac{2\mathsf{m}}{s} + o(s^{-1})$, which becomes $z\partial^A\partial_A\left(\zeta_\tau^\pm + \zeta_z^\pm\right) = \partial^A\zeta_A^\pm + o(s^{-1})$. Since the right-hand side is independent of z, each side must vanish independently at leading order, implying

$$\partial^A\partial_A\left(\zeta_\tau + \zeta_z\right) = o(s^{-2}) \quad \text{for } z \neq 0, \tag{3.16}$$

$$\partial^A\zeta_A = o(s^{-1}). \tag{3.17}$$

In words, at leading order the sum $\zeta_\tau^\pm + \zeta_z$ must be a harmonic function of x^A, and $\zeta_A^\pm$ must not diverge in the 2D flat space charted by x^A.

Note that the terms involving $\zeta_\alpha^\pm$ in the general solutions (3.13)–(3.15) represent "homogeneous" solutions to the gauge transformation Eq. (3.9) and trace-free condition, namely, solutions to $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$ and $\xi^\alpha_{,\alpha} = 0$. They therefore arise from the freedom to perform gauge transformations within the family of RGs.

The solutions $\xi_\alpha^\pm$ in Eqs. (3.13)–(3.15) are completely general. We will show that any particular solution falls into one of three classes, each with its own distinct type of irregularity away from the particle.

3.2.3 Half-string solutions

Let us set the arbitrary functions $\zeta_\alpha^\pm = 0$ in Eqs. (3.13) and (3.15). This corresponds to a particular choice of gauge. These solutions obviously diverge on Γ (where $s = 0 = z$), but they also diverge *away* from the particle. Recall $s \pm z = (\varrho^2 + z^2)^{1/2} \pm z$, so $s + z$ vanishes on the ("radial") half-ray $\varrho = 0$, $z < 0$, while $s - z$ vanishes on the half-ray $\varrho = 0$, $z > 0$. Hence, ξ_α^+ is singular on the $z < 0$ half-ray, and ξ_α^- is singular on the $z > 0$ half-ray. Taking the limit $\varrho \to 0$ at fixed $z \neq 0$, on the singular half-ray, gives for the remaining components

$$\xi_\tau^\pm \sim \pm 4\mathsf{m}\ln\varrho, \qquad \xi_A^\pm \sim \frac{4\mathsf{m}|z|x^A}{\varrho^2}. \tag{3.18}$$

In words, (i) the component of $\xi_\alpha^\pm$ tangent to Γ diverges logarithmically on a half-ray emanating radially from the particle either inward (for ξ_α^+) or outward (for ξ_α^-), and (ii) the component of $\xi_\alpha^\pm$ orthogonal to both Γ and ℓ^α diverges like the inverse distance to the corresponding half-rays (with a directional dependence).

These solutions (diverging either inwards or outwards) have the general structure of what we shall refer as *half-string* solutions.

The remaining gauge freedom given by $\zeta_\alpha^\pm$ can be used to switch between the two half-string solutions by choosing $\zeta_\tau^\pm(\tau, x^A) = \mp 2\mathsf{m}\ln\varrho^2$ and $\zeta_z^\pm = 0 = \zeta_A^\pm$. We get

$$\xi_\tau^\pm = \pm 2\mathsf{m}\ln\frac{s \pm z}{\varrho^2} + o(1) = \mp 2\mathsf{m}\ln(s \mp z) + o(1), \tag{3.19}$$

and

$$\xi^{\pm}_A = \frac{2\mathsf{m}x^A}{s \pm z} \pm 2\mathsf{m}z\partial_A \ln \varrho^2 + o(1) = \frac{2\mathsf{m}x^A}{s \mp z} + o(1), \tag{3.20}$$

where we have used $\partial_A \varrho = x_A/\varrho$, and $\varrho^2 = (s+z)(s-z)$. One can easily verify that this choice of $\zeta^{\pm}_\alpha$ satisfies the constraints (3.16) and (3.17).

However, switching between half-string singularities in this way requires ζ_α to diverge along $x^A = 0$. If we restrict $\zeta^{\pm}_\alpha(\tau, x^A)$ to be continuous, then the string singularity is fixed on one side. Furthermore, restricting $\zeta^{\pm}_\alpha(\tau, x^A)$ to be C^0 functions of x^A implies $\zeta^{\pm}_\alpha(\tau, x^A) = \zeta^{\pm}_\alpha(\tau, 0) + O(s)$, making the term $z\partial_A(\zeta^{\pm}_\tau + \zeta^{\pm}_z)$ in Eq. (3.15b) of order s. The half-string solutions are then given by

$$\xi^{\pm}_\alpha = \xi^{0\pm}_\alpha(x^a) + Z^{\pm}_\alpha(\tau) + o(1), \tag{3.21}$$

where

$$\xi^{0\pm}_\tau = \pm 2\mathsf{m} \ln(s \pm z), \tag{3.22a}$$

$$\xi^{0\pm}_z = 0, \tag{3.22b}$$

$$\xi^{0\pm}_A = \frac{2\mathsf{m}x^A}{s \pm z}, \tag{3.22c}$$

and with $Z^{\pm}_\alpha(\tau) \equiv \zeta^{\pm}_\alpha(\tau, 0)$. For simplicity, we consider $Z^{\pm}_\alpha(\tau)$ to be smooth.

Equation (3.21) defines a family of half-string solutions where $\xi^{\pm}_\tau$ diverges like $\ln \varrho$ when $\varrho \to 0$, and $\xi^{\pm}_A$ diverges as $1/\varrho$ in the half of spacetime described above.

We note that the half-string solutions $\xi^{\pm}_\alpha$ of Eq. (3.21) have *no definite parity*, since $\xi^{0\pm}_A(-x^a) \neq \pm\xi^{0\pm}_A(x^a)$. To see this note that under a transformation $x^a \to -x^a$ we have $(z, x^A) \to (-z, -x^A)$ and $s \to s$. Hence Eq. (3.22) under $x^a \to -x^a$ reads

$$\xi^{0\pm}_\tau(-x^a) = \pm 2\mathsf{m} \ln(s \mp z), \tag{3.23a}$$

$$\xi^{0\pm}_z(-x^a) = 0, \tag{3.23b}$$

$$\xi^{0\pm}_A(-x^a) = -\frac{2\mathsf{m}x^A}{s \mp z}. \tag{3.23c}$$

3.2.4 Full-string solutions

The half-string fields ξ^+_α and ξ^-_α in Eq. (3.21) correspond to independent trace-free solutions to Eq. (3.9). Any linear combination $n\xi^+_\alpha + (1-n)\xi^-_\alpha$, $n \in \mathbb{R} - \{0, 1\}$, is also a solution. Such solutions are singular on the ray $\varrho = 0$, on both sides of the particle, and we will call them *full-string*. We write the gauge vector as $\xi^{(n)}_\alpha = \xi^{0(n)}_\alpha + Z_\alpha(u) + o(1)$, where $Z^0_\alpha(\tau)$ is arbitrary, and $\xi^{0(n)}_\alpha = n\xi^{0+}_\alpha + (1-n)\xi^{0-}_\alpha$. In words: the divergences on each side of the particle has different magnitudes, and is proportional to n and $1-n$.

Let us consider the case where the divergences are weighted identically, namely by choosing $n = 1/2$. This solution is

$$\xi_\alpha = \xi^0_\alpha(x^a) + Z_\alpha(\tau) + o(1), \tag{3.24}$$

where

$$\xi^0_\tau = \mathsf{m} \ln \frac{s+z}{s-z}, \tag{3.25a}$$

$$\xi^0_z = 0, \tag{3.25b}$$

$$\xi^0_A = \frac{2\mathsf{m}sx^A}{\varrho^2}, \tag{3.25c}$$

and we have defined $Z_\alpha(\tau) = \frac{1}{2}Z^+_\alpha(\tau) + \frac{1}{2}Z^-_\alpha(\tau)$. We again assume $Z_\alpha(\tau)$ to be smooth. These solutions inherit the singular form of the two half-string solutions from which they were constructed. Explicitly ξ_α diverges along the entire ray $\varrho = 0$, for both $z > 0$ and $z < 0$: in the limit $\varrho \to 0$ at fixed $z \neq 0$ we have

$$\xi_\tau \sim -2\mathsf{m}\,\mathrm{sign}(z)\ln\varrho, \qquad \xi_A \sim \frac{2\mathsf{m}|z|x^A}{\varrho^2}. \tag{3.26}$$

Unlike the half-string solutions, these solutions are parity-regular: ξ_a at leading order is comprised of an odd-parity piece $\xi^0_a(x^b)$ that is discontinuous at $x^b = 0$, plus a piece Z_α that is independent of the limit we approach the worldline..

3.2.5 No-string solutions

In a similar construction as the one we just used for the full-string gauges, namely combining two half-string solutions, we can obtain a *no-string* solution. Let us consider combining two half-string solutions by gluing together the regular regions of each. The gluing surface can be chosen almost arbitrarily, as long as the two half-strings lie on opposite sides of it. As a simple choice, let us take the gluing surface to be smooth. This way the leading-order term can be approximately a plane intersecting the particle at each given τ. Each plane can be written as $p_a(\tau)x^a = 0$, for some p_a perpendicular to the plane. In Chapter 5 we will take $p_a x^a = 0$ to be the leading-order approximation to a sphere of constant Boyer–Lindquist (t, r), in the Kerr case.

We define the no-string solution $\xi_\alpha = \xi^+\theta(p_a x^a) + \xi^-\theta(-p_a x^a) + o(1)$ as

$$\xi_\alpha = \xi^0_\alpha(x^a) + Z_\alpha(\tau, z) + o(1), \tag{3.27}$$

where

$$\xi^0_\tau = 2\mathsf{m}\ln(s+z)\theta(p_a x^a) - 2\mathsf{m}\ln(s-z)\theta(-p_a x^a), \tag{3.28a}$$

$$\xi^0_z = 0, \tag{3.28b}$$

$$\xi^0_A = \frac{2\mathsf{m}x^A}{s+z}\theta(p_a x^a) + \frac{2\mathsf{m}x^A}{s-z}\theta(-p_a x^a), \tag{3.28c}$$

and

$$Z_\alpha = Z^+_\alpha(\tau)\theta(p_a x^a) + Z^-_\alpha(\tau)\theta(-p_a x^a). \tag{3.29}$$

We again assume each $Z^\pm_\alpha$ is a smooth function of τ, but in general we let $Z^+_\alpha \neq Z^-_\alpha$. The no-string solutions, considered as distributions, solve the transformation Eq. (3.11) together with the trace-free condition, even on the surface $p_a x^a = 0$, at the relevant order: the delta-function terms arising from differentiating (3.27) are formally sub-leading, and are contained within the $o(s^{-1})$ terms in these equations.

The no-string solutions constructed this way are smooth for both $p^a x_a > 0$ and $p^a x_a < 0$, but the divergences have been removed at the cost of introducing a jump discontinuity at $p^a x_a = 0$. Like the equal-weight full-string solutions and unlike the half-string ones, these solutions are parity-regular (since they have odd-parity). More accurately, they are very nearly, but not quite parity-regular. They come in the correct general form $\xi_a = \xi^0_a(n^i) + Z_a + o(1)$, where $\partial_a\xi^0_b \sim 1/s$, $\partial_a Z_b \sim s^0$, and ξ^0_a is odd under $n^i \to -n^i$. But here Z_a is not necessarily continuous at $x^a = 0$, this will have

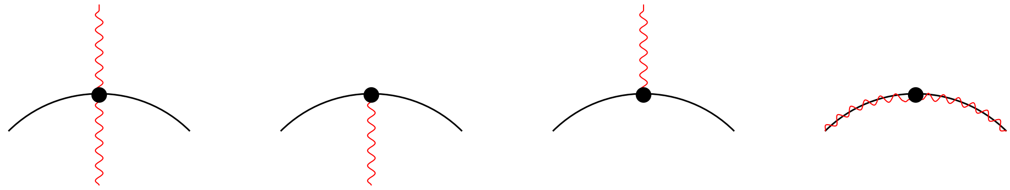

Figure 3.2: Singular structure of the RGs. From left to right: full-string solutions are singular along ℓ^α; half-string solutions are either regular in the exterior or the interior of a closed surface intersecting the particle; no-string solutions are regular everywhere off-the-particle but discontinuous at a surface intersecting the particle.

important implications in later sections.

3.2.6 Singular form of the metric perturbation

We can now describe the local singular form of the MP in the completed RGs. As we recall the singular structure is determined by $h^{(\mathrm{rec})}_{\alpha\beta}$ only (since we have assumed that $h^{(\mathrm{compl})}_{\alpha\beta}$ is regular enough). Each of the above classes of gauge transformations will have a distinct singular MP. By inverting Eq. (3.9) to obtain $h^{\mathrm{Rad}}_{\alpha\beta}$ and substituting Eq. (3.8), we have

$$h^{\mathrm{Rad}}_{\alpha\beta} = \frac{2\mathsf{m}}{s}\delta_{\alpha\beta} - \xi_{\alpha;\beta} - \xi_{\beta;\alpha} + o(s^{-1}). \tag{3.30}$$

Where ξ^α is given, in turn, by Eqs. (3.21), (3.24) and (3.27). This way we obtain expressions for the leading-order term of the half-string, equal-weight full-string, and no-string RG perturbation, see Figure 3.2.

Table 3.1: The leading-order singular form of the RG perturbation near the particle. The half-string solutions in the left column, $h^\pm_{\alpha\beta}$, corresponds to $\xi^\pm_\alpha$ of Eq. (3.21). The full-string and no-string solutions, middle and right columns, are constructed from the corresponding gauge transformations ξ_α, given in Eqs. (3.24) and (3.27), respectively. We used $\theta^\pm = \theta(\pm p_a x^a)$, and omitted the label 'Rad' from the MP for brevity.

Half-string solutions	Full-string solution	No-string solution
$h^\pm_{\tau\tau} = \frac{2\mathsf{m}}{s}$	$h_{\tau\tau} = \frac{2\mathsf{m}}{s}$	$h_{\tau\tau} = \frac{2\mathsf{m}}{s}$
$h^\pm_{\tau z} = -\frac{2\mathsf{m}}{s}$	$h_{\tau z} = -\frac{2\mathsf{m}}{s}$	$h_{\tau z} = -\frac{2\mathsf{m}}{s}$
$h^\pm_{zz} = \frac{2\mathsf{m}}{s}$	$h_{zz} = \frac{2\mathsf{m}}{s}$	$h_{zz} = \frac{2\mathsf{m}}{s}$
$h^\pm_{\tau A} = \mp\frac{2\mathsf{m}x_A}{s(s\pm z)}$	$h_{\tau A} = \frac{2\mathsf{m}zx_A}{s\varrho^2}$	$h_{\tau A} = h^+_{\tau A}\theta^+ + h^-_{\tau A}\theta^-$
$h^\pm_{zA} = \pm\frac{2\mathsf{m}x_A}{s(s\pm z)}$	$h_{zA} = -\frac{2\mathsf{m}zx_A}{s\varrho^2}$	$h_{zA} = h^+_{zA}\theta^+ + h^-_{zA}\theta^-$
$h^\pm_{AB} = \frac{2\mathsf{m}}{s(s\pm z)^2}\left(2x_Ax_B - \varrho^2\delta_{AB}\right)$	$h_{AB} = \frac{2\mathsf{m}(s^2+z^2)}{s\varrho^4}(2x_Ax_B - \varrho^2\delta_{AB})$	$h_{AB} = h^+_{AB}\theta^+ + h^-_{AB}\theta^-$

The MP inherits the string singularities of the gauge transformation vector from which it was constructed, see Table 3.1. The divergences on the MP are stronger than they were in the gauge vector. Near the singular strings we have, as $\varrho \to 0$ with fixed $z \neq 0$,

$$h^\pm_{\tau A} \sim \mp\frac{4\mathsf{m}x_A}{\varrho^2}, \quad h^\pm_{zA} \sim \pm\frac{4\mathsf{m}x_A}{\varrho^2}, \quad h^\pm_{AB} \sim \frac{8\mathsf{m}|z|(2x_Ax_B - \varrho^2\delta_{AB})}{\varrho^4}, \tag{3.31}$$

for the half-string solutions, and

$$h^{\pm}_{\tau A} \sim \frac{2\mathsf{m}\,\mathrm{sign}(z)x_A}{\varrho^2}, \quad h^{\pm}_{zA} \sim -\frac{2\mathsf{m}\,\mathrm{sign}(z)x_A}{\varrho^2}, \quad h^{\pm}_{AB} \sim \frac{4\mathsf{m}|z|(2x_Ax_B - \varrho^2\delta_{AB})}{\varrho^4}, \tag{3.32}$$

for the full-string solutions. The leading-order singularity for the three types of RG are summarized in Table 3.1.

3.2.7 Re-expressing the gauge transformation in a covariant form

So far we have relied on our Fermi-like coordinates to obtain expressions for the gauge transformations and the relevant MP. In the following sections we will use those results to tackle the SF problem, and we will require the transformation from the local coordinates to an arbitrary coordinate system (like BL coordinates). We will do it in two steps, first writing the gauge transformation in covariant form, and then expanding that covariant form in arbitrary coordinates.

Starting from the covariant definition of three scalar fields

$$x^a \equiv -e^a_{\bar\alpha}\sigma^{;\bar\alpha} \tag{3.33}$$

together with the condition

$$\sigma_{;\bar\alpha}u^{\bar\alpha} = 0, \tag{3.34}$$

which states that the point x off-the-worldline is connected to a point $\bar{x} = x_0(\tau)$ on the worldline by a geodesic that intersects the worldline orthogonally. We will also make use of the fact that the triad $e^a_{\bar\alpha}$ satisfies

$$\frac{De^{\bar\alpha}_a}{d\tau} = \omega_a{}^b e^{\bar\alpha}_b. \tag{3.35}$$

The quantity $\sigma(x,\bar{x})$ in these expressions is one-half the squared geodesic distance from x to $\bar{x}$.

Now, since the point $\bar{x}$ depends on the point x, when differentiating a function of the two points, say $f(x,\bar{x})$, we have

$$\frac{df(x,\bar{x}(x))}{dx^\alpha} = \frac{\partial f}{\partial x^\alpha} + \frac{\partial f}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dx^\alpha} = \frac{\partial f}{\partial x^\alpha} + \frac{\partial f}{\partial \bar{x}^\beta}u^{\bar\beta}\frac{d\tau}{dx^\alpha}. \tag{3.36}$$

In terms of one-forms, this reads

$$df = \frac{\partial f}{\partial x^\alpha}dx^\alpha + \frac{\partial f}{\partial \bar{x}^\beta}u^{\bar\beta}d\tau. \tag{3.37}$$

By applying the same principle, we can differentiate Eq. (3.34) to find

$$d\tau = \nu\sigma_{;\bar\alpha\beta}u^{\bar\alpha}dx^\beta \tag{3.38}$$

where $\nu \equiv -(\sigma_{;\bar\alpha\bar\beta}u^{\bar\alpha}u^{\bar\beta})^{-1}$. We can differentiate Eq. (3.33) in the same manner to find

$$dx^a = -\frac{De^a_{\bar\alpha}}{d\tau}\sigma^{;\bar\alpha} - e^a_{\bar\alpha}\left(\sigma^{;\bar\alpha}{}_\alpha dx^\beta + \sigma^{;\bar\alpha}{}_{\bar\beta}u^{\bar\beta}d\tau\right). \tag{3.39}$$

Substituting Eqs. (3.35) and (3.38) into this equation returns

$$dx^a = -e^b_{\bar\alpha}\left[\delta^a_b\sigma^{;\bar\alpha}{}_\alpha + \nu\left(\omega^a{}_b\sigma^{;\bar\alpha} + \delta^a_b\sigma^{;\bar\alpha}{}_{\bar\beta}u^{\bar\beta}\right)\sigma_{;\alpha\bar\gamma}u^{\bar\gamma}\right]dx^\alpha. \tag{3.40}$$

We can now write any one-form $\xi_\alpha = (\xi_\tau, \xi_a)$ in covariant form using $\xi_\alpha = \xi_\tau \frac{d\tau}{dx^\alpha} + \xi_a \frac{dx^a}{dx^\alpha}$. All of these expressions are exact.

Since we require only leading-order behaviour in the transformation, we can use the standard covariant expansions [18]

$$\sigma_{\bar{\alpha}\bar{\beta}} = g_{\bar{\alpha}\bar{\beta}} + O(s^2), \quad \sigma_{\alpha\bar{\beta}} = -g^{\bar{\alpha}}_{\alpha} g_{\bar{\alpha}\bar{\beta}} + O(s^2), \tag{3.41}$$

where $g^{\bar{\alpha}}_{\alpha}$ is the parallel propagator from $\bar{x}^\alpha = x_0^\alpha(\tau)$ to x^α and $\sigma^{;\bar{\beta}} \sim s$. With these expansions, at leading-order we find

$$\frac{dx^a}{dx^\alpha} = g^{\bar{\alpha}}_{\alpha} e^a_{\bar{\alpha}} + O(\lambda), \tag{3.42}$$
$$\frac{d\tau}{dx^\alpha} = -g^{\bar{\alpha}}_{\alpha} u_{\bar{\alpha}} + O(\lambda^2). \tag{3.43}$$

For a covector with components $\sim s^0$, this allows us to write

$$\xi_\alpha = g^{\bar{\alpha}}_{\alpha} \left(-\xi_\tau u_{\bar{\alpha}} + \xi_a e^a_{\bar{\alpha}}\right) + O(s\xi). \tag{3.44}$$

Notice that because we work at leading-order, we do not require the rotation ${\omega_a}^b$. Equation (3.44) is the key result of this section, and it can be used to obtain the gauge vector in any set of coordinates.

Let us now consider the gauge-generator we found for the RGs, substituting the form of ξ_α [Eqs. (3.21), (3.24) and (3.27)] in (3.44) gives

$$\xi_\alpha = g^{\bar{\alpha}}_{\alpha}(-\xi^0_\tau u_{\bar{\alpha}} + \xi^0_A e^A_{\bar{\alpha}} + Z_{\bar{\alpha}}) + o(1), \tag{3.45}$$

where we have used $\xi^0_z = 0$. The triad vector in Eq. (3.4) can be simplified by noting that $P_{\mu\nu}\ell^\mu\ell^\nu = (u_\mu\ell^\mu)^2$:

$$e^\alpha_3 = -\frac{P^\alpha_\beta \ell^\beta}{u_\mu \ell^\mu}. \tag{3.46}$$

The remaining two legs of the triad can be determined by the orthonormality condition between the tetrad and the four velocity, $g_{\alpha\beta}u^\alpha e^\beta_a = 0$ and $g_{\alpha\beta}e^\alpha_a e^\beta_b = \delta_{ab}$, to give

$$u_\alpha e^\alpha_A = 0, \quad \ell_\alpha e^\alpha_A = 0, \quad e_{\alpha A} e^\alpha_B = \delta_{AB}. \tag{3.47}$$

The expression for $P_{\alpha\beta}$ can be found in terms of the triad legs inverting the completeness relation $-u_\alpha u_\beta + e_{a\alpha}e^a_\beta = g_{\alpha\beta}$, namely $P_{\alpha\beta} = e_{a\alpha}e^a_\beta$. We express the product $e_{\alpha A}e^A_\beta$ in terms of e^α_3 as

$$e_{\alpha A}e^A_\beta = e_{a\alpha}e^a_\beta - e_{3\alpha}e^3_\beta = Q_{\alpha\beta}, \tag{3.48}$$

where we defined

$$Q_{\alpha\beta} \equiv P_{\alpha\beta} - \frac{P_{\alpha\mu}P_{\beta\nu}\ell^\mu\ell^\nu}{(u_\gamma \ell^\gamma)^2}. \tag{3.49}$$

The non vanishing spatial components of ξ^0_α are proportional to $x^A = -e^A_{\bar{\alpha}}\bar{\sigma}^{;\bar{\alpha}}$ and can be written as

$$\xi^0_A = -\xi e^A_{\bar{\alpha}} \sigma^{;\bar{\alpha}}, \tag{3.50}$$

where according to Eqs. (3.22c), (3.25c) and (3.28c) define

$$\xi^{\pm} \equiv \frac{2\mathsf{m}}{s \pm z} \qquad \text{in the half-string case,} \tag{3.51a}$$
$$\xi \equiv \frac{2\mathsf{m}s}{\varrho^2} \qquad \text{in the full-string case,} \tag{3.51b}$$
$$\xi \equiv \xi^+\theta^+ + \xi^-\theta^- \qquad \text{in the no-string case.} \tag{3.51c}$$

Substituting back in Eq. (3.45) and using (3.48) we find

$$\xi_\alpha = -g^{\bar\alpha}_{\alpha}\left(\xi^0_\tau u_{\bar\alpha} + \xi Q_{\bar\alpha\bar\beta}\sigma^{;\bar\beta} - Z_{\bar\alpha}\right) + o(1). \tag{3.52}$$

Which is the covariant form of the gauge transformation relating the LG and RGs.

3.2.8 Coordinate expansion

We now wish to express the covariant expansion of Eq. (3.52) in arbitrary coordinates, or rather in terms of the coordinate differences $\delta x^{\alpha'} \equiv x^\alpha - x^{\alpha'}$. This will allows us to move from the coordinate system centred on the worldline to any other system, for example that where the origin coincides with the centre of the background BH of the EMRI system. The differences $\delta x^{\alpha'}$ give the distance from a point of coordinates x^α relative to a point of nearby coordinates $x^{\alpha'}$ on the worldline. We will use the coordinate expansions [66]

$$g^{\alpha'}_{\beta}(x,x') = \delta^{\alpha'}_{\beta} + O(s), \tag{3.53a}$$
$$\sigma^{;\alpha'}(x,x') = -\delta x^{\alpha'} + O(s^2), \tag{3.53b}$$
$$\sigma_{;\alpha'\beta'}(x,x') = g_{\alpha'\beta'} + O(s^2). \tag{3.53c}$$

To relate our Fermi-like coordinates (τ, x^a) with the differences $\delta x^{\alpha'}$, we replace the dependence on $\bar x$ with the coordinates of the particle $x^{\alpha'} = x^\alpha_0(\tau')$ at some other location on Γ. Let us choose τ' to be the proper time on which $\delta t = t(x) - t(x') = 0$, where t is the BL time-coordinate, which is practical in explicit coordinate calculations. This replacement involves defining $x^a(\tau) = -e^a_\alpha(x_0(\tau))\nabla^\alpha\sigma(x, x_0(\tau))$ and expanding $x^a(\tau)$ about $\tau' = \tau - \delta\tau$. We get

$$\begin{aligned} x^a(\tau) =& x^a(\tau') + u^{\alpha'}x^a_{;\alpha'}(\tau')\delta\tau + O(s^2) = -e^a_{\alpha'}\left[\sigma^{;\alpha'}(x,x') + u^{\beta'}\sigma^{;\alpha'}_{\beta'}(x,x')\delta\tau + O(s^2)\right] \\ =& -e^a_{\alpha'}\left[\sigma^{;\alpha'}(x,x') + u^{\alpha'}\delta\tau + O(s^2)\right] = e^a_{\alpha'}\delta x^{\alpha'} + O(s^2), \end{aligned} \tag{3.54}$$

where the primes denote the coordinates associated with x'. We have used the expansions in Eq. (3.53) and $e^a_{\alpha'}u^{\alpha'} = 0$. Combining Eq. (3.54) with Eq. (3.46) gives $z = z_0 + O(s^2)$, where

$$z_0 \equiv -u_{\alpha'}\delta x^{\alpha'} - \frac{\ell_{\alpha'}\delta x^{\alpha'}}{\ell_{\beta'}u^{\beta'}}. \tag{3.55}$$

In the same way we can obtain expressions for the distances s and ϱ^2; using Eq. (3.1) to get $s = s_0 + O(s^2)$, with

$$s_0^2 = P_{\alpha'\beta'}\delta x^{\alpha'}\delta x^{\beta'}. \tag{3.56}$$

Straightforwardly for $\varrho^2 = s^2 - z^2$ we get $\varrho^2 = s_0^2 - z_0^2 + O(s^3)$. We now can expand ξ_α in terms of the coordinate differences by substituting the expansions for s, σ and ϱ in Eq. (3.52). Expanding ξ_α about x' we get

$$\xi_\alpha = -g_\alpha^{\alpha'}\left(\xi_\tau^0 u_{\alpha'} + \xi Q_{\alpha'\beta'}\sigma^{;\beta'} - Z_{\alpha'}\right) + o(1). \tag{3.57}$$

Using Eq. (3.53) we arrive at

$$\xi_\alpha = -\xi_\tau^0 u_{\alpha'} + \xi Q_{\alpha'\beta'}\delta x^{\beta'} - Z_{\alpha'} + o(1). \tag{3.58}$$

We note that the first term of Eq. (3.58) is parallel to the four-velocity, while the second term is orthogonal to both $u^{\alpha'}$ and $\ell^{\alpha'}$. This allows us to split $\xi^0_{\alpha'}$ into a parallel and perpendicular component:

$$\xi^0_{\alpha'} = \xi_{\|\alpha'} + \xi_{\perp\alpha'}, \tag{3.59}$$

with $\xi_{\|\alpha'} \equiv -\xi^0_\tau u_{\alpha'}$ and $\xi_{\perp\alpha'} \equiv \xi Q_{\alpha'\beta'}\delta x^{\beta'}$. We use, in turn, the corresponding expressions for ξ [Eq. (3.51) for each type of gauge transformation] to get

$$\xi_\tau^{0\pm} = \pm\, 2\mathsf{m}\ln(s_0 \pm z_0), \qquad \xi^\pm = \frac{2\mathsf{m}}{s_0 \pm z_0} \qquad \text{(half-string)}, \tag{3.60a}$$

$$\xi_\tau^0 = \mathsf{m}\ln\frac{s_0+z_0}{s_0-z_0}, \qquad \xi = \frac{2\mathsf{m}s_0}{s_0^2 - z_0^2} \qquad \text{(full-string)}, \tag{3.60b}$$

$$\xi_\tau^0 = \xi_\tau^{0+}\theta^+ + \xi_\tau^{0-}\theta^- \qquad \xi = \xi^+\theta^+ + \xi_-\theta^- \qquad \text{(no-string)}. \tag{3.60c}$$

We observe that, at leading-order, if all the components $\xi_a^\pm$ have the same definite-parity under $x^a \to -x^a$, then all the components of $\xi_{\alpha\perp}$ have that same parity under the transformation $\delta x^{\alpha'} \to -\delta x^{\alpha'}$, regardless of the choice of coordinates. This can be seen from Eq. (3.44) together with the facts that $g_\alpha^{\bar\alpha} e_{\bar\alpha}^a$ does not alter the parity, and that each x^a is a linear combination of $\delta x^{\alpha'}$.

3.3 Self-force in a Locally deformed radiation gauge

To define what we mean by a locally-Lorenz (LL) gauge, we first recall the form of the globally Lorenz MP near the particle, given in our Fermi-like coordinates in Eq. (3.8). In arbitrary coordinates, the expression reads [18]

$$h^{\rm Lor}_{\alpha\beta} = \frac{2\mathsf{m}}{s}(g_{\alpha\beta} + 2\tilde u_\alpha \tilde u_\beta) + O(1), \tag{3.61}$$

where s is the geodesic distance to the worldline, and $\tilde u_\beta$ corresponds to any smooth extension of the four velocity u_α off Γ. The terms $O(1)$ are finite but not necessarily continuous on Γ. By an LL gauge, we mean any gauge in which the metric possesses the same leading-order singularity structure as $h^{\rm Lor}_{\alpha\beta}$; that is,

$$h^{\rm LL}_{\alpha\beta} = \frac{2\mathsf{m}}{s}(g_{\alpha\beta} + 2\tilde u_\alpha \tilde u_\beta) + o(s^{-1}). \tag{3.62}$$

The terms $o(s^{-1})$ may diverge at the particle, but not as strongly as does the leading-order singularity. In particular, we shall need to allow logarithmic divergences, which potentially arise in the RG at sub-leading order, as our analysis in the previous section suggests.

We wish to start from $h^{\rm Rad}_{\alpha\beta}$ and locally transform it to some $h^{\rm LL}_{\alpha\beta}$. The gauge transformation

$\xi_\alpha = \xi_\alpha^{\text{Rad}\to\text{LL}}$ must satisfy

$$\ell^\beta(\xi_{\alpha;\beta} + \xi_{\beta;\alpha}) = \frac{2\mathsf{m}}{s}(\ell_\alpha + 2\tilde{u}_\alpha\tilde{u}_\beta\ell^\beta) + o(s^{-1}), \tag{3.63}$$

and

$$\xi^\alpha{}_{;\alpha} = \frac{2\mathsf{m}}{s} + o(s^{-1}). \tag{3.64}$$

Finding an LL gauge is now a matter of solving Eqs. (3.63) and (3.64) for ξ^α, which was done in the previous section.

Starting from the MP $h_{\alpha\beta}^{\text{Rad}}$ we can obtain the corresponding MP in the LL gauge in any of the three categories (full-, half-, and no-string gauges) from the corresponding gauge vector $\xi_\alpha = \xi_\alpha^0 + Z_\alpha + o(1)$ given in Sec. 3.2. The sub-leading terms $Z_\alpha + o(1)$ correspond to different choices of LL gauge, and this choice is left arbitrary.

In the context of SF calculations, recalling that the force is gauge dependent, we require to give its value and to fully specify the MP in which it was calculated. A numerical implementation of the CCK-reconstruction and completion will give the MP in a particular RG. In our analysis we will choose a specific LL gauge. We set

$$\xi_\alpha^{\text{Rad}\to\text{LL}} = \xi_\alpha^0, \tag{3.65}$$

where ξ_α^0 in arbitrary coordinates is obtained from Eqs. (3.58) and (3.60).

3.3.1 Mode-sum formula for the SF in an LL gauge

From the local singularity structures of Eqs. (3.61) and (3.62), it follows that the generator $\hat{\xi}_\alpha \equiv \xi_\alpha^{\text{Lor}\to\text{LL}}$ of the gauge transformation from $h_{\alpha\beta}^{\text{Lor}}$ to $h_{\alpha\beta}^{\text{LL}}$ satisfies

$$\hat{\xi}_{\alpha;\beta} + \hat{\xi}_{\beta;\alpha} = o(s^{-1}) \tag{3.66}$$

near the worldline. The $o(s^{-1})$ term in Eq. (3.66) imply that $\hat{\xi}_\alpha$ may fall outside of the Barack-Ori class of gauge; this term could, for example, give jump discontinuities. We shall demand $\hat{\xi}_\alpha$ to be continuous.

With this restriction, these LL gauges fall within the class of gauges studied by Barack and Ori [19], in which the LG mode-sum of Eq. (2.55) and the corresponding regularization parameters are gauge invariant. Namely for the LL gauges we can write directly

$$F_\alpha^{\text{LL}} = \sum_{\ell=0}^{\infty}\left[(\tilde{F}_\alpha^{\text{LL}})_\pm^\ell - A_\alpha^\pm L - B_\alpha - C_\alpha/L\right] - D_\alpha, \tag{3.67}$$

where the ℓ-independent parameters $A_\alpha^\pm$, B_α, C_α, and D_α take their Lorenz-gauge values[4] given in the Appendix B and in Refs. [97, 125]. The quantities $(\tilde{F}_\alpha^{\text{LL}})^\ell$ are the multipole modes of the retarded force in the LL-gauge evaluated at the particle limit $x \to x_0$. If $h_{\alpha\beta}^{\text{LL}}$ is known in advance, $(\tilde{F}_\alpha^{\text{LL}})_\pm^\ell$ are calculated using Eq. (2.46) mode by mode. We have used $\tilde{F}_\alpha$ to denote the retarded force instead of $F_\alpha^{(\text{ret})}$ (the notation of Chapter 2) to simplify notation.

Given $\tilde{F}_\alpha^{\text{LL}}$, the ℓ modes $(\tilde{F}_\alpha^{\text{LL}})_\pm^\ell$ are constructed by expanding each coordinate component of this field (artificially considered as a scalar field) in spherical-harmonic functions on a surface of constant BL radius r, then adding up all azimuthal numbers m for a given multipole number ℓ, and finally

[4]The LG retarded force $(\tilde{F}_\alpha^{\text{LL}})_\pm^\ell$ and regularization parameters depend on the off-worldline extension of the four-velocity and the affine connections.

evaluating the result at the particle's limit. This limit will generally be direction-dependent, and one must ensure that it is taken from the same direction as the one used to derive the regularization parameters. In the mode-sum formula (3.67) the limit is taken from one of the radial directions, $r \to r_0^{\pm}$, holding t, θ, φ fixed. $(\tilde{F}_\alpha^{\rm LL})_\pm^\ell$ and $A_\alpha^\pm$ denote the corresponding one-sided values (the values of the parameters B_α, C_α and D_α turn out not to depend on the direction).

Let us now rewrite Eq. (3.67) in terms of the modes of the retarded force in the RG, $(\tilde{F}_\alpha^{\rm Rad})^\ell$, which are the modes we will be calculating in practice. The difference between the two gauges due to $\xi_\alpha^{\rm Rad\to LL}$ can be obtained according to Eq. (2.58). Let $\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL}$ be the change in the retarded force induced by transforming to the LL gauge, and denote its ℓ-modes by $(\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL})_\pm^\ell$, where we allow for a directional dependence corresponding to $r \to r_0^\pm$. We can rewrite Eq. (3.67) as

$$F_\alpha^{\rm LL} = \sum_{\ell=0}^{\infty} \left[(\tilde{F}_\alpha^{\rm Rad})_\pm^\ell + (\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL})_\pm^\ell - A_\alpha^\pm L - B_\alpha - C_\alpha/L\right] - D_\alpha, \tag{3.68}$$

where both $(\tilde{F}_\alpha^{\rm Rad})_\pm^\ell$ and $(\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL})_\pm^\ell$ must be calculated via the same directional limit to the particle as were the regularization parameters, and all terms must be defined with the same off-worldline extension of u^α and $P_\alpha{}^\beta$.

We assume, tentatively, that $(\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL})_\pm^\ell$ admits a large-ℓ asymptotic with a similar form to that of $(\tilde{F}_\alpha^{\rm Lor})_\pm^\ell$, namely

$$(\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL})_\pm^\ell = \delta A_\alpha^\pm L + \delta B_\alpha + \delta C_\alpha/L + O(1/L^2), \tag{3.69}$$

where $\delta A_\alpha^\pm$, δB_α and δC_α are ℓ-independent parameters [we will verify this form with an explicit calculation in next subsection, showing that the parameter values are in fact zero through $O(1/L)$]. With this assumption, Eq. (3.68) becomes

$$F_\alpha^{\rm LL} = \sum_{l=0}^{\infty} \left[(\tilde{F}_\alpha^{\rm Rad})_\pm^\ell - (A_\alpha^\pm - \delta A_\alpha^\pm)L - (B_\alpha - \delta B_\alpha) - (C_\alpha - \delta C_\alpha)/L\right] - (D_\alpha - \delta D_\alpha), \tag{3.70}$$

where

$$\delta D_\alpha \equiv \sum_{\ell=0}^{\infty} \left[(\delta_\xi \tilde{F}_\alpha^{\rm Rad\to LL})_\pm^\ell - \delta A_\alpha^\pm L - \delta B_\alpha - \delta C_\alpha/L\right]. \tag{3.71}$$

Since the argument in the last sum is $O(L^{-2})$ at large ℓ, the sum should be convergent. And since we started with a convergent sum in Eq. (3.68), the sum in Eq. (3.70) should therefore also be convergent.

Eq. (3.70) is the mode-sum formula for the SF in the LL gauge. It requires three pieces of input: (i) the modes $(\tilde{F}_\alpha^{\rm Rad})^\ell$, which are constructed from the MP obtained numerically via CCK-reconstruction and completion; (ii) the standard, LG regularization parameters $\{A_\alpha^\pm, B_\alpha, C_\alpha, D_\alpha\}$, given in Appendix B for generic orbits in Kerr and for a particular choice of extension; and (iii) the corrections to the LG parameters $\{\delta A_\alpha^\pm, \delta B_\alpha, \delta C_\alpha, \delta D_\alpha\}$ associated with the particular LL-gauge chosen. The latter will be obtained analytically in Sec. 3.3.2 via a local analysis.

Having three types of RGs (full-, half-, and no-string gauges) leads to considering which of them are suitable as input for the mode-sum formula (3.70). As we argued above, the CCK-reconstruction probably cannot be used to compute the full-string MP, so this class of solutions is irrelevant in practice. The retarded-force modes $(\tilde{F}_\alpha^{\rm Rad})_\pm^\ell$ could be derived from either "halves" of a no-string MP, by taking the corresponding limits $r \to r_0^\pm$. However, the gauge vector $\hat{\xi}_\alpha = \xi_\alpha^{\rm Lor\to LL}$ associated with the no-string solution would not have a well defined limit to the particle [due to the unmodelled discontinuous term $Z_\alpha(\tau, z)$ in $\xi_\alpha^{\rm Rad\to LL}$; recall Eq. (3.27)], which we do not allow here: a discontinuous LL gauge would fall outside the Barack-Ori class, and there would be no guarantee that the mode-sum formula (3.70) applies in that form.

Rather, the retarded-force modes $(\tilde{F}_\alpha^{\rm Rad})_\pm^\ell$ should be derived from a half-string MP, with the limit $r \to r_0^\pm$ taken from the regular side of $p_a x^a = 0$. Gauge vectors $\xi_\alpha^{\rm Lor\to LL}$ associated with half-string solutions *are* continuous, because the corresponding vector $\xi_\alpha^{\rm Rad\to LL}$ accounts explicitly for the full discontinuity in $h_{\alpha\beta}^{\rm Rad}$ at the relevant order. Hence, an LL gauge derived from a half-string RG belongs to the Barack-Ori class as required. A CCK reconstruction (and completion) gives only the "regular half" of a half-string solution (as shown in [1] for the flat case toy model), so fixing the string's direction (by fixing the half-string gauge) dictates the direction from which the limit $r \to r_0^\pm$ should be taken when computing $(\tilde{F}_\alpha^{\rm Rad})_\pm^\ell$ and $A_\alpha^\pm$ in Eq. (3.70): for a string extending over $r > r_0$ take $r \to r_0^-$; for a string extending over over $r < r_0$ take $r \to r_0^+$.

3.3.2 Regularization parameters

Let us now calculate expressions for δA_α, δB_β, δC_α and δD_α appearing in Eq. (3.70). This will be done for the general setup of a particle in geodesic motion in Kerr spacetime. We will stress the importance of the choice of extension and comment on the impact of different choices of LL gauges.

Let us assume we have obtained (numerically) the reconstructed modes $(\tilde{F}_\alpha^{\rm Rad})_+^\ell$ and/or the modes $(\tilde{F}_\alpha^{\rm Rad})_-^\ell$ in a half-string RG and wish to obtain the SF in an LL gauge related to this RG by the gauge vector $\xi_\alpha^\pm = \xi_\alpha^{0\pm}$ given in Eq. (3.21). The calculation of δA_α, δB_β, δC_α and δD_α follows the method first implemented by Barack-Ori to derived the LG regularization parameters [31, 97, 125, 139].

In BL coordinates the particle is at $x_0^\alpha = (t, r_0, \theta_0, \varphi_0)$. We introduce new polar coordinates $(\tilde{\theta}, \tilde{\varphi})$, so that the particle is located at the pole ($\tilde{\theta}_0 = 0$) of the new system, and $\tilde{\varphi}$ is chosen so that the particle's velocity at x_0 (projected onto the 2-sphere) points along the $\tilde{\varphi}_0 = 0$ longitudinal line. This construction simplifies the multipole decomposition required for the mode-sum formula since the value of each ℓ-mode of the retarded force at the particle has a sole contribution from the axially-symmetric, $m = 0$ azimuthal mode. We use locally Cartesian coordinates $\hat{x} = \rho\cos\tilde{\varphi}$, $\hat{y} = \rho\sin\tilde{\varphi}$, where $\rho = \rho(\tilde{\theta})$ is some smooth function with the property $\rho = \tilde{\theta} + O(s^2)$ near the particle. In terms of these variables, we have $\delta\theta = \tilde{\theta} - \tilde{\theta}_0 = \hat{x} + O(s^2)$ and $\delta\varphi = \tilde{\varphi} - \tilde{\varphi}_0 = \hat{y}/\sin\theta_0 + O(s^2)$ [97]. At leading-order, we can write $\delta_\xi \tilde{F}_\alpha^\pm(x', \delta x')$ as $\delta_\xi \tilde{F}_\alpha^\pm(\delta r, \hat{x}, \hat{y}; x_0)$. We have chosen $\delta t = 0$ as before.

The ℓ modes of $\delta_\xi F_\alpha^\pm$ in Eq. (3.68) are calculated by evaluating the Legendre integral [31, 97]

$$(\delta_\xi \tilde{F}_\alpha)_\pm^\ell = \frac{L}{2\pi} \lim_{\delta r\to 0^\pm} \int_{-1}^{1} d(\cos\tilde{\theta}) \mathsf{P}_\ell(\cos\tilde{\theta}) \int_0^{2\pi} d\tilde{\varphi}\, \delta_\xi \tilde{F}_\alpha^\pm(\delta r, \hat{x}, \hat{y}), \tag{3.72}$$

where P_ℓ is the Legendre polynomial. Notice that Eq. (3.72) depends on the off-worldline extension via $\delta_\xi \tilde{F}_\alpha^\pm$ [see Eq. (2.58) in Chapter 2]. Let us recall that the singularity of $\tilde{F}_\alpha^\pm$ is inherited from the local behaviour of $\xi_\alpha^\pm$: it 'starts' at x_0 and extends into the $\mp p_a x^a > 0$ part of spacetime. The analysis by Barack and Ori [139] showed that the only contribution to the integral in Eq. (3.72) comes from the immediate neighbourhood of the singularity. Therefore only a regular neighbourhood around the particle is needed to evaluate $(\delta_\xi \tilde{F}_\alpha)_\pm^\ell$. Note that the integral in Eq. (3.72) is calculated in the side of spacetime where the RG is regular, before taking the limit to the particle, to avoid encountering the string-like singularity.

To simplify the integral in Eq. (3.72) we recall that in general $\xi_\alpha^\pm$ contains pieces that are parallel $\xi_{\alpha\parallel}^\pm$ and perpendicular $\xi_{\alpha\perp}^\pm$ to u^α. $\xi_\perp$ in Eq. (3.22) is bounded, and so is the corresponding $\delta_{\xi_\perp} \tilde{F}_\alpha^\pm$.

Using the fact that the integrand is bounded to exchange the order in which we evaluate the integral and the limit we get

$$(\delta_\xi \tilde{F}_\alpha)^\ell_\pm = \frac{L}{2\pi}\int_{-1}^{1} d(\cos\tilde{\theta})\mathsf{P}_\ell(\cos\tilde{\theta})\int_0^{2\pi} d\tilde{\varphi}\lim_{\delta r\to 0^\pm}\delta_{\xi_\perp}\tilde{F}^\pm_\alpha(\delta r,\hat{x},\hat{y}), \tag{3.73}$$

where we have also used the fact that $\delta_{\xi_\parallel}\tilde{F}^\pm_\alpha$ does not contribute to $\delta_\xi \tilde{F}^\pm_\alpha$ as was mentioned in Sec. 2.2 and Appendix C.

Equation (3.73) is valid for any extension. We will choose the rigid extension $\tilde{u}^\alpha(x) \equiv u^\alpha(x_0)$ and $\tilde{\Gamma}^\alpha_{\beta\gamma}(x) \equiv \Gamma^{\alpha'}_{\beta'\gamma'}(x_0)$ expressed in BL coordinates. The effect of the choice of extension in the force is discussed in Appendix C. This way the components of the four velocity and the Christoffel symbols do not depend on the coordinates of the field point, namely they are constant when taking derivatives with respect to $\delta x'$. This allows us to write $(\delta_{\xi_\perp}\tilde{F}_\alpha)^\ell = \delta_{(\xi_\perp)^\ell}\tilde{F}_\alpha$ and obtain the ℓ modes of $\delta_\xi \tilde{F}_\alpha$ directly from those of $\xi^\pm_{\alpha\perp}$.

These modes are calculated from

$$(\xi_{\alpha\perp})^\ell_\pm = \frac{\mathsf{m}L}{\pi}Q_{\alpha\beta}\lim_{\delta r\to 0^\pm}\int_{-1}^{1} d(\cos\tilde{\theta})\mathsf{P}_\ell(\cos\tilde{\theta})\int_0^{2\pi} d\tilde{\varphi}\frac{\delta x^\beta}{s_0 \pm z_0}, \tag{3.74}$$

where $Q_{\alpha\beta}$, s_0 and z_0 are given in Eqs. (3.49), (3.55) and (3.56), respectively. We then note that at $\delta t = \delta r = 0$, both the numerator and denominator of the integrand scale linearly with ρ. The integral over $\cos\tilde{\theta}$ therefore reduces to $2\delta^\ell_0$, leaving us with

$$(\xi_{\alpha\perp})^\ell_\pm = \frac{\mathsf{m}}{\pi}\delta^\ell_0\int_0^{2\pi} d\tilde{\varphi}\frac{Q_{\alpha\theta}\cos\tilde{\varphi} + Q_{\alpha\varphi}\sin\tilde{\varphi}/\sin\theta_0}{R^\pm(x_0,\tilde{\varphi})}, \tag{3.75}$$

where $R^\pm(x_0,\tilde{\varphi})$ is $(s_0 \pm z_0)/\rho$ evaluated at $\delta t = 0 = \delta r$.

The general form of the integral in Eq. (3.75) is valid for any orbit. In the example of *equatorial* orbits ($\theta_0 = \pi/2$), we find

$$R^\pm = r_0\left[1 + (P_{\varphi\varphi}/r_0^2 - 1)\sin^2\tilde{\varphi}\right]^{1/2} \mp \left(u_\varphi + \frac{\ell_\varphi}{\ell_\alpha u^\alpha}\right)\sin\tilde{\varphi}, \tag{3.76}$$

and

$$(\xi_{\alpha\perp})^\ell_\pm = \pm\frac{\mathsf{m}}{r_0}\delta^\ell_0 Q_{\alpha\varphi}\frac{2c}{b-c^2}\left(1 - \frac{1}{\sqrt{1+b-c^2}}\right), \tag{3.77}$$

where $b \equiv P_{\varphi\varphi}/r_0^2 - 1$ and $c \equiv \frac{1}{r_0}[u_\varphi + \ell_\varphi/(\ell_\alpha u^\alpha)]$ are the factors appearing in Eq. (3.76), and we have used the fact that $P_{\theta\theta} = r_0^2$ for equatorial orbits.

Given $(\xi_{\alpha\perp})^\ell_\pm$, calculating $(\delta_\xi \tilde{F}_\alpha)^\ell_\pm$ is a straightforward matter of substituting Eq. (3.77) into Eq. (2.58). In Appendix C we explore the choice of extension in Eq. (2.58), and we write it explicitly for two different extensions. Since $\xi_{\alpha\perp}$ is ℓ-independent and only contains the $\ell = 0$ mode, by comparing with Eq. (3.69) we can read off

$$\delta A_\alpha = \delta B_\alpha = \delta C_\alpha = 0. \tag{3.78}$$

We compare Eq. (3.77) with Eq. (3.71) to write

$$\delta D^\pm_\alpha = \sum_\ell (\delta_\xi \tilde{F}_\alpha)^\ell_\pm = (\delta_\xi \tilde{F}_\alpha)^{\ell=0}_\pm = \delta_{(\xi_\perp)^{\ell=0}_\pm}\tilde{F}_\alpha. \tag{3.79}$$

To shorten the discussion of this section we present the relevant expressions for δD_α in Appendix D. With the explicit value of δD_α we can now calculate the SF in an LL gauge from the reconstructed modes of a half-string RG. Let us recall that the computation of the retarded force will depend on the chosen extension, and that in particular the chosen rigid-extension might not be the best choice for practical schemes. For a different extension (for example a rigid extension of u^α leaving $\Gamma^\alpha_{\beta\gamma}$ as a field) we may get different values of δD_α. An important fact is that, regardless of its actual value, we will have $\delta D^+_\alpha = -\delta D^-_\alpha$ in general. This property of $\delta D^\pm_\alpha$ is proven in Appendix D, and it will be useful in the next section.

3.3.3 Alternative choices of LL gauge

In our construction of the LL gauges, we made a specific choice: a particular half-, full-, or no-string RG related to the LL gauge by the gauge vector $\xi_\alpha = \xi^0_\alpha$. Adding terms of $o(1)$ to ξ_α has no impact on the GSF in the LL gauge, meaning such terms are not worth considering for our purposes. But adding an $O(1)$ term does affect the GSF, and we could have made the alternative choice $\xi_\alpha = \xi^0_\alpha + Z_\alpha(\tau)$, with $Z_\alpha(\tau)$ left arbitrary. Suppose we had done so, and then Eq. (3.70) would have become

$$F^{\rm LL}_\alpha = \sum_{\ell=0}^{\infty}\left[(\tilde{F}^{\rm Rad}_\alpha)^\ell - A_\alpha L - B_\alpha - C_\alpha/L\right] + \delta D^{\rm new}_\alpha, \tag{3.80}$$

where the new δD_α parameter is

$$\delta D^{\rm new}_\alpha = \sum_{\ell=0}^{\infty}(\delta_\xi \tilde{F}_\alpha)^\ell = \sum_{\ell=0}^{\infty}\left[(\delta_{\xi^0}\tilde{F}_\alpha)^\ell + (\delta_Z \tilde{F}_\alpha)^\ell\right]. \tag{3.81}$$

The first term is the δD_α that we have already calculated, and the second term is the change to it due to the nonzero Z_α. From this new term, one can see that the freedom to choose Z_α allows us to almost arbitrarily alter δD_α. The question then arises of whether we have made the best choice in setting Z_α to zero. For example, we might try to choose a Z_α for which $\delta D^{\rm new}_\alpha = 0$. To do so, we note that $\delta_Z \tilde{F}_\alpha$ is smooth at the worldline, allowing us to write $\sum_\ell(\delta_Z \tilde{F}_\alpha)^\ell$ simply as

$$\delta_Z \tilde{F}_\alpha = -\mathsf{m}\left(P^\lambda_\alpha \frac{D^2 Z_\lambda}{d\tau^2} + R_{\alpha\mu\lambda\nu}u^\mu Z^\lambda u^\nu\right), \tag{3.82}$$

where here all quantities are evaluated on the worldline. Finding a Z_α for which $\delta D^{\rm new}_\alpha = 0$ simply requires solving the ordinary differential equation

$$\mathsf{m}\left(P^\lambda_\alpha \frac{D^2 Z_\lambda}{d\tau^2} + R_{\alpha\mu\lambda\nu}u^\mu Z^\lambda u^\nu\right) = \delta D_\alpha, \tag{3.83}$$

with δD_α given by $\sum_{\ell=0}^{\infty}(\delta_{\xi^0}\tilde{F}_\alpha)^\ell$ as before.

Let us stress once more that since the SF is gauge dependent, when we calculate the SF we must fully specify the LL gauge in which we are working. For that reason, there is no apparent advantage to knowing that there might exist an LL gauge in which $\delta D^{\rm new}_\alpha$ vanishes; finding such a gauge would still require us to calculate $\sum_\ell(\delta_{\xi^0}\tilde{F}_\alpha)^\ell$ analytically, and it would only add the extra step of solving an ODE for Z_α.

3.4 Self-force in an undeformed radiation-gauge

Let us now work in an undeformed RG, namely we now seek to obtain a way to calculate the SF using directly the CCK-reconstructed modes of the RG. We will begin in the LG, where the first-order deviation from geodesic motion z_1^μ is governed by Eq. (2.50). We will transform to a no-string gauge and find the corrected equation of motion via Eq. (2.59). We will need to write the SF using the Quinn-Wald-Gralla angle-averaged form of Eq. (2.51) to derive a new mode-sum formula. We will rely on the results of the half-string analysis—in particular the fact that $\delta D_\alpha^+ + \delta D_\alpha^- = 0$— to show that this new mode-sum is also applicable in the LL gauge. We refer to Appendix D.4, where $\delta D_\alpha^+ + \delta D_\alpha^- = 0$ is established as an extension-independent property.

3.4.1 Equation of motion formulated in an undeformed radiation-gauge

The gauge vector that brings a global LG to a no-string gauge is given by $\xi_\alpha = -\xi_\alpha^0 - Z_\alpha + o(1)$, where ξ_α^0 and Z_α are found in (3.28) and (3.29). Substituting ξ_α into Eq. (2.59), we find that the transformation induces a change in position

$$\Delta z_1^a = \frac{3}{4\pi} \lim_{s\to 0} \int n^a n^b (\xi_b^0 + Z_b) d\Omega, \tag{3.84}$$

where the integral is over a sphere of radius s around the particle, and $d\Omega \equiv \sin\theta d\theta d\phi$. As before, the angles (θ, ϕ) on the unit sphere around the particle are defined in the usual way from $x^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The first term of Eq. (3.84) vanishes, since $n^\alpha n^\beta$ has even parity while ξ_β^0 has odd parity. We write

$$\Delta z_1^a = \frac{3}{4\pi}\left(Z_b^+ \int_{\frac{1}{2}S^2} n^a n^b d\Omega + Z_b^- \int_{\frac{1}{2}S^2} n^a n^b d\Omega \right), \tag{3.85}$$

where integrals are evaluated in half of the two-sphere $\frac{1}{2}S^2$ in which $Z_b^\pm$ is regular. Using $\int n^a n^b d\Omega = \frac{4\pi}{3}\delta^{ab}$ and the even parity of the integrand, we arrive at

$$\Delta z_1^a = \frac{1}{2}\left[Z_+^a(\tau) + Z_-^a(\tau) \right]. \tag{3.86}$$

Because the term ξ_α^0 of the no-string gauge vector has odd parity, it does not produce a change in position; instead, we have a simple average of the shifts in position induced by the smooth functions $Z_a^\pm$. The odd parity of ξ_α^0 also allows us to write our result for Δz_1^α in terms of the full gauge-vector as

$$\Delta z_1^a = -\frac{1}{2}\lim_{x^b\to 0}\left[\xi_+^a(\tau, x^b) + \xi_-^a(\tau, -x^b) \right], \tag{3.87}$$

where x^b is a point chosen so that the two terms are regular in the region $\pm p_a x^a > 0$, where $\xi_\pm^a$ are regular respectively. With this coordinated choice of limit to the particle, the singular pieces of ξ_α^+ and ξ_α^- cancel. If the limit were not coordinated in this way, it would be ill-defined, since ξ_α^+ and ξ_α^- do not separately have unique limits at the particle.

For a point x on the worldline, $\tilde{P}^{\alpha\beta}\xi_\beta(x+\delta x)$ has the same parity (at leading-order) under $\delta x \to -\delta x$ as does ξ_α under $x^a \to -x^a$. The change in position is then

$$\Delta z_1^\alpha = -\frac{1}{2}\lim_{\delta x\to 0}\left[\tilde{P}^{\alpha\beta}\xi_\beta^+ + \tilde{P}^{\alpha\beta}\xi_\beta^- \right], \tag{3.88}$$

where we have multiplied Eq. (3.87) by e_a^α and used $e_a^\alpha \xi^a = \tilde{P}^{\alpha\beta}\xi_\beta + O(s\xi)$ for any smooth extension.

$\xi_\alpha^\pm$ are evaluated at $x^\alpha \pm \delta x^\alpha$ in the corresponding side of the spacetime where they are regular. From the shift in position, the acceleration can be found simply by taking two derivatives along the worldline, leading to

$$\begin{aligned}\mathsf{m}\frac{D^2\Delta z_1^\alpha}{d\tau^2} =& -\frac{1}{2}\mathsf{m}\lim_{\delta x\to 0}\left[P^{\alpha\beta}u^\mu\nabla_\mu\left(u^\nu\nabla_\nu\xi_\alpha^+\right)+P^{\alpha\beta}u^\mu\nabla_\mu\left(u^\nu\nabla_\nu\xi_\alpha^-\right)\right]\\ =& -\mathsf{m}R^\alpha{}_{\mu\beta\nu}u^\mu\Delta z_1^\beta u^\nu\\ &+\frac{1}{2}\mathsf{m}\lim_{\delta x\to 0}\left[\delta_{\xi^+}\tilde{F}^\alpha - P^{\alpha\beta}(u^\mu\nabla_\mu u^\nu)\nabla_\nu\xi_\beta^+ + \delta_{\xi^-}\tilde{F}^\alpha - P^{\alpha\beta}(u^\mu\nabla_\mu u^\nu)\nabla_\nu\xi_\beta^-\right].\end{aligned} \tag{3.89}$$

The first line holds for any smooth extensions of u^μ, $P^{\alpha\beta}$, and ∇ off the worldline, [see Eqs. (C.9)-(C.10)]. In the second line, we have expressed $P^{\alpha\beta}u^\mu\nabla_\mu\left(u^\nu\nabla_\nu\xi_\alpha^\pm\right)$ in terms of $\delta_{\xi^\pm}\tilde{F}^\beta$, where a tilde denotes the retarded-force off the worldline as before. The contribution from ξ_α^0 has odd parity, which causes the terms involving $(u^\mu\nabla_\mu u^\nu)$ to cancel one another, hence

$$\mathsf{m}\frac{D^2\Delta z_1^\alpha}{d\tau^2} = -\mathsf{m}R^\alpha{}_{\mu\beta\nu}u^\mu\Delta z_1^\beta u^\nu + \frac{1}{2}\mathsf{m}\lim_{\delta x\to 0}\left[\delta_{\xi^+}\tilde{F}^\alpha + \delta_{\xi^-}\tilde{F}^\alpha\right]. \tag{3.90}$$

Comparing with the SF Eq. (2.50), we find that

$$\Delta_\xi F^\alpha = \frac{1}{2}\mathsf{m}\lim_{\delta x\to 0}\left[\delta_{\xi^+}\tilde{F}^\alpha + \delta_{\xi^-}\tilde{F}^\alpha\right], \tag{3.91}$$

and it corresponds to the average of the change in the retarded-force computed on two opposite sides of the particle. The contributions from ξ_α^0 cancel [see Eq. (3.28)] and we can write Eq. (3.91) as

$$\Delta_\xi F^\alpha = \frac{1}{2}\left(\delta_{Z^+}\tilde{F}^\alpha + \delta_{Z^-}\tilde{F}_\alpha\right), \tag{3.92}$$

where we have taken the limit, on the corresponding side where ξ^+ and ξ^- are regular, so that all quantities are evaluated at the worldline. The total SF in the no-string gauge can be written in terms of the SF computed in the LG and an extra term corresponding to the gauge transformation:

$$F^\alpha = F^\alpha_{\rm Lor} + \Delta_\xi F^\alpha. \tag{3.93}$$

3.4.2 Mode-sum formula

In practice Eq. (3.93) would still require the previously obtained value of the Lorenz SF, which is exactly what we seek to avoid. We now can express the term $F^\alpha_{\rm Lor}$ in terms of the LG mode-sum Eq. (2.55), namely

$$F^\alpha = F^\alpha_{\rm Lor} + \frac{1}{2}\lim_{\delta x\to 0}\left[\delta_{\xi^+}\tilde{F}^\alpha + \delta_{\xi^-}\tilde{F}^\alpha\right] \tag{3.94}$$

$$\begin{aligned}=& \frac{1}{2}\sum_\ell\left[(\tilde{F}^\alpha_{\rm Lor+})^\ell - A^\alpha_+ L - B^\alpha - C^\alpha/L\right]\\ &+\frac{1}{2}\sum_\ell\left[(\tilde{F}^\alpha_{\rm Lor-})^\ell - A^\alpha_- L - B^\alpha - C^\alpha/L\right] + \frac{1}{2}\sum_\ell\left[(\delta_{\xi^+}\tilde{F}^\alpha)^\ell + (\delta_{\xi^-}\tilde{F}^\alpha)^\ell\right].\end{aligned} \tag{3.95}$$

To arrive at this, we have made simple manipulations: we wrote $F^\alpha_{\rm Lor}$ as an average of the two one-sided limit mode-sums $\sum_\ell\left[(\tilde{F}^\alpha_{\rm Lor\pm})^\ell - A^\alpha_\pm L - B^\alpha - C^\alpha/L\right]$ and decomposed $\delta_{\xi^\pm}\tilde{F}^\alpha$ into ℓ-modes. It is understood that the same extension must be chosen for the LG mode-sums and the modes of the extra terms from the gauge transformation. We now can note that the combination $(\tilde{F}^\alpha_{\rm Lor\pm})^\ell +$

$(\delta_{\xi^\pm}\tilde{F}^\alpha)^\ell$ gives $(\tilde{F}^\alpha_\pm)^\ell$, the mode of the retarded-force in the no-string gauge. This leads to the simple mode-sum formula

$$F^\alpha = \sum_\ell \left[\frac{1}{2}(\tilde{F}^\alpha_+)^\ell + \frac{1}{2}(\tilde{F}^\alpha_-)^\ell - B^\alpha - C^\alpha/L\right], \tag{3.96}$$

where B^α and C^α are the regularization parameters in the LG for the chosen extension, and we have used the fact that in general $A^\alpha_+ = -A^\alpha_-$ [97, 125].

In summary, Eq. (3.96) can be applied to calculate the SF in an undeformed no-string RG. As input it requires the modes of the retarded-force calculated from the completed RG perturbations and the standard LG regularization parameters. We will present a numerical implementation of Eq. (3.96) for a particle orbiting a Schwarzschild BH in Chapter 4.

We could repeat the calculations of Sec. 3.4.1 with the locally deformed no-string gauge. In other words, we would take the average of the two-sided half-string-mode-sums of Eq. (3.70). By virtue of $\delta D^+_\alpha = -\delta D^-_\alpha$ we would find that the GSF in the LL gauge can be obtained with the same mode-sum as in the undeformed gauge Eq. (3.96). The reason for this is that the gauge vector ξ^0_α, which relates the no-string gauge to its LL version, has odd parity around the particle. Therefore, the SF calculated from the mode-sum or Eq. (3.96) can be interpreted equally well as the SF in the undeformed no-string gauge or in its LL counterpart.

3.4.3 Summary

Before proceeding to the numerical implementation, let us summarise the main outcomes of the analysis presented in this Chapter. Using Fermi-like coordinates we solved for the local gauge-transformation relating the LG and the RG. This led us to identify three types of RGs, according to the singular structure of the transformation. The first class is the half-string gauges where the singularity is not confined to the particle, but rather extends radially in half of the spacetime from the particle to either the EH or infinity. The components of the gauge transformation to the half-string gauges are given in Eq. (3.22). The second class corresponds to gauges with a full-string, where the singularity extends along a null direction from the EH across the particle and reaches infinity. The gauge transformation to the full-string gauges is Eq. (3.25). The third class is the no-string gauges, which is constructed by gluing the two halves of spacetime where the half-string gauges are regular. This construction introduces a discontinuity on a closed surface containing the particle as seen from Eq. (3.28). The singular structure of these gauges permeates to the components of the MP, which are summarised in Fermi-like coordinates in Table 3.1.

We considered two practical methods to compute the SF. The first one involves working in a deformed RG or LL-gauge. This deformation takes place near the particle so that the RG perturbation agrees to leading order with the LG singularity. In this LL gauge the standard LG mode-sum formula is still valid and the regularization parameters take their LG values. The MP in the LL gauge, or directly the modes of the LL retarded-force can be calculated from the corresponding RG modes. The GSF can be calculated using the mode-sum formula (3.70). This new mode-sum includes corrections to the LG parameters and it is valid on the regular side of spacetime, opposite to the half-string singularity.

The second method allows us to calculate the SF directly from the RG perturbation. We used the angle-average representation of the SF to work in the no-string RG. We related the GSF in the no-string RG with its LG counterpart expressed in terms of the standard LG mode-sum. We expanded the extra contribution from the gauge transformation in harmonic modes, and rewrote everything in terms of the modes of the retarded-force in the undeformed RG. The outcome was another mode-sum formula, Eq. (3.96), which uses the values of the LG regularization parameters, and involves the average of the two-sided retarded-force. This is the method we will numerically implement. We will consider the test case of a Schwarzschild background in the next Chapter.

Chapter 4

Numerical implementation for circular orbits in Schwarzschild spacetime

In this Chapter we present the numerical implementation of the method described in Chapter 3 to obtain the GSF. We specialize to circular orbits around a Schwarzschild BH. The main goals of this implementation are:

1. study the applicability of a GSF calculation based on curvature scalars,
2. correct results in the literature [83],
3. provide an insight into the challenges we may find in the Kerr case,
4. compare the computational efficiency with respect to LG calculations.

The numerical results of this Chapter were previously published in [2]. The results from the MST method where provided by Abhay G. Shah.

Section 4.1 describes the frequency-domain algorithm we follow. In Sec. 4.2 we will give more details of our specific calculation. Our implementation takes advantage of the formalism presented in Secs. 2.1.3 and 3.4. We start in Sec. 4.2.1 with a short description of the static modes, which will be also relevant in Chapter 5. In Sec. 4.2.2 we give the explicit form of the Sasaki-Nakamura transformation in Schwarzschild, namely the $a \to 0$ limit of what we reviewed in Sec. 2.1.4. With the reconstructed perturbations as an input, we calculate the retarded-force modes using Eq. 2.46, and finally regularize. This last step will be done using the averaged version of the mode-sum formula Eq. (3.96) as derived in Sec. 3.4. We obtain the SF values in both the ingoing and outgoing RGs. In Sec. 4.2.3 we include the explicit Teukolsky sources and expressions for the retarded force. In Sec. 4.2.4 we describe the inclusion of the low multipoles, this will be done in the LG. We will briefly argue why this can be done in this particular case. The large-ℓ modes that are not computed numerically are included by performing a fitting of the regularised modes to an analytical power series. In Sec. 4.2.5 this tail fitting is described.

As we argued in Chapter 1, the mode-sum formula is a robust method to check the consistency of the calculated value of the SF; in Sec. 4.3.2 we show the convergence plots of the mode-sum for the radial and temporal components of the GSF. Among other consistency checks we performed to our code, in Sec. 4.3.3 we show the energy fluxes at the EH and at infinity, and the red-shift invariant H. We compare the GSF values obtained from two methods: the first one using numerically obtained values of the Sasaki-Nakamura field; and the second one using the analytical MST method described in Sec. 2.1.5. In Sec. 4.3.5 we compare the efficiency of our numerical code with that of Barack-Sago in the LG [36]. We also present in Sec. 4.3.4 a large-r comparison with the LG-gauge values for the radial component of the SF, again comparing with [36].

4.1 Algorithm

The algorithm to numerically obtain the GSF in a Schwarzschild background for circular orbits follows the one used by Shah *et al.* [83], except where stated. Our method requires to numerically obtain ψ_0 for the ORG and ψ_4 for the IRG. We outline the steps of our numerical implementation here.

- Choose the orbit at radius r_0. Obtain the relevant orbital parameters $\mathcal{E} = \frac{r_0 f_0}{\sqrt{r_0^2 - 3Mr_0}}$, $\mathcal{L} = \sqrt{\frac{r_0^2 M}{r_0 - 3M}}$ and $\Omega^2 = \frac{M}{r_0^3}$, with $f_0 \equiv 1 - 2M/r_0$. We fix the maximum number of modes to compute, $\ell_{\max} = 80$. This choice of $\ell_{\max}$ comes as a trade-off between controlling the numerical error of the large-ℓ modes and keeping the computational time manageable.

- For each static mode ($m = 0$) with $\ell \geq 2$ we analytically calculate the radial function $R_0(r)$ via Eq. (4.2). We obtain $R_4(r)$ using $R_4(r) = r^4 f^2 \bar{R}_0(r)$ with $f \equiv 1 - 2M/r$.

- For each $m \neq 0$ we numerically integrate the radial Sasaki-Nakamura equation in r_* with suitable boundary-conditions [95] (see Sec. 4.2.2). The numerical value of the boundaries is set to $r_* = -95M$ for the EH and $r_* = 6000M$ for infinity. The integration is done using a modification from real to complex variables with quadruple precision of the adaptive stepsize Bulirsch-Stoer routine described in [140]. The integration routine returns the value of the function and the first derivative with respect to r_*. We algebraically relate the Sasaki-Nakamura field with $R_4(r)$ and $R_4'(r)$ at the particle's location using Eq. (4.7) and an analytical first derivative of it. To calculate second-order (and higher) derivatives of $R_4(r)$ we use the Teukolsky equation. The field $R_0(r)$ and its derivatives are obtained using $R_0(r) = r^{-4} f^{-2} \bar{R}_4(r)$ and the corresponding Teukolsky equation. The homogeneous solutions can also be found using the MST method described Sec. 2.1.5 as shown in [2]. The agreement between the two methods will be discussed in Sec. 4.3.4.

- We construct the inhomogeneous solutions using the standard variation of parameters method. We explicitly impose junction conditions for the homogeneous solutions and their first derivatives at $r = r_0$, using the gravitational source. Shah *et al.* [83] performed a formal integration of the Green's function over the source terms to construct the particular inhomogeneous solution ψ_0. The resulting fields $\psi_0(r)$ and $\psi_4(r)$ are discontinuous at the location of the particle.

- With the modes of fields $\psi_0(r)$ and $\psi_4(r)$ we find the harmonic modes of the Hertz potential $\Psi^{\mathrm{ORG}}_{\ell m}(r)$ and $\Psi^{\mathrm{IRG}}_{\ell m}(r)$, respectively. This is done by inverting the frequency-domain version of Eq. (2.20). Each mode of the Hertz potential is fully constructed by attaching the appropriate angular and time dependence: ${}_sY_{\ell m}(\theta, \varphi)\, e^{-i\omega t}$.

- The MP can be recovered on each of the regular sides of the no-string RGs using in turn Eq. (2.19a) and Eq. (2.19b), for the ORG and the IRG respectively.

- We analytically calculate the ℓ-modes of the retarded-force $F^{\ell}_{(\mathrm{ret})}$ (for each $\ell \geq 2$) by taking derivatives of the CCK-reconstructed MP expressed in terms of the Hertz potential. This is a convenient way of analytically identifying the different angular dependence on ${}_sY_{\ell m}(\theta,\varphi)$ with $s=\pm 2, \pm 1, 0$. This helps in the posterior re-expansion in terms of the usual scalar spherical harmonics. The explicit expressions of the ℓ-modes for the retarded force are given in Eq. (4.15) and Eq. (4.14).
- The remaining modes $\ell = 0, 1$ are added in the LG as discussed in Sec. 4.2.4. A method for including the gauge-invariant content of the non-radiative modes in the case of eccentric orbits around Kerr will be presented in Chapter 5.
- We use the definitions of spin-weighted spherical harmonics in terms of derivatives of scalar spherical harmonics [see Eq. (E.8) in the Appendix]. This way we can implement the appropriate coupling formulas [36] to re-express the r component of the retarded force in the basis of the scalar spherical harmonics where the mode-sum was derived [125, 139]. In Schwarzschild the coupling is finite and it relates a given ℓ-mode with its four nearest "neighbours", namely, contributions to a given ℓ spherical harmonic mode come from the $\ell \pm 2, \ell \pm 1$ and ℓ spin-weighted modes. The latter implies that we need to calculate $\ell_{\max}+2$ modes to have all the contributions to the $\ell_{\max}$ term in the mode-sum. This coupling and the implementation of the average mode-sum formula were missing in the prescription described in [83].
- After all the contributions to a single ℓ-mode are considered we apply the mode-sum regularization formula given by Eq. (3.96) to obtain the radial component of the GSF.
- We extrapolate the remaining $\ell > \ell_{\max}$ modes doing a fitting of the regularized modes as described in Sec. 4.2.5.
- We use the mode-sum formula Eq. (2.69) for the red-shift invariant H^R.
- We calculate the temporal component of the GSF with

$$F^t = \sum_{\ell,m} \frac{im\Omega \mathsf{m}}{2f} u^\alpha u^\beta h^{\ell m}_{\alpha\beta}, \tag{4.1}$$

  where $h^{\ell m}_{\alpha\beta}$ are the harmonic modes of the retarded MP in the basis of spin-weighted spherical harmonics in either the IRG or the ORG. The sum in Eq. (4.1) converges exponentially fast and does not require regularization.

4.2 Details of the implementation

4.2.1 Static modes

For the static modes ($m = 0 = \omega$) we have two linearly-independent solutions of Eq. (2.12a), which are proportional to associated Legendre polynomials of first (P_ℓ) and second (Q_ℓ) kind:

$$R_{0-}(r) \equiv \frac{\mathsf{P}^2_\ell(x)}{r^2 f} = -\frac{\Gamma(\ell+3)}{8M^4\Gamma(\ell-1)} {}_2F_1\left[2-\ell, \ell+3; 3, -\frac{r-2M}{2M}\right], \tag{4.2a}$$

$$R_{0+}(r) \equiv \frac{\mathsf{Q}^2_\ell(x)}{r^2 f} = \frac{2^\ell M^{\ell+1}\Gamma(\ell+3)\Gamma(\ell+1)}{r^{\ell+3} f^2} {}_2F_1\left[\ell-1, \ell+1; 2\ell+1, \frac{2M}{r-2M}\right], \tag{4.2b}$$

where ${}_2F_1$ are hypergeometric functions and $x \equiv \frac{r-M}{M}$.

The leading-order term of the asymptotic expansions of $R_{0-}(r)$ are given by

$$R_{0-} \propto \begin{cases} \dfrac{(2M)^\ell\,(2\ell)!}{(\ell-2)!\ell!} r^{\ell-2} & \text{when} \quad r \to \infty \\ \dfrac{(\ell-1)\ell(\ell+1)(\ell+2)(\ell^2+\ell-6)}{48M^3}(r-2M) & \text{when} \quad r \to 2M \end{cases} . \tag{4.3}$$

$R_{0-}(r)$ is regular at the EH but it fails to give the expected r^{-5} behaviour to have purely outgoing radiation at infinity. The regularity of R_{0-} is easily seen by transforming Eq. (2.12a) to $\tilde{R}_{0-} \equiv r^4 f^2 R_{0-}$, and moving to a coordinate system which is regular at the EH [141] (for example Kruskal coordinates[1]). The leading-order term for the expansions of $R_{0+}(r)$ are

$$R_{0+} \propto \begin{cases} r^{-\ell-3} & \text{when} \quad r \to \infty \\ \dfrac{(\ell+2)!}{2(\ell-2)!}(r-2M)^{-2} & \text{when} \quad r \to 2M \end{cases} , \tag{4.5}$$

$R_{0+}(r)$ is not regular at the EH since it includes a sub-leading logarithmic-term [141]. The asymptotic behaviour of these solutions was previously discussed by Barack and Ori near the EH [141] and by Poisson [142] and Keild *et al.* [143].

4.2.2 Chandrasekhar-Sasaki-Nakamura transformation

In the Schwarzschild case the radial part of Sasaki-Nakamura equation [Eq. (2.24) of Chapter 2 with $s=-2$] reduces to

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_{-2}(r)\right] X_{\ell m}(r) = 0, \quad \text{with} \quad V_{-2}(r) \equiv f\left(\frac{r\lambda_0 - 6M}{r^3}\right), \tag{4.6}$$

with $\lambda_s = (\ell-s)(\ell+s+1)$ as before. The relation between the solutions of the homogeneous Teukolsky equation with $s=-2$ and the function $X(r)$ was first found in [114]. In Schwarzschild it reads

$$R_4(r) = 2rf(r-3M+ir^2\omega)\frac{X'(r)}{\eta} + \left[rf\lambda_0 - 6Mf - 2r\omega(3iM - ir + r^2\omega)\right]\frac{X(r)}{\eta}, \tag{4.7}$$

where $\eta = \lambda_0\lambda_1 - 12iM\omega$, the prime denotes derivatives with respect of r and we have omitted the harmonic indices (ℓm) of $R_4(r)$ and $X(r)$. The field $R_0(r)$ is obtained, by virtue of the symmetries of the homogeneous Teukolsky equation [109], using

$$R_0 = \frac{\bar{R}_4}{r^4 f^2}. \tag{4.8}$$

To integrate Eq. (4.6) we set physical boundary-conditions. These are such as to give outgoing radiation at the EH and ingoing radiation at infinity [83]:

$$X^H = e^{i\omega r_*}\sum_{n=0}^{n_{\max}} c_n\left(\frac{r}{M}-2\right)^n \quad \text{and} \quad X^\infty = e^{-i\omega r_*}\sum_{n=0}^{n_{\max}} d_n\left(\frac{M}{r}\right)^n, \quad \text{respectively}, \tag{4.9}$$

with $c_n = 0 = d_n$ for $n<0$. The values of the coefficients c_n and d_n are calculated according to the

[1] Kruskal coordinates V, U are defined by

$$V \equiv e^{(t+r_*)/(4M)}, \qquad U \equiv -e^{(r_*-t)/(4M)}, \tag{4.4}$$

and the same angular coordinates $\{\theta, \varphi\}$.

recurrence relations [83]

$$
\begin{aligned}
c_n =& -\frac{i(n-3)M\omega}{2n(n+4iM\omega)}c_{n-3} + \frac{\ell(\ell+1)-(n-2)(n-3+12iM\omega)}{4n(n+4iM\omega)}c_{n-2} \\
&+ \frac{\ell(\ell+1)-2n^2+5n-6-12i(n-1)M\omega}{2n(n+4iM\omega)}c_{n-1},
\end{aligned}
\tag{4.10a}
$$

$$
d_n = \frac{-i}{2nM\omega}\left[(n-3)(n+1)d_{n-2} + (\ell+n)(\ell-n+1)d_{n-1}\right]. \tag{4.10b}
$$

Eq. (4.10) is obtained by substituting the expressions of Eq. (4.9) as ansätze for Eq. (4.6) (with $a=0$). The value of $n_{\max}$ is chosen so that the relative difference between the $n+1$ and the accumulated sum is smaller than a cut-off set to 10^{-15}.

4.2.3 Explicit expressions for the source and the force using IRG and ORG modes

The source and self-acceleration in the ORG were previously presented in [81, 83] while the IRG are included here for the first time. We have identified and corrected typos in the sources —in particular we have noticed an incompatibility between the corresponding equations for the source in [81] and [83]. The authors of [83] chose $\theta=\pi/2$ for the self-acceleration, which makes it difficult to read the full angular dependence of their expressions. The knowledge of this dependence is needed to change the basis from spin-weighted spherical harmonics to the usual spherical harmonics. Let us recall that the mode-sum scheme guarantees to give the right value of the GSF only on the basis were the regularization parameters are given. Only recently, these parameters became available in the basis of tensor harmonics [54]. Therefore, we keep the explicit dependence on θ.

We write the source of Teukolsky equation as a sum of three terms $T_{\pm 2} = T^{(0)} + T^{(1)} + T^{(2)}$, according to the angular dependence on the particle's location of each term. The explicit form —in the Schwarzschild case— of the source terms in the IRG is

$$
T^{(0)} = -\sum_{\ell m} \frac{\mathsf{m} u^t f_0^2}{4}\delta(r-r_0)\left(\lambda_0\lambda_1\right)^{1/2} {}_{-2}Y_{\ell m}(\theta,\varphi)\,\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t_0\right), \tag{4.11a}
$$

$$
\begin{aligned}
T^{(1)} =& \sum_{\ell m} \frac{\mathsf{m}\Omega u^t f_0 r_0^2}{2}\left[i f_0 \delta'(r-r_0) - \left(m\Omega + \frac{4iM}{r_0^2}\right)\delta(r-r_0)\right]\lambda_1^{1/2} \\
& {}_{-2}Y_{\ell m}(\theta,\varphi)\,{}_{-1}\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t_0\right),
\end{aligned}
\tag{4.11b}
$$

$$
\begin{aligned}
T^{(2)} =& \sum_{\ell m} \frac{\mathsf{m}\Omega^2 u^t r_0^4}{4}\left[f_0^2\delta''(r-r_0) + \left(2im\Omega f_0 - \frac{2(r_0+2M)f_0}{r_0^2}\right)\delta'(r-r_0)\right. \\
& \left. -\left(m^2\Omega^2 + \frac{2im\Omega(r_0+M)}{r_0^2} - \frac{2(4M-r_0)}{r_0^3}\right)\delta(r-r_0)\right] {}_{-2}Y_{\ell m}(\theta,\varphi)\,{}_{-2}\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t_0\right).
\end{aligned}
\tag{4.11c}
$$

The corresponding source of the ORG is

$$
T^{(0)} = -\sum_{\ell m} \frac{\mathsf{m} u^t}{r_0^4}\delta(r-r_0)\left(\lambda_0\lambda_1\right)^{1/2}\; {}_{2}Y_{\ell m}(\theta,\varphi)\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t_0\right), \tag{4.12a}
$$

$$
T^{(1)} = \sum_{\ell m} 2\frac{\mathsf{m}\Omega u^t}{r_0^2}\left[i\delta'(r-r_0) + \left(\frac{m\Omega}{f_0} + \frac{4i}{r_0}\right)\delta(r-r_0)\right]\lambda_1^{1/2}\,{}_{2}Y_{\ell m}(\theta,\varphi)\,{}_{1}\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t_0\right), \tag{4.12b}
$$

$$T^{(2)} = \sum_{\ell m} \mathsf{m}\Omega^2 u^t \left[\delta''(r-r_0) + \left(\frac{6}{r_0} - \frac{2im\Omega}{f_0} \right) \delta'(r-r_0) \right.$$
$$\left. - \left(\frac{m^2\Omega^2}{f_0^2} + \frac{2im\Omega(3r_0 - 5M)}{r_0^2 f_0^2} - \frac{10}{r_0^2} \right) \delta(r-r_0) \right] {}_2Y_{\ell m}(\theta,\varphi) {}_2\bar{Y}_{\ell m}\left(\frac{\pi}{2}, \Omega t_0\right). \tag{4.12c}$$

The radial component of the retarded force[2] in the IRG can be computed, from Eq. (2.46), as

$$F^r_{\text{ret}} = \sum_{i=1}^{6} F^r_i, \tag{4.13}$$

where the frequency-domain modes of F^r_i are

$$F^r_{1\,\ell m} = \frac{1}{4r_0^2}(u^t)^2 \mathsf{m}(\lambda_0\lambda_1)^{1/2} \left[f_0\partial_r + 2\partial_t - \frac{2}{r_0}\left(f_0 - \frac{M}{r_0}\right) \right] (\Psi_{\ell m} + \bar{\Psi}_{\ell m}) Y_{\ell m}(\theta,\varphi), \tag{4.14a}$$

$$F^r_{2\,\ell m} = \frac{1}{4f_0 r_0^4}(u^t)^2 M \mathsf{m}(\lambda_0\lambda_1)^{1/2} (r_0 f_0 \partial_r + r_0\partial_t - 4f_0)(\Psi_{\ell m} + \bar{\Psi}_{\ell m}) \sin^2\theta\, Y_{\ell m}(\theta,\varphi), \tag{4.14b}$$

$$F^r_{3\,\ell m} = \frac{1}{4r_0^2}(u^t)^2 \Omega i \mathsf{m} \lambda_0 \lambda_1^{1/2} (\Psi_{\ell m} - \bar{\Psi}_{\ell m}) \sin\theta \left[{}_1Y_{\ell m}(\theta,\varphi) + {}_{-1}Y_{\ell m}(\theta,\varphi) \right], \tag{4.14c}$$

$$F^r_{4\,\ell m} = -\frac{1}{2f_0}(u^t)^2 \mathsf{m}\Omega i \lambda_1^{1/2} \left[\partial_t^2 + 2f_0\partial_t\partial_r + f_0^2\partial_r^2 - \frac{2}{r_0^2}(M + r_0 f_0)\partial_t - \frac{2f_0^2}{r_0}\partial_r + \frac{2f_0^2}{r_0^2} \right] \times$$
$$(\Psi_{\ell m} - \bar{\Psi}_{\ell m}) \sin\theta \, {}_{-1}Y_{\ell m}(\theta,\varphi), \tag{4.14d}$$

$$F^r_{5\,\ell m} = \frac{1}{4f_0 r_0^4}(u^t)^2 M \mathsf{m} \lambda_1 (r_0 f_0\partial_r + r_0\partial_t - 2f_0)(\Psi_{\ell m} + \bar{\Psi}_{\ell m}) \sin^2\theta \, {}_{-2}Y_{\ell m}(\theta,\varphi), \tag{4.14e}$$

$$F^r_{6\,\ell m} = -\frac{1}{4f_0^2 r_0^5}(u^t)^2 M \mathsf{m} \left[r_0^4 f_0 \partial_t^2 \partial_r + 2r_0^4 f_0^2 \partial_t \partial_r^2 + r_0^4 f_0^3 \partial_r^3 + 2r_0^3 f_0^2 \partial_t^2 + 2r_0^2 f_0 (r_0 - 5M)\partial_t\partial_r \right.$$
$$\left. -2(r_0^2 - 6Mr_0 + 4M^2)\partial_t - 2r_0^2 f_0^3 \partial_r \right] (\Psi_{\ell m} + \bar{\Psi}_{\ell m}) \sin^2\theta \, {}_{-2}Y_{\ell m}(\theta,\varphi), \tag{4.14f}$$

where we have omitted to specify that Ψ is the IRG hertz potential. The corresponding terms for the ORG are

$$F^r_{1\,\ell m} = -\frac{1}{16} r_0 f_0^2 (u^t)^2 \mathsf{m}(\lambda_0\lambda_1)^{1/2} \left[r_0 f_0 \partial_r - 2r_0\partial_t + 2\left(f_0 + \frac{3M}{r_0}\right) \right] \times$$
$$(\Psi_{\ell m} + \bar{\Psi}_{\ell m}) Y_{\ell m}(\theta,\varphi), \tag{4.15a}$$

$$F^r_{2\,\ell m} = -\frac{1}{16} f_0 (u^t)^2 M \mathsf{m}(\lambda_0\lambda_1)^{1/2} \left[r_0 f_0 \partial_r - r_0\partial_t + 2(1 + f_0) \right] (\Psi_{\ell m} + \bar{\Psi}_{\ell m}) \times$$
$$\sin^2\theta Y_{\ell m}(\theta,\varphi), \tag{4.15b}$$

$$F^r_{3\,\ell m} = \frac{1}{16} r_0^2 f_0^2 (u^t)^2 \Omega i \mathsf{m} \lambda_0 \lambda_1^{1/2} (\Psi_{\ell m} - \bar{\Psi}_{\ell m}) \sin\theta \left[{}_1Y_{\ell m}(\theta,\varphi) + {}_{-1}Y_{\ell m}(\theta,\varphi) \right], \tag{4.15c}$$

$$F^r_{4\,\ell m} = -\frac{1}{8} f_0 (u^t)^2 \mathsf{m} r_0^4 \Omega i \lambda_1^{1/2} \left[\partial_t^2 - 2f_0\partial_t\partial_r + f_0^2\partial_r^2 - \frac{3}{r_0}(1 + f_0)\partial_t + \frac{2f_0}{r_0^2}(3r_0 - 2M)\partial_r \right.$$
$$\left. + \frac{2}{r_0^2}(1 + 2f_0) \right] (\Psi_{\ell m} - \bar{\Psi}_{\ell m}) \sin\theta \, {}_1Y_{\ell m}(\theta,\varphi), \tag{4.15d}$$

$$F^r_{5\,\ell m} = -\frac{1}{16} f_0 (u^t)^2 M \mathsf{m} \lambda_1 (r_0\partial_t - r_0 f_0 \partial_r - 2)(\Psi_{\ell m} + \bar{\Psi}_{\ell m}) \sin^2\theta \, {}_2Y_{\ell m}(\theta,\varphi), \tag{4.15e}$$

[2] In [2] we showed the formulae to obtain this component from the components of the MP projected along the Newman-Penrose tetrad. This is just an extra step that we choose not to include here.

$$F^r_{6\,\ell m} = \frac{1}{16} f_0 (u^t)^2 M \mathsf{m} \left[r_0^3 \partial_t^2 \partial_r - 2 r_0^3 f_0 \partial_t \partial_r^2 + r_0^3 f_0^2 \partial_r^3 + 6 r_0^2 \partial_t^2 - 2 r_0 (9 r_0 - 13M) \partial_t \partial_r \right.$$
$$+ 12 r_0^2 f_0 (r_0 - M) \partial_r^2 - 6(5 r_0 - 4M) \partial_t + \frac{2}{r_0} (17 r_0^2 - 32 r_0 M + 8M^2) \partial_r$$
$$\left. - \frac{16}{r_0^2} \left(M^2 - r_0^2 \right) \right] (\Psi_{\ell m} + \bar{\Psi}_{\ell m}) \sin^2\theta \; {}_2Y_{\ell m}(\theta, \varphi). \tag{4.15f}$$

The spin-weighted spherical harmonics ${}_sY_{\ell m}(\theta, \varphi)$ appearing in Eqs. (4.14) and (4.15) are re-expanded in terms of scalar spherical harmonics using Eq. (E.8). For a given spherical harmonic we get

$$F^r_{\ell m} = Y_{\ell m}(\theta, \varphi) \left[\mathcal{F}^r_{(-2)\ell-2,m} + \mathcal{F}^r_{(-1)\ell-1,m} + \mathcal{F}^r_{(0)\ell m} + \mathcal{F}^r_{(+1)\ell+1,m} + \mathcal{F}^r_{(+2)\ell+2,m} \right], \tag{4.16}$$

where

$$\mathcal{F}^r_{(\pm2)\ell m} = \alpha^{\ell m}_{(\pm2)} f^r_{2\,\ell m} + (f^r_{5\,\ell m} + f^r_{6\,\ell m}) \frac{\gamma^{\ell m}_{(\pm2)} - \beta^{\ell m}_{(\pm2)}}{(\lambda_0 \lambda_1)^{1/2}} \pm \frac{\beta^{\ell m}_{(\pm2)}}{\lambda_0^{1/2}} f^r_{4\,\ell m},$$
$$\mathcal{F}^r_{(\pm1)\ell m} = \pm \frac{\delta^{\ell m}_{(\pm1)}}{\lambda_0^{1/2}} f^r_{4\,\ell m} + \frac{2m\epsilon^{\ell m}_{(\pm1)}}{(\lambda_0 \lambda_1)^{1/2}} f^r_{6\,\ell m},$$
$$\mathcal{F}^r_{(0)\ell m} = f^r_{1\,\ell m} + f^r_{2\,\ell m} \alpha^{\ell m}_{(0)} + (2 f^r_{3\,\ell m} \mp f^r_{4\,\ell m}) \frac{m}{\lambda_0^{1/2}} + (f^r_{5\,\ell m} + f^r_{6\,\ell m}) \frac{m^2 - \beta_0^{\ell m} + \gamma^{\ell m}_{(0)}}{(\lambda_0 \lambda_1)^{1/2}}. \tag{4.17}$$

The functions $f^r_{i\,\ell m}$ are the angle-independent coefficients of Eqs. (4.14) or (4.15): for example, $F^r_{1\,\ell m} = f^r_{1\,\ell m} Y_{\ell m}(\theta, \varphi)$. Notice the sign dependence of the coefficient multiplying $f^r_{4\,\ell m}$ — the upper sign is for the IRG modes while the lower sign for the ORG modes. The coupling coefficients $\alpha^{\ell m}$, $\beta^{\ell m}$, $\gamma^{\ell m}$, $\delta^{\ell m}$ and $\epsilon^{\ell m}$ are given explicitly in [36] and included as Eq. (E.9) of Appendix E.

4.2.4 Completion of the reconstruction

In Schwarzschild the CCK-reconstruction from Weyl scalars recovers the $\ell \geq 2$ 'spin-weighted' sector. Wald showed that the solution needs to be completed by including corrections to the Kerr mass and AM [98] (and perturbations to C-metrics and Kerr-NUT metrics, which he proved not to be physical in vacuum). Friedman *et al.* showed that the C and Kerr-NUT perturbations can be ruled out in the vacuum spacetime outside the trajectory of a point particle [83]. In Chapter 5, we give a full discussion of the inclusion of the completion piece in Kerr.

In Schwarzschild we know that the remaining part of the solution is the monopole and dipole of the linearised EFE. We include these low modes in the LG, where gauge discontinuities are avoided.

The shift in the mass parameter across the $r = r_0$ surface is encoded in the monopole part of the solution (the $\ell = 0$, $m = 0$ mode). In the LG the nonvanishing components of these perturbations are [77]

$$h^{\ell=0}_{tt}(r \leq r_0) = -\frac{AfMP(r)}{r^3}, \quad h^{\ell=0}_{rr}(r \leq r_0) = \frac{AMQ(r)}{r^3 f}, \tag{4.18a}$$
$$h^{\ell=0}_{\theta\theta}(r \leq r_0) = \sin^{-2}\theta \, h^{\ell=0}_{\varphi\varphi}(r \leq r_0) = AfMP(r), \tag{4.18b}$$

where

$$A = \frac{2\mathsf{m}\mathcal{E}}{3Mr_0 f_0}\left[M - (r_0 - 3M)\ln f_0\right], \tag{4.19}$$

$$P(r) = r^2 + 2Mr + 4M^2, \qquad Q(r) = r^3 - Mr^2 - 2M^2 r + 12M^3. \tag{4.20}$$

The external components are

$$\begin{aligned}
h_{tt}^{\ell=0}(r \geq r_0) =& \frac{2\mathsf{m}\mathcal{E}}{3r^4 r_0 f_0}\Big\{ 3r^3(r_0 - r) + M^2(r_0^2 - 12Mr_0 + 8M^2) + \\
& (r_0 - 3M)\left[rP(r) f \ln f - rM(r + 4M) + 8M^3 \ln\left(\frac{r_0}{r}\right)\right]\Big\}, & (4.21a)\\
h_{rr}^{\ell=0}(r \geq r_0) =& -\frac{2\mathsf{m}\mathcal{E}}{3Mr^4 r_0 f_0 f^2}\Big\{ 3M^2\left(r_0^2 - 12Mr_0 + 8M^2\right) - 2Mr\left(r_0^2 - 6Mr_0 - 10M^2\right) \\
& -r^3 r_0 + (r_0 - 3M)\left[5Mr^2 + \frac{r}{M}Q(r) f \ln f - 8M^2(2r - 3M)\ln\left(\frac{r_0}{r}\right)\right]\Big\}, & (4.21b)\\
h_{\theta\theta}^{\ell=0}(r \geq r_0) =& \sin^{-2}\theta h_{\varphi\varphi}^{\ell=0}(r \geq r_0) = -\frac{2\mathsf{m}\mathcal{E}}{9rr_0 f_0}\Big\{ 3r_0^2 M - 80M^2 r_0 + 156M^3 \\
& +(r_0 - 3M)\left[44M^2 + 3\frac{r}{M}P(r) f \ln f - 3r^2 - 12Mr + 24M^2 \ln\left(\frac{r_0}{r}\right)\right]\Big\}. & (4.21c)
\end{aligned}$$

Detweiler and Poisson showed [117] that the LG metric given by Eqs. (4.18) and (4.21) is unique and any gauge transformation within the class of LGs would make the metric singular at infinity, at the EH or in both limits at the same time. Notice that as $r \to \infty$ the tt component of the metric tends to a constant value, i.e., the metric is not asymptotically flat. This pathology of the gauge can be cured by moving away from the LG by performing a shift $t \to t(1+\alpha)$ with constant $\alpha \sim O(\mathsf{m})$. It is straightforward to show using Eq. (6) of [19] that this gauge transformation does not contribute to the values of the radial component of the GSF.

For $\ell = 1$, $m = 0$ there is only one non-vanishing component of the MP, namely [77]

$$h_{t\varphi}^{\ell=1,m=0}(r) = -2\mathsf{m}\mathcal{L}\sin^2\theta\left[\frac{r^2}{r_0^3}\Theta(r_0 - r) + \frac{1}{r}\Theta(r - r_0)\right], \tag{4.22}$$

where Θ is the usual step function.

We calculate the contribution to the retarded force from the $\ell = 0, 1$ solutions by directly substituting (4.18), (4.21) and (4.22) in Eq. (2.46). The resulting contribution to the force agrees with the values first obtained by Detweiler and Poisson at $\theta = \frac{\pi}{2}$ [117].

The $\ell = 1$, $m = 1$ mode is added numerically using the prescription described in [117]. This mode is related to the motion around the centre of mass of the BH-particle system. A detailed physical interpretation and comparison with a PN calculation can be found in [117].

4.2.5 Fitting the large-ℓ tail

Let us now describe how we include the contribution from remaining $\ell > \ell_{\max}$ modes. We include the large-ℓ tail for each of the one-side limits of the retarded force; instead of taking the average and only afterwards including the large-ℓ tail. This allows for an intermediate comparison with the values of [83], where only one of the sided values was included. Each of the side-dependent values is computed according to

$$F^\alpha_\pm = \sum_{\ell=0}^{\ell_{\max}}\left[\left(F^\alpha_{(\mathrm{ret})}\right)^\ell_\pm \mp A^\alpha L - B^\alpha\right] - D^\alpha_\pm + \sum_{\ell_{\max}+1}^{\infty}\left[\sum_{k=2}^{k_{\max}}\frac{\tilde{E}_k^\pm}{L^k}\right] + O\left(\frac{1}{\ell_{\max}^{k_{\max}}}\right), \tag{4.23}$$

where the $\pm$ superscripts indicate the limit $r \to r_0^\pm$ from which is calculated, and in general $\tilde{E}_k^+ \neq \tilde{E}_k^-$. We extract the coefficients $\tilde{E}_k^\pm$ by matching $\left[\left(F^\alpha_{(\mathrm{ret})}\right)^\ell_\pm - A^\alpha_\pm L - B^\alpha\right]$ (from a certain $\ell_{\min}$ to $\ell_{\max}$) to a power series of the form[3]

$$\sum_{k=2}^{k_{\max}} \frac{\tilde{E}_k^\pm}{L^k} \equiv \frac{\tilde{E}_2^\pm}{L^2} + \frac{\tilde{E}_4^\pm}{L^4} + \frac{\tilde{E}_5^\pm}{L^5} + \frac{\tilde{E}_6^\pm}{L^6} + \cdots + \frac{\tilde{E}_{k_{\max}}^\pm}{L^{k_{\max}}}. \tag{4.24}$$

The best-fit values of $\tilde{E}_k^\pm$ are extracted using the least-squares fit (implemented in Mathematica), the errors are estimated by modifying $\ell_{\min}$ and $k_{\max}$ using the procedure described in [83]. We find that the singular part of the SF contains odd, negative powers of $L = (\ell + 1/2)$ on either side of r_0. If the tail was fitted using the averaged modes of the retarded force, only even powers of L would appear. The SF is then calculated using Eq. (3.96), where the $\ell > \ell_{\max}$ tail is included using the best numerical fit.

An interesting detail to be noted here is that we numerically find F^α_{self}, unlike the sided-limits $F^\alpha_\pm$, to be independent of its mode decomposition: we get the same average if we write the GSF as a sum over different spin-weighted spherical harmonics, as done in Eqs. (4.14) -(4.15), or as a sum over ordinary spherical harmonics, as done in Eq. (4.16). In other words, our numerical experiment suggests that for this setup the value of the regularization parameters is not affected by the choice of harmonic basis. We will further comment on this numerical result in Sec. 4.3.4.

4.3 Numerical results

4.3.1 Sources of numerical error

Let is identify four independent sources of errors in the results we will present: (i) from the integration of Sasaki-Nakamura equation, (ii) from the MST method, (iii) from the inclusion of the large-ℓ tail, and (iv) from the numerical dipole-mode. These are further discussed next.

We experimented with the location of the boundaries to test the robustness of this choice and found that even a more modest choice ($r_* = -60M$ for the EH and $r_* = 1000M$ for infinity) gives the same result. The computational difference from moving the boundaries comes only in the number of calculated terms required for the boundary conditions. We allow a relative error of $1/10^{15}$ on each step of the numerical integration. These errors propagate to give a relative error $\sim 1/10^{12}$ in the value of each harmonic of the Sasaki-Nakamura field and its first derivative. However these systematic errors turn out to be subdominant to those from the large-ℓ tail.

The error in calculating solutions for the radial part of the homogeneous Teukolsky equation using the MST-method can be reduced by, first, numerically calculating ν with a very high accuracy (usually higher than the one mentioned in Table I), and second, by choosing a high enough $n_{\max}$, the cut-off in n-series of the hypergeometric and confluent hypergeometric series in Eq. (2.32). To reduce the computation time, one can find relations between the derivatives of the hypergeometric and confluent hypergeometric functions appearing in Eq. (2.32) using a combination of various Gauss's relations for contiguous functions.

[3] In [45], a series of the form $E_2/((2\ell-1)(2\ell+3)) + E_4/((2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5))...$ is used to fit the singular part of the force and increase the convergence rate [62]. The sum from $\ell = 0$ to infinity of each term in the series is zero and does not contribute to the SF. Analytical expression for E_2, E_4, E_6 were given in [45] and we verify that they have different values than the parameters we would obtain by fitting the averaged modes to a similar series. Namely the coefficients E_k are gauge dependent.

The total value of the radial component of the SF has two pieces, as it was explained in Sec. 4.2.5. The first one $F_r^{\ell\leq\ell_{\max}}$ is obtained by the methods described in Sec. 4.2. The remaining large-ℓ tail $F_r^{\ell>\ell_{\max}}$ is extrapolated numerically as described in Sec. 4.2.5 using $\tilde{N} = \ell_{\max} - \ell_{\min} + 1$ of the regularized large-ℓ-modes. We varied $\ell_{\max}$, $\ell_{\min}$ and $\tilde{N}$ to estimate the total error of the large-ℓ tail.

Each of the two methods to solve Teukolsky equation (numerical integration and MST) give a different large-ℓ tail, with their corresponding errors. The accuracy with which the coefficients $\tilde{E}_k^{\pm}$ in Eq. (4.23) can be extracted depends on $\tilde{N}$ and the accuracy of the regularized modes. Due to its high accuracy, the MST method allows a very accurate extrapolation of the tail. With respect of the values reported in Table 4.1, the total tail is responsible for the agreement in the last 4-5 digits between the Sasaki-Nakamura and MST methods. The relative difference of the two methods is within the error bars reported for the numerical-integration computation. These error bars were estimated by varying the numerical parameters of the fitting.

The error in the MST method is dominated by the even-dipole mode, which is estimated from varying the inner boundary of the integration $[r_{\min} = (2+\epsilon)M]$ from $\epsilon = 10^{-9}$ to $\epsilon = 10^{-6}$. The error of this piece is below that of the large-ℓ tail for the numerical-integration method. As explained in Sec. 4.2.4, this piece is included in the LG. The difficulty of obtaining this mode with higher accuracy is related to a numerical matrix-inversion, which is needed to obtain the solution to a coupled system of ordinary-differential equations [117].

4.3.2 Convergence of the mode sums for F^r and F^t

A feature of the mode-sum regularization procedure is that it provides an immediate validity test of the results. If the retarded values of the force *and* the implementation of the coupling formulae to express the force as purely spherical harmonics contain a systematic error, then the sum over ℓ-modes after regularization may not converge, see Fig. 4.1. We recall that it is also required to consistently use the off-the-particle extension of the four velocity (we used the same extension as the LG regularization parameters A^α and B^α of Appendix B) and metric when calculating the retarded-force and the regularization parameters, otherwise the mode-sum will not give the correct value.

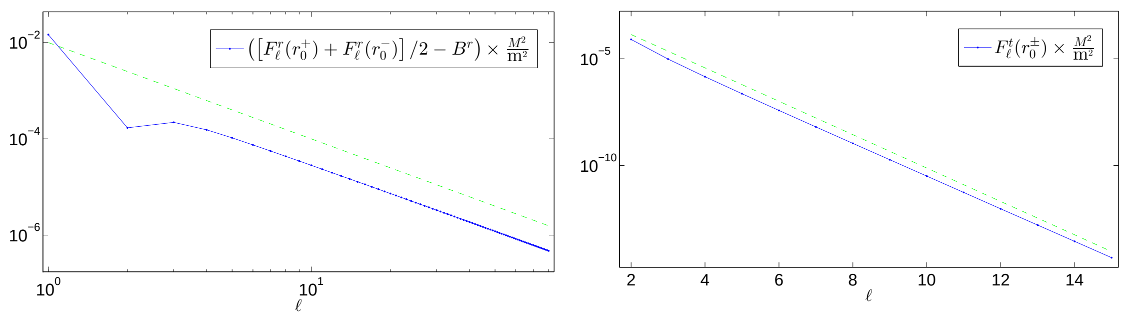

Figure 4.1: Left Panel shows the large-ℓ behaviour for the modes of the r component of the SF (solid blue line in log-log scale) computed using the average version of the mode-sum formula [Eq. (3.96) with $\ell_{\max} = 80$, only A^rL and B^r are subtracted]. The $1/\ell^2$ reference line (green dashed) confirms the expected fall-off at large ℓ. The right panel shows the convergence of the t component (solid blue line in semi-log scale) of the SF. In this case the reference line (green dashed) shows exponential convergence. In both cases the results correspond to an orbital radius of $r_0 = 10M$.

For the radial component (left panel of Fig. 4.1) we found that the sum over ℓ modes of the average $\frac{1}{2}\left[F^r(r_0^+) + F^r(r_0^-)\right]$ converges as $1/\ell$, with the green (dashed) line as reference, as expected. In the case of the time component (right panel of Fig. 4.1), we show the exponential convergence of the sum, also as expected.

4.3.3 Flux of energy and invariant red-shift

The total flux of emitted energy $\mathsf{m}\dot{\mathcal{E}}$ is directly proportional to the t component of the GSF [97, 118]:

$$\mathsf{m}\dot{\mathcal{E}} \equiv \mathsf{m}\frac{d\mathcal{E}}{dt} = -\frac{F_t}{u^t}. \tag{4.25}$$

Let us denote the fluxes at infinity by $\frac{d\mathcal{E}_\infty}{dt}$ and at the EH by $\frac{d\mathcal{E}_{\rm EH}}{dt}$. We calculate them following the procedure given in [95]. We verify numerically that

$$-\mathsf{m}\frac{d\mathcal{E}}{dt} = \mathsf{m}\frac{d\mathcal{E}_{\rm EH}}{dt} + \mathsf{m}\frac{d\mathcal{E}_\infty}{dt} \tag{4.26}$$

is satisfied up to $\sim 10^{-5}$ of relative difference for all radii considered, within the range of $6M - 150M$. The discrepancy comes from the numerical error accumulated during the long integration of the field in the tortoise coordinate r_* from the EH to where we set our numerical infinity.

Our results are consistent with previous works by Barack and Sago [36], and more recently Gundlach *et al.* [144]. Our calculation shows that at the innermost stable circular-orbit (ISCO) the ratio $\dot{\mathcal{E}}_{\rm EH}/\dot{\mathcal{E}}_\infty$ has a value of 3.27×10^{-3} and decreases monotonically with r_0 up to 2.06×10^{-9} when $r_0 = 150M$, in agreement with [36, 144]

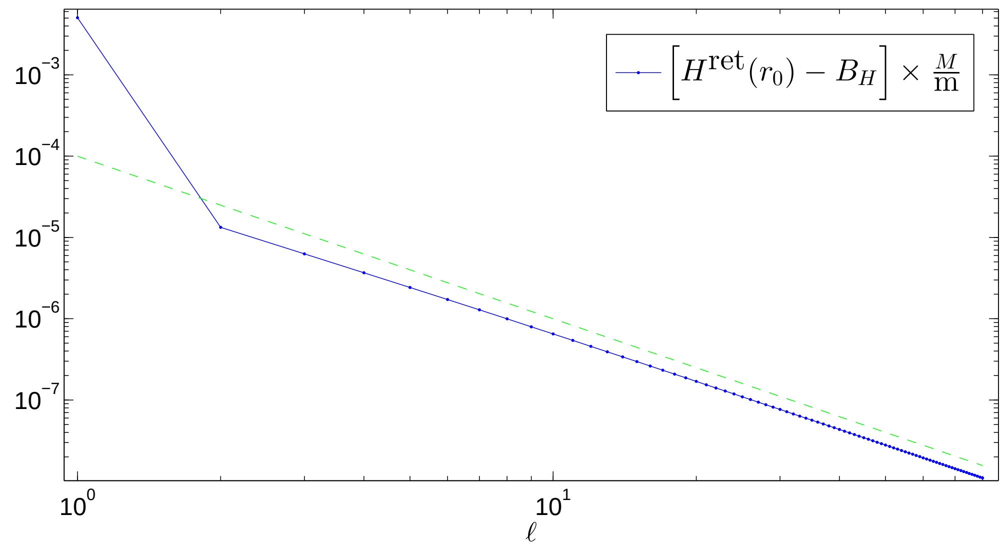

Figure 4.2: Convergence of the regularized ℓ modes of $H^R = \frac{1}{2}h^R_{\alpha\beta}u^\alpha u^\beta$ at orbital radius of $r_0 = 200M$. We calculate numerically $\ell_{\rm max} = 80$ modes and regularise with the analytical value of B_H [given in Eq. (2.71)]. The $1/\ell^2$ reference line (green dashed) confirms the expected fall-off at large ℓ.

We obtain numerical results for the red-shift invariant H^R defined in Eq. (2.66), see Fig. 4.2. The tail is included in the same way as that of the force (and described in Sec. 4.2.5). For an orbital radius of $r_0 = 6M$ we get $H^{R\pm} = -0.52362 \times M/\mathsf{m}$ while at $r_0 = 200M$ we obtain $H^{R\pm} = -0.010076 \times M/\mathsf{m}$ which is consistent with the values given by Sago *et al.* [82] to all significant digits shown.

r_0/M	$F^{r\,\mathrm{Num}}(r_0) \times \frac{M^2}{\mathrm{m}^2}$	$F^{r\,\mathrm{MST}}(r_0) \times \frac{M^2}{\mathrm{m}^2}$	$F^{r\,\mathrm{Lor}}(r_0) \times \frac{M^2}{\mathrm{m}^2}$
6	0.03350126(1)	0.033501265(1)	0.0244661
7	0.026070691(5)	0.0260706936(1)	0.0214989
8	0.020941671(3)	0.02094167456(7)	0.0183577
9	0.017214435(1)	0.01721443676(8)	0.0156369
10	0.0144093850(9)	0.01440938542(6)	0.0133895
12	0.0105299277(5)	0.01052992732(2)	0.0100463
14	0.008031952(1)	0.00803195180(1)	0.00777307
16	0.006328227(1)	0.006328226988(6)	–
18	0.005114225(1)	0.005114225196(3)	–
20	0.0042187145(9)	0.004218713944(1)	0.00415706
24	0.003011654(1)	0.0030116542558(6)	–
28	0.002257118(5)	0.0022571178017(2)	–
32	0.001754261(4)	0.0017542618884(1)	–
36	0.001402452(3)	0.00140245195919(6)	–
40	0.0011467454(5)	0.00114674532583(3)	0.00114288
50	0.0007465337(2)	0.00074653378046(1)	0.000744949
60	0.00052437948(8)	0.000524379436446(3)	0.000523616
70	0.00038842358(5)	0.000388423560775(1)	0.00038801
80	0.00029922175(3)	0.0002992217373675(7)	0.000298979
90	0.00023755802(2)	0.0002375580134958(4)	0.000237406
100	0.00019316231(2)	0.0001931623007419(2)	0.000193063
120	0.00013491660(1)	0.00013491660149634(8)	0.000134868
140	0.000099532396(7)	0.00009953239215925(3)	–
160	0.000076441055(5)	0.00007644105294526(1)	–
180	0.000060543785(4)	0.00006054378560513(1)	–
200	0.000049135297(3)	0.000049135296208105(1)	–

Table 4.1: Comparison between the radial component of the GSF, for different values of r_0/M. The second column shows the values computed using numerical integration of Sasaki-Nakamura equation while the values in the third column are calculated in the ORG using the MST method. The quantities in parenthesis indicate the estimated error on the last quoted decimal shown. The error in the second column is estimated by changing the numerical parameters of the fitting that contributes to the large-ℓ tail. The error quoted in the third column is estimated from moving the inner boundary when numerically solving the $\ell = 1$, $m = 1$ multipole. The LG values are taken from [36] where the corresponding error estimation can be found. Note the asymptotic agreement for large r between the RG values (first and second column) with the LG values in the third column, as discussed.

4.3.4 Analysis of results and further consistency checks

We now present a comparison between two calculations of the radial component of the GSF: one using the MST method, and another one using numerical integration of the Sasaki-Nakamura field. Fig. 4.3 shows in blue (solid line) the fractional difference in $F^r(r_0)$ for a sample of radii. The values of F^r are obtained using Eq. (3.96) with $\ell_{\max} = 80$ calculated modes and a fitted tail of the form given by Eq. (4.24) on each side-limit. In Fig. 4.3 the red (dashed line) shows the fractional difference between the IRG and the ORG values, before including the large-ℓ tail. In this case both results were obtained by using the Sasaki-Nakamura method. The values used to generate the plot can be found in Table 4.1.

The method of Shah *et al.* [83] does not follow a rigorous method to implement the mode-sum. They used the LG mode-sum for their implementation and guessed (incorrectly) that $\delta D^{\alpha}_{\pm} = 0$, independently of the gauge. The authors of [83] submitted an *erratum* clarifying the issues we have raised in [1, 2].

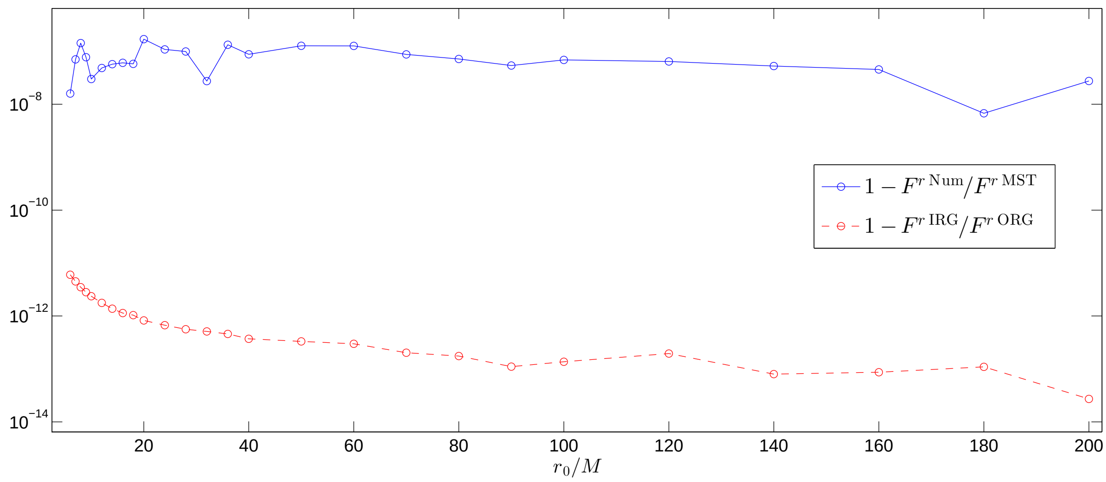

Figure 4.3: Relative difference for the averaged r component of the SF. The blue (solid) line compares the values in the ORG computed through numerical integration of the Sasaki-Nakamura field against the values calculated using the MST analytical method. The estimated error of the numerical method is dominated by the $\ell > \ell_{\max}$ fitted term, while the error of the MST method is dominated by the inclusion of the even dipole mode as discussed in Sec. 4.3.1, which is below the difference of the two methods. These errors are shown explicitly in Table 4.1. The red (dashed) line compares the relative difference between the force calculated from the IRG and the ORG modes (using numerical integration) before including the large-ℓ tail. This difference appears consistent with the numerical error.

In principle the GSF in the ORG and the IRG could have different values. In fact by just looking at Eqs. (4.14) and (4.15) it is not obvious that the results would agree. The Hertz potential Ψ takes a different form when calculated in the ORG and IRG. However, for circular orbits of Schwarzschild it turns out that the MP and the values of the SF in the IRG and ORG give the same value, as shown in Fig. 4.3. The equality of the MP in both gauges can be shown analytically using the symmetries of Teukolsky equation.

Our GSF values agree at large r with those in the LG [36], See Table 4.1 and Fig. 4.4. To see this let us consider the change in the GSF due to the gauge transformation from LG to ORG generated by $\xi^r(r)$ which is given by Eq. (A25) of [81]:

$$\delta_\xi F_r^{\mathrm{Lor}\to\mathrm{Rad}} = \frac{3M\mathsf{m}}{r_0^3 - 3Mr_0^2}\xi^r(r_0). \tag{4.27}$$

Assuming ξ^r falls off at least as r^{-1} then Eq. (4.27) would fall off as r^{-3}. In fact, the numerical data shows that the difference goes as r^{-4}, see Fig. 4.4.

4.3.5 Comparison of computational cost

A LG code for circular orbits in Schwarzschild calculates the GSF, in the strong-field regime, running on a standard desktop machine in approximately 2 hours with $\ell_{\max} \sim 25$ and a fractional accuracy of $\lesssim 10^{-4}$ for the higher modes [36]. Our numerical integration can achieve an accuracy of $\lesssim 10^{-12}$, for each mode, running on a single core of a standard desktop machine in about 30 minutes calculating the same number of modes. To integrate and calculate $\ell_{\max} = 85$ modes, and achieve the same accuracy $\lesssim 10^{-12}$ for each mode, it takes ~ 14.5 hours running on a single core.

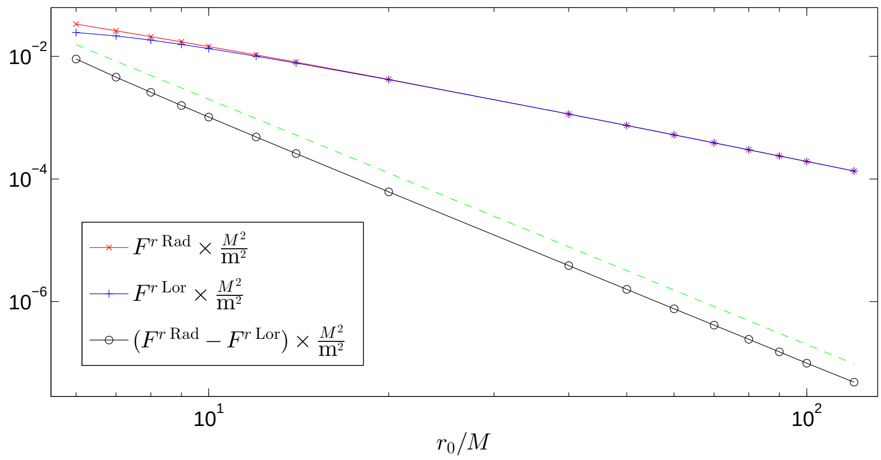

Figure 4.4: Comparison between the radial component of the SF in the LG and that of in RG (in log-log scale). The red line (with data points denoted by $\times$) corresponds to the values in the radiation gauge given in Table 4.1. The blue line (with data points denoted by $+$) are the values in the Lorenz gauge from Barack-Sago [36]. The black line (solid line with data points denoted by $\circ$) is the difference between $F^{r\mathrm{Rad}} - F^{r\mathrm{Lor}}$ and the green (dashed) line is a reference line $\sim r_0^{-4}$.

We expect the savings in computational time to be greater for eccentric orbits around Kerr. In particular, for eccentric orbits of Schwarzschild the time of frequency-domain calculation is comparable to that of the corresponding time-domain implementation [145].

Chapter 5

Completion of metric reconstruction for a particle orbiting a Kerr black hole

In Chapter 3 we obtained a practical method to calculate the GSF using the reconstructed modes of the RG perturbation. However, as we stated in Chapter 2, the CCK reconstruction gives only part of the MP. The remaining part of the solution, namely the completion piece, needs to be included separately. Wald showed that the completion piece is a pure mass and AM perturbation of Kerr, up to gauge [98] (and up to C-metric or Kerr-NUT, which are however irregular and thus unphysical). In the Schwarzschild case, the invariant content of the completion piece is purely in the $\ell = 0, 1$ modes. A simple LG completion piece can then be constructed following the works of Detweiler and Poisson [117], and Barack and Lousto [77], as we discussed in Chapter 4. However, in Kerr these perturbations are not confined to these multipoles only, since the EFE can not be decomposed in harmonic modes. This has been a longstanding problem of BH perturbation theory.

In this Chapter we address the problem of including the completion piece for orbits around Kerr. An initial investigation of the problem was carried out by Price [100]. Price constructed the completion piece as a sum of mass and AM perturbations of the metric, obtained by varying the Kerr metric (written in BL coordinates) with respect to the mass and spin parameters, respectively. The problem then reduces to determining the amplitudes of these resulting homogeneous solutions of the EFEs. He proposed to do so by requiring continuity of the completion piece (plus a gauge piece) off the particle, however he did not take into account the discontinuity coming from the reconstructed part of the MP. Moreover, he only went as far as implementing this idea in the Schwarzschild case. We note that Price's goal of obtaining a completed metric that is *smooth* (off the particle) is rather ambitious; for certain practical purposes it may be sufficient to require only that *invariant* quantities are smooth. This is the line that we will take in our analysis.

The problem of determining the amplitudes of the homogeneous mass and AM completion solutions may be said to be equivalent to the problem of calculating the invariant mass and AM content of the CCK-reconstructed metric. Dolan and Barack [49] proposed a method, based on the work of Abbot and Desner [146], to "measure" these quantities for a given MP, in a quasi-local fashion, by evaluating certain surface integrals. The method relies only on the existence of time-translation and rotational Killing-symmetries in the *background*, and it is applicable to the Kerr case. However, the evaluation of the necessary integrals (and summation over modes) turns out to be extremely difficult for the RG perturbations.

Our procedure will take advantage of some auxiliary gauge-invariant quantities. These invariants will be derived in Sec. 5.1. The completion pieces with arbitrary amplitudes will be constructed using BL coordinates and variations of the mass and spin parameters of the Kerr metric, following Price. We determine these amplitudes by imposing continuity of the full invariants (a sum of the reconstructed and completion pieces) across a surface intersecting the particle. We will show how our explicit calculation agrees with the standard picture of the CCK reconstruction for circular orbits of Schwarzschild [1, 49, 92, 100], namely that by fixing the total mass and AM of the system we can determine the amplitudes of the completion piece. In Sec. 5.4.1 we extend the method for eccentric equatorial orbits around Kerr. The main results of this Chapter are given in Secs. 5.2.4, 5.3.4 and 5.4.3, where the completion pieces are respectively given for circular orbits around Schwarszchild and Kerr and for eccentric equatorial orbits around Kerr. The results of Sec. 5.4.3 fully solve the completion problem for equatorial orbits around Kerr.

A related completion problem was also recently considered by Sano and Tagoshi. They studied the case of a ring of particles around Schwarzschild [101] and around Kerr [102]. They looked at an homogeneous part of the Hertz potential in the IRG with certain free parameters. They numerically obtained those parameters by imposing continuity of certain (gauge-dependent) Weyl scalars and MP components. In Kerr it is not clear if this homogeneous part of the Hertz potential corresponds to any physical perturbation.

This chapter is the result of work in collaboration with Amos Ori, Leor Barack, Adam Pound and Maarten van de Meent. We acknowledge that Amos Ori first derived the auxiliary gauge-invariant quantities. Maarten van de Meent provided the analytical form for the integrals of Appendix F.5.

5.1 Auxiliary gauge-invariant quantities

The ten independent components of the Weyl curvature tensor are encapsulated in five complex scalars [see Eq. (2.8)]. For the Kerr background ψ_0 and ψ_4 vanish, therefore they are gauge-invariant. However ψ_2 is not gauge invariant. Consider a gauge transformation $x^\alpha \to x^{\alpha\prime} = x^\alpha + \xi^\alpha$ of $O(\mathsf{m})$. Since ψ_2 is a scalar field, it changes according to

$$\psi_2' \to \psi_2 + \Delta\psi_2, \tag{5.1}$$

where $\Delta\psi_2$ can be expressed in terms of the Lie derivative acting on ψ_2:

$$\Delta\psi_2 = -\xi^\alpha \psi^{(0)}_{2,\alpha} = -\xi^t \psi^{(0)}_{2,t} - \xi^r \psi^{(0)}_{2,r} - \xi^\theta \psi^{(0)}_{2,\theta} - \xi^\varphi \psi^{(0)}_{2,\varphi} + O(\mathsf{m}^2). \tag{5.2}$$

In the case of the Kerr background ψ_2 is given by $\psi_2^{(0)} = \varrho^3 M$, according to Eq. (2.8c), and $\varrho = -\frac{1}{r - ia\cos\theta}$ is a spin-coefficient in the Newman-Penrose formalism as given in (2.7). Thus

$$\psi^{(0)}_{2,r} = 3M\varrho^4 \quad \text{and} \quad \psi^{(0)}_{2,\theta} = 3iaM\sin\theta\varrho^4, \quad \text{with} \quad \psi^{(0)}_{2,t} = \psi^{(0)}_{2,\phi} = 0, \tag{5.3}$$

where M and a are the mass and spin parameters of the BH respectively,

Eq. (5.1) also implies that the perturbation of $\psi_2^{(0)}$, denoted by $\delta\psi_2$, transforms as

$$\delta\psi_2' \to \delta\psi_2 + \Delta\psi_2. \tag{5.4}$$

with the same $\Delta\psi_2$ appearing in Eq. (5.2).

5.1.1 Our preferred gauge

Suppose that we are given the perturbation $h_{\alpha\beta}$, in whichever gauge[1]. We now change the gauge to a *preferred gauge* $\tilde{h}_{\alpha\beta}$, which we choose to be a gauge where $\delta\tilde{\psi}_2$ (namely the gauge-transformed $\delta\psi_2$) vanishes: $\Delta\tilde{\psi}_2 \equiv \tilde{\psi}_2 - \psi_2 = -\delta\psi_2$. We denote the gauge vector which takes us from the original $h_{\alpha\beta}$ to $\tilde{h}_{\alpha\beta}$ by $\tilde{\xi}^\alpha$. In virtue of Eq. (5.3) we have

$$\Delta\tilde{\psi}_2 = -\left(\tilde{\xi}^r \psi^{(0)}_{2,r} + \tilde{\xi}^\theta \psi^{(0)}_{2,\theta}\right). \tag{5.5}$$

The requirement $\delta\tilde{\psi}_2 = 0$, or equivalently $\Delta\tilde{\psi}_2 = -\delta\psi_2$, thus reads

$$\tilde{\xi}^r \psi^{(0)}_{2,r} + \tilde{\xi}^\theta \psi^{(0)}_{2,\theta} = \delta\psi_2, \tag{5.6}$$

which we can solve for $\tilde{\xi}^r$ and $\tilde{\xi}^\theta$, leaving $\tilde{\xi}^t$ and $\tilde{\xi}^\phi$ arbitrary (this arbitrariness will not concern us). Eq. (5.6) is a complex algebraic equation—which actually amounts to a set of two equations, for the two *real* unknowns $\tilde{\xi}^r$, $\tilde{\xi}^\theta$.

We alternatively write Eq. (5.6) as

$$\tilde{\xi}^r + (ia\sin\theta)\tilde{\xi}^\theta = \Phi, \tag{5.7}$$

by defining $\Phi \equiv \frac{\varrho^{-4}}{3M}\delta\psi_2$ and using Eq. (5.3).

We shall now consider the general Kerr case $a \neq 0$ (the special Schwarzschild case $a = 0$ is simpler, but needs be treated separately; this will be done in Sec. 5.1.4). We obtain the solution

$$\tilde{\xi}^r = \mathrm{Re}(\Phi), \qquad \tilde{\xi}^\theta = \frac{\mathrm{Im}(\Phi)}{a\sin\theta}. \tag{5.8}$$

The covariant components are constructed with contractions of the background metric, $\tilde{\xi}_\alpha = g_{\alpha\beta}\tilde{\xi}^\beta$, explicitly

$$\tilde{\xi}_r = g_{rr}\tilde{\xi}^r \quad , \qquad \tilde{\xi}_\theta = g_{\theta\theta}\tilde{\xi}^\theta, \tag{5.9}$$

where $g_{\alpha\beta}$ are the unperturbed metric functions in BL coordinates [see Eq. (2.1)]. The remaining components $\tilde{\xi}_t$ and $\tilde{\xi}_\phi$ are left arbitrary.

5.1.2 Auxiliary Invariants

The new components of the MP in the *preferred gauge*, $\tilde{h}_{\alpha\beta}$, are given by

$$\tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \left(\tilde{\xi}_{\alpha;\beta} + \tilde{\xi}_{\beta;\alpha}\right) = h_{\alpha\beta} - \left(\tilde{\xi}_{\alpha,\beta} + \tilde{\xi}_{\beta,\alpha}\right) + 2\Gamma^\gamma_{\alpha\beta}\tilde{\xi}_\gamma. \tag{5.10}$$

Let us use indices a and b to denote the r and θ BL coordinates. Since $\Gamma^t_{ab} = 0 = \Gamma^\phi_{ab}$ in the Kerr background, we can write

$$\tilde{h}_{ab} = h_{ab} - \left(\tilde{\xi}_{a,b} + \tilde{\xi}_{b,a}\right) + 2\Gamma^c_{ab}\tilde{\xi}_c, . \tag{5.11}$$

We define, for notation convenience,

$$\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\} \equiv \{\tilde{h}_{rr}, \tilde{h}_{\theta\theta}, \tilde{h}_{r\theta}\}. \tag{5.12}$$

We claim that $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ are gauge invariant, which we show in the next section.

[1]We will later choose $h_{\alpha\beta}$ to be in turn the reconstructed MP, and the completion piece.

5.1.3 Direct verification of gauge invariance

Suppose that we apply a gauge transformation associated with a certain gauge vector ξ'^{α} to the original perturbation $h_{\alpha\beta}$. Then the various gauge-dependent fields (e.g. $h_{\alpha\beta}$, $\delta\psi_2$, ...) change. We will denote these changes by $'$ (i.e. $h'_{\alpha\beta}$, $\delta\psi'_2$, ...). In particular for the perturbation of ψ_2 we have

$$\delta\psi'_2 = -\left(\xi^{r\prime}\psi^{(0)}_{2,r} + \xi^{\theta\prime}\psi^{(0)}_{2,\theta}\right). \tag{5.13}$$

We now seek to find the changes in $\tilde{h}'_{ab}$ and show they vanish. Starting from Eq. (5.11) we have

$$\tilde{h}'_{ab} = h'_{ab} - \left[\left(\tilde{\xi}'_{a,b} + \tilde{\xi}'_{b,a}\right) - 2\Gamma^{c}_{ab}\tilde{\xi}'_{c}\right]. \tag{5.14}$$

The first term h'_{ab} is given by the standard gauge-transformation rule:

$$h'_{ab} = -\left[\left(\xi'_{a,b} + \xi'_{b,a}\right) - 2\Gamma^{\mu}_{ab}\xi'_{\mu}\right] = -\left[\left(\xi'_{a,b} + \xi'_{b,a}\right) - 2\Gamma^{c}_{ab}\xi'_{c}\right], \tag{5.15}$$

where we have used again $\Gamma^{t}_{ab} = 0 = \Gamma^{\varphi}_{ab}$. To evaluate the second term of Eq. (5.14) (namely the term in squared brackets), we need to calculate $\tilde{\xi}^{a\prime}$. This can be done by combining of Eq. (5.6) and Eq. (5.13), which yields

$$\tilde{\xi}^{r}\psi^{(0)}_{2,r} + \tilde{\xi}^{\theta}\psi^{(0)}_{2,\theta} = \delta\psi'_2 = -\left(\xi^{r\prime}\psi^{(0)}_{2,r} + \xi^{\theta\prime}\psi^{(0)}_{2,\theta}\right). \tag{5.16}$$

The solution to the equation above is trivial:

$$\tilde{\xi}^{a} = -\xi^{a\prime}. \tag{5.17}$$

This solution for $\tilde{\xi}^{a\prime}$ is unique and naturally follows from the uniqueness of $\tilde{\xi}^{a}$, established in Eq. (5.8). Even more it has a straightforward interpretation, which actually allows its derivation without any calculations: let us denote the new quantities $\tilde{\xi}^{a}$ (after the $\xi^{\alpha\prime}$ gauge transformation) by $\tilde{\xi}^{a\prime}$. Recall that by definition $\tilde{\xi}^{a\prime}$ is the gauge vector ξ^{a}, if only the r,θ components, required for transforming from the h'-gauge (namely after the $\xi^{\alpha\prime}$ transformation) to the preferred-gauge. Obviously this gauge transformation can be done in two stages, $\tilde{\xi}' = \xi_1 + \xi_2$: in the first stage we simply undo the ξ' gauge transformation, namely $\xi_1 = -\xi'$; the second stage entails the transition to the *preferred gauge* from the original h, namely $\xi_2 = \tilde{\xi}$. Overall we obtain $\tilde{\xi}' = \tilde{\xi} - \xi'$, namely $\tilde{\xi}' \equiv \tilde{\xi}' - \tilde{\xi} = -\xi'$.

Substituting the trivial result of Eq. (5.17) in the squared-bracketed term in the right-hand side of Eq. (5.14), one finds

$$\tilde{h}'_{ab} = -\left[\left(\xi'_{a,b} + \xi'_{b,a}\right) - 2\Gamma^{c}_{ab}\xi'_{c}\right] + \left[\left(\xi'_{a,b} + \xi'_{b,a}\right) - 2\Gamma^{c}_{ab}\xi'_{c}\right]. \tag{5.18}$$

yielding the desired gauge-invariance result $\tilde{h}'_{ab} = 0$.

5.1.4 Auxiliary invariants in Schwarzschild

As we mentioned in Sec. 5.1.1 the Schwarzschild case has to be consider separately. This is due to the fact that in Eq. (5.8), ξ_θ diverges as $a \to 0$. Instead of considering the Kerr solutions in Eq. (5.8), let us consider the components of the gradient of $\psi^{(0)}_2$ when $a \to 0$. Setting $a = 0$ we have

$\varrho = -\frac{1}{r}$. Eq. (5.3) then reads

$$\psi^{(0)}_{2,r} = 3M/r^4, \qquad \psi^{(0)}_{2,\theta} = 0. \tag{5.19}$$

Therefore Eq. (5.2) reduces to

$$\Delta\psi_2 = -\xi^r \psi^{(0)}_{2,r} = -\frac{3M}{r^4}\xi^r. \tag{5.20}$$

Since ξ^r is real, this equation tells us at once (recalling $\Delta\delta\psi_2 = \Delta\psi_2$) that $\mathrm{Im}(\Delta\psi_2) = 0$ — namely the quantity

$$\mathcal{I}_2^{(\mathrm{Sch})} \equiv \mathrm{Im}(\delta\psi_2) \tag{5.21}$$

is *gauge-invariant*.

Unlike the situation in the Kerr case, one of our invariants can be just $\mathrm{Im}(\delta\psi_2)$. However $\mathrm{Re}(\delta\psi_2)$ is still gauge-dependent. We shall thus choose our preferred gauge $\tilde{h}_{\alpha\beta}$ to be a gauge in which

$$\mathrm{Re}(\delta\tilde{\psi}_2) = 0, \tag{5.22}$$

namely $\Delta\tilde{\psi}_2 = -\mathrm{Re}(\delta\psi_2)$. Noting that $\Delta\tilde{\psi}_2 = -(3M/r^4)\tilde{\xi}^r$, we obtain

$$\tilde{\xi}^r = \frac{r^4}{3M}\mathrm{Re}(\delta\psi_2), \tag{5.23}$$

which is in full agreement with the more general expression of Eq. (5.8) for $\tilde{\xi}^r$. Note that in the Schwarzschild case $\tilde{\xi}^\theta$ (just like $\tilde{\xi}^t$ and $\tilde{\xi}^\phi$) is left arbitrary.

Again, we define the covariant components $\tilde{\xi}_\alpha = g_{\alpha\beta}\tilde{\xi}^\beta$, and in particular

$$\tilde{\xi}_r = g_{rr}\tilde{\xi}^r. \tag{5.24}$$

Let us consider again the components of the MP in the new gauge,

$$\tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \left(\tilde{\xi}_{\alpha,\beta} + \tilde{\xi}_{\beta,\alpha}\right) + 2\Gamma^\gamma_{\alpha\beta}\tilde{\xi}_\gamma. \tag{5.25}$$

Where we only require to obtain the $\tilde{h}_{rr}$ component. Since in the Schwarzschild case Γ^γ_{rr} vanishes for any $x^\gamma \neq r$, we obtain the explicit expression for this component:

$$\mathcal{I}_1^{(\mathrm{Sch})} \equiv \tilde{h}_{rr} = h_{rr} - 2\tilde{\xi}_{r,r} + 2\Gamma^r_{rr}\tilde{\xi}_r. \tag{5.26}$$

Notice that in Schwarzschild we have labelled the invariants not by the order we have derived them but rather to make $\tilde{h}_{rr}$ the "first" invariant both in Schwarzschild and Kerr. The gauge invariance of Eq. (5.26) does follow from the invariance of its Kerr counterpart.

5.2 Circular orbits in Schwarzschild spacetime

Before considering the problem of determining the completion piece for Kerr, let us consider the more simple Schwarzschild case. Equipped with the invariants obtained in the previous section we will now obtain the completion pieces. The problem reduces to determining the gauge-invariant amplitudes of the mass and AM perturbation. This piece is whatever is needed to add to the reconstructed piece for satisfying the linearised EFE. The amplitudes of the completion piece can be determined by imposing continuity off-the-particle of the auxiliary invariants. In Schwarzschild the expressions for the CCK-reconstructed MP are easier to handle, which makes for a more pedagogical implementation of our method. Also, in this case we know in advance what the amplitudes of the completion piece should give, which serves to test our method.

5.2.1 Strategy

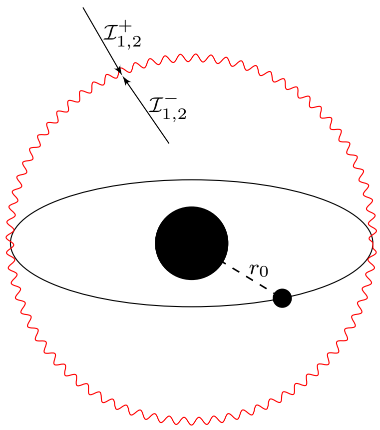

Figure 5.1: Each of the auxiliary gauge-invariant quantities has a CCK-reconstructed part $\mathcal{I}^{(\mathrm{rec})}$ and a completion part $\mathcal{I}^{(\mathrm{comp})}$. The reconstructed MP has a gauge discontinuity on the sphere intersecting the particle (red line). The amplitudes δM and δJ of the completion pieces are determined by imposing continuity of $\mathcal{I}^{(\mathrm{rec})}_{1,2} + \mathcal{I}^{(\mathrm{comp})}_{1,2}$ off the particle. The full invariant should be continuous everywhere off the particle.

We consider a particle of mass m moving in a circular orbit at $r = r_0$ around a Schwarzschild BH of mass M, where $\mathsf{m} \ll M$. The reconstructed part of the MP in a no-string IRG can be obtained according to the CCK procedure. As described in Sec. 2.1, we start by solving Teukolsky equation mode by mode (we choose $s = -2$ and solve for $\psi_{s=-2} \equiv \varrho^{-4}\psi_4$). We assume that the completion piece is stationary and axially symmetric (since it should only include corrections to the mass and AM), and this way we restrict ourselves to analytically finding only the $m = 0 = \omega$ modes. The appropriate Hertz potential is then obtained by inverting a fourth-order differential equation Eq. (2.20b). With only the $m = 0$ of this equation, we perform an analytical angular-inversion. We recover the ℓ-modes of the reconstructed MP (the $\ell \geq 2$ sector) on each of the two sides of the sphere $\mathcal{S}$ with $r = r_0$, see Fig. 5.1.

The modes of the MP are the required input to obtain the invariants on each side of $\mathcal{S}$. We take the limits $r \to r_0^\pm$ to obtain the jump on each mode of the invariants. These jumps are then summed analytically up to $\ell \to \infty$ in a distributional way. The completion pieces are constructed as mass and AM perturbations of the Schwarzschild metric (keeping the BL coordinates fixed). This gives two homogeneous solutions with arbitrary amplitudes $\delta M^\pm$ and $\delta J^\pm$. The jumps on the amplitudes are determined by imposing continuity of the auxiliary invariants across $\mathcal{S}$ off the particle, namely at $\theta \neq \pi/2$, to obtain a system of two equations, each of them determines the jump on one of the missing amplitudes.

5.2.2 Analytic solutions of Teukolsky equation

The radial part of Teukolsky equation that describes the perturbations due to the particle moving around the BH has the form

$$\hat{\mathcal{T}}\psi_{\ell m\omega}(r) = \tilde{T}_{\ell m\omega}(r), \tag{5.27}$$

where $\hat{\mathcal{T}}$ is the radial Teukolsky operator given explicitly in Eq. (2.12a) (with $s = -2$), $\psi_{\ell m\omega}$ is the radial part of the relevant Weyl scalar (we calculate ψ_4 but it is possible to obtain ψ_0 instead), and $\tilde{T}_{\ell m\omega}$ is the frequency-domain source. We have defined $\tilde{T} \equiv 4\pi\Sigma T_s$, with T_s as given by Eq. (2.16), to simplify notation. The frequency-domain source has the form

$$\tilde{T}_{\ell m\omega}(r) = \sum_{k=0}^{2} \tilde{t}_{\ell m\omega[k]}(r_0)\delta^{(k)}(r - r_0), \tag{5.28}$$

where $\tilde{t}_{\ell m\omega[k]}(r_0)$ are three functions determined by Eq. (2.16), and $\delta^{(k)}(r-r_0)$ is the k order derivative of the delta-function with respect to its argument.

The procedure starts by solving the homogeneous part of Eq. (5.27) (for $m = 0$ only) with retarded boundary-conditions. This can be done analytically to obtain the homogeneous solutions for $\psi_{s=-2}$, these solutions are related algebraically to those given in Sec. 4.2.1 according to $R_{4\pm}(r) = r^4 f^2 \bar{R}_{0\pm}(r)$, due to the symmetries of Teukolsky equation. For the static modes ($m = 0 = \omega$) the two linearly-independent solutions are explicitly

$$R_{4-}(r) \equiv r^2 f \mathsf{P}_\ell^2\left(\frac{r-M}{M}\right), \tag{5.29a}$$

$$R_{4+}(r) \equiv r^2 f \mathsf{Q}_\ell^2\left(\frac{r-M}{M}\right), \tag{5.29b}$$

where P_ℓ and Q_ℓ are Legendre polynomials as before. The regularity of this solutions follows that of the solutions in Sec. 4.2.1.

The inhomogeneous solutions of the radial part of Teukolsky equation $\psi_{4,\ell 00}(r)$ are constructed using the variation of parameters method (the expressions for the circular orbit Schwarzschild limit can be recovered from the general expressions in Appendix F.1). The total inhomogeneous solution $\psi_4(r,\theta)$ is given by

$$\psi_4(r,\theta) = \sum_{\ell=2}^{\infty} \psi_{4,\ell 00}(r)\, {}_{-2}Y_{\ell 00}(\theta), \tag{5.30}$$

where ${}_{-2}Y_{\ell 00}(\theta)$ is the appropriate angular function (an $\omega = 0 = m$ and $s = -2$ spin-weighted spherical harmonic).

The modes of the Hertz potential $\Psi_{\ell 00}(r)$ are computed from $\psi_{4,\ell 00}(r)$, using Eq. (2.20). We use the algebraic inversion of the angular relation in Eq. (2.20b), and work in the IRG. Alternatively we might have chosen to construct ψ_0 and integrate the radial equation of (2.20b) or use the corresponding equations to work in the ORG using (2.20a).

5.2.3 Analytical reconstruction: MP and auxiliary-invariants

The reconstructed piece of the MP is obtained via

$$h^{(\mathrm{rec})}_{\alpha\beta}(r,\theta) = \sum_{\ell\geq 2} \hat{H}_{\alpha\beta}\Psi_{\ell 00}(r)\, {}_{-2}Y_{\ell 00}(\theta), \tag{5.31}$$

where $\hat{H}$ is the reconstruction operator [the Schwarzschild limit of Eq. (2.19)], and we have omitted the $\pm$ denoting on which side of $\mathcal{S}$ the MP is calculated. Given $h^{(\mathrm{rec})}_{\alpha\beta}$, one constructs the two gauge-invariant quantities $\mathcal{I}^{(\mathrm{rec})}_{1,2}(r,\theta) = \hat{I}^{\alpha\beta}_{1,2} h^{(\mathrm{rec})}_{\alpha\beta}(r,\theta)$ (where $\hat{I}^{\alpha\beta}_{1,2}$ are certain differential operators), on each side of $\mathcal{S}$. We then obtain the jumps

$$\left[\mathcal{I}^{(\mathrm{rec})}_{1,2}\right](\theta) \equiv \lim_{\epsilon\to 0} \hat{I}^{\alpha\beta}_{1,2}\left(h^{(\mathrm{rec})}_{\alpha\beta}(r_0+\epsilon,\theta) - h^{(\mathrm{rec})}_{\alpha\beta}(r_0-\epsilon,\theta)\right). \tag{5.32}$$

In the $m = 0$ sector the ℓ-modes of $\mathcal{I}^{(\mathrm{rec})}_1$ in terms of the homogeneous solutions of Eq. (5.29) and spherical harmonics are

$$\mathcal{I}_{1\pm}^{(\mathrm{rec})}(r,\theta) = \sum_{\ell=2}^{\infty} -\frac{4\pi}{3f^3\lambda_1 M r^2 r_0 \mathcal{W}} \left\{R'_{4\pm}(r)\left[\lambda_1 r^2 - 2(\lambda_0 - 3) Mr - 6M^2\right] + \lambda_1 R_{4\pm}(r)(3M - 2r)\right\}$$
$$\times \left\{\lambda_1 R_{4\mp}(r_0)\left[2M - r_0\left(r_0^2\Omega^2 + 1\right)\right] - 2Mr_0^3\Omega^2 R'_{4\mp}(r_0)\right\} u^t Y_\ell(\theta)\bar{Y}_\ell(\theta_0) \tag{5.33}$$

where Ω is the orbital frequency (computed using Eq. (2.11) with $a = 0$), $\lambda_s = (\ell - s)(\ell + s + 1)$ as before, and the prime denotes derivatives with respect to r. We have left indicated the Wronskian of the homogeneous solutions $\mathcal{W} \equiv R_{4-}(r)R'_{4+}(r) - R_{4+}(r)R'_{4-}(r)$. It is useful to notice that $\mathcal{I}_{1-}^{(\mathrm{rec})} \leftrightarrow \mathcal{I}_{1+}^{(\mathrm{rec})}$ under $R_{4-} \leftrightarrow R_{4+}$ (keeping $\mathcal{W}$ fixed). At the limit $r \to r_0$ the terms proportional to $R_{4+}(r_0)R_{4-}(r_0)$ and $R'_{4-}(r_0)R'_{4+}(r_0)$ are continuous and do not contribute to the jump. The remaining terms combine to obtain the relevant $\mathcal{W}(r_0)$ (or derivatives of it). Note that we have used the definitions of the spin-weighted spherical harmonics Eq. (E.8), to write Eq. (5.33) in terms of the usual scalar spherical harmonics. The total jump in the $m = 0$ sector then works out to be

$$[\mathcal{I}_1^{(\mathrm{rec})}](\theta) = \sum_{\ell=2}^{\infty} \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2} Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \sum_{\ell=2}^{\infty} \frac{4\mathsf{m}\mathcal{E}\pi(r_0 - M)}{3Mr_0 f_0^3} Y_\ell(\theta)\bar{Y}''_\ell(\theta_0), \tag{5.34}$$

where we have used the identity $u^t = \mathcal{E}/f_0$, and $\lambda_0 \bar{Y}_\ell(\theta_0) = -\bar{Y}''_\ell(\theta_0)$ on the equator.

We rewrite Eq. (5.34) as a sum starting from $\ell = 0$:

$$\begin{aligned}[\mathcal{I}_1^{(\mathrm{rec})}](\theta) =& \sum_{\ell=0}^{\infty} \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2} Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \sum_{\ell=0}^{\infty} \frac{4\mathsf{m}\mathcal{E}\pi(r_0 - M)}{3Mr_0 f_0^3} Y_\ell(\theta)\bar{Y}''_\ell(\theta_0) \\ &- \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2} Y_0(\theta)\bar{Y}_0(\theta_0) - \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2} Y_1(\theta)\bar{Y}_1(\theta_0), \\ =& \sum_{\ell=0}^{\infty} \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2} Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \sum_{\ell=0}^{\infty} \frac{4\mathsf{m}\mathcal{E}\pi(r_0 - M)}{3Mr_0 f_0^3} Y_\ell(\theta)\bar{Y}''_\ell(\theta_0) - \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2} Y_0(\theta)\bar{Y}_0(\theta_0),\end{aligned} \tag{5.35}$$

where we have added and subtracted the $\ell = 0, 1$ terms that are not contained in the first sum of Eq. (5.34), the second sum of Eq. (5.34) is trivially extended since $Y''_0(\theta) = 0 = Y''_1(\theta)$ on the equator. To write the second equality we have also used $Y_1(\theta)\bar{Y}_1(\theta_0) = 0$ on the equator. The infinite sums in Eq. (5.35) are done analytically in a distributional way (see [99] for a detailed proof of the validity of the sums). The first term on the right-hand-side of Eq. (5.35) sums to a delta-function supported on $\theta = \theta_0$, while the second term gives the second θ-derivative of a delta-function:

$$[\mathcal{I}_1^{(\mathrm{rec})}](\theta) = \frac{8\mathsf{m}\mathcal{E}\pi}{3Mf_0^2}\delta(\theta - \theta_0) - \frac{4\mathsf{m}\mathcal{E}\pi(r_0 - M)}{3Mr_0 f_0^3}\delta''(\theta - \theta_0) - \frac{2\mathsf{m}\mathcal{E}}{3Mf_0^2}. \tag{5.36}$$

The reconstructed piece of the first invariant only gives information about the energy of the orbiting particle.

We look now at the reconstructed piece of the second invariant $\mathcal{I}_2$ using Eq. (5.21), and again we express it in terms of scalar spherical harmonics using Eq. (E.8):

$$\mathcal{I}_{2\pm}^{(\mathrm{rec})}(r,\theta) = \sum_{\ell\geq 2} \frac{4\pi\mathsf{m} r_0 f_0 \mathcal{L}}{\mathcal{W}\lambda_0 r_0^2 r^5 f}\left[r_0 R'_{4\mp}(r_0) - 2R_{4\mp}(r_0)\right] \times$$
$$\left\{\left[r(\lambda_0 + 4) - 12M\right] R_{4\pm}(r) - 2r(r - 3M)R'_{4\pm}(r)\right\} Y_\ell(\theta)\bar{Y}'_\ell(\theta_0), \tag{5.37}$$

and the jump across the sphere with $r = r_0$ yields

$$\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) = \sum_{\ell=2}^{\infty} \frac{4\pi \mathsf{m}\mathcal{L}}{r_0^4} Y_\ell(\theta)\bar{Y}'_\ell(\theta_0), \tag{5.38}$$

which again can be summed analytically as a distribution adding and subtracting the missing $\ell = 0, 1$ pieces. The result is

$$\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) = -\frac{4\pi \mathsf{m}\mathcal{L}}{r_0^4}\delta'(\theta - \theta_0) + \frac{3\mathsf{m}\mathcal{L}}{r_0^4}\cos\theta, \tag{5.39}$$

which has information only about the AM of the particle.

5.2.4 Determination of the completion piece

The "extra" contribution to the jumps in the invariants $\left[\mathcal{I}_{1,2}^{(\mathrm{comp})}\right]$ is constructed from the completion piece of the MP. This completion piece has the form

$$h_{\alpha\beta}^{(\mathrm{comp})\pm}(r,\theta) = \delta M^{\pm} h_{\alpha\beta}^{(\delta M)}(r,\theta) + \delta J^{\pm} h_{\alpha\beta}^{(\delta J)}(r,\theta), \tag{5.40}$$

where $\delta M^{\pm}$ and $\delta J^{\pm}$ are unknown amplitudes at this stage. In Eq. (5.40) the superscripts $+$ and $-$ are used to distinguish between the completion solutions for $r > r_0$ and $r < r_0$, respectively. The components of the two MP perturbations $h_{\alpha\beta}^{(\delta M)}(r)$ and $h_{\alpha\beta}^{(\delta J)}(r)$ are obtained via

$$h_{\alpha\beta}^{(\delta M)}(r) \equiv \left.\frac{\partial g_{\alpha\beta}(M,J)}{\partial M}\right|_{J\to 0}, \qquad h_{\alpha\beta}^{(\delta J)}(r,\theta) \equiv \left.\frac{\partial g_{\alpha\beta}(M,J)}{\partial J}\right|_{J\to 0}, \tag{5.41}$$

where $g_{\alpha\beta}$ is the Kerr metric in BL coordinates, ∂_M is taken with fixed $J = Ma$ (and fixed BL coordinates) and ∂_J is taken with fixed M (and fixed BL coordinates). Given $h_{\alpha\beta}^{(\mathrm{comp})\pm}$, the jumps $\left[\mathcal{I}_{1,2}^{(\mathrm{comp})}\right]$ are given by

$$\left[\mathcal{I}_{1,2}^{(\mathrm{comp})}\right](\theta) \equiv \lim_{\epsilon\to 0} \hat{I}_{1,2}^{\alpha\beta}\left(h_{\alpha\beta}^{(\mathrm{comp})+}(r_0+\epsilon,\theta) - h_{\alpha\beta}^{(\mathrm{comp})-}(r_0-\epsilon,\theta)\right). \tag{5.42}$$

To obtain

$$\left[\mathcal{I}_1^{(\mathrm{comp})}\right](r,\theta) = \frac{2[\delta M]}{3Mf^2}, \qquad \text{and} \qquad \left[\mathcal{I}_2^{(\mathrm{comp})}\right](r,\theta) = -\frac{3[\delta J]\cos\theta}{r^4}, \tag{5.43}$$

where $[\delta M] \equiv \delta M^+ - \delta M^-$ and $[\delta J] \equiv \delta J^+ - \delta J^-$.

5.2.5 Solution for the amplitudes $[\delta M]$, $[\delta J]$

The jumps $[\delta M]$ and $[\delta J]$ are determined from the two regularity conditions

$$\left[\mathcal{I}_{1,2}^{(\mathrm{rec})}\right](\theta) + \left[\mathcal{I}_{1,2}^{(\mathrm{comp})}\right](\theta) = 0 \quad \text{for } \theta \neq \pi/2. \tag{5.44}$$

We now impose the regularity conditions of Eqs. (5.44) at $r = r_0$ and $\theta \neq \pi/2$, explicitly

$$\begin{aligned}
\left[\mathcal{I}_1\right](\theta \neq \pi/2) &= -\frac{2\mathsf{m}\mathcal{E}}{3Mf_0^2} + \frac{2[\delta M]}{3Mf_0^2} = 0, \\
\left[\mathcal{I}_2\right](\theta \neq \pi/2) &= \frac{3\mathsf{m}\mathcal{L}}{r_0^4}\cos\theta - \frac{3[\delta J]}{r_0^4}\cos\theta = 0.
\end{aligned} \tag{5.45}$$

Note that in the Schwarzschild case, the amplitudes of the mass and AM perturbations decouple and we have one unknown amplitude for each regularity condition, as expected from the separability of the EFE. In general this is not expected in the Kerr case. The solutions of Eq. (5.45) give

$$[\delta M] = \mathsf{m}\mathcal{E}, \quad \text{and} \quad [\delta J] = \mathsf{m}\mathcal{L}. \tag{5.46}$$

Namely the jumps of the mass and AM perturbations in the invariants are simply the energy and AM of the particle, as expected.

The fact that these amplitudes are independent of θ is a strong test of our result. By virtue of Wald's theorem, at infinity the completed MP should read as the metric components for a linear in δa Kerr solution with total mass $M + \mathsf{m}\mathcal{E}$ and spin-parameter $\delta a = \mathsf{m}\mathcal{L}/M$, hence

$$h^{(\text{comp})+}_{tt} = 1 + \frac{2(M+\mathsf{m}\mathcal{E})}{r}, \quad h^{(\text{comp})+}_{t\varphi} = -\frac{\mathsf{m}\mathcal{L}\sin^2\theta}{r} \tag{5.47a}$$

$$h^{(\text{comp})+}_{rr} = \frac{1}{f^2} - \frac{2(M+\mathsf{m}\mathcal{E})}{rf^2} \tag{5.47b}$$

$$h^{(\text{comp})+}_{\theta\theta} = r^2, \qquad h^{(\text{comp})+}_{\varphi\varphi} = r^2\sin^2\theta, \tag{5.47c}$$

Together Eq. (5.46) and Eq. (5.47) fix the completion in the interior $h^{(\text{comp})-}_{\alpha\beta} = 0$.

5.3 Circular equatorial orbits in Kerr spacetime

5.3.1 Strategy

Let us now consider the particle is moving on a circular equatorial-orbit in a Kerr background. The strategy will be the same as in the Schwarzschild case. We follow the CCK-reconstruction procedure mode by mode to obtain the MP. The reconstructed sector of the invariants is constructed mode by mode, and the limit $r \to r_0$ is taken to read off the jump across S. The sums in the Kerr case are considerably more complicated. All the sums appearing in the reconstructed part of the invariants are evaluated analytically as distributions using the formulae in Appendix F.3. The completion piece is given by Eqs. (5.40) and (5.41), without taking $J \to 0$. We impose the same regularity condition [Eq. (5.44) in the previous section] to the Kerr auxiliary invariants [Eq. (5.12) of Sec. 5.1.2], in the same way as in the Schwarzschild case. Alternatively, $[\delta M]$ and $[\delta J]$ may be obtained from a single invariant—either $\mathcal{I}_1$ or $\mathcal{I}_2$—by evaluating either one of the two conditions in Eq. (5.44) at two different values of $\theta \neq \pi/2$. We obtain a system of two linearly-independent equations, which determines the jumps in the missing amplitudes.

5.3.2 Analytic solutions of Teukolsky equation

The homogeneous solutions to the radial part of the spin $s = -2$ Teukolsky equation in Kerr (with $m = 0$), Eq. (2.9), are hypergeometric functions [147]:

$$R_{4-}(r) = \Delta^2\, {}_2F_1\left(-\ell+2, \ell+3; 3; -z_+\right), \tag{5.48a}$$

$$R_{4+}(r) = A_{\ell s}\, z_-^{-\ell+1}\, {}_2F_1\left(\ell-1, \ell+1; 2\ell+2; z_-^{-1}\right), \tag{5.48b}$$

where $A_{\ell s} \equiv \frac{\ell!(\ell+2)!}{(2\ell+1)!(2-\ell)!}$, $z_\pm \equiv \frac{r-r_\pm}{r_+-r_-}$. We have again used '$-$' to denote the solution that is regular at the EH and '+' for the solution that is regular at infinity. The inhomogeneous solutions of the radial part of Teukolsky equation $\psi_{s=-2} = \varrho^{-4}\psi_4$ are constructed with the method of variations of parameters of Appendix F.1.

5.3.3 Metric reconstruction and auxiliary invariants

We algebraically invert Eq. (2.20) to find the Hertz potential and obtain the modes of the reconstructed MP in the IRG. We only require to obtain explicitly the relevant components required to compute the Kerr invariants, namely the h_{ab} of Eq. (5.11).

Having the Hertz potential we directly compute $\delta\psi_2$ without having to obtain all the components of the MP, Riemann and Weyl tensors. Let us recall that $\delta\psi_2$ is required to obtain $\mathcal{I}_{1,2,3}$, from which two of them suffice to evaluate the amplitudes of the jumps. $\delta\psi_2$ is calculated[2,3] using:

$$
\begin{aligned}
\delta\psi_2 =& \frac{1}{12}\left[(\boldsymbol{D}+2\varrho-\bar{\varrho})(\boldsymbol{D}+2\varrho-\bar{\varrho})(\bar{\boldsymbol{\delta}}+\alpha+3\bar{\beta}-\bar{\tau})(\bar{\boldsymbol{\delta}}+4\bar{\beta}+3\bar{\tau})\right.\\
&+(\boldsymbol{D}+2\varrho-\bar{\varrho})(\bar{\boldsymbol{\delta}}+2\bar{\beta}-\pi-\bar{\tau})(\boldsymbol{D}+\varrho-\bar{\varrho})(\bar{\boldsymbol{\delta}}+4\bar{\beta}+3\bar{\tau})\\
&+(\boldsymbol{D}+2\varrho-\bar{\varrho})(\bar{\boldsymbol{\delta}}+2\bar{\beta}-\pi-\bar{\tau})(\bar{\boldsymbol{\delta}}-\alpha+3\bar{\beta}-\pi-\bar{\tau})(\boldsymbol{D}+3\bar{\varrho})\\
&+(\bar{\boldsymbol{\delta}}-\alpha+\bar{\beta}-2\pi-\bar{\tau})(\bar{\boldsymbol{\delta}}-2\alpha+2\bar{\beta}-2\pi-\bar{\tau})(\boldsymbol{D}-\bar{\varrho})(\boldsymbol{D}+3\bar{\varrho})\\
&+(\bar{\boldsymbol{\delta}}-\alpha+\bar{\beta}-2\pi-\bar{\tau})(\boldsymbol{D}+\varrho-\bar{\varrho})(\bar{\boldsymbol{\delta}}-\alpha+3\bar{\beta}-\pi-\bar{\tau})(\boldsymbol{D}+3\bar{\varrho})\\
&\left.+(\bar{\boldsymbol{\delta}}-\alpha+\bar{\beta}-2\pi-\bar{\tau})(\boldsymbol{D}+\varrho-\bar{\varrho})(\boldsymbol{D}+\varrho-\bar{\varrho})(\bar{\boldsymbol{\delta}}+4\bar{\beta}+3\bar{\tau})\right]\bar{\Psi},
\end{aligned}
\tag{5.49}
$$

where we have omitted the vanishing spin-coefficient ϵ.

We leave the Wronskian of the homogeneous solutions unevaluated as we take the limit $r \to r_0$ in the inhomogeneous solutions, just like in the Schwarzschild case, and calculate the side-values of $h_{ab}^{(\mathrm{rec})}$ (see Appendix F.4 for the full expressions). The jumps $\left[\mathcal{I}_{1,2}^{(\mathrm{rec})}\right]$ are independent of the explicit form of the homogeneous solutions, since all remaining discontinuous terms are proportional to $\mathcal{W}$ (or derivatives of $\mathcal{W}$ which are further expressed in terms of $\mathcal{W}$), just like we found in the Schwarzschild case.

The jump across the sphere with $r = r_0$ is given by a sum of all the $\ell \geq 2$ modes. We choose to split the sums according to their ℓ and θ dependence, schematically

$$
\begin{aligned}
\left[\mathcal{I}_1^{(\mathrm{rec})}\right](\theta) =& \sum_{\ell=2}^{\infty}\left\{c_0^\circ Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + c_1^\circ\lambda_1 Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \frac{c_2^\circ}{\lambda_0\lambda_1}\left[\lambda_0 Y_\ell(\theta) + 2\cot\theta Y_\ell'(\theta)\right]\bar{Y}_\ell(\theta_0)\right.\\
&+c_3^\circ Y_\ell(\theta)\bar{Y}_\ell'(\theta_0) + \frac{c_4^\circ}{\lambda_0}Y_\ell(\theta)\bar{Y}_\ell'(\theta_0) + \frac{c_5^\circ}{\lambda_0}Y_\ell'(\theta)\bar{Y}_\ell(\theta_0) + \frac{c_6^\circ\lambda_1}{\lambda_0}Y_\ell'(\theta)\bar{Y}_\ell(\theta_0)\\
&\left.+\frac{c_7^\circ}{\lambda_0}Y_\ell'(\theta)\bar{Y}_\ell'(\theta_0) + \frac{c_8^\circ}{\lambda_0^2}Y_\ell'(\theta)\bar{Y}_\ell'(\theta_0)\right\},
\end{aligned}
\tag{5.50a}
$$

$$
\begin{aligned}
\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) =& \sum_{\ell=2}^{\infty}\left\{d_0^\circ Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \frac{d_1^\circ}{\lambda_0\lambda_1}\left[\lambda_0 Y_\ell(\theta) + 2\cot\theta Y_\ell'(\theta)\right]\bar{Y}_\ell(\theta_0) + d_2^\circ Y_\ell(\theta)\bar{Y}_\ell'(\theta_0)\right.\\
&+\frac{d_3^\circ}{\lambda_0}Y_\ell(\theta)\bar{Y}_\ell'(\theta_0) + \frac{d_4^\circ}{\lambda_0}Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \frac{d_5^\circ\lambda_1}{\lambda_0}Y_\ell'(\theta)\bar{Y}_\ell(\theta_0) + d_6^\circ Y_\ell'(\theta)\bar{Y}_\ell'(\theta_0)\\
&\left.+\frac{d_7^\circ}{\lambda_0}Y_\ell'(\theta)\bar{Y}_\ell'(\theta_0) + \frac{d_8^\circ}{\lambda_0^2}Y_\ell'(\theta)\bar{Y}_\ell'(\theta_0)\right\},
\end{aligned}
\tag{5.50b}
$$

[2] The expression that directly relates ψ_2 with Ψ in the original Cohen-Kegeles work [94] requires a factor 2. This is a longstanding error of the CCK formalism as mentioned in previous works by Keidl *et al.* [81] and Pound *et al.* [1]

[3] This expression is equivalent to the one independently obtained by Sano and Tagoshi [102]

$$\delta\psi_2 = \frac{1}{2}\left[\boldsymbol{D}\boldsymbol{D}\varrho(\bar{\boldsymbol{\delta}}+2\bar{\beta})\frac{1}{\varrho}(\bar{\boldsymbol{\delta}}+4\bar{\beta})\bar{\Psi} - 4\pi(\boldsymbol{D}+\varrho)\boldsymbol{D}(\bar{\boldsymbol{\delta}}+4\bar{\beta})\bar{\Psi} + 6\pi\boldsymbol{D}\pi\boldsymbol{D}\bar{\Psi}\right],$$

which was published after we have implemented our method.

where the long coefficients c_i°, d_i° of the sums are ℓ-independent and $\lambda_1 = \lambda_0 - 2$ [these can be recovered from the coefficients in the eccentric orbits case c_i, d_i given explicitly in Eqs. (F.4) and (F.5) of Appendix F]. All the sums in Eq. (5.50) can be evaluated analytically as distributions at $\theta = \theta_0$ by including (and subtracting) the $\ell = 0, 1$ terms missing in the sums:

$$\begin{aligned}
\left[\mathcal{I}_1^{(\mathrm{rec})}\right](\theta) =& c_0^\circ\delta(\theta-\theta_0) - c_1^\circ\delta''(\theta-\theta_0) - 2c_1^\circ\delta(\theta-\theta_0) - c_3^\circ\delta'(\theta-\theta_0) + c_6^\circ\delta'(\theta-\theta_0) + c_7^\circ\delta(\theta-\theta_0)\\
&- c_0^\circ Y_0(\theta)\bar{Y}_0(\theta_0) + 2c_1^\circ Y_0(\theta)\bar{Y}_0(\theta_0) - c_3^\circ Y_1(\theta)\bar{Y}_1'(\theta_0) - c_7^\circ {}_{-1}Y_1(\theta) {}_{-1}\bar{Y}_1(\theta_0)\\
&- \frac{c_2^\circ}{\sin\theta}\int_{-1}^{1} d(\cos\theta'')\int_{-1}^{1} d(\cos\theta')\left[\delta(\theta'-\theta_0) - Y_0(\theta')\bar{Y}_0(\theta_0)\right]\\
&- c_4^\circ\int_{-1}^{1} d(\cos\theta_0')\left[\delta(\theta-\theta_0') - Y_0(\theta)\bar{Y}_0(\theta_0') - Y_1(\theta)\bar{Y}_1(\theta_0')\right]\\
&- \frac{c_5^\circ - 2c_6^\circ}{\sin\theta}\int_{-1}^{1} d(\cos\theta')\left[\delta(\theta'-\theta_0) - Y_0(\theta')\bar{Y}_0(\theta_0)\right]\\
&+ \frac{c_8^\circ}{\sin\theta}\int_{-1}^{1}\int_{-1}^{1} d(\cos\theta')d(\cos\theta_0')\left[\delta(\theta'-\theta_0') - Y_0(\theta')\bar{Y}_0(\theta_0') - Y_1(\theta')\bar{Y}_1(\theta_0')\right], \quad (5.51\mathrm{a})
\end{aligned}$$

$$\begin{aligned}
\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) =& d_0^\circ\delta(\theta-\theta_0) - d_2^\circ\delta'(\theta-\theta_0) + d_5^\circ\delta'(\theta-\theta_0) - d_6^\circ\delta''(\theta-\theta_0) + d_7^\circ\delta(\theta-\theta_0)\\
&- d_0^\circ Y_0(\theta)\bar{Y}_0(\theta_0) - d_2^\circ Y_1(\theta)\bar{Y}_1(\theta_0) - d_6^\circ Y_1'(\theta)\bar{Y}_1'(\theta_0) - d_7^\circ {}_{-1}Y_1(\theta) {}_{-1}\bar{Y}_1(\theta_0)\\
&- \frac{d_1^\circ}{\sin\theta}\int_{-1}^{1} d(\cos\theta'')\int_{-1}^{1} d(\cos\theta')\left[\delta(\theta'-\theta_0) - Y_0(\theta')\bar{Y}_0(\theta_0)\right]\\
&- d_3^\circ\int_{-1}^{1} d(\cos\theta_0')\left[\delta(\theta-\theta_0') - Y_0(\theta)\bar{Y}_0(\theta_0') - Y_1(\theta)\bar{Y}_1(\theta_0')\right]\\
&- \frac{d_4^\circ - 2d_5^\circ}{\sin\theta}\int_{-1}^{1} d(\cos\theta')\left[\delta(\theta'-\theta_0) - Y_0(\theta')\bar{Y}_0(\theta_0)\right]\\
&+ \frac{d_8^\circ}{\sin\theta}\int_{-1}^{1}\int_{-1}^{1} d(\cos\theta')d(\cos\theta_0')\left[\delta(\theta'-\theta_0') - Y_0(\theta')\bar{Y}_0(\theta_0') - Y_1(\theta')\bar{Y}_1(\theta_0')\right], \quad (5.51\mathrm{b})
\end{aligned}$$

where we have used $Y_\ell'(\theta) = \lambda_0^{1/2}\ {}_{-1}Y_\ell(\theta)$. The integrals can be evaluated analytically by considering separately the regions $0 < \theta < \pi/2$ and $\pi/2 < \theta < \pi$ (see Appendix F.3 for the details) to give

$$\begin{aligned}
\left[\mathcal{I}_1^{(\mathrm{rec})}\right](\theta) =& -\frac{2u^t\mu\Sigma_0}{3Mr_0^3\Delta_0^2}\left\{2a^6M\Omega^2 + ar_0^3(4Mr_0 - 6M^2 - 3r_0^2)\Omega - 3a^3r_0(r_0^2f_0 - 2M^2)\Omega\right.\\
&-4a^5M\Omega - a^4\left[3M^2r_0\Omega^2 + 2r_0^3\Omega^2 + M(r_0^2\Omega^2 - 2)\right] + a^2r_0\left[3M^2(r_0^2\Omega^2 - 1)\right.\\
&\left.\left.+r_0^2(5 + r_0^2\Omega^2) + Mr_0(2r_0^2\Omega^2 - 5)\right] + r_0^3\left[3M^2 + r_0^2 + r_0^4\Omega^2 - 3Mr_0(1 + r_0^2\Omega^2)\right]\right\},
\end{aligned} \quad (5.52\mathrm{a})$$

$$\begin{aligned}
\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) =& \frac{u^t\mu\Sigma_0}{3aMr_0^3}\left\{3\left[2a^4M\Omega + 2r_0^5\Omega + 2a^2r_0(r_0^2f_0 - 3M^2)\Omega - a^5M\Omega^2 + a^3(r_0^3\Omega^2\right.\right.\\
&\left.-M + 3M^2r_0\Omega^2 + Mr_0^2\Omega^2) - ar_0(r_0^4\Omega^2 - 3M^2 - 3Mr_0 + 3r_0^2 + 3Mr_0^3\Omega^2)\right]\\
&+a\left[a^4M\Omega^2 - 2a^3M\Omega + 2aMr_0(2r_0 - 3M)\Omega + a^2(M + 3M^2r_0\Omega^2 - Mr_0^2\Omega^2\right.\\
&\left.\left.-r_0^3\Omega^2) + r_0\left(3M^2 + r_0^2 + r_0^4\Omega^2 - 3Mr_0(1 + r_0^2\Omega^2)\right)\right]\cos(2\theta)\right\}\csc^2\theta \quad (5.52\mathrm{b})
\end{aligned}$$

where $\Sigma_0 \equiv \Sigma(r_0) = r_0 + a^2\cos^2\theta$.

5.3.4 Determination of the completion piece

The completion pieces are given by Eq. (5.41), without evaluating $J \to 0$, with arbitrary amplitudes $\delta M^{\pm}$, $\delta J^{\pm}$. In Kerr we obtain

$$h_{tt}^{(\mathrm{comp})} = \frac{r}{\Sigma^2}\left\{\delta M\left[3\cos(2\theta)a^2 + 3a^2 + 2r^2\right] - 4a\delta J\cos^2\theta\right\}, \tag{5.53a}$$

$$h_{t\varphi}^{(\mathrm{comp})} = -\frac{r}{\Sigma^2}\left[\delta M\sin^2(2\theta)a^3 + 2\delta J\left(r^2 - a^2\cos^2\theta\right)\sin^2\theta\right], \tag{5.53b}$$

$$h_{rr}^{(\mathrm{comp})} = \frac{r}{M\Delta^2}\left\{\left[2(3M-r)\cos^2\theta a^2 + 2ra^2 + 2Mr^2\right]\delta M + \left[arf\cos(2\theta) - 2Ma - ra\right]\delta J\right\}, \tag{5.53c}$$

$$h_{\theta\theta}^{(\mathrm{comp})} = \frac{2a\cos^2\theta}{M}(\delta J - a\delta M), \tag{5.53d}$$

$$\begin{aligned} h_{\varphi\varphi}^{(\mathrm{comp})} = -\frac{a\sin^2\theta}{4M\Sigma^2}\Big[&3\delta Ma^5 - 3\delta Ja^4 - Mr\delta Ma^3 + \left(\delta Ma^2 - \delta Ja + Mr\delta M\right)\cos(4\theta)a^3 \\ &+8r^2\delta Ma^3 - 8r^2\delta Ja^2 + 8r^4\delta Ma + 4Mr^3\delta Ma - 8r^4\delta J - 8Mr^3\delta J + 4\big(\delta Ma^5 \\ &-\delta Ja^4 + 2r^2\delta Ma^3 - 2r^2\delta Ja^2 - Mr^3\delta Ma + 2Mr^3\delta J\big)\cos(2\theta)\Big], \end{aligned} \tag{5.53e}$$

where we omitted the $\pm$ for brevity.

The completion part of the invariants, namely $\mathcal{I}_{1,2\pm}^{(\mathrm{comp})}(r,\theta)$, is constructed using Eq. (5.53) according to Eq. (5.12). Explicitly we find

$$\left[\mathcal{I}_1^{(\mathrm{comp})}\right](r,\theta) = \frac{\Sigma}{3M\Delta^2}\left((5a^2 + r^2)[\delta M] - 3a[\delta J]\right), \tag{5.54a}$$

$$\left[\mathcal{I}_{2\pm}^{(\mathrm{comp})}\right](r,\theta) = -\frac{\Sigma}{6aM}\left\{\left(a\cos(2\theta) - 9a\right)[\delta M] + 6[\delta J]\right\}\csc^2\theta. \tag{5.54b}$$

Notice that $\left[\mathcal{I}_1^{(\mathrm{comp})}\right]$ depends only on δM as we take the $a \to 0$ limit and we recover Eq. (5.43). For circular orbits around Kerr we find that those amplitudes are (just like in the Schwarzschild case) identically

$$[\delta M] = \mathsf{m}\mathcal{E}, \qquad \text{and} \qquad [\delta J] = \mathsf{m}\mathcal{L}. \tag{5.55}$$

Notice that our resulting jumps are, just like in the Schwarzschild case, independent of the chosen point on $\mathcal{S}$. Namely the jumps are independent of θ, which is a strong test of our results.

This result means that by fixing the total mass and AM of the system the amplitudes of the completion pieces are fully determined as for Schwarzschild.

5.4 Eccentric-equatorial orbits in Kerr

5.4.1 Strategy

We consider a bound geodesic around a Kerr BH, parametrized by the specific energy and AM $\{\mathcal{E},\mathcal{L}\}$. The BL radial-position of an orbiting particle of mass m is $r = r_0(t_0)$, with $r_{\min} \leq r_0 \leq r_{\max}$. t_0 denotes the instantaneous BL time of the particle. We denote the radial component of its four-velocity with respect of the proper time of the particle τ by $\dot{r}_0 \equiv dr_0/d\tau$.

The CCK procedure is still valid for the setup we just described in the time-domain. Instead of looking at the full time-domain problem we will decompose the source in partial rings, each labelled by t_0, see Fig. 5.2. Under this construction the problem will reduce to a sum of circular orbits, treated similarly as in Sec. 5.3.1, and the reconstructed part of the invariants are recovered analytically at each t_0 inside the region $r_{\min} \leq r_0 \leq r_{\max}$. The completion pieces are constructed simply by allowing a t_0-dependence of the arbitrary amplitudes in their circular-orbits counterparts. We solve for the 'partial' amplitudes of the mass and AM perturbations by imposing the regularity conditions of the completed invariants at a given time t_0. The 'full' amplitudes of the completion pieces are then recovered by integrating over t_0.

5.4.2 Metric reconstruction and auxiliary invariants

The radial part of Teukolsky equation is

$$\begin{aligned}\hat{\mathcal{T}}\psi_{\ell m\omega}(r) =& \tilde{T}_{\ell m\omega}(r) \\ =& \int \tilde{T}_{\ell m}(r; r_0(\tau), \dot{r}_0(\tau))e^{i\omega t}d\tau,\end{aligned} \tag{5.56}$$

where $\hat{\mathcal{T}}$ is the same as in the circular orbits case. In the second equality we expressed $\tilde{T}$ in terms of the original (2d) time-domain source $\tilde{T}_{\ell m}(r; r_0(\tau), \dot{r}_0(\tau))$. The integral is an inverse Fourier transform, which should be taken over a radial period with a suitable normalization that is being absorbed in $\tilde{T}_{\ell m}$ for the time being, but it can be recovered from the expressions in Sec. 2.1. The time-domain source has the form

$$\begin{aligned}&\tilde{T}_{\ell m}(r; r_0(\tau), \dot{r}_0(\tau)) \\ &= \sum_{k=0}^{2} \tilde{t}_{\ell m[k]}(r_0(\tau), \dot{r}_0(\tau))\delta^{(k)}(r - r_0(\tau)),\end{aligned} \tag{5.57}$$

where the coefficients $\tilde{t}_{\ell m[k]}$, like $r_0(\tau)$ and $\dot{r}_0(\tau)$, depend on the chosen parametrization.

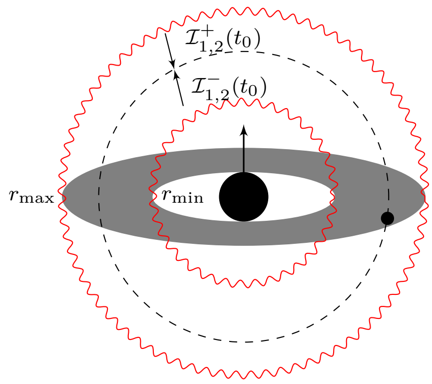

Figure 5.2: The particle moves in a precessing eccentric orbit, covering all the libration region (in gray). The idea to determine the completion pieces follows from the circular-orbit case. We impose continuity for the total invariants (reconstructed plus completion) at the level of partial rings in the libration region. By integrating over partial amplitudes of the completion piece over t_0 we read the total jump of the amplitudes in the completion pieces.

The particular solution to Eq. (5.56) satisfying retarded boundary-conditions is given by

$$\psi_{\ell m\omega}(r) = \int G_{\ell m\omega}(r; t_0)dt_0, \tag{5.58}$$

where we have changed the parametrization from τ to t_0 using $d\tau = \frac{d\tau}{dt_0}dt_0$ and reabsorbing the $\frac{d\tau}{dt_0}$ in $G_{\ell m\omega}(r; t_0)$. The retarded Green's function $G_{\ell m\omega}(r; t_0)$ satisfies

$$\hat{\mathcal{T}}G_{\ell m\omega}(r; t_0) = \tilde{T}_{\ell m}(r; r_0(t_0), \dot{r}_0(t_0))e^{i\omega t_0}, \tag{5.59}$$

with the same boundary conditions as $\psi_{\ell m\omega}$, and $G_{\ell m\omega}(r; t_0)$ depends on time only parametrically, not functionally.

We assume that just like in the circular-orbit case the completion piece is stationary and axially symmetric, and we look at $G_{\ell 00}$ only. $G_{\ell 00}$ satisfies

$$\hat{\mathcal{T}}G_{\ell 00}(r; t_0) = \tilde{T}_{\ell 0}(r; r_0(t_0), \dot{r}_0(t_0)). \tag{5.60}$$

Note that the source $\tilde{T}_{\ell 0}$ has the same general form as for a circular orbit, but with different coefficients $\tilde{t}_{\ell 0[k]}$ (in particular, these coefficients involve $\dot{r}_0(t_0)$). We now think of $G_{\ell 00}(r;t_0)$ as the field due to a partial ring of radius $r = r_0(t_0)$ that expands/contracts at momentary radial velocity $\dot{r}_0(t_0)$. We can then proceed with the completion procedure as in the circular-orbit case (see Sec. 5.2.1), holding t_0 (hence also r_0 and $\dot{r}_0$) fixed.

Given $h^{(\mathrm{rec})}_{\alpha\beta}$ one then constructs, at each t_0, the two gauge-invariant quantities $\mathcal{I}^{(\mathrm{rec})}_{1,2}(r,\theta;t_0) = \hat{I}^{\alpha\beta}_{1,2} h^{(\mathrm{rec})}_{\alpha\beta}(r,\theta;t_0)$ and obtains their jumps across $r = r_0(t_0)$:

$$\left[\mathcal{I}^{(\mathrm{rec})}_{1,2}\right](\theta;t_0) \equiv \lim_{\epsilon\to 0} \hat{I}^{\alpha\beta}_{1,2}\left(h^{(\mathrm{rec})}_{\alpha\beta}(r_0(t_0)+\epsilon,\theta;t_0) - h^{(\mathrm{rec})}_{\alpha\beta}(r_0(t_0)-\epsilon,\theta;t_0)\right). \tag{5.61}$$

The completion pieces at each t_0 are constructed in the same way as in the circular-orbit case Eq. (5.53), including the t_0 dependence in the amplitudes of the perturbations, namely $\delta M^{\pm} \to \delta M^{\pm}(t_0)$ and $\delta J^{\pm} \to \delta J^{\pm}(t_0)$.

Given $h^{(\mathrm{comp})}_{\alpha\beta}$, the jumps $\left[\mathcal{I}^{(\mathrm{comp})}_{1,2}\right]$ are obtained via

$$\left[\mathcal{I}^{(\mathrm{comp})}_{1,2}\right](\theta;t_0) \equiv \hat{I}^{\alpha\beta}_{1,2}\left(h^{(\mathrm{comp})+}_{\alpha\beta}(r\to r_0(t_0),\theta;t_0) - h^{(\mathrm{comp})-}_{\alpha\beta}(r\to r_0(t_0),\theta;t_0)\right). \tag{5.62}$$

These t_0-dependent jumps $[\delta M](t_0) \equiv \delta M^+(t_0) - \delta M^-(t_0)$ and $[\delta J](t_0) \equiv \delta J^+(t_0) - \delta J^-(t_0)$ are now determined instantaneously from the two regularity conditions

$$\left[\mathcal{I}^{(\mathrm{rec})}_{1,2}\right](\theta;t_0) + \left[\mathcal{I}^{(\mathrm{comp})}_{1,2}\right](\theta;t_0) = 0 \quad \text{for } \theta \neq \pi/2. \tag{5.63}$$

Now consider the true, time-domain completion piece $h^{(\mathrm{comp})}_{\alpha\beta}(r,\theta)$. Outside the libration domain it is given by

$$h^{(\mathrm{comp})}_{\alpha\beta}(r > r_{\max},\theta) = \int dt_0\, h^{(\mathrm{comp})+}_{\alpha\beta}(r,\theta;t_0) = \delta\tilde{M}^+ h^{(\delta M)}_{\alpha\beta}(r,\theta) + \delta\tilde{J}^+ h^{(\delta J)}_{\alpha\beta}(r,\theta), \tag{5.64a}$$

$$h^{(\mathrm{comp})}_{\alpha\beta}(r < r_{\min},\theta) = \int dt_0\, h^{(\mathrm{comp})-}_{\alpha\beta}(r,\theta;t_0) = \delta\tilde{M}^- h^{(\delta M)}_{\alpha\beta}(r,\theta) + \delta\tilde{J}^- h^{(\delta J)}_{\alpha\beta}(r,\theta), \tag{5.64b}$$

where $\delta\tilde{M}^{\pm}, \delta\tilde{J}^{\pm}$ are constant amplitudes given by

$$\delta\tilde{M}^{\pm} = \int \delta M^{\pm}(t_0)dt_0, \qquad \delta\tilde{J}^{\pm} = \int \delta J^{\pm}(t_0)dt_0, \tag{5.65}$$

and all t_0 integrals are over a full radial-period. Note that, assuming analyticity off the particle, the time-domain solutions (5.64a) must extend smoothly all the way to the worldline, on either side:

$$h^{(\mathrm{comp})}_{\alpha\beta}(r > r_0(t),\theta) = \delta\tilde{M}^+ h^{(\delta M)}_{\alpha\beta}(r,\theta) + \delta\tilde{J}^+ h^{(\delta J)}_{\alpha\beta}(r,\theta), \tag{5.66a}$$

$$h^{(\mathrm{comp})}_{\alpha\beta}(r < r_0(t),\theta) = \delta\tilde{M}^- h^{(\delta M)}_{\alpha\beta}(r,\theta) + \delta\tilde{J}^- h^{(\delta J)}_{\alpha\beta}(r,\theta). \tag{5.66b}$$

We are interested in the jumps

$$[\delta\tilde{M}] \equiv \delta\tilde{M}^+ - \delta\tilde{M}^- = \int [\delta M](t_0)dt_0, \qquad [\delta\tilde{J}] \equiv \delta\tilde{J}^+ - \delta\tilde{J}^- = \int [\delta J](t_0)dt_0. \tag{5.67}$$

The integrals in Eq. (5.67) have to be evaluated over an orbital period, as before.

5.4.3 Determination of the completion piece

The explicit calculation of the jumps in the reconstructed part of the invariants at a given t_0 follows directly from the circular-orbit case [with the same homogeneous solutions as Eq. (5.48)]. In particular the ℓ-dependence that appears with the spherical harmonics $Y_\ell(\theta)$ remains unchanged and no additional sums (from the ones used for the circular-orbit case and given in Appendix F.3) are needed to analytically evaluate the jumps across the sphere. In other words, the expression for jumps at each time t_0 have the same structure as Eq. (5.51) in the previous section, with the replacement of the circular-orbit coefficients c_i°, d_i° by the more general expressions c_i, d_i given explicitly in Eqs. (F.4) and (F.5) of Appendix F.

The completion pieces also have the same structure as in the circular orbits case Eq. (5.53), with the appropriate replacement of the amplitudes $\delta M^\pm \to \delta M^\pm(t_0)$ and $\delta J^\pm \to \delta J^\pm(t_0)$. At each time t_0 we impose the regularity condition Eq. (5.63) and solve for the arbitrary amplitudes $\delta M(t_0)$ and $\delta J(t_0)$ in terms of $[\mathcal{I}^{(\mathrm{rec})}](t_0)$. We perform the integrals Eq. (5.65) by choosing $\{p, e\}$ as the orbital parameters [148], which are defined in terms of the two turning-points ($r_{\min}$ for periastron and $r_{\max}$ for apastron) as

$$r_{\min} = \frac{Mp}{1+e}, \quad \text{and} \quad r_{\max} = \frac{Mp}{1-e}. \tag{5.68}$$

This way the specific energy $\mathcal{E}$ and AM $\mathcal{L}$ are given by

$$\mathcal{L} = x - a\mathcal{E}, \quad \mathcal{E} = \left[1 - \left(\frac{M}{p}\right)(1-e^2)\left\{1 - \frac{x^2}{p^2}(1-e^2)\right\}\right]^{1/2}, \quad \text{with} \quad x^2 = \frac{-N \mp \Delta_x^{1/2}}{2F}, \tag{5.69}$$

where the upper sign corresponds to prograde orbits, and

$$\begin{aligned}
F =& \frac{1}{p^3}[p^3 - 2M(3+e^2)p^2 + M^2(3+e^2)^2 p - 4Ma^2(1-e^2)^2], && (5.70)\\
N =& \frac{1}{p}\left\{-Mp^2 + [M^2(3+e^2) - a^2]p - Ma^2(1+3e^2)\right\}, && (5.71)\\
C =& (a^2 - Mp)^2, && (5.72)\\
\Delta_x =& N^2 - 4FC. && (5.73)
\end{aligned}$$

The position of the particle is given in BL coordinates as

$$r(t_0) = r_0(\chi) = \frac{pM}{1 + e\cos\chi}, \tag{5.74}$$

where the parameter χ increases monotonically along the orbit. We set $t_0(\chi = 0) = 0$ to be the time the particle is at periastron and define the radial period (the t-time interval between one periastron and a consecutive one) by $T_r \equiv t_0(\chi = 2\pi) = 2t_0(\chi = \pi)$.

We next need to evaluate the integrals in Eqs. (5.67). In practice it is easier to consider the deviation of the integrals from the value of the specific energy and AM:

$$[\delta\tilde{M}] - \mathsf{m}\mathcal{E} = \int_{-\pi}^{\pi}\left\{[\delta M](t_0) - \frac{\mathsf{m}\mathcal{E}}{T_r}\right\}\frac{dt_0}{d\chi}d\chi, \text{ and } [\delta\tilde{J}] - \mathsf{m}\mathcal{L} = \int_{-\pi}^{\pi}\left\{[\delta J](t_0) - \frac{\mathsf{m}\mathcal{L}}{T_r}\right\}\frac{dt_0}{d\chi}d\chi. \tag{5.75}$$

The integrands are evaluated analytically in Appendix F.5.

After evaluating the integrals Eq. (5.67) gives

$$\left[\delta\tilde{M}\right] = \mathsf{m}\mathcal{E}, \qquad \text{and} \qquad \left[\delta\tilde{J}\right] = \mathsf{m}\mathcal{L}. \tag{5.76}$$

The total jump of the amplitudes in the completion pieces correspond to the energy and the AM of the orbiting particle. These amplitudes turned out to be functionally independent of θ, just like in the circular-orbit case. By fixing the total energy and AM of the system we can determine, using Eq. (5.76), straightforwardly the contributions on each side of $\mathcal{S}$. This will give a Kerr metric at infinity parametrized by $M + \mathsf{m}\mathcal{E}$ and $a + \mathsf{m}\mathcal{L}/M$. The amplitudes of the completion piece in the interior are then fixed to zero, just as we did in the circular-orbit case.

5.5 Summary

The inclusion of the completion piece remained a long-standing open problem in BH-perturbation theory. This piece together with the CCK-reconstructed perturbation should satisfy the full EFE.

In this Chapter we have addressed the completion problem in Kerr using a new, rigorous and practical method. We took advantage of the auxiliary invariants (5.12) and continuity conditions (5.44) and (5.63). We tested our method for the simple case of circular orbits around Schwarzschild where the completion piece was previously known. In Kerr the expression are considerably more complicated, but still analytical. With the jumps of Eqs. (5.55) and (5.76), and by fixing the total mass and AM of the system, the completion pieces in Eq. (5.53) are fully determined. This fully solves the completion problem for any equatorial orbit around Kerr.

An extension to perform a similar calculation for non-equatorial (inclined) orbits around Kerr will follow the same conceptual approach: imposing regularity on the invariants at a given time and integrating over the libration region to find the amplitudes of the completion. This new calculation will have to deal with the longitudinal modes that arise due to the fact that the particle is no longer contained in the equatorial plane, and exhibits an 'extra' angular velocity $\dot{\theta}_0$. This gives (for $m = 0$) a two-dimensional spectrum for the orbital frequencies. This might result in the appearance of different products of the angular functions (from those we give in Appendix F.3) with more complicated mode-dependence. We may be able to evaluate those sums analytically, but if that is not possible our numerical experiments on circular orbits of Kerr suggest the sums would converge slowly, giving 'large' numerical errors.

Chapter 6

Concluding remarks

In this thesis we sought to develop a GSF formalism for BH perturbation theory in the RGs. We expect the method based on the RG-reconstructed modes and completion will become the workhorse of SF calculations.

We have analysed the local singular-structure of the RGs (either ingoing or outgoing) in a practical basis of Fermi-like coordinate. The leading-order singularity of the gauge vector relating the LG and RGs permeates to the singular structure of the RG perturbation. This structure provided a natural classification of RGs and we identified three categories. Table 3.1 summarizes the local form of the singularity (in local Fermi-like coordinates) for each category. The RGs with a full-string singularity are not suitable for numerical implementations and orbital evolutions.

Based on the singular structure of the RGs, we considered two methods to calculate the GSF for eccentric orbits around Kerr. The first method considered a local deformation of the RG near the particle, so that its leading-order term would corresponds to the LG singularity. Such a gauge is regular in the class of gauges considered by Barack and Ori [19] where the standard LG mode-sum gives the desired value of the SF. The retarded force in this LL gauge can be obtained from the corresponding half-string completed RG force (or alternatively from a no-string completed RG), where the CCK-reconstruction procedure is defined and practical. To regularize this retarded force using (3.70), on each side-limit of the particle's location, the inclusion of a non trivial correction to the Lorenz D_α regularization parameter is required. The calculation of this correction was done in Fermi-like coordinates in Sec. 3.3.2 and expression in BL coordinates appear in Appendix D for a rigid off-the-particle extension of the four velocity and connections.

In Sec. 3.4 the second method was formulated, and it considered an undeformed RG. This method takes advantage of the spatial average form of the SF (2.51). The result was the averaged version of the mode-sum formula (3.96) for this type of gauges. This new mode-sum is also applicable in an LL gauge since the δD_α, mentioned above, is antisymmetric with respect of the direction the limit to the particle is taken. In other words, δD_α flips sign across the particle, and by taking the average $\delta D_\alpha^\pm$ cancel each other.

It should be noted that the final SF value obtained using the '+' half-string solution should by no means agree with the final SF value obtained using the '$-$' solution, or with the one obtained using the no-string solution (the average of the former two): the three values are given in different gauges.

Let us recall that a complete gauge-invariant description of the motion involves the SF as well as the associated MP, given in the same gauge. In the case of the half-string scheme, the prescription is simple: take the CCK-reconstructed (and completed) half-string RG perturbation, and add to it the corresponding gauge perturbation $2\xi_{(\alpha;\beta)}$, [given according to Eq. (3.60a) with $Z^{\pm}_{\alpha} = 0$]; this perturbation can be attenuated in any convenient way to suppress its support away from the particle. This will produce an LL perturbation in a corresponding LL gauge. In the case of the no-string scheme, the situation is a bit more subtle: the force is given in the same gauge as the reconstructed (and completed) MP, but the MP in that gauge has a discontinuity across a surface through the particle, which might complicate calculations of some gauge-invariant aspects of the motion.

A detailed numerical implementation to calculate the SF for a massive particle in a circular orbit around a Schwarzschild BH was presented in Chapter 4. We considered the regular sides of the RG given by the CCK-reconstruction, namely we worked in a no-string gauge. This computation is first of its kind: the first calculation of the GSF using reconstructed RG-perturbations and regularised using the average-mode-sum formula. We showed the equivalence (at the level of GSF calculations in our particular setup) of working in an IRG or an ORG. We made a successful comparison between the MST method and numerical integration of Teukolsky equation. The numerical code also recovers well known quantities available in the literature, such as the energy fluxes, the red-shift H^R and the t component of the SF. Even more, the SF calculated this way agrees asymptotically with the LG values, with the difference between them falling off with r^{-4}, as expected from the gauge transformation equation for the SF, this was shown in Fig. 4.4.

In Chapter 5 we considered the completion part of the solution, namely the piece that is required to add to the CCK-reconstruction perturbation to satisfy the linearised EFE. Our solution to the completion problem took advantage of certain gauge-invariant-auxiliary quantities which are related to the components of the MP and ψ_2. Each invariant was constructed by adding two contributions: one due to the MP reconstructed with the CCK procedure, and another obtained from the completion piece. We argued that such invariants must be smooth off the particle across a sphere intersecting it (even though in practice we only imposed continuity). In this way we determine the missing amplitudes of the completion piece.

We are working to extend our GSF numerical-implementation to calculate the GSF using the MST-method, completion of the RG and the average-mode-sum formula for generic orbits around Kerr [145]. The more general computation will follow the basic algorithm we introduced in Sec. 4.1. Teukolsky equation remains separable in Kerr —unlike the tensorial equations in the LG— and the MP-reconstruction procedure is well understood. One of the remaining challenges in SF calculations for inclined orbits in Kerr is the completion piece of the MP. A second challenge in the Kerr calculations is the re-expansion of the ℓ-modes into the spin-0 spherical harmonics (where the regularization parameter are known [125, 139]). This involves a numerical projection of the spin-weighted spheroidal harmonics (where the harmonics modes of the retarded-force are obtained), which might not have the finite coupling they exhibited in the Schwarzchild case. The coupling will be simpler if a suitable off-worldline-extension of the four velocity is chosen. An alternative to implementing this coupling would be to obtain regularization parameters in the basis of spin-weighed spheroidal harmonics similarly to [54].

In order to make comparisons between PN and perturbation theory, a delicate issue must be addressed. PN calculations are done in coordinates with the origin coinciding with the centre of mass of the BH-particle system, while the coordinates for SF computations coincide with those of the background BH. This means that for the comparison between the two methods the SF requires the dipole moment associated with the displacement from the centre of the BH to the centre of mass of the full system. In Schwarzschild this contribution was obtained numerically by Detweiler and Poisson [117], but in Kerr the problem has not been addressed to date to our knowledge.

The full calculation of the GSF for generic orbits around Kerr was not achieved in this work, however we have provided in this thesis all the tools for it. We developed two practical methods to calculate the GSF using the reconstructed RG perturbation. In this method the most computationally expensive task involves obtaining curvature scalars. This is done by solving scalar-wave equations which are separable even in Kerr. We have numerically implemented one of the methods in the simple case Schwarzschild, and obtained the GSF in an undeformed no-string RG. This serves as a test of the applicability and computational cost of the method. Our values for the GSF also correct those previously computed in [83]. Along the way we have cleared one open problem of BH-perturbation theory, namely the inclusion of the completion piece for any equatorial orbit around Kerr.

Appendix A

Motion as defined in matched asymptotic expansions

The standard method of deriving an equation of motion in the context of the SF is matched asymptotic expansions [14, 16, 119–123, 149]. In this Appendix we present a review of the basic ideas used in this method, and in particular how to obtain equations of motion in gauges outside the LG class of gauges, such as the RGs that we addressed in Chapter 3.

A.1 Centre of mass

In the method of matched asymptotic expansions, see Fig. A.1, one assumes that the particle is actually a small, compact object. Let $\mathsf{g}_{\mu\nu}(x,\varepsilon)$ be the exact solution to the full, non-linear EFE for the spacetime including that small object, where ε is used to count powers of m but will be set to 1 at the end of the calculation. Also let $\mathcal{R}$ denote the other lengthscales of the system, which are much larger than m.

Suppose we work in the local Fermi-like coordinates (τ, x^a) centred on Γ, introduced in Sec. 3.1. We do not begin with any definite association between Γ and the bulk motion of the small object, but we start by assuming that the object is only a small distance from Γ. At distances $s \gg \mathsf{m}$, far from the object, one can expand the exact metric as $\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + O(\varepsilon^3)$, which is the form of the expansion assumed throughout Chapter 2. We call this the outer expansion. In this expansion the first-order perturbation, $h^{(1)}_{\mu\nu} \equiv h_{\mu\nu}$, is that of a point particle moving on Γ in the background $g_{\alpha\beta}$ [16].

At distances $s \sim \mathsf{m}$, near the object, the outer expansion fails because in that region the metric is dominated not by $g_{\mu\nu}$, but by the gravity of the small object. The method of matched asymptotic expansions overcomes that problem by adopting a second expansion near the object. Rather than taking the limit of small *mass and size* by keeping external distances fixed while sending the mass and size to zero, we take the limit by keeping the mass and size of the object fixed while sending other distances to infinity. This second limit is achieved by writing the metric components in terms of scaled variables $\bar{x}^a = x^a/\mathsf{m}$. Holding these scaled variables fixed while expanding for small m, we have

$$\mathsf{g}_{\mu\nu}(\bar{x},\varepsilon) = g^{(0)}_{\mu\nu}(\tau,\bar{x}^a) + \varepsilon g^{(1)}_{\mu\nu}(\tau,\bar{x}^a) + O(\varepsilon^2), \tag{A.1}$$

where $g^{(0)}_{\mu\nu}(\tau,\bar{x}^a)$ is the metric of the small body if it were isolated. We call this the inner expansion.

The motion of the small object is defined by examining the metric in a buffer region $\mathsf{m} \ll s \ll \mathcal{R}$

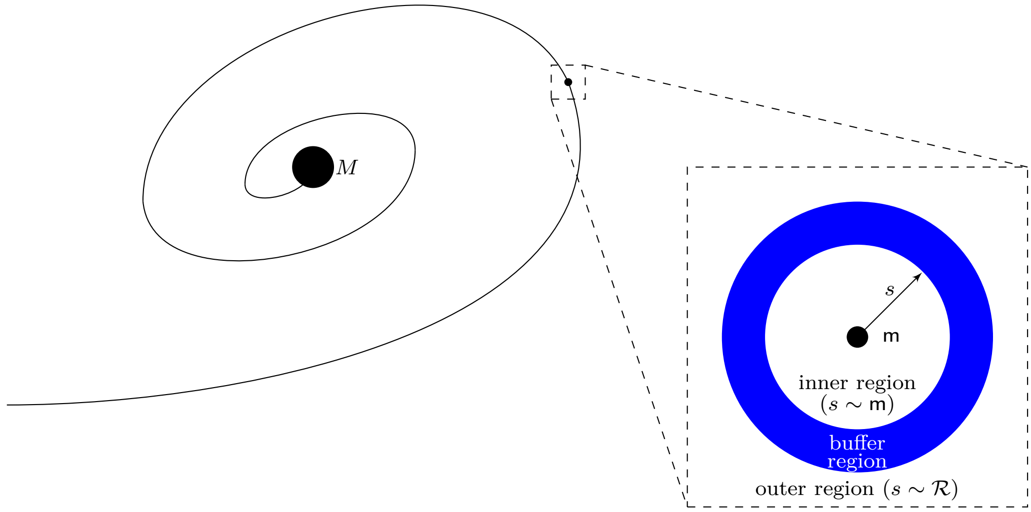

Figure A.1: Matched asymptotic expansions: In the outer region, far from m, the metric of the physical spacetime g is expanded. In the inner region, near m, the metric of the small object g is dominant. In this region the outer expansion fails and a second expansion is obtained. In the buffer region the two expansions are valid so they must agree (up to a gauge).

around the body. Because $s \gg \mathsf{m}$, we can expect the outer expansion to be valid here; because $s \ll \mathcal{R}$, we can expect the inner expansion to also be valid here; and because they are both expansions of the same metric $\mathsf{g}_{\mu\nu}$, the two expansions must agree (up to gauge). This allows us to infer information about the outer expansion from information about the inner expansion in the buffer region. The first thing we infer is that the existence of an inner expansion requires the outer expansion to have the local form [16, 119]

$$h_{\mu\nu} \sim 1/s, \qquad h^{(2)}_{\mu\nu} \sim 1/s^2, \tag{A.2}$$

near the wordline.

Furthermore, we note that while the buffer region is asymptotically small from the perspective of the outer expansion, it corresponds to asymptotic infinity from the perspective of the inner expansion. Using that fact, we can define multipole moments of the inner expansion, and those multipole moments become the kernels of the outer expansion. As an example, we note that the Arnowitt-Deser-Misner (ADM) mass of $\bar{g}^{(0)}_{\tau\tau}$ in the inner expansion defines the point-particle mass m in the outer expansion [16].

For the particular purpose of defining the object's motion, we will be interested in the mass dipole moment of the object's unperturbed metric:

$$M^a = \frac{3}{8\pi} \lim_{\bar{s}\to\infty} \int \bar{g}^{(0)}_{\tau\tau}(\tau, \bar{x}^b) n^a dS, \tag{A.3}$$

where the integration is over a sphere of radius $\bar{s}$ around the object, and n^a is the unit vector $x^a/s = \bar{x}^a/\bar{s}$ normal to the sphere. Using this formula, we can meaningfully define the object's motion. Per unit-mass, a mass-dipole moment has the interpretation of the position of the centre of mass relative to the origin of the coordinates. Since we work in coordinates centred on the worldline

Γ, the mass-dipole per unit-mass can be interpreted as the position relative to Γ. More explicitly, imagine the object's motion is described by a worldline $z^\alpha(\tau,\varepsilon)$ with the expansion

$$z^\alpha(\tau,\varepsilon) = z_0^\alpha(\tau) + \varepsilon z_1^\alpha(\tau) + O(\varepsilon^2), \tag{A.4}$$

where $z_0^\mu(\tau)$ are the coordinates on the geodesic Γ, and $z_1^\alpha(\tau)$ is a vector field on Γ. Then we define the leading-order correction z_1^α to the object's position as[1]

$$z_1^\alpha \equiv \frac{M^\alpha}{\mathsf{m}}, \tag{A.5}$$

where $M^\alpha \equiv e_a^\alpha M^a$. This was the method used by Gralla and Wald in the first rigorous derivation of the first-order GSF, and modifications of it have since been the basis for derivations of the second-order GSF [121, 149].

We can relate M^α to the perturbations in the outer expansion by appealing to the assumed agreement between the two expansions in the buffer region. In that region, we can expand $\bar{g}^{(0)}_{\tau\tau}$ as

$$\bar{g}^{(0)}_{\tau\tau}(\tau,\bar{s},n^a) = \frac{1}{\bar{s}} g^{(0,1)}_{\tau\tau}(\tau) + \frac{1}{\bar{s}^2} g^{(0,2)}_{\tau\tau}(\tau,n^a) + O(\bar{s}^{-3}). \tag{A.6}$$

It is easy to see that only the term $\frac{1}{\bar{s}^2} g^{(0,2)}_{\tau\tau}$ contributes to Eq. (A.3). Written in terms of the unscaled variable s, this term becomes $\frac{\mathsf{m}^2}{s^2} g^{(0,2)}_{\tau\tau}$, and we can see it must correspond to a $1/s^2$ term in $h^{(2)}_{\tau\tau}$ in the outer expansion. Therefore, noting Eq. (A.2), we can write

$$M^a = \frac{3}{8\pi} \lim_{s\to 0} \int h^{(2)}_{\tau\tau} n^a dS, \tag{A.7}$$

or

$$\mathsf{m} z_1^a = \frac{3}{8\pi} \lim_{s\to 0} \int h^{(2)}_{\tau\tau} n^a dS, \tag{A.8}$$

where now the integral is over a sphere of radius s. Eq. (A.8) gives the first-order correction to the position in terms of the second-order perturbation in the outer expansion.

A.2 Equation of motion in any sufficiently regular gauge

In the work of Gralla-Wald [16], a first-order self-forced equation of motion was found by solving the Einstein equation to sufficiently high order to establish a formula for $\partial_\tau^2 M^a$. The result was

$$\mathsf{m}\frac{D^2 z^\alpha_{1\mathrm{Lor}}}{d\tau^2} = -\mathsf{m} R^\alpha{}_{\mu\beta\nu} u^\mu z^\beta_{1\mathrm{Lor}} u^\nu + F^\alpha_{\mathrm{Lor}}. \tag{A.9}$$

The first term, $-R^\alpha{}_{\mu\beta\nu} u^\mu z^\beta_{1\mathrm{Lor}} u^\nu$ describes the acceleration due to the background curvature. The second term is the LG force, F^α_{Lor}. It can be written in alternative forms, we will require the Quinn-Wald-Gralla spherical-average form

$$F^\alpha_{\mathrm{Lor}} = \frac{1}{4\pi} \lim_{s\to 0} \int \tilde{F}^\alpha_{\mathrm{Lor}} d\Omega. \tag{A.10}$$

Using the result (A.9) for the motion in the LG, we can find the motion in a different gauge

[1] An alternative method, called the self-consistent method, instead defines a mass dipole relative to the accelerated worldline $z^\alpha(\tau,\varepsilon)$, deriving an equation of motion for z^α by ensuring that that mass dipole vanishes [119]. That method is designed to maintain uniform accuracy on long timescales by avoiding an expansion of $z^\alpha(\tau,\varepsilon)$. Here, for simplicity, we work with the expanded worldline.

by referring to how the mass-dipole moment is altered by a gauge transformation. Under a gauge transformation generated by a first-order gauge vector ξ_α, the second-order perturbation is altered according to $h^{(2)}_{\mu\nu} \to h^{(2)}_{\mu\nu} + \Delta h^{(2)}_{\mu\nu}$, where [1]

$$\Delta h^{(2)}_{\mu\nu} = h_{\mu\nu;\rho}\xi^\rho + 2h_{\rho(\mu}\xi^\rho{}_{;\nu)} + \xi^\rho\xi_{(\mu;\nu)\rho} + \xi^\rho{}_{;\mu}\xi_{\rho;\nu} + \xi^\rho{}_{;(\mu}\xi_{\nu);\rho}, \tag{A.11}$$

and again we have used $h_{\mu\nu}$ to be the first-order perturbation $h^{(1)}_{\mu\nu}$. We restrict our attention to gauge transformations preserving the form (A.2) for all τ, to maintain compatibility with the existence of an inner expansion. Straightforward analysis of the transformation laws $\Delta h_{\alpha\beta} = 2\xi_{(\alpha;\beta)}$ and Eq. (A.11) shows that this compatibility requirement imposes the following conditions on ξ_α in the limit $s \to 0$ [1]:

1. $\xi_\tau = f_1(\tau)\ln s + o(\ln s)$,

2. $\xi_a = f_2(\tau, n^a) + o(1)$,

3. τ derivatives do not increase the degree of singularity; e. g. , $\partial_\tau \xi_\alpha = O(\xi_\alpha)$,

4. spatial derivatives increase the degree of singularity by at most one order of s ; e. g. , $\partial_a\xi_\alpha = O(\xi_\alpha/s)$.

The functions f_1 and f_2 must be at least twice-differentiable but otherwise are chosen arbitrarily. Let us note that all the gauge transformations between LG and RG, see Sec. 3.2, satisfy the four conditions given above, but this conditions are not restricted to the RG. Given these conditions, a simple calculation shows that if we begin in the LG, where $h_{\mu\nu} = \frac{2\mathsf{m}}{s}\delta_{\mu\nu} + O(1)$, the change in the time-time component of the second-order MP due to ξ_α is

$$\Delta h^{(2)}_{\tau\tau} = -\frac{2\mathsf{m}}{s^3}x^a\xi_a + o(\lambda^{-2}). \tag{A.12}$$

Of all the terms in Eq. (A.11), $h_{\mu\nu;\rho}\xi^\rho$ is the only one that contributes to this result. From Eq. (A.7) we get the change in mass-dipole moment and substituting $\Delta z^a_1 = \Delta M^a/\mathsf{m}$ Eq. (A.8) reads

$$\Delta z^a_1 = -\frac{3}{4\pi}\lim_{s\to 0}\int n^a n^b \xi_b d\Omega. \tag{A.13}$$

This is the change in position under a gauge transformation as considered in Sec. 2.5.

Once the change in position is in hand, the change in the GSF can be calculated in a few short steps. First, we write the result covariantly using $\Delta z^\alpha = e^\alpha_a \Delta z^a_1$. Next, we calculate the acceleration of Δz^α_1 by taking two covariant derivatives along the worldline, yielding

$$\frac{D^2\Delta z^\alpha_1}{d\tau^2} = \Delta a^\alpha \equiv -\frac{3}{4\pi}\frac{D^2}{d\tau^2}\left[e^\alpha_a \lim_{s\to 0}\int n^a n^b\xi_b d\Omega\right], \tag{A.14}$$

Finally, we add and subtract $R^\alpha{}_{\mu\beta\nu}u^\mu\Delta z^\beta u^\nu$, leading to the evolution equation

$$\mathsf{m}\frac{D^2\Delta z^\alpha_1}{d\tau^2} = -\mathsf{m}R^\alpha{}_{\mu\beta\nu}u^\mu\Delta z^\beta_1 u^\nu + \Delta F^\alpha, \tag{A.15}$$

where we have identified

$$\begin{aligned}\Delta F^\alpha &\equiv \mathsf{m}\Delta a^\alpha + \mathsf{m}R^\alpha{}_{\mu\beta\nu}u^\mu \Delta z_1^\beta u^\nu \\ &= -\frac{3}{4\pi}\frac{D^2}{d\tau^2}\left[e_a^\alpha \lim_{s\to 0}\int n^a n^b \xi_b d\Omega\right] + \mathsf{m}R^\alpha{}_{\mu\beta\nu}u^\mu \Delta z_1^\beta u^\nu,\end{aligned} \tag{A.16}$$

as the change in the GSF under the transformation generated by ξ_α. Our reason for adding zero in the form of Riemann terms is that doing so allows us to write the evolution equation for $z^\alpha_{1\mathrm{Lor}} + \Delta z^\alpha_1$ in terms of a geodesic-deviation term plus a SF term, as in Eq. (A.9):

$$\mathsf{m}\frac{D^2}{d\tau^2}(z^\alpha_{1\mathrm{Lor}} + \Delta z^\alpha_1) = -\mathsf{m}R^\alpha{}_{\mu\beta\nu}u^\mu(z^\beta_{1\mathrm{Lor}} + \Delta z^\beta_1)u^\nu + F^\alpha_{\mathrm{Lor}} + \Delta F^\alpha. \tag{A.17}$$

This provides a method of finding the SF in a broad class of gauges, beginning in the LG and then transforming to the desired gauge. If the transformation satisfies conditions 1–4 enumerated above, *and* it is such that the integral (A.13) yields a well-defined, C^2 function of τ along Γ, then we say that the end gauge is *sufficiently regular* to define the GSF. We calculate the change in force, ΔF^α, generated by such a transformation, using Eq. (A.16). The total GSF in the end-gauge is then given by the force in the LG plus the change due to the gauge transformation. This is the method used for deriving expressions for the SF in the RG in Chapter 3.

Appendix B

Lorenz-Gauge regularization parameters

We include the analytical expressions required to regularize the GSF in the LG, for a particle at an arbitrary point z along its orbit around a Kerr BH. The Kerr values for regularization parameters were first given in [31] for the scalar field and [97] in the EM and gravitational cases.

Let us assume that z has BL coordinates $(t_0, r_0, \theta_0, \varphi_0)$. In the LG the regularization parameters C^α and D^α are zero:

$$C^\alpha = D^\alpha = 0. \tag{B.1}$$

The components of the parameter A^α are given by

$$A^r = -\frac{\mathsf{m}^2}{V}\left(\frac{\sin^2\theta_0}{g_{rr}g_{\theta\theta}g_{\varphi\varphi}}\right)^{1/2}(V + u_r^2/g_{rr})^{1/2}, \tag{B.2a}$$

$$A^t = -(u_r/u_t)A^r, \qquad A^\theta = A^\varphi = 0, \tag{B.2b}$$

where

$$V \equiv 1 + u_\theta^2/g_{\theta\theta} + u_\varphi^2/g_{\varphi\varphi}, \tag{B.3}$$

and the four-velocity u^α and the metric components of the Kerr background $g_{\alpha\beta}$ are evaluated at z.

The components of the B^α parameter are more complicated. In general they can be expressed as

$$B^\alpha = \mathsf{m}^2(2\pi)^{-1}P^\alpha_{abcd}I^{abcd}, \tag{B.4}$$

where the Roman indices run over angular coordinates θ, φ only. The coefficients P^α_{abcd} are given by

$$P^\alpha_{abcd} = \frac{1}{2}\left[P^\alpha_d(3P_{abc} + 2P_{ab}P_c) - P^{\alpha\lambda}(2P_{\lambda ab} + P_{ab\lambda})P_{cd}\right] + (3P^\alpha_a P_{be} - P^\alpha_e P_{ab})C^e_{cd}, \tag{B.5}$$

with

$$P_\alpha \equiv u^\lambda u^\rho g_{\lambda\rho,\alpha}, \quad P_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta, \quad P_{\alpha\beta\lambda} \equiv \Gamma^\lambda_{\alpha\beta}P_{\lambda\gamma}, \tag{B.6}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the background connection-coefficients at the location of the particle. The remaining non-vanishing coefficients

$$C^\theta_{\varphi\varphi} = \frac{1}{2}\sin\theta_0\cos\theta_0, \quad C^\varphi_{\theta\varphi} = C^\varphi_{\varphi\theta} = -\frac{1}{2}\cot\theta_0. \tag{B.7}$$

The quantities I^{abcd} are

$$I^{abcd} = (\sin\theta_0)^{-N} \int_0^{2\pi} G(\chi)^N (\cos\chi)^{4-N} d\chi, \tag{B.8}$$

where

$$G(\chi) = P_{\theta\theta}\cos^2\chi + 2P_{\theta\varphi}\sin\chi\cos\chi/\sin\theta_0 + P_{\varphi\varphi}\sin^2\chi/\sin^2\theta_0, \tag{B.9}$$

and $N = N(abcd)$ is the number of times the coordinate φ occurs in the combination (a, b, c, d):

$$N = \delta^a_\varphi + \delta^b_\varphi + \delta^c_\varphi + \delta^d_\varphi. \tag{B.10}$$

In terms of standard complete Elliptic-integrals we can write

$$I^{abcd} = \frac{(\sin\theta_0)^{-N}}{d}\left[QI_k^{(N)}\hat{K}(w) + I_E^{(N)}\hat{E}(w)\right], \tag{B.11}$$

where we introduced the parameters

$$\begin{aligned}
Q =& \alpha + 2 - (\alpha^2+\beta^2)^{1/2}, && \text{(B.12a)}\\
d =& 3P_{\varphi\varphi}^{5/2}(\sin\theta_0)^{-5}(\alpha^2+\beta^2)(4\alpha+4-\beta^2)^{3/2}(Q/2)^{1/2}, && \text{(B.12b)}\\
\alpha \equiv& \sin^2\theta_0 P_{\theta\theta}/P_{\varphi\varphi} - 1, && \text{(B.12c)}\\
\beta \equiv& 2\sin\theta_0 P_{\theta\varphi}/P_{\varphi\varphi}. && \text{(B.12d)}
\end{aligned}$$

$\hat{K}(w) \equiv \int_0^{\pi/2}(1-w\sin^2 x)^{-1/2}dx$ and $\hat{E}(w) \equiv \int_0^{\pi/2}(1-w\sin^2 x)^{1/2}dx$ are the complete Elliptic-integrals of the first and second kinds respectively, with argument

$$w = \frac{2(\alpha^2+\beta^2)^{1/2}}{\alpha+2+(\alpha^2+\beta^2)^{1/2}}. \tag{B.13}$$

The ten coefficients $I_K^{(N)}$, $I_E^{(N)}$ are given by

$$\begin{aligned}
I_K^{(0)} =& 4[12\alpha^3 + \alpha^2(8-3\beta^2) - 4\alpha\beta^2 + \beta^2(\beta^2-8)], &&\\
I_E^{(0)} =& -16[8\alpha^3 + \alpha^2(4-7\beta^2) + \alpha\beta^2(\beta^2-4) - \beta^2(\beta^2+4)], && \text{(B.14a)}\\
I_K^{(1)} =& 8\beta[9\alpha^2 - 2\alpha(\beta^2-4) + \beta^2], &&\\
I_E^{(1)} =& -4\beta[12\alpha^3 - \alpha^2(\beta^2-52) + \alpha(32-12\beta^2) + \beta^2(3\beta^2+4)], && \text{(B.14b)}\\
I_K^{(2)} =& -4[8\alpha^3 - \alpha^2(\beta^2-8) - 8\alpha\beta^2 + \beta^2(3\beta^2-8)], &&\\
I_E^{(2)} =& 8[4\alpha^4 + \alpha^3(\beta^2+12) - 2\alpha^2(\beta^2-4) + 3\alpha\beta^2(\beta^2-4) + 2\beta^2(3\beta^2-4)], && \text{(B.14c)}\\
I_K^{(3)} =& 8\beta[\alpha^3 - 7\alpha^2 + \alpha(3\beta^2-8) + \beta^2], &&\\
I_E^{(3)} =& -4\beta[8\alpha^4 - 4\alpha^3 + \alpha^2(15\beta^2-44) + 4\alpha(5\beta^2-8) + \beta^2(3\beta^2+4)], && \text{(B.14d)}\\
I_K^{(4)} =& -4[4\alpha^4 - 4\alpha^3 + \alpha^2(7\beta^2-8) + 12\alpha\beta^2 - \beta^2(\beta^2-8)], &&\\
I_E^{(4)} =& 16[4\alpha^5 + 4\alpha^4 + \alpha^2(7\beta^2-4) + \alpha^2(11\beta^2-4) + (2\alpha+1)\beta^2(\beta^2+4)]. && \text{(B.14e)}
\end{aligned}$$

Appendix C

The choice of off-worldline extension

In Chapter 3 we calculated corrections to the LG regularization parameters. These corrections are required to implement the mode-sum formula in the half-string RGs. We now explore how the choice of extension may affect our results

From the coordinate expansion of the gauge vector (Sec. 3.2.7), we now consider the expansion of the change $\delta_\xi \tilde{F}^\pm_\alpha$ in the retarded-force generated by that vector. For concreteness, let us define x' to be the position on the worldline at BL time t, such that $\delta t = 0$.

Under a gauge transformation generated by $\xi^\pm_\mu$, the retarded-force $\tilde{F}^\pm_\mu$ off the worldline transforms according to Eq. (2.58), which we rewrite here in the slightly different form

$$\delta_\xi \tilde{F}^\pm_\alpha = -\mathsf{m}\tilde{P}_\alpha{}^\beta \left[\tilde{u}^\mu \tilde{\nabla}_\mu \left(\tilde{u}^\nu \tilde{\nabla}_\nu \xi^\pm_\beta\right) - \left(\tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^\nu\right) \tilde{\nabla}_\nu \xi^\pm_\beta + \tilde{R}_{\beta\mu}{}^\gamma{}_\nu \tilde{u}^\mu \xi^\pm_\gamma \tilde{u}^\nu\right]. \tag{C.1}$$

Here $\tilde{u}^\alpha$, $\tilde{P}_\alpha{}^\beta$, and $\tilde{\nabla}_\mu$ are smooth-off-the-worldline extensions of the four-velocity u^α, projection operator $P_\alpha{}^\beta$, and covariant derivative ∇_μ, and $\tilde{R}_{\beta\mu}{}^\gamma{}_\nu$ is the Riemann tensor corresponding to $\tilde{\nabla}_\mu$. We wish here to allow any smooth extension, and in general the fields will have expansions of the form

$$\tilde{u}^\alpha = u^{\alpha'} + \tilde{u}^{\alpha'}{}_{,\mu'}\delta x^{\mu'} + O(s^2), \tag{C.2a}$$
$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^{\alpha'}_{\beta'\gamma'} + \tilde{\Gamma}^{\alpha'}_{\beta'\gamma',\mu'}\delta x^{\mu'} + O(s^2), \tag{C.2b}$$
$$\tilde{P}_\alpha{}^\beta = P_{\alpha'}{}^{\beta'} + O(s), \tag{C.2c}$$
$$\tilde{R}_{\alpha\beta}{}^\gamma{}_\delta = R_{\alpha'\beta'}{}^{\gamma'}{}_{\delta'} + O(s). \tag{C.2d}$$

In these expansions, each of the quantities on the left is a function of the field point $x = x' + \delta x'$, and the expansion coefficients on the right are functions of the worldline point x'.

To evaluate Eq. (C.1) for these arbitrary extensions, we first determine the action of $\tilde{\nabla}_\mu$ on a bivector $w_\alpha(x', \delta x')$ that is a function of $x^{\alpha'}$ and $\delta x^{\alpha'}$.

Both $x^{\alpha'}$ and $\delta x^{\alpha'}$ are implicitly functions of x^α: $x^{\alpha'} = x^{\alpha'}(t)$, and $\delta x^{\alpha'} = x^\alpha - x^{\alpha'}(t)$. When we act with a derivative at x^α, we must differentiate these quantities as

$$\partial_\alpha x^{\mu'} = \delta^t_\alpha \frac{u^{\mu'}}{u^{t'}}, \tag{C.3a}$$

$$\partial_\alpha \delta x^{\mu'} = \delta^{\mu'}_\alpha - \delta^t_\alpha \frac{u^{\mu'}}{u^{t'}}. \tag{C.3b}$$

Now define $\partial_{\mu'}$ to be a partial derivative with respect to $x^{\mu'}$, holding $\delta x^{\mu'}$ fixed, and define $\delta_{\mu'}$ to be a partial derivative with respect to $\delta x^{\mu'}$, holding $x^{\mu'}$ fixed. Using Eq. (C.3), we find

$$\partial_\mu w_\alpha(x', \delta x') = \frac{\partial x^{\beta'}}{\partial x^\mu}\partial_{\beta'} w_\alpha + \frac{\partial \delta x^{\beta'}}{\partial x^\mu}\delta_{\beta'} w_\alpha = \delta^t_\mu \frac{u^{\beta'}}{u^{t'}}\partial_{\beta'} w_\alpha + \left(\delta^{\beta'}_\mu - \delta^t_\mu \frac{u^{\beta'}}{u^{t'}}\right)\delta_{\beta'} w_\alpha. \tag{C.4}$$

Combining this with the expansion of the Christoffel symbols, we arrive at

$$\tilde\nabla_\mu w_\alpha(x', \delta x') = \left[\delta^\gamma_\alpha \delta^t_\mu \frac{u^{\beta'}}{u^{t'}}\partial_{\beta'} + \delta^\gamma_\alpha\left(\delta^{\beta'}_\mu - \delta^t_\mu \frac{u^{\beta'}}{u^{t'}}\right)\delta_{\beta'} - \Gamma^{\gamma'}_{\mu'\alpha'} + O(s)\right] w_\gamma. \tag{C.5}$$

Notice that in this expression, $\partial_{\beta'}$ and $\Gamma^{\alpha'}_{\beta'\gamma'}$ do not affect w_α's parity or its scaling with s, while $\delta_{\beta'}$ both reverses the parity and reduces the order by one power of s.

From these results and the expansion of $\tilde u^\mu$ in Eq. (C.2a), we immediately find

$$\tilde u^\mu \tilde\nabla_\mu w_\alpha(x', \delta x') = \left[\delta^\gamma_\alpha u^{\beta'}\partial_{\beta'} + \delta^\gamma_\alpha\left(\tilde u^{\beta'}{}_{,\delta'} - \tilde u^{t'}{}_{,\delta'}\frac{u^{\beta'}}{u^{t'}}\right)\delta x^{\delta'}\delta_{\beta'} - u^{\mu'}\Gamma^{\gamma'}_{\mu'\alpha'} + O(s)\right] w_\gamma. \tag{C.6}$$

Here we see that for any w_α, the operator $\tilde u^\mu \tilde\nabla_\mu$ does not increase the singular behavior of the leading-order term, and it preserves the parity at that order; as we would expect, even though we work off the worldline, there is a sense in which a derivative "along the worldline" changes neither the parity nor the order. Therefore, in particular, $\tilde u^\mu\tilde\nabla_\mu \xi_\beta$ and $\tilde u^\nu \tilde\nabla_\nu\left(\tilde u^\mu \tilde\nabla_\mu \xi_\beta\right)$ have the same parity as ξ_β at that order.

Using Eq. (C.6), we can straightforwardly evaluate the first term in the transformation (C.1). We now move to the second term, $(\tilde u^\mu \tilde\nabla_\mu \tilde u^\nu)\tilde\nabla_\nu \xi_\beta$. An explicit calculation, using the expansions (C.2a) and (C.2b) and the differentiation rules (C.3), yields

$$\tilde u^\nu \tilde\nabla_\nu \tilde u^\mu = \left[u^{\alpha'}\tilde u^{\mu'}{}_{,\beta'\alpha'} + 2u^{\alpha'}\Gamma^{\mu'}_{\alpha'\gamma'}\tilde u^{\gamma'}{}_{,\beta'} + \tilde u^{\alpha'}{}_{,\beta'}\tilde u^{\mu'}{}_{,\alpha'} + u^{\alpha'}\tilde\Gamma^{\mu'}_{\alpha'\gamma',\beta'}u^{\gamma'}\right]\delta x^{\beta'} + O(s^2). \tag{C.7}$$

We note that this expression is the only place in which the choice of extension $\tilde\Gamma^\alpha_{\beta\gamma}$ enters into our calculation. Defining $\tilde a^\mu \equiv \tilde u^\nu \tilde\nabla_\nu \tilde u^\mu$, the above result can be written compactly as $\tilde a^\mu = a^{\mu'}{}_{,\nu'}\delta x^{\nu'} + O(s^2)$. Combining this with Eq. (C.5), we find

$$\left(\tilde u^\mu \tilde\nabla_\mu \tilde u^\nu\right)\tilde\nabla_\nu \xi_\beta = \left(a^{\gamma'}{}_{,\mu'} - a^{t'}{}_{,\mu'}\frac{u^{\gamma'}}{u^{t'}}\right)\delta x^{\mu'}\delta_{\gamma'}\xi_\beta + O(s). \tag{C.8}$$

Notice that this term preserves the parity under and order of ξ_β.

The final expression for the change in retarded force can be found by substituting the expansions (C.6) and (C.8), together with (C.2c), (C.2d), and (C.2a), into Eq. (C.1). We note that, regardless of extension, the resulting expression for $\delta_\xi \tilde F^\pm_\alpha$ receives no contribution from the parallel component $\xi^\pm_{\alpha\|}$ at leading-order. To see this, replace $\tilde P_\alpha{}^\beta$ in Eq. (C.1) with its leading-order term $P_{\alpha'}{}^{\beta'}$, and observe that (i) $u^{\mu'}\nabla_{\mu'}P_{\alpha'}{}^{\beta'}=0$ (so, at leading-order, the projection operator commutes with the derivatives along Γ); (ii) $P_{\alpha'}{}^{\beta'}\xi^{\pm}_{\beta\|}=0$; and (iii) $R_{\beta'\mu'}{}^{\gamma'}{}_{\nu'}\tilde{u}^{\mu'}\xi^{\pm}_{\gamma\|}\tilde{u}^{\nu'}=0$ by virtue of the symmetries of the Riemann tensor.

In the following two subsections, we write $\delta_\xi\tilde{F}^{\pm}_{\alpha}$ explicitly for two choices of extension. However, as we have noted along the course of the calculation, regardless of the choice of extension, the change in the retarded-force has the same parity and scaling with s as does ξ_α itself. Since $\delta_{\xi_\|}\tilde{F}^{\pm}_{\alpha}$ does not contribute to $\delta_\xi\tilde{F}^{\pm}_{\alpha}$, we may focus on $\delta_{\xi_\perp}\tilde{F}^{\pm}_{\alpha}$, which we now conclude is of order s^0 and possesses the same parity as $\xi_{\alpha\perp}$ under $\delta x^{\alpha'}\to-\delta x^{\alpha'}$. Since we have also shown that $\xi_{\alpha\perp}$ inherits the parity of ξ_a, we now have the following: if ξ_a has a definite parity under $x^a\to-x^a$, then $\delta_\xi\tilde{F}^{\pm}_{\alpha}$ has that same parity under $\delta x^{\alpha'}\to-\delta x^{\alpha'}$.

C.1 Example 1: rigid extension

In the simplest extension, which we call "rigid", the coordinate components of both $\tilde{u}^\alpha$ and $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are extended as constant fields, i.e., they are taken to have the same coordinate values at x as at x'. If we adopt this extension, then the partial derivatives of these quantities in the $\delta x'$ direction (i.e., $\tilde{u}^{\alpha'}{}_{,\beta'}\delta x^{\beta'}$ and $\tilde{\Gamma}^{\alpha'}_{\gamma'\delta,\beta'}\delta x^{\beta'}$) all vanish. We immediately find

$$\delta_\xi\tilde{F}^{\pm}_{\alpha}=-\mathsf{m}P_{\alpha'}{}^{\beta'}u^{\mu'}\nabla_{\mu'}(u^{\nu'}\nabla_{\nu'}\xi^{\pm}_{\beta})-\mathsf{m}R_{\alpha'\mu'\gamma'\nu'}u^{\mu'}\xi^{\gamma}_{\pm}u^{\nu'}+O(s), \tag{C.9}$$

where $\nabla_{\mu'}$ is the covariant derivative that acts on the x' dependence of its argument while holding the $\delta x'$ dependence fixed.

The rigid extension might not be the most useful in practice, since it is not an extension for which the LG parameters A_α, B_α, C_α are available [19, 97]. But it affords a simple demonstration of our main conclusions. It is also useful when comparing with the existing literature, because it is implicitly the one used by Shah *et al.* in their calculation of the RG GSF [81, 83]. We use Eq. (C.9) in Chapter 3 to derive corrections to the LG regularization parameters.

C.2 Example 2: rigid extension of u^α, natural extension of metric-related quantities

Another obvious option is to use a rigid extension of the four-velocity while allowing all metric-related quantities to retain their natural values at the field point x; e.g., $\tilde{\Gamma}^\alpha_{\beta\gamma}=\Gamma^\alpha_{\beta\gamma}$. With this choice, we find

$$\begin{aligned}\delta_\xi\tilde{F}^{\pm}_{\alpha}=&-\mathsf{m}P_{\alpha'}{}^{\beta'}u^{\mu'}\nabla_{\mu'}(u^{\nu'}\nabla_{\nu'}\xi^{\pm}_{\beta})-\mathsf{m}R_{\alpha'\mu'\gamma'\nu'}u^{\mu'}\xi^{\gamma}_{\pm}u^{\nu'}\\&+\mathsf{m}P_{\alpha'}{}^{\beta'}\left[\Gamma^{\mu'}_{\gamma'\delta',\nu'}-\frac{u^{\mu'}}{u^{t'}}\Gamma^{t}_{\gamma'\delta',\nu'}\right]u^{\gamma'}u^{\delta'}\delta x^{\nu'}\delta_{\mu'}\xi^{\pm}_{\beta}+O(s).\end{aligned} \tag{C.10}$$

This extension is the one used to derive the LG regularization parameters in [19, 97]. It is also used in our numerical implementation of Chapter 4. If one wants to calculate the GSF in a half-string-LL gauge, Eq. (C.10) would give the correct δD^α parameters for the mode-sum formula of Eq. (3.70).

Appendix D

Corrections to the Lorenz-Gauge regularization parameters

We present the corrections to the standard LG regularization parameter D_α for specific orbital setups. These were derived according to the discussion of Sec. 3.3.2 and published in [1]. All the corrections were calculated using the "rigid" extension defined in Eq. (C.9).

D.1 Arbitrary geodesic orbit in Schwarzschild geometry

Specializing first to the Schwarzschild background, let M denote the BH mass, and $\mathcal{E}$ and $\mathcal{L}$ stand for the particle's specific energy and AM. Without loss of generality we set the particle to move on the equatorial plane. The expressions below are understood to be evaluated at $r = r_0$ and $\dot{r} = \pm[\mathcal{E}^2 - f_0(1 + \mathcal{L}^2/r_0^2)]$. The four velocity is $u^\alpha = (\mathcal{E}/f, \dot{r}, 0, \mathcal{L}/r_0^2)$, and the principal null-vector is $\ell^\alpha = (f^{-1}, 1, 0, 0)$, namely the MP is given in an IRG as defined by Eq. (2.18a).

Following the procedure described in Sec. 3.3.2 we find, using computer algebra,

$$\delta D_t^\pm = \pm \frac{\mathsf{m}^2\mathcal{L}^2 C_t(\mathcal{E}, r, \dot{r})}{r^7(\mathcal{E} - \dot{r})^3}, \qquad \delta D_r^\pm = \pm \frac{\mathsf{m}^2\mathcal{L}^2 C_r(\mathcal{E}, r, \dot{r})}{r^7(\mathcal{E} - \dot{r})^3 f},$$
$$\delta D_\theta^\pm = 0, \qquad \delta D_\varphi^\pm = \pm \frac{2\mathsf{m}^2\mathcal{L} C_\varphi(\mathcal{E}, r, \dot{r})}{r^4(\mathcal{E} - \dot{r})^2}, \tag{D.1}$$

where

$$C_t(\mathcal{E}, r, \dot{r}) = 2rf[r^2(1-\mathcal{E}^2) + Mr(3\mathcal{E}^2 - 4) + 4M^2] + [3r^2(1-\mathcal{E}^2) + 4Mr(\mathcal{E}^2 - 4) + 20M^2]r\mathcal{E}\dot{r}$$
$$+ [r^2(9\mathcal{E}^2 - 1) + 6Mr(1 - 2\mathcal{E}^2) - 8M^2]r\dot{r}^2 + (3r - 4M)(r^2\dot{r}^4 - 3r^2\mathcal{E}\dot{r}^3), \tag{D.2a}$$

$$C_r(\mathcal{E}, r, \dot{r}) = r^3(\mathcal{E}^2 + \mathcal{E}^4 - 2) - 6Mr^2(\mathcal{E}^2 - 2) - [r^2(1 + 3\mathcal{E}^2) - 8Mr + 12M^2]r\mathcal{E}\dot{r}$$
$$+ r(3r^2\mathcal{E}^2 - 2Mr + 4M^2)\dot{r}^2 - r^3\mathcal{E}\dot{r}^3 + 8M^2r(\mathcal{E}^2 - 3) + 16M^3, \tag{D.2b}$$

$$C_\varphi(\mathcal{E}, r, \dot{r}) = r^2(\mathcal{E}^2 - 1) - Mr(3\mathcal{E}^2 - 4) - 4M^2 + [r(\mathcal{E}^2 - 1) + 4M]rE\dot{r}$$
$$- (2r\mathcal{E}^2 + M)r\dot{r}^2 + r^2\mathcal{E}\dot{r}^3. \tag{D.2c}$$

D.2 Circular geodesic orbit in Schwarzschild geometry

From the general expressions we just gave for geodesics of Schwarzschild we consider the special case of circular motion, for which $\dot{r}_0 = 0$, $\mathcal{E} = f_0(1-3M/r_0)^{-1/2}$ and $\mathcal{L} = (Mr_0)^{1/2}(1-3M/r_0)^{-1/2}$. In this case Eq. (D.1) simplifies to

$$\delta D_r^{\pm} = \pm \frac{3\mathsf{m}^2 M^2}{r_0^{5/2}(r_0 - 3M)^{3/2}}, \qquad \text{and} \qquad \delta D_t^{\pm} = \delta D_\theta^{\pm} = \delta D_\varphi^{\pm} = 0. \tag{D.3}$$

D.3 Circular equatorial orbits in Kerr geometry

We now generalize to Kerr but immediately specialize to circular equatorial orbits, for simplicity. We denote by M and aM the mass and spin of the BH, and introduce

$$\Delta_0 \equiv r_0^2 - 2Mr_0 + a^2, \qquad v \equiv \sqrt{M/r_0}. \tag{D.4}$$

The specific energy and angular momentum are given in terms of the BL orbital-radius as

$$\mathcal{E} = \frac{1 - 2v^2 + av^3/M}{\sqrt{1 - 3v^2 + 2av^3/M}}, \quad \mathcal{L} = r_0 v \frac{1 - 2av^3/M + a^2v^4/M^2}{\sqrt{1 - 3v^2 + 2av^3/M}}. \tag{D.5}$$

We find

$$\delta D_\alpha^{\pm} = \pm \mathcal{Q}_\alpha \frac{2\mathsf{m}^2 c}{r_0(b - c^2)} \left(1 - \frac{1}{\sqrt{1 + b - c^2}}\right), \tag{D.6}$$

where

$$b = r_0^{-3}\left[\mathcal{L}^2 r_0 + a^2(2M + r_0)\right], \quad c = \frac{a^2\mathcal{E}\mathcal{L} + \mathcal{E}\mathcal{L}r_0^2 - a\mathcal{L}^2 - a\Delta_0}{r_0\left(a^2\mathcal{E} - a\mathcal{L} + \mathcal{E}r_0^2\right)}, \tag{D.7}$$

and

$$\mathcal{Q}_t = \mathcal{Q}_\theta = \mathcal{Q}_\varphi = 0, \qquad \mathcal{Q}_r = \frac{3M}{r_0^3}\frac{vr_0^2 - a(r_0 - M) - a^2 v}{r_0 - 3M + 2av}. \tag{D.8}$$

We note that δD_α as written is not defined at $a = 0$, where $b - c^2$ vanishes. However, the limit $a \to 0$ of δD_α is well defined, and it agrees (of course) with the Schwarzschild result displayed in Eq. (D.3).

D.4 Parity and $\delta D_\alpha^{\pm}$

From the expressions in the previous section we see that the corrections to δD_α are in general nonzero. The second important fact to notice is that the corresponding $\delta D_\alpha^{\pm}$ are equal in magnitude but opposite in sign. We might think this feature is a consequence of the choice of extension, rather than a general result for δD. We now seek to establish the latter result in full generality, in other words we want to consider that, for any smooth extension, $\delta D_\alpha^+ = -\delta D_\alpha^-$.

This result follows from the relationship between the parities of the '±' solutions. Let us recall that for a no-string gauge, the components of the gauge vector $\xi^0_{\alpha\perp}$ have odd parity. Naturally the half-string gauge vectors from which the no-string solutions were constructed relate to one another according to $\xi^{0+}_{\alpha\perp}(x^a) = -\xi^{0-}_{\alpha\perp}(-x^a)$, except at $p_a x^a = 0$. This relationship is most easily visualized on a small sphere of constant geodesic-distance from the particle, with half the sphere in the regular

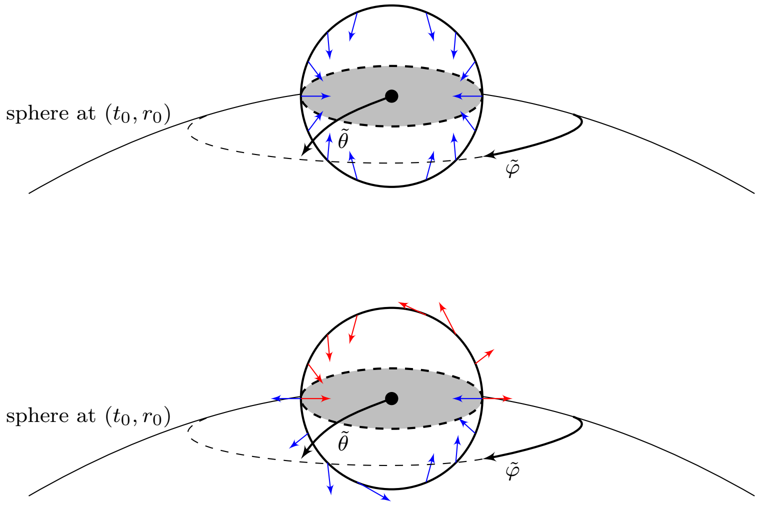

Figure D.1: Parity of vector fields around the particle. The particle, indicated by the black ball, sits at the north pole $\tilde{\theta} = 0$ of the BL coordinate sphere defined by $(t, r) = (t_0, r_0)$. It is surrounded by a much smaller sphere of radius s. The shaded disc is tangent, at $\tilde{\theta} = 0$, to the large sphere. Upper panel: a smooth vector field with odd parity is shown on the surface of the smaller sphere. Its restriction to the shaded disk inherits odd parity under reflection through the centre of the disk. Lower panel: a discontinuous vector field with odd parity is shown. The field exhibits a jump continuity across the disk, although it possesses odd parity on the small sphere. Its limit to the disc, either from above (shown in red) or from below (in blue), does not inherit that parity. Reproduced from [1].

half of the '+' solution and half in the regular half of the '$-$' solution, as shown in Fig. D.1. At antipodal points, the $\pm$ gauge vectors point in opposite directions with equal magnitude.

Now return to Eq. (3.72) and consider the parity. From the gauge vectors, we see that $\delta_{\xi^0+}\tilde{F}_{\alpha}(x', \delta x') = -\delta_{\xi^0-}\tilde{F}_{\alpha}(x', -\delta x')$, which follows from the results of Appendix C. In terms of the variables in Eq. (3.72), the relation becomes $\delta_{\xi^0+}\tilde{F}_{\alpha}(\delta r, \hat{x}, \hat{y}) = -\delta_{\xi^0-}\tilde{F}_{\alpha}(-\delta r, -\hat{x}, -\hat{y})$. We can see that

$$\begin{aligned} \lim_{\delta r \to 0^+} \int d\cos\tilde{\theta} d\tilde{\varphi} \mathsf{P}_{\ell}(\cos\tilde{\theta}) \delta_{\xi^0+}\tilde{F}_{\alpha}(\delta r, \hat{x}, \hat{y}) \\ = - \lim_{\delta r \to 0^-} \int d\cos\tilde{\theta} d\tilde{\varphi} \mathsf{P}_{\ell}(\cos\tilde{\theta}) \delta_{\xi^0-}\tilde{F}_{\alpha}(\delta r, -\hat{x}, -\hat{y}) \\ = - \lim_{\delta r \to 0^-} \int d\cos\tilde{\theta} d\tilde{\varphi} \mathsf{P}_{\ell}(\cos\tilde{\theta}) \delta_{\xi^0-}\tilde{F}_{\alpha}(\delta r, \hat{x}, \hat{y}), \end{aligned} \tag{D.9}$$

where the first equality follows from the odd parity of $\tilde{F}_{\alpha}(\delta r, \hat{x}, \hat{y})$ under $(\delta r, \hat{x}, \hat{y}) \to (-\delta r, -\hat{x}, -\hat{y})$, and the second follows from the invariance of the integral under the change of integration variables $(\hat{x}, \hat{y}) \to (-\hat{x}, -\hat{y})$, (which corresponds to a rotation $\tilde{\varphi} \to \tilde{\varphi} + \pi$).

The result of Eq. (D.9) shows that the corrections to the regularization parameters in the '$\pm$' solutions are precisely opposite, for generic orbits in Kerr and regardless of the choice of extension.

Appendix E

Angular functions

In this Appendix we give a short review of the angular functions which are eigenfunctions of the angular part of Teukolsky Eq. (2.12b). We start with the solutions for the Schwarzschild ($a = 0$) case, or spin-weighted spherical harmonics. For completeness we include a method to calculate the spin-weighted spheroidal harmonics, even though they were not explicitly used in this thesis.

E.1 Spin-weighted spherical harmonics

The spin-weighted spherical harmonics ${}_sY_{\ell m}(\theta, \varphi)$ are functions defined on a sphere just like the ordinary spherical harmonics $Y_{\ell m}(\theta, \varphi)$. For a given value of the 'spin' parameter s they satisfy the orthogonality relation

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta)\ {}_s\bar{Y}_{\ell m}(\theta, \varphi)\ {}_sY_{\ell' m'}(\theta, \varphi) = \delta_{\ell\ell'}\delta_{mm'}. \tag{E.1}$$

They also satisfy the completeness relation

$$\sum_{\ell,m} {}_s\bar{Y}_{\ell m}(\theta, \varphi)\ {}_sY_{\ell m}(\theta', \varphi') = \delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi'), \tag{E.2}$$

for each integer value of s. This means that for each value of s the functions ${}_sY_{\ell m}(\theta, \varphi)$ form a complete set of orthonormal functions on the unit sphere. The complex conjugated ${}_s\bar{Y}_{\ell m}(\theta, \varphi)$ can be calculated using

$${}_s\bar{Y}_{\ell m}(\theta, \varphi) = (-1)^{s+m}\ {}_{-s}Y_{\ell,-m}(\theta, \varphi). \tag{E.3}$$

We define $\eth$ and its complex conjugated operator $\bar{\eth}$ in terms of how they raise (or lower) the spin of the function ${}_sY_{\ell m}(\theta, \varphi)$:

$$\eth_s\ {}_sY_{\ell m}(\theta, \varphi) = (\lambda_s)^{1/2}\ {}_{s+1}Y_{\ell m}(\theta, \varphi), \tag{E.4a}$$

$$\bar{\eth}_s\ {}_sY_{\ell m}(\theta, \varphi) = -\ (\lambda_{-s})^{1/2}\ {}_{s-1}Y_{\ell m}(\theta, \varphi), \tag{E.4b}$$

where $\lambda_s = (\ell - s)(\ell + s + 1)$ as before. This can be defined in terms of angular derivatives acting on a quantity η, of spin s, as

$$\eth_s\eta = -\ (\sin\theta)^s \left[\partial_\theta + i\csc\theta\partial_\varphi\right] (\sin\theta)^{-s}\eta, \tag{E.5a}$$

$$\bar{\eth}_s\eta = -\ (\sin\theta)^{-s} \left[\partial_\theta - i\csc\theta\partial_\varphi\right] (\sin\theta)^{s}\eta. \tag{E.5b}$$

On the equator the spin-weighted spherical harmonics are given by

$$Y_{\ell m}\left(\frac{\pi}{2},0\right) = i^{\ell+m}\frac{\sqrt{(2\ell+1)(\ell-m)!(\ell+m)!}}{\sqrt{4\pi}(\ell-m)!!(\ell+m)!!}e_{\ell m}, \tag{E.6a}$$

$${}_1Y_{\ell m}\left(\frac{\pi}{2},0\right) = i^{\ell+m}\frac{\sqrt{(2\ell+1)(\ell-m)!(\ell+m)!}}{\sqrt{4\pi\lambda_0}}\begin{cases}\frac{m e_{\ell m}}{(\ell-m)!!(\ell+m)!!}, \\ -\frac{i e_{\ell,m+1}}{(\ell-m-1)!!(\ell+m-1)!!}\end{cases}, \tag{E.6b}$$

$${}_2Y_{\ell m}\left(\frac{\pi}{2},0\right) = i^{\ell+m}\frac{\sqrt{(2\ell+1)(\ell-m)!(\ell+m)!}}{\sqrt{4\pi\lambda_0\lambda_1}}\begin{cases}\frac{(2m^2-\lambda_0) e_{\ell m}}{(\ell-m)!!(\ell+m)!!}, \\ -\frac{2im e_{\ell,m+1}}{(\ell-m-1)!!(\ell+m-1)!!}\end{cases}, \tag{E.6c}$$

with

$$e_{\ell m} \equiv \begin{cases}1, & \text{for} \quad \ell+m \quad \text{even} \\ 0, & \text{for} \quad \ell+m \quad \text{odd}\end{cases}. \tag{E.7}$$

In terms of the usual spherical harmonics,

$$(\lambda_0)^{1/2}\,{}_1Y_{\ell m}(\theta,\varphi) = -\left(\partial_\theta - m\csc\theta\right)Y_{\ell m}(\theta,\varphi), \tag{E.8a}$$

$$(\lambda_0)^{1/2}\,{}_{-1}Y_{\ell m}(\theta,\varphi) = \left(\partial_\theta + m\csc\theta\right)Y_{\ell m}(\theta,\varphi), \tag{E.8b}$$

$$(\lambda_0\lambda_1)^{1/2}\,{}_2Y_{\ell m}(\theta,\varphi) = \left(\partial_\theta^2 - \cot\theta\partial_\theta + 2m\cot\theta\csc\theta - 2m\csc\theta\partial_\theta + m^2\csc^2\theta\right)Y_{\ell m}(\theta,\varphi), \tag{E.8c}$$

$$(\lambda_0\lambda_1)^{1/2}\,{}_{-2}Y_{\ell m}(\theta,\varphi) = \left(\partial_\theta^2 - \cot\theta\partial_\theta - 2m\cot\theta\csc\theta + 2m\csc\theta\partial_\theta + m^2\csc^2\theta\right)Y_{\ell m}(\theta,\varphi), \tag{E.8d}$$

where we have used $\partial_\varphi Y_{\ell m}(\theta,\varphi) \equiv imY_{\ell m}(\theta,\varphi)$.

The coefficients that allow the re-expansion of the spin-weighted spherical harmonics in terms of the usual scalar spherical harmonics [appearing in Eq. (4.17) and given in [36]] are

$$\alpha^{\ell m}_{(+2)} = -C_{\ell+1,m}C_{\ell+2,m}, \quad \alpha^{\ell m}_{(0)} = 1 - C^2_{\ell m} - C^2_{\ell+1,m}, \quad \alpha^{\ell m}_{(-2)} = -C_{\ell m}C_{\ell-1,m}, \tag{E.9a}$$

$$\beta^{\ell m}_{(+2)} = \ell C_{\ell+1,m}C_{\ell+2,m}, \quad \beta_{(0)} = \ell C^2_{\ell+1,m} - (\ell+1)C^2_{\ell m}, \quad \beta_{(-2)} = -(\ell+1)C_{\ell m}C_{\ell-1,m}, \tag{E.9b}$$

$$\gamma^{\ell m}_{(+2)} = \ell^2 C_{\ell+1,m}C_{\ell+2,m}, \quad \gamma^{\ell m}_{(0)} = m^2 - \ell(\ell+1) + \ell^2 C^2_{\ell+1,m} + (\ell+1)^2 C^2_{\ell m}, \\ \gamma^{\ell m}_{(-2)} = (\ell+1)^2 C_{\ell m}C_{\ell-1,m}, \tag{E.9c}$$

$$\delta^{\ell m}_{(+1)} = \ell C_{\ell+1,m}, \qquad \delta^{\ell m}_{(-1)} = -(\ell+1)C_{\ell m}, \tag{E.9d}$$

$$\epsilon^{\ell m}_{(+1)} = (1-\ell)C_{\ell+1,m}, \qquad \epsilon^{\ell m}_{(-1)} = (\ell+2)C_{\ell m}, \tag{E.9e}$$

$$\begin{aligned}\zeta^{\ell m}_{(+3)} &= -\,\ell C_{\ell+1,m}C_{\ell+2,m}C_{\ell+3,m}, \\ \zeta^{\ell m}_{(+1)} &= C_{\ell+1,m}\left[\ell(1 - C^2_{\ell+1,m} - C^2_{\ell+2,m}) + (\ell+1)C^2_{\ell m}\right], \\ \zeta^{\ell m}_{(-1)} &= -\,C_{\ell m}\left[(\ell+1)(1 - C^2_{\ell-1,m} - C^2_{\ell m}) + \ell C^2_{\ell+1,m}\right], \\ \zeta^{\ell m}_{(-3)} &= (\ell+1)C_{\ell m}C_{\ell-1,m}C_{\ell-2m},\end{aligned} \tag{E.9f}$$

$$\begin{aligned}
\xi^{\ell m}_{(+3)} =& \ell^2 C_{\ell+1,m} C_{\ell+2,m} C_{\ell+3,m}, \\
\xi^{\ell m}_{(+1)} =& C_{\ell+1,m} \left[m^2 - \ell(\ell+1) + \ell^2 C^2_{\ell+1,m} + (\ell+1)^2 C^2_{\ell m} + \ell^2 C^2_{\ell+2,m} \right], \\
\xi^{\ell m}_{(-1)} =& C_{\ell m} \left[m^2 - \ell(\ell+1) + \ell^2 C^2_{\ell+1,m} + (\ell+1)^2 C^2_{\ell m} + (\ell+1)^2_{\ell-1,m} \right], \\
\xi^{\ell m}_{(-3)} =& (\ell+1)^2 C_{\ell m} C_{\ell-1,m} C_{\ell-2,m},
\end{aligned} \tag{E.9g}$$

with

$$C_{\ell m} = \left[\frac{\ell^2 - m^2}{4\ell^2 - 1} \right]^{1/2}. \tag{E.10}$$

E.2 Spin-weighted spheroidal harmonics

The eigenfunctions of the angular part of Teukolsky's equation (2.12b) are the spin-weighted spheroidal harmonics, which reduce to the spin-weighted spherical harmonics of the previous section in the case of Schwarzschild, $a = 0$. The spin-weighted spheroidal harmonics can be written as a spectral sum in terms of the spin-weighted spherical harmonics as

$${}_sS_{\ell m\omega}(\theta) = \sum_{j=\ell_{\min}}^{\infty} b_{j\omega}\, {}_sY_{jm}(\theta), \tag{E.11}$$

where $\ell_{\min} = \max(|s|, |m|)$, and it is important to note that we have excluded the factor $e^{im\varphi}$ in the definition of the spin-weighted spherical harmonics.

Let us now summarize the method to obtain the coefficients $b_{j\omega}$. We substitute the expansion of Eq. (E.11) in Eq. (2.12b). We identify the terms of Eq. (2.12b) that are independent of a, and combine them to write an equation for the spin-weighted spherical harmonics, with eigenvalue $\ell(\ell+1)$. Hence we obtain

$$\sum_{j=\ell_{\min}}^{\infty} b_{j\omega} [(a\omega)^2 \cos^2\theta - 2a\omega s \cos\theta - j(j+1)] \, |sjm\rangle = -\Lambda_{\ell m} \sum_{j=\ell_{\min}}^{\infty} b_{j\omega} \, |sjm\rangle, \tag{E.12}$$

where we have used ${}_sY_{jm}(\theta) = |sjm\rangle$ according to Dirac's notation and $\Lambda_{\ell m}$ is the eigenvalue of the spin-weighted spheroidal harmonic. The inner product is defined as

$$\langle sjm|f(\theta)|sjm\rangle \equiv \int_0^{\pi} {}_s\bar{Y}_{jm}(\theta) f(\theta) {}_sY_{jm}(\theta) \sin\theta d\theta. \tag{E.13}$$

We now multiply Eq. (E.12) by $\langle s\ell m|$, which corresponds to the complex conjugate of ${}_sY_{\ell m}(\theta)$, and evaluate the inner products [150]

$$\langle s\ell m| \cos^2\theta \, |sjm\rangle = \frac{1}{3}\delta_{j\ell} + \frac{2}{3}\sqrt{\frac{2\ell+1}{2j+1}} \, \langle j, 2, m, 0|\ell m\rangle \, \langle j, 2, -s, 0|\ell, -s\rangle \equiv c^m_{j\ell 2}, \tag{E.14a}$$

$$\langle s\ell m| \cos\theta \, |sjm\rangle = \sqrt{\frac{2\ell+1}{2j+1}} \, \langle j, 1, m, 0|\ell m\rangle \, \langle j, 1, -s, 0|\ell, -s\rangle \equiv c^m_{j\ell 1}, \tag{E.14b}$$

$$\langle s\ell m|sjm\rangle = \delta_{j\ell}. \tag{E.14c}$$

Since $\langle j_1, m_1; j_2, m_2 | j, m\rangle$ are the Clebsch-Gordan coefficients we can simplify the sum in Eq. (E.12) in virtue of the fact that $c^m_{j,\ell,2} \neq 0$ only for $j \in [\ell-2, \ell-1, \ell, \ell+1, \ell+2]$, and $c^m_{j,\ell,1} \neq 0$ only for $j \in [\ell-1, \ell, \ell+1]$. This leads to a finite sum:

$$b_{\ell-2,\omega}(a\omega)^2 c^m_{\ell-2,\ell,2} + b_{\ell-1,\omega}\left[(a\omega)^2 c^m_{\ell-1,\ell,2} - 2a\omega s c^m_{\ell-1,\ell,1}\right] + b_{\ell,\omega}\left[(a\omega)^2 c^m_{\ell,\ell,2} - 2a\omega s c^m_{\ell,\ell,1} - \lambda_0\right] \\ + b_{\ell+1,\omega}\left[(a\omega)^2 c^m_{\ell+1,\ell,2} - 2a\omega s c^m_{\ell+1,\ell,1}\right] + b_{\ell+2,\omega}(a\omega)^2 c^m_{\ell+2,\ell,2} = -\Lambda_{\ell m} b_{\ell\omega}. \quad \text{(E.15)}$$

This last equation can be written as a matrix equation for the $b_{\ell\omega}$ coefficients, with $\Lambda_{\ell m}$ eigenvalue. The matrix is band-diagonal, and it may be solved numerically.

Appendix F

Completion for eccentric orbits

In this Appendix we include explicit analytical-expressions that complement the calculations appearing in Chapter 5 for Kerr. While all of the content of this Appendix is essential in determining the amplitudes of the completion piece, the expression are too long. We avoided them in Chapter 5 in an attempt to keep the discussion of the method clear. We use $u^\alpha \equiv \{\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}\}$ to denote the components of the four velocity, which is different from the notation used in rest of this work.

In Sec. F.1 we describe the method of variation of parameters used to obtain inhomogeneous solutions of the radial part of Teukolsky Eq. (2.12a), for eccentric orbits. For the axially symmetric part (with $m = 0$) we give analytical expressions for the coefficients of the inhomogeneous solutions. In Sec. F.2 we give the ℓ-independent coefficients appearing in the mode-sums of Eq. (5.50) which give the jump of the auxiliary invariants. In Sec. F.3 we give the summation formulae to go from Eq. (5.50) to Eq. (5.53), in which the sums are performed as distributions. In Sec. F.4 we give the explicit expressions for the side-limit of the reconstructed part of the invariants, for the case of circular orbits of Kerr. The method to evaluate the integrals of Eq. (5.75), or the jump in the amplitudes of the completion across the libration region, appears in Sec. F.5.

F.1 Variation of Parameters

Let $R_{4\pm}(r)$ be the ℓ-modes of the homogeneous solutions to the radial part of the spin $s = -2$ Teukolsky equation. The regular solutions R_{4+} and R_{4-} satisfy retarded boundary-conditions at infinity and at the EH respectively, and they are given explicitly in Eq. (5.29). We use these solutions to construct inhomogeneous solutions (since they form a basis of linearly independent solutions) using the standard variation of parameters method. Explicitly, the inhomogeneous solutions (with the ℓ indices omitted for simplicity) are given by

$$\varrho^{-4}\psi_4(r) = R_{4+}(r)\int_{r_{\min}}^{r} \frac{Z_-(r')}{\mathcal{W}}\delta(r' - r_0(\tau))dr' + R_{4-}(r)\int_{r}^{r_{\max}} \frac{Z_+(r')}{\mathcal{W}}\delta(r' - r_0(\tau))dr', \quad \text{(F.1)}$$

where $\mathcal{W}$ is the Wronskian of the homogeneous solutions as before, and the functions $Z_\pm(r(\tau))$ are

$$
\begin{aligned}
Z_\pm(r(\tau)) =& \frac{\pi}{\Delta r^3 \dot{t}} \left(R_{4\mp}(r) \left({}_{-2}\bar{Y}_\ell(\theta_0) r \left(\Delta \left(a^4\lambda_0\dot{\varphi}^2 - 2a^3\lambda_0\dot{t}\dot{\varphi} + a^2\left(\lambda_0\left(2r^2\dot{\varphi}^2+\dot{t}^2\right) - 2r\dot{\varphi}^2(2M+r)\right)\right.\right.\right.\\
& -2ar\dot{t}\dot{\varphi}(\lambda_0 r - 4M) + r\left(\lambda_1 r^3\dot{\varphi}^2 - 4M\dot{t}^2 + 2r\dot{t}^2\right)\Big) - 4\Delta\dot{r}r^2(a\dot{\varphi}-\dot{t}) + 2\dot{r}^2r^4\Big)\\
& - \left(\dot{r}r^2 - \Delta(a\dot{\varphi}-\dot{t})\right)\left(r\,{}_{-2}Y_\ell''(\theta_0)\left(\dot{r}r^2 - \Delta(a\dot{\varphi}-\dot{t})\right) - 2i\,{}_{-2}\bar{Y}_\ell'(\theta_0)\left(a^4\dot{\varphi} - a^3\dot{t}\right.\right.\\
& \left.\left.\left. +a^2r\dot{\varphi}(3r-2M) + ar(\dot{r}r + 2M\dot{t} - r\dot{t}) + 2r^4 f\dot{\varphi}\right)\right)\right) - 2R_{4\mp}'(r)\left(a^2\dot{\varphi} - a\dot{t} + r^2\dot{\varphi}\right)\times\\
& \left({}_{-2}\bar{Y}_\ell(\theta_0)\left(a^4 + a^2r(r-3M) - Mr^3 f\right)\left(a^2\dot{\varphi} - a\dot{t} + r^2\dot{\varphi}\right)\right.\\
& \left.\left. +i\Delta r\,{}_{-2}\bar{Y}_\ell'(\theta_0)\left(\dot{r}r^2 - \Delta(a\dot{\varphi}-\dot{t})\right)\right)\right), \qquad \text{(F.2)}
\end{aligned}
$$

where $\lambda_s = (\ell - s)(\ell + s + 1)$, $\Delta = r^2 + a^2 - 2Mr$ and $f = 1 - 2M/r$, and M and a are the mass and spin parameters of the Kerr BH respectively, as before. Eq. (F.2) should be evaluated at $r = r_0(\tau)$ and $\theta_0 = \pi/2$. Since we are dealing with only the axially symmetric part of the solution we used ${}_{-2}Y_\ell(\theta) \equiv {}_{-2}Y_{\ell 0}(\theta, \varphi)$. The circular-orbit expressions are recovered by setting $\dot{r} = 0$, and considering r_0 as a constant along the orbit:

$$
Z^\circ_\pm = \int_{r_<}^{r_>} \frac{Z_\pm(r')}{\mathcal{W}(r')}\delta(r' - r_0)dr' = \frac{Z_\pm(r_0)}{\mathcal{W}(r_0)}. \qquad \text{(F.3)}
$$

where $r_<$ is $\min(r_0, r')$, and $r_>$ is $\max(r_0, r')$.

F.2 Coefficients appearing in the mode-sums in the reconstructed auxiliary invariants

All the expressions in this section are taken at $r = r_0$ and $\theta \neq \pi/2$. The ℓ independent coefficients appearing in Eq. (5.50) and (5.51) are

$$
\begin{aligned}
c_0 =& \frac{4\pi\Sigma\dot{t}}{3Mr^3\Delta^4}\left((M+r)\dot{\varphi}^2a^{10} - 2(M+r)\dot{t}\dot{\varphi}a^9 + \left(\dot{\varphi}^2r^3 - 6M\dot{\varphi}^2r^2 + \dot{t}^2r - 5M^2\dot{\varphi}^2r\right.\right.\\
& \left. +M\dot{t}^2\right)a^8 + 2Mr(5M+7r)\dot{t}\dot{\varphi}a^7 - r\left(-8r\dot{\varphi}^2M^3 + 5\left(\dot{t}^2 - 4r^2\dot{\varphi}^2\right)M^2\right.\\
& \left. +4\left(5\dot{\varphi}^2r^3 + 2\dot{t}^2r\right)M + r^2\dot{t}^2\right)a^6 + 2r^2\left(\dot{r}r\left(M^2+3r^2\right) + \left(-8M^3 - 24rM^2 + 13r^2M\right.\right.\\
& \left.\left. +3r^3\right)\dot{t}\right)\dot{\varphi}a^5 + r^2\left(\dot{\varphi}^2r^5 - 5\dot{t}^2r^3 - 4M^4\dot{\varphi}^2r - 2\dot{r}\left(M^2+3r^2\right)\dot{t}r + 4M^3\left(2\dot{t}^2 - 9r^2\dot{\varphi}^2\right)\right.\\
& \left. +M^2\left(75\dot{\varphi}^2r^3 + 28\dot{t}^2r\right) - M\left(36\dot{\varphi}^2r^4 + 5\dot{t}^2r^2\right)\right)a^4 + 2r^3\left(\dot{r}r\left(3r^3 - 2M^3 - 5rM^2 - 4r^2M\right)\right.\\
& \left. + \left(4M^4 + 40rM^3 - 55r^2M^2 + 13r^3M + 2r^4\right)\dot{t}\right)\dot{\varphi}a^3 + r^3\left(-\dot{r}^2\left(M^2+3r^2\right)r^2 + 2\dot{r}\left(2M^3\right.\right.\\
& \left. +5rM^2 + 4r^2M - 3r^3\right)\dot{t}r - rf\left(3\dot{t}^2r^3 - \dot{\varphi}^2r^5 - 2M^3\left(\dot{t}^2 - 6r^2\dot{\varphi}^2\right) - M^2\left(32\dot{\varphi}^2r^3 + 23\dot{t}^2r\right)\right.\\
& \left.\left.\left. +M\left(27\dot{\varphi}^2r^4 + 4\dot{t}^2r^2\right)\right)\right)a^2 + \left((M+7r)\dot{\varphi}^2a^8 - 2(M+7r)\dot{t}\dot{\varphi}a^7 + \left(23\dot{\varphi}^2r^3 + 7\dot{t}^2r\right.\right.\\
& \left. -5M^2\dot{\varphi}^2r + M\left(\dot{t}^2 - 30r^2\dot{\varphi}^2\right)\right)a^6 + 2r\left(\left(5M^2 + 31rM - 20r^2\right)\dot{t} - 4\dot{r}r^2\right)\dot{\varphi}a^5\\
& +r\left(28\dot{\varphi}^2r^4 + 17\dot{t}^2r^2 + 8\dot{r}\dot{t}r^2 + 8M^3\dot{\varphi}^2r + M^2\left(38r^2\dot{\varphi}^2 - 5\dot{t}^2\right) - 2M\left(35\dot{\varphi}^2r^3\right.\right.\\
& \left.\left. +16\dot{t}^2r\right)\right)a^4 + 2r^2\left(\dot{r}r\left(M^2 - 7r^2f\right) - \left(8M^3 + 42rM^2 - 61r^2M + 19r^3\right)\dot{t}\right)\dot{\varphi}a^3\\
& +r^2\left(4\dot{r}^2r^3 - 2\dot{r}\left(M^2 - 7r^2f\right)\dot{t}r + rf\left(15\dot{\varphi}^2r^4 + 13\dot{t}^2r^2 + 2M^3\dot{\varphi}^2r + M^2\left(7r^2\dot{\varphi}^2\right.\right.\right.\\
& \left.\left.\left. -4\dot{t}^2\right) - 5M\left(4\dot{\varphi}^2r^3 + 5\dot{t}^2r\right)\right)\right)a^2 - 2fr^4\left(\left(2M^3 + 9rM^2 - 17r^2M + 6r^3\right)\dot{t}\right.\\
& \left. -\dot{r}r\left(M^2 + 6rM - 3r^2\right)\right)\dot{\varphi}a + r^3\left(-\dot{r}^2\left(M^2 + 6rM - 3r^2\right)r^2 + 2\dot{r}\left(2M^3 + 11rM^2\right.\right.\\
& \left.\left.\left. -12r^2M + 3r^3\right)\dot{t}r - r^2f^2\left(-3\left(\dot{t}^2 + r^2\dot{\varphi}^2\right)r^2 + M^2\dot{t}^2 + M\left(6r\dot{t}^2 - r^3\dot{\varphi}^2\right)\right)\right)\right)\cos(2\theta)a^2\\
& +4Mfr^6\left(\dot{r}r(r-3M) + \left(6M^2 - 11rM + 4r^2\right)\dot{t}\right)\dot{\varphi}a + 2Mr^5\left(\dot{r}^2(3M-r)r^2\right.
\end{aligned}
$$

$$-2\dot{r}\left(6M^2-5rM+r^2\right)\dot{t}r+r^2f^2\left(3M\dot{t}^2-r\left(\dot{t}^2+3r^2\dot{\varphi}^2\right)\right)\Big)\Big), \tag{F.4a}$$

$$c_1=\frac{4\pi\Sigma^2}{3Mr^2\Delta^3\dot{t}}\Big(\dot{r}^2r^4-2\dot{r}(a\dot{\varphi}-\dot{t})\Delta r^2+\Big(2\dot{\varphi}^2a^4-4\dot{t}\dot{\varphi}a^3+\left(2\dot{t}^2+r(3r-2M)\dot{\varphi}^2\right)a^2$$
$$+4(M-r)r\dot{t}\dot{\varphi}a+r\left(\dot{\varphi}^2r^3+\dot{t}^2rf\right)\Big)\Delta\Big), \tag{F.4b}$$

$$c_2=-\frac{16a^2\pi\Sigma\sin^2\theta}{Mr^3\Delta^4\dot{t}}\Big((M+r)\dot{\varphi}^2a^8-2(M+r)\dot{t}\dot{\varphi}a^7+\left(2\dot{\varphi}^2r^3+\dot{t}^2r-5M^2\dot{\varphi}^2r+M\dot{t}^2\right)a^6$$
$$-2r\left(\dot{r}r^2-\left(5M^2+rM-2r^2\right)\dot{t}\right)\dot{\varphi}a^5+r\Big(\dot{\varphi}^2r^4+2\dot{t}^2r^2+2\dot{r}\dot{t}r^2+8M^3\dot{\varphi}^2r$$
$$-5M^2\left(\dot{t}^2+2r^2\dot{\varphi}^2\right)+2M\left(r^3\dot{\varphi}^2-r\dot{t}^2\right)\Big)a^4+2r^2\Big(\dot{r}r\left(M^2-r^2f\right)+\Big(6rM^2-8M^3$$
$$+r^2M-r^3\Big)\dot{t}\Big)\dot{\varphi}a^3+r^2\Big(\dot{r}^2r^3+2\dot{r}\left(r^2f-M^2\right)\dot{t}r+rf\Big(2r\dot{\varphi}^2M^3-\left(4\dot{t}^2+5r^2\dot{\varphi}^2\right)M^2$$
$$+\left(4r^3\dot{\varphi}^2-r\dot{t}^2\right)M+r^2\dot{t}^2\Big)\Big)a^2-2Mfr^4\left(\left(2M^2-3rM+r^2\right)\dot{t}-\dot{r}Mr\right)\dot{\varphi}a$$
$$-Mr^3\left(\dot{r}^2Mr^2+2\dot{r}Mf\dot{t}r^2+r^2f^2\left(M\dot{t}^2-r^3\dot{\varphi}^2\right)\right)\Big), \tag{F.4c}$$

$$c_3=\frac{8a\pi\Sigma\cos\theta}{3Mr^2\Delta^3\dot{t}}\Big(\dot{\varphi}^2a^7-2\dot{t}\dot{\varphi}a^6+\left(\dot{t}^2+r(3r-2M)\dot{\varphi}^2\right)a^5+r(\dot{r}r-3\dot{t}r+4M\dot{t})\dot{\varphi}a^4$$
$$-r\left(-3\dot{\varphi}^2r^3+\dot{r}\dot{t}r+2M\left(\dot{t}^2+4r^2\dot{\varphi}^2\right)\right)a^3-2r^3(2\dot{r}M+(r-3M)\dot{t})\dot{\varphi}a^2$$
$$+r^3\left(-2r\dot{r}^2+(4M-3r)\dot{t}\dot{r}-rf\left(\dot{t}^2+(4M-r)r\dot{\varphi}^2\right)\right)a+(4M-r)r^4(\dot{r}r+rf\dot{t})\dot{\varphi}\Big), \tag{F.4d}$$

$$c_4=-\frac{16a\pi\Sigma\cos\theta}{3Mr^3\Delta^4\dot{t}}\Big((M+r)\dot{\varphi}^2a^9-2(M+r)\dot{t}\dot{\varphi}a^8+\Big(3\dot{\varphi}^2r^3-3M\dot{\varphi}^2r^2+\dot{t}^2r-4M^2\dot{\varphi}^2r$$
$$+M\dot{t}^2\Big)a^7-2r\left(\dot{r}r^2+\left(3r^2-4M^2-4rM\right)\dot{t}\right)\dot{\varphi}a^6+r\Big(5\dot{\varphi}^2r^4-13M\dot{\varphi}^2r^3+3\dot{t}^2r^2$$
$$+4M^2\dot{\varphi}^2r^2+2\dot{r}\dot{t}r^2-5M\dot{t}^2r+4M^3\dot{\varphi}^2r-4M^2\dot{t}^2\Big)a^5+2fr^3\Big(\left(2M^2+5rM-4r^2\right)\dot{t}$$
$$-\dot{r}r^2\Big)\dot{\varphi}a^4+r^2\Big(\dot{r}^2(r-M)r^2+2\dot{r}(r-3M)\dot{t}r^2-rf\Big(-5\dot{\varphi}^2r^4-3\dot{t}^2r^2+2M^2\left(\dot{t}^2-r^2\dot{\varphi}^2\right)$$
$$+M\left(11\dot{\varphi}^2r^3+7\dot{t}^2r\right)\Big)\Big)a^3-2fr^5\left(\dot{r}r^2+\left(4M^2-8rM+3r^2\right)\dot{t}\right)\dot{\varphi}a^2+r^4\Big(\dot{r}^2(M-r)r^2$$
$$-2\dot{r}Mf\dot{t}r^2-r^2f^2\left(-2\dot{\varphi}^2r^3+4M\dot{\varphi}^2r^2-\dot{t}^2r+3M\dot{t}^2\right)\Big)a-2r^8f^2(\dot{r}r+rf\dot{t})\dot{\varphi}\Big), \tag{F.4e}$$

$$c_5=-\frac{8a^2\pi\Sigma\sin(2\theta)}{3Mr^3\Delta^4\dot{t}}\Big(2(M+4r)\dot{\varphi}^2a^8-4(M+4r)\dot{t}\dot{\varphi}a^7+\Big(25\dot{\varphi}^2r^3+8\dot{t}^2r-10M^2\dot{\varphi}^2r$$
$$+2M\left(\dot{t}^2-15r^2\dot{\varphi}^2\right)\Big)a^6+2r\left(2\left(5M^2+16rM-11r^2\right)\dot{t}-5\dot{r}r^2\right)\dot{\varphi}a^5+r\Big(29\dot{\varphi}^2r^4$$
$$+19\dot{t}^2r^2+10\dot{r}\dot{t}r^2+16M^3\dot{\varphi}^2r-34M\left(\dot{t}^2+2r^2\dot{\varphi}^2\right)r+M^2\left(28r^2\dot{\varphi}^2-10\dot{t}^2\right)\Big)a^4$$
$$+4r^2\left(\dot{r}r\left(M^2-4r^2f\right)-\left(8M^3+18rM^2-31r^2M+10r^3\right)\dot{t}\right)\dot{\varphi}a^3+r^2\Big(5\dot{r}^2r^3$$
$$-4\dot{r}\left(M^2-4r^2f\right)\dot{t}r+rf\Big(15\dot{\varphi}^2r^4+14\dot{t}^2r^2+4M^3\dot{\varphi}^2r+M^2\left(2r^2\dot{\varphi}^2-8\dot{t}^2\right)$$
$$-2M\left(8\dot{\varphi}^2r^3+13\dot{t}^2r\right)\Big)\Big)a^2+2fr^4\Big(\dot{r}r\left(2M^2-3r^2f\right)-2\Big(2M^3+3rM^2$$
$$-8r^2M+3r^3\Big)\dot{t}\Big)\dot{\varphi}a+r^3\Big(\dot{r}^2\left(-2M^2+3r^2f\right)r^2+2\dot{r}\left(4M^3+10rM^2-12r^2M+3r^3\right)\dot{t}r$$
$$-r^2f^2\left(-3\left(\dot{t}^2+r^2\dot{\varphi}^2\right)r^2+2M^2\dot{t}^2+M\left(6r\dot{t}^2-2r^3\dot{\varphi}^2\right)\right)\Big)\Big), \tag{F.4f}$$

$$c_6=-\frac{8a^2\pi\Sigma\sin(2\theta)}{3Mr^2\Delta^3\dot{t}}\Big(\dot{r}^2r^4-2\dot{r}(a\dot{\varphi}-\dot{t})\Delta r^2+\Big(2\dot{\varphi}^2a^4-4\dot{t}\dot{\varphi}a^3+\left(2\dot{t}^2+r(3r-2M)\dot{\varphi}^2\right)a^2$$
$$+4(M-r)r\dot{t}\dot{\varphi}a+r\left(\dot{\varphi}^2r^3+\dot{t}^2r-2M\dot{t}^2\right)\Big)\Delta\Big), \tag{F.4g}$$

$$c_7=-\frac{32a\pi\Sigma\sin\theta}{3Mr\Delta^3\dot{t}}\Big(M\dot{\varphi}^2a^5+(\dot{r}r+rf\dot{t})\dot{\varphi}a^4-\left(-M\dot{t}^2+r\dot{t}^2+\dot{r}r\dot{t}+2M^2r\dot{\varphi}^2\right)a^3$$
$$+r\left(\dot{r}r(r-3M)+\left(4M^2-3rM+r^2\right)\dot{t}\right)\dot{\varphi}a^2-r\Big(\dot{r}^2r^2+\dot{r}(2r-3M)\dot{t}r$$
$$+rf\left(-M\dot{t}^2+r\dot{t}^2+Mr^2\dot{\varphi}^2\right)\Big)a+Mr^3(\dot{r}r+rf\dot{t})\dot{\varphi}\Big), \tag{F.4h}$$

$$c_8=-\frac{32\pi a\Sigma\sin\theta}{3\Delta^3Mr\dot{t}}\Big(a^5M\dot{\varphi}^2+a^4\dot{\varphi}(\dot{r}r+\dot{t}rf)-a^3\left(\dot{r}r\dot{t}+2M^2r\dot{\varphi}^2-M\dot{t}^2+r\dot{t}^2\right)$$
$$+a^2r\dot{\varphi}\left(\dot{r}r(r-3M)+\dot{t}\left(4M^2-3Mr+r^2\right)\right)-ar\Big(\dot{r}^2r^2+\dot{r}r\dot{t}(2r-3M)$$
$$+rf\left(Mr^2\dot{\varphi}^2-M\dot{t}^2+r\dot{t}^2\right)\Big)+Mr^3\dot{\varphi}(\dot{r}r+\dot{t}rf)\Big), \tag{F.4i}$$

for $[\mathcal{I}_1^{(\mathrm{rec})}]$ and

$$\begin{aligned}
d_0 =& -\frac{4\pi\Sigma\csc^2\theta}{3Mr^2\Delta^2\dot{t}}\left(\dot{r}^2(r-M)r^5+2\dot{r}(M-r)(a\dot{\varphi}-\dot{t})\Delta r^3+\left(\dot{\varphi}^2a^4-2\dot{t}\dot{\varphi}a^3\right.\right.\\
&+\left.\left(\dot{t}^2+r(3r-M)\dot{\varphi}^2\right)a^2+2(M-2r)r\dot{t}\dot{\varphi}a+r\left(\dot{\varphi}^2r^3+\dot{t}^2r-M\dot{t}^2\right)\right)\Delta^2\\
&-\left(5\dot{\varphi}^2a^8-10\dot{t}\dot{\varphi}a^7+\left(5\dot{t}^2+r(13r-25M)\dot{\varphi}^2\right)a^6-2r(4\dot{r}r+12\dot{t}r-25M\dot{t})\dot{\varphi}a^5\right.\\
&+r\left(12\dot{\varphi}^2r^3-42M\dot{\varphi}^2r^2+11\dot{t}^2r+40M^2\dot{\varphi}^2r+8\dot{r}\dot{t}r-25M\dot{t}^2\right)a^4\\
&+2r^2\left(\dot{r}(13M-5r)r+\left(38rM-40M^2-9r^2\right)\dot{t}\right)\dot{\varphi}a^3+r^2\left(4\dot{r}^2r^2+2\dot{r}(5r-13M)\dot{t}r\right.\\
&+\left.rf\left(5\dot{\varphi}^2r^3-11M\dot{\varphi}^2r^2+7\dot{t}^2r+10M^2\dot{\varphi}^2r-20M\dot{t}^2\right)\right)a^2-2fr^4\left(\dot{r}r(r-5M)\right.\\
&+\left(10M^2-9rM+2r^2\right)\dot{t}\right)\dot{\varphi}a+r^3\left(\dot{r}^2(r-5M)r^2+2\dot{r}\left(10M^2-7rM+r^2\right)\dot{t}r\right.\\
&\left.\left.\left.-r^2f^2\left(5M\dot{t}^2-r\left(\dot{t}^2+r^2\dot{\varphi}^2\right)\right)\right)\right)\cos(2\theta)\right), \qquad \text{(F.5a)}\\
d_1 =& -\frac{16\pi\Sigma}{Mr^3\Delta^2\dot{t}}\left((M+r)\dot{\varphi}^2a^8-2(M+r)\dot{t}\dot{\varphi}a^7+\left(2\dot{\varphi}^2r^3+\dot{t}^2r-5M^2\dot{\varphi}^2r+M\dot{t}^2\right)a^6\right.\\
&-2r\left(\dot{r}r^2-\left(5M^2+rM-2r^2\right)\dot{t}\right)\dot{\varphi}a^5+r\left(\dot{\varphi}^2r^4+2\dot{t}^2r^2+2\dot{r}\dot{t}r^2+8M^3\dot{\varphi}^2r\right.\\
&\left.-5M^2\left(\dot{t}^2+2r^2\dot{\varphi}^2\right)+2M\left(r^3\dot{\varphi}^2-r\dot{t}^2\right)\right)a^4+2r^2\left(\dot{r}r\left(M^2-r^2f\right)+\left(6rM^2\right.\right.\\
&\left.\left.-8M^3+r^2M-r^3\right)\dot{t}\right)\dot{\varphi}a^3+r^2\left(\dot{r}^2r^3+2\dot{r}\left(-M^2+r^2f\right)\dot{t}r+rf\left(2r\dot{\varphi}^2M^3\right.\right.\\
&\left.\left.-\left(4\dot{t}^2+5r^2\dot{\varphi}^2\right)M^2+\left(4r^3\dot{\varphi}^2-r\dot{t}^2\right)M+r^2\dot{t}^2\right)\right)a^2-2Mfr^4\left(\left(2M^2-3rM+r^2\right)\dot{t}\right.\\
&\left.\left.-\dot{r}Mr\right)\dot{\varphi}a-Mr^3\left(\dot{r}^2Mr^2+2\dot{r}Mr^2f\dot{t}+r^2f^2\left(M\dot{t}^2-r^3\dot{\varphi}^2\right)\right)\right), \qquad \text{(F.5b)}\\
d_2 =& \frac{8\pi\Sigma\cot\theta\csc\theta}{3aMr^2\Delta\dot{t}}\left(\dot{\varphi}a^2-\dot{t}a+r^2\dot{\varphi}\right)\left[(a\dot{\varphi}-\dot{t})\Delta-\dot{r}r^2\right]\left[a^2\cos(2\theta)-r^2\right], \qquad \text{(F.5c)}\\
d_3 =& -\frac{16\pi\Sigma\cot\theta\csc\theta}{3Mr^3\Delta^2\dot{t}}\left[\dot{r}r^2-(a\dot{\varphi}-\dot{t})\Delta\right]\left(r\left(\dot{t}a^4-\dot{\varphi}a^5+(5M-r)r\dot{\varphi}a^3+r(2\dot{r}r+2\dot{t}r-5M\dot{t})a^2\right.\right.\\
&\left.+3Mr^3f\dot{\varphi}a-(3M-r)r^2(\dot{r}r+\dot{t}r-2M\dot{t})\right)+a\left(M\dot{t}a^3-M\dot{\varphi}a^4+r\left(2M^2-3rM+r^2\right)\dot{\varphi}a^2\right.\\
&\left.\left.+r(\dot{r}r(r-M)+Mrf\dot{t})a+r^5f^2\dot{\varphi}\right)\cos(2\theta)\right), \qquad \text{(F.5d)}\\
d_4 =& -\frac{16\pi\Sigma\cot\theta}{3Mr^2\Delta^2\dot{t}}\left(6\dot{\varphi}^2a^8-12\dot{t}\dot{\varphi}a^7+3\left(2\dot{t}^2+5r^2f\dot{\varphi}^2\right)a^6-2r(5\dot{r}r+14\dot{t}r-30M\dot{t})\dot{\varphi}a^5\right.\\
&+r\left(48M^2r\dot{\varphi}^2+10\dot{r}r\dot{t}+13r\left(\dot{t}^2+r^2\dot{\varphi}^2\right)-6M\left(5\dot{t}^2+8r^2\dot{\varphi}^2\right)\right)a^4+4r^2\left(\dot{r}(8M-3r)r\right.\\
&+\left.\left(22rM-24M^2-5r^2\right)\dot{t}\right)\dot{\varphi}a^3+r^2\left(5\dot{r}^2r^2-4\dot{r}(8M-3r)\dot{t}r+rf\left(5\dot{\varphi}^2r^3+8\dot{t}^2r\right.\right.\\
&\left.\left.+12M^2\dot{\varphi}^2r-12M\left(2\dot{t}^2+r^2\dot{\varphi}^2\right)\right)\right)a^2-2fr^4\left(\dot{r}r(r-6M)+2\left(6M^2-5rM+r^2\right)\dot{t}\right)\dot{\varphi}a\\
&\left.+r^3\left(\dot{r}^2(r-6M)r^2+2\dot{r}\left(12M^2-8rM+r^2\right)\dot{t}r-r^2f^2\left(6M\dot{t}^2-r\left(\dot{t}^2+r^2\dot{\varphi}^2\right)\right)\right)\right), \qquad \text{(F.5e)}\\
d_5 =& -\frac{16\pi\Sigma\cot\theta}{3Mr^2\Delta^2\dot{t}}\left(a^2-Mr\right)\left[\dot{r}r^2-(a\dot{\varphi}-\dot{t})\Delta\right]^2, \qquad \text{(F.5f)}\\
d_6 =& -\frac{4\pi\Sigma\csc\theta}{3aMr^2\Delta\dot{t}}\left(\dot{\varphi}a^2-\dot{t}a+r^2\dot{\varphi}\right)\left[(a\dot{\varphi}-\dot{t})\Delta-\dot{r}r^2\right]\left[\cos(2\theta)a^2+a^2-2r^2\right], \qquad \text{(F.5g)}\\
d_7 =& \frac{8\pi\Sigma\csc\theta}{3Mr^3\Delta^2\dot{t}}\left[\dot{r}r^2-(a\dot{\varphi}-\dot{t})\Delta\right]\left((r-M)\dot{\varphi}a^5+(M-r)\dot{t}a^4+r\left(2M^2+rM+5r^2\right)\dot{\varphi}a^3\right.\\
&+r\left(\dot{r}r(5r-M)+\left(r^2-2M^2-3rM\right)\dot{t}\right)a^2+4r^3\left(r^2-2M^2-rM\right)\dot{\varphi}a+(M-r)\left(-\dot{\varphi}a^4\right.\\
&\left.\left.+\dot{t}a^3+(2M-3r)r\dot{\varphi}a^2+r^2(\dot{t}f-\dot{r})a-2r^4f\dot{\varphi}\right)a\cos(2\theta)-2(3M-r)r^3(\dot{r}r+rf\dot{t})\right), \qquad \text{(F.5h)}\\
d_8 =& \frac{16\pi a\Sigma(\cos(2\theta)+3)\csc(\theta)}{3\Delta^2Mr^3\dot{t}}\left(a^7M\dot{\varphi}^2-2a^6M\dot{t}\dot{\varphi}+a^5\left(M\left(4r^2\dot{\varphi}^2+\dot{t}^2\right)-4M^2r\dot{\varphi}^2-r^3\dot{\varphi}^2\right)\right.\\
&+a^4r\dot{\varphi}\left(8M^2\dot{t}-6Mr\dot{t}+r^2(\dot{t}-\dot{r})\right)+a^3r\left(r^2\left(\dot{r}\dot{t}-12M^2\dot{\varphi}^2\right)+2Mr\left(2M^2\dot{\varphi}^2+\dot{t}^2\right)-4M^2\dot{t}^2\right.\\
&\left.+9Mr^3\dot{\varphi}^2-2r^4\dot{\varphi}^2\right)-2a^2f^2r^2r^2\dot{t}\dot{\varphi}(M-r)+ar^2\left(4M^3\left(2r^2\dot{\varphi}^2+\dot{t}^2\right)-4M^2r\left(3r^2\dot{\varphi}^2+\dot{t}^2\right)\right.\\
&\left.\left.+Mr^2\left(6r^2\dot{\varphi}^2-\dot{r}^2-2\dot{r}\dot{t}+\dot{t}^2\right)+r^3\left(\dot{r}^2-r^2\dot{\varphi}^2+\dot{r}\dot{t}\right)\right)+f^2r^2r^4\dot{\varphi}(r(\dot{r}+\dot{t})-2M\dot{t})\right), \qquad \text{(F.5i)}
\end{aligned}$$

for $[\mathcal{I}_2^{(\mathrm{rec})}]$. The circular orbits coefficients are recovered from Eqs. (F.4) and (F.5) by taking $\dot{r}=0$.

F.3 Summation formulae

The sums in Eq. (5.50) are analytically obtained. They are

$$\begin{aligned}
\sum_{\ell\geq 2}\left[\lambda_0 Y_\ell(\theta)+2\cot\theta Y_\ell'(\theta)\right]\frac{\bar{Y}_\ell(\theta_0)}{\lambda_0\lambda_1} &= \sum_{\ell\geq 2}-\frac{{}_{-2}Y_\ell(\theta)\bar{Y}_\ell(\theta_0)}{(\lambda_0\lambda_1)^{1/2}} \\
&= \sum_{\ell\geq 2}\csc^2\theta\int d(\cos\theta'')\int d(\cos\theta')Y_\ell(\theta')\bar{Y}_\ell(\theta_0) \\
&= -\frac{1}{4\pi}\left\{\frac{1}{2}-\csc^2\theta\left[1-\cos\theta\,\mathrm{sign}(\pi-2\theta)\right]\right\}, \qquad \text{(F.6a)} \\
\sum_{\ell\geq 2}\frac{Y_\ell(\theta)\bar{Y}_\ell'(\theta_0)}{\lambda_0} &= -\sum_{\ell\geq 2}\int d(\cos\theta_0')Y_\ell(\theta)\bar{Y}_\ell(\theta_0') \\
&= \frac{1}{4\pi}\left[\mathrm{sign}(\pi-2\theta)+\frac{3}{2}\cos\theta\right], \qquad \text{(F.6b)} \\
\sum_{\ell\geq 2}\frac{Y_\ell'(\theta)\bar{Y}_\ell(\theta_0)}{\lambda_0} &= -\sum_{\ell\geq 2}\csc\theta\int d(\cos\theta')Y_\ell(\theta')\bar{Y}_\ell(\theta_0) \\
&= \frac{1}{4\pi}\left[\mathrm{sign}(\pi-2\theta)\csc\theta-\cot\theta\right], \qquad \text{(F.6c)} \\
\sum_{\ell\geq 2}\frac{Y_\ell'(\theta)\bar{Y}_\ell(\theta_0)}{\lambda_0^2} &= \sum_{\ell\geq 2}\csc\theta\iint d(\cos\theta')d(\cos\theta'')Y_\ell(\theta'')\bar{Y}_\ell(\theta_0) \\
&= \frac{1}{4\pi}\left\{\left[1-\mathrm{sign}(\pi-2\theta)\cos\theta\right]\csc\theta-\frac{3}{4}\sin\theta\right\}, \qquad \text{(F.6d)}
\end{aligned}$$

where we have exchanged the order of summation and integration, and the distributions are omitted since the sums are to be evaluated at $\theta\neq\theta_0$. In Eq. (F.6) we used

$$\sum_{\ell\geq 2}Y_\ell(\theta)\bar{Y}_\ell(\theta_0)\equiv\delta(\theta-\theta_0)-Y_0(\theta)\bar{Y}_0(\theta_0)-Y_1(\theta)\bar{Y}_1(\theta_0). \qquad \text{(F.7)}$$

F.4 Sided expressions for the auxiliary invariants: circular-equatorial orbits of Kerr

We include the general form of the sided values of the two invariants we use in the Kerr case:

$$\mathcal{I}_{k\pm}^{(\mathrm{rec})}(r,\theta)=\sum_{i,j}A_{k;ij}^{\pm}R_{4\pm}^{(i)}(r)\,{}_{-2}Y_\ell^{(j)}(\theta), \quad \text{with} \quad k=\{1,2\}. \qquad \text{(F.8)}$$

These expressions are required to prove that the sums converge as distributions on the sphere (see [99] for the explicit proof, which we have left out of this work). For circular orbits the coefficients $A_{ij}^{\pm}$ are

$$
\begin{aligned}
A^{\pm}_{1;00} =& \frac{\Sigma \csc^2\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0 \dot{t}} \left[\lambda_1 a_{01}(r_0) R_{4\mp}(r_0) + a_{02}(r_0) R'_{4\mp}(r_0)\right] \left\{16a^2 r - 40a^2 M + 72Mr^2 - 48r^3\right. \\
& + (a^2 M + 3a^2 r - 12Mr^2 + 8r^3)\lambda_0 + 4\left[(3M-2r)r^2(2+\lambda_0) + a^2\left\langle 2M - r(4+\lambda_0)\right\rangle\right]\cos(2\theta) \\
& \left. - a^2(M-r)\lambda_0\cos(4\theta)\right\} + \frac{\Sigma\cot\theta\csc\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0\lambda_1\dot{t}} \left[a_{03}(r_0)R_{4\mp}(r_0) + a_{04}(r_0)R'_{4\mp}(r_0)\right] \times \\
& \left\{a^2(32 + 10\lambda_0 - \lambda_0^2) - r\left[4M(6 + 11\lambda_0 - \lambda_0^2) + r(8 - 34\lambda_0 + 3\lambda_0^2)\right]\right. \\
& \left. + \lambda_1\left[-4Mr(\lambda_0 - 1) + a^2\lambda_0 + r^2(3\lambda_0 - 4)\right]\cos(2\theta)\right\}, \qquad \text{(F.9a)}
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{1;10} =& \frac{\Sigma \csc^2\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0 \dot{t}} \left[a_{11}(r_0) R_{4\mp}(r_0) + a_{12}(r_0) R'_{4\mp}(r_0)\right] \left\{48a^4 - 80a^2M^2 + 16a^2 Mr - 32a^2r^2\right. \\
& + 144M^2r^2 - 144Mr^3 + 48r^4 + (32a^4 + 2a^2M^2 - 60a^2Mr + 10a^2r^2 - 24M^2r^2 + 72Mr^3 \\
& - 32r^4)\lambda_0 + (2a^2Mr - a^4 + 3a^2r^2 - 8Mr^3 + 4r^4)\lambda_0^2 + 4\left\langle 4a^4 + r^2\left[6M^2 + 2Mr(\lambda_0 - 3)\right.\right. \\
& \left.\left. - r^2\lambda_1\right](2+\lambda_0) + a^2\left[4M^2 - 2Mr(10+\lambda_0) - r^2(\lambda_0^2 - 8 + 2\lambda_0)\right]\right\rangle\cos(2\theta) \\
& \left. + a^2\lambda_0(r^2\lambda_1 - 2M^2 - 2Mr\lambda_1 + a^2\lambda_0)\cos(4\theta)\right\} \\
& + \frac{\Sigma\cot\theta\csc\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0\lambda_1\dot{t}} \left[a_{13}(r_0)R_{4\mp}(r_0) + a_{14}(r_0)R'_{4\mp}(r_0)\right] \times \\
& \left\{a^2\left[M - r(1+\lambda_0)\right] + r^2\left[r - r\lambda_0 + M(-1 + 2\lambda_0)\right]\right\}\left[10 - \lambda_0 + \lambda_1\cos(2\theta)\right], \qquad \text{(F.9b)}
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{1;10} =& \frac{\Sigma \cot\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0 \dot{t}} \left[\lambda_1 a_{21}(r_0) R_{4\mp}(r_0) + a_{22}(r_0) R'_{4\mp}(r_0)\right] \left[2(3M - 2r)r^2 + a^2(r - 3M)\right. \\
& \left. + a^2(M-r)\cos(2\theta)\right] + \frac{\Sigma\cot\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0\dot{t}} \left[a_{23}(r_0)R_{4\mp}(r_0) + a_{24}(r_0)R'_{4\mp}(r_0)\right] \times \\
& \left[r(4M + 2r + 10M\lambda_0 - 7r\lambda_0) - 3a^2(2+\lambda_0) + \Delta\cos(2\theta)\right], \qquad \text{(F.9c)}
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{1;11} =& \frac{\Sigma \cot\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0\lambda_1 \dot{t}} \left[\lambda_1 a_{31}(r_0) R_{4\mp}(r_0) + a_{32}(r_0) R'_{4\mp}(r_0)\right] \left\{2r^2\left[r^2\lambda_1 - 6M^2 - 2Mr(\lambda_0 - 3)\right]\right. \\
& \left. - a^4(4 + 3\lambda_0) + a^2(6M^2 - r^2\lambda_1 + 6Mr\lambda_0) + a^2(a^2\lambda_0 - 2M^2 - 2Mr\lambda_1 + r^2\lambda_1)\cos(2\theta)\right\} \\
& + \frac{\Sigma\csc\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0\lambda_1\dot{t}} \left[a_{33}(r_0)R_{4\mp}(r_0) + a_{34}(r_0)R'_{4\mp}(r_0)\right]\left\{a^2\left[M - r(1+\lambda_0)\right]\right. \\
& \left. + r^2\left[r - r\lambda_0 + M(2\lambda_0 - 1)\right]\right\}, \qquad \text{(F.9d)}
\end{aligned}
$$

and the coefficients $B^{\pm}_{ij}$ are

$$
\begin{aligned}
A^{\pm}_{2;00} =& \frac{\Sigma \csc^4\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0 \dot{t}} \left[\lambda_1 b_{01}(r_0) R_{4\mp}(r_0) + b_{02}(r_0) R'_{4\mp}(r_0)\right] \left\{44r - 18M + 5r\lambda_0\right. \\
& \left. + 4\left[6M - r(\lambda_0 - 4)\right]\cos(2\theta) - \left[6M + r(\lambda_0 - 4)\right]\cos(4\theta)\right\} + \frac{\Sigma\cot\theta\csc^3\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0\lambda_1\dot{t}} \times \\
& \left[b_{03}(r_0)R_{4\mp}(r_0) + b_{04}(r_0)R'_{4\mp}(r_0)\right]\left\{a^2(\lambda_0 - 8) - r\left[r(4+\lambda_0) - 12M\right]\right\} \times \\
& \left[\lambda_1\cos(2\theta) - 6 - \lambda_0\right], \qquad \text{(F.10a)}
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{2;10} =& \frac{\Sigma \csc^4\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0 \dot{t}} \left[b_{11}(r_0) R_{4\mp}(r_0) + b_{12}(r_0) R'_{4\mp}(r_0)\right] \left\{38a^2 - 18M^2 - 14Mr - 6r^2\right. \\
& + 8a^2\lambda_0 - 11Mr\lambda_0 + 3r^2\lambda_0 - 4\left[Mr(14 - 3\lambda_0) - 6M^2 + 2a^2(\lambda_0 - 3) + r^2\lambda_1\right]\cos(2\theta) \\
& \left. + \left[2a^2 - 6M^2 - Mr(\lambda_0 - 6) + r^2\lambda_1\right]\cos(4\theta)\right\} + \frac{\Sigma\cot\theta\csc^3\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0\lambda_1\dot{t}} \times \\
& \left[b_{13}(r_0)R_{4\mp}(r_0) + b_{14}(r_0)R'_{4\mp}(r_0)\right]\left[r^2(r - 3M) - a^2(M - 3r)\right]\left[\lambda_1\cos(2\theta) - 6 - \lambda_0\right], \\
& \qquad \text{(F.10b)}
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{2;01} =& \frac{\Sigma \cot\theta \csc^2\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0 \dot{t}} \left[\lambda_1 b_{21}(r_0) R_{4\mp}(r_0) + b_{22}(r_0) R'_{4\mp}(r_0)\right] \left[(\lambda_0 - 1)\cos(2\theta) - 3 - \lambda_0\right] \\
&+ \frac{\Sigma \csc^3\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0 \dot{t}} \left[b_{23}(r_0) R_{4\mp}(r_0) + b_{24}(r_0) R'_{4\mp}(r_0)\right] \big\{76a^2 - 96Mr + 32r^2 + (12a^2 \\
&-48Mr + 24r^2)\lambda_0 + (4r^2 - a^2)\lambda_0^2 - 4\lambda_1 \big\langle 6a^2 + r\left[r(4+\lambda_0) - 12M\right]\big\rangle \cos(2\theta) \\
&+a^2\lambda_1^2 \cos(4\theta)\big\}\,, \qquad \text{(F.10c)} \\
A^{\pm}_{2;11} =& \frac{\Sigma \cot\theta \csc^2\theta\, {}_{-2}\bar{Y}_\ell(\theta_0)}{\mathcal{W}\Delta^3\lambda_0\lambda_1 \dot{t}} \left[\lambda_1 b_{31}(r_0) R_{4\mp}(r_0) + b_{32}(r_0) R'_{4\mp}(r_0)\right] (Mr - a^2)\times \\
&\left[(\lambda_0 - 1)\cos(2\theta) - 3 - \lambda_0\right] + \frac{\Sigma \csc^3\theta\, {}_{-2}\bar{Y}'_\ell(\theta_0)}{\mathcal{W}\Delta\lambda_0\lambda_1 \dot{t}} \left[b_{33}(r_0) R_{4\mp}(r_0) + b_{34}(r_0) R'_{4\mp}(r_0)\right] \times \\
&\big\{30a^2 r - 14a^2 M - 24Mr^2 + 8r^3 + (9a^2 r - a^2 M - 12Mr^2 + 4r^3)\lambda_0 \\
&-4r\left[2a^2 + r(r - 3M)\right]\lambda_1 \cos(2\theta) + a^2(M - r)\lambda_1 \cos(4\theta)\big\}\,. \qquad \text{(F.10d)}
\end{aligned}
$$

The eccentric orbits expressions are not included here due to their complexity. Furthermore the proof of the distributional convergence of the sums in the eccentric orbits case follows directly from the circular orbits results.

F.5 Evaluation of integrals for the case of eccentric orbits around Kerr

To obtain the result of Eq. (5.76), namely the jump in the amplitudes of the completion pieces, we need to evaluate the integrals of Eq. (5.75). First, we observe that only the part of the integrands that is even under $\chi \mapsto -\chi$ will contribute to the final integral. After symmetrizing and setting $\mathsf{m} = 1$, we find that the integrals take the following form,

$$
\begin{aligned}
\left[\delta\tilde{M}\right] - \mathcal{E} &= \frac{1}{T_r}\int_{-1}^{1} \frac{C^M_{14} y^{14} + C^M_{12} y^{12} + C^M_{10} y^{10} + C^M_8 y^8 + C^M_6 y^6 + C^M_4 y^3 + C^M_2 y^2}{B\sqrt{1-y^2}\sqrt{1-ky^2}(1-h_1 y^2)^2(1-h_2 y^2)^2(1-h_3 y^2)^2} dy \\
\left[\delta\tilde{J}\right] - \mathcal{L} &= \frac{1}{T_r}\int_{-1}^{1} \frac{C^J_{14} y^{14} + C^J_{12} y^{12} + C^J_{10} y^{10} + C^J_8 y^8 + C^J_6 y^6 + C^J_4 y^3 + C^J_2 y^2}{B\sqrt{1-y^2}\sqrt{1-ky^2}(1-h_1 y^2)^2(1-h_2 y^2)^2(1-h_3 y^2)^2} dy
\end{aligned} \qquad \text{(F.11)}
$$

with common factors

$$y \equiv \sin\left(\frac{\chi}{2}\right), \qquad \text{(F.12)}$$

$$
\begin{aligned}
B \equiv& p(1+e)^2\left(a^2(1+e)^2 + p(p - 2(1+e))\right)^2 \\
&\times\sqrt{\frac{a^2\left[p - \mathcal{E}^2(2e+p+6)\right] + 4a(e+3)\mathcal{E}\mathcal{L} + \mathcal{L}^2(p-2e-6)}{4p}}, \qquad \text{(F.13)}
\end{aligned}
$$

$$k \equiv \frac{-4e(\mathcal{L} - a\mathcal{E})^2}{a^2\left[p - \mathcal{E}^2(2e+p+6)\right] + 4a(e+3)\mathcal{E}\mathcal{L} + \mathcal{L}^2(p-2e-6)}, \qquad \text{(F.14)}$$

$$h_1 \equiv \frac{2e}{1+e}, \qquad \text{(F.15)}$$

$$h_2 \equiv \frac{2a^2 e}{a^2(1+e) - p(1+\sqrt{1-a^2})}, \qquad \text{(F.16)}$$

$$h_3 \equiv \frac{2a^2 e}{a^2(1+e) - p(1-\sqrt{1-a^2})}. \qquad \text{(F.17)}$$

$$\text{(F.18)}$$

where the correct units are recovered by replacing $p \to p/M$. The coefficients in the mass integrand are given by

$$
\begin{aligned}
C^M_{14} =& -256a^4e^7(\mathcal{L}-a\mathcal{E})^2, && \text{(F.19)}\\
C^M_{12} =& 128a^2e^6\left(7a^2(e+1)-9p\right)(\mathcal{L}-a\mathcal{E})^2, && \text{(F.20)}\\
C^M_{10} =& -64e^5\Big(a^4\left(21(e+1)^2(\mathcal{L}-a\mathcal{E})^2+4p^2\right)-6p^2(\mathcal{L}-a\mathcal{E})^2\\
&+a^2p(\mathcal{L}-a\mathcal{E})(3(p-18(e+1))(\mathcal{L}-a\mathcal{E})+10\mathcal{L}p)\Big), && \text{(F.21)}\\
C^M_{8} =& 32e^4\Big(a^4\left(35(e+1)^3(\mathcal{L}-a\mathcal{E})^2+p^2(20e-3p+12)\right)+p^2(7p-30(e+1))(\mathcal{L}-a\mathcal{E})^2\\
&+a^2p\Big[\left(-127e^2+15e(p-18)+p(3p+23)-111\right)(\mathcal{L}-a\mathcal{E})^2\\
&+2\mathcal{L}p(25e-3p+17)(\mathcal{L}-a\mathcal{E})-\left(\mathcal{L}^2+3\right)p^2\Big]\Big), && \text{(F.22)}\\
C^M_{6} =& 16e^3\Big(a^4\left(6p^2(2e(-3e+p-4)+p-2)-35(e+1)^4(\mathcal{L}-a\mathcal{E})^2\right)\\
&+a^2p\Big[2p^2\left((12e+6)\mathcal{L}(\mathcal{L}-a\mathcal{E})+(-6e-3)(\mathcal{L}-a\mathcal{E})^2+2(e+1)\mathcal{L}^2+6e+3\right)\\
&+(e+1)(23e+11)\mathcal{L})-4p(\mathcal{L}-a\mathcal{E})((e(7e+23)+10)(\mathcal{L}-a\mathcal{E})\\
&+4(e+1)(e(37e+86)+25)(\mathcal{L}-a\mathcal{E})^2-3p^3\Big]+2p^2\left((33e^2+e(60-14p)\right.\\
&\left.+(p-11)p+39)(\mathcal{L}-a\mathcal{E})^2-\mathcal{L}p(p+6)(\mathcal{L}-a\mathcal{E})+\mathcal{L}^2p^2\right)\Big), && \text{(F.23)}\\
C^M_{4} =& 8e^2\Big(a^4(e+1)\left(21(e+1)^4(\mathcal{L}-a\mathcal{E})^2-3(5e+1)p^3+4(7e+1)(e+1)p^2\right)\\
&+a^2p\Big[-3(e+1)p^2\left((10e+2)\mathcal{L}(\mathcal{L}-a\mathcal{E})+(-5e-1)(\mathcal{L}-a\mathcal{E})^2+2(e+1)\mathcal{L}^2+5e+1\right)\\
&+4(e+1)p(\mathcal{L}-a\mathcal{E})(6(e(e+4)+1)(\mathcal{L}-a\mathcal{E})+(e+1)(19e+7)\mathcal{L})\\
&-3(e+1)^2(e(29e+74)+13)(\mathcal{L}-a\mathcal{E})^2+(9e+3)p^3-p^4\Big]+p^2\Big[p\left(45e^2-6e(p-11)\right.\\
&\left.+p^2+45\right)(\mathcal{L}-a\mathcal{E})^2+2\mathcal{L}p\left(3e^2+3e(p+6)-p(p+3)+15\right)(\mathcal{L}-a\mathcal{E})\\
&-6(e+1)(e(13e+22)+17)(\mathcal{L}-a\mathcal{E})^2+\mathcal{L}^2p^2(-6e+p-6)+3\mathcal{E}^2p^3\Big]\Big), && \text{(F.24)}\\
C^M_{2} =& 4e\Big(2(e+1)\mathcal{L}^2p^4(3e-p+3)\Big)-7a^4(e+1)^6(\mathcal{L}-a\mathcal{E})^2-2a^4e(e+1)^2p^2(4e-3p+4)\\
&+a^2(e+1)^2p\left(e\left(22e^2+e(82-9p)-2p(3p+19)+66\right)-5p+6\right)(\mathcal{L}-a\mathcal{E})^2\\
&-2a^2(e+1)^2\mathcal{L}p^2(e(13e-6p+18)+5)(\mathcal{L}-a\mathcal{E})+2a^2ep^3\left(p^2-3(e+1)p+3(e+1)^2\right)\\
&+p^2\left(\left(5e^2+3\right)p^2-2ep^3-2(e+1)(e(17e+25)+20)p\right.\\
&\left.+12(e+1)^2(e(4e+7)+5)\right)(\mathcal{L}-a\mathcal{E})^2+4\mathcal{L}p^3\left(ep^2+3(e+1)p\right.\\
&\left.-3(e+2)(e+1)^2\right)(\mathcal{L}-a\mathcal{E})+\mathcal{E}^2p^5(p-6e-6)+4a^2(e+1)^3\mathcal{L}^2p^3. && \text{(F.25)}
\end{aligned}
$$

The coefficients for the angular momentum integrand are

$$
\begin{aligned}
C^J_{14} =& -256a^5e^7(\mathcal{L}-a\mathcal{E})^2, && \text{(F.26)}\\
C^J_{12} =& 128a^3e^6\left(7a^2(e+1)-12p\right)(\mathcal{L}-a\mathcal{E})^2, && \text{(F.27)}\\
C^J_{10} =& -64ae^5\left(7a^4\left(3(e+1)^2(\mathcal{L}-a\mathcal{E})^2+p^2\right)\right.\\
&\left.-8a^2p(\mathcal{L}-a\mathcal{E})(9(e+1)(\mathcal{L}-a\mathcal{E})-2\mathcal{L}p)-12p^2(\mathcal{L}-a\mathcal{E})^2\right), && \text{(F.28)}\\
C^J_{8} =& 32ae^4\Big(a^4\left(35(e+1)^3(\mathcal{L}-a\mathcal{E})^2+p^2(35e-3p+21)\right)+4p^2(4p-15(e+1))(\mathcal{L}-a\mathcal{E})^2\Big)\\
&+a^2p\left((-2e(83e+180)+p(3p+14)-138)(\mathcal{L}-a\mathcal{E})^2\right.
\end{aligned}
$$

$$+2\mathcal{L}p(40e-3p+26)(\mathcal{L}-a\mathcal{E})-\left(\mathcal{L}^2+6\right)p^2\Big), \tag{F.29}$$

$$\begin{aligned}
C_6^J =&16ae^3\Big(a^4\left(-35(e+1)^4(\mathcal{L}-a\mathcal{E})^2+6(2e+1)p^3-21(3e+1)(e+1)p^2\right)\\
&+a^2p\Big[2p^2\left((12e+6)\mathcal{L}(\mathcal{L}-a\mathcal{E})-(6e+3)(\mathcal{L}-a\mathcal{E})^2+2(e+1)\mathcal{L}^2+12e+6\right)\\
&+2(e+1)(73e+31)\mathcal{L})-p(\mathcal{L}-a\mathcal{E})((e(e+56)+31)(\mathcal{L}-a\mathcal{E})+4(e+1)(e(46e+113)\\
&+25)(\mathcal{L}-a\mathcal{E})^2-3p^3\Big]+p^2\left(2\left(66e^2+8e(15-4p)+(p-26)p+78\right)(\mathcal{L}-a\mathcal{E})^2\right.\\
&\left.-2\mathcal{L}p(p+12)(\mathcal{L}-a\mathcal{E})+5\mathcal{L}^2p^2\right)\Big),
\end{aligned} \tag{F.30}$$

$$\begin{aligned}
C_4^J =&8e^2\Big(a^5(e+1)\left(21(e+1)^4(\mathcal{L}-a\mathcal{E})^2-3(5e+1)p^3+7(7e+1)(e+1)p^2\right)+6a\mathcal{E}^2p^5\Big)\\
&+a^3p\Big[(9e+3)p^3-p^4-3(e+1)p^2\left((10e+2)\mathcal{L}(\mathcal{L}-a\mathcal{E})-(5e+1)(\mathcal{L}-a\mathcal{E})^2\right.\\
&\left.+2(e+1)\mathcal{L}^2+10e+2\right)+(e+1)p(\mathcal{L}-a\mathcal{E})(3(e(e+24)+7)(\mathcal{L}-a\mathcal{E})\\
&+2(e+1)(59e+17)\mathcal{L})-12(e+1)^2(e(8e+23)+1)(\mathcal{L}-a\mathcal{E})^2\Big]\\
&+ap^2\Big[2\mathcal{L}p\left(3(e-1)p+6(e+1)(e+5)-p^2\right)(\mathcal{L}-a\mathcal{E})+\mathcal{L}^2p^2(p-15(e+1))\\
&+\left(-6ep^2+12(e(8e+13)+7)p-12(e+1)(e(13e+22)+17)+p^3\right)(\mathcal{L}-a\mathcal{E})^2\Big],
\end{aligned} \tag{F.31}$$

$$\begin{aligned}
C_2^J =&4e\Big(a^5(e+1)^2\left(-7(e+1)^4(\mathcal{L}-a\mathcal{E})^2-2ep^2(7e-3p+7)\right)+a\mathcal{E}^2p^5(p-12(e+1))\\
&+a^3p\Big[(e+1)^2\left(-6ep^2-(e(3e+32)+5)p+4(e+1)(e(4e+15)-3)\right)(\mathcal{L}-a\mathcal{E})^2\\
&+4(e+1)^3\mathcal{L}^2p^2-2(e+1)^2\mathcal{L}p(e(19e-6p+24)+5)(\mathcal{L}-a\mathcal{E})+12e^3p^2-6e^2p^3\\
&+24e^2p^2+2ep^4-6ep^3+12ep^2\Big]+ap^2\Big[\left(\left(5e^2+3\right)p^2-2ep^3-8(e+1)(e(8e+13)+8)p\right.\\
&\left.+24(e+1)^2(e(4e+7)+5)\right)(\mathcal{L}-a\mathcal{E})^2+4\mathcal{L}p\left(ep^2+3(e+1)p\right.\\
&\left.-6(e+2)(e+1)^2\right)(\mathcal{L}-a\mathcal{E})+(e+1)\mathcal{L}^2p^2(15e-2p+15)\Big]\Big).
\end{aligned} \tag{F.32}$$

The integrals (F.11) can be recognized as elliptic integrals. Consequently, they can be evaluated using standard techniques for elliptic integrals (see for example Sec. 17 of [151]). The first step is to expand the integrands in partial fractions. The result is a sum of integrals of the form

$$I_{n,m}(k,h_i)=\int_0^1\frac{y^{2n}}{\sqrt{1-y^2}\sqrt{1-ky^2}(1-h_iy^2)}dy. \tag{F.33}$$

The integrals $I_{n,m}$ satisfy the following recurrence relations

$$I_{n,m}(k,h_i)=\frac{I_{n-1,m}(k,h_i)-I_{n-1,m}(k,h_i)}{h_i},\text{ and} \tag{F.34}$$

$$I_{n,0}(k,h_i)=I_{n,0}(k)=\frac{(k+1)(2n-2)I_{n-1,0}(k)-(2n-3)A_{n-2,0}(k)}{2n-1}. \tag{F.35}$$

Using these relations, the integrals (F.11) can be further reduced to a linear combination of five basic integrals

$$I_{0,0}(k)=K(k), \tag{F.36}$$

$$I_{1,0}(k)=\frac{K(k)-E(k)}{k},\text{ and} \tag{F.37}$$

$$I_{0,1}(k,h_i)=\Pi(h,k)\quad\text{(once for each } h_i), \tag{F.38}$$

where K, E, and Π are the elliptic integrals of the first, second and third kind, respectively. The coefficient of each integral is a complicated expression involving a, p, e, $\mathcal{E}$ and $\mathcal{L}$. Substituting the expressions (5.69) for $\mathcal{E}$ and $\mathcal{L}$ in terms of a, p, and e, these vanish after some straightforward, but tedious algebra. This establishes the result given in Eq. (5.76).

Bibliography

[1] A. Pound, C. Merlin, and L. Barack, "Gravitational self-force from radiation-gauge metric perturbations," *Phys. Rev. D*, vol. 89, p. 024009, 2014.

[2] C. Merlin and A. G. Shah, "Self-force from reconstructed metric perturbations: Numerical implementation in schwarzschild spacetime," *Phys. Rev. D*, vol. 91, p. 024005, Jan 2015.

[3] R. Hulse and J. Taylor, "Discovery of a pulsar in a binary system," *Ap. J.*, vol. 195, pp. L51–L53, Jan. 1975.

[4] J. Droste, "The field of a single centre in einstein's theory of gravitation, and the motion of a particle in that field," *General Relativity and Gravitation*, vol. 34, no. 9, pp. 1545–1563, re-print: 2002.

[5] A. Einstein, L. Infeld, and B. Hoffmann, "The Gravitational Equations and the Problem of Motion," *Annals of Mathematics*, vol. 39, pp. 65–100, Jan. 1938.

[6] R. Geroch and J. Traschen, "Strings and other distributional sources in general relativity," *Phys. Rev. D*, vol. 36, pp. 1017–1031, Aug 1987.

[7] A. Pound and E. Poisson, "Osculating orbits in Schwarzschild spacetime, with an application to extreme mass-ratio inspirals," *Phys. Rev. D*, vol. 77, p. 044013, 2008.

[8] A. Pound and E. Poisson, "Multi-scale analysis of the electromagnetic self-force in a weak gravitational field," *Phys. Rev. D*, vol. 77, p. 044012, 2008.

[9] P. Amaro-Seoane, S. Aoudia, S. Babak, P. Binetruy, E. Berti, *et al.*, "Low-frequency gravitational-wave science with eLISA/NGO," *Class. Quant. Grav.*, vol. 29, p. 124016, 2012.

[10] J. Veitch, I. Mandel, B. Aylott, B. Farr, V. Raymond, *et al.*, "Estimating parameters of coalescing compact binaries with proposed advanced detector networks," *Phys. Rev. D*, vol. 85, p. 104045, 2012.

[11] C. L. Rodriguez, I. Mandel, and J. R. Gair, "Verifying the no-hair property of massive compact objects with intermediate-mass-ratio inspirals in advanced gravitational-wave detectors," *Phys. Rev. D*, vol. 85, p. 062002, Mar. 2012.

[12] S. Hild, S. Chelkowski, and A. Freise, "Pushing towards the ET sensitivity using 'conventional' technology," 2008.

[13] J. R. Gair, I. Mandel, M. C. Miller, and M. Volonteri, "Exploring intermediate and massive black-hole binaries with the Einstein Telescope," *Gen. Rel. Grav.*, vol. 43, pp. 485–518, 2011.

[14] Y. Mino, M. Sasaki, and T. Tanaka, “Gravitational radiation reaction to a particle motion,” *Phys. Rev. D*, vol. 55, pp. 3457–3476, 1997.

[15] T. C. Quinn and R. M. Wald, “An axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved spacetime,” *Phys. Rev. D*, vol. 56, pp. 3381–3394, 1997.

[16] S. E. Gralla and R. M. Wald, “A Rigorous Derivation of Gravitational Self-force,” *Class. Quant. Grav.*, vol. 25, p. 205009, 2008.

[17] A. Pound, “Self-consistent gravitational self-force,” *Phys. Rev. D*, vol. 81, p. 024023, 2010.

[18] E. Poisson, “The Motion of point particles in curved space-time,” *Living Rev. Rel.*, vol. 7, p. 6, 2004.

[19] L. Barack and A. Ori, “Gravitational self-force and gauge transformations,” *Phys. Rev. D*, vol. 64, p. 124003, Oct 2001.

[20] S. L. Detweiler and B. F. Whiting, “Self force via a Green’s function decomposition,” *Phys. Rev. D*, vol. 67, p. 024025, 2003.

[21] W. G. Anderson and A. G. Wiseman, “ A Matched expansion approach to practical self-force calculations,” *Class. Quant. Grav.*, vol. 22, pp. S783–S800, 2005.

[22] A. C. Ottewill and B. Wardell, “Quasilocal contribution to the scalar self-force: Geodesic motion,” *Phys. Rev. D*, vol. 77, p. 104002, 2008.

[23] A. C. Ottewill and B. Wardell, “Quasi-local contribution to the scalar self-force: Non-geodesic Motion,” *Phys. Rev. D*, vol. 79, p. 024031, 2009.

[24] B. S. DeWitt and C. M. DeWitt *Physics*, vol. (Long Island City, NY), 1, 3, 1964.

[25] M. J. Pfenning and E. Poisson, “Scalar, electromagnetic, and gravitational selfforces in weakly curved space-times,” *Phys. Rev. D*, vol. 65, p. 084001, 2002.

[26] M. Casals, S. Dolan, A. C. Ottewill, and B. Wardell, “Pade Approximants of the Green Function in Spherically Symmetric Spacetimes,” *Phys. Rev. D*, vol. 79, p. 124044, 2009.

[27] M. Casals, S. Dolan, A. C. Ottewill, and B. Wardell, “Self-Force and Green Function in Schwarzschild spacetime via Quasinormal Modes and Branch Cut,” *Phys. Rev. D*, vol. 88, p. 044022, 2013.

[28] C. Kavanagh, “Scalar Green function in Kerr spacetime: Branch cut contribution.” 17th Capra Meeting on Radiation Reaction in General Relativity.

[29] L. Barack and A. Ori, “Mode sum regularization approach for the selfforce in black hole space-time,” *Phys. Rev. D*, vol. 61, p. 061502, 2000.

[30] L. Barack, “Self force on a scalar particle in spherically symmetric space-time via mode sum regularization: Radial trajectories,” *Phys. Rev. D*, vol. 62, p. 084027, 2000.

[31] L. Barack and A. Ori, “Gravitational selfforce on a particle orbiting a Kerr black hole,” *Phys. Rev. Lett.*, vol. 90, p. 111101, 2003.

[32] J. Thornburg, "Highly accurate and efficient self-force computations using time-domain methods: Error estimates, validation, and optimization," 2010.

[33] I. Vega and S. L. Detweiler, "Regularization of fields for self-force problems in curved spacetime: Foundations and a time-domain application," *Phys. Rev. D*, vol. 77, p. 084008, 2008.

[34] S. Akcay, "A Fast Frequency-Domain Algorithm for Gravitational Self-Force: I. Circular Orbits in Schwarzschild Spacetime," *Phys. Rev. D*, vol. 83, p. 124026, 2011.

[35] P. Zimmerman, I. Vega, E. Poisson, and R. Haas, "Self-force as a cosmic censor," 2012.

[36] L. Barack and N. Sago, "Gravitational self force on a particle in circular orbit around a Schwarzschild black hole," *Phys. Rev. D*, vol. 75, p. 064021, 2007.

[37] L. Barack and N. Sago, "Gravitational self-force on a particle in eccentric orbit around a Schwarzschild black hole," *Phys. Rev. D*, vol. 81, p. 084021, 2010.

[38] N. Warburton and L. Barack, "Self force on a scalar charge in Kerr spacetime: circular equatorial orbits," *Phys. Rev. D*, vol. 81, p. 084039, 2010.

[39] N. Warburton and B. Wardell, "Applying the effective-source approach to frequency-domain self-force calculations," *Phys. Rev. D*, vol. 89, no. 4, p. 044046, 2014.

[40] N. Warburton, "Self force on a scalar charge in Kerr spacetime: inclined circular orbits," *Phys. Rev. D*, vol. 91, no. 2, p. 024045, 2015.

[41] N. Warburton and L. Barack, " Self force on a scalar charge in Kerr spacetime: eccentric equatorial orbits," *Phys. Rev. D*, vol. 83, p. 124038, 2011.

[42] R. Haas, "Time domain calculation of the electromagnetic self-force on eccentric geodesics in Schwarzschild spacetime," 2011.

[43] T. M. Linz, J. L. Friedman, and A. G. Wiseman, "Self force on an accelerated particle," *Phys. Rev. D*, vol. 90, no. 2, p. 024064, 2014.

[44] P. Zimmerman and E. Poisson, "Gravitational self-force in nonvacuum spacetimes," *Phys. Rev. D*, vol. 90, no. 8, p. 084030, 2014.

[45] A. Heffernan, A. Ottewill, and B. Wardell, "High-order expansions of the Detweiler-Whiting singular field in Schwarzschild spacetime," *Phys. Rev. D*, vol. 86, p. 104023, 2012.

[46] A. Heffernan, A. Ottewill, and B. Wardell, " High-order expansions of the Detweiler-Whiting singular field in Kerr spacetime," *Phys. Rev. D*, vol. 89, no. 2, p. 024030, 2014.

[47] L. Barack, D. A. Golbourn, and N. Sago, "m-Mode Regularization Scheme for the Self Force in Kerr Spacetime," *Phys. Rev. D*, vol. 76, p. 124036, 2007.

[48] S. R. Dolan, L. Barack, and B. Wardell, "Self force via m-mode regularization and 2+1D evolution: II. Scalar-field implementation on Kerr spacetime," *Phys. Rev. D*, vol. 84, p. 084001, 2011.

[49] S. R. Dolan and L. Barack, "Self-force via m-mode regularization and 2+1D evolution: III. Gravitational field on Schwarzschild spacetime," 2012.

[50] I. Vega, P. Diener, W. Tichy, and S. L. Detweiler, “Self-force with (3+1) codes: A Primer for numerical relativists,” *Phys. Rev. D*, vol. 80, p. 084021, 2009.

[51] P. Canizares, C. F. Sopuerta, and J. L. Jaramillo, “Pseudospectral Collocation Methods for the Computation of the Self-Force on a Charged Particle: Generic Orbits around a Schwarzschild Black Hole,” *Phys. Rev. D*, vol. 82, p. 044023, 2010.

[52] B. Wardell, I. Vega, J. Thornburg, and P. Diener, “A Generic effective source for scalar self-force calculations,” *Phys. Rev. D*, vol. 85, p. 104044, 2012.

[53] S. R. Dolan and L. Barack, “ Self force via m-mode regularization and 2+1D evolution: Foundations and a scalar-field implementation on Schwarzschild,” *Phys. Rev. D*, vol. 83, p. 024019, 2011.

[54] B. Wardell and N. Warburton, “Applying the effective-source approach to frequency-domain self-force calculations: Lorenz-gauge gravitational perturbations,” 2015.

[55] S. Dolan, “Approaches to Self-Force Calculations on Kerr spacetime.” 16th Capra Meeting on Radiation Reaction in General Relativity.

[56] B. Wardell, “Self-force: Computational Strategies,” 2015.

[57] B. Wardell, C. R. Galley, A. Zenginoğlu, M. Casals, S. R. Dolan, *et al.*, “Self-force via Green functions and worldline integration,” *Phys. Rev. D*, vol. 89, no. 8, p. 084021, 2014.

[58] A. Zenginoglu and C. R. Galley, “Caustic echoes from a Schwarzschild black hole,” *Phys. Rev. D*, vol. 86, p. 064030, 2012.

[59] M. Casals, E. Poisson, and I. Vega, “Regularization of static self-forces,” *Phys. Rev. D*, vol. 86, p. 064033, 2012.

[60] L. Barack and L. M. Burko, “Radiation reaction force on a particle plunging into a black hole,” *Phys. Rev. D*, vol. 62, p. 084040, 2000.

[61] L. M. Burko, “Self force on particle in orbit around a black hole,” *Phys. Rev. Lett.*, vol. 84, p. 4529, 2000.

[62] S. L. Detweiler, E. Messaritaki, and B. F. Whiting, “Selfforce of a scalar field for circular orbits about a Schwarzschild black hole,” *Phys. Rev. D*, vol. 67, p. 104016, 2003.

[63] L. M. Diaz-Rivera, E. Messaritaki, B. F. Whiting, and S. L. Detweiler, “Scalar field self-force effects on orbits about a Schwarzschild black hole,” *Phys. Rev. D*, vol. 70, p. 124018, 2004.

[64] P. Canizares and C. F. Sopuerta, “An Efficient Pseudospectral Method for the Computation of the Self-force on a Charged Particle: Circular Geodesics around a Schwarzschild Black Hole,” *Phys. Rev. D*, vol. 79, p. 084020, 2009.

[65] R. Haas, “Scalar self-force on eccentric geodesics in Schwarzschild spacetime: A Time-domain computation,” *Phys. Rev. D*, vol. 75, p. 124011, 2007.

[66] R. Haas and E. Poisson, “Mode-sum regularization of the scalar self-force: Formulation in terms of a tetrad decomposition of the singular field,” *Phys. Rev. D*, vol. 74, p. 044009, 2006.

[67] C. O. Lousto and H. Nakano, " A New method to integrate (2+1)-wave equations with Dirac's delta functions as sources," *Class. Quant. Grav.*, vol. 25, p. 145018, 2008.

[68] I. Vega, B. Wardell, P. Diener, S. Cupp, and Roland, "Scalar self-force for eccentric orbits around a Schwarzschild black hole," *Phys. Rev. D*, vol. 88, p. 084021, 2013.

[69] P. Diener, I. Vega, B. Wardell, and S. Detweiler, "Self-consistent orbital evolution of a particle around a Schwarzschild black hole," *Phys. Rev. Lett.*, vol. 108, p. 191102, 2012.

[70] N. Warburton and L. Barack, "Self force on a scalar charge in Kerr spacetime: circular equatorial orbits," *Phys. Rev. D*, vol. 81, p. 084039, 2010.

[71] L. M. Burko and Y. T. Liu, "Selfforce on a scalar charge in the space-time of a stationary, axisymmetric black hole," *Phys. Rev. D*, vol. 64, p. 024006, 2001.

[72] A. C. Ottewill and P. Taylor, "Static Kerr Green's Function in Closed Form and an Analytic Derivation of the Self-Force for a Static Scalar Charge in Kerr Space-Time," *Phys. Rev. D*, vol. 86, p. 024036, 2012.

[73] J. Thornburg, "Scalar self-force for highly eccentric orbits in Kerr spacetime." 17th Capra Meeting on Radiation Reaction in General Relativity.

[74] J. Kuchar, E. Poisson, and I. Vega, "Electromagnetic self-force on a static charge in Schwarzschild–de Sitter spacetimes," *Class. Quant. Grav.*, vol. 30, p. 235033, 2013.

[75] W. G. Anderson, E. E. Flanagan, and A. C. Ottewill, "Quasi-local contribution to the gravitational self-force," *Phys. Rev. D*, vol. 71, p. 024036, 2005.

[76] L. Barack and C. O. Lousto, "Computing the gravitational self-force on a compact object plunging into a Schwarzschild black hole," *Phys. Rev. D*, vol. 66, p. 061502, 2002.

[77] L. Barack and C. O. Lousto, " Perturbations of Schwarzschild black holes in the Lorenz gauge: Formulation and numerical implementation," *Phys. Rev. D*, vol. 72, p. 104026, 2005.

[78] S. R. Dolan, N. Warburton, A. I. Harte, A. Le Tiec, B. Wardell, *et al.*, "Gravitational self-torque and spin precession in compact binaries," *Phys. Rev. D*, vol. 89, no. 6, p. 064011, 2014.

[79] S. R. Dolan, P. Nolan, A. C. Ottewill, N. Warburton, and B. Wardell, "Tidal invariants for compact binaries on quasi-circular orbits," *Phys. Rev. D*, vol. 89, no. 6, p. 064011, 2014.

[80] S. E. Field, J. S. Hesthaven, and S. R. Lau, "Persistent junk solutions in time-domain modeling of extreme mass ratio binaries," *Phys. Rev. D*, vol. 81, p. 124030, 2010.

[81] T. S. Keidl, A. G. Shah, J. L. Friedman, D.-H. Kim, and L. R. Price, "Gravitational Self-force in a Radiation Gauge," *Phys. Rev. D*, vol. 82, p. 124012, 2010.

[82] N. Sago, L. Barack, and S. Detweiler, "Two approaches for the gravitational self-force in black hole spacetime: Comparison of numerical results," *Phys. Rev. D*, vol. 78, p. 124024, Dec 2008.

[83] A. G. Shah, T. S. Keidl, J. L. Friedman, D.-H. Kim, and L. R. Price, "Conservative, gravitational self-force for a particle in circular orbit around a Schwarzschild black hole in a Radiation Gauge," *Phys. Rev. D*, vol. 83, p. 064018, 2011.

[84] L. Barack, A. Ori, and N. Sago, "Frequency-domain calculation of the self force: The High-frequency problem and its resolution," *Phys. Rev. D*, vol. 78, p. 084021, 2008.

[85] N. Sago, "Gravitational self-force effects on a point mass moving around a Schwarzschild black hole," *Class. Quant. Grav.*, vol. 26, p. 094025, 2009.

[86] S. E. Field, J. S. Hesthaven, and S. R. Lau, "Discontinuous Galerkin method for computing gravitational waveforms from extreme mass ratio binaries," *Class. Quant. Grav.*, vol. 26, p. 165010, 2009.

[87] S. Hopper and C. R. Evans, "Gravitational perturbations and metric reconstruction: Method of extended homogeneous solutions applied to eccentric orbits on a Schwarzschild black hole," *Phys. Rev. D*, vol. 82, p. 084010, 2010.

[88] S. Hopper and C. R. Evans, "Metric perturbations from eccentric orbits on a Schwarzschild black hole: I. Odd-parity Regge-Wheeler to Lorenz gauge transformation and two new methods to circumvent the Gibbs phenomenon," *Phys. Rev. D*, vol. 87, no. 6, p. 064008, 2013.

[89] S. Akcay, N. Warburton, and L. Barack, " Frequency-domain algorithm for the Lorenz-gauge gravitational self-force," *Phys. Rev. D*, vol. 88, p. 104009, Nov. 2013.

[90] T. Osburn, E. Forseth, C. R. Evans, and S. Hopper, "Lorenz gauge gravitational self-force calculations of eccentric binaries using a frequency domain procedure," *Phys. Rev. D*, vol. 90, no. 10, p. 104031, 2014.

[91] N. Warburton, S. Akcay, L. Barack, J. R. Gair, and N. Sago, "Evolution of inspiral orbits around a Schwarzschild black hole," *Phys. Rev. D*, vol. 85, p. 061501, 2012.

[92] A. G. Shah, J. L. Friedman, and T. S. Keidl, "EMRI corrections to the angular velocity and redshift factor of a mass in circular orbit about a Kerr black hole," *Phys. Rev. D*, vol. 86, p. 084059, 2012.

[93] P. L. Chrzanowski, "Vector potential and metric perturbations of a rotating black hole," *Phys. Rev. D*, vol. 11, pp. 2042–2062, Apr 1975.

[94] L. S. Kegeles and J. M. Cohen, "Constructive procedure for perturbations of spacetimes," *Phys. Rev. D*, vol. 19, pp. 1641–1664, Mar 1979.

[95] S. A. Hughes, "Evolution of circular, nonequatorial orbits of Kerr black holes due to gravitational-wave emission," *Phys. Rev. D*, vol. 61, p. 084004, Apr. 2000.

[96] M. van de Meent, "Resonantly enhanced kicks from equatorial small mass-ratio inspirals," *Phys. Rev. D*, vol. 90, p. 044027, 2014.

[97] L. Barack, "Gravitational self force in extreme mass-ratio inspirals," *Class. Quant. Grav.*, vol. 26, p. 213001, 2009.

[98] R. M. Wald, "On perturbations of a Kerr black hole," *Journal of Mathematical Physics*, vol. 14, no. 10, 1973.

[99] C. Merlin, A. Ori, A. Pound, L. Barack, and M. van de Meent, "Completion of metric reconstruction for a particle orbiting a Kerr black-hole," *in preparation*, 2015.

[100] L. R. Price, *Developments in the perturbation theory of algebraically special spacetimes.* PhD thesis, University of Florida, 2007.

[101] Y. Sano and H. Tagoshi, "Gravitational field of a Schwarzschild black hole and a rotating mass ring," 2014.

[102] Y. Sano and H. Tagoshi, "Gravitational perturbation induced by a rotating ring around a Kerr black hole," 2014.

[103] R. M. Wald, *General Relativity.* University Of Chicago Press, 1st ed., June 1984.

[104] S. Mano, H. Suzuki, and E. Takasugi, "Analytic Solutions of the Teukolsky Equation and Their Low Frequency Expansions," *Prog. Theor. Phys.*, vol. 95, no. 6, p. 1079, 1996.

[105] E. Newman and R. Penrose, "An approach to gravitational radiation by a method of spin coefficients," *Journal of Mathematical Physics*, vol. 3, no. 3, pp. 566–578, 1962.

[106] S. Chandrasekhar, *The Mathematical Theory of Black Holes (Oxford Classic Texts in the Physical Sciences).* Oxford University Press, USA, Nov. 1998.

[107] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* 1973.

[108] S. Drasco and C. Cutler, "One black hole is not like the other: binary black holes with extreme mass ratios." http://www.tapir.caltech.edu/ sdrasco/animations/index.html.

[109] S. A. Teukolsky, "Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations," *Phys. Rev. Lett.*, vol. 29, pp. 1114–1118, Oct 1972.

[110] J. M. Cohen and L. S. Kegeles, "Electromagnetic fields in curved spaces: A constructive procedure," *Phys. Rev. D*, vol. 10, pp. 1070–1084, Aug 1974.

[111] J. Cohen and L. Kegeles, "Space-time perturbations," *Physics Letters A*, vol. 54, no. 1, pp. 5 – 7, 1975.

[112] B. Whiting and L. Price, "Metric reconstruction from Weyl scalars," *Class. Quant. Grav.*, vol. 22, pp. S589–S604, 2005.

[113] S. A. Hughes, "Computing radiation from Kerr black holes: Generalization of the Sasaki-Nakamura equation," *Phys. Rev. D*, vol. 62, p. 044029, Jul 2000.

[114] M. Sasaki and T. Nakamura, "A class of new perturbation equations for the Kerr geometry," *Physics Letters A*, vol. 89, no. 2, pp. 68 – 70, 1982.

[115] A. G. Shah, J. L. Friedman, and B. F. Whiting, "Finding high-order analytic post-Newtonian parameters from a high-precision numerical self-force calculation," *Phys. Rev. D*, vol. 89, p. 064042, 2014.

[116] S. Mano and E. Takasugi, " Analytic solutions of the Teukolsky equation and their properties," *Prog. Theor. Phys.*, vol. 97, pp. 213–232, 1997.

[117] S. L. Detweiler and E. Poisson, "Low multipole contributions to the gravitational selfforce," *Phys. Rev. D*, vol. 69, p. 084019, 2004.

[118] S. L. Detweiler, "A Consequence of the gravitational self-force for circular orbits of the Schwarzschild geometry," *Phys. Rev. D*, vol. 77, p. 124026, 2008.

[119] A. Pound, "Self-consistent gravitational self-force," *Phys. Rev. D*, vol. 81, p. 024023, 2010.

[120] A. Pound, "Singular perturbation techniques in the gravitational self-force problem," *Phys. Rev. D*, vol. 81, p. 124009, 2010.

[121] A. Pound, "Second-order gravitational self-force," *Phys. Rev. Lett.*, vol. 109, p. 051101, 2012.

[122] A. Pound, "Nonlinear gravitational self-force. I. Field outside a small body," *Phys. Rev. D*, vol. 86, p. 084019, 2012.

[123] S. E. Gralla, "Gauge and Averaging in Gravitational Self-force," *Phys. Rev. D*, vol. 84, p. 084050, 2011.

[124] L. Barack, "Gravitational self force by mode sum regularization," *Phys. Rev. D*, vol. 64, p. 084021, 2001.

[125] L. Barack and A. Ori, "Regularization parameters for the self force in Schwarzschild spacetime. 2. Gravitational and electromagnetic cases," *Phys. Rev. D*, vol. 67, p. 024029, 2003.

[126] S. E. Gralla and R. M. Wald, "A Note on the Coordinate Freedom in Describing the Motion of Particles in General Relativity," *Class.Quant.Grav.*, vol. 28, p. 177001, 2011.

[127] T. Hinderer and E. E. Flanagan, "Two timescale analysis of extreme mass ratio inspirals in Kerr. I. Orbital Motion," *Phys. Rev. D*, vol. 78, p. 064028, 2008.

[128] L. Barack and N. Sago, "Beyond the geodesic approximation: conservative effects of the gravitational self-force in eccentric orbits around a Schwarzschild black hole," *Phys. Rev. D*, vol. 83, p. 084023, 2011.

[129] S. Akcay, A. L. Tiec, L. Barack, N. Sago, and N. Warburton, "Comparison Between Self-Force and Post-Newtonian Dynamics: Beyond Circular Orbits," 2015.

[130] L. Barack and N. Sago, "Gravitational self-force correction to the innermost stable circular orbit of a Schwarzschild black hole," *Phys.Rev.Lett.*, vol. 102, p. 191101, 2009.

[131] S. Akcay, L. Barack, T. Damour, and N. Sago, "Gravitational self-force and the effective-one-body formalism between the innermost stable circular orbit and the light ring," *Phys.Rev.*, vol. D86, p. 104041, 2012.

[132] S. Isoyama, L. Barack, S. R. Dolan, A. Le Tiec, H. Nakano, *et al.*, "Gravitational Self-Force Correction to the Innermost Stable Circular Equatorial Orbit of a Kerr Black Hole," *Phys. Rev. Lett.*, vol. 113, no. 16, p. 161101, 2014.

[133] A. Le Tiec, A. H. Mroue, L. Barack, A. Buonanno, H. P. Pfeiffer, *et al.*, "Periastron Advance in Black Hole Binaries," *Phys. Rev. Lett.*, vol. 107, p. 141101, 2011.

[134] A. I. Harte, "Mechanics of extended masses in general relativity," *Class. Quant. Grav.*, vol. 29, p. 055012, 2012.

[135] D. Bini and T. Damour, "Two-body gravitational spin-orbit interaction at linear order in the mass ratio," *Phys. Rev. D*, vol. 90, no. 2, p. 024039, 2014.

[136] P. Nolan, C. Kavanagh, S. R. Dolan, A. C. Ottewill, N. Warburton, *et al.*, "Octupolar invariants for compact binaries on quasi-circular orbits," 2015.

[137] C. Merlin, “Gravitational self-force from curvature scalars.” 16th Capra Meeting on Radiation Reaction in General Relativity.

[138] E. Poisson, A. Pound, and I. Vega, “The motion of point particles in curved spacetime,” *ArXiv e-prints*, Feb. 2011.

[139] L. Barack and A. Ori, “Regularization parameters for the self-force in Schwarzschild spacetime: Scalar case,” *Phys. Rev. D*, vol. 66, p. 084022, Oct 2002.

[140] Press, William H. and Flannery, Brian P. and Teukolsky, Saul A. and Vetterling, William T., *Numerical recipes in Fortran 77 : the art of scientific computing.*

[141] L. Barack and A. Ori, “Late time decay of gravitational and electromagnetic perturbations along the event horizon,” *Phys. Rev. D*, vol. 60, p. 124005, 1999.

[142] E. Poisson, “Gravitational radiation from infall into a black hole: Regularization of the Teukolsky equation,” *Phys. Rev. D*, vol. 55, pp. 639–649, 1997.

[143] T. S. Keidl, J. L. Friedman, and A. G. Wiseman, “Finding fields and self-force in a gauge appropriate to separable wave equations,” *Phys. Rev. D*, vol. 75, p. 124009, June 2007.

[144] C. Gundlach, S. Akcay, L. Barack, and A. Nagar, “ Critical phenomena at the threshold of immediate merger in binary black hole systems: The extreme mass ratio case,” *Phys. Rev. D*, vol. 86, p. 084022, Oct. 2012.

[145] M. van de Meent and A. G. Shah, “Metric perturbations produced by eccentric equatorial orbits around a Kerr black hole,” 2015.

[146] L. Abbott and S. Deser, “Stability of gravity with a cosmological constant,” *Nuclear Physics B*, vol. 195, no. 1, pp. 76 – 96, 1982.

[147] L. Barack, “Late time decay of scalar, electromagnetic, and gravitational perturbations outside rotating black holes,” *Phys. Rev. D*, vol. 61, p. 024026, 2000.

[148] W. Schmidt, “Celestial mechanics in Kerr space-time,” *Class.Quant.Grav.*, vol. 19, p. 2743, 2002.

[149] S. E. Gralla, “Second Order Gravitational Self Force,” *Phys. Rev. D*, vol. 85, p. 124011, 2012.

[150] W. B. Campbell and T. Morgan, *Debye Potentials for the Gravitational Field.* Orange aid preprint series in nuclear, atomic & relativistic astrophysics, 1970.

[151] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* New York: Dover, ninth dover printing, tenth gpo printing ed., 1964.