IMPROVED DIRECT ESTIMATORS FOR SMALL AREAS

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ABSTRACT

Unbiased direct estimators for small area quantities are usually considered too variable to be of any practical use. In this paper we propose a class of model-based direct estimators for small area quantities that appears to overcome this objection, in the sense that these estimators are comparable in efficiency to the indirect model-based small area estimators (e.g. empirical best linear unbiased predictors, or EBLUPs) that are now widely used. There are many practical advantages associated with such model-based direct (MBD) estimators, arising from the fact that they are computed as weighted linear combinations of the actual sample data from the small areas of interest. Note that in this case the weights 'borrow strength' via a model that explicitly allows for small area effects. One particular advantage that we explore in this paper is that estimation of mean squared error (MSE) is then straightforward, using well-known methods that are in common use for population level estimates. Empirical results reported in this paper show that the MBD estimator represents a real alternative to the EBLUP, with the simple MSE estimator associated with the MBD estimator providing good coverage performance. We also report results that indicate that the MBD estimator may be more robust than the EBLUP when the small area model is incorrectly specified. Furthermore, the MBD approach is easily extended to provide multi-purpose weights that are efficient across a range of variables, including variables that are unsuitable for EBLUP, e.g. variables that contain a significant proportion of zeros.

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ABSTRACT

Unbiased direct estimators for small area quantities are usually considered too variable to be of any practical use. In this paper we propose a class of model-based direct estimators for small area quantities that appears to overcome this objection, in the sense that these estimators are comparable in efficiency to the indirect model-based small area estimators (e.g. empirical best linear unbiased predictors, or EBLUPs) that are now widely used. There are many practical advantages associated with such model-based direct (MBD) estimators, arising from the fact that they are computed as weighted linear combinations of the actual sample data from the small areas of interest. Note that in this case the weights 'borrow strength' via a model that explicitly allows for small area effects. One particular advantage that we explore in this paper is that estimation of mean squared error (MSE) is then straightforward, using well-known methods that are in common use for population level estimates. Empirical results reported in this paper show that the MBD estimator represents a real alternative to the EBLUP, with the simple MSE estimator associated with the MBD estimator providing good coverage performance. We also report results that indicate that the MBD estimator may be more robust than the EBLUP when the small area model is incorrectly specified. Furthermore, the MBD approach is easily extended to provide multi-purpose weights that are efficient across a range of variables, including variables that are unsuitable for EBLUP, e.g. variables that contain a significant proportion of zeros.

Key Words: Small Area Estimation; Model-based estimation; Multipurpose sample weights; MSE estimation, Mixed model, EBLUP.

1. Introduction

The dominant paradigm in survey estimation for populations is weighted linear estimation, typically based on linear regression models, while the rapidly expanding field of small area estimation is currently dominated by a model-based predictive approach (EBLUP) where the survey weights have little or no relevance. See Rao (2003). Many of the practical advantages of weighted linear estimation are lost when one adopts EBLUP. Perhaps the most important of these are the simplicity of both the estimation process and estimation of mean squared error, and the fact that one can use multi-purpose weights for straightforward analysis of survey data sets that contain many variables (Chambers, 1996). A further advantage is that calibration constraints are readily included in an estimation method that uses weights, allowing survey analysts who prefer a design-based approach to inference to obtain estimates that have good design-based properties (Hidiroglou *et al*, 2000).

In the following section we review the use of regression-based survey weighting for population level quantities. In Section 3 we discuss issues that arise when survey weights that also reflect small area or local characteristics are required. Section 4 introduces survey weights based on the linear mixed model used in many small area estimation applications. These weights lead naturally to the model-based direct estimator (MBD) for small areas, which is then contrasted with the EBLUP under the same model. In section 5 we provide illustrative empirical results that compare the EBLUP and MBD approaches. Finally, in Section 6 we discuss some important issues that arise when a weighting approach is used in small area estimation and identify related topics that require further attention.

2. Regression-Based Sample Weighting for Population Estimation

In this section we briefly review regression-based sample weighting for estimation of population level quantities. To start, we fix our notation. Let Y_U denote an N-vector of population values of a characteristic of interest, and suppose that our primary aim is estimation of the total T_y of the values in Y_U (or their mean M_y). In order to assist us in this objective, we shall assume that we have 'access' to X_U, an $N \times p$ matrix of values of p auxiliary variables that are related, in some sense, to the values in Y_U. In particular, we assume that the individual sample values in X_U are known. The non-sample values in X_U may not be individually known, but are assumed known at some aggregate level. At a minimum, we know the population totals T_x of the columns of X_U. Given this set up, it is standard to estimate the total and mean of the values in Y_U by

$$\hat{T}_{wy} = \sum_s w_i y_i \tag{1}$$

and

$$\hat{M}_{wy} = \sum_s w_i y_i / \sum_s w_i \tag{2}$$

respectively. Here s is a sample of size n from a population of size N and the weights $\{w_i; i \in s\}$ are $O(Nn^{-1})$. Many survey applications require weights that are calibrated on X, in the sense that they exactly reproduce the known population totals defined by the columns of X_U, i.e.

$$\sum_s w_i x_i = \hat{T}_{wx} = T_x . \tag{3}$$

Weights that satisfy (3) can be constructed under the assumption that Y_U and X_U are related by the linear regression model

$$Y_U = X_U \beta + \varepsilon_U \tag{4}$$

where ε_U is random error vector of dimension N with $E(\varepsilon_U) = 0$ and $Var(\varepsilon_U) = \sigma^2 V$, where V is a known positive definite matrix of order N. Without loss of generality, we arrange the vector Y_U so that its first n elements correspond to the sample units. We can then conformably partition Y_U, X_U and V according to sample and non-sample units as

$$Y_U = \begin{bmatrix} Y_s \\ Y_r \end{bmatrix}, \; X_U = \begin{bmatrix} X_s \\ X_r \end{bmatrix} \text{ and } V = \begin{bmatrix} V_{ss} & V_{sr} \\ V_{rs} & V_{rr} \end{bmatrix}.$$

Here Y_s is the $n \times 1$ vector defined by the sample values in Y_U, X_s is the corresponding $n \times p$ matrix of sample values of the auxiliary variable and V_{ss} is the $n \times n$ component of V associated with Y_s. A subscript of r is used to denote corresponding quantities defined by the $N - n$ non-sample units, e.g. V_{rs} is the $(N-n) \times n$ matrix defined by $Cov(Y_r, Y_s) = \sigma^2 V_{rs}$.

Given this set-up, and assuming (4) holds, the Best Linear Unbiased Predictor (BLUP) of the population total of Y is given by (1) with weights defined by

$$w_{BLUP} = 1_n + H'\left(X_U' 1_N - X_s' 1_n\right) + \left(I_n - H' X_s'\right) V_{ss}^{-1} V_{sr} 1_{N-n} \tag{5}$$

where I_n is the identity matrix of order n, 1_N, 1_n, 1_r are vectors of one's with dimensions N, n and N - n respectively, and $H = \left(X_s' V_{ss}^{-1} X_s\right)^{-1} X_s' V_{ss}^{-1}$. See Royall (1976).

It is easy to see that the BLUP weights (5) are calibrated on the variables defining the columns of X_U, i.e. $X_s' w_{BLUP} = X_U' 1_N = T_x$. Furthermore, this calibration property is equivalent to unbiased prediction under the linear regression model (4), since for any vector of weights w that satisfies the calibration constraints (3) we have

$$E(\hat{T}_{wy} - T_y) = E(w'Y_s - 1_N' Y_U) = E(w'X_s - 1_N' X_U)\beta = 0 .$$

3. Sample Weighting for Small Area Estimation

The primary target of most surveys is estimation of population level quantities, and so sample weights are usually calculated so that they lead to efficient population level inference. We refer to this as population weighting. In particular, small area and individual level variation are assumed to 'average out' over the population, in the sense that if in fact $Y = X\beta + Zu + e$ where $X\beta$ denotes the contribution from population level effects, Zu denotes the contribution from small area effects and e denotes the contribution from individual effects, then $1'X\beta >> 1'(Zu + e)$ so that weights based on the model $y = X\beta + \varepsilon$ (i.e. population weighting) will still give almost unbiased estimates at population level. However, estimation at small area level is typically an increasingly important secondary objective of many sample surveys, and in this context the above argument fails. This is because small area effects do not average out at small area level. For example, using population weights $\{w_i; i \in s\}$ for estimating the mean M_{yj} of the survey variable *Y* in small area *j* via the weighted mean of the survey values in area *j* will be inefficient, maybe even biased. Here s_j denotes the sampled units in small area *j*. This estimator is often referred to as the (weighted) direct estimator of M_{yj}.

An immediate consequence is that some form of local weighting is required if survey weights are used to construct small area estimates, where we define local weighting as weights that reflect the local characteristics of the small areas that make up the population. This requirement is in addition to the calibration constraints typically imposed for population estimation, resulting in more variable sample weights and leading to greater mean squared errors when the resulting small area estimates are aggregated to the population level.

The simplest way to take account of differences in the distribution of *Y* across the *J* small areas of interest is to assume that area effects are constant within a small area. This suggests we extend (4) to

$$Y_j = X_j\beta + Z_j 1_{N_j} + \varepsilon_j \qquad (7)$$

where a subscript of *j* denotes restriction to small area *j*. It is easy to see that unbiased estimation under this model requires weights that are calibrated both on *X* and on the small area population counts N_j. Assuming *X* contains an intercept term, this equates to $p + J - 1$ calibration constraints, i.e. an additional $J - 1$ constraints.

There are two problems with (7). The first is that it implicitly contains the assumption that the relationship between *Y* and *X* is essentially the same in each small area. The second is that *J* is sometimes so large that fitting (7) becomes difficult using the sample data. If we believe that the relationship between *Y* and *X* varies between areas we could consider extending (7) (again assuming *X* contains an intercept term) to

$$Y_j = X_j\beta_j + \varepsilon_j . \qquad (8)$$

This is the small area post-stratification model, and is equivalent to calibrating on *X* at small area, rather than population, level (i.e. *pJ* constraints). It can only be used if we know the area level values of the calibration constraints and is clearly even more problematic than (7) when *J* is large.

However, we can also build small area effects into survey weights by basing them on mixed models. That is, we use the BLUP specification (5), with *V* defined by an appropriate model that allows for the possibility of correlations between individuals, both within small areas and between small areas.

4. Small Area Estimation Based on a Linear Mixed Model

The most commonly used class of models in small area inference is the class of linear mixed models. Let Y_j be the $N_j \times 1$ vector of values of variable of interest in small area *j* and let X_j be the $N_j \times p$ matrix of values of the auxiliary variables associated with. We consider the following specification for the distribution of Y_j given X_j:

$$Y_j = X_j\beta + Z_j u_j + e_j . \qquad (9)$$

Here β is a $p \times 1$ vector of fixed effects, Z_j is a $N_j \times q$ matrix of known covariates characterising differences between the J small areas, u_{j} is a random area effect associated with the j^{th} small area and e_j is a $N_j \times 1$ vector of individual level random errors. The random vectors u_j and e_j are assumed to be independently distributed, with zero means and with variances $Var(u_j) = \Sigma$ and $Var(e_j) = \sigma_e^2 I_{N_j}$ respectively, so that the covariance matrix of Y_j is then $Var(Y_j) = V_j = \sigma_e^2 I_{N_j} + Z_j \Sigma Z_j'$, which depends on a $k \times 1$ vector of parameters θ, and which together with σ_e^2 are usually called the variance components of the model. Finally, it is usually assumed that sampling is uninformative given the values of the auxiliary variables, so the sample data also follow the population model (9).

By aggregating the area-specific models (9) over the J small areas, we are led to the population level model

$$Y = X\beta + Zu + e \tag{10}$$

where $Y = (Y_1', \ldots\ldots, Y_J')'$, $X = (X_1', \ldots\ldots, X_J')'$, $Z = diag(Z_j; 1 \le j \le J)$, $u = (u_1', \ldots\ldots, u_J')'$ and $e = (e_1', \ldots\ldots, e_J')'$. The variance-covariance matrix of Y is $V = diag(V_j; 1 \le j \le J)$. We assume that X has full column rank p. This is the general linear mixed model, which includes most of the small area models used in practice (Rao, 2003, page 107). Again, we consider the decomposition of Y, X, Z and V into sample and non-sample components as mentioned after (4). We use similar notation at the small area level by introducing an extra subscript j to denote small area. For example, we denote by s_j the set of n_j sample units in area j, r_j the corresponding $N_j - n_j$ non-sampled units in the area and put $V_{jss} = \sigma_e^2 I_{n_j} + Z_{js} \Sigma Z_{js}'$ and $V_{jsr} = Z_{js} \Sigma Z_{jr}'$. In practice the variance components that define V are unknown and must be estimated from the sample data using suitable estimation methods such as maximum likelihood (ML), restricted maximum likelihood (REML) or method of moments. We use a 'hat' to denote an estimate and put $\hat{V} = diag(\hat{V}_j; 1 \le j \le J)$, with $\hat{V}_j = \hat{\sigma}_e^2 I_{N_j} + Z_j \hat{\Sigma} Z_j'$.

Given this notation, and assuming (9) holds, we first note that the EBLUP for the j^{th} small area mean M_{yj} is

$$\hat{M}_{yj}^{EBLUP} = f_j \bar{Y}_{js} + (1 - f_j)[\bar{X}_{jr}' \hat{\beta} + \bar{Z}_{jr}' \hat{\Sigma}\, Z_{js}' \hat{V}_{jss}^{-1} (Y_{js} - X_{js} \hat{\beta})] \tag{11}$$

where $f_j = n_j / N_j$ and $\bar{X}_{jr}$ and $\bar{Z}_{jr}$ are vectors of means for the $N_j - n_j$ non-sampled units in small area *j*. An approximately unbiased estimator of the MSE of (11) is

$$v(\hat{M}_{yj}^{EBLUP}) = (1-f_j)^2 \left[g_{1j}(\hat{\theta}) + g_{2j}(\hat{\theta}) + 2g_{3j}(\hat{\theta}) \right] + N_j^{-1}(1-f_j)\hat{\sigma}_e^2 \qquad (12)$$

where

$$g_{1j}(\hat{\theta}) = \bar{Z}'_{jr}\left(\hat{\Sigma} - \hat{\Sigma}Z'_{js}\hat{V}_{jss}^{-1}Z_{js}\hat{\Sigma}\right)\bar{Z}_{jr},$$

$$g_{2j}(\hat{\theta}) = \left(\bar{X}'_{jr} - b'_j X_{js}\right)\left(\sum_j X'_{js}\hat{V}_{jss}^{-1}X_{js}\right)^{-1}\left(\bar{X}'_{jr} - b'_j X_{js}\right)'$$

$$g_{3j}(\hat{\theta}) = tr\left\{\left(\nabla b'_j\right)\hat{V}_{jss}\left(\nabla b_j\right)v(\hat{\theta})\right\}$$

with $b'_j = \bar{Z}'_{jr}\hat{\Sigma}Z'_{js}\hat{V}_{jss}^{-1}$, $\nabla b'_j = \partial b'_j / \partial\theta$ and where $v(\hat{\theta})$ is the estimate of the asymptotic covariance matrix of $\hat{\theta}$ defined by the inverse of the relevant observed information matrix. See Prasad and Rao (1990) and Rao (2003, pp. 107-110).

In contrast, under the population level linear mixed model (10), the sample weights that define the EBLUP for the population total of *Y* are

$$w_{EBLUP} = 1_n + \hat{H}'\left(X'1_N - X'_s 1_n\right) + \left(I_n - \hat{H}'X'_s\right)\hat{V}_{ss}^{-1}\hat{V}_{sr}1_r \qquad (13)$$

where $\hat{H} = \left(X'_s\hat{V}_{ss}^{-1}X_s\right)^{-1}X'_s\hat{V}_{ss}^{-1} = \left(\sum_j X'_{js}\hat{V}_{jss}^{-1}X_{js}\right)^{-1}\left(\sum_j X'_{js}\,\hat{V}_{jss}^{-1}\right)$. It is easy to see that these 'EBLUP' weights are the empirical version of the BLUP weights (5) under (10). Furthermore, since they only depend on the random area effects structure of the mixed model (10) via the covariance structure in the sample/population, extension to more complex covariance structures (e.g. spatial correlation between population units) only requires $\hat{V}_{ss}^{-1}$ and $\hat{V}_{sr}$ to be computed under these more complex models. We do not pursue this extension in this paper however.

The model-based direct (MBD) estimator of the j^{th} small area mean M_{yj} is the direct estimator of this quantity based on the EBLUP weights (13). That is, it is defined as

$$\hat{M}_{yj}^{MBD} = \sum_{s_j} w_i y_i \Big/ \sum_{s_j} w_i \qquad (14)$$

where the weights used in (14) are those associated with the sample units in small area *j* in (13). Note that we refer to (14) as a direct estimator because it is a weighted mean of the sample data from the small area of interest. However, this does not mean that it can be calculated just using these data. The EBLUP sample weights (13) will be a function of the data from the entire sample. That is, they ‘borrow strength’ from other areas through the model (10). Another important point that needs be made at this stage is that the MBD estimator (14) is not the same as EBLUP (11), even though both sum to the same population level EBLUP. This is because there is no unique representation of (11) as a weighted mean of the sample data values from small area *j*.

An important consideration in small area estimation is estimation of the mean squared error (MSE) of the small area estimator. We can easily adapt straightforward methods of MSE estimation for population level estimators to estimation of the MSE of (14). To start, observe that when small area effects are part of the mean structure of a linear model for *Y*, e.g. via fixed area effects, see (8) and (9), MSE estimation is relatively straightforward. Well known results indicate that robust model-based methods as well as appropriately conditioned design-based methods lead to MSE estimators $v(\hat{M}_y) = \sum_s w_i^2 (y_i - \hat{y}_i)^2 + \text{lower order terms}$, where $\hat{y}_i$ denotes the fitted value for y_i under the linear model implied by the calibration constraints.

In order to estimate the mean squared error of (15), we note that the implied population level model (10) includes random area effects and so one needs to consider whether it is appropriate to condition on these effects when estimating this MSE. For example, the rather complicated MSE estimator (12) of the EBLUP does involve this conditioning. On the other hand, estimation of the MSE of (15) is straightforward if we do not condition on random area effects, treat the EBLUP weights (13) as fixed and use standard methods for estimating the MSE of a weighted linear estimator of a domain mean under the population model (4). See Royall and Cumberland (1978). The choice between these two approaches is largely philosophical and depends on how much one ‘believes’ the linear mixed model (10). In particular, in this paper we treat this model as a vehicle for generating estimation weights, but then base inference on (4), which is consistent with the way mean squared errors are estimated at population level. Thus, we write down a first order approximation to prediction variance for the area *j* weighted mean (14) as

$$Var(\hat{M}_{yj}^{MBD} - M_{yj}) = Var\left\{ \left(\sum_{s_j} w_i \right)^{-1} \left(\sum_{s_j} w_i y_i \right) - N_j^{-1} \left(\sum_{s_j} y_i + \sum_{r_j} y_i \right) \right\}$$

$$\approx N_j^{-2} \left(\sum_{s_j} a_i^2 Var(y_i) + \sum_{r_j} Var(y_i) \right) \qquad (15)$$

where $a_i = \left(\sum_{s_j} w_k\right)^{-1}\left(N_j w_i - \sum_{s_j} w_k\right)$. A robust model-based estimate of (15) is obtained by substituting the squared residual $(y_i - x_i'\hat{\beta})^2$ for $Var(y_i)$ in the first (leading) term on the right hand side of (15). If these squared sample residuals are also used to estimate the second term, the resulting estimator of (15) is

$$v(\hat{M}_{yj}^{MBD}) = \sum_{s_j} \lambda_i (y_i - x_i'\hat{\beta})^2 \qquad (16)$$

where $\lambda_i = N_j^{-2}\left(a_i^2 + (N_j - n_j)/(n_j - 1)\right)$. Using (16) to estimate the prediction mean squared error of $\hat{M}_{yj}^{MBD}$ implicitly assumes that this weighted mean is unbiased for M_{yj}. However, this is not generally the case, since $E(\hat{M}_{yj}^{MBD} - M_{yj}) \approx (\hat{M}_{xj}^{MBD} - M_{xj})'\beta$ under (10), where $\hat{M}_{xj}^{MBD}$ denotes the weighted average of the sample values of the auxiliary variables in area *j*. Calibration on *X* ensures that this term vanishes at population level, but not necessarily at small area level. A simple estimate of this bias is

$$b(\hat{M}_{yj}^{MBD}) = (\hat{M}_{xj}^{MBD} - M_{xj})'\hat{\beta}. \qquad (17)$$

Our suggested estimator of the mean squared error of (14) is therefore

$$m\hat{s}e(\hat{M}_{yj}^{MBD}) = v(\hat{M}_{yj}^{MBD}) + \left(b(\hat{M}_{yj}^{MBD})\right)^2 \qquad (18)$$

Note that one could alternatively 'bias correct' $\hat{M}_{yj}^{MBD}$ directly using $b(\hat{M}_{yj}^{MBD})$. However, this is not recommended since this correction increases the variability of our estimator much more than it reduces its bias. Using it in (18) is a more conservative, and safer, approach.

Like the EBLUP (11), the EBLUP weights (13) are variable specific since they depend on the estimated variance components for *Y* via the matrices $\hat{V}_{sr}$ and $\hat{V}_{ss}$. This can be a limitation if a true 'multipurpose' approach to small area estimation is required. In the context of weighted linear estimation via (14), this translates into the use of the same sample weights across a wide range of variable types. In this paper we investigate two approaches to deriving multi-purpose weights based on (13), the first based on averaging the variance components associated with a select group of variables and the second based on averaging the sample weights (13) generated for these variables. We also investigated a third approach based on averaging the intra-area correlations associated with these variables. However, this led to rather unstable results, and so was not pursued further.

In what follows we use a subscript of k to index the group of K variables that define the multipurpose weights. In our first approach, we average the estimated covariance matrices $\hat{V}_{k,j}$ for each variable and each small area

$$\bar{V}_j = \frac{1}{K}\sum_{k=1}^{K}\hat{V}_{k,j} = \frac{1}{K}\sum_{k=1}^{K}\left(\hat{\sigma}_{e,k}^2 I_{N_j} + Z_{k,j}\hat{\Sigma}_k Z'_{k,j}\right).$$

The corresponding multipurpose version of the EBLUP sample weights (13) is then

$$w_{EBLUP}^{(I)} = 1_n + \bar{H}'(X'1_N - X'_s 1_n) + (I_n - \bar{H}'X'_s)\bar{V}_{ss}^{-1}\bar{V}_{sr}1_r \tag{19}$$

where $\bar{H} = \left(\sum_j X'_{js}\bar{V}_{jss}^{-1}X_{js}\right)^{-1}\left(\sum_j X'_{js}\bar{V}_{jss}^{-1}\right)$ and $\bar{V}_{jss}$, $\bar{V}_{jsr}$ are defined by the sample/non-sample decomposition of $\bar{V}_j$. Our second approach simply defines the multipurpose weights as the average of the variable specific weights (13) across the group of K variables. That is

$$w_{EBLUP}^{(II)} = \frac{1}{K}\sum_{k=1}^{K} w_{k,EBLUP}\ . \tag{20}$$

Under either (19) or (20), the MBD estimator (14) of the j^{th} small area mean for a variable of interest Y is then calculated using these multi-purpose sample weights. Similarly, when using (18) to estimate the MSE of this estimator we use these weights to define a_i (and hence λ_i) in (16). Note, however, that implementation of this formula requires calculation of $\hat{\beta}$, which depends on the particular variable of interest. Under (19) we have the option of either using the 'average' $\bar{V}_{jss}$ in this calculation or using the actual $\hat{V}_{jss}$ for this variable. For (20), there is no alternative but to use a variable specific $\hat{\beta}$. The empirical investigations reported in the next section indicated that there was almost no difference in MSE estimation performance for the MBD estimator defined by (19) depending on which of these alternative ways of defining $\hat{\beta}$ was used. Our empirical study therefore used variable specific values of $\hat{\beta}$ to define the residuals underpinning MSE estimation for the MBD estimators based on both (19) and (20).

The MBD estimator (14) is easy to interpret and to build into a survey processing system. Furthermore, its mean squared error is easily estimated via a straightforward generalisation of the standard robust estimator of the mean squared error of the EBLUP for the population mean of Y. This is in contrast to the rather complicated estimator (12) of the conditional prediction variance of the area j EBLUP (11). However, this does not mean that the MBD estimator (14) is superior to the EBLUP (11). As noted earlier, both (11) and (14) sum to the population EBLUP under the linear mixed model (10). Furthermore, under this model it is clear that the EBLUP must be more efficient asymptotically, since it approximates the best linear predictor when (10) actually holds. For example, in the special case where $X = Z = 1_N$, the weight associated with sampled unit *i* in area *j* under the MBD approach is

$$w_i = \frac{N}{n}\left\{1 + \frac{1}{1+n_j\hat{\phi}}\left[(N_j - n_j)\hat{\phi} + \frac{\bar{N}-\bar{n}}{\bar{n}}\right]\right\}$$

where $\hat{\phi} = \hat{\Sigma}/\hat{\sigma}_e^2$, $\bar{N} = \sum_j N_j(1+n_j\hat{\phi})^{-1} / \sum_j (1+n_j\hat{\phi})^{-1}$ and $\bar{n}$ is defined similarly. That is, (14) reduces to the area *j* sample mean, which is well known to have high variability in small samples. In contrast, (11) is then a linear combination of the overall sample mean and the area *j* sample mean, and has much less variability. In the next section we provide some simulation results that illustrate the loss of efficiency when the linear mixed model (9) holds for the small areas of interest and the MBD rather than the EBLUP is used to predict the small area means.

It is sometimes claimed that a disadvantage of any direct estimator (including the MBD estimator) is that it is not defined when there is no sample in small area *j*. In contrast, the EBLUP (11) then equals the synthetic estimator $M'_{xj}\hat{\beta}$. However, no sample data in an area also means that the validity of any estimator for that area is completely model-dependent. In particular, we cannot check to see if (9) holds. There is also the problem that different areas are then treated unequally in estimation. Areas with sample data have their means estimated via EBLUP, while those without have their means estimated via synthetic estimators. Furthermore, in such a case the weighted average of these estimates across all small areas does not equal the EBLUP of the population mean (a property of the MBD estimators). A standard work-around when this occurs is to rescale all the small area estimates to sum to this population estimate (or some other acceptable value). However, this is rather arbitrary. For example, if most of the small areas have no sample, then such a rescaling exercise could substantially change the final predicted value of the area *j* mean of *Y* for a ‘sample area’ relative to its EBLUP value (11), in which case one has to wonder about the efficiency of the final result.

5. Some Empirical Results

In this section we illustrate the performance of small area estimation based on the MBD approach via design-based simulation. Our basic data come from the same sample of 1652

Australian broadacre farms that were used in the simulation study reported in Chambers (1996). Here however we used these sample farms to generate a target population of 81982 farms by sampling with replacement from them with probabilities proportional to their sample weights. We then drew 1000 independent stratified random samples from this (fixed) population, with total sample size in each simulation equal to the original sample size (1652) and with strata defined by the 29 different Australian broadacre agricultural regions. Sample sizes within these strata were fixed to be the same as in the original sample. Note that these varied from a low of 6 to a high of 117, allowing an evaluation of the performance of different small area estimation methods across a range of realistic small area sample sizes. Table 1 shows the stratum population and sample sizes for this population.

We considered the 29 regions as small areas, with 8 variables of interest. These are (i) TCC = total cash costs (A$) of the farm business over the surveyed year, (ii) TCR = total cash receipts (A$) of the farm business over the surveyed year, (iii) FCI = farm cash income (A$), defined as TCR – TCC, (iv) Crops = area under crops (in hectares), (v) Cattle = number of beef cattle on the farm, (vi) Sheep = number of sheep on the farm, (vii) Equity = total farm equity (A$), and (viii) Debt = total farm debt (A$). Our aim was to estimate the average of these variables in each of the 29 different regions. In doing so, we used the fact that these regions can be grouped into three zones (Pastoral, Mixed Farming, and Coastal), with farm area (hectares) known for each farm in the population. This auxiliary variable is referred to as Size in what follows.

Although the linear relationship between the eight target variables and Size is rather weak in the original sample data, this improves when separate linear models are fitted within six post strata. These post-strata are defined by splitting each zone into small farms (farm area less than zone median) and large farms (farm area greater than or equal to zone median). The matrix X of auxiliary variable values in (10) was then defined so as to include an effect for Size, effects for the post-strata and effects for interactions between Size and the post strata. Two different specification for X (corresponding to whether an intercept was included or not) and two different specifications for Z (corresponding to whether a random slope on Size was included or not) were then used to specify (10) and hence the EBLUP and MBD estimators based on this model. These four specifications are set out in Table 2.

For the farm data, models I and II are appropriate (with II fitting marginally better) while models III and IV are badly specified. We use REML estimates of random effects parameters throughout, obtained via the *lme* function in R (Bates and Pinheiro, 1998). For each model, four different estimators of the 29 regional means were computed, along with corresponding estimators of their mean squared error. These were the EBLUP (11) with MSE estimator (12), referred to as EBLUP below; the MBD estimator (14) based on variable specific weights (13) and with MSE estimator (18), referred to as MBD0 below; the MBD estimator (14) based on multipurpose weights (19) and with MSE estimator (18), referred to as MBD1 below; and the MBD estimator (14) based on multipurpose weights (20) and with MSE estimator (18), referred to as MBD2 below. Note that three of the eight target variables in the study (Crops, Equity and Debt) were not suited to linear modelling via (10) because of large numbers of zeros, so the weights used in MBD1 and MBD2 were based on the $K = 5$ remaining variables (TCC, TCR, FCI, Cattle and Sheep).

The simulation study was carried out in two stages. In the first, we contrasted the performance of MBD0 with EBLUP under models I to IV using TCC as the variable of interest. Results from this stage are set out in Table 3 and in Figures 1 – 3. In the second stage of the study we investigated the performance all four methods for all eight response variables under the 'reasonably specified' models I and II. Results from this stage are set out in Tables 4 – 6 and in Figures 4 – 5.

Three measures of estimation performance were computed using the estimates generated in the simulation study. These were the relative mean error and the relative root mean squared error (RMSE), both expressed as percentages, of regional mean estimates and the coverage rate of nominal 95 per cent confidence intervals for regional means. Table 3 presents the average and median values of these measures (all computed over the 29 regions) generated by EBLUP and MBD0 under models I – IV for the variable TCC. We note that the average relative mean errors under MBD0 are smaller than those under EBLUP for all models except model IV. However, the average relative RMSEs for MBD0 are marginally higher than those for EBLUP under models I and II and smaller for models III and IV. Average coverage rates for MBD0 are relatively higher than those for EBLUP under all models. Although neither dominates, it seems clear that for TCC, MBD0 is more robust to model misspecification than EBLUP.

Figures 1 – 3 show the region-specific performances generated by EBLUP and MBD0 (ordered by increasing population size). Figure 1 shows the better relative mean error performances of both EBLUP and MBD0 under models I and II and their worse relative mean error performance under model IV. Figure 2 shows that the relative RMSEs of regional estimates generated by MBD0 are comparable with those generated under EBLUP, with neither approach dominating. Overall, with the exception of two regions (3 and 21), it seems that MBD0 under model II performs marginally better overall.

In the two regions (3 and 21) where MBD0 fails, inspection of the population and sample data indicated that this is because of a few outlying estimates. In fact, the outlying values of MBD0 for region 21 are all caused by the presence of a single massive outlier (TCC > A$30,000,000) in the original sample. This outlier was included in the simulation population (twice) and then selected (in one case, twice) in 37 of the 1000 simulation samples. If we discard the outlier driven estimates in regions 3 and 21 then the MBD approach seems the method of choice for regional estimation in our simulation study. This is confirmed when we return to Table 3 and now consider the columns containing the median values of relative mean error and relative RMSE.

Figure 3 summarizes region-specific variation in the nominal 95 percent confidence interval coverage rates generated by EBLUP and MBD0. If we ignore the outlier driven results for regions 3 and 21, the results displayed in Figure 5 show that MBD0 approach gives marginally better coverage rates under Models I and II. A close look at these results also indicates that in the event of model misspecification (e.g. under Models III and IV) the MBD0 coverage rate is more robust.

In the second stage of the simulation study, we compared the two variable specific estimators EBLUP and MBD0 with the two multi-purpose estimators MBD1 and MBD2. Table 4 presents the average and median relative mean errors and relative RMSEs, as well as the average coverage rates, generated by these four estimators for the five variables TCC, TCR, FCI, Cattle and Sheep under the 'reasonably specified' Models I and II. These results show that under the better fitting Model II, there is little, if any, difference in the average relative mean errors of the multi-purpose estimators MBD1 and MBD2 compared with the average relative mean error of the variable specific estimator MBD0, with all three often substantially better than EBLUP. Under Model I, the two multipurpose estimators MBD1 and MBD2 are substantially better than MBD0 and EBLUP. In terms of relative RMSE, the results are more equivocal. Under Model I there is little to choose between MBD0, MBD1 and MBD2 in terms of average relative RMSE, with the corresponding performance of EBLUP rather more fragile. When one turns to the better fitting Model II, however, it is clear that the better multipurpose approach is MBD1. By considering median, rather than average, values of relative mean error and relative RMSE, we also see that the estimation performances of the multipurpose estimators MBD1 and MBD2 appear to be more robust than those of the variable specific estimators MBD0 and EBLUP. Finally, we note that the average coverage rates of all three direct estimators are quite similar under both Models I and II and dominate the corresponding average coverage performance of EBLUP. Overall it seems clear that for our data set the multi-purpose estimator MBD1 is the estimator of choice for these five variables.

Figure 4 shows the region-specific relative mean errors, relative RMSEs and coverage rates for TCC under Models I and II for EBLUP, MBD0, MBD1 and MBD2. The superior efficiency of all estimators under Model II (after allowing for the outliers in regions 3 and 21) is evident, as is the superior performance of MBD2. A similar pattern of results was observed for TCR, FCI, Cattle and Sheep.

The unstable performance of EBLUP for the Cattle and Sheep variables in Table 4 is noteworthy. Upon investigation we found that the anomalous results for Cattle were caused by the presence of negative estimates for this variable in two regions (11 and 14), which were themselves the result of zero values in the data. In particular, in region 11 there were 1283 zeros in the simulated population of 1586 values. This resulted in 185 negative estimates out of the 1000 simulated for this region. Similarly in the region 14, there were 1972 zeros in the 2182 values in the simulated population, leading to 354 negative estimates. A similar reason lay behind the EBLUP results for Sheep. In this case, however, in region 3 there were only 11 non-zero values for Sheep in a simulated population of size 189, leading to 223 negative estimates, while in region 18 a majority of zero values for Sheep lead to 323 negative estimates.

As noted earlier, our results indicate that multi-purpose estimation based on MBD1 is preferable to that based on MBD2. Consequently, in Table 5 we contrast the performances of the variable specific estimators EBLUP and MBD0 with the multi-purpose estimator MBD1 for the three variables (Crops, Equity and Debt) that contain a large number of zeros. The superior performance of MBD1 is obvious, as is the poor performance of EBLUP for these variables. Note that these results are based on Model I, since Model II cannot be fitted to these variables. In Table 6 we show that there is little change in the average performance of MBD1 when the set of variables determining the multi-purpose weights used by this estimator is extended from the original $K = 5$ variable set (TCC, TCR, FCI, Cattle, Sheep) to the entire $K = 8$ variable set (TCC, TCR, FCI, Cattle, Sheep, Crops, Equity, Debt). Again, note that this extension is only possible under Model I. Finally, in Figure 5 we show the overall region-specific superior performance of MBD1 (under either $K = 5$ or $K = 8$) for the variable Debt. Similar region-specific performances (not shown here) were observed for Crops and Equity.

6. Discussion and Further Research

The empirical results reported in the previous section are evidence that the MBD estimator (14), particularly when combined with the multipurpose weights (19), can perform well and represents a real alternative to the EBLUP, with the associated easy to calculate MSE estimator (18) providing good coverage performance. Furthermore, they indicate that the MBD approach may be more robust than EBLUP in the realistic situation where (10) is a working model, rather than the (unknown) true model.

These results should not be taken as a blanket recommendation for MBD over EBLUP, however. As noted in section 4, if one sets practical considerations aside, then EBLUP must be the estimation method of choice when (9) actually holds. In such a case, the extent of the efficiency gain over MBD will depend on both the distribution of the auxiliary variables as well as the sample distribution across the small areas. To illustrate this, we return to the Australian broadacre farm population used in the previous section, but this time carry out a model-based simulation, first generating population values for TCC under the random intercepts model (Model 1 in Table 2) with β and σ_e^2 set at their fitted population values and with different values of σ_u^2 chosen in order to obtain a range of values for the intra area correlation $\gamma = \sigma_u^2 / (\sigma_e^2 + \sigma_u^2)$, and then sampling from this simulated population using the same regionally stratified design as used in the simulation study reported in the previous section. Table 7 sets out the results of this simulation, in terms of the square root of the ratio of the average empirical MSE of the Horvitz-Thompson estimator (HTE) of a regional total to that of the corresponding EBLUP under the model used to generate the data (denoted

HTE/EBLUP), and the corresponding ratio (denoted MBD/EBLUP) of the average empirical MSE of the MBD estimator of a regional total to that of the same EBLUP. Note that values of these ratios for averages over all 29 regions as well as over regions with smaller sample sizes and those with larger sample sizes are given. These clearly show that in the case where all model assumptions are valid, the EBLUP, as one would expect, dominates both the MBD as well as the conventional direct estimator (HTE). However, the extent of this dominance decreases significantly as the strength of the regional effect increases, particular for regions with larger sample sizes. The MBD in turn dominates the HTE except where the regional effect is small, in which case we see that the EBLUP weights used in the MBD introduce slightly more variance than they eliminate bias.

Before closing, we also mention a number of issues that impact on the utility of the MBD estimator that remain unresolved. For example, negative weights, which occurred in some regions in the simulation study reported in the previous section, can lead to impossible (i.e. negative) estimates. Since such values are easily identified, they should not cause problems in real life. However, the problem remains of how to modify the weights (13) to ensure they are strictly positive. A related issue that has already been noted is the impact of outlier *Y*-values on (14). Certainly this estimator, because it is a linear combination of just the small area data values, is more susceptible to outliers in these values than the EBLUP estimator (11). Methods for dealing with negative weights under 'standard' regression models have been discussed in the literature (Huang and Fuller, 1978; Bardsley and Chambers, 1984; Deville and Sarndal, 1992; Chambers, 1996) but their application in the context of mixed models remains to be explored. Further, the data set used in section 5 involved skewed data as well as a potential nonlinear relationship between the survey and auxiliary variables. It is possible to adapt the MBD approach for small area estimation when variables are linear on a transformed scale. The authors will report on this research in another paper.

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Table 1 Regional population and sample sizes

Region	*N*	*n*	Region	*N*	*n*
1	79	6	16	2683	60
2	115	10	17	2689	60
3	189	30	18	2847	34
4	330	25	19	3056	74
5	388	36	20	3139	51
6	465	19	21	3910	73
7	604	36	22	4486	117
8	729	40	23	4550	80
9	737	30	24	4587	95
10	964	30	25	5368	83
11	1586	51	26	5528	103
12	1778	62	27	6489	108
13	1984	55	28	6980	81
14	2182	47	29	10933	77
15	2607	79			

Table 2 Different mixed model specifications considered in the simulations

Model	Model Type	*X*	*Z*
I	Random Intercepts	Intercept included	Intercept only
II	Random Slopes	Intercept included	Intercept + Size
III	Random Slopes with fixed intercept	Intercept included	Size only
IV	Random Slopes with zero intercept	Intercept excluded	Size only

Table 3 Average (ARME) and median (MRME) values of relative mean error, average (ARRMSE) and median (MRRMSE) values of relative RMSE and average (ACR) coverage rates for TCC

Model	Method	ARME	MRME	ARRMSE	MRRMSE	ACR
I	EBLUP	4.24	1.55	19.92	15.74	0.90
	MBD0	-2.49	-0.82	20.56	14.45	0.92
II	EBLUP	2.98	0.61	19.87	16.40	0.85
	MBD0	-2.13	-0.47	20.15	13.16	0.93
III	EBLUP	4.52	1.95	23.89	19.94	0.69
	MBD0	-3.84	0.13	21.14	14.44	0.94
IV	EBLUP	1.17	-2.63	23.38	19.73	0.65
	MBD0	2.20	2.06	22.35	20.61	0.97

Table 4 Average and median relative mean error (ARME, MRME), average and median relative RMSE (ARRMSE, MRRMSE) and average coverage rate (ACR) for five variables best suited to linear mixed modelling

Model	Criterion	Method	TCC	TCR	FCI	Beef	Sheep
I	ARME	EBLUP	4.24	5.48	6.93	138.48	304.24
		MBD0	-2.49	-9.25	-13.80	-15.05	-7.33
		MBD1	-1.54	-1.30	-0.50	-1.78	0.69
		MBD2	-1.29	-1.02	-0.04	-1.35	0.98
	MRME	EBLUP	1.55	0.55	-2.08	0.95	-0.23
		MBD0	-0.82	-3.87	-2.83	-4.79	-4.48
		MBD1	-0.61	-0.42	-0.56	-0.97	-0.35
		MBD2	-0.52	-0.39	-0.54	-0.75	-0.30
	ARRMSE	EBLUP	19.92	21.76	63.93	304.74	906.18
		MBD0	20.56	23.34	54.42	37.45	24.88
		MBD1	20.86	21.77	59.72	33.29	30.24
		MBD2	20.85	21.77	60.07	33.36	30.64
	MRRMSE	EBLUP	15.74	14.83	40.41	25.97	13.00
		MBD0	14.45	16.20	35.85	30.34	15.50
		MBD1	14.69	13.41	42.09	30.55	14.67
		MBD2	14.74	13.46	42.45	30.56	14.67
	ACR	EBLUP	0.90	0.88	0.87	0.86	0.91
		MBD0	0.92	0.91	0.94	0.93	0.94
		MBD1	0.92	0.92	0.94	0.95	0.96
		MBD2	0.92	0.92	0.94	0.95	0.96
II	ARME	EBLUP	2.98	2.85	16.70	131.66	2.63
		MBD0	-2.13	-1.25	0.50	-0.29	3.66
		MBD1	-1.67	-1.29	0.74	-1.95	1.10
		MBD2	-1.30	-0.72	3.17	-1.29	0.93
	MRME	EBLUP	0.61	1.37	3.98	0.62	0.00
		MBD0	-0.47	-0.51	0.35	-0.31	0.00
		MBD1	-0.65	-0.50	0.24	-0.30	-0.15
		MBD2	-0.52	0.01	0.53	-0.22	-0.09
	ARRMSE	EBLUP	19.87	20.28	68.85	231.08	630.01
		MBD0	20.15	21.46	65.43	30.80	37.82
		MBD1	19.06	21.03	64.03	30.09	32.04
		MBD2	27.13	34.84	129.29	45.16	34.99
	MRRMSE	EBLUP	16.40	15.61	33.89	22.64	11.73
		MBD0	13.16	12.39	37.64	28.79	14.68
		MBD1	12.84	12.18	37.92	24.84	14.77
		MBD2	12.84	12.71	37.62	24.93	14.72
	ACR	EBLUP	0.85	0.86	0.84	0.86	0.89
		MBD0	0.93	0.93	0.90	0.95	0.96
		MBD1	0.93	0.93	0.94	0.95	0.96
		MBD2	0.93	0.93	0.94	0.95	0.96

Table 5 Average relative mean error (ARME), average relative RMSE (ARRMSE) and average coverage rate (ACR) for EBLUP, MBD0 and MBD1 for variables with many zeros. Model I is assumed.

Variable	ARME			ARRMSE			ACR		
	EBLUP	MBD0	MBD1	EBLUP	MBD0	MBD1	EBLUP	MBD0	MBD1
Crops	90.31	0.003	-0.21	123.96	23.53	22.92	0.95	0.96	0.96
Equity	4.36	-9.32	-1.20	18.51	19.14	17.05	0.88	0.92	0.94
Debt	8.39	-4.94	-0.96	29.02	27.71	28.57	0.91	0.93	0.93

Table 6 Average relative mean error (ARME), average relative RMSE (ARRMSE) and average coverage rate (ACR) for multi-purpose weighting (MBD1) based on original $K = 5$ and extended $K = 8$ variable sets. Model I is assumed.

Variable	$K = 5$			$K = 8$		
	ARME	ARRMSE	ACR	ARME	ARRMSE	ACR
TCC	-1.54	20.86	0.92	-1.08	20.91	0.92
TCR	-1.30	21.77	0.92	-0.80	21.83	0.92
FCI	-0.50	59.72	0.94	0.21	60.22	0.94
Cattle	-1.78	33.29	0.95	-1.05	33.49	0.95
Sheep	0.69	30.24	0.96	1.24	31.06	0.96
Crops	-0.21	22.92	0.96	-0.20	22.97	0.96
Equity	-1.20	17.05	0.94	-0.72	17.14	0.94
Debt	-0.96	28.57	0.93	-0.68	28.74	0.93

Table 7 Ratio of the square root of the average mean squared errors of Horvitz-Thompson (HTE) and MBD estimates of regional totals to EBLUP-based estimates of the same totals. Sample design is stratified by region, with SRSWOR within regions and sample allocations as in Table 1. The data were generated using Model 1 of Table 4, and this model was also assumed by both the EBLUP and MBD methods.

Average over	RMSE Ratio	$\gamma = \sigma_u^2 / (\sigma_e^2 + \sigma_u^2)$				
		0.1	0.2	0.3	0.4	0.5
All 29 regions	HTE/EBLUP	2.10	1.58	1.41	1.36	1.36
	MBD/EBLUP	2.27	1.56	1.31	1.19	1.14
7 smaller regions	HTE/EBLUP	3.14	2.54	2.29	2.22	2.22
($n \leq 30$)	MBD/EBLUP	3.84	2.40	1.81	1.52	1.39
22 larger regions	HTE/EBLUP	1.39	1.17	1.12	1.10	1.08
($n > 30$)	MBD/EBLUP	1.44	1.18	1.10	1.06	1.04

Figure 1 Regional relative mean errors for EBLUP (dashed line) and MBD0 (solid line) for TCC under models I (top left), II (top right), III (bottom left) and IV (bottom right).

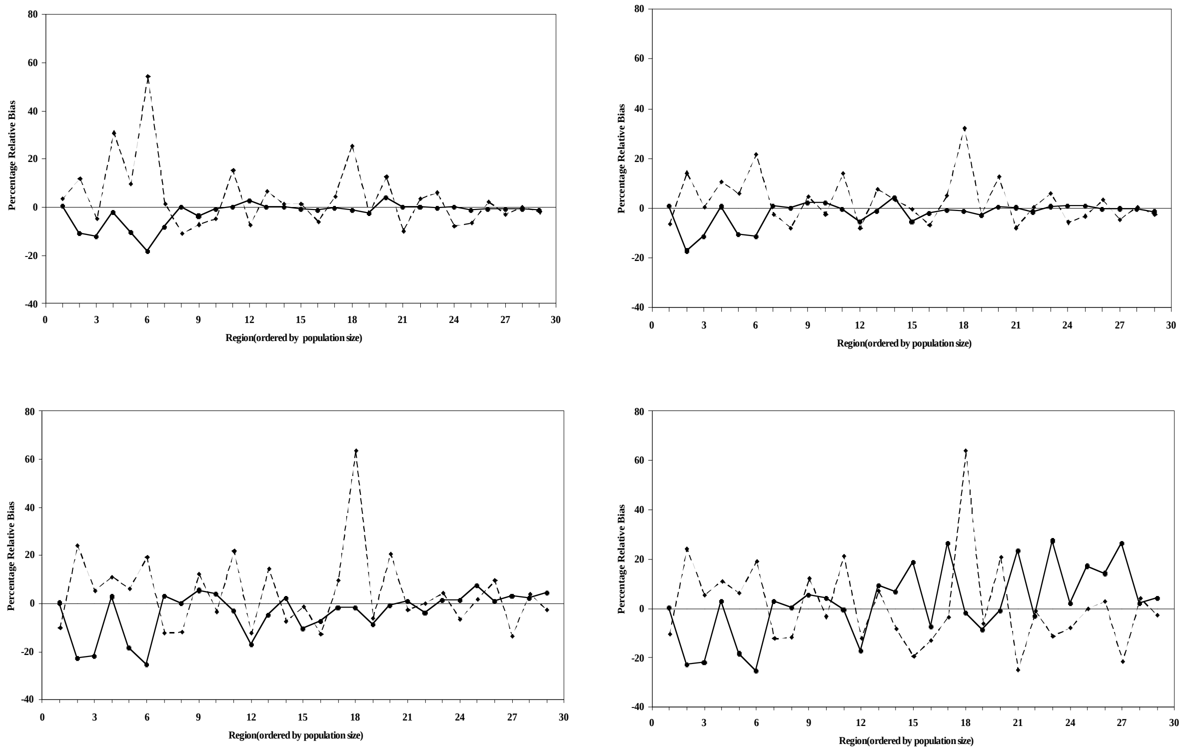


Figure 2 Regional relative RMSEs for EBLUP (dashed line) and MBD0 (solid line) for TCC under models I (top left), II (top right), III (bottom left) and IV (bottom right).

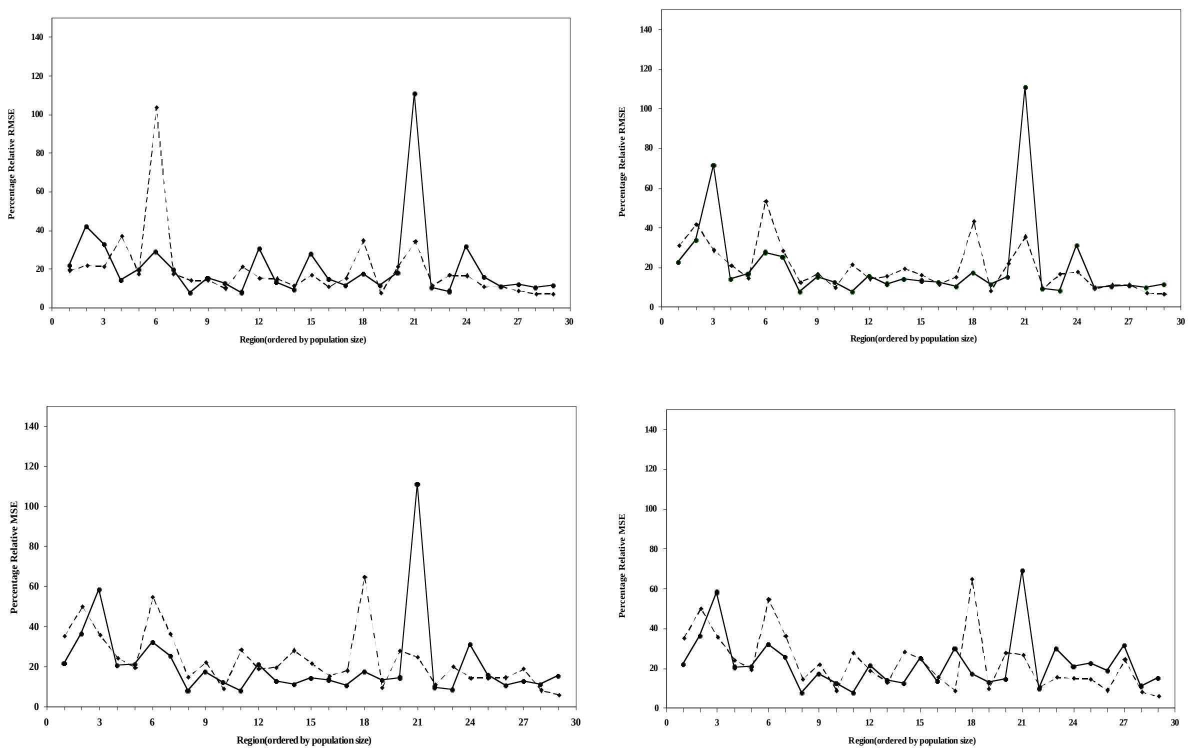


Figure 3 Regional coverage rates for EBLUP (dashed line) and MBD0 (solid line) for TCC under models I (top left), II (top right), III (bottom left) and IV (bottom right).

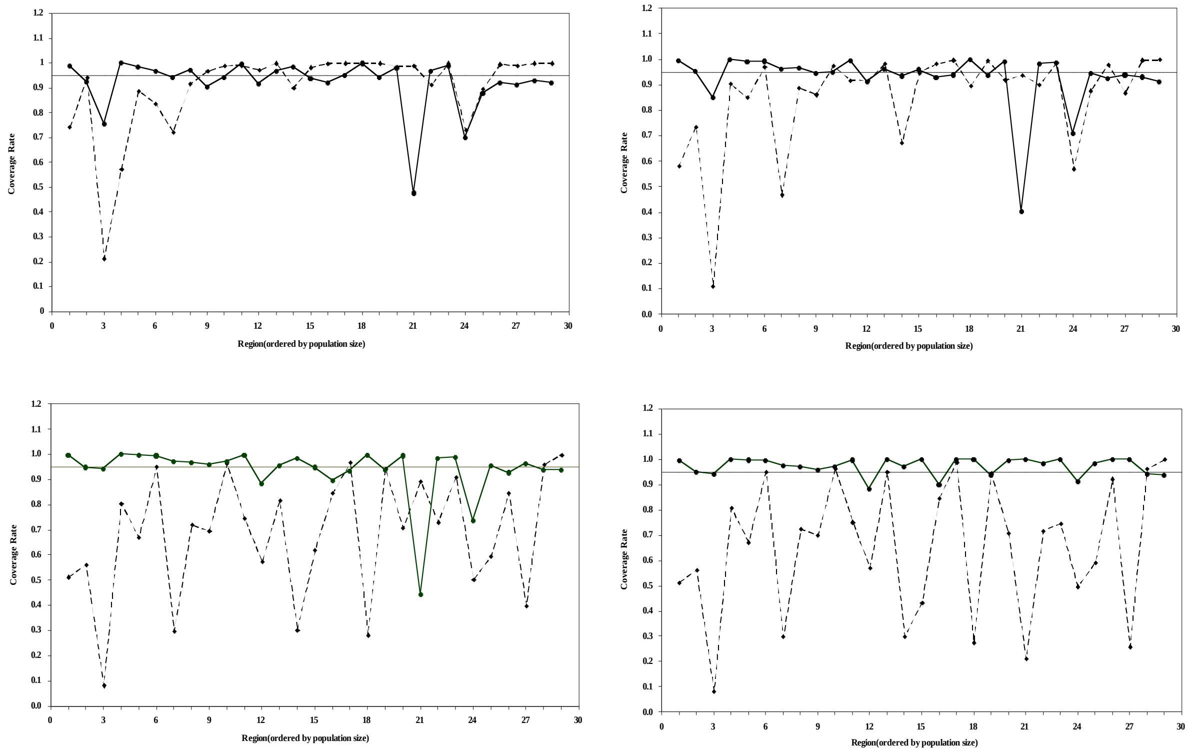
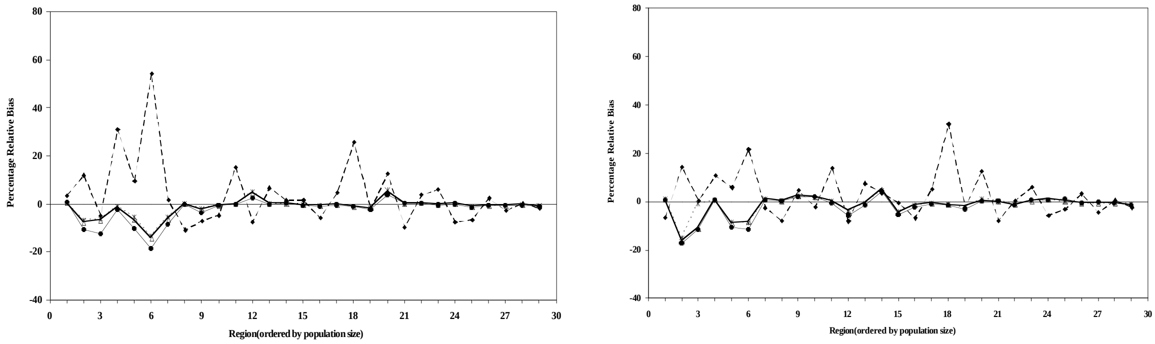
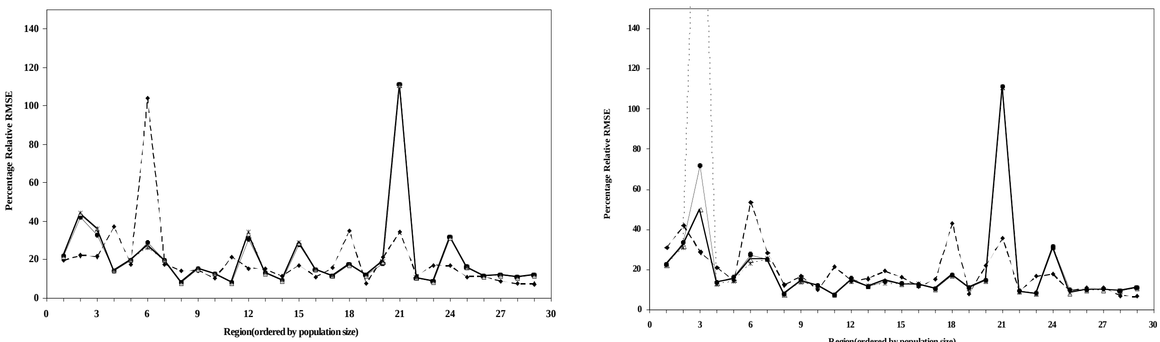


Figure 4 Regional performances of EBLUP (dashed line), MBD0 (thin line), MBD1 (thick line) and MBD2 (dotted line) for TCC under models I (left) and II (right).

Relative Mean Error



Relative RMSE



Coverage Rate

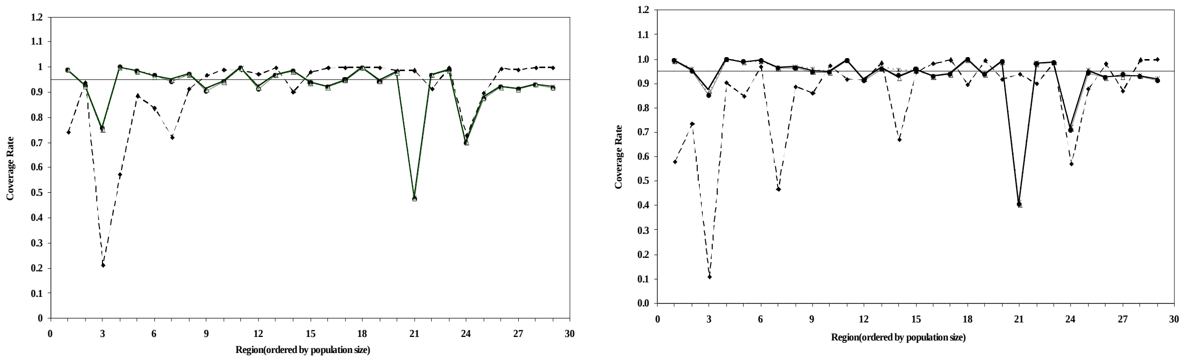
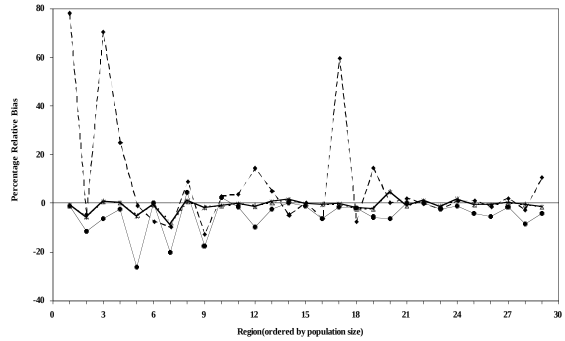
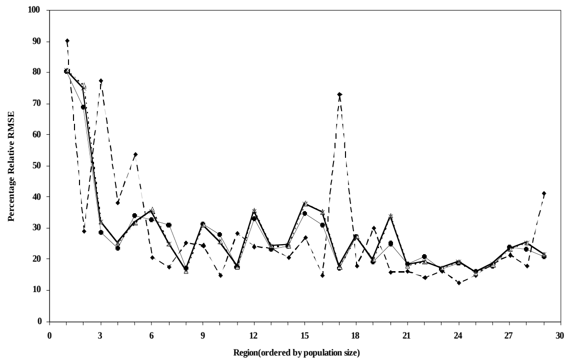


Figure 5 Regional performances of EBLUP (dashed line), MBD0 (thin line), MBD1 under *K* = 5 (thick line) and MBD1 under *K* = 8 (dotted line) for Debt under model I.

Relative Mean Error



Relative RMSE



Coverage Rate