UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

Quantised vortices, mutual friction and elasticity in superfluid neutron stars.

by

Stuart Wells

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ABSTRACT

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This thesis investigates and builds upon the well known approaches to modelling superfluid neutron stars. We build single and multiple fluid systems in Newtonian gravity, introducing mutual friction and elasticity. We then move into general relativity, detailing how to build a single superfluid system using a quantised vorticity vector. We introduce multiple interacting fluids into the model, producing the calculation of mutual friction in general relativity. We then use the variational approach to incorporate elasticity of the vortex array into the model in order to follow the format of Newtonian calculations.

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Declaration of Authorship

I, Stuart Wells , declare that the thesis entitled *Quantised vortices, mutual friction and elasticity in superfluid neutron stars.* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission

Signed:...

Date:..

Acknowledgements

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Introduction

1

Introduction

Neutron stars are formed during the collapse of main sequence stars. When a massive star has come to the end of its life, it goes out with a bang. After the star uses up its supply of hydrogen and helium, heavy elements collect in the core of the star. Once the mass of the core exceeds the Chandrasekhar limit of $1.44M_\odot$ ($M_\odot \approx 2 \times 10^{33}$g is the mass of the Sun), the core starts to collapse [7, 8, 9, 10]. This collapse causes a shock wave which throws off the outer layers of the star, while the core continues to compress. At this point, the resulting structure will depend upon the mass of the stars compressed core. If its mass is below $2 - 3M_\odot$, known as the Oppenheimer-Volkoff limit [11], the core will compress to form a structure similar to a giant nucleus. However, if the mass in the core exceeds this limit, then it has no choice other than to collapse to form a black hole. If the star does not end up as a black hole, then the giant nucleus formed in the collapse is what we call a neutron star. This explosive ending to a stars life is known as a supernova. Not all stars experience supernova, as the star must initially have a mass $M \gtrsim 8M_\odot$ in order to do so. Stars with mass which is below this threshold will turn into a giant star and eventually end up as a white dwarf.

As you can imagine, a neutron star formed from the violent supernova explosion described above will be incomparable to anything we have or can create on Earth. Neutron stars have masses in the range $1 - 2M_{\odot}$ and all of this mass is compressed into a sphere of radius $10 - 14$ km, which is approximately $60,000$ times smaller than the radius of the Sun. Due to neutron stars having such extreme densities, which we cannot reproduce on Earth to study further, we still do not know for sure what the composition of this type of star is. However, observations of these types of stars allow us to predict what some of their composition must be [12, 13, 14, 15].

We delve deeper into neutron star composition in the following Chapter, as it is important to have an understanding of the environment one is trying to model. We discuss the history of neutron stars, including their discovery and how a special type of neutron star, a pulsar, provided evidence supporting Einstein's Theory of Relativity. Chapter 2 also investigates how the early models of neutron stars progressed and details what the stars are thought to be comprised of in today's understanding. A crucial component of neutron stars, and the reasoning behind their name, is their large body of fluid neutrons. The star is comprised mostly of neutrons and once you get below the surface of the star, this fluid is thought to be in a superfluid state.

This leads us on to the second part of Chapter 2, in which we consider what a superfluid is and how they behave. Superfluids have been studied via experiments with liquid helium and most of our knowledge about them come from such experiments. We discuss why a fluid may enter a superfluid state and the properties that these fluids exhibit. One of the important features of superfluids, and an important topic for this thesis, is that they do not rotate like a conventional fluid. When a container of superfluid helium is rotated, the fluid does not follow the container as it has zero friction. Instead, vortices form in the fluid and the faster the container is rotated, the greater number of vortices are formed. These quantised vortices are a key component of our models and are also likely to play a large role in many of the behaviours observed in neutron stars.

After we have gained knowledge about the environment we are studying, we would like to understand how one would usually model these systems and what tools are commonly used. In Chapter 3 we discuss different approaches to modelling fluid systems. Firstly, we consider the variational approach for a general fluid and then consider turning this fluid into a superfluid. We introduce quantised vortices in order to specify that the fluid is indeed superfluid. This gives rise to complications that come with a vortex array, that you don't usually find in a normal fluid. We must add in the behaviours seen in experiments with superfluid helium, such as the Magnus force, mutual friction and elasticity.

Once we understand these forces, we introduce a second fluid component to represent the normal fluid in the system. In the case of a neutron star, this normal fluid would consist of superconducting protons and electrons, as we will discuss in Chapter 2. We consider perturbations of this two fluid model, using Lagrangian perturbation theory and plane wave perturbations. The investigation into plane wave perturbations of the system is in order to find Tkachenko waves and see if they are likely to play an important role in neutron stars.

As neutron stars are such extreme environments, the most accurate models of them must be formulated in general relativity. In Chapter 4 we start to consider how we can model our system in the framework of general relativity. We first consider existing work on relativistic fluid dynamics, in order to understand the different methods for creating fluid models in relativity as opposed to Newtonian gravitation.

With an understanding of relativistic fluid dynamics, we proceed to recreate our Newtonian superfluid model in general relativity. Chapter 5 discusses how we use quantised vortices in relativity, following the Newtonian logic of a quantised vorticity vector rather than the relativistic vorticity tensor. We build a multiple fluid system including superfluid neutrons, quantised vortices and a second 'normal' fluid component. We include the Magnus force and mutual friction in this relativistic model, as we did in our Newtonian model in Chapter 3.

In Newtonian theory, we created a multiple fluid system including mutual friction and elasticity. Up to this point in our general relativistic model, we have only introduced mutual friction. Chapter 6 addresses the remaining components in order for our Newtonian and relativistic models to match. We consider relativistic elasticity in three dimensions and create a two dimensional vortex elasticity. Once we create an elastic contribution to the model, we have completed our transition of the Newtonian multifluid model with mutual friction and vortex elasticity to the framework of general relativity.

2

Neutron Stars

2.1 The discovery of neutrons and neutron stars

In order for us to learn about or even consider the existence of neutron stars, we first need to know that neutrons themselves exist. Thankfully, the neutron was discovered experimentally by Sir James Chadwick in 1932, while working at the Cavendish Laboratory in Cambridge [16, 17]. Shortly after this discovery, there were discussions that stars could be comprised of these new neutrons by Rosenfeld, Bohr and Landau [18]. It didn't take long before Walter Baade and Fritz Zwicky proposed a star which was composed of neutrons and would be born from supernova explosions [19]. They stated that such a star would be very dense and much smaller than most stars.

The first modelling of neutron star cores was attempted by Oppenheimer and Volkoff in 1939 [11]. They assumed that the neutron star matter would be comprised of an ideal gas of free neutrons at high density. It was thought at the time that massive normal stars may also have neutron cores. However, as our understanding of nuclear fusion increased, we realised this would not be the case and research into neutron cores faded somewhat. The few people still working with neutron stars were contemplating topics such as the composition and equation of state of these stars [20]. Another reason that neutron stars were generally neglected was that they would be too difficult to observe with the technology of the time, due to their very small size.

Interest in neutron stars was given a boost following the discovery of non solar x-ray sources in 1962 [21, 22]. This was due to the thought that the source of these x-rays could be a young, warm neutron star. This triggered investigations to find out more about neutron star cooling. Interest continued to build after the observation of the first quasar, but it was shown that there was no connection between neutron stars and quasars. Even with a great interest developing in the field of compact objects, many people did not take the work seriously. This could have been due to the vast extrapolation from familiar physics at the time [18].

Neutron star research was brought back to life after the discovery of pulsars in 1967 [23]. The original observations were made by Jocelyn Bell while working as part of a team studying quasars using a radio telescope. She noticed a signal which was pulsing regularly at a rate of approximately one pulse every 1.3 seconds. Due to the regularity and consistent power of this source, it resembled a signal from a beacon and was nicknamed LGM-1 for Little Green Men [17]. This discovery was of such significance that Bell's PhD supervisor, Anthony Hewish, received the Nobel Prize in 1974 for leading the team that made the discovery. After the discovery of this regularly pulsing signal, it was almost immediately proposed by Thomas Gold and Sir Fred Hoyle that the source could be a rapidly rotating neutron star [24]. Due to this discovery of a pulsar, research into neutron stars exploded, much like the supernova they themselves are created by. Already by 1971, the structure of such stars was thought to be comprised of a superfluid core, solid crust, an atmosphere and a strong magnetic field, as seen in Figure 2.1 taken from [1]. This description should sound very familiar to current researchers, over 40 years after this original structure was proposed.

In 1969 it was noticed that there was a significant sudden decrease in the periods of

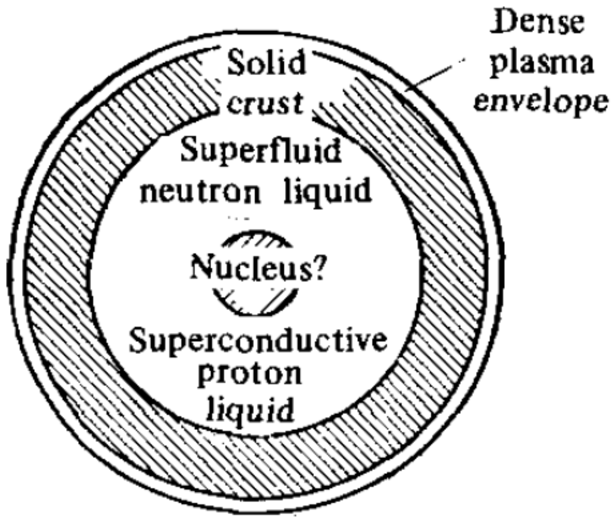

Figure 2.1: *A figure demonstrating the basic structure of a pulsar from a review published in 1971 [1]. Shortly after the discovery of the first pulsar in 1967, neutron star models had most of the components we still consider to be true today.*

the Vela pulsar [25]. This behaviour was observed in many other sources and is today referred to as glitches. This sparked research into the cause of this unusual behaviour and two explanations stand out from the crowd. The first explanation is that the stars glitch due to starquakes, much like we have earthquakes here on Earth [26, 27]. The second explanation was that of vortex unpinning, which relies on properties of the superfluid interior of the star to cause the glitches [28, 29, 30]. Of course it is likely to be a combination of several factors which combine to cause glitches at various points in a stars life.

In 1974 Hulse and Taylor discovered PSR B1913+16 [31], which is a binary system comprising of a pulsar and a neutron star. A binary system is a pair of objects orbiting around a common centre of mass. The pulse that we detect from pulsars is a jet of electromagnetic radiation, which is emitted from the star due to it rotating very rapidly and being extremely magnetised [32]. This jet is aligned with the magnetic axis of the star, which is not necessarily in the same direction as its rotation axis. Hence as the star spins, we see the jet as a pulse as it crosses our line of sight.

The PSR B1913+16 system was the first binary system to be detected containing a pulsar. Hulse and Taylor were awarded the 1993 Nobel Prize in Physics for their discovery. Analysis of this binary system has shown that the orbits of the stars are decaying, which implies that the system is losing energy somewhere. The amount of energy that the system is losing can be explained by Einstein's General Theory of Relativity, as the energy lost by the system due to gravitational wave emission. Figure 2.2 shows the observed orbital decay (points with error bars), along with the decay predicted by general relativity due to gravitational wave emission (solid line).

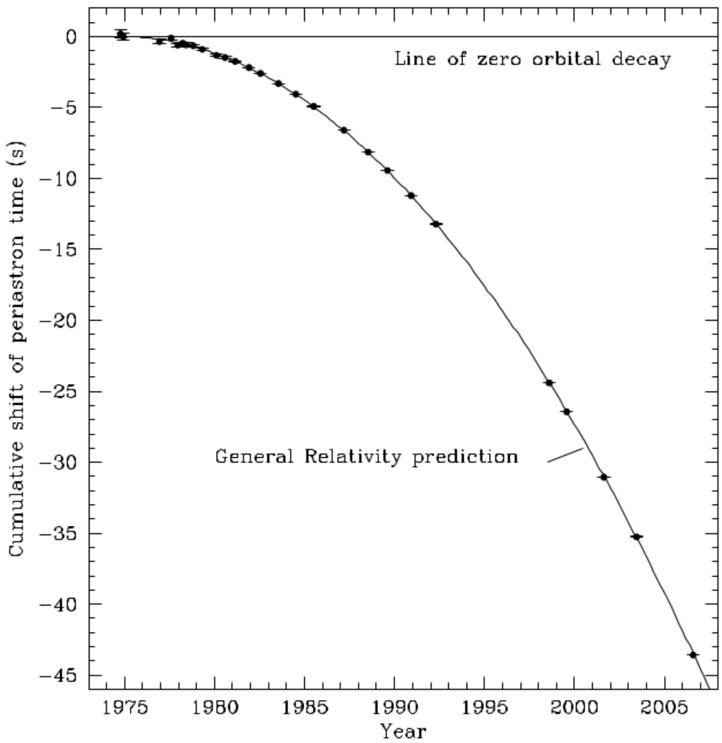

Figure 2.2: *This plot shows the observed orbital decay of PSR B1913+16 with error bars, along with the decay predicted from general relativity due to gravitational wave emission [2]. The observations coincide with the prediction, giving strong evidence for Einstein's General Theory of Relativity.*

The observations from PSR B1913+16 match the prediction from general relativity. This is seen as positive evidence for general relativity to be correct and for gravitational waves to exist, as predicted by the theory. Hence, we arrive at the main reason we wish to understand neutron stars in greater detail. If we can accurately model neutron stars, we can aid the detection of gravitational waves from neutron star systems and determine that general relativity is the most accurate theory of gravitation.

2.2 Composition of neutron stars

As we saw in the previous section, the standard model of neutron stars was established in the 1970s. There are many papers and textbooks concerning the composition of neutron stars. I will use information from [12], [13], [14] and [15] in the following. The neutron star can be split into four main sections, we call these the outer crust, inner crust, outer core and inner core. As we can see from Figure 2.3, the crust is quite thin relative to the radius of the whole star and the bulk of the star is the core. The contents of the four sections of a neutron star are described below:

Outer crust

The outer crust is relatively thin compared to the radius of the star, approximately 500m. It is comprised mainly of heavy nuclei in the form of a solid lattice, and a degenerate electron gas.

Inner crust

The inner crust is approximately twice as thick as the outer crust. Throughout this layer, the pressure increases and neutrons start to be squeezed out of the nuclei. The neutrons, which drip from the solid lattice, start to form a superfluid as we go deeper into the star.

Outer core

The outer core is the largest section of the neutron star. Due to the immense pressure as we proceed into the star, the nuclei we saw in the crust cannot exist any more. For this reason, the core is flooded with superfluid neutrons, along with a small proportion of superconducting protons and electrons. The neutron superfluid makes up approximately 90% of the outer core with the remaining 10% comprised mainly of protons. The density in this section of the star is comparable to that of atomic nuclei.

Inner core

The inner core may or may not exist in a different state to that of the outer core, this will depend on the mass of the star and how matter behaves at this density. The density here could be $\sim 10^{15}$ g/cm^3 and we are currently unsure what happens to the neutrons in this environment. They could be remain as a superfluid, as they are in the outer core, in which case there will not be two separate sections to the core. Another outcome could be that they split, similar to how they exited the nuclei, into more exotic forms of matter such as hyperons or a quark-gluon plasma.

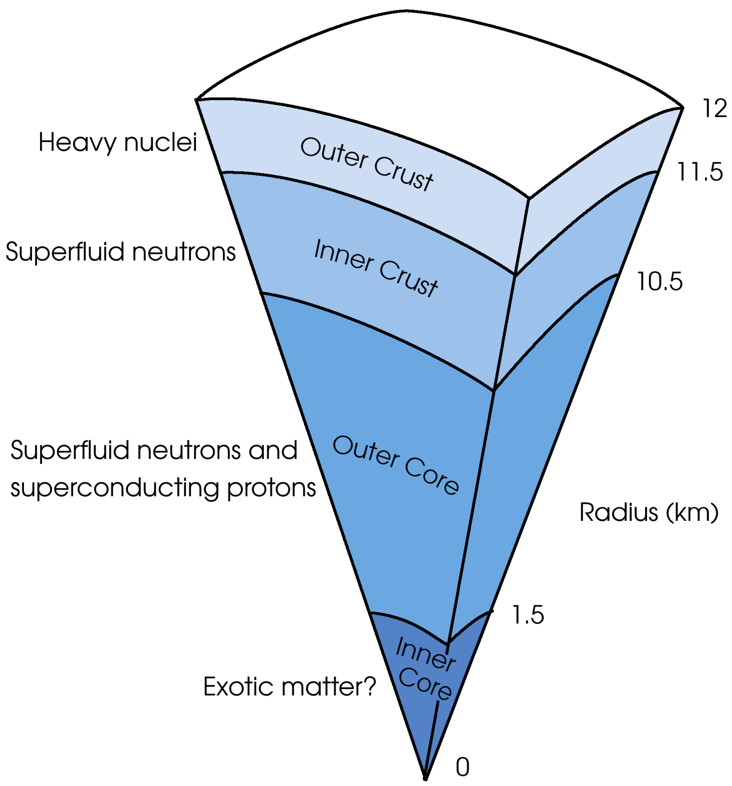

Figure 2.3: *This illustration gives an idea of the depths of each of the different layers of a neutron star. The crust is solid and made primarily of heavy nuclei. As you proceed further into the star, neutrons start leaving the nuclei to form a superfluid. The main component of the core is the neutron superfluid, around 90%, with approximately 10% of the core being comprised of protons. The centre of the star is extremely dense so there may be some exotic matter such as quark-gluon plasma or colour superconductor.*

2.3 Superfluids and vortices

As the main component of neutron stars, superfluids are a necessity if you wish to realistically model a neutron star. We can't travel to a neutron star and observe how the superfluid neutrons behave, so we have to find ways of investigating them here on Earth. In order to understand more about superfluids, we can experiment with liquid helium. Helium is particularly useful because it enters a superfluid state when it is cooled to $T < 2.17$ K, a temperature obtainable experimentally. This is shown in Figure 2.4, the peak at T_λ shows the phase transition of the fluid helium as it is cooled. The superfluid phase of helium was discovered by Pyotr Kapitsa, John F. Allen and Don Misener in 1937 [33] [34]. They noticed that instead of the liquid helium cooling down and freezing as its temperature is decreased, it remained liquid and exhibited some strange properties. The point in the specific heat capacity of helium at which it undergoes this transition looks somewhat like the Greek letter Lambda (λ), as can be seen in Figure 2.4, and hence was denoted the Lambda point. These two states of helium are denoted helium I and helium II. Helium above the Lambda point is referred to as helium I and the superfluid helium, below 2.17 K, is denoted helium II.

After the discovery of the superfluid phase of helium, many experiments were developed in order to understand this new phenomenon. Such experiments with helium [4, 35, 36] help to develop models for superfluids and show us which behaviours they exhibit. With this information we can then start to understand how the neutron superfluid within a neutron star behaves.

So, what *is* a superfluid? Simply put, a superfluid is a fluid which exhibits zero viscosity, which means it acts like a fluid but does not experience friction. As superfluids do not experience friction, they behave very differently in comparison to normal fluids. For example, in Figure 2.5, the superfluid helium is seen to be creeping up the side of its container and dripping from the bottom. This is due to the absence of friction between the superfluid and the container.

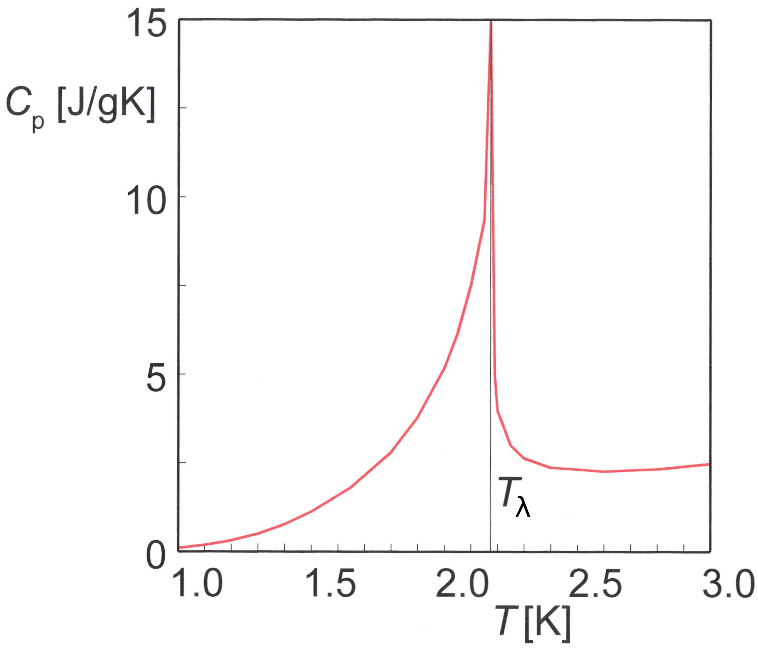

Figure 2.4: *Heat capacity of liquid helium as a function of temperature. The peak at the Lambda point* $T_\lambda = 2.17$ *K marks the phase transition. Helium is often referred to as helium I above* T_λ *and as helium II in the superfluid phase.*

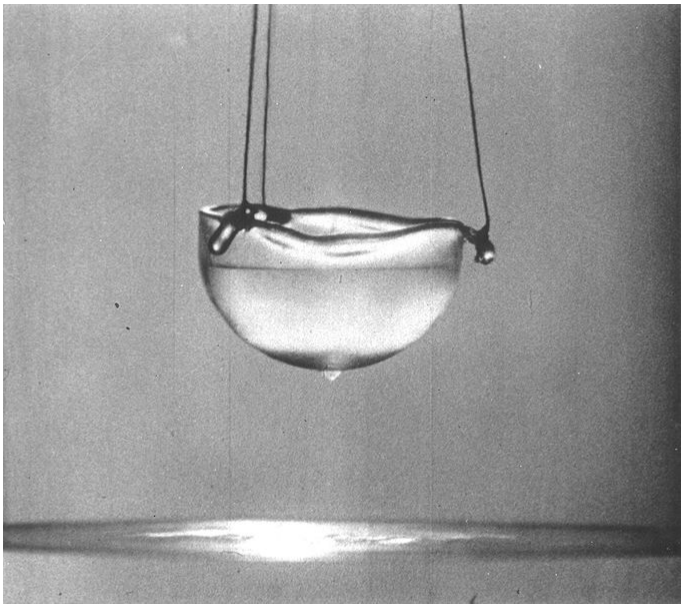

Figure 2.5: *An experiment in which superfluid helium escapes its container. It travels up the sides and drips off of the bottom, due to zero friction [3].*

This zero viscosity behaviour was first seen by J. Reppy and D. Depatie [37], however, viscous behaviour had also been seen under certain conditions by A. D. B. Woods and A. C. H. Hallett [38]. This contradictory behaviour was explained by introducing a two fluid model for the helium. It was proposed that helium II consists of two inseparable fluids. The first fluid component exhibits the observed zero viscosity and frictionless flow, whereas the second component has viscosity and hence exhibits the viscous behaviour.

The two fluid model for helium was advanced by L. Landau [39], who was later awarded the Nobel Prize in Physics in 1962. Landau proposed that if a fluid is at absolute zero then it must be in a perfect, frictionless state. When its temperature is increased, this causes the excitation of phonons, quantised collisionless sound waves and quasi-particles of higher momentum and energy which he called rotons [40]. These excitations behave like an ordinary gas and are responsible for any heat transport. Hence, they are the viscous fluid in the two fluid model. An experiment proposed by Landau measured the superfluid fraction of rotating helium II as a function of temperature. It was seen that almost all of the sample was in the superfluid state when the temperature was below 1 K. This experiment was performed by E. Andronikashvili in 1946 and provided key evidence to help establish the two fluid model of superfluidity.

A key feature of superfluids which experiments with helium have shown us, is that vortices form in the superfluid when it is rotated. The fluid itself doesn't move but it 'rotates' through the act of forming vortices. These vortices are extremely small and are also quantised. This means that when the superfluid rotation is increased by a specific amount, a new vortex will form. The vortices collect in an array that behaves in a similar way to an elastic lattice and they also exhibit tension as they bend. As rotation is increased and more vortices are created in the superfluid, the shape of the lattice changes. Figure 2.6 demonstrates the triangular shape of the vortex array as the rotation of the superfluid is increased from (a) to (l).

L. Onsager and R. Feynman both independently showed that vorticity enters the superfluid through the formation of vortex lines [41, 42]. This lead to the Onsager-Feynman quantisation condition, which we will use when averaging over a number of vortices later on. The first measurement of quantised vortex lines in rotating helium II was by H. Hall and W. Vinen in 1956 [36]. Experiments were undertaken to see how these vortices behave and interact with each other and their surroundings under various conditions

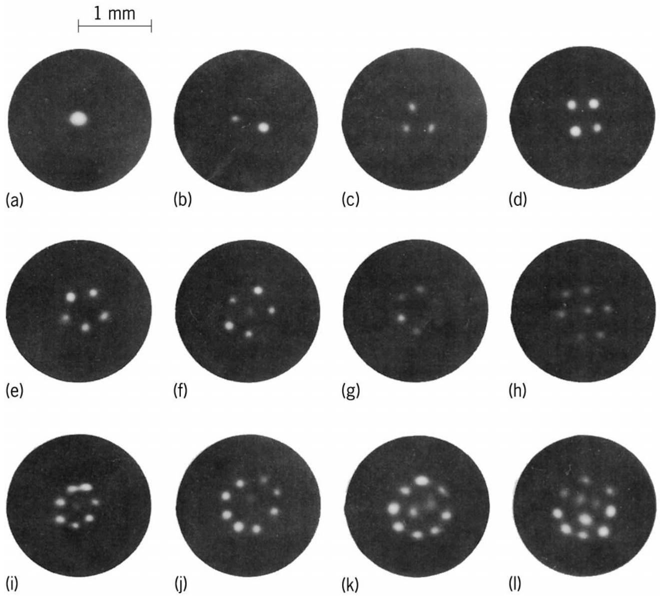

Figure 2.6: *The vortex array that forms in superfluid helium due to rotation, with rotation speed increasing from (a) to (l) [4]. The lattice formed is triangular but also varies slightly depending upon the number of vortices present.*

[43, 44, 45]. One effect of rotating superfluid helium, which we will encounter later on in this thesis, is mutual friction [46, 47, 48]. In the 1950s, many scientists thought that the quantised vortices formed in a rotating superfluid would act as scattering centres for the excitations constituting the normal fluid and would give rise to a mutual friction force. This force was investigated by Hall and Vinen in their papers mentioned above [35, 36]. We can use the results from experiments to formulate a mathematical model for a superfluid including its vortices.

2.4 Neutron star superfluidity

On Earth, we need extremely cold temperatures to enable the superfluid state to occur. This has been demonstrated by the experiments with supercooled helium described in the previous section. Liquid helium has to be cooled to $T < 2.17$ K in order for it to enter the superfluid state. In contrast, neutron stars have very high temperatures, around $10^{10} - 10^{11}$ K at birth. So, is it evidence of madness that we are considering the occurrence of superfluidity within a neutron star?

Neutron stars are comprised of neutrons, protons and electrons, which are fermions (particles with half-integer spin). Fermions must obey the Pauli exclusion principle, which states that two identical fermions cannot occupy the same quantum state simultaneously [49]. However, bosons (particles with integer spin) do not obey the Pauli exclusion principle, which means that any number of bosons can occupy the same state. This is demonstrated pictorially in Figure 2.7.

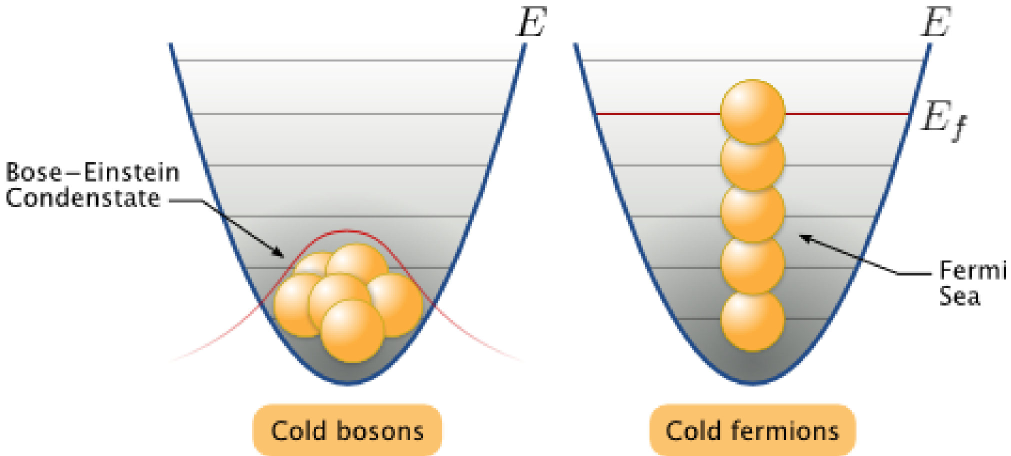

Figure 2.7: *A comparison of the possible energy states of bosons and fermions. For bosons, there is no limit to the number of particles that occupy each energy level. However, in the case of fermions, only one fermion of a particular spin may occupy each state. The highest occupied state for fermions is referred to as the Fermi energy, seen above as* E_f *[5].*

For the neutron fluid to become superfluid, we need it to be possible to send the particles to the lowest energy state. In 1957, L. Cooper, J. Bardeen and J. Schrieffer discovered that it is possible for two fermions to combine and form a Cooper pair, which itself is a composite boson. As we know from Figure 2.7, this would enable multiples of these Cooper paired fermions to occupy the lowest energy state and allow for superfluidity. In 1972, the Nobel Prize in Physics was awarded to Cooper, Bardeen and Schrieffer for their BCS theory. They initially developed the theory to explain superconductivity, but it was later found that it also applied to nucleons. Hence, it is indeed possible for superfluidity to occur in nuclear matter [50, 51, 52, 53]. It is estimated that the critical temperature for neutron superfluidity is $T_c \simeq 5.5 \times 10^8$ K within a neutron star [54]. This estimation comes from observations of neutron star glitches and cooling, both of which are expected to be caused, or heavily influenced, by the neutron superfluid within the star. The critical temperature for a superfluid or superconductor is related to BCS theory via the BCS energy gap. There is a specific amount of energy required to break the Cooper pairs formed in the fluid or material. If this energy is supplied to the system, then the composite bosons will separate to fermions and the superfluid or superconducting behaviour will no longer exist. You can see from Figure 2.8 that there is a particular Fermi wavenumber for which the neutrons require a significant amount of energy to break the Cooper pairing and hence the superfluidity. This is then translated into a critical temperature for the neutron fluid, demonstrated in Figure 2.9.

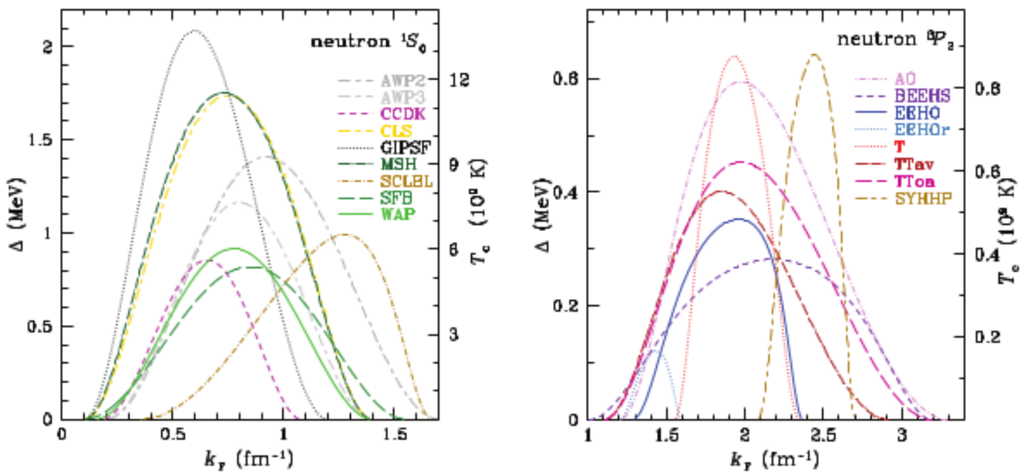

Figure 2.8: *Left: Neutron singlet gap energy (left axis) and critical temperature (right axis). Right: Neutron triplet gap energy and critical temperature. See [6] for more detail.*

2.5 General relativity

Given the evidence that general relativity is the correct framework in which to model our system, we would like to know some of the basics. Information on relativity can be found in a number of sources such as [55, 56, 57, 58, 59, 60]. As we are considering relativity, we will assume the Einstein Equivalence Principle. This means that [61, 62, 63, 64]

- test bodies fall with the same acceleration independently of their internal structure or composition;

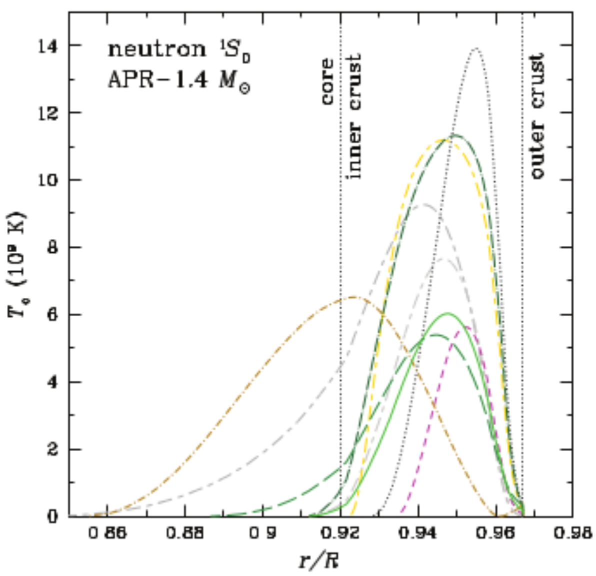

Figure 2.9: *Critical temperature T_c for neutron singlet superfluidity as a function of fractional radius of a neutron star using the APR equation of state ($M = 1.4M_\odot, R = 11.6$ km). Different curves correspond to different gap models which are shown in Figure 2.8. Vertical dotted lines denote the boundaries between the core, inner crust, and outer crust of the neutron star. See [6] for more detail.*

- the outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed;
- the outcome of any local non-gravitational experiment is independent of where and when in the Universe it is performed.

If the Equivalence Principle holds, then gravitation must be described by a metric-based theory [64]. Which means

1. spacetime is endowed with a symmetric metric,
2. the trajectories of freely falling bodies are geodesics of that metric, and
3. in local freely falling reference frames, the non-gravitational laws of physics are those of special relativity.

The spacetime metric is denoted $g_{ab} = g_{ba}$ and we take its signature to be $- + ++$, demonstrated in (2.2) below. The metric provides the structure of the spacetime we are working within. The symmetry of the metric implies that there are in general ten independent components. The metric can be determined from the line element [61, 65]

$$ds^2 = g_{ab} dx^a dx^b, \tag{2.1}$$

which, in a local set of Minkowski coordinates $\{t, x, y, z\}$, looks like

$$ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2. \tag{2.2}$$

Here we see the $-+++$ signature mentioned above. The metric has an inverse g^{ab} such that

$$g^{ab} g_{bc} = \delta^a{}_c, \tag{2.3}$$

where $\delta^a{}_c$ is the unit tensor. In relativity, one should take into consideration the covariance and contravariance of vectors, whereas in non-relativistic models this can be ignored. Covariant vectors are denoted using a lowered index V_a and contravariant vectors are denoted V^a with the raised index. The metric g_{ab} is used to raise and lower the indices of the vectors

$$V_a = g_{ab} V^b \qquad\qquad V^a = g^{ab} V_b. \tag{2.4}$$

A trajectory parametrised by proper time $x^a(\tau)$ where

$$d\tau^2 = -\frac{1}{c^2} ds^2, \tag{2.5}$$

has a corresponding unit tangent vector

$$u^a = \frac{dx^a}{d\tau}, \tag{2.6}$$

where $u^a = \gamma v^a$ and $u^0 = dt/d\tau$ with $\gamma = (1 - v^2/c^2)^{-1/2}$. The magnitude of u^a is given by

$$g_{ab} u^a u^b = g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} = \frac{ds^2}{d\tau^2} = -c^2, \tag{2.7}$$

which can be seen from the equations given above. It is quite often the case that the speed of light is taken to be $c = 1$, which we will use in our calculations. There are three classes of curves: timelike, spacelike and null. A vector is considered timelike if $g_{ab}V^aV^b < 0$, spacelike if $g_{ab}V^aV^b > 0$ and null if $g_{ab}V^aV^b = 0$. We can see that u^a is a timelike curve and it will be particularly useful later on.

Covariant and contravariant vectors transform differently when we change coordinates, from x^a to $\overline{x}^a$ say. The different transformations are given by

$$\overline{V}^a = \frac{\partial \overline{x}^a}{\partial x^b} V^b \tag{2.8}$$

and

$$\overline{V}_a = \frac{\partial x^b}{\partial \overline{x}^a} V_b, \tag{2.9}$$

for contravariant and covariant vectors respectively. Tensors with greater rank transform in the same way by acting linearly on each index with the transformations above.

Before diving straight into relativity, we will first consider how fluids are modelled in the Newtonian framework. We will return to relativity later on in order to build our superfluid model.

Newtonian Theory

3

Newtonian Models

If we wish to model the superfluid system within a neutron star by using general relativity, we need to be able to confirm that our equations are correct. Hence, we start by considering superfluid systems in Newtonian gravity, in order to have something to compare and confirm our relativistic models with later on.

3.1 Variational principles and Lagrange's equations

3.1.1 Hamilton's principle

We consider the motion of a system between two times t_1 and t_2 and small variations of this motion from the actual motion. The configuration of a system is described by values of the coordinates $q_1, \ldots, q_n$. These correspond to a particular point in a Cartesian hyperspace where the qs form the n coordinate axes, this is known as the configuration space. As time evolves, the system changes and a curve is traced out in configuration space, which is the path of motion of the system. Hence, we can parametrise the curve by time such that each point on the curve has an associated value, or values, of time.

The path given in configuration space is not the motion in space of a particle, as each point in configuration space represents the whole system at a given time [66].

Hamilton's principle describes the motion of systems for which all forces are derivable from a scalar potential, which can be a function of the coordinates, velocities and time. These systems are called monogenic and if the potential is a function of only the position coordinates, then it is also conservative. For a monogenic system, Hamilton's principle states that the motion of a system from time t_1 to t_2 is such that the line integral (called the action or the action integral),

$$I = \int_{t_1}^{t_2} L dt, \tag{3.1}$$

where $L = T - V$ is the difference between kinetic T and potential V energies, has a stationary value for the actual path of the motion. This means that for a system travelling from t_1 to t_2, out of all possible paths between these two times, it travels along the one for which (3.1) is stationary. Stationary means that the integral along the given path has the same value as along the neighbouring paths (cf. Figure 3.1). This corresponds to the first derivative vanishing. Hence, we can say that the motion of the system is such that the variation of the line integral I for fixed t_1 and t_2 is zero

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) dt = 0. \tag{3.2}$$

3.1.2 Some techniques of the calculus of variations

Before showing that Lagrange's equations can be found from (3.2), we investigate the calculus of variations. A main problem is to find the curve for which some given line integral has a stationary value.

We consider a simple problem of the function $f(y, \dot{y}, x)$ defined on a path $y = y(x)$ between two values x_1 and x_2. Here we denote the derivative with respect to x by a

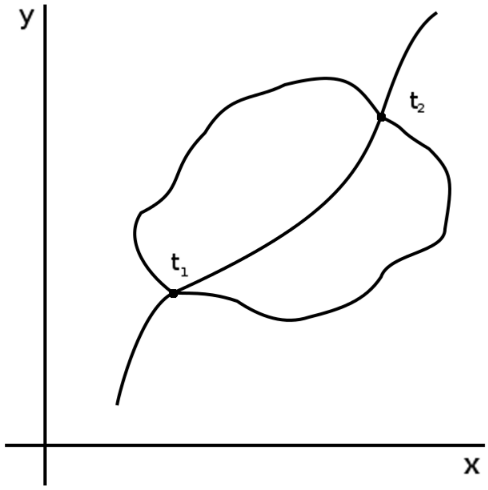

Figure 3.1: *Path of the system point in configuration space.*

dot, for example $\dot{y}$. We want to find the path $y(x)$ such that the line integral J of the function f between x_1 and x_2

$$\dot{y} = \frac{dy}{dx}, \qquad J = \int_{x_1}^{x_2} f(y, \dot{y}, x) dx, \tag{3.3}$$

has a stationary value relative to paths which differ infinitesimally from the correct function $y(x)$. Here the variable x takes the same role as the parameter t which we saw previously. We consider paths for which $y(x_1) = y_1$ and $y(x_2) = y_2$ (cf. Figure 3.2). Note that this problem is one dimensional and y is not a coordinate, just a function of x.

We will use differential calculus to find the stationary points of our function. J must have a stationary value for the correct path relative to any neighbouring path. So, the variation must be zero relative to some particular set of paths labelled by an infinitesimal parameter α. This set of paths will be denoted $y(x, \alpha)$, whereas $y(x, 0)$ represents the correct path. If a function $\eta(x)$ vanishes at x_1 and x_2, then a possible set of paths will be given by

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x). \tag{3.4}$$

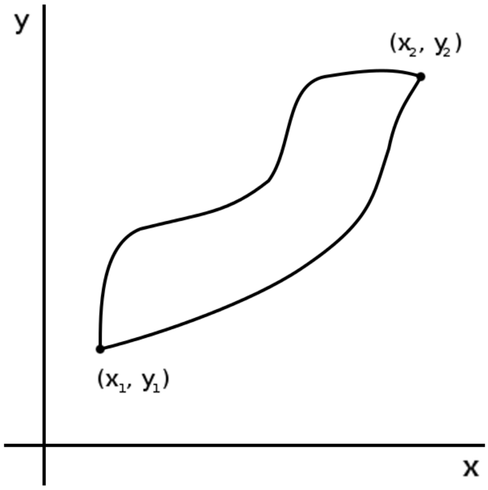

Figure 3.2: *Varied paths of the function of $y(x)$ in the one dimensional extremum problem.*

We assume that the functions $y(x)$ and $\eta(x)$ are both well behaved, continuous and non singular between x_1 and x_2, with continuous first and second derivatives between the same two points. For this family of curves, J, seen in (3.3), is also a function of α

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)dx. \tag{3.5}$$

The condition for finding a stationary point is given by

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0. \tag{3.6}$$

Taking the derivative of (3.5), we see that

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx. \tag{3.7}$$

The second half of this integral

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial \alpha \partial x}dx, \tag{3.8}$$

integrates by parts to

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial \alpha \partial x}dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}dx. \tag{3.9}$$

We know that all of the varied curves pass through (x_1, y_1) and (x_2, y_2), so the partial derivative of y with respect to α at x_1 and x_2 must vanish. We can then use (3.9) to rewrite (3.7) in the form

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right) \frac{\partial y}{\partial \alpha} dx, \tag{3.10}$$

as the first terms on the right hand side of equation (3.9) vanish. Hence, looking back to (3.6) we see that the condition for a stationary value can be written

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right) \left(\frac{\partial y}{\partial \alpha} \right)_{\alpha=0} dx = 0. \tag{3.11}$$

The partial derivative of y with respect to α is an arbitrary function of x, with continuity and conditions on the end points. Hence for the integral to vanish it must be the case that the other term vanishes. Hence, for J to have a stationary value we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0. \tag{3.12}$$

The differential quantity

$$\left(\frac{\partial y}{\partial \alpha} \right)_{\alpha=0} d\alpha \equiv \delta y, \tag{3.13}$$

is the infinitesimal departure of the varied path from the correct path $y(x)$, at the point x. As this corresponds to the virtual displacement discussed in Chapter 1 of [66], it will have the notation δy. We can write the variation of J about the correct path in a similar fashion

$$\left(\frac{dJ}{d\alpha} \right)_{\alpha=0} d\alpha \equiv \delta J. \tag{3.14}$$

So, if y satisfies the condition (3.12), then we can write the condition that J is stationary for the correct path as

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right) \delta y dx. \tag{3.15}$$

3.1.3 Derivation of Lagrange's equations from Hamilton's principle

Now we consider the case in which f is a function of many independent variables y_i and their derivatives $\dot{y}_i$. All of these are considered as functions of the parametric variable x. Then, a variation of the integral J between points 1 and 2

$$\delta J = \delta \int_1^2 f(y_1(x), y_2(x), \ldots, \dot{y}_1(x), \dot{y}_2(x), \ldots, x) dx, \tag{3.16}$$

is obtained by considering J as a function of a parameter α, which labels a possible set of curves $y_1(x, \alpha)$. We introduce α by setting

$$\begin{aligned} y_1(x, \alpha) &= y_1(x, 0) + \alpha \eta_1(x), \\ y_2(x, \alpha) &= y_2(x, 0) + \alpha \eta_2(x), \\ \vdots \quad & \quad \vdots \quad \quad \vdots \end{aligned} \tag{3.17}$$

where $y_1(x, 0), y_2(x, 0), \ldots$ are the solutions to the extremum problem and $\eta_1, \eta_2, \ldots$ are independent functions of x. These vanish at the end points and are continuous through the second derivative, but otherwise are arbitrary.

Following the previous section, the variation of J is given by

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha} d\alpha \right) dx. \tag{3.18}$$

Integration by parts is used for the second sum of the above equation

$$\int_1^2 \frac{\partial f}{\partial \dot{y}_i} \frac{\partial^2 \dot{y}_i}{\partial \alpha \partial x} dx = \left. \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha} \right|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_i} \right) dx, \tag{3.19}$$

where the first term vanishes again, as all curves pass through the fixed end points. Hence, by combining (3.18) and (3.19) we find that the variation of J becomes

$$\delta J = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i} \right) \right) \delta y_i dx, \tag{3.20}$$

where the the variation δy_i is given in the familiar way

$$\delta y_i = \left(\frac{\partial y_i}{\partial \alpha} \right)_{\alpha=0} d\alpha. \tag{3.21}$$

As the y variables are independent, the variations δy_i are independent. This means that, as before, for the variation of J to vanish we must have that the terms multiplying the δy_i must vanish. Which tells us that for all $i = 1, 2, \ldots, n$,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i} \right) = 0. \tag{3.22}$$

The equations above, (3.22), are the generalisation of the equation (3.12) seen in the previous section to many variables. These are known as the Euler-Lagrange equations.

For Hamilton's principle, which we saw at the beginning of the chapter,

$$\delta I = \delta \int_1^2 L(q_i, \dot{q}_i, t) dt = 0. \tag{3.23}$$

This has the same form as (3.16). All we need to do to relate the two is consider the transformation

$$\begin{aligned} x &\to t \\ y_i &\to q_i \\ f(y_i, \dot{y}_i, x) &\to L(q_i, \dot{q}_i, t). \end{aligned} \tag{3.24}$$

Then, as the y_i variables are independent, the q_i must also be independent in Hamilton's principle. If this is indeed the case, then we know that the Euler-Lagrange equations for the integral I become the Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad \text{for } i = 1, 2, \ldots, n. \tag{3.25}$$

Hence we have seen that Lagrange's equations of motion follow from Hamilton's principle.

3.2 Variational approach for a fluid system

In the previous section we saw a simple example of how one would use the variational approach to find equations of motion. Next, we will steer towards our goal by applying it to a multifluid system. Although this method is used by many people to model fluids, we will later consider a different approach in Section 3.3. Our approach will use a method whereby we average over a conserved number of vortices. For now we demonstrate the variational approach [67, 68, 69, 70, 71].

We know from previous discussions about neutron stars and superfluid helium, that we require a model comprised of multiple fluids. One fluid component will refer to the superfluid neutrons and the second fluid component refers to the normal fluid which, in the case of a neutron star, will be the proton fluid. We denote the separate fluids by the general labels x and y in the equations below. The variables for a multifluid system are the number density n_{x}, the kinematic velocity v^i_{x} and the mass m_{x}, for each of the x fluid components. From these, we define the mass density

$$\rho_{\mathrm{x}} = m_{\mathrm{x}} n_{\mathrm{x}} \tag{3.26}$$

and the number current

$$n^i_{\mathrm{x}} = n_{\mathrm{x}} v^i_{\mathrm{x}}. \tag{3.27}$$

In order to find the total mass density we sum over each of the fluid components

$$\rho = \sum_{\mathrm{x}} \rho_{\mathrm{x}}, \tag{3.28}$$

where, for example, if we are modelling a two fluid system of neutrons and protons, then $\mathrm{x} \in \{\mathrm{n}, \mathrm{p}\}$. The system is described by a Lagrangian $\mathcal{L}$, which is a function of the number densities n_{x}, number currents n^i_{x} and the gravitational potential Φ. We can write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_{\mathrm{H}}(n_{\mathrm{x}}, n^i_{\mathrm{x}}) + \mathcal{L}_{\mathrm{grav}} - \rho\Phi, \tag{3.29}$$

where the hydrodynamical term is given by

$$\mathcal{L}_{\mathrm{H}} = \sum_{\mathrm{x}} \frac{m_{\mathrm{x}} n^{\mathrm{x}}_i n^i_{\mathrm{x}}}{2n_{\mathrm{x}}} - \mathcal{E} \tag{3.30}$$

and the gravitation contribution is given by

$$\mathcal{L}_{\mathrm{grav}} = -\frac{1}{8\pi G}(\nabla\Phi)^2. \tag{3.31}$$

The energy $\mathcal{E}$ term in the hydrodynamical piece of the Lagrangian represents the equation of state. This generally has the form $\mathcal{E} = \mathcal{E}(n_{\mathrm{x}}, w^2_{\mathrm{xy}})$, where $w^i_{\mathrm{xy}} = v^i_{\mathrm{x}} - v^i_{\mathrm{y}}$ is the velocity difference between the two fluid components.

If we take the variation of our Lagrangian $\mathcal{L}$ with respect to the gravitational potential Φ, we find from (3.25) that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) - \frac{\partial L}{\partial \Phi} = 0, \tag{3.32}$$

is calculated to be

$$\nabla\left(-\frac{2}{8\pi G}\nabla\Phi\right) + \rho = 0, \tag{3.33}$$

which gives us the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho. \tag{3.34}$$

Then, we vary $\mathcal{L}$ with respect to the fluid variables n_{x} and n^i_{x} in the same way as above, in order to find the hydrodynamical equations of motion. We assume that the total number of particles is conserved, which gives us the continuity equation

$$\partial_t n_{\mathrm{x}} + \nabla_i n^i_{\mathrm{x}} = 0. \tag{3.35}$$

Then the conservation of momentum is given by the collective Euler equation

$$\sum_{\mathrm{x}} \left(n_{\mathrm{x}} \partial_t \pi^i_{\mathrm{x}} - n_{\mathrm{x}} \nabla^i \pi^{\mathrm{x}}_0 - \epsilon^{ijk} n^{\mathrm{x}}_j \epsilon_{klm} \nabla^l \pi^m_{\mathrm{x}} \right) = 0, \tag{3.36}$$

where π^{x}_i is the canonical momenta

$$\pi^{\mathrm{x}}_i = \frac{\partial \mathcal{L}}{\partial n^i_{\mathrm{x}}} = p^{\mathrm{x}}_i \tag{3.37}$$

and p^{x}_i is the hydrodynamical momenta

$$p^{\mathrm{x}}_i = \frac{\partial \mathcal{L}_{\mathrm{H}}}{\partial n^i_{\mathrm{x}}} = m_{\mathrm{x}} \left(v^{\mathrm{x}}_i + \sum_{\mathrm{x}} \varepsilon_{\mathrm{xy}} w^{\mathrm{yx}}_i \right). \tag{3.38}$$

In the case of magnetohydrodynamics, as in [68], the canonical momenta (3.37) have an extra term which originates from the electromagnetic contributions in the Lagrangian. But, here we are only considering the hydrodynamical case without an electromagnetic field, so these do not occur in our calculation. The momenta above show the entrainment term $\varepsilon_{\mathrm{xy}}$ entering the equations. The entrainment between the different fluids

$$\varepsilon_{\mathrm{xy}} = \frac{2}{\rho_{\mathrm{x}}} \left(\frac{\partial \mathcal{E}}{\partial w^2_{\mathrm{xy}}} \right)_{n_{\mathrm{x}}}, \tag{3.39}$$

can cause the misalignment of the momentum p^i_{x} and the corresponding flux n^i_{x}. Also seen above is the scalar function π^{x}_0, which is given by

$$\pi^{\mathrm{x}}_0 = -\mu_{\mathrm{x}} - \frac{1}{2} m_{\mathrm{x}} v^2_{\mathrm{x}} - m_{\mathrm{x}} \Phi, \tag{3.40}$$

where there chemical potential μ_{x} for each fluid is given by

$$\mu_{\mathrm{x}} = \left(\frac{\partial \mathcal{E}}{\partial n_{\mathrm{x}}} \right)_{n_{\mathrm{y}}, w^2_{\mathrm{xy}}}. \tag{3.41}$$

The momentum equation (3.36) can be written in terms of a force balance equation

$$\sum_{\mathrm{x}} \left(F_{\mathrm{Hx}^i} + \rho \nabla^i \Phi \right) = 0, \tag{3.42}$$

where any external forces acting on the system would appear on the right hand side of the above equation. The hydrodynamical force density seen above is given by [72]

$$F^i_{\mathrm{Hx}} = n_{\mathrm{x}} \left(\left(\partial_t + v^j_x \nabla_j \right) p^i_{\mathrm{x}} + \nabla^i \mu_{\mathrm{x}} + m_{\mathrm{x}} \sum_{\mathrm{y}} (\varepsilon_{\mathrm{xy}} w^{\mathrm{yx}}_j) \nabla^i v^j_{\mathrm{x}} \right). \tag{3.43}$$

3.3 Single fluid system

As we learnt in Chapter 2, our aim is to model a superfluid system. The previous section introduced a multifluid model but we are yet to include information regarding the vortex array within the system. Here we will formulate the superfluid equation of motion for a single superfluid, including some interaction from a second fluid. We will consider this fluid to be rotating, which means that we will need to include details about the vortex array. To do this, we say that the vortices are quantised and we use a quantised vorticity vector to determine the equations of motion for the fluid [61, 68, 73].

3.3.1 Quantised vortices

We will consider a system of superfluid neutrons, as one would expect to find such a fluid in the interior of a neutron star. A neutron star will be rotating about its axis and can be thought of as a rotating container for the superfluid neutrons. Hence, due to the stars rotation, the superfluid will form an array of tiny quantised vortices. The number of vortices in this array depends upon the magnitude of the rotation of the star, but we will say that the direction of the array is along $\hat{\kappa}_i$. As the vortices are quantised, we will say that they have a number density $\mathcal{N}_{\mathrm{v}}$ and we can write the macroscopically averaged vorticity of the system as

$$\mathcal{W}_{\rm n}^i = \frac{1}{m}\epsilon^{ijk}\nabla_j p_k^{\rm n} = \mathcal{N}_{\rm v}\kappa^i. \tag{3.44}$$

Here we see the canonical momentum $p_{\rm n}^i$, the quantum of circulation $\kappa = h/2m$ is contained within the vortex direction $\kappa_i = \kappa\hat{\kappa}_i$ and the mass is denoted m. This equation for the averaged vorticity, (3.44), is known as the Onsager-Feynman condition. If the rotation of the star, and hence the fluid, is constant, then no vortices will be destroyed or created. Hence, we will assume here that the number of vortices is conserved, which gives us a continuity equation for the number density

$$\partial_t \mathcal{N}_{\rm v} + \nabla_j\left(\mathcal{N}_{\rm v} u^j\right) = 0, \tag{3.45}$$

where u^i is the collective vortex velocity within a fluid element. Our task is to find out how the vortices behave, so we take the time derivative of (3.44) and see that

$$\partial_t \mathcal{W}_{\rm n}^i = -\kappa^i\nabla_j\left(\mathcal{N}_{\rm v} u^j\right) + \mathcal{N}_{\rm v}\partial_t\kappa^i. \tag{3.46}$$

Combining this with the fact that $\nabla_j \mathcal{W}_{\rm n}^j = 0$, we find

$$\partial_t \mathcal{W}_{\rm n}^i = \nabla_j\left(\mathcal{W}_{\rm n}^j u^i\right) - \nabla_j\left(\mathcal{W}_{\rm n}^i u^j\right) + \mathcal{N}_{\rm v}\left(\partial_t\kappa^i + u^j\nabla_j\kappa^i - \kappa^j\nabla_j u^i\right). \tag{3.47}$$

The motion of a single vortex is represented by the Lie transportation of the vector κ^i, which denotes the direction of the vortex, by the flow u^i. This is written

$$\partial_t\kappa^i + \mathcal{L}_u\kappa^i = 0, \tag{3.48}$$

where the Lie derivative is defined by

$$\mathcal{L}_u\kappa^i = u^j\nabla_j\kappa^i - \kappa^j\nabla_j u^i. \tag{3.49}$$

Using this equation of motion, we can rewrite (3.47) as

$$\partial_t \mathcal{W}_{\rm n}^i + \epsilon^{ijk}\nabla_j\left(\epsilon_{klm}\mathcal{W}_{\rm n}^l u^m\right) = 0, \tag{3.50}$$

which shows that canonical vorticity $\mathcal{W}^i_{\rm n}$, is locally conserved and advected by the u^i flow. If we rewrite the evolution equation (3.50) as a total outer product, we find that

$$\partial_t p^i_{\rm n} - \epsilon^{ijk}\epsilon_{klm} u_j \nabla^l p^m_{\rm n} = \nabla^i \Psi_{\rm n}, \tag{3.51}$$

where $\Psi_{\rm n}$ is an unspecified scalar potential.

In reality, it is unlikely that the vortices and the superfluid will be moving together. There are a number of forces which would disturb the system and cause the vortices to travel with a different velocity to that of the fluid, which we will see later on. In light of this, we will now assume that the vortices do not move with the flow by introducing a velocity difference $w^i_{\rm nv} = v^i_{\rm n} - u^i$, where $v^i_{\rm n}$ denotes the velocity of the superfluid neutrons. This enables us to rewrite (3.51), using the velocity difference and the definition of the vorticity (3.44), as

$$n_{\rm n}\partial_t p^i_{\rm n} - n_{\rm n}\epsilon^{ijk}\epsilon_{klm} v^{\rm n}_j \nabla^l p^m_{\rm n} - n_{\rm n}\nabla^i\Psi_{\rm n} = \mathcal{N}_{\rm v}\rho_{\rm n}\epsilon^{ijk}\kappa_j w^{\rm nv}_k. \tag{3.52}$$

We see that we now have a force appearing the right hand side of (3.52), due to the velocity difference between the vortices and the superfluid neutrons. This is in fact the Magnus force exerted on the vortices by the fluid. Obviously this only enters the picture when $w^{\rm nv}_k$ is non zero, or in other words, when the vortices and the fluid are not moving together. A more common example of the Magnus effect is when a rotating sphere moving through a fluid curves to one side, as seen in any sport involving a struck ball. In our case, the Magnus force displaces vortices from their equilibrium position due to the fluid flow past them. The Magnus force is given by

$$f^i_{\rm M} = -\mathcal{N}_{\rm v}\rho_{\rm n}\epsilon^{ijk}\kappa_j w^{\rm nv}_k, \tag{3.53}$$

which has a negative sign because it is defined as a lifting force. Hence, we have arrived at

$$n_{\rm n}\partial_t p^i_{\rm n} - n_{\rm n}\epsilon^{ijk}\epsilon_{klm} v^{\rm n}_j \nabla^l p^m_{\rm n} - n_{\rm n}\nabla^i\Psi_{\rm n} = -f^i_{\rm M} \tag{3.54}$$

as our equation of motion for the single neutron superfluid. This incorporates the quantised nature of the vortices and also takes into consideration that the vortices will not be travelling along with the neutron superfluid.

3.3.1.1 A second look at vortex number conservation

We note here that the conservation of number density used above,

$$\partial_t \mathcal{N}_\mathrm{v} + \nabla_j \left(\mathcal{N}_\mathrm{v} u^j \right) = 0, \tag{3.55}$$

is usually assumed to hold. The same is true for the motion of a single vortex,

$$\partial_t \kappa^i + \mathcal{L}_u \kappa^i = 0. \tag{3.56}$$

However, it could be the case that they are only conserved in the vortex plane. If we consider this new planar approach, then the calculation would be as follows. If the vortex number density is conserved only in the vortex plane, it would be written

$$\partial_t \mathcal{N}_\mathrm{v} + \nabla_j^\perp \left(\mathcal{N}_\mathrm{v} u^j \right) = 0, \tag{3.57}$$

where the projection is given by $\nabla_j^\perp = \perp^k{}_j \nabla_k = \left(\delta_j^k - \hat{\kappa}_j \hat{\kappa}^k \right) \nabla_k$. We now take the time derivative of the vorticity to see that

$$\partial_t \mathcal{W}_\mathrm{n}^i = -\kappa^i \nabla_j^\perp \left(\mathcal{N}_\mathrm{v} u^j \right) + \mathcal{N}_\mathrm{v} \partial_t \kappa^i. \tag{3.58}$$

Expanding and manipulating this equation leads to

$$\partial_t \mathcal{W}_\mathrm{n}^i = -\nabla_j^\perp (u^j \mathcal{W}^i) + \mathcal{N}_\mathrm{v} u^j \nabla_j^\perp \kappa^i + \mathcal{N}_\mathrm{v} \partial_t \kappa^i. \tag{3.59}$$

Here we will say that the motion of a single vortex is given by

$$\partial_t \kappa^i + \perp^i{}_j \mathcal{L}_u \kappa^j = 0, \tag{3.60}$$

where the Lie derivative is defined as before. Remembering that $\nabla_j \mathcal{W}_\mathrm{n}^j = 0$ while plugging the equation of motion of a single vortex into (3.59), leads us to

$$\partial_t \mathcal{W}_\mathrm{n}^i = \nabla_k (u^i \mathcal{W}_\mathrm{n}^k) - \nabla_j (u^j \mathcal{W}_\mathrm{n}^i). \tag{3.61}$$

This should look familiar as it gives (3.50) from the previous calculation.

Hence, we see that whether the vortex number density conservation and single vortex motion is restricted to the vortex plane or not, the equations of motion are the same. We will see this projected version of the conservation equation and motion of a single vortex later on in Chapter 5.

3.3.2 Mutual friction

As we have discussed previously, the second fluid interacts with the vortices through a mutual friction force [74, 75]. This is due to scattering of the second fluid component off of the vortex cores. We will now add mutual friction into our system to balance with the Magnus force. The mutual friction force is proportional to the velocity difference between the vortices u^i and the second fluid v_p^i. Hence it is written in the form

$$f_i^F = C \left(v_i^\mathrm{p} - u_i \right), \tag{3.62}$$

for the friction force on a single vortex. Ignoring vortex inertia, these two forces balance each other such that

$$f_i^\mathrm{M} + f_i^\mathrm{F} = 0, \tag{3.63}$$

which allows us to rewrite the vortex velocity in the form

$$u_i = v_i^\mathrm{p} - \frac{\rho_\mathrm{n}}{C} \epsilon_{ijk} \kappa^j \left(v_\mathrm{n}^k - u^k \right). \tag{3.64}$$

Next, we take the cross product of this form of the vortex velocity with κ^i to find

$$\epsilon_{ijk} \kappa^j u^k = \epsilon_{ijk} \kappa^j v_\mathrm{p}^k - \frac{\rho_\mathrm{n}}{C} \epsilon_{ijk} \epsilon^{klm} \kappa^j \kappa_l \left(v_m^\mathrm{n} - u_m \right), \tag{3.65}$$

and cross it again to give

$$\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l u_m = \epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l v^{\mathrm{p}}_m - \frac{\rho_{\mathrm{n}}}{C}\epsilon_{ijk}\epsilon^{klm}\epsilon_{mqr}\kappa^j\kappa^l\kappa_q\left(v^r_{\mathrm{n}} - u^r\right). \tag{3.66}$$

If we focus on the final term of (3.66) for a moment, we can expand it to see that

$$\frac{\rho_{\mathrm{n}}}{C}\epsilon_{ijk}\epsilon^{klm}\epsilon_{mqr}\kappa^j\kappa^l\kappa_q\left(v^r_{\mathrm{n}} - u^r\right) = -\frac{\rho_{\mathrm{n}}}{C}\epsilon_{ijk}\kappa^j\kappa^2\left(v^k_{\mathrm{n}} - u^k\right), \tag{3.67}$$

remembering that $\epsilon_{ijk}\kappa^j\kappa^k = 0$. This means that (3.66) becomes

$$\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l u_m = \epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l v^{\mathrm{p}}_m + \frac{\rho_{\mathrm{n}}\kappa^2}{C}\epsilon_{ijk}\kappa^j\left(v^k_{\mathrm{n}} - u^k\right), \tag{3.68}$$

which we can substitute into the first equation we crossed with κ^i (3.65) to give

$$\begin{aligned}\epsilon_{ijk}\kappa^j u^k = \epsilon_{ijk}\kappa^j v^k_{\mathrm{p}} - \frac{\rho_{\mathrm{n}}}{C}\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l v^{\mathrm{n}}_m + \frac{\rho_{\mathrm{n}}}{C}\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l v^{\mathrm{p}}_m \\ + \frac{\rho^2_{\mathrm{n}}\kappa^2}{C^2}\epsilon_{ijk}\kappa^j\left(v^k_{\mathrm{n}} - u^k\right).\end{aligned} \tag{3.69}$$

We can collect the vortex velocities

$$\begin{aligned}\left(1 + \frac{\rho^2_{\mathrm{n}}\kappa^2}{C^2}\right)\epsilon_{ijk}\kappa^j u^k = \epsilon_{ijk}\kappa^j v^k_{\mathrm{p}} + \frac{\rho_{\mathrm{n}}}{C}\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l\left(v^{\mathrm{p}}_m - v^{\mathrm{n}}_m\right) \\ + \frac{\rho^2_{\mathrm{n}}\kappa^2}{C^2}\epsilon_{ijk}\kappa^j v^k_{\mathrm{n}},\end{aligned} \tag{3.70}$$

which we do in order to plug it into our first equation (3.64). This gives

$$\begin{aligned}u_i = v^{\mathrm{p}}_i + \frac{C}{\rho_{\mathrm{n}}\kappa^2}\left(\frac{1}{1 + C^2/\rho^2_{\mathrm{n}}\kappa^2}\right)\epsilon_{ijk}\kappa^j\left(v^k_{\mathrm{p}} - v^k_{\mathrm{n}}\right) \\ + \frac{1}{\kappa^2}\left(\frac{1}{1 + C^2/\rho^2_{\mathrm{n}}\kappa^2}\right)\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l\left(v^{\mathrm{p}}_m - v^{\mathrm{n}}_m\right).\end{aligned} \tag{3.71}$$

Finally, we plug this into our original form of the mutual friction force (3.62), to see that

$$f^F_i = \left(\frac{C\rho_{\mathrm{n}}}{\rho^2_{\mathrm{n}}\kappa^2 + C^2}\right)\left[C\epsilon_{ijk}\kappa^j w^k_{\mathrm{np}} + \rho_{\mathrm{n}}\epsilon_{ijk}\epsilon^{klm}\kappa^j\kappa_l w^{\mathrm{np}}_m\right]. \tag{3.72}$$

Since this is the force per vortex, we can multiply by the vortex number density in order to get the mutual friction force on the whole array of vortices

$$f_i^F = \mathcal{N}_\mathrm{v}\left(\frac{C\rho_\mathrm{n}}{\rho_\mathrm{n}^2\kappa^2 + C^2}\right)\left[C\epsilon_{ijk}\kappa^j w_\mathrm{np}^k - \rho_\mathrm{n}\kappa^2 \perp_{ij} w_\mathrm{np}^j\right], \tag{3.73}$$

where $\perp_{ij} = g_{ij} - \hat{\kappa}_i\hat{\kappa}_j$ is the projection orthogonal to $\hat{\kappa}^i = \kappa^i/\kappa$. The first term in the mutual friction force (3.73) is analogous to the Magnus force, although it is now expressed in terms of the velocity difference w_i^np instead of w_i^nv. The second term introduces dissipation into the flow. From the equations we've used above, we can determine that [74]:

1. in the limit $C \to \infty$ we have $u_i \to v_i^\mathrm{p}$. This means that the vortices are strongly coupled to the second fluid.
2. in the opposite limit $C \to 0$ we find that $u_i \to v_i^\mathrm{n}$. The vortices must flow with the neutron superfluid as they are weakly coupled to the second fluid.
3. the dissipative part of the mutual friction force vanishes in both the $C \to \infty$ and the $C \to 0$ limit.

The factor of C is usually rewritten in terms of a dimensionless parameter $\mathcal{B}$. The analysis of this and the estimates for these parameters can be found in [74]. The dimensionless parameter is related to C via

$$\mathcal{B} = \frac{C}{\rho_\mathrm{n}\kappa}, \tag{3.74}$$

which leads to the mutual friction force

$$f_i^F = \rho_\mathrm{n}\kappa\mathcal{N}_\mathrm{v}\mathcal{B}\left(v_i^\mathrm{p} - u_i\right) = \mathcal{N}_\mathrm{v}\left(\frac{\mathcal{B}}{1+\mathcal{B}^2}\right)\left[\mathcal{B}\rho_\mathrm{n}\epsilon_{ijk}\kappa^j w_\mathrm{np}^k - \rho_\mathrm{n}\kappa \perp_{ij} w_\mathrm{np}^j\right]. \tag{3.75}$$

3.3.3 Elasticity

We have seen in previous sections that there are two forces which move the vortex array out of equilibrium. The first is the Magnus force, caused by the fluid flow past the vortices, and the second is the mutual friction force, caused by interactions between the second fluid component and the vortices. We will now add a restoring force, which attempts to return the vortex array to equilibrium and cancel out the Magnus and mutual friction effects. We treat this restoring force as an elasticity, acting between each of vortices. This elastic behaviour of the vortices is seen in experiments using superfluid helium [4, 35, 36], so it is logical to include it in our model. We say that this elastic force balances the Magnus and mutual friction forces, enabling us to write a new force balance equation

$$f^i_{\mathrm{M}} + f^i_{\mathrm{F}} + f^i_{\mathrm{E}} = 0. \tag{3.76}$$

The elastic force is usually denoted σ_i, and is found from the elastic stress tensor [76],

$$\sigma_i{}^k = K\xi_l{}^l \delta_i{}^k + 2\mu \left(\xi_i{}^k - \frac{1}{3}\delta_i{}^k \xi_l{}^l \right), \tag{3.77}$$

where K is the bulk modulus and is related to Lamé coefficients by $K = \lambda + (2/3)\mu$. Also, the two index $\xi_i{}^k$ is related to the elastic displacement ξ_i^{E}, by

$$\xi_i{}^k = \frac{1}{2}\left(\frac{\partial \xi_i^{\mathrm{E}}}{\partial x_k} + \frac{\partial \xi^k_{\mathrm{E}}}{\partial x^i} + \frac{\partial \xi_l^{\mathrm{E}}}{\partial x_k}\frac{\partial \xi^l_{\mathrm{E}}}{\partial x^i} \right). \tag{3.78}$$

The elastic displacement ξ_i^{E} of a vortex that we see here is the difference between its displaced position and equilibrium position, $x'_i - x_i$. We will denote this elastic vortex displacement ξ_i^{v} from now on.

As the displacement of each vortex is extremely small, we ignore higher order terms and instead use

$$\xi_i{}^k = \frac{1}{2}\left(\frac{\partial \xi_i^{\mathrm{v}}}{\partial x_k} + \frac{\partial \xi^k_{\mathrm{v}}}{\partial x^i} \right). \tag{3.79}$$

In our single fluid system (3.54), the force term that entered our equation was $-f_i^{\mathrm{M}}$, so we will balance this and the mutual friction force with the elastic term σ_i. This gives us

$$-f_i^{\mathrm{M}} - f_i^{\mathrm{F}} = f_i^{\mathrm{E}} = \sigma_i, \tag{3.80}$$

where the elasticity σ_i is found by taking the derivative of the stress tensor (3.77),

$$\sigma_i = \nabla_k \sigma_i^{\ k}. \tag{3.81}$$

The elasticity of the vortex array acts in the plane of the vortices. There is no elasticity acting along the vortex direction, so

$$\sigma_i^{\ j} \hat{\kappa}_j = 0. \tag{3.82}$$

Using this with (3.81) we see that the elasticity is given by

$$\sigma_i = \nabla_k^{\perp} \sigma_i^{\ k}, \tag{3.83}$$

where $\nabla_i^{\perp}$ is the gradient perpendicular to the direction of the array. We could also define the perpendicular gradient using the projection $(g_{ij} - \hat{\kappa}_i \hat{\kappa}_j)\nabla^j = \nabla_i^{\perp}$ as we saw previously. From equation (3.83) we find that the full elastic term, in terms of the vortex displacement, is given by

$$\sigma_i = \mu \left[\left(\frac{K}{\mu} + \frac{1}{3} \right) \nabla_i^{\perp} \left(\nabla_{\perp}^k \xi_k^{\mathrm{v}} \right) + \nabla_k^{\perp} \left(\nabla_{\perp}^k \xi_i^{\mathrm{v}} \right) \right]. \tag{3.84}$$

As well as exhibiting elasticity in the vortex array, there may also be a form of elasticity along the vortex lines themselves. It is possible to consider this scenario of elasticity along the vortices by including an averaged vortex tension. The tension term would be of the form [68],

$$\mathcal{T}_{\mathrm{n}}^i = \frac{\mathcal{E}_{\mathrm{vn}}}{\kappa} \mathcal{W}_{\mathrm{n}}^j \nabla_j \hat{\mathcal{W}}_{\mathrm{n}}^i, \tag{3.85}$$

where $\mathcal{E}_{\mathrm{vn}}$ is the energy per unit length of a single vortex, however, we will not be considering vortex tension in this calculation.

3.3.3.1 A more general elasticity

We note here that although equation (3.84) has the familiar and recognisable form of the two dimensional vortex elasticity, this assumes that $\hat{\kappa}^i$ will be fixed along a coordinate direction. Later on we may prefer a more general form for the elasticity, whereby the direction of the array is not necessarily along a coordinate axis. Hence we need to use projections to form a general elasticity term. If we start from (3.81), we can use the condition (3.82) to find an equation similar to (3.83). Using the projection $\perp_{ij} = g_{ij} - \hat{\kappa}_i\hat{\kappa}_j$, this is given by

$$\sigma_i = \nabla_j^{\perp}\sigma_i{}^j + \sigma_i{}^j\nabla_k\perp_j^k, \tag{3.86}$$

which can be written in terms of $\hat{\kappa}^i$

$$\sigma_i = \nabla_j^{\perp}\sigma_i{}^j - \sigma_i{}^j\hat{\kappa}^k\nabla_k\hat{\kappa}_j. \tag{3.87}$$

In order for (3.83) to hold, we must assume that $\hat{\kappa}^i$ is a Killing vector. This means that it must satisfy the Killing equation $\nabla_i\hat{\kappa}_j + \nabla_j\hat{\kappa}_i = 0$, which will kill the second term in (3.87). We can now see that the elastic term takes the form

$$\sigma_i = \mu\left[\left(\frac{K}{\mu} - \frac{2}{3}\right)\nabla_i^{\perp}\nabla_k\xi_{\rm v}^k + \nabla_k^{\perp}\nabla_i^{\perp}\xi_{\rm v}^k + \nabla_k^{\perp}(\perp_{ij}\nabla_{\perp}^k\xi_{\rm v}^j)\right]. \tag{3.88}$$

As $\hat{\kappa}^i$ is no longer along a coordinate direction, we cannot commute the orthogonal derivatives. This is due to the fact that $\nabla_i\perp_{jk} \neq 0$. This is the form of elasticity orthogonal to a general vector $\hat{\kappa}^i$.

3.4 Two fluid system

As we wish to improve our model in order to more closely replicate what we expect to be occurring inside of a neutron star, we need to specify an equation for the second fluid. As we have discussed previously, the interior of a neutron star is thought to be mainly comprised of neutrons. However, there are also other particles present such as protons and electrons, which we will now incorporate into our model. We will therefore consider a two fluid system comprising of both neutrons and protons, where we are assuming that the electrons are strongly coupled to the proton fluid. In the previous section we discussed the mutual friction force which arises due to the presence of a second fluid and here we will show the equations governing this fluid. The Euler equations for a multiple fluid system, in a frame rotating at angular velocity Ω, are given by

$$(\partial_t + v^{\rm x}_j \nabla^j)(v^{\rm x}_i + \varepsilon_{\rm x} w^{\rm yx}_i) + 2\epsilon_{ijk}\Omega^j v^k_{\rm x} + \nabla(\tilde{\mu}_{\rm x} + \Phi) + \varepsilon_{\rm x} w^j_{\rm yx} \nabla_i v^{\rm x}_j = 0. \quad (3.89)$$

Where ${\rm x}, {\rm y} \in \{{\rm n}, {\rm p}\}$ denote the components associated to the neutron or proton fluid, $v^{\rm x}_i$ is the velocity of the specified fluid, $w^{\rm yx}_i = v^{\rm y}_i - v^{\rm x}_i$ gives the difference between the fluid velocities, $\varepsilon_{\rm x}$ is the entrainment parameter, $\tilde{\mu}_{\rm x}$ is the chemical potential per unit mass and $\Phi_{\rm x}$ is the gravitational potential. We assume that the neutrons and protons have equal mass, $m_n = m_p = m$. Since we are considering a rotating star, we wish to include the vorticity in order to impose the neutrons be superfluid. Under rotation, the superfluid neutrons form an array of singly quantised vortices. We will average over them, as before, to acquire the macroscopic vorticity

$$\mathcal{W}^i = \mathcal{N}_{\rm v} \kappa \hat{\kappa}^i = \frac{1}{m} \epsilon^{ijk} \nabla_j (v^{\rm n}_k + \varepsilon^{\rm n} w^{\rm pn}_k), \quad (3.90)$$

where $\hat{\kappa}^i$ is a unit vector in the direction of the vortex array, $\kappa = 1.99 \times 10^{-3}$ cm^{2}s^{-1} is the quantum of circulation and $\mathcal{N}_{\rm v}$ is the number density of the vortices. We assume that the number of vortices is conserved, which gives us

$$\partial_t \mathcal{N}_{\rm v} + \nabla^{\perp}_i (\mathcal{N}_{\rm v} u^i) = 0, \quad (3.91)$$

where u^i is the macroscopically averaged vortex velocity and $\perp$ denotes orthogonality to the vortex direction $\hat{\kappa}^i$, as before. We also find, from the definition of the vorticity (3.90), that

$$\partial_t \mathcal{W}_i + \epsilon_{ijk} \epsilon^{klm} \nabla^j \left(\mathcal{W}_l u_m\right) = 0. \quad (3.92)$$

By using (3.91) and (3.92) in the same way as our formulation in Section 3.3, we can find an equation of motion for the neutron superfluid,

$$(\partial_t + v^{\rm n}_j \nabla^j)(v^{\rm n}_i + \varepsilon_{\rm n} w^{\rm pn}_i) + 2\epsilon_{ijk}\Omega^j v^k_{\rm n} + \nabla(\tilde{\mu}_{\rm n} + \Phi) + \varepsilon_{\rm n} w^j_{\rm pn} \nabla_i v^{\rm n}_j = \mathcal{N}_{\rm v} \kappa \epsilon_{ijk} \hat{\kappa}^j (v^k_{\rm n} - u^k). \quad (3.93)$$

If we were to consider the case without entrainment, when $\varepsilon_i^{\rm n}$ is zero, this equation would match (3.54) up to a rotational term. The extra terms appearing in (3.93) are due to the presence of a second fluid in the system, when compared to the single fluid system in Section 3.3. The term on the right hand side of (3.93) should be recognisable as the Magnus force, which we saw in the single fluid case. The vortices will have an effect on the movement of the proton fluid, which we will write in the form of a drag force $\rho_{\rm n}\kappa\mathcal{N}_{\rm v}\mathcal{R}(u^i - v_{\rm p}^i)$. As we have seen, this is the mutual friction caused by interactions between the second fluid and the vortices. $\mathcal{R}$ is a dimensionless drag parameter and varies depending on what region of the neutron star we are looking at (from $\mathcal{R} \approx 10^{-10}$ to $\mathcal{R} \approx 1$). In the core, drag occurs due to the scattering of electrons off of the vortex core, whereas in the crust there will be interactions with the lattice phonons. Hence, we would like to investigate how the variation of the drag parameter, $\mathcal{R}$, affects the modes in this system.

Now that we know how the vortices interact with the proton fluid via mutual friction, we can write the second component equation of motion for the protons

$$\begin{aligned}(\partial_t + v_j^{\rm p}\nabla^j)(v_i^{\rm p} - \varepsilon_{\rm p} w_i^{\rm pn}) + 2\epsilon_{ijk}\Omega^j v_{\rm p}^k + \nabla(\tilde{\mu}_{\rm p} + \Phi) \\ - \varepsilon_{\rm p} w_{\rm pn}^j \nabla_i v_j^{\rm p} = \kappa\mathcal{N}_{\rm v}\frac{(1-x_{\rm p})}{x_{\rm p}}\mathcal{R}(u_i - v_i^{\rm p}),\end{aligned} \tag{3.94}$$

where the proton number $x_{\rm p} = \rho_{\rm p}/(\rho_{\rm n}+\rho_{\rm p})$ and the entrainment parameter $\varepsilon_{\rm p} = \varepsilon_{\rm n}(1-x_{\rm p})/x_{\rm p}$. Now that we have formulated the equations both for the superfluid neutrons and the proton fluid, we would like to have an equation governing the vortices. This comes in the form of the force balance equation between the Magnus force, mutual friction and the elastic force of the vortex lattice which we saw previously

$$\mathcal{N}_{\rm v}\kappa\epsilon_{ijk}\hat{\kappa}^j(v_{\rm n}^k - u^k) + \kappa\mathcal{N}_{\rm v}\mathcal{R}(u_i - v_i^{\rm p}) + \sigma_i = 0, \tag{3.95}$$

where σ_i is the elasticity given by

$$\sigma_i = \frac{\mu_{\rm v}}{\rho_{\rm n}}\left[2\nabla_i^{\perp}(\nabla_{\perp}^j\xi_j^{\rm v}) - (\nabla_{\perp}^2)\xi_i^{\rm v}\right]. \tag{3.96}$$

Here, $\xi_i^{\rm v}$ is the displacement of the vortex line from its equilibrium position,

$$\mu_{\rm v} = \frac{\rho_{\rm n}\kappa^2\mathcal{N}_{\rm v}}{16\pi} \tag{3.97}$$

is the shear modulus of the lattice and $\nabla_i^{\perp}$ is again the gradient perpendicular to the direction of the vortex array. This elasticity only takes into consideration linear order lattice displacements, as they are assumed to be very small. As before, we will not consider the contribution due to vortex tension here but it could be included using a term of the form [77]

$$\sigma_i^T = -\frac{\rho_{\rm n}\kappa^2\mathcal{N}_v}{8\pi}\ln\left(\frac{b}{a}\right)\frac{\partial^2\epsilon_i}{\partial z^2}, \tag{3.98}$$

where a is the vortex core radius, b is the intervortex spacing for a triangular lattice and the rotation axis of the star is taken to be in the z direction.

We can write the continuity equations for the neutrons and protons as

$$\partial_t\rho_{\rm n} + \nabla^i(\rho_{\rm n}v_i^{\rm n}) = 0, \tag{3.99}$$

$$\partial_t\rho_{\rm p} + \nabla^i(\rho_{\rm p}v_i^{\rm p}) = 0, \tag{3.100}$$

and the gravitational potential obeys the Poisson equation

$$\nabla^2\Phi = 4\pi G(\rho_{\rm n} + \rho_{\rm p}), \tag{3.101}$$

where G is the gravitational constant. Next, we shall consider how this multifluid system of neutrons and protons behaves when it is perturbed.

3.4.1 Perturbations

We are only going to consider linear order perturbations, in a background in which the two fluids rotate together with angular velocity Ω. In this background we have

$$\kappa\mathcal{N}_{\rm v} = 2\Omega. \tag{3.102}$$

We now perturb the equations of motion for each of the fluids, which we found in the previous section, to find

$$\partial_t(\delta v_i^{\rm n} + \varepsilon_{\rm n}\delta w_i^{\rm pn}) + 2\epsilon_{ijk}\Omega^j\delta v_{\rm n}^k + \nabla\delta\tilde{\mu}_{\rm n} = -2\Omega\mathcal{R}(\delta u_i - \delta v_i^{\rm p}) - \delta\sigma_i \tag{3.103}$$

and

$$\partial_t(\delta v_i^{\rm p} - \varepsilon_{\rm p}\delta w_i^{\rm pn}) + 2\epsilon_{ijk}\Omega^j\delta v_{\rm p}^k + \nabla\delta\tilde{\mu}_{\rm p} = 2\Omega\frac{(1-x_{\rm p})}{x_{\rm p}}\mathcal{R}(\delta u_i - \delta v_i^{\rm p}), \tag{3.104}$$

where we have ignored the perturbations of the gravitational potential $\delta\Phi$, using Cowling approximation. Here we have made use of the force balance equation (3.95) and you can see that the elasticity only acts on the superfluid neutrons. The perturbed elastic force, $\delta\sigma_i$, is written as

$$\delta\sigma_i = c_T^2\left[2\nabla_i^{\perp}(\nabla_{\perp}^j\delta\xi_j^{\rm v}) - (\nabla_{\perp}^2)\delta\xi_i^{\rm v}\right], \tag{3.105}$$

where we have defined the Tkachenko wave speed $c_T^2 = \kappa\Omega/8\pi$. We have also assumed that the vortices are in equilibrium in the background, which means that the displacement will vanish unless the system is perturbed. Hence, all of the following displacements will be perturbed quantities and we will drop the perturbation symbol for them. We would like to consider Lagrangian perturbations of the two fluid system using $\Delta_{\rm v}u^i = \partial_t\xi_{\rm v}^i$, as we are already dealing with the displacement vector $\xi_{\rm v}^i$. But, since we are in a rotating frame and the fluids moving together in the background, we find that $\Delta_{\rm v}u^i = \delta u^i$. Hence we proceed by using Eulerian perturbations, however, we will revisit Lagrangian perturbations in later sections. When we perturb the force balance equation (3.95), we find

$$2\Omega\epsilon_{ijk}\hat{\kappa}^j\delta v_{\rm p}^k - 2\Omega\epsilon_{ijk}\hat{\kappa}^j\delta w_{\rm pn}^k - 2\Omega\epsilon_{ijk}\hat{\kappa}^j\delta u^k + 2\Omega\mathcal{R}\delta u_i - 2\Omega\mathcal{R}\delta v_i^{\rm p} + \delta\sigma_i = 0, \tag{3.106}$$

and the perturbed continuity equations are given by

$$\partial_t\delta\rho_{\rm n} + \nabla^i(\rho_{\rm n}\delta v_i^{\rm n}) = 0 \tag{3.107}$$

and

$$\partial_t \delta\rho_{\rm p} + \nabla^i(\rho_{\rm p}\delta v_i^{\rm p}) = 0. \tag{3.108}$$

3.4.2 Incompressible fluids

To make our problem simpler, we will consider the case in which the fluids are incompressible, which tells us that $\delta\rho_{\rm x} = 0$. So, from the continuity equations (3.107) and (3.108), we see that

$$\nabla^i \delta v_i^{\rm p} = \nabla^i \delta v_i^{\rm n} = 0. \tag{3.109}$$

Next, we take plane wave perturbations and say that perturbed quantities take the form

$$\delta f_i(\mathbf{x}, t) = \bar{f}_i \exp(ik_j x^j - i\omega t), \tag{3.110}$$

where $\bar{f}_i$ is a constant amplitude. We choose coordinates such that the rotation axis of the system aligns with the z axis and the wave vector k^i is in the x-z plane, giving $k^i = (k\sin\theta, 0, k\cos\theta)$.

Now that we have specified the type of perturbation we will use, we can plug this into the perturbed equations of motion (3.103), (3.104) and (3.106) and get

$$- i\omega\bar{v}_i^{\rm n}(1-\varepsilon_{\rm n}) - i\omega\varepsilon_{\rm n}\bar{v}_i^{\rm p} + 2\epsilon_{ijk}\Omega^j\bar{v}_{\rm n}^k + ik_i\bar{\mu}_{\rm n} = 2\Omega\mathcal{R}(i\omega\bar{\xi}_i^{\rm v} + \bar{v}_i^{\rm p}) - \bar{\sigma}_i, \tag{3.111}$$

$$- i\omega\bar{v}_i^{\rm p}(1-\varepsilon_{\rm p}) - i\omega\varepsilon_{\rm p}\bar{v}_i^{\rm n} + 2\epsilon_{ijk}\Omega^j\bar{v}_{\rm p}^k + ik_i\bar{\mu}_{\rm p} = -2\Omega\frac{(1-x_{\rm p})}{x_{\rm p}}\mathcal{R}(i\omega\bar{\xi}_i^{\rm v} + \bar{v}_i^{\rm p}) \tag{3.112}$$

and

$$\epsilon_{ijk}\hat{\kappa}^j\bar{v}_{\rm p}^k - \epsilon_{ijk}\hat{\kappa}^j\bar{w}_{\rm pn}^k + i\omega\epsilon_{ijk}\hat{\kappa}^j\bar{\xi}_{\rm v}^k - i\omega\mathcal{R}\bar{\xi}_i^{\rm v} - \mathcal{R}\bar{v}_i^{\rm p} + \tilde{\sigma}_i = 0, \tag{3.113}$$

where we have defined $\tilde{\sigma}_i = \bar{\sigma}_i/2\Omega$ and we write $\bar{\mu}_{\rm x}$ as the amplitude of $\delta\tilde{\mu}_{\rm x}$. From the continuity equations (3.109), we can also see that

$$k_j\bar{v}_{\rm p}^j = k_j\bar{v}_{\rm n}^j = 0. \tag{3.114}$$

Next, we contract each of the above equations (3.111), (3.112) and (3.113) with k^i, Ω^i and $\epsilon^{ilm}\Omega_l k_m$. We do this to obtain nine scalar equations, which we can then solve for the frequency ω^2. Contracting the three equations with k^i gives us

$$2\epsilon_{ijk}k^i\Omega^j\bar{v}_{\rm n}^k + ik^2\bar{\mu}_{\rm n} = 2\Omega\mathcal{R}i\omega\bar{\xi}_i^{\rm v}k^i - \bar{\sigma}_i k^i, \tag{3.115}$$

$$2\epsilon_{ijk}k^i\Omega^j\bar{v}_{\rm p}^k + ik^2\bar{\mu}_{\rm p} = -2\Omega\frac{(1-x_{\rm p})}{x_{\rm p}}\mathcal{R}i\omega\bar{\xi}_i^{\rm v}k^i \tag{3.116}$$

and

$$\epsilon_{ijk}k^i\hat{\kappa}^j\bar{v}_{\rm p}^k - \epsilon_{ijk}k^i\hat{\kappa}^j\bar{w}_{\rm pn}^k + i\omega\epsilon_{ijk}k^i\hat{\kappa}^j\bar{\xi}_{\rm v}^k - i\omega\mathcal{R}\bar{\xi}_i^{\rm v}k^i + \tilde{\sigma}_i k^i = 0. \tag{3.117}$$

Then we repeat the process, contracting instead with Ω^i, to give the following three equations

$$-i\omega\bar{v}_i^{\rm n}\Omega^i(1-\varepsilon_{\rm n}) - i\omega\varepsilon_{\rm n}\bar{v}_i^{\rm p}\Omega^i + ik_i\Omega^i\bar{\mu}_{\rm n} = 2\Omega\mathcal{R}\Omega^i(i\omega\bar{\xi}_i^{\rm v} + \bar{v}_i^{\rm p}) - \bar{\sigma}_i\Omega^i, \tag{3.118}$$

$$-i\omega\bar{v}_i^{\rm p}\Omega^i(1-\varepsilon_{\rm p}) - i\omega\varepsilon_{\rm p}\bar{v}_i^{\rm n}\Omega^i + ik_i\Omega^i\bar{\mu}_{\rm p} = -2\Omega\frac{(1-x_{\rm p})}{x_{\rm p}}\mathcal{R}\Omega^i(i\omega\bar{\xi}_i^{\rm v} + \bar{v}_i^{\rm p}) \tag{3.119}$$

and

$$i\omega\mathcal{R}\bar{\xi}_i^{\rm v}\Omega^i + \mathcal{R}\bar{v}_i^{\rm p}\Omega^i - \tilde{\sigma}_i\Omega^i = 0. \tag{3.120}$$

Finally we contract the equations with $\epsilon^{ilm}\Omega_l k_m$, to obtain the final set of three equations

$$\begin{aligned}&-i\omega\bar{v}_i^{\rm n}\epsilon^{ilm}\Omega_l k_m(1-\varepsilon_{\rm n}) - i\omega\varepsilon_{\rm n}\bar{v}_i^{\rm p}\epsilon^{ilm}\Omega_l k_m\\ &+2\epsilon_{ijk}\epsilon^{ilm}\Omega^j\bar{v}_{\rm n}^k\Omega_l k_m = 2\Omega\mathcal{R}\epsilon^{ilm}\Omega_l k_m(i\omega\bar{\xi}_i^{\rm v} + \bar{v}_i^{\rm p}) - \bar{\sigma}_i\epsilon^{ilm}\Omega_l k_m,\end{aligned} \tag{3.121}$$

$$\begin{aligned}&-i\omega\bar{v}_i^{\rm p}\epsilon^{ilm}\Omega_l k_m(1-\varepsilon_{\rm p}) - i\omega\varepsilon_{\rm p}\bar{v}_i^{\rm n}\epsilon^{ilm}\Omega_l k_m + 2\epsilon_{ijk}\epsilon^{ilm}\Omega^j\bar{v}_{\rm p}^k\Omega_l k_m\\ &\quad = -2\Omega\frac{(1-x_{\rm p})}{x_{\rm p}}\mathcal{R}\epsilon^{ilm}\Omega_l k_m(i\omega\bar{\xi}_i^{\rm v} + \bar{v}_i^{\rm p})\end{aligned} \tag{3.122}$$

and

$$\begin{aligned}&\epsilon_{ijk}\epsilon^{ilm}\Omega_l k_m\hat{\kappa}^j\bar{v}_{\rm p}^k - \epsilon_{ijk}\epsilon^{ilm}\Omega_l k_m\hat{\kappa}^j\bar{w}_{\rm pn}^k + i\omega\epsilon_{ijk}\epsilon^{ilm}\Omega_l k_m\hat{\kappa}^j\bar{\xi}_{\rm v}^k\\ &\quad - i\omega\mathcal{R}\bar{\xi}_i^{\rm v}\epsilon^{ilm}\Omega_l k_m - \mathcal{R}\bar{v}_i^{\rm p}\epsilon^{ilm}\Omega_l k_m + \tilde{\sigma}_i\epsilon^{ilm}\Omega_l k_m = 0.\end{aligned} \tag{3.123}$$

In order to simplify the equations above, we will need to plug in the components of each of the terms in the nine equations. In cylindrical coordinates, they are as follows:

- $k_i = (k \sin\theta, 0, k\cos\theta)$
- $\bar{v}^{\mathrm{x}}_i = (\bar{v}^{\mathrm{x}}_1, \bar{v}^{\mathrm{x}}_2, -\bar{v}^{\mathrm{x}}_1 \tan\theta)$ from (3.114)
- $\bar{\xi}^{\mathrm{v}}_i = (\bar{\xi}^{\mathrm{v}}_1, \bar{\xi}^{\mathrm{v}}_2, \bar{\xi}^{\mathrm{v}}_3)$
- $\bar{\sigma}_i = (-c_T^2 k^2 \bar{\xi}^{\mathrm{v}}_1 \sin^2\theta, c_T^2 k^2 \bar{\xi}^{\mathrm{v}}_2 \sin^2\theta, 0)$ from (3.96)
- $\Omega_i = (0, 0, \Omega)$
- $\hat{\kappa}_i = (0, 0, 1)$

Next we plug these into equations (3.115) to (3.123), in order to find the nine equations that need to be solved to find ω^2. We find that

$$-2\Omega\bar{v}^{\mathrm{n}}_2 \sin\theta + ik\bar{\mu}_{\mathrm{n}} = 2\Omega\mathcal{R}i\omega(\bar{\xi}^{\mathrm{v}}_1 \sin\theta + \bar{\xi}^{\mathrm{v}}_3 \cos\theta) - \bar{\sigma}_1 \sin\theta, \tag{3.124}$$

$$-2\Omega\bar{v}^{\mathrm{p}}_2 \sin\theta + ik\bar{\mu}_{\mathrm{p}} = -2\Omega\frac{(1-x_{\mathrm{p}})}{x_{\mathrm{p}}}\mathcal{R}i\omega(\bar{\xi}^{\mathrm{v}}_1 \sin\theta + \bar{\xi}^{\mathrm{v}}_3 \cos\theta), \tag{3.125}$$

$$-\bar{v}^{\mathrm{p}}_2 \sin\theta + \bar{w}^{\mathrm{pn}}_2 \sin\theta - i\omega\bar{\xi}^{\mathrm{v}}_2 \sin\theta - i\omega\mathcal{R}(\bar{\xi}^{\mathrm{v}}_1 \sin\theta + \bar{\xi}^{\mathrm{v}}_3 \cos\theta) + \tilde{\sigma}_1 \sin\theta = 0, \tag{3.126}$$

$$i\omega\bar{v}^{\mathrm{n}}_1(1-\varepsilon_{\mathrm{n}})\tan\theta + i\omega\varepsilon_{\mathrm{n}}\bar{v}^{\mathrm{p}}_1 \tan\theta + ik\bar{\mu}_{\mathrm{n}}\cos\theta = 2\Omega\mathcal{R}(i\omega\bar{\xi}^{\mathrm{v}}_3 - \bar{v}^{\mathrm{p}}_1 \tan\theta), \tag{3.127}$$

$$i\omega\bar{v}^{\mathrm{p}}_1(1-\varepsilon_{\mathrm{p}})\tan\theta + i\omega\varepsilon_{\mathrm{p}}\bar{v}^{\mathrm{n}}_1 \tan\theta + ik\bar{\mu}_{\mathrm{p}}\cos\theta = -2\Omega\frac{(1-x_{\mathrm{p}})}{x_{\mathrm{p}}}\mathcal{R}(i\omega\bar{\xi}^{\mathrm{v}}_3 - \bar{v}^{\mathrm{p}}_1 \tan\theta), \tag{3.128}$$

$$i\omega\bar{\xi}^{\mathrm{v}}_3 - \bar{v}^{\mathrm{p}}_1 \tan\theta = 0, \tag{3.129}$$

$$-i\omega\bar{v}^{\mathrm{n}}_2(1-\varepsilon_{\mathrm{n}}) - i\omega\varepsilon_{\mathrm{n}}\bar{v}^{\mathrm{p}}_2 + 2\Omega\bar{v}^{\mathrm{n}}_1 = 2\Omega\mathcal{R}(i\omega\bar{\xi}^{\mathrm{v}}_2 + \bar{v}^{\mathrm{p}}_2) - \bar{\sigma}_2, \tag{3.130}$$

$$-i\omega\bar{v}^{\mathrm{p}}_2(1-\varepsilon_{\mathrm{p}}) - i\omega\varepsilon_{\mathrm{p}}\bar{v}^{\mathrm{n}}_2 + 2\Omega\bar{v}^{\mathrm{p}}_1 = -2\Omega\frac{(1-x_{\mathrm{p}})}{x_{\mathrm{p}}}\mathcal{R}(i\omega\bar{\xi}^{\mathrm{v}}_2 + \bar{v}^{\mathrm{p}}_2) \tag{3.131}$$

and

$$\bar{v}^{\mathrm{p}}_1 - \bar{w}^{\mathrm{pn}}_1 + i\omega\bar{\xi}^{\mathrm{v}}_1 - i\omega\mathcal{R}\bar{\xi}^{\mathrm{v}}_2 - \mathcal{R}\bar{v}^{\mathrm{p}}_2 + \tilde{\sigma}_2 = 0. \tag{3.132}$$

3.4.2.1 Without mutual friction

Firstly we consider the undamped case, in which we ignore the effects of drag by setting $\mathcal{R} = 0$. We will also say that the entrainment vanishes, $\varepsilon_{\mathrm{x}} = 0$. In doing this we find two sets of solutions. There are the inertial modes given by

$$\omega^2 = 4\Omega^2 \cos^2\theta, \tag{3.133}$$

and the Tkachenko waves

$$\omega^2 = 4\Omega^2 \cos^2\theta + c_T^2 k^2 \sin^4\theta - \frac{c_T^4 k^4}{4\Omega^2}\sin^4\theta. \tag{3.134}$$

If we now assume $c_T^2 k^2 \ll \Omega^2$ and that the propagation is perpendicular to the rotation axis ($\cos\theta = 0$), then we find the Tkachenko wave dispersion relation

$$\omega = \pm c_T k. \tag{3.135}$$

By taking $\cos\theta = 0$, we are considering how waves propagate within the plane of the vortex array. If we look at the bigger picture of a neutron star, this plane would be the equatorial plane of the star.

3.4.2.2 With mutual friction

We would now like to investigate what impact the mutual friction, or drag, in the system has on the propagation of the waves. This is controlled through the drag parameter $\mathcal{R}$, which was previously set to zero. For this section we take the entrainment to be zero, so we have $\varepsilon_{\rm n} = \varepsilon_{\rm p} = 0$ but keep the drag parameter non zero. Firstly, we consider the propagation along the rotation axis, the z axis, by setting $\theta = 0$. We find two sets of modes, one which is undamped with dispersion relation

$$\omega = \pm 2\Omega \tag{3.136}$$

and one that is affected by the mutual friction

$$\omega = \pm 2\Omega\left(1 - \frac{\mathcal{R}^2}{x_{\rm p}(1+\mathcal{R}^2)}\right) - i2\Omega\frac{\mathcal{R}}{x_{\rm p}(1+\mathcal{R}^2)}. \tag{3.137}$$

These agree with the two classes of modes in [78].

Now we consider the wave propagation perpendicular to the rotation axis, where $\cos\theta = 0$. This will tell us how the waves behave within the vortex plane. For this we consider a typical pulsar with spin rate $\nu_{\rm star} = 10$ Hz and long wavelength oscillations across the whole superfluid region, such that $k = 10^{-6}$ cm^{-1}. We plot the frequency of the modes as a function of the drag parameter $\mathcal{R}$. In Figure 3.3 we see plots over a range of proton fractions, from large values of the proton fraction $x_{\rm p} = 0.96$ down to smaller but more realistic values $x_{\rm p} = 0.1$.

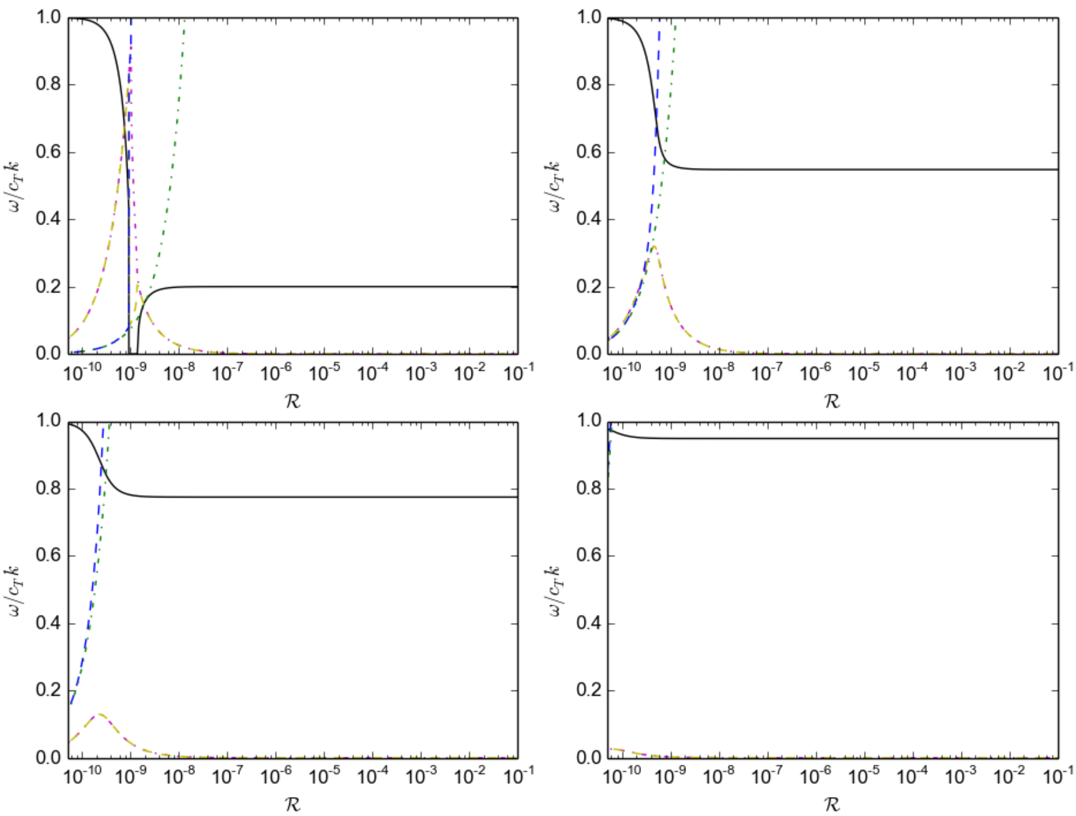

Figure 3.3: *The real part (solid lines) and the modulus of the imaginary part (dotted lines) of the modes, for proton fraction $x_{\rm p} = 0.96$ (top left), $x_{\rm p} = 0.7$ (top right), $x_{\rm p} = 0.4$ (bottom left) and $x_{\rm p} = 0.1$ (bottom right), with $k = 10^{-6}$. You can see that, for the high proton fraction case, the real part of the frequency vanishes where there is an imaginary root and is then damped as the drag parameter is increased. This imaginary root occurs for values of $\mathcal{R}$ such that the damping time scale $\tau_m \approx 1/2\Omega\mathcal{R}$ is approximately equal to the Tkachenko wave period $P_T = 2\pi/\omega_T$ with $\omega_T = k\sqrt{\kappa\Omega/\pi}$ (yellow, red and blue curves). Also, close to this, there is another imaginary root which can be seen in the yellow curve. As we decrease the proton fraction, from top left to bottom right, and tend towards the realistic neutron star core fraction $x_{\rm p} = 0.1$, the real part does not vanish and there are no imaginary roots.*

We see that, for large values of the proton fraction, the real part of the frequency vanishes and then the damping becomes large for large $\mathcal{R}$. Also, close to this value, there is an imaginary root. As we tend towards smaller, more realistic, values of the proton fraction the real part does not vanish and there are no imaginary roots.

3.5 Lagrangian perturbation theory

Modelling a stationary and well behaved system doesn't usually give any interesting insight to real world problems. In nature, systems are rarely perfectly behaved, so we want to see how the star behaves when it is perturbed. We saw previously what happens to our fluid system when it is perturbed using a specific perturbation, namely a plane wave perturbation. We will now investigate what happens when we perturb our single fluid system from Section 3.3 using Lagrangian perturbation theory. This enables us to write the equations in terms of a displacement vector relating to the component being perturbed. The Lagrangian perturbation ΔQ of a quantity Q is related to the Eulerian variation δQ by

$$\Delta Q = \delta Q + \mathcal{L}_\xi Q, \tag{3.138}$$

where ξ_i is the Lagrangian displacement. We see from [61, 73, 79] that the Lagrangian perturbation of the fluid velocity is given by

$$\Delta v^i = \partial_t \xi^i. \tag{3.139}$$

Given that the perturbation of the metric is

$$\Delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i, \tag{3.140}$$

we can then deduce that

$$\Delta v_i = \partial_t \xi_i + v^j \nabla_i \xi_j + v^j \nabla_j \xi_i. \tag{3.141}$$

In order to perturb our system, we will assume that there there is no entrainment in the system, which allows us to rewrite (3.54) in the form

$$(\partial_t + \mathcal{L}_{v_\mathrm{n}})v_i^\mathrm{n} - \nabla_i \left(\frac{\Psi_\mathrm{n}}{m} + v_\mathrm{n}^j v_j^\mathrm{n} \right) = -\frac{f_i^\mathrm{M}}{\rho_\mathrm{n}}, \tag{3.142}$$

using $p_\mathrm{n}^i = m(v_\mathrm{n}^i + \varepsilon_\mathrm{n} w_\mathrm{nv}^i)$ with the entrainment parameter ε_n equal to zero. Now we specify that Ψ_n is of the form $-\mu_\mathrm{n} - m\Phi_\mathrm{n} - mv_\mathrm{n}^j v_j^\mathrm{n}/2$ where μ_n and Φ_n are the chemical and gravitational potentials respectively. We do this in order for the left hand side of (3.142) to match that of a standard Euler equation for a single fluid. This form of the equation is also useful because the Lagrangian variation Δ commutes with $(\partial_t + \mathcal{L}_{v_\mathrm{n}})$ [80, 81]. After applying the perturbation to (3.142), we find that

$$(\partial_t + \mathcal{L}_{v_\mathrm{n}})\Delta v_i^\mathrm{n} + \nabla_i \left(\Delta\tilde{\mu}_\mathrm{n} + \Delta\Phi_\mathrm{n} - \frac{1}{2}\Delta\left(v_\mathrm{n}^j v_j^\mathrm{n}\right) \right) = -\Delta\left(\frac{f_i^\mathrm{M}}{\rho_\mathrm{n}} \right), \tag{3.143}$$

where $\tilde{\mu}_\mathrm{n} = \mu_\mathrm{n}/m$. We wish to find an equation in terms of the displacement vector ξ_i, so using the perturbations of quantities seen above, we can rewrite (3.143) as

$$\begin{aligned} \partial_t^2 \xi_i + 2v_\mathrm{n}^j \nabla_j \partial_t \xi_i + \left(v_\mathrm{n}^j \nabla_j\right)^2 \xi_i + \nabla_i \delta\Phi_\mathrm{n} + \xi^j \nabla_i \nabla_j \Phi_\mathrm{n} & \\ - \left(\nabla_i \xi^j\right) \nabla_j \tilde{\mu}_\mathrm{n} + \nabla_i \Delta\tilde{\mu}_\mathrm{n} = -\Delta\left(\frac{f_i^\mathrm{M}}{\rho_\mathrm{n}} \right). & \end{aligned} \tag{3.144}$$

3.5.1 Magnus force perturbations

We also wish to write the Magnus force term, on the right hand side of (3.143), in terms of displacement vectors. When the system is relaxed and has no forces applied to it, the vortices travel along with the fluid. So we know that, in the unperturbed state, the velocity difference $w_\mathrm{nv}^i = v_\mathrm{n}^i - u^i$ vanishes, which gives us

$$\Delta\left(\frac{f_i^\mathrm{M}}{\rho_\mathrm{n}} \right) = -\mathcal{N}_\mathrm{v} \epsilon_{ijk} \kappa^j \Delta w_\mathrm{nv}^k. \tag{3.145}$$

As a consequence of w_nv^i vanishing in the background, it is true that $\Delta w_\mathrm{nv}^i = \delta w_\mathrm{nv}^i$.

Currently we have only used one displacement vector, which represents the displacement of the fluid. It is likely the case that the displacement of the vortices when perturbed is not equal to the displacement of the fluid, so we will use $\xi^i_{\rm v}$ to denote the corresponding displacement vector of the vortices. This vortex displacement vector emerges from the Lagrangian perturbation of the vortex velocity u^i, in the same way that the fluid displacement ξ^i is related to the fluid velocity $v^i_{\rm n}$ in equation (3.139). In general, we know for a certain fluid component $\rm x \in \{n, p, v, \ldots\}$ that the Lagrangian displacement vector for that fluid will be given by

$$\Delta_{\rm x} v^i_{\rm x} = \delta v^i_{\rm x} + \mathcal{L}_{\xi_{\rm x}} v^i_{\rm x} = \partial_t \xi^i_{\rm x}. \tag{3.146}$$

This tells us that the perturbation of a velocity $v^i_{\rm x}$ with respect to a different fluid perturbation $\Delta_{\rm y}$ is given by

$$\Delta_{\rm y} v^i_{\rm x} = \delta v^i_{\rm x} + \mathcal{L}_{\xi_{\rm y}} v^i_{\rm x} = \Delta_{\rm x} v^i_{\rm x} - \mathcal{L}_{\xi_{\rm x}} v^i_{\rm x} + \mathcal{L}_{\xi_{\rm y}} v^i_{\rm x}, \tag{3.147}$$

by rearranging (3.146) for $\delta v^i_{\rm x}$ and plugging it into (3.147). Hence, in the case of our Magnus force, we are considering the perturbation of the vortex velocity u^i with respect to the neutron fluid perturbation $\Delta_{\rm n}$, denoted Δ above. Using the general perturbation identities above, we can find that

$$\Delta u^i = \Delta_{\rm v} u^i - \mathcal{L}_{\xi_{\rm v}} u^i + \mathcal{L}_{\xi} u^i. \tag{3.148}$$

Hence, the perturbed Magnus force, (3.145), takes the form

$$\Delta \left(\frac{f^{\rm M}_i}{\rho_{\rm n}} \right) = -\mathcal{N}_{\rm v} \epsilon_{ijk} \kappa^j \left(\Delta v^k_{\rm n} - \mathcal{L}_{\xi} u^k - \Delta_{\rm v} u^k + \mathcal{L}_{\xi_{\rm v}} u^k \right), \tag{3.149}$$

which we can simplify, remembering that the fluid and the vortices are travelling together in the background, to give

$$\Delta \left(\frac{f^{\rm M}_i}{\rho_{\rm n}} \right) = -\mathcal{N}_{\rm v} \epsilon_{ijk} \kappa^j \left[\partial_t (\xi^k - \xi^k_{\rm v}) - \mathcal{L}_{\xi - \xi_{\rm v}} v^k_{\rm n} \right]. \tag{3.150}$$

Now that we have determined all of our perturbed terms, we discover that our superfluid equation (3.142) behaves according to

$$\begin{aligned}\partial_t^2\xi_i + 2v_{\rm n}^j\nabla_j\partial_t\xi_i + \left(v_{\rm n}^j\nabla_j\right)^2\xi_i + \nabla_i\delta\Phi_{\rm n} + \xi^j\nabla_i\nabla_j\Phi_{\rm n} \\ - \left(\nabla_i\xi^j\right)\nabla_j\tilde{\mu}_{\rm n} + \nabla_i\Delta\tilde{\mu}_{\rm n} = \mathcal{N}_{\rm v}\epsilon_{ijk}\kappa^j\left[\partial_t(\xi^k - \xi_{\rm v}^k) - \mathcal{L}_{\xi-\xi_{\rm v}}v_{\rm n}^k\right],\end{aligned} \tag{3.151}$$

when our system of superfluid neutrons and vortex array is perturbed.

3.5.2 Mutual friction perturbations

In order to write down the complete perturbed force balance equation, we need to find the perturbed form of the mutual friction force and also the elasticity. Here we will see how the mutual friction force transforms when we apply the Lagrangian perturbation. In the background, the vortices and the second fluid are moving together, which causes the mutual friction to vanish in a similar way to the Magnus force before. Hence, the perturbed mutual friction force is given by

$$\Delta f_i^F = \rho_{\rm n}\kappa\mathcal{N}_{\rm v}\mathcal{B}\left(\Delta v_i^{\rm p} - \Delta u_i\right). \tag{3.152}$$

We have already seen the form of Δu_i and we can use (3.147) from the previous section to find that

$$\Delta v_i^{\rm p} = \Delta_{\rm p}v_i^{\rm p} - \mathcal{L}_{\xi_{\rm p}}v_i^{\rm p} + \mathcal{L}_\xi v_i^{\rm p}. \tag{3.153}$$

Putting these back into the mutual friction expression (3.152) we see that

$$\Delta f_i^F = \rho_{\rm n}\kappa\mathcal{N}_{\rm v}\mathcal{B}\left(\Delta_{\rm p}v_i^{\rm p} - \mathcal{L}_{\xi_{\rm p}}v_i^{\rm p} - \Delta_{\rm v}u_i + \mathcal{L}_{\xi_{\rm v}}u_i + \mathcal{L}_\xi v_i^{\rm p} - \mathcal{L}_\xi u_i\right), \tag{3.154}$$

but as $v_i^{\rm p} = u_i$ in the background, this simplifies to

$$\Delta f_i^F = \rho_{\rm n}\kappa\mathcal{N}_{\rm v}\mathcal{B}\left(\Delta_{\rm p}v_i^{\rm p} - \Delta_{\rm v}u_i - \mathcal{L}_{\xi_{\rm p}-\xi_{\rm v}}v_i^{\rm p}\right), \tag{3.155}$$

and hence we arrive at

$$\Delta f_i^F = \rho_{\rm n}\kappa\mathcal{N}_{\rm v}\mathcal{B}\left(\partial_t\left(\xi_i^{\rm p} - \xi_i^{\rm v}\right) - \mathcal{L}_{\xi_{\rm p}-\xi_{\rm v}}v_i^{\rm p} + v_{\rm p}^j\left[\nabla_i\left(\xi_j^{\rm p} - \xi_j^{\rm v}\right) + \nabla_j\left(\xi_i^{\rm p} - \xi_i^{\rm v}\right)\right]\right). \tag{3.156}$$

Now that we have the perturbed form of the Magnus force and mutual friction, all that remains in our force balance equation is the elasticity, which we will consider next.

3.5.3 Elasticity perturbations

We have already discussed that, in the background, the vortices are in equilibrium. This means that the elastic displacement $\xi_i^{\rm v}$ vanishes in the unperturbed system and only arises when the system is perturbed, along with the Magnus force and mutual friction.

Using the force balance equation between the Magnus force, mutual friction and elasticity, we can rewrite the superfluid equation (3.143) to include the vortex elasticity. In doing this, we find a slightly different version of our perturbed superfluid equation, given by

$$\begin{aligned}\partial_t^2\xi_i + 2v_{\rm n}^j\nabla_j\partial_t\xi_i + \left(v_{\rm n}^j\nabla_j\right)^2\xi_i + \nabla_i\delta\Phi_{\rm n} + \xi^j\nabla_i\nabla_j\Phi_{\rm n} \\ - \left(\nabla_i\xi^j\right)\nabla_j\tilde{\mu}_{\rm n} + \nabla_i\Delta\tilde{\mu}_{\rm n} = \Delta\left(\frac{f_i^{\rm E} + f_i^{\rm F}}{\rho_{\rm n}}\right).\end{aligned} \tag{3.157}$$

Yet again we need to consider what form the right hand side will take when the system is perturbed. However, as we have discussed, the vortices are in equilibrium in the background, which means that the elastic displacement will be zero unless the system is perturbed. Hence, the perturbed elastic force remains in the same form that we have seen previously

$$\Delta f_i^{\rm E} = \mu\left[\left(\frac{K}{\mu} + \frac{1}{3}\right)\nabla_i^\perp\left(\nabla_\perp^j\xi_j^{\rm v}\right) + \nabla_j^\perp\left(\nabla_\perp^j\xi_i^{\rm v}\right)\right]. \tag{3.158}$$

We have determined the form of the Magnus force, mutual friction and elasticity when the system is perturbed. Hence, we note that the perturbed force balance equation for the vortices

$$\Delta(f_i^{\mathrm{M}} + f_i^{\mathrm{F}} + f_i^{\mathrm{E}}) = 0, \tag{3.159}$$

takes the form

$$\begin{aligned}\mathcal{N}_{\mathrm{v}}\rho_{\mathrm{n}}\epsilon_{ijk}\kappa^j \left[\partial_t(\xi^k - \xi_{\mathrm{v}}^k) - \mathcal{L}_{\xi-\xi_{\mathrm{v}}} v_{\mathrm{n}}^k\right] &= \mu \left[\left(\frac{K}{\mu} + \frac{1}{3}\right) \nabla_i^{\perp} \left(\nabla_{\perp}^j \xi_j^{\mathrm{v}}\right) + \nabla_j^{\perp} \left(\nabla_{\perp}^j \xi_i^{\mathrm{v}}\right)\right] \\ &+\rho_{\mathrm{n}}\kappa\mathcal{N}_{\mathrm{v}}\mathcal{B}\left(\partial_t\left(\xi_i^{\mathrm{p}} - \xi_i^{\mathrm{v}}\right) - \mathcal{L}_{\xi_{\mathrm{p}}-\xi_{\mathrm{v}}} v_i^{\mathrm{p}} + v_{\mathrm{p}}^j \left[\nabla_i\left(\xi_j^{\mathrm{p}} - \xi_j^{\mathrm{v}}\right) + \nabla_j\left(\xi_i^{\mathrm{p}} - \xi_i^{\mathrm{v}}\right)\right]\right).\end{aligned} \tag{3.160}$$

So, (3.157) and (3.160) determine how this single superfluid neutron system and its vortex array behave when they are perturbed, using Lagrangian perturbation instead of Eulerian perturbation.

General Relativity

4

Relativistic Models

4.1 Relativistic fluid dynamics and vorticity conservation

Now we will begin to consider the approach we need to take in order to model our system in general relativity. In the case of a single relativistic fluid, we can formulate an Euler equation by considering the continuity equation of the stress-energy tensor [61, 82, 83]

$$\nabla_a T^{ab} = 0. \tag{4.1}$$

When considering a perfect fluid, we know that the stress-energy tensor is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{4.2}$$

where u_a is the fluid velocity, p is the pressure and ρ is the total energy density. Given a relation $p = p(\rho)$, there are four independent variables. The equations of motion are $\nabla_a T^a{}_b = 0$, which follows from the Einstein equations and the fact that $\nabla_a G^a{}_b = 0$. We take an equation of state of the form $\rho = \rho(n)$ where n is the particle number density.

The chemical potential μ is given by

$$d\rho = \frac{\partial \rho}{\partial n} dn \equiv \mu dn, \tag{4.3}$$

and from the Euler relation we get

$$\mu n = p + \rho. \tag{4.4}$$

We now eliminate the free index of $\nabla_a T^a{}_b = 0$ in two ways. Firstly, we contract it with u^b and secondly we project it with $\perp^b_c$. Using $u^a u_a = -1$ we can see that

$$\nabla_a (u^b u_b) = 0 \quad \Rightarrow \quad u_b \nabla_a u^b = 0. \tag{4.5}$$

So, when we contract the equations of motion with u^b we find

$$u^a \nabla_a \rho + (\rho + p) \nabla_a u^a = 0. \tag{4.6}$$

Then, by using the chemical potential and Euler relation, it becomes

$$u^a \mu \nabla_a n + \mu n \nabla_a u^a = 0 \quad \Rightarrow \quad \nabla_a n^a = 0, \tag{4.7}$$

where $n^a = n u^a$. If instead, we use the projection $\perp^b_c$, we see that the equations of motion give

$$D_c p = -(\rho + p) a_c, \tag{4.8}$$

where $D_c \equiv \perp^b_c \nabla_b$ is a purely spatial derivative and $a_c \equiv u^a \nabla_a u_c$ is the acceleration. This is similar to the Euler equation for Newtonian fluids. Another way of understanding $\nabla_a T^a{}_b = 0$ is if we define

$$\mu_b = \mu u_b, \tag{4.9}$$

and note that $u_a du^a = 0$, then

$$d\rho = -\mu_b dn^b. \tag{4.10}$$

The stress-energy-momentum tensor can now be written in the form

$$T^a{}_b = p\delta^a{}_b + n^a\mu_b. \tag{4.11}$$

The fluid element momentum μ_b is conjugate to the particle number density current n^a. If we now consider the continuity equation $\nabla_a T^a{}_b = 0$, we can determine that

$$n^a\omega_{ab} + \mu_b\nabla_a n^a = 0, \tag{4.12}$$

where the vorticity ω_{ab} is defined to be the anti-symmetrised derivative of the momentum $\mu_a = \mu u_a$

$$\omega_{ab} \equiv 2\nabla_{[a}\mu_{b]} = \nabla_a\mu_b - \nabla_b\mu_a. \tag{4.13}$$

We now contract equation (4.12) with n^b and use the fact that the vorticity is antisymmetric to find that

$$\nabla_a n^a = 0, \tag{4.14}$$

which tells us that the particle flux is conserved. Clearly this result also tells us that

$$n^a\omega_{ab} = 0, \tag{4.15}$$

from equation (4.12). This is our relativistic Euler equation, where $n^a = nu^a$ is the particle flux, u^a is the fluid velocity and ω_{ab} is the vorticity two form.

The definition of the vorticity, equation (4.13), implies that its exterior derivative vanishes,

$$\nabla_{[a}\omega_{bc]} = 0. \tag{4.16}$$

In the case that the Euler equation (4.15) holds, the vorticity is conserved along the flow. That is, we have

$$\mathcal{L}_u\omega_{ab} = 0. \tag{4.17}$$

Hence, the equations of motion (4.15) can be seen as an integrability condition for the vorticity. We will investigate these conservation laws further in Section 4.3.

4.2 Conventional formulation of perfect fluid and simple superfluid theory

The equations of motion found in Section 4.1 are also derivable from a Lagrangian variation principle [65, 84, 85, 86]. The 4-momentum π_a, a combination of the 3-momentum $\pi_i = p_i$ and the energy $\pi_0 = E$, is defined to be

$$\pi_a = \frac{\partial L}{\partial u^a}, \tag{4.18}$$

where L is the relevant position and velocity dependent Lagrangian function for the system and

$$u^a = \frac{dx^a}{d\tau}, \tag{4.19}$$

which was described previously in Chapter 2. We obtain an equation of motion from the momentum and Lagrangian by taking

$$\frac{d\pi_a}{d\tau} = \frac{\partial L}{\partial x^a}. \tag{4.20}$$

For barotropic perfect fluid models, the Lagrangian will have the form

$$L = \frac{1}{2}\mu g_{ab} u^a u^b - \frac{1}{2}\mu c^2, \tag{4.21}$$

where μ gives the mass in the first term and the potential energy in the second term. The 4-momentum will be given by the relation

$$\pi_a = \mu u_a, \tag{4.22}$$

so that we have $E = \gamma\mu c^2$ and $p_a = \gamma\mu v_a$ where μ can be interpreted as the relevant effective mass. We now want to specify an equation of state giving the pressure p in terms of the density ρ. In order to do this, we first write ρ in terms of the conserved number density n, in the form

$$\rho = mn + \frac{\epsilon}{c^2}, \tag{4.23}$$

where m is the mass of the particle we are considering and ϵ is an extra energy contribution due to compression. We can then write the pressure in terms of the density as

$$p = (n\mu - \rho)\, c^2, \tag{4.24}$$

where the effective mass μ is given by

$$\mu = \frac{d\rho}{dn} = m + \frac{1}{c^2}\frac{d\epsilon}{dn}. \tag{4.25}$$

This will be the effective mass term which appeared in the Lagrangian (4.21).

As we are dealing with a fluid flow rather than a single particle, we will convert the equation of motion (4.20) from a particle evolution equation to the equivalent field evolution equations. We rewrite the left hand side of the equation using the gradient of the momentum $\nabla_a \pi_b$ in the form $d\pi_a/d\tau = u^b \nabla_b \pi_a$. Similarly, we rewrite the right hand side of the equation in terms of the gradient of the Lagrangian $\nabla_a L = \partial L/\partial x^a + (\partial L/\partial \pi_b)\nabla_a \pi_b$. Hence, we arrive at the field evolution equation of the form

$$u^b \nabla_b \pi_a + \pi_b \nabla_a u^b = \nabla_a L. \tag{4.26}$$

We can also start from the Hamiltonian function in terms of the position and momentum variables [87], which comes from the Legendre transformation

$$H = \pi_a u^a - L. \tag{4.27}$$

In doing this, we can find our previous equations (4.19) and (4.20) when we specify that

$$\frac{dx^a}{d\tau} = \frac{\partial H}{\partial \pi_a} \quad \text{and} \quad \frac{d\pi_a}{d\tau} = -\frac{\partial H}{\partial x^a}. \tag{4.28}$$

As we wish to model a fluid, not a single particle, we find another field equation in evolving the Hamiltonian this time instead of the Lagrangian. This takes the form

$$2u^b \nabla_{[b}\pi_{a]} = -\nabla_a H, \tag{4.29}$$

where the square brackets denote antisymmetrisation over the bracketed indices. Contracting (4.29) with u^a we obtain the relation

$$u^a \nabla_a H = 0, \tag{4.30}$$

as the left hand side vanished due to being antisymmetric. This tells us that the Hamiltonian is conserved along the flow u^a.

We can find the Hamiltonian for our perfect fluid by using the Lagrangian (4.21) seen previously. This determines that the Hamiltonian is given by

$$H = \frac{1}{2\mu} g^{ab} u_a u_b + \frac{\mu c^2}{2}. \tag{4.31}$$

In order for u^a to be correctly normalised, the Hamiltonian must vanish. If this is the case we get

$$H = 0 \quad \Rightarrow \quad u_a u^a = -c^2, \tag{4.32}$$

as required. Hence, once we use this information in our Hamiltonian field equations (4.29), we find that

$$u^a \nabla_{[a}\pi_{b]} = 0. \tag{4.33}$$

We found previously that equation (4.15) can be obtained from the Lagrangian equations of motion given by the Lagrangian in (4.21). Next we need to find the conservation of number current (4.14) from the action integral

$$\mathcal{I} = \int \mathcal{L} d\mathcal{S}^{(4)}, \quad d\mathcal{S}^{(4)} = \frac{||g||^{1/2}}{c} d^4 x \tag{4.34}$$

where $\mathcal{S}^{(4)}$ is the four dimensional background manifold and $\mathcal{L}$ is the scalar Lagrangian functional, in order to complete the system of equations. We simplify things by looking at the irrotational case, where $\omega_{ab} = 0$. In this case, we take the independent variable to be the phase φ from the mesoscopic phase factor $e^{i\varphi}$ and the action is taken to be the pressure p expressed as a function of the effective mass μ. The mass is proportional to the magnitude of the momentum

$$\mu^2 c^2 = \pi_a \pi^a, \tag{4.35}$$

where the momentum is given by

$$\pi_a = \hbar \nabla_a \varphi, \tag{4.36}$$

which applies in the irrotational case. Hence, if we set $\mathcal{L} = p$ and use the pressure variation $\delta p = c^2 n \delta\mu$, the variation of the Lagrangian is given by

$$\delta\mathcal{L} = -n^a \delta\pi_a = -\hbar n^a \nabla_a (\delta\varphi). \tag{4.37}$$

If we say that the action integral (4.34) must be invariant with respect to infinitesimal variations of φ, then we find the required conservation law (4.14).

4.3 Vorticity conservation and flux 2-surfaces

We are interested in finding conservation laws which tell us that certain properties remain the same after being displaced. The variation given by a displacement vector ξ^a is written in terms of the Lie derivative along ξ^a, denoted $\mathcal{L}_\xi X$ for some quantity X. The Lie derivative of the metric is given by

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a, \tag{4.38}$$

which vanishes if the displacement vector ξ^a is a Killing vector. If the displacement is not a Killing vector, the Lie derivative clearly does not commute with the raising and lowering of indices. The Lie derivative of a vector X^a along ξ^a is given by

$$\mathcal{L}_\xi X^a = \xi^b \nabla_b X^a - X^b \nabla_b \xi^a, \tag{4.39}$$

and we can use (4.38) to see that this changes to

$$\mathcal{L}_\xi X_a = \xi^b \nabla_b X_a + X_b \nabla_a \xi^b, \tag{4.40}$$

for a vector $X_a = g_{ab} X^b$. If we consider our momentum π_a, we find the Lie derivative to be

$$\mathcal{L}_\xi \pi_a = \xi^b \nabla_b \pi_a + \pi_b \nabla_a \xi^b = \xi^b \omega_{ba} + \nabla_a \left(\pi_b \xi^b \right), \tag{4.41}$$

where $\omega_{ba} = 2\nabla_{[b}\pi_{a]}$ is the relativistic vorticity tensor. Due to this definition of the vorticity tensor, its exterior derivative will vanish

$$\nabla_{[a}\omega_{bc]} = 0, \tag{4.42}$$

as we saw previously in (4.16). Lie differentiation commutes with the exterior derivative, which means that if we take the exterior derivative of (4.41), we find

$$\mathcal{L}_\xi \omega_{ab} = -2\nabla_{[a} \left(\omega_{b]c} \xi^c \right). \tag{4.43}$$

If we now look back to the Lagrangian equation (4.26), this can be rewritten in terms of the Lie derivative as

$$\mathcal{L}_u \pi_a = \nabla_a L. \tag{4.44}$$

So, as before, if we apply the exterior derivative to this equation we get a conservation equation for the vorticity

$$\mathcal{L}_u \omega_{ab} = 0, \tag{4.45}$$

which tells us that the vorticity is conserved along the flow u^a. We know from (4.33) that u^a is a zero eigenvalue eigenvector of the vorticity tensor ω_{ab}. As ω_{ab} has a zero eigenvalue eigenvector, it must satisfy the degeneracy condition

$$\omega_{a[b}\omega_{cd]} = 0, \tag{4.46}$$

which also implies that it has rank two. Hence the flow u^a is just one case of a whole tangent subspace of eigenvectors e^a satisfying the same condition as (4.33)

$$e^a\omega_{ab} = 0. \tag{4.47}$$

This subspace is spanned by a unit worldsheet bivector $\mathcal{E}^{ab}$ [65], as long as the vorticity does not vanish $\omega = \left(\frac{1}{2}\omega_{ab}\omega^{ab}\right)^{1/2} \neq 0$. The bivector is proportional to the dual vorticity tensor

$$W^{ab} = \frac{1}{2}\epsilon^{abcd}\omega_{cd}, \tag{4.48}$$

and is written

$$\mathcal{E}^{ab} = \frac{1}{\omega}W^{ab}. \tag{4.49}$$

We can see that the tangent bivector $\mathcal{E}^{ab}$ satisfies

$$\mathcal{E}^{ab}\mathcal{E}_{ab} = -2c^2 \quad \text{and} \quad \mathcal{E}^{ab}\omega_{bc} = 0. \tag{4.50}$$

The equation above means that if we contract any covector with the bivector, this new term will be a solution of (4.47). An example of this can be seen from the helicity h^a, which is given by [88, 89, 90, 91]

$$h^a = \omega\mathcal{E}^{ab}\pi_b. \tag{4.51}$$

As well as being a solution of the vortex worldsheet tangentiality condition, it also satisfies the helicity conservation law

$$\nabla_a h^a = 0. \tag{4.52}$$

4.4 Variational fluid model

Next we will introduce the concept of a matter space into our model and use it within the variational approach. A more in depth discussion of matter space components will occur in Chapter 6. We start by reviewing the standard approach, as per the Living Reviews article [61] and [92, 93, 94]. For simplicity we will consider a single matter component, represented by a conserved flux n^a. For an isotropic system, the matter Lagrangian Λ should be a relativistic invariant and hence should only depend on $n^2 = -g_{ab}n^a n^b$. In effect, this means that the Lagrangian depends on the flux and the space-time metric. An arbitrary variation of $\Lambda = \Lambda(n^2) = \Lambda(n^a, g_{ab})$ then gives (ignoring terms that can be written as total derivatives, that is, the 'surface terms' in the action)

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\mu_a \delta n^a + \frac{1}{2}\left(\Lambda g^{ab} + n^a \mu^b\right)\delta g_{ab}\right], \tag{4.53}$$

where g is the determinant of the metric and μ_a is the canonical momentum

$$\mu_a = \frac{\partial \Lambda}{\partial n^a} = -2\frac{\partial \Lambda}{\partial n^2} g_{ab} n^b. \tag{4.54}$$

We have also made use of

$$\delta\sqrt{-g} = \frac{1}{2} g^{ab}\delta g_{ab}. \tag{4.55}$$

Equation (4.53) illustrates why a variational derivation of fluid dynamics is non trivial. As it stands, the variation of Λ suggests that the equations of motion would be $\mu_a = 0$, in essence, none of the fluids carry energy or momentum. This problem is resolved by imposing constraints on the fluxes. A natural way to do this is to make use of a three dimensional 'matter space'.

The coordinates of matter space, X^A where $A \in \{1, 2, 3\}$, serve as labels that distinguish fluid element worldlines. These labels are assigned at the initial time of the evolution, say $t = 0$. The matter space coordinates can be considered as scalar fields on spacetime, with a unique map obtained by a pull-back construction relating them to the spacetime coordinates.

The variational construction then involves three steps. Firstly, we note that the conservation of the individual fluxes is ensured provided that the dual three-form

$$n_{abc} = \epsilon_{abcd} n^d, \quad n^a = \frac{1}{3!}\epsilon^{abcd} n_{bcd}, \tag{4.56}$$

where ϵ_{abcd} is the usual volume form associated with the spacetime, is closed. That is

$$\nabla_{[a} n_{bcd]} = 0 \quad \longrightarrow \quad \nabla_a n^a = 0. \tag{4.57}$$

In the second step, we make use of the matter space to construct three-forms that are automatically closed on spacetime

$$n_{abc} = \psi^A{}_{[a} \psi^B{}_b \psi^C{}_{c]} n_{ABC}, \tag{4.58}$$

where the map is given by

$$\psi^A{}_a = \frac{\partial X^A}{\partial x^a} \tag{4.59}$$

and the Einstein summation convention applies to repeated matter space indices A, B, C. The volume form n_{ABC}, which is assumed to be antisymmetric, provides matter space with a geometric structure. If integrated over a volume in matter space it provides a measure of the number of particles in that volume. Specifically, we have

$$n_{ABC} = n\epsilon_{ABC}. \tag{4.60}$$

With this definition, the three-form (4.58) is closed provided n_{ABC} is a function of the X^A. In other words, the scalar fields X^A are taken to be fundamental variables.

The final step involves introducing the Lagrangian displacement ξ^a and linking back to the spacetime perturbations. This displacement tracks the movement of a given fluid element. From the standard definition of Lagrangian variations in the relativistic context, we have

$$\Delta X^A = \delta X^A + \mathcal{L}_\xi X^A = 0 \ , \tag{4.61}$$

where δX^A is the Eulerian variation and $\mathcal{L}_\xi$ is the Lie derivative along ξ^a, as we have seen in Chapter 3. This means that convective variations are such that

$$\delta X^A = -\mathcal{L}_\xi X^A = -\xi^a \frac{\partial X^A}{\partial x^a} = -\xi^a \psi^A{}_a, \tag{4.62}$$

as X^A acts as a scalar field on spacetime. For later benefit, it is also worth noting that this leads to

$$\Delta \psi^A{}_a = 0 \tag{4.63}$$

and after some algebra, one finds that

$$\Delta n_{abc} = 0, \tag{4.64}$$

which in turn implies

$$\delta n^a = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a \left(\nabla_b \xi^b + \frac{1}{2} g^{bc} \delta g_{bc} \right) . \tag{4.65}$$

This is the key result. By expressing the variations of the matter Lagrangian in terms of the displacement ξ^a, we ensure that the flux conservation is accounted for in the equations of motion. The variation of Λ now leads to

$$\delta \left(\sqrt{-g} \Lambda \right) = \sqrt{-g} \left\{ \frac{1}{2} \left[\left(\Lambda - n^c \mu_c \right) g^{ab} + n^a \mu^b \right] \delta g_{ab} + f_a \xi^a \right\} + S.T. \tag{4.66}$$

From the constrained variation it thus follows that the equations of motion are simply given by

$$f_b \equiv n^a \left(\nabla_b \mu_a - \nabla_a \mu_b \right) = 2 n^a \nabla_{[b} \mu_{a]} = 0. \tag{4.67}$$

Equation (4.67) above should look familiar as we derived it in the previous section in the form (4.33). In the previous derivation we denoted the momentum π_a (4.22) whereas here we denote it μ_a. We can read off the stress-energy tensor

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\Lambda\right)}{\delta g_{ab}} = \left(\Lambda - n^c \mu_c\right) g^{ab} + n^a \mu^b. \tag{4.68}$$

Introducing the matter four-velocity such that $n^a = nu^a$ and $\mu_a = \mu u_a$, where μ is the chemical potential, we see that the energy is given by

$$\varepsilon = u_a u_b T^{ab} = -\Lambda. \tag{4.69}$$

Moreover, from the first law of thermodynamics, we identify the pressure as

$$p = -\varepsilon + n\mu = \Lambda - n^c \mu_c. \tag{4.70}$$

This means that the stress-energy tensor takes the form

$$T^{ab} = pg^{ab} + n^a \mu^b = \varepsilon u^a u^b + ph^{ab}, \tag{4.71}$$

where we have used the standard projection

$$h^{ab} = g^{ab} + u^a u^b. \tag{4.72}$$

Finally, we can determine from (4.68) that

$$\nabla_a T^{ab} = -f^b + \nabla^b \Lambda - \mu_a \nabla^b n^a = -f^b = 0. \tag{4.73}$$

The second equality follows from i) the fact that Λ is a function only of n^a and g_{ab}, and ii) the definition of the momentum μ_a.

4.5 Introduction of the Kalb-Ramond gauge field

For a macroscopic model which will allow for an averaged effect of a number of vortices in the fluid, we will investigate the Kalb-Ramond gauge field [95]. We wish to start by writing the current in terms of an antisymmetric tensor field $B_{ab} = -B_{ba}$. We define the exterior derivative of this tensor to be

$$N_{abc} = 3\nabla_{[a}B_{bc]}, \tag{4.74}$$

and its dual is the number current, but written in the form

$$n^a = \frac{1}{3!}\epsilon^{abcd}N_{bcd}. \tag{4.75}$$

This will be invariant under the gauge transformations given by $B_{ab} \mapsto B_{ab} + 2\nabla_{[a}\chi_{b]}$. The closure condition

$$\nabla_{[a}N_{bcd]} = 0, \tag{4.76}$$

is the equivalent condition to the conservation of the number current (4.14).

We will now perform a Legendre transformation $\mathcal{L} \mapsto \Lambda$, where we replace the independent scalar field φ, seen in Section 5 of [95], with the antisymmetric gauge tensor B_{ab}. This will create a new yet similar model, in terms of a Lagrangian function Λ, which takes the form

$$\Lambda = -\frac{c^2}{12\Phi^2}N^{abc}N_{abc} - V\{\Phi\}, \tag{4.77}$$

in which the potential energy density term V is some suitably chosen algebraic function of the dilatonic amplitude Φ [95]. We can find the irrotationality condition $\omega_{ab} = 2\nabla_{[a}\pi_{b]} = 0$ from the new Lagrangian (4.77) in the equivalent dual form

$$\nabla_a\left(\Phi^{-2}N^{abc}\right) = 0, \tag{4.78}$$

by requiring invariance with respect to the independent variations of the gauge 2-form B_{ab}.

We replace the scalar field φ by the tensor field B_{ab} to take our model to the more general perfect fluid case. We want to keep the particle conservation equation (4.76) but get rid of the condition of irrotationality (4.78). The way to do this is to introduce a Lagrangian of the form

$$\mathcal{L} = \Lambda - \frac{1}{4}\epsilon^{abcd}B_{ab}\omega_{cd}, \tag{4.79}$$

where the vorticity 2-form is constructed from independent gauge fields in a similar way to the current 3-form. The vorticity is constructed such that the property (4.42) remains true. The independent gauge fields are taken to be two independent scalars $\chi^{\pm}$ and the vorticity ω_{ab} is then written in terms of these as

$$\omega_{ab} = 2\left(\nabla_{[a}\chi^{+}\right)\nabla_{b]}\chi^{-}. \tag{4.80}$$

If we plug the Lagrangian (4.79) into the action integral (4.34), with the requirement of invariance with respect to variations of B_{ab}, $\chi^{\pm}$ and Φ, it returns our original momentum equation (4.15). Hence, as we already have the particle conservation equation, we now have a complete set of equations which govern the motion of a perfect fluid, which are the same as those derived from the stress-energy tensor in (4.1).

4.6 Going forward

In this chapter we have summarised the standard approaches one would take in order to model fluids in relativity. As we proceed into the development of our model, we will call on and adapt these methods as we create our own multifluid system including quantised vortices, mutual friction and elasticity in general relativity.

5

Quantised Vortices and Mutual Friction

So far we have only seen how to model fluids in relativity. In order for our system to model superfluids, we must input some information regarding superfluidity. Hence, as in the Newtonian calculations, we will input the characteristic of quantised vortices into our model to specify that it be superfluid.

5.1 Thin vortex string limit

Following on from Section 4.3, we will now consider that the vorticity of the system is concentrated within the neighbourhood of some vorticity flux 2-surface $\mathcal{S}^{(2)}$ [95]. This could correspond to zero values of the scalar gauge fields $\chi^{\pm}$ seen previously. We will describe this vorticity 2-surface in terms of two coordinates σ^0 and σ^1.

The worldsheet embedding, $\{\sigma^0, \sigma^1\} \mapsto x^a = \overline{x}^a\{\sigma\}$, with induce a 2-surface metric

γ_{ab} on the worldsheet which is given by

$$\gamma_{AB} = g_{ab}\overline{x}^a_{,A}\overline{x}^b_{,B}, \tag{5.1}$$

where the comma denotes partial differentiation with respect to the 2-surface coordinates. This will in turn specify a worldsheet measure

$$d\mathcal{S}^{(2)} = \frac{||\gamma||^{1/2}}{c} d\sigma^0 d\sigma^1, \tag{5.2}$$

of the 2-surface element spanned by the coordinate variations $d\sigma^0$ and $d\sigma^1$.

We assume that the vorticity distribution is confined within a small region given by the displacements $\delta\chi^+$ and $\delta\chi^-$ of the scalar fields. Then we take the thin string limit as the size of these displacements tend to zero. The dual vorticity (4.48) in this limit, is given by an integral over the two dimensional worldsheet by

$$W^{ab} = \frac{c}{||g||^{1/2}} \int \overline{W}^{ab} \delta\left[x^a - \overline{x}^a\{\sigma\}\right] d\mathcal{S}^{(2)}, \tag{5.3}$$

where the term $\overline{W}^{ab}$ is the antisymmetric vorticity flux tensor on the two dimensional worldsheet.

If we are considering a continuous vorticity distribution, the vorticity conservation equation (4.42) can be expressed in this dual form as

$$\nabla_a W^{ab} = 0. \tag{5.4}$$

But, if we are considering the thin string limit in which the vorticity dual is of the form (5.3), the conservation equation above (5.4) will give us a similar condition for the 2-surface vorticity flux $\overline{W}^{ab}$ [95]. We say that $\eta^a{}_b$ is the first fundamental tensor of the worldsheet, which projects a vector at a point on the worldsheet onto its tangential part in the worldsheet. This is given by

$$\eta^a{}_b = c^{-2}\mathcal{E}^{ac}\mathcal{E}_{cb}. \tag{5.5}$$

Then we can say that the covariant derivative on the worldsheet is given by

$$\overline{\nabla}_a = \eta^b{}_a \nabla_b. \tag{5.6}$$

Given these, we can note that the conservation condition for the vorticity on the worldsheet will be in the form

$$\overline{\nabla}_a \overline{W}^{ab} = 0. \tag{5.7}$$

We can see that the worldsheet tangent bivector $\mathcal{E}^{ab}$ will also satisfy an equation of the same form

$$\overline{\nabla}_a \mathcal{E}^{ab} = 0. \tag{5.8}$$

Hence, (5.7) tells us that the worldsheet vorticity flux tensor $\overline{W}^{ab}$ is of the form

$$\overline{W}^{ab} = \kappa \mathcal{E}^{ab}, \tag{5.9}$$

where κ is constant on the worldsheet, $\overline{\nabla}_a \kappa = 0$.

For the thin string limit, the constant κ defined by (5.9) is the 2-surface integral of the vorticity across any spacelike section through the world tube. By Stokes' theorem, this is equal to the value of the Jacobi action around a closed curve surrounding the vortex string. Hence we will have

$$\kappa = \oint dS = \oint \pi_a dx^a = 2\pi\hbar n, \tag{5.10}$$

where n is an integer representing the number of individual quantised vortices carrying the flux. So, in the case in which we are considering a single quantised vortex, we will have $n = +1$ and find that

$$\kappa = 2\pi\hbar. \tag{5.11}$$

5.2 Quantised vortices

Now we will start our novel calculation in relativity, in which we will follow the Newtonian formulation of a superfluid system as in Chapter 3. In order to follow the Newtonian formulation of the superfluid equations, we will need to use a quantised vorticity vector instead of the relativistic vorticity tensor ω_{ab}. By adapting the kinematic rotation vector in [88], we form the vorticity vector from the vorticity tensor by saying

$$\mathcal{W}^a = \frac{1}{2}\epsilon^{abcd}u_b\omega_{cd}, \tag{5.12}$$

which we can reverse to find the vorticity tensor in terms of the vector

$$\omega_{ab} = -\epsilon_{abcd}u^c\mathcal{W}^d. \tag{5.13}$$

We can see from (5.12) that the vorticity vector is orthogonal to the flow, $u_a\mathcal{W}^a = 0$, and also from (5.13) that the Euler equation (4.15) holds. Next, we use the conservation of vorticity (4.17) to find an evolution equation for the vorticity

$$\mathcal{L}_u\mathcal{W}^a + \mathcal{W}^a\left(\nabla_b u^b\right) - u^a\left(\mathcal{W}^b\dot{u}_b\right) = 0, \tag{5.14}$$

which can be written as

$$h^a{}_b\left[\mathcal{L}_u\mathcal{W}^b + \mathcal{W}^b\left(\nabla_c u^c\right)\right] = 0, \tag{5.15}$$

where the projection is given by

$$h^a{}_b = \delta^a_b + u^a u_b. \tag{5.16}$$

In order to find (5.14) and hence (5.15), we have made use of

$$\mathcal{L}_u\epsilon^{abcd} = -\epsilon^{abcd}\left(\nabla_e u^e\right). \tag{5.17}$$

Next, we express the vorticity in terms of a collection of vortex lines, in the same way as the Newtonian formulation. So we say that the quantised vorticity vector can be written

$$\mathcal{W}^a = \mathcal{N}\kappa^a, \tag{5.18}$$

where $\mathcal{N}$ is the number density of the vortices, $\kappa^a = \kappa\hat{\kappa}^a$ is the direction of the vortex array and $\kappa = h/2$ is the quantum of circulation. The conservation of the vortex line density is found to be

$$\perp^b{}_a \nabla_b \left(\mathcal{N} u^a\right) = 0, \tag{5.19}$$

where the projection is given by

$$\perp^b{}_a = \delta^b_a - \hat{\kappa}_a \hat{\kappa}^b. \tag{5.20}$$

We can then find that the equation of motion for a single vortex is

$$\left(\delta^a_b + u_b u^a - \hat{\kappa}_b \hat{\kappa}^a\right) \mathcal{L}_u \kappa^b = 0. \tag{5.21}$$

To check that we are indeed reproducing the Newtonian formulation in general relativity, we take the Newtonian limit of (5.15) and (5.21). We find that (5.15) equates to

$$\partial_t \mathcal{W}^i + \epsilon^{ijk} \nabla_j (\epsilon_{klm} \mathcal{W}^l u^m_{\rm v}) = 0, \tag{5.22}$$

which is the Newtonian evolution equation for the vorticity (3.50). Also, (5.21) produces

$$\partial_t \kappa^i + \perp^i{}_j \mathcal{L}_{u_{\rm v}} \kappa^j = 0, \tag{5.23}$$

which is the Newtonian equation for the motion of a single vortex (3.60). As we use u^a to denote the fluid velocity in this chapter, we use $u^a_{\rm v}$ to denote the collective vortex velocity in the two equations above.

5.2.1 Bivector and vector comparison

Although the two methods above are very different, the calculation presented in Section 5.2 for a single superfluid is consistent with that of Carter in [65] and Section 5.1. In Chapter 4 we used the continuity equation of the stress-energy tensor $\nabla^a T_{ab} = 0$ to find the system of equations for a relativistic perfect fluid. The equations for such a system are the conservation of particle flux

$$\nabla_a n^a = 0 \tag{5.24}$$

and the relativistic Euler equation

$$n^a \omega_{ab} = 0. \tag{5.25}$$

Both of these are shown in the same way in Carter's calculations. However, from here, Carter proceeds by using the dual vorticity tensor

$$W_{ab} = \frac{1}{2}\epsilon^{abcd}\omega_{cd} \tag{5.26}$$

and the unit worldsheet element tangent bivector

$$\mathcal{E}^{ab} = \frac{1}{\omega}W^{ab}, \tag{5.27}$$

where $\omega = (\frac{1}{2}\omega_{ab}\omega^{ab})^{1/2}$ is the vorticity magnitude. The equation for the conservation of vorticity is expressible in terms of the vorticity dual as

$$\overline{\nabla}_b \overline{W}^{ab} = 0. \tag{5.28}$$

The bars in equation (5.28) denote the restriction of the dual vorticity tensor and the covariant derivative to a two dimensional surface. The worldsheet tangent bivector satisfies a conservation condition of the same form as equation (5.28)

$$\overline{\nabla}_b \mathcal{E}^{ab} = 0. \tag{5.29}$$

We determine from these equations that the surface vorticity flux tensor $\overline{W}^{ab}$ must have the form

$$\overline{W}^{ab} = k\mathcal{E}^{ab}, \tag{5.30}$$

where k is a constant on the worldsheet, $\overline{\nabla}_a k = 0$. The constant k defined by (5.30) is interpretable as the value of the 2-surface integral of the vorticity.

Our system is related to the one summarised above through the final two equations (5.29) and (5.30). The 2-surface integral of the vorticity k in Carter's work relates to our quantum of circulation κ. Also, it can be shown that the conservation condition for the worldsheet tangent bivector, equation (5.29), is equivalent to our equation of motion for a single vortex

$$\left(\delta^a_b + u_b u^a - \hat{\kappa}_b \hat{\kappa}^a\right) \mathcal{L}_u \kappa^b = 0. \tag{5.31}$$

Hence, our relativistic single fluid system built using the quantised vorticity vector matches the similar system built by Carter using the dual vorticity tensor.

5.3 The Magnus force and mutual friction

Up until now, the vortices and the superfluid are considered to be moving together. In this section we will assume that the fluid and vortex velocities are no longer equal, implying that there is a velocity difference between them. We will then introduce a frictional force between the vortices and the normal fluid component, causing a second velocity difference.

5.3.1 Two-fluid model with friction

We now consider a system comprised of two fluids. Let the first fluid component have particle density n and the second component s. The corresponding fluxes are $n^a = nu^a$ and $s^a = su^a_{\mathrm{s}}$. We say that the first fluid represents massive particles and is conserved

$$\nabla_a n^a = 0. \tag{5.32}$$

The second fluid represents entropy, and is in general not conserved

$$\nabla_a s^a = \Gamma_{\rm s}. \tag{5.33}$$

We assume that the two fluids are coupled by friction, which gives us the equations of motion

$$n^a \omega_{ab} = \mathcal{R}^{\rm n}_b \tag{5.34}$$

and

$$s^a \omega^{\rm s}_{ab} + \mu^{\rm s}_b \Gamma_{\rm s} = \mathcal{R}^{\rm s}_b, \tag{5.35}$$

where

$$\omega^{\rm s}_{ab} = 2\nabla_{[a}\mu^{\rm s}_{b]} \tag{5.36}$$

and $\mu^{\rm s}_a = \mu^{\rm s} u^{\rm s}_a$. If we assume that there are no external forces acting on the system then we can write down a force balance equation

$$\mathcal{R}^{\rm n}_a + \mathcal{R}^{\rm s}_a = 0. \tag{5.37}$$

Due to ω_{ab} being antisymmetric, we can see from (5.34) that

$$n^a \mathcal{R}^{\rm n}_a = 0. \tag{5.38}$$

We also use (5.35) and (5.37) to see that

$$\left(s^a \mu^{\rm s}_a\right) \Gamma_{\rm s} = s^a \mathcal{R}^{\rm s}_a = -s^a \mathcal{R}^{\rm n}_a = -s^a n^b \omega_{ba}. \tag{5.39}$$

Next, we define the temperature to be $T = \mu_{\rm s} = -u^a_{\rm s} \mu^{\rm s}_a$ and as the two fluids do not move together we can say that

$$u^a_{\rm s} = \gamma \left(u^a + w^a\right), \qquad u^a w_a = 0, \qquad \gamma = \left(1 - w^2\right)^{-1/2}. \tag{5.40}$$

Combining the above equations with (5.39), we find that

$$sT\Gamma_{\rm s} = s\gamma w^a n^b \omega_{ba} \geq 0, \tag{5.41}$$

in accordance with the second law of thermodynamics. We can also see from (5.39) that we need

$$s^a \mathcal{R}_a^{\mathrm{n}} = s\gamma w^a \mathcal{R}_a^{\mathrm{n}} \geq 0, \tag{5.42}$$

which can be satisfied by assuming a friction force

$$\mathcal{R}_a^{\mathrm{n}} = \alpha w_a \ , \qquad \text{where} \qquad \alpha > 0. \tag{5.43}$$

5.3.2 The Magnus force

We will now impose that the vortices do not move with the flow. We say that there is friction between the vortices and the second component in our two fluid model. This means that we can write the vortex velocity as

$$u_{\mathrm{v}}^a = \tilde{\gamma}\left(u^a + v^a\right). \tag{5.44}$$

In Sections 4.1 and 5.2 the vortices moved with the flow, so we had $u_{\mathrm{v}}^a = u^a$ and wrote the equations using the fluid velocity for simplicity. We will now use the Euler equation (4.15) in the form

$$u_{\mathrm{v}}^a \omega_{ab} = 0 \tag{5.45}$$

and introduce the velocity difference stated in equation (5.44). This enables us to rewrite the Euler equation as

$$u^a \omega_{ab} = -v^a \omega_{ab} \equiv f_b^{\mathrm{M}}, \tag{5.46}$$

where the right hand side defines the relativistic analogue of the Magnus force. It is clearer to see this once we plug in the definition for ω_{ab}, (5.13), to get

$$f_b^{\mathrm{M}} = -v^a \omega_{ab} = \mathcal{N} \epsilon_{abcd} v^a u_{\mathrm{v}}^c \kappa^d = \mathcal{N} \epsilon_{bad} \kappa^a v^d, \tag{5.47}$$

where we have defined

$$\epsilon_{abc} = \epsilon_{dabc} u_{\mathrm{v}}^d. \tag{5.48}$$

Hence, from (5.46) and (5.47), we arrive at the final equation of motion

$$n^a \omega_{ab} = n\mathcal{N} \epsilon_{bac} \kappa^a v^c. \tag{5.49}$$

Taking the Newtonian limit of this equation of motion gives us the correct Newtonian superfluid equation (3.142),

$$(\partial_t + \mathcal{L}_{v_{\rm n}}) v_i^{\rm n} + \nabla_i \left(\tilde{\mu}_{\rm n} + \Phi_{\rm n} - \frac{1}{2} v_{\rm n}^j v_j^{\rm n} \right) = \mathcal{N}_{\rm v} \epsilon_{ijk} \kappa^j w_{\rm nv}^k. \tag{5.50}$$

We can see using (5.34) that the superfluid equation of motion (5.49) means we must have

$$R_a^{\rm n} = n\mathcal{N} \epsilon_{abc} \kappa^b v^c. \tag{5.51}$$

5.3.3 Mutual friction

Microphysically one would expect the mutual friction to arise from the scattering of the second component off of the vortex core. We represent this by using the relative velocity between the vortices and the normal fluid

$$u_{\rm v}^a = \hat{\gamma} \left(u_{\rm s}^a + q^a \right). \tag{5.52}$$

Combining the relative velocities (5.40), (5.44) and (5.52) we find that

$$\tilde{\gamma} = \hat{\gamma} \gamma \tag{5.53}$$

and

$$q^a = \gamma \left(v^a - w^a \right). \tag{5.54}$$

Mesoscopically, the vortices move under the influence of two forces. The Magnus force is balanced by dissipative scattering of the normal component. This leads to an equation of motion of the form

$$\beta q_a = -R_a^{\rm n} = -\epsilon_{dabc} u_{\rm v}^d \kappa^b v^c, \tag{5.55}$$

assuming that the vortices are taken to be massless. We can rewrite this as

$$w_a = v_a + \frac{1}{\eta}\epsilon_{dabc}u^d\kappa^b v^c, \tag{5.56}$$

where

$$\eta = \beta\gamma/\tilde{\gamma} \tag{5.57}$$

is the friction coefficient. In fact, is useful to decompose κ^a into its components parallel and orthogonal to the flow u^a,

$$\kappa^a = \kappa_{\|}u^a + \kappa^a_{\perp}, \tag{5.58}$$

in which case we have

$$w_a = v_a + \frac{1}{\eta}\epsilon_{dabc}u^d\kappa^b_{\perp} v^c. \tag{5.59}$$

We have enough information to keep track of the vortices as the system evolves. However, there is a more practical approach in which we do not need to keep track of the vortices. To do this, we start by rearranging (5.59) to find an expression for v^a in terms of w^a. Then, we can plug it back into the expression for $\mathcal{R}^{\rm n}_a$. The method we use to rearrange (5.59) is exactly the same as in the Newtonian problem [75]. In the first step, we find that

$$\epsilon^{eafg}u_e\kappa^{\perp}_f w_a = \eta\left(v^g - w^g\right) + \frac{1}{\eta}\kappa^2_{\perp}\tilde{\perp}^g_c v^c, \tag{5.60}$$

where

$$\tilde{\perp}^g_c = \delta^g_c - \hat{\kappa}^g_{\perp}\hat{\kappa}^{\perp}_c, \tag{5.61}$$

with $\kappa^a_{\perp} = \kappa_{\perp}\hat{\kappa}^a_{\perp}$. The second step leads to

$$\epsilon_{bgcd}\epsilon^{eafg}u^b\kappa^c_{\perp}u_e\kappa^{\perp}_f w_a = -\kappa^2_{\perp}\tilde{\perp}^c_d w_c = -\eta\epsilon_{bgcd}u^b\kappa^c_{\perp}w^g - \left(\eta^2 + \kappa^2_{\perp}\right)(w_d - v_d). \tag{5.62}$$

Then we reach the final result, which is

$$v_d = w_d + \left(\frac{1}{\eta^2 + \kappa^2_{\perp}}\right)\left[\eta\epsilon_{bgcd}u^b\kappa^c_{\perp}w^g - \kappa^2_{\perp}\tilde{\perp}^c_d w_c\right]. \tag{5.63}$$

As mentioned above, we use this in our Magnus force expression (5.51) to find that

$$f_a^{\mathrm{M}} = \mathcal{N}\tilde{\gamma}\left[\left(\frac{\eta^2}{\eta^2+\kappa_\perp^2}\right)\epsilon_{dabc}u^d\kappa_\perp^b w^c + \left(\frac{\eta}{\eta^2+\kappa_\perp^2}\right)\kappa_\perp^2\,\tilde{\perp}^c_a w_c\right], \tag{5.64}$$

which gives us the equation of motion containing the Magnus force and mutual friction. Note that we no longer have to worry about the vortices after rearranging in this way. We can evaluate $\tilde{\gamma}$ by using (5.63) to show that

$$v^2 = \frac{\eta^2 w^2 + (\kappa_\perp^a w_a)^2}{\eta^2+\kappa_\perp^2}. \tag{5.65}$$

Finally, from (5.42), we find that the second law requires us to have $\eta \geq 0$.

We have now completed our relativistic formulation of a multifluid system including mutual friction between the vortices and the second fluid component. This model could now be used when considering various mechanisms relating to neutron stars in which mutual friction could be a factor. Some features of neutron stars which mutual friction could contribute towards are pulsar glitch relaxation and damping of neutron star oscillations, as stated in the Newtonian mutual friction calculations [74].

6

Relativistic Elasticity

We have created a two fluid system including quantised vortices and mutual friction in relativity. This system almost matches the multifluid system we detailed in Chapter 3. However, we are missing one component which is the vortex elasticity. Relativistic elasticity, or relasticity, is used to model neutron stars in work such as [96]. In this work, the neutron star is modelled as an elastic ball. We will firstly consider how to work with elasticity in relativity, before proceeding to look at elasticity specifically related to the vortices in our model.

6.1 Matter space

6.1.1 The configuration gradient

We have already brushed upon the concept of matter space previously and here we will go into a little more detail. Following the relativistic framework of [96], [97] and [98, 99, 100, 101] we create a map from the four dimensional spacetime to three dimensional matter space

$$\chi : M_4 \rightarrow X_3. \tag{6.1}$$

In the local coordinates x^a on spacetime and $\tilde{X}^A$ on matter space, this is written as

$$x^a \mapsto \tilde{X}^A = \chi^A(x^a). \tag{6.2}$$

We will be dealing with derivatives of this map χ, hence we will simplify things by denoting its derivative $d\chi$ by ψ. Hence, ψ has the properties

$$\psi : \quad M_4 \rightarrow TX_3 \otimes T^*M_4, \tag{6.3}$$

and in coordinates

$$x^a \mapsto \psi^A{}_a := \frac{\partial \tilde{X}^A}{\partial x^a}. \tag{6.4}$$

We now introduce a time foliation of spacetime, such that

$$\chi : \quad R \times M_3 \rightarrow X_3, \tag{6.5}$$

with

$$(t, x^i) \mapsto \tilde{X}^A. \tag{6.6}$$

This has spatial and time derivatives given by

$$\psi^A{}_i := \frac{\partial \tilde{X}^A}{\partial x^i} \text{ and } \psi^A{}_t := \frac{\partial \tilde{X}^A}{\partial t}. \tag{6.7}$$

We call χ the configuration and both $\psi^A{}_a$ and $\psi^A{}_i$ the configuration gradient.

The matter space coordinates $\tilde{X}^A$ label particles and are constant along particle world lines. This means that

$$u^a \psi^A{}_a = 0, \tag{6.8}$$

where the four-velocity u^a is tangential to the matter world lines. The four-velocity can be parametrised as

$$u^a = (u^t, u^i) = \alpha^{-1} W \left(1, \hat{v}^i\right), \quad W = \left(1 - v^j v_j\right)^{-1/2}, \tag{6.9}$$

which then tells us that

$$\psi^A{}_t = -\hat{v}^i \psi^A{}_i. \tag{6.10}$$

6.1.2 Matter space metric and particle number current

The volume form n_{ABC} on matter space gives us the number of particles in a given volume, when integrated over the volume. A conformal metric is required, in order to define angles on matter space, and hence compare them to spacetime in order to see deformations. These two properties define a Riemannian metric which we call k_{AB}. Assuming n_{ABC} is compatible with our metric k_{AB}, we know that in matter space coordinates $\tilde{X}^a$, we have

$$n_{ABC} = \sqrt{k_{\tilde{X}}}\,\delta_{ABC}, \tag{6.11}$$

where the determinant is given by

$$k_{\tilde{X}} := \frac{1}{3!}\delta^{ABC}\delta^{DEF} k_{AD} k_{BE} k_{CF}. \tag{6.12}$$

The $\tilde{X}$ notation reminds us that this quantity depends upon the matter space $\tilde{X}^A$ coordinates.

Matter space has no time by construction, but one can use k_{AB} as a function on spacetime by considering $k_{ab}\left(\chi^C\left(x^d\right)\right)$. The pushforward of k_{AB} to spacetime is denoted k_{ab} and it satisfies the conditions

$$\mathcal{L}_u k_{ab} = 0, \text{ and } u^a k_{ab} = u^b k_{ab} = 0. \tag{6.13}$$

Any tensor field on matter space could be pushed forward to spacetime and defined by its contractions with u^a and its Lie derivative along u^a vanishing. We could also consider the components of $k_{ab}\left(\chi^C\left(x^d\right)\right)$ as scalars on spacetime, which are constant along particle world lines. This tell us that

$$u^a k_{AB,a} = 0, \tag{6.14}$$

which is given in coordinate form by

$$k_{AB,t} + \hat{v}^i k_{AB,i} = 0. \tag{6.15}$$

The authors of [96] note that they prefer to work with k_{AB} over k_{ab}, due to it having fewer components and a simpler evolution equation, whereas [98] continues with k_{ab} in spacetime.

Next we consider the pushforward of our volume form n_{ABC} to a three dimensional n_{abc} on spacetime. Applying the map to n_{ABC}, we see that

$$n_{abc} := \psi^A{}_a \psi^B{}_b \psi^C{}_c n_{ABC}. \tag{6.16}$$

Spacetime already has its own volume form, denoted ϵ_{abcd} which is compatible with the metric g_{ab}. This is defined by

$$\epsilon_{abcd} = \sqrt{g_x}\delta_{abcd}, \tag{6.17}$$

where the determinant is given in the same way as before, by

$$g_x := \frac{1}{4!}\delta^{abcd}\delta^{efgh} g_{ae} g_{bf} g_{cg} g_{dh}. \tag{6.18}$$

We can then define the particle number current in spacetime

$$n^a = \frac{1}{3!}\epsilon^{abcd} n_{bcd}, \tag{6.19}$$

which is timelike and conserved

$$\nabla_a n^a = \epsilon^{abcd}\nabla_a n_{bcd} = 0, \tag{6.20}$$

where ∇_a is the covariant derivative. The right hand side vanishes because it is the pushforward of $n_{[BCD,A]}$. We split the number current n^a into a matter four-velocity and a particle density by saying

$$n^a = nu^a, \tag{6.21}$$

where the velocity u^a is normalised as

$$u^a u_a = -1. \tag{6.22}$$

This tells us that $n^2 = -n^a n_a$ and $n = -u_a n^a$.

6.2 Relativistic dynamics

6.2.1 Action and stress energy tensor

We now consider how elasticity is modelled in a three dimensional fluid system [96]. We start with the matter action

$$S = \int e(g^{ab}, \psi^A{}_a, k_{AB}, ...) g_x^{1/2} d^4x, \tag{6.23}$$

where the dots represent any other tensors on matter space. We can vary this with respect to the metric

$$\delta S = \frac{1}{2} \int T_{ab} \delta g^{ab} g_x^{1/2} d^4x, \tag{6.24}$$

to find that the stress-energy tensor T_{ab} is given by

$$T_{ab} = 2\frac{\partial e}{\partial g^{ab}} - e g_{ab}. \tag{6.25}$$

We then define h_{ab}, which is the projection tensor that projects orthogonally to the four-velocity

$$h_{ab} = u_a u_b + g_{ab}. \tag{6.26}$$

This allows us to rewrite the stress-energy tensor as

$$T_{ab} = 2\frac{\partial e}{\partial g^{ab}} - e h_{ab} + e u_a u_b, \tag{6.27}$$

or, if we define the pressure term

$$p_{ab} = 2\frac{\partial e}{\partial g^{ab}} - e h_{ab}, \tag{6.28}$$

we get

$$T_{ab} = e u_a u_b + p_{ab}. \tag{6.29}$$

We wish to take some of the spacetime terms with us to matter space. Hence, we define the pullback of the spacetime metric to the three dimensional matter space as

$$g^{AB} = \psi^A{}_a \psi^B{}_b g^{ab}, \tag{6.30}$$

which has a matrix inverse g_{AB}. This means that we now have two Riemannian metrics on matter space, k_{AB} and g_{AB}. We will say that k_{AB} is the matter space metric, but raise and lower indices with the g_{AB} and g^{AB}. This tells us that $k^{AB} = g^{AC} g^{BD} k_{CD}$, but, this is not the matrix inverse of k_{AB}.

We will say that the quantity

$$\psi_A{}^a = \psi^B{}_b g^{ab} g_{AB}, \tag{6.31}$$

is the inverse of $\psi^A{}_a$ in the sense that

$$\psi^A{}_a \psi_B{}^a = \delta^A{}_B, \tag{6.32}$$

and

$$\psi^A{}_a \psi_A{}^b = h_a{}^b. \tag{6.33}$$

These can be shown by making use of the properties mentioned above and remembering that $h_a{}^c h_c{}^b = h_a{}^b$.

We can now consider the energy as function of g^{AB}, as we have

$$e(\psi^A{}_a, g^{ab}) = e(g^{AB}). \tag{6.34}$$

Hence, we see that the derivative required for the pressure term in (6.28) now takes the form

$$\frac{\partial e}{\partial g^{ab}} = \frac{\partial e}{\partial g^{AB}} \frac{\partial g^{AB}}{\partial g^{ab}} = \frac{\partial e}{\partial g^{AB}} \psi^A{}_a \psi^B{}_b. \tag{6.35}$$

So, we can see from the property of the map (6.8) that $p_{ab}u^a = 0 = u_a h_{bc} T^{ab}$. This tells us that there is no energy flux relative to the matter and that we are considering ideal, or non dissipative, elastic matter. As we have mentioned, p_{ab} is the pressure tensor, which for a perfect fluid takes the form $p_{ab} = ph_{ab}$, where p is the pressure. The Lagrangian e in the matter action (6.23) is the total energy density in the rest frame of the matter.

Next, we can find that the number density can be written

$$n^2 = \frac{1}{3!} n^{abc} n_{abc} = \frac{1}{3!} g^{ad} g^{be} g^{cf} n_{abc} n_{def}. \tag{6.36}$$

Since n_{abc} is only comprised of $\psi^A{}_a$ and n_{ABC} (6.16), it is independent of g^{ab}. Hence the derivative of (6.36) gives

$$\frac{\partial n}{\partial g^{ab}} = \frac{1}{2} n h_{ab}, \tag{6.37}$$

which is considered a function of g^{ab}, $\psi^A{}_a$ and the matter tensors. If we then define the internal energy per rest mass ϵ such that

$$e = n(1 + \epsilon), \tag{6.38}$$

we see

$$p_{ab} = 2n \frac{\partial \epsilon}{\partial g^{ab}}. \tag{6.39}$$

In a similar way to that of (6.35), we write the pressure as

$$p_{ab} = n \tau_{AB} \psi^A{}_a \psi^B{}_b, \tag{6.40}$$

where τ_{AB} is defined to be

$$\tau_{AB} = 2\frac{\partial \epsilon}{\partial g^{AB}}. \tag{6.41}$$

6.2.2 Isotropic matter

We will consider the case in which the internal energy ϵ depends upon g^{ab}, $\psi^A{}_a$ and the matter space metric k_{AB}. The energies e and ϵ should transform as scalars on spacetime and on matter space. Hence we need to find all double scalars that can be made from g^{AB} and k_{AB}.

We see from (6.30) that g^{AB} transforms as a (2,0)-tensor on matter space and as a scalar on spacetime. We define the mixed index metric

$$k^A{}_B = g^{AC}k_{BC} = g^{ac}\psi^A{}_a\psi^C{}_c k_{BC}, \tag{6.42}$$

which transforms as a (1,1)-tensor on matter space and as a scalar on spacetime. The eigenvalues of $k^A{}_B$ transform as scalars on matter space and are the required double scalars we are looking for.

We can split our $k^A{}_B$ matrix into its determinant k and a unit determinant matrix

$$\eta^A{}_B = k^{-1/3}k^A{}_B, \tag{6.43}$$

where the determinant $k = n^2$ is found from the definition of the determinant and the fact that n_{ABC} is the volume form of k_{AB}.

We can now consider the internal energy ϵ as a function of n and $\eta^A{}_B$. It must depend on the unit matrix $\eta^A{}_B$ via the two scalar invariants associated to it. Hence we can say that

$$\epsilon(k^A{}_B) = \epsilon(k, \eta^A{}_B) = \epsilon(n, I_1, I_2), \tag{6.44}$$

where we have defined the scalar invariants of ${\eta^A}_B$ to be

$$I_1 = {\eta^A}_A = k^{-1/3} g^{AB} k_{AB} \tag{6.45}$$

and

$$I_2 = {\eta^A}_B {\eta^B}_A = k^{-2/3} g^{AB} g^{CD} k_{AC} k_{BD}. \tag{6.46}$$

We note that

$$\frac{\partial k}{\partial g^{AB}} = k g_{AB}, \tag{6.47}$$

which tells us

$$\frac{\partial n}{\partial g^{AB}} = \frac{1}{2} n g_{AB}. \tag{6.48}$$

With these we can find the form of τ_{AB} from the pressure term (6.40), which is

$$\tau_{AB} = n g_{AB} \frac{\partial \epsilon}{\partial n} + 2 \left[\left(\eta_{AB} - \frac{1}{3} g_{AB} I_1 \right) \frac{\partial \epsilon}{\partial I_1} + 2 \left(\eta_{AC} {\eta^C}_B - \frac{1}{3} g_{AB} I_2 \right) \frac{\partial \epsilon}{\partial I_2} \right], \tag{6.49}$$

or

$$\tau_{AB} = \frac{p}{n} g_{AB} + 2 \left(f_1 \pi^1_{AB} + f_2 \pi^2_{AB} \right), \tag{6.50}$$

where we have said that

$$p = n^2 \frac{\partial \epsilon}{\partial n}, \tag{6.51}$$

$$f_Y = \frac{\partial \epsilon}{\partial I_Y}, \quad (\text{for } Y = 1, 2) \tag{6.52}$$

$$\pi^1_{AB} = \eta_{AB} - \frac{1}{3} g_{AB} I_1 \tag{6.53}$$

and

$$\pi^2_{AB} = 2 \left(\eta_{AC} {\eta^C}_B - \frac{1}{3} g_{AB} I_2 \right). \tag{6.54}$$

Hence, plugging this version of τ_{AB} (6.50) into (6.40), can write the pressure tensor in the form

$$p_{ab} = p h_{ab} + \pi_{ab}, \tag{6.55}$$

where the first term is the same as that for a perfect fluid and the second represents the anisotropic stress. This stress is given by

$$\pi_{ab} = \psi^A{}_a \psi^B{}_b \pi_{AB}, \tag{6.56}$$

where

$$\pi_{AB} = 2n \left(f_1 \pi^1_{AB} + f_2 \pi^2_{AB} \right), \tag{6.57}$$

with f_1, f_2, π^1_{AB} and π^2_{AB} given by (6.52)-(6.54). We can see that π_{ab} is a trace free spatial tensor, ie.

$$\pi_{ab} u^a = 0 \;\text{ and }\; \pi_{ab} h^{ab} = 0, \tag{6.58}$$

as $\psi^A{}_a u^a = 0$. The full anisotropic term, written in terms of the unit determinant matrix $\eta^A{}_B$, is

$$\pi_{ab} = 2n\psi^A{}_a \psi^B{}_b \left[f_1 \left(\eta_{AB} - \frac{1}{3} g_{AB} \eta^C{}_C \right) + 2f_2 \left(\eta_{AC} \eta^C{}_B - \frac{1}{3} g_{AB} \eta^C{}_D \eta^D{}_C \right) \right] \tag{6.59}$$

6.2.3 Hookean model

Instead of treating the energy as a function of n and the individual invariants, we can instead use a strain scalar comprised of a combination of the invariants. This scalar must vanish in the relaxed state, when $\eta_{AB} = g_{AB}$. An example of such a strain scalar can be found in [98] in the form

$$s^2 = \frac{1}{36} \left(I_1^3 - I_3 - 24 \right), \tag{6.60}$$

where the invariants are given by

$$I_1 = \eta^A{}_A \;, \qquad I_2 = \eta^A{}_B \eta^B{}_A \;, \qquad I_3 = \eta^A{}_B \eta^B{}_C \eta^C{}_A. \tag{6.61}$$

It is worth noting that s^2 is a function of n as well as g^{AB} and k_{AB}. We need to keep this in mind when we work out various partial derivatives in the following.

In general, the energy is $\varepsilon = \varepsilon(n, s^2)$, but as we tend to be mostly interested in slight deformations, it is natural to consider a Hookean model. Thus we consider the energy

$$\varepsilon = \check{\varepsilon}(n) + \check{\mu}(n)s^2, \tag{6.62}$$

where the checks indicate that the quantity is determined in the relaxed state, and $\check{\mu}$ is the shear modulus. Using this, we find that the derivative of the energy with respect to the matter space metric is given by

$$\frac{\partial \varepsilon}{\partial g^{AB}} = \frac{\partial \varepsilon}{\partial n}\frac{\partial n}{\partial g^{AB}} + \frac{\partial \varepsilon}{\partial s^2}\frac{\partial s^2}{\partial g^{AB}} = \frac{1}{2} n g_{AB}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right) + \check{\mu}\frac{\partial s^2}{\partial g^{AB}}. \tag{6.63}$$

Hence, we find the stress-energy tensor

$$T_{ab} = n h_{ab}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right) - \varepsilon g_{ab} + 2\check{\mu}\psi^A{}_a\psi^B{}_b\frac{\partial s^2}{\partial g^{AB}}. \tag{6.64}$$

Next, let us assume that s^2 depends on the simplest invariant I_1. Then we need

$$I_1 = \eta^A{}_A = n^{-2/3} g^{AB} k_{AB}, \tag{6.65}$$

where we recall that k_{AB} is fixed and does not depend on n. This allows us to work out the partial derivatives required

$$\frac{\partial s^2}{\partial g^{AB}} = \frac{\partial s^2}{\partial I_1}\frac{\partial I_1}{\partial g^{AB}} = \frac{\partial s^2}{\partial I_1}\eta_{AB} \tag{6.66}$$

and

$$\frac{\partial s^2}{\partial n} = \frac{\partial s^2}{\partial I_1}\frac{\partial I_1}{\partial n} = -\frac{2}{3n}\frac{\partial s^2}{\partial I_1} I_1. \tag{6.67}$$

This means that we get a contribution to the stress-energy tensor

$$\pi^1_{ab} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\psi^A{}_a\psi^B{}_b\left(\eta_{AB} - \frac{1}{3}g_{AB}I_1\right) = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\psi^A{}_a\psi^B{}_b\eta_{\langle AB\rangle} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\eta_{\langle ab\rangle}, \tag{6.68}$$

where the $\langle \ldots \rangle$ brackets indicate the removal of the trace. A similar analysis for the other two invariants I_2 and I_3 leads to

$$\pi^2_{ab} = 4\check{\mu}\frac{\partial s^2}{\partial I_2}\psi^A{}_a\psi^B{}_b\eta_{C\langle A}\eta_{B\rangle}^C = 4\check{\mu}\frac{\partial s^2}{\partial I_2}\eta_{c\langle a}\eta_{b\rangle}^c \tag{6.69}$$

and

$$\pi^3_{ab} = 6\check{\mu}\frac{\partial s^2}{\partial I_3}\psi^A{}_a\psi^B{}_b\eta^{CD}\eta_{C\langle A}\eta_{B\rangle D} = 6\check{\mu}\frac{\partial s^2}{\partial I_3}\eta^{cd}\eta_{c\langle a}\eta_{b\rangle d}. \tag{6.70}$$

The general expression would be, depending on the choice of strain scalar s^2, some linear combination of these quantities. We note that each π^N_{ab}, where $(N = 1-3)$, is trace-free and orthogonal to u^a.

As a check of these results, note that when using the strain scalar mentioned above (6.60), we find that

$$\pi_{ab} = \sum_N \pi^N_{ab} = \frac{\check{\mu}}{6}\left[\left(\eta^c{}_c\right)^2\eta_{\langle ab\rangle} - \eta^{cd}\eta_{c\langle a}\eta_{b\rangle d}\right], \tag{6.71}$$

which agrees with the result from [98]. The final stress-energy tensor is

$$T_{ab} = nh_{ab}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right) - \varepsilon g_{ab} + \pi_{ab}. \tag{6.72}$$

6.2.4 The unsheared state

Elastic matter has an unsheared state, when nothing has disturbed it from its preferred or equilibrium position. Hence, for some number density n, there is a minimum value for the energy ϵ which corresponds to this unsheared state.

The unsheared state corresponds to $\eta^A{}_B = \delta^A{}_B$. It is seen from (6.59) that the anisotropic term π_{ab} will vanish when it is the case that $\eta^A{}_B = \delta^A{}_B$. This tells us that η_{AB} is the matrix inverse of g^{AB}

$$\eta_{AB} = g_{AB}. \tag{6.73}$$

Hence, in the unsheared state we have that

$$k_{AB} = n^{2/3} g_{AB}. \tag{6.74}$$

We will be taking into consideration the unsheared state when looking at perturbations of the system.

6.2.5 Perturbations

Now that we have an understanding of relasticity, we will extend our relativistic model which we built in Chapter 5 to include vortex elasticity. In Section 6.2.2 we formulated the anisotropic pressure term π_{ab}. In the unsheared or relaxed state this term will vanish, leaving us with the perfect fluid equations. Hence, we would like to have a closer look at this term when the system is perturbed and therefore when the anisotropic term is non zero. We could try to think of this anisotropic term as the elasticity in the superfluid vortices. This three dimensional elasticity would have to be split into a two dimensional piece in the plane of the vortex array and a tension term along the direction of the vortices. Our starting point will be our pressure term (6.59), which we recall is of the form

$$\begin{aligned}\pi_{ab} = \psi^A{}_a \psi^B{}_b \pi_{AB} =& 2n \Bigg[f_1 \left(\psi^A{}_a \psi^B{}_b \eta_{AB} - \frac{1}{3} \psi^A{}_a \psi^B{}_b g_{AB} \eta^C{}_C \right) \\ &+ 2 f_2 \left(\psi^A{}_a \psi^B{}_b \eta_{AC} \eta^C{}_B - \frac{1}{3} \psi^A{}_a \psi^B{}_b g_{AB} \eta^C{}_D \eta^D{}_C \right) \Bigg].\end{aligned} \tag{6.75}$$

So far, we have been considering this nonlinear elastic term. We will look at linear perturbations of the anisotropic term using the Lagrangian perturbation Δ, as seen in previous chapters.

If we call the pushforward of η_{AB} to spacetime $\eta_{ab} = \psi^A{}_a \psi^B{}_b \eta_{AB}$, then equation (6.75) can be written in terms of spacetime tensors as

$$\begin{aligned}\Delta\pi_{ab} =&\Delta\Bigg\{2n\Bigg[f_1\Bigg(\eta_{ab}-\frac{1}{3}h_{ab}\eta_{cd}h^{cd}\Bigg)\\&+2f_2\Bigg(h_a{}^c h_b{}^d h^{ef}\eta_{df}\eta_{ce}-\frac{1}{3}h_{ab}h^{cd}h^{ef}\eta_{de}\eta_{cf}\Bigg)\Bigg]\Bigg\},\end{aligned}\tag{6.76}$$

where we have applied the Lagrangian perturbation Δ to the elastic term. Expanding this to see all of the perturbed quantities we get

$$\begin{aligned}\Delta\pi_{ab} =&\Delta(2n)\Bigg[f_1\Bigg(\eta_{ab}-\frac{1}{3}h_{ab}\eta_{cd}h^{cd}\Bigg)\\&+2f_2\Bigg(h_a{}^c h_b{}^d h^{ef}\eta_{df}\eta_{ce}-\frac{1}{3}h_{ab}h^{cd}h^{ef}\eta_{de}\eta_{cf}\Bigg)\Bigg]\\&+2n\Bigg[\Delta(f_1)\Bigg(\eta_{ab}-\frac{1}{3}h_{ab}\eta_{cd}h^{cd}\Bigg)\\&+f_1\Bigg(\Delta\eta_{ab}-\frac{1}{3}\Delta(h_{ab}\eta_{cd}h^{cd})\Bigg)\\&+2\Delta(f_2)\Bigg(h_a{}^c h_b{}^d h^{ef}\eta_{df}\eta_{ce}-\frac{1}{3}h_{ab}h^{cd}h^{ef}\eta_{de}\eta_{cf}\Bigg)\\&+2f_2\Bigg(\Delta(h_a{}^c h_b{}^d h^{ef}\eta_{df}\eta_{ce})-\frac{1}{3}\Delta(h_{ab}h^{cd}h^{ef}\eta_{de}\eta_{cf})\Bigg)\Bigg].\end{aligned}\tag{6.77}$$

Any terms outside of the perturbation are in the relaxed state. In the relaxed state, we know that $\eta_{AB}=g_{AB}$ from (6.73). Hence, when we apply the map $\psi^A{}_a$ to this, we see that $\eta_{ab}=h_{ab}$. Using this on the unperturbed terms simplifies the equation to

$$\begin{aligned}\Delta\pi_{ab} =&2n\Bigg[f_1\Bigg(\Delta\eta_{ab}-\frac{1}{3}\Delta(h_{ab}\eta_{cd}h^{cd})\Bigg)\\&+2f_2\Bigg(\Delta(h_a{}^c h_b{}^d h^{ef}\eta_{df}\eta_{ce})-\frac{1}{3}\Delta(h_{ab}h^{cd}h^{ef}\eta_{de}\eta_{cf})\Bigg)\Bigg].\end{aligned}\tag{6.78}$$

Next we expand the equation using the product rule on the perturbations. Simultaneously, we will be able to rewrite the relaxed η_{ab} as h_{ab} as before. This gives us an equation of the form

$$\begin{aligned}\Delta\pi_{ab} =& 2n\Bigg[f_1\left(\Delta\eta_{ab} - \frac{1}{3}h_{ab}h_{cd}\Delta h^{cd} - \frac{1}{3}h_{ab}h^{cd}\Delta\eta_{cd} - \Delta h_{ab}\right) \\ &+ 2f_2\bigg(2h_a{}^c h_b{}^d\Delta\eta_{cd} + h_{ac}h_{bd}\Delta h^{cd} + h_{ac}\Delta h_b{}^c \\ &+ h_{bc}\Delta h_a{}^c - \frac{2}{3}h_{ab}h^{cd}\Delta\eta_{cd} - \frac{2}{3}h_{ab}h_{cd}\Delta h^{cd} - \Delta h_{ab}\bigg)\Bigg].\end{aligned} \tag{6.79}$$

We will make use of several identities, which we show in Appendix A, namely

$$h_{ab}\Delta g^{ab} = -h^{ab}\Delta g_{ab}, \tag{6.80}$$

$$h_{ac}h_{bd}\Delta g^{cd} = -h_a{}^c h_b{}^d\Delta g_{cd}, \tag{6.81}$$

$$\Delta u_a = u^b\Delta g_{ab} + \frac{1}{2}u_a u^c u^d\Delta g_{cd}, \tag{6.82}$$

$$\Delta\eta_{ab} = h_a{}^c h_b{}^d\Delta\eta_{cd}, \tag{6.83}$$

as well as the definition of the projection $h_{ab} = g_{ab} + u_a u_b$ such that $u^a h_{ab} = 0$ and finally that $\Delta\delta_a{}^b = 0$. These enable us, after some algebra, to write our elastic term π_{ab} in the convenient form

$$\begin{aligned}\Delta\pi_{ab} =& 2n\Bigg[f_1\left(h_a{}^c h_b{}^d(\Delta\eta_{cd} - \Delta g_{cd}) - \frac{1}{3}h_{ab}h^{cd}(\Delta\eta_{cd} - \Delta g_{cd})\right) \\ &+ 4f_2\left(h_a{}^c h_b{}^d(\Delta\eta_{cd} - \Delta g_{cd}) - \frac{1}{3}h_{ab}h^{cd}(\Delta\eta_{cd} - \Delta g_{cd})\right)\Bigg],\end{aligned} \tag{6.84}$$

which clearly simplifies a step further to

$$\Delta\pi_{ab} = 2n\left(f_1 + 4f_2\right)\left(h_a{}^c h_b{}^d(\Delta\eta_{cd} - \Delta g_{cd}) - \frac{1}{3}h_{ab}h^{cd}(\Delta\eta_{cd} - \Delta g_{cd})\right). \tag{6.85}$$

We have been using the pushforward of the unit determinant matrix η_{ab} in our equations so far, but we need to consider this term in a little more detail. This originally came from the matter space metric k_{AB} in that

$$\eta_{ab} = \psi^A{}_a \psi^B{}_b \eta_{AB} = n^{-2/3} \psi^A{}_a \psi^B{}_b k_{AB}. \tag{6.86}$$

But, the matter space metric $k_{ab} = \psi^A{}_a \psi^B{}_b k_{AB}$ corresponds to the three dimensional elastic matter in an unstrained state. So in our equation (6.85), we see that we have the difference between the current state of matter in the g_{ab} term and the unstrained state $\eta_{ab} \sim k_{ab}$. Hence, since the matter space metric k_{ab} is fixed in matter space and also flow line orthogonal, since it came from matter space, the Lagrangian perturbation Δk_{ab} will vanish.

Since $\Delta k_{ab} = 0$, we can find out what will happen to our $\Delta\eta_{ab}$ term in the elasticity (6.85). We know from (6.86) that

$$\Delta\eta_{ab} = k_{ab} \Delta n^{-2/3}, \tag{6.87}$$

by using $\Delta k_{ab} = 0$. Next we use the perturbation of n from [61], given by

$$\Delta n = -\frac{n}{2} h^{ab} \Delta g_{ab}, \tag{6.88}$$

to rewrite (6.87) in the form

$$\Delta\eta_{ab} = \frac{1}{3} h_{ab} h^{cd} \Delta g_{cd}. \tag{6.89}$$

We can then use this to simplify (6.85) a little to

$$\Delta\pi_{ab} = -2n\left(f_1 + 4f_2\right)\left(h_a{}^c h_b{}^d \Delta g_{cd} - \frac{1}{3} h_{ab} h^{cd} \Delta g_{cd}\right). \tag{6.90}$$

6.2.6 Hookean model perturbation

If we use the same method of perturbation on our Hookean model, we arrive at a similar equation to that above. The perturbation is given by

$$\Delta\pi_{ab} = -\check{\mu}\left(h_a{}^c h_b{}^d \Delta g_{de} - \frac{1}{3} h_{ab} h^{cd} \Delta g_{cd}\right), \tag{6.91}$$

which means that for the models to be equivalent, the shear modulus must be

$$\check{\mu} = 2n(f_1 + 4f_2). \tag{6.92}$$

6.2.7 Newtonian limit

If we consider the Newtonian limit of the divergence of our elastic term above, we find that it matches the expected linear elasticity term, which we saw earlier in Chapter 3. From Appendix A we see the limit is given by

$$\nabla^j \Delta\pi_{ij} = -2n\left(f_1 + 4f_2\right)\left(\frac{1}{3}\nabla_i\nabla^j\xi_j + \nabla^2\xi_i\right). \tag{6.93}$$

So this relativistic elasticity term does correspond to Newtonian elasticity.

6.3 Vortex tension and elasticity

In the previous section, we created a three dimensional elasticity term $\Delta\pi_{ab}$. However, in the case of superfluid vortices, we know from experiments discussed previously that they exhibit elasticity within their array and also tension along the vortices. So in order to model these behaviours, we would like to have both a tension piece along the vortex direction $\hat{\kappa}^a$ and also a two dimensional vortex array elasticity orthogonal to $\hat{\kappa}^a$.

In order to find these two components of the three dimensional elasticity $\Delta\pi_{ab}$, we project away the $\hat{\kappa}^a$ direction from both indices to get the vortex array elasticity E^{v}_{ab} and then project along the $\hat{\kappa}^a$ direction to find the tension T^{v}_{ab}. However, there are still some remaining terms which are not part of the tension or elasticity which we will denote $S_{ab} = \Delta\pi_{ab} - E^{\mathrm{v}}_{ab} - T^{\mathrm{v}}_{ab}$. So, after applying the projection $\delta_a{}^b - \hat{\kappa}_a\hat{\kappa}^b$ to (6.90), we find a vortex elasticity term in the plane orthogonal to $\hat{\kappa}^a$

$$E^{\mathrm{v}}_{ab} = -2n\left(f_1 + 4f_2\right)\left[\left(h_a{}^c - \hat{\kappa}_a\hat{\kappa}^c\right)\left(h_b{}^d - \hat{\kappa}_b\hat{\kappa}^d\right)\Delta g_{cd} - \frac{1}{3}\left(h_{ab} - \hat{\kappa}_a\hat{\kappa}_b\right)h^{cd}\Delta g_{cd}\right]. \tag{6.94}$$

Then, by projecting into the direction of the vortices using $\hat{\kappa}_a\hat{\kappa}_b\hat{\kappa}^c\hat{\kappa}^d$, we can also find the vortex tension

$$T^{\mathrm{v}}_{ab} = 2n\left(f_1 + 4f_2\right)\hat{\kappa}_a\hat{\kappa}_b\left[\frac{1}{3}h^{cd} - \hat{\kappa}^c\hat{\kappa}^d\right]\Delta g_{cd}. \tag{6.95}$$

As we mentioned above, these are not all of the terms that comprise $\Delta\pi_{ab}$. The 'leftover' terms are

$$S_{ab} = 2\left(f_1 + 4f_2\right)\left[2\hat{\kappa}_a\hat{\kappa}_b\hat{\kappa}^c\hat{\kappa}^d - h_a{}^c\hat{\kappa}_b\hat{\kappa}^d - \hat{\kappa}_a\hat{\kappa}^c h_b{}^d\right]\Delta g_{cd}. \tag{6.96}$$

6.3.1 Newtonian limit

Taking the Newtonian limit of the elasticity tensor (6.94) doesn't quite give the two dimensional Newtonian elasticity in common usage, such as (3.96). As we discussed before, it is usually the case that the vortex direction is taken to be along one of the coordinate axes in Newtonian calculations. Hence, it makes more sense to compare our limit to (3.88). If we assume again that $\hat{\kappa}^a$ satisfies the Killing equation, we arrive at the Newtonian limit

$$\nabla^j E^{\mathrm{v}}_{ij} = 2n\left(f_1 + 4f_2\right)\left[\frac{2}{3}\nabla^{\perp}_i\nabla_k\xi^k - \nabla^{\perp}_k\nabla^{\perp}_i\xi^k - \nabla^j_{\perp}(\perp_{ik}\nabla^{\perp}_j\xi^k)\right]. \tag{6.97}$$

The limit of the tension term is found to be

$$\nabla^j T^{\mathrm{v}}_{ij} = 2n\left(f_1 + 4f_2\right)\left[\frac{2}{3}\nabla^{\parallel}_i\nabla^k\xi_k - 2\nabla^{\parallel}_i\nabla^k_{\parallel}\xi_k\right]. \tag{6.98}$$

If we were to use a tension term in our Newtonian calculation produced by projecting the Newtonian elastic stress tensor (3.77), then this limit matches. However, it does not match what would commonly be used as vortex tension in the Newtonian case, (3.98). This is due to the assumptions made in Newtonian calculations mentioned previously.

Due to our formulation in general relativity, it is difficult to expand our three dimensional elastic term $\Delta\pi_{ab}$ into a planar elasticity and vortex tension. In Newtonian calculations it is easy to pick and choose which terms you would like to keep and also to set unwanted terms to zero. We will now focus on building the vortex elasticity without including the vortex tension. This approach should eliminate any need for using projections to obtain an elastic term and should also mean that there will no longer be unwanted terms such as those in S_{ab} above.

6.4 Two dimensional formulation

The three dimensional formulation in the previous section gave us a two dimensional elasticity. However, it also produced unwanted terms which we are not interested in. Hence we will now try a new approach and create only a vortex elasticity term, using a similar method and tools to those of the previous sections. Instead of a three dimensional matter space, we will use a two dimensional version in order to create the vortex elasticity and we shall exclude the tension. We consider single component isotropic matter, which represents the 'vortex fluid'. The stress energy tensor obtained from the energy ε is of the familiar form

$$T_{ab} = 2\frac{\partial \varepsilon}{\partial g^{ab}} - \varepsilon g_{ab} = \varepsilon u_a u_b + p_{ab}, \tag{6.99}$$

where the p_{ab} contribution, which encodes the stresses, is given by

$$p_{ab} = 2\frac{\partial \varepsilon}{\partial g^{ab}} - \varepsilon h_{ab} \tag{6.100}$$

and the projection is the usual

$$h_{ab} = g_{ab} + u_a u_b. \tag{6.101}$$

We will soon be concerned with 'two dimensional' versions of three dimensional terms. These two dimensional terms will be denoted with a bar. Our first encounter of a term of this form is the projection

$$\bar{h}_{ab} = h_{ab} - \hat{\kappa}_a \hat{\kappa}_b. \tag{6.102}$$

We would like to work with matter space variables $\tilde{X}^A$ which, as we know, are scalar fields in spacetime. We also want to use the matter space map

$$\psi^A{}_a = \frac{\partial \tilde{X}^A}{\partial x^a}. \tag{6.103}$$

As before, the map is such that

$$\psi^A{}_a \psi^b{}_A = h_a{}^b \tag{6.104}$$

and

$$\psi^A{}_a \psi^a{}_B = \delta_B{}^A. \tag{6.105}$$

It is also the case that

$$u^a \psi^A{}_a = 0. \tag{6.106}$$

The vortex number flux follows from the usual variation argument, and is given by

$$N^a = N u^a = \frac{1}{3!} \epsilon^{abcd} N_{bcd}. \tag{6.107}$$

It is easy to see that this flux is conserved, as in the previous discussions,

$$\nabla_a (N u^a) = 0. \tag{6.108}$$

Next we will consider one of the matter space directions to be related to our vortex direction, by specifying that

$$\hat{\kappa}_a = \frac{\partial \tilde{X}^A}{\partial x^a} \delta_A{}^3 = \psi^3{}_a, \tag{6.109}$$

which tells us that

$$\psi^a{}_B \hat{\kappa}_a = \psi^a{}_B \psi^A{}_a \delta_A{}^3 = \delta_B{}^3 = \hat{\kappa}_B. \tag{6.110}$$

This leaves us with two other dimensions in the matter space map. We denote the remaining dimensions of the map with a bar such that

$$\bar{\psi}^A{}_a \hat{\kappa}^a = 0 = \bar{\psi}^A{}_a \hat{\kappa}_A \tag{6.111}$$

and $A \in \{1,2\}$ when using $\bar{\psi}^A{}_a$. Considering this new information, we can rewrite the number density (6.107) in the form

$$N^a = \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\hat{\kappa}^e = \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_b \bar{N}_{cd}, \tag{6.112}$$

see Appendix A for this manipulation. With this new form of N^a, the conservation equation (6.108) can be written in the form, see Appendix A,

$$\nabla^{\perp}_a N^a = 0, \tag{6.113}$$

where the derivative is orthogonal to $\hat{\kappa}^a$. This form of the number density conservation equation matches that in Chapter 5 and is also familiar from our discussion in the Newtonian formulation. We also make use of a two dimensional version of the pullback of the spacetime metric which we saw in (6.30)

$$\bar{g}_{AB} = g_{AB} - \hat{\kappa}_A\hat{\kappa}_B, \tag{6.114}$$

which implies that

$$\psi^A{}_a\psi^B{}_b\bar{g}_{AB} = \bar{h}_{ab} = \bar{\psi}^A{}_a\bar{\psi}^B{}_b\bar{g}_{AB}. \tag{6.115}$$

For isotropic matter, we have the energy as a function of the matter space map and spacetime metric

$$\varepsilon(\bar{\psi}^A{}_a, g^{ab}) = \varepsilon(\bar{g}^{AB}), \tag{6.116}$$

where

$$\bar{g}^{AB} = \bar{\psi}^A{}_a\bar{\psi}^B{}_b g^{ab}. \tag{6.117}$$

Equation (6.116) has to hold since this is the only way to contract the spacetime indices if the building blocks are the map and the spacetime metric. In the following, we will raise and lower matter space indices with $\bar{g}^{AB}$ and its inverse $\bar{g}_{AB}$.

To build the stress-energy tensor, we need

$$\frac{\partial \varepsilon}{\partial g^{ab}} = \frac{\partial \varepsilon}{\partial \bar{g}^{AB}} \frac{\partial \bar{g}^{AB}}{\partial g^{ab}} = \bar{\psi}^A{}_a \bar{\psi}^B{}_b \frac{\partial \varepsilon}{\partial \bar{g}^{AB}}. \tag{6.118}$$

This shows that $u^a p_{ab} = 0$ which means that ε is indeed the energy. For isotropic matter, we have

$$\frac{\partial N}{\partial \bar{g}^{AB}} = \frac{1}{2} N \bar{g}_{AB}, \tag{6.119}$$

which is shown in Appendix A. Equation (6.119) tells us that the stress-energy tensor can now be written

$$T_{ab} = \left[\bar{\psi}^A{}_a \bar{\psi}^B{}_b \bar{g}_{AB}\right] N \frac{\partial \varepsilon}{\partial N} - \varepsilon g_{ab}. \tag{6.120}$$

Let us now consider elastic matter. To do this we will need to compare the current shape of the matter to a relaxed reference shape. A key component to this construction is the matter space metric

$$\bar{k}_{ab} = \bar{\psi}^A{}_a \bar{\psi}^B{}_b \bar{k}_{AB}, \tag{6.121}$$

which is a fixed matter space tensor. As long as the deformation is conformal, which we will assume, $\bar{k}_{ab}$ does not depend on the number density N. The strain associated with the deformation is quantified by the difference

$$s_{ab} = \frac{1}{2}(\bar{h}_{ab} - \bar{\eta}_{ab}) \tag{6.122}$$

where

$$\bar{\eta}_{ab} = \bar{k}^{-1/2} \bar{k}_{ab}, \tag{6.123}$$

see Appendix A for details.

Equivalently, we can compare the matter space quantities $\bar{\eta}_{AB}$ and $\bar{g}_{AB}$. We know that in the relaxed state they are equal and this will come into play later on. As we are dealing with isotropic matter, it is natural to work with invariants. From $\bar{\eta}_{AB}$ we can construct the invariants

$$I_1 = \bar{\eta}^A{}_A \quad \text{and} \quad I_2 = \bar{\eta}^A{}_B \bar{\eta}^B{}_A. \tag{6.124}$$

However, the number density N^2 is also invariant. This follows from the fact that the determinant of $\bar{k}_{AB}$ is given by

$$\bar{k} = N^2 = \frac{1}{2!}\left(N^2 I_1^2 - N^2 I_2\right). \tag{6.125}$$

It is natural to work with N as one of the variables, as this retains the connection with the standard fluid problem. In addition, we will need another invariant, or some combination of them. The invariants available to us so far are

$$\bar{k} = N^2, \quad I_1 = \bar{\eta}^A{}_A \quad \text{and} \quad I_2 = \bar{\eta}^A{}_B \bar{\eta}^B{}_A. \tag{6.126}$$

In general, we have the energy $\varepsilon = \varepsilon(N, s^2)$ where the strain scalar s is some combination of the remaining invariants, such that it vanishes when the matter is in the relaxed state. As we tend to be mostly interested in slight deformations, it is natural to consider a Hookean model, such that

$$\varepsilon = \check{\varepsilon}(N) + \check{\mu} s^2, \tag{6.127}$$

where $\check{\mu}$ is the shear modulus and the checks indicate that the quantity is determined in the relaxed state. Let us now return to the stress-energy tensor (6.120). Considering our energy of the form (6.127), we now have

$$\frac{\partial \varepsilon}{\partial \bar{g}^{AB}} = \frac{\partial \varepsilon}{\partial N}\frac{\partial N}{\partial \bar{g}^{AB}} + \frac{\partial \varepsilon}{\partial s^2}\frac{\partial s^2}{\partial \bar{g}^{AB}} = \frac{1}{2} N \bar{g}_{AB}\left(\frac{\partial \check{\varepsilon}}{\partial N} + \frac{\partial \check{\mu}}{\partial N} s^2 + \check{\mu}\frac{\partial s^2}{\partial N}\right) + \check{\mu}\frac{\partial s^2}{\partial \bar{g}^{AB}}. \tag{6.128}$$

Thus we find that the stress-energy tensor is now given by

$$T_{ab} = N \bar{\psi}^A{}_a \bar{\psi}^B{}_b \bar{g}_{AB}\left(\frac{\partial \check{\varepsilon}}{\partial N} + \frac{\partial \check{\mu}}{\partial N} s^2 + \check{\mu}\frac{\partial s^2}{\partial N}\right) - \varepsilon g_{ab} + 2\check{\mu} \bar{\psi}^A{}_a \bar{\psi}^B{}_b \frac{\partial s^2}{\partial \bar{g}^{AB}}, \tag{6.129}$$

or

$$T_{ab} = N\bar{\psi}^A{}_a\bar{\psi}^B{}_b\bar{g}_{AB}\left(\frac{\partial\check{\varepsilon}}{\partial N} + \frac{\partial\check{\mu}}{\partial N}s^2\right) - \varepsilon g_{ab} + 2\check{\mu}\bar{\psi}^A{}_a\bar{\psi}^B{}_b\frac{\partial s^2}{\partial\bar{g}^{AB}} + N\check{\mu}\bar{\psi}^A{}_a\bar{\psi}^B{}_b\bar{g}_{AB}\frac{\partial s^2}{\partial N}, \tag{6.130}$$

where the final two terms give us the two dimensional anisotropic stress $\bar{\pi}_{ab}$. We will now assume that the strain scalar s^2 depends on the simplest invariant I_1. Then we need

$$I_1 = \bar{\eta}^A{}_A = N^{-1}\bar{g}^{AB}\bar{k}_{AB}, \tag{6.131}$$

where we recall that $\bar{k}_{AB}$ is fixed and does not depend on N. This allows us to work out the partial derivatives we need

$$\frac{\partial s^2}{\partial\bar{g}^{AB}} = \frac{\partial s^2}{\partial I_1}\frac{\partial I_1}{\partial\bar{g}^{AB}} = \frac{\partial s^2}{\partial I_1}\bar{\eta}_{AB} \tag{6.132}$$

and

$$\frac{\partial s^2}{\partial N} = \frac{\partial s^2}{\partial I_1}\frac{\partial I_1}{\partial N} = -\frac{1}{N}\frac{\partial s^2}{\partial I_1}I_1. \tag{6.133}$$

This means that the anisotropic contribution to the stress-energy tensor is

$$\bar{\pi}^1_{ab} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\bar{\psi}^A{}_a\bar{\psi}^B{}_b\left(\bar{\eta}_{AB} - \frac{1}{2}\bar{g}_{AB}I_1\right) = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\bar{\psi}^A{}_a\bar{\psi}^B{}_b\bar{\eta}_{\langle AB\rangle} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\bar{\eta}_{\langle ab\rangle}, \tag{6.134}$$

where the $\langle\ldots\rangle$ brackets indicate the removal of the trace. A similar analysis for the other invariant I_2 tells us that the corresponding partial derivatives are

$$\frac{\partial s^2}{\partial\bar{g}^{AB}} = \frac{\partial s^2}{\partial I_2}\frac{\partial I_2}{\partial\bar{g}^{AB}} = \frac{\partial s^2}{\partial I_2}2\bar{\eta}_A{}^C\bar{\eta}_{CB} \tag{6.135}$$

and

$$\frac{\partial s^2}{\partial N} = \frac{\partial s^2}{\partial I_2}\frac{\partial I_2}{\partial N} = -\frac{2}{N}\frac{\partial s^2}{\partial I_2}I_2. \tag{6.136}$$

These give another contribution to the stress-energy tensor

$$\begin{aligned}\bar{\pi}^2_{ab} &= 4\check{\mu}\bar{\psi}^A{}_a\bar{\psi}^B{}_b\frac{\partial s^2}{\partial I_2}\left(\bar{\eta}_A{}^C\bar{\eta}_{CB} - \frac{1}{2}\bar{g}_{AB}I_2\right) \\ &= 4\check{\mu}\frac{\partial s^2}{\partial I_2}\bar{\psi}^A{}_a\bar{\psi}^B{}_b\bar{\eta}_{C\langle A}\bar{\eta}_{B\rangle}{}^C = 4\check{\mu}\frac{\partial s^2}{\partial I_2}\bar{\eta}_{c\langle a}\bar{\eta}_{b\rangle}{}^c.\end{aligned} \tag{6.137}$$

The general expression would be, depending on the chosen scalar s^2, some linear combination of these quantities. We note that each $\bar{\pi}^Y_{ab}$, where $Y \in \{1, 2\}$, is trace-free and orthogonal to both u^a and $\hat{\kappa}^a$. The full anisotropic term is given by

$$\bar{\pi}_{ab} = \sum_Y \bar{\pi}^Y_{ab} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\bar{\eta}_{\langle ab\rangle} + 4\check{\mu}\frac{\partial s^2}{\partial I_2}\bar{\eta}_{c\langle a}\bar{\eta}_{b\rangle}{}^c. \tag{6.138}$$

6.4.1 The unsheared state

As in the previous three dimensional formulation, the matter has a relaxed state in which the anisotropic term will vanish. The unsheared state corresponds to $\bar{\eta}^A{}_B = \bar{\delta}^A{}_B$, in which case the anisotropic term $\bar{\pi}_{ab}$ will vanish. This tells us that $\bar{\eta}_{AB}$ is the matrix inverse of $\bar{g}^{AB}$

$$\bar{\eta}_{AB} = \bar{g}_{AB}. \tag{6.139}$$

Hence it will be the case that

$$\bar{k}_{AB} = N\bar{g}_{AB}. \tag{6.140}$$

We also know that in the unstrained state, the strain scalar s^2 must also vanish. This is an important feature to consider when constructing a strain scalar. We will see in the following section that we must choose a scalar such that it vanishes when the system is unperturbed.

6.4.2 Perturbations

As the anisotropic term derived above vanishes in the relaxed state, we need to consider what happens when the system is not relaxed. We do this, once again, by considering perturbations of $\bar{\pi}_{ab}$. This anisotropic stress will give us the two dimensional vortex elasticity. Firstly, we need to use a strain scalar which vanishes when the system is relaxed. We will use the scalar

$$s^2 = \left(I_2 - \frac{I_1^2}{2}\right), \tag{6.141}$$

which implies that the relevant derivatives are

$$\frac{\partial s^2}{\partial I_1} = -I_1 \quad \text{and} \quad \frac{\partial s^2}{\partial I_2} = 1. \tag{6.142}$$

Putting these into (6.138) tells us that the anisotropic stress is

$$\bar{\pi}_{ab} = \sum_N \bar{\pi}^N_{ab} = 4\check{\mu}\left(\bar{\eta}_{c\langle a}\bar{\eta}_{b\rangle}{}^c - \frac{1}{2}\bar{\eta}^c{}_c\bar{\eta}_{\langle ab\rangle}\right). \tag{6.143}$$

Applying the perturbation Δ to this equation gives us

$$\Delta\pi_{ab} = 4\check{\mu}\Delta\left(\eta_{c\langle a}\eta_{b\rangle}{}^c - \frac{1}{2}\eta^c{}_c\eta_{\langle ab\rangle}\right). \tag{6.144}$$

We now expand all of the perturbations using the product rule and remember that in the relaxed or unperturbed state $\bar{\eta}_{ab} = \bar{h}_{ab} = g_{ab} + u_a u_b - \hat{\kappa}_a\hat{\kappa}_b$. This implies, after some simplification, that

$$\Delta\bar{\pi}_{ab} = 4\check{\mu}\left(\bar{h}_{ac}\Delta\bar{\eta}_b{}^c + \bar{h}_b{}^c\Delta\bar{\eta}_{ca} - \bar{h}_{ab}\bar{h}_c{}^d\Delta\bar{\eta}_d{}^c - \Delta\bar{\eta}_{ab} + \frac{1}{2}\bar{h}_{ab}\Delta\bar{\eta}^c{}_c\right). \tag{6.145}$$

We will now rewrite this expression using identities similar to those in the previous three dimensional calculation. We also note that the perturbation of the matter space metric will again vanish $\Delta\bar{k}_{ab} = 0$, as this represents matter in the unstrained state and is fixed in matter space. As the perturbation of $\bar{k}_{ab}$ vanishes, we can find that the perturbation of $\bar{\eta}_{ab}$ is given by

$$\Delta\bar{\eta}_{ab} = \frac{1}{2}\bar{h}_{ab}h^{cd}\Delta g_{cd}. \tag{6.146}$$

Hence, using the information and identities discussed above, we arrive at the equation

$$\Delta\bar{\pi}_{ab} = -\,4\check{\mu}\left(\bar{h}_a{}^c\bar{h}_b{}^d - \frac{1}{2}\bar{h}_{ab}\bar{h}^{cd}\right)\Delta g_{cd}. \tag{6.147}$$

This is our two dimensional anisotropic stress contribution to the stress-energy tensor, from which vortex elasticity arises.

6.4.3 Newtonian limit

If we consider the Newtonian limit of our two dimensional elasticity term (6.147), we find

$$\nabla^j \Delta\bar{\pi}_{ij} = 4\check{\mu}\left[\nabla_i^\perp \nabla_\perp^k \xi_k - \nabla_k^\perp \nabla_i^\perp \xi^k - \nabla_\perp^j (\perp_{ik} \nabla_j^\perp \xi^k)\right]. \tag{6.148}$$

As we found with the three dimensional case, this does not match the commonly used Newtonian vortex elasticity. However, it does follow the form of the general elasticity (3.88). You can see that all of the derivatives are orthogonal to $\hat{\kappa}^a$, as in the Newtonian vortex elasticity, however we have different terms due to not being able to commute the derivatives as discussed before.

Combining our new relativistic model from Chapter 5 with our new two dimensional vortex elasticity built in this chapter completes our construction of a multiple fluid model including quantised vortices, mutual friction and elasticity in general relativity.

6.5 Variational approach redux

We now return to the variational description in spacetime, with the intention of extending it to account for elasticity. The main motivation for this exercise is that force balance equations, such as (4.67), are more readily adapted to multi-fluid settings, where it is useful to have individual momentum equations for the different constituents. These equations can, to a certain extent, be extracted from the equations obtained from the total stress energy tensor, but this route is not as elegant.

The main lesson from the matter space derivation we produced in the previous sections, is that we need to allow the Lagrangian to depend on k_{ab}. Recall that k_{ab} is independent of n and that the contraction $n^a k_{ab}$ vanishes. This leads to

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\mu_a \delta n^a + \left(\frac{1}{2}\Lambda g^{ab} + \frac{\partial\Lambda}{\partial g_{ab}}\right)\delta g_{ab} + K^{ab}\delta k_{ab}\right], \tag{6.149}$$

where we say that

$$K^{ab} = \frac{\partial \Lambda}{\partial k_{ab}}. \tag{6.150}$$

As usual, we replace δn^a with the Lagrangian displacement ξ^a. As we have mentioned previously, k_{ab} is constructed from a fixed matter space tensor which means that its perturbation vanishes, $\Delta k_{ab} = 0$. This tells us that the Eulerian perturbation takes the form

$$\delta k_{ab} = -\xi^c \nabla_c k_{ab} - k_{cb} \nabla_a \xi^c - k_{ac} \nabla_b \xi^c. \tag{6.151}$$

Again ignoring surface terms and keeping in mind that k_{ab} is symmetric, we have

$$K^{ab} \delta k_{ab} = \xi^a \left[2\nabla_b \left(K^{bc} k_{ac} \right) - K^{bc} \nabla_a k_{bc} \right]. \tag{6.152}$$

Making use of this result, we arrive at

$$\delta \left(\sqrt{-g} \Lambda \right) = \sqrt{-g} \left\{ \left[\frac{1}{2} \left(\Lambda - n^c \mu_c \right) g^{ab} + \frac{\partial \Lambda}{\partial g_{ab}} \right] \delta g_{ab} + \tilde{f}_a \xi^a \right\}, \tag{6.153}$$

where

$$\tilde{f}_a = 2n^b \nabla_{[a} \mu_{b]} + 2\nabla_b \left(K^{bc} k_{ac} \right) - K^{bc} \nabla_a k_{bc} = 0. \tag{6.154}$$

From this we find that the stress-energy tensor takes the form

$$T^{ab} = \left(\Lambda - n^c \mu_c \right) g^{ab} + 2 \frac{\partial \Lambda}{\partial g_{ab}}. \tag{6.155}$$

In order to make contact with the matter space derivation we saw previously, let us focus on the Hookean model, where

$$\Lambda = -\check{\varepsilon}(n) - \check{\mu}(n) s^2 = -\varepsilon. \tag{6.156}$$

Again, we build the strain scalar s^2 out of the invariants of

$$\eta_{ab} = n^{-2/3} k_{ab}. \tag{6.157}$$

These invariants,

$$I_1 = \eta^a{}_a, \quad I_2 = \eta^a{}_b \eta^b{}_a, \quad \ldots, \tag{6.158}$$

still depend on n and hence on n^a and g_{ab}, as well as k_{ab}.

We find that

$$\mu_a = \frac{\partial \Lambda}{\partial n^a} = \frac{\partial n^2}{\partial n^a}\frac{\partial \Lambda}{\partial n^2} = -\frac{1}{n}\frac{\partial \Lambda}{\partial n} g_{ab} n^b = \frac{1}{n}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right) g_{ab} n^b, \tag{6.159}$$

$$\frac{\partial \Lambda}{\partial g_{ab}} = -\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right)\frac{\partial n}{\partial g_{ab}} - \check{\mu}\frac{\partial s^2}{\partial g_{ab}}, \tag{6.160}$$

where, noting that n^a is held fixed in the partial derivative,

$$\frac{\partial n}{\partial g_{ab}} = -\frac{1}{2n} n^a n^b. \tag{6.161}$$

It is also useful to note that

$$\frac{\partial s^2}{\partial g_{ab}} = -g^{ac} g^{bd} \frac{\partial s^2}{\partial g^{cd}}. \tag{6.162}$$

Also, when working out this derivative, we need to hold n fixed (as is clear from (6.160)). This means that that stress-energy tensor takes the form

$$\begin{aligned} T^{ab} = &\left[\Lambda + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right)\right] g^{ab} \\ &+ \frac{1}{n}\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right) n^a n^b + 2\check{\mu} g^{ac} g^{bd} \frac{\partial s^2}{\partial g^{cd}}, \end{aligned} \tag{6.163}$$

which condenses to

$$T^{ab} = \Lambda g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2 + \check{\mu}\frac{\partial s^2}{\partial n}\right) h^{ab} + 2\check{\mu} g^{ac} g^{bd} \frac{\partial s^2}{\partial g^{cd}}. \tag{6.164}$$

Now, as before, let us consider the explicit case when s^2 depends on the first invariant, I_1. For this we need

$$I_1 = \eta^a{}_a = n^{-2/3} g^{ab} k_{ab}, \tag{6.165}$$

$$\frac{\partial s^2}{\partial n} = -\frac{2I_1}{3n}\frac{\partial s^2}{\partial I_1}, \tag{6.166}$$

$$K^{ab} = \frac{\partial \Lambda}{\partial k_{ab}} = -\check{\mu}\frac{\partial s^2}{\partial k_{ab}} = -\check{\mu}n^{-2/3}g^{ab}\frac{\partial s^2}{\partial I_1}, \tag{6.167}$$

and finally

$$\frac{\partial s^2}{\partial g^{cd}} = \frac{\partial s^2}{\partial I_1}\eta_{cd}. \tag{6.168}$$

Using these results, we arrive at the stress-energy tensor

$$\begin{aligned} T^{ab} &= -\varepsilon g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right)h^{ab} + 2\check{\mu}\frac{\partial s^2}{\partial I_1}\left(\eta^{ab} - \frac{1}{3}I_1 h^{ab}\right) \\ &= -\varepsilon g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right)h^{ab} + 2\check{\mu}\frac{\partial s^2}{\partial I_1}\eta^{\langle ab\rangle} \end{aligned} \tag{6.169}$$

This should look familiar as it is the result we obtained from our matter space derivation previously. In the same way, we also obtain results for the cases in which s^2 depends on I_2

$$I_2 = \eta^a{}_b\eta^b{}_a = n^{-4/3}g^{ab}k_{bc}g^{cd}k_{da}, \tag{6.170}$$

$$\frac{\partial s^2}{\partial n} = -\frac{4I_2}{3n}\frac{\partial s^2}{\partial I_2}, \tag{6.171}$$

$$K^{ab} = \frac{\partial \Lambda}{\partial k_{ab}} = -\check{\mu}\frac{\partial s^2}{\partial k_{ab}} = -2\check{\mu}n^{-4/3}g^{ac}g^{bd}k_{cd}\frac{\partial s^2}{\partial I_2}, \tag{6.172}$$

$$\frac{\partial s^2}{\partial g^{cd}} = 2\frac{\partial s^2}{\partial I_2}\eta_{bc}\eta_d{}^b \tag{6.173}$$

and finally on I_3

$$I_3 = \eta^a{}_a = n^{-6/3}g^{bd}k_{ad}g^{ec}k_{be}g^{af}k_{cf}, \tag{6.174}$$

$$\frac{\partial s^2}{\partial n} = -\frac{6I_3}{3n}\frac{\partial s^2}{\partial I_3}, \tag{6.175}$$

$$K^{ab} = \frac{\partial \Lambda}{\partial k_{ab}} = -\check{\mu}\frac{\partial s^2}{\partial k_{ab}} = -3\check{\mu}n^{-6/3}g^{cd}g^{ae}g^{bf}k_{df}k_{ce}\frac{\partial s^2}{\partial I_3}, \tag{6.176}$$

$$\frac{\partial s^2}{\partial g^{cd}} = 3\frac{\partial s^2}{\partial I_3}\eta^{ab}\eta_{ac}\eta_{db}. \tag{6.177}$$

These tell us that the stress-energy contribution, when s^2 depends upon all three invariants, is given by

$$T^{ab} = -\varepsilon g^{ab} + n\left(\frac{d\check{\varepsilon}}{dn} + \frac{d\check{\mu}}{dn}s^2\right)h^{ab} + \pi^{ab}, \tag{6.178}$$

where

$$\pi^{ab} = 2\check{\mu}\frac{\partial s^2}{\partial I_1}\eta^{\langle ab\rangle} + 4\check{\mu}\frac{\partial s^2}{\partial I_2}\eta_c{}^{\langle a}\eta^{b\rangle c} + 6\check{\mu}\frac{\partial s^2}{\partial I_3}\eta_{cd}\eta^{c\langle a}\eta^{b\rangle d}. \tag{6.179}$$

We can now extend (4.73) to show that the equations of motion are, indeed, given by (6.154). To see this, we will first rewrite (6.164) in the form

$$\begin{aligned} T^{ab} &= (\Lambda - n^c\mu_c)\, g^{ab} + n^a\mu^b + 2\check{\mu}\frac{\partial s^2}{\partial I_1}\eta^{ab} + 4\check{\mu}\frac{\partial s^2}{\partial I_2}\eta_c{}^a\eta^{bc} + 6\check{\mu}\frac{\partial s^2}{\partial I_3}\eta_{cd}\eta^{ca}\eta^{bd} \\ &= (\Lambda - n^c\mu_c)\, g^{ab} + n^a\mu^b - 2g^{ac}K^{bd}k_{cd}. \end{aligned} \tag{6.180}$$

From this, it follows that

$$\nabla_a T^{ab} = -2g^{cb}n^a\nabla_{[c}\mu_{a]} + K^{ac}\nabla^b k_{ac} - 2\nabla^c\left(K^{bd}k_{cd}\right) = -\tilde{f}^b = 0, \tag{6.181}$$

using the fact that k_{ab} is symmetric.

6.6 Vorticity variation

We shall now consider that the Lagrangian depends upon the antisymmetric tensor ω_{ab}. This comes from the matter space tensor ω_{AB}, i.e

$$\omega_{ab} = \psi^A{}_a\psi^B{}_b\omega_{AB}. \tag{6.182}$$

This matter space tensor is fixed provided

$$u^a\omega_{ab} = 0 \tag{6.183}$$

and

$$\mathcal{L}_u\omega_{ab} = 0. \tag{6.184}$$

This means that ω_{AB} only depends on the matter space coordinates X^A. The equations above imply that

$$\nabla_{[a}\omega_{bc]} = 0 \tag{6.185}$$

and as we have seen previously, we also have

$$\Delta\omega_{ab} = 0. \tag{6.186}$$

Now, let us consider a Lagrangian $\Lambda = \Lambda(\mathcal{N}^2)$ where

$$\mathcal{N}^2 = \frac{1}{2}\omega^{ab}\omega_{ab} = \frac{1}{2}g^{ac}g^{bd}\omega_{ab}\omega_{cd}. \tag{6.187}$$

Then we see that

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\left(\frac{1}{2}\Lambda g^{ab} + \frac{\partial\Lambda}{\partial g_{ab}}\right)\delta g_{ab} + \frac{\partial\Lambda}{\partial\omega_{ab}}\delta\omega_{ab}\right]. \tag{6.188}$$

Introducing the Lagrangian displacement as before and ignoring surface terms we have

$$\delta\omega_{ab} = -\xi^c\nabla_c\omega_{ab} - \omega_{ac}\nabla_b\xi^c - \omega_{cb}\nabla_a\xi^c. \tag{6.189}$$

From this we arrive at

$$\frac{\partial\Lambda}{\partial\omega_{ab}}\left[\nabla_c\omega_{ab} + \nabla_b\omega_{ca} + \nabla_a\omega_{bc}\right] - 2\omega_{ac}\nabla_b\left(\frac{\partial\Lambda}{\partial\omega_{ab}}\right) = 0. \tag{6.190}$$

Notice that the first bracket vanishes due to the exterior derivative of ω_{ab} vanishing (6.185). This means that we are left with

$$\omega_{ac}\nabla_b\left(\frac{\partial\Lambda}{\partial\omega_{ab}}\right) = 0. \tag{6.191}$$

This could be seen as the equation of motion for the vortices in the absence of the superfluid condensate.

Let us now ask what would happen if Λ depended on both n_{abc} and ω_{ab}. We would have

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\left(\frac{1}{2}\Lambda g^{ab} + \frac{\partial\Lambda}{\partial g_{ab}}\right)\delta g_{ab} + \frac{\partial\Lambda}{\partial n_{abc}}\delta n_{abc} + \frac{\partial\Lambda}{\partial\omega_{ab}}\delta\omega_{ab}\right]. \tag{6.192}$$

From $\Delta n_{abc} = 0$ we can find that

$$\delta n_{abc} = -\xi^d \nabla_d n_{abc} - n_{dbc} \nabla_a \xi^d - n_{adc} \nabla_b \xi^d - n_{abd} \nabla_c \xi^d. \tag{6.193}$$

Using this and (6.190) we can determine that

$$\begin{aligned} \frac{\partial \Lambda}{\partial n_{abc}} [\nabla_d n_{abc} - \nabla_c n_{dab} - \nabla_a n_{bcd} + \nabla_b n_{cda}] - 3 n_{dbc} \nabla_a \left(\frac{\partial \Lambda}{\partial n_{abc}} \right) \\ - 2\omega_{ad} \nabla_b \left(\frac{\partial \Lambda}{\partial \omega_{ab}} \right) = 0. \end{aligned} \tag{6.194}$$

Then since $\nabla_{[a} n_{bcd]} = 0$, the first term vanishes and we are left with

$$-\frac{1}{2} n_{dbc} \nabla_a \mu^{abc} + 2\omega_{ad} \nabla_b \left(\frac{\partial \Lambda}{\partial \omega_{ab}} \right) = 0, \tag{6.195}$$

using

$$\frac{\partial \Lambda}{\partial n_{abc}} = -\frac{1}{3!} \mu^{abc}. \tag{6.196}$$

The first term of (6.195) can be simplified in the follow way. Firstly we write it as

$$-\frac{1}{2} n_{dbc} \nabla_a \mu^{abc} = -\frac{1}{2} \epsilon_{dbce} \epsilon^{abcf} n^e \nabla_a \mu_f, \tag{6.197}$$

which can then be simplified to

$$-\frac{1}{2} \epsilon_{dbce} \epsilon^{abcf} n^e \nabla_a \mu_f = n^e \nabla_d \mu_e - n^e \nabla_e \mu_d = n^e \omega_{de}. \tag{6.198}$$

Hence, we arrive at the equation

$$n^a \omega_{ab} = 2\omega_{ab} \nabla_c \left(\frac{\partial \Lambda}{\partial \omega_{ac}} \right). \tag{6.199}$$

There is a contradiction here in that the variation of ω_{ab} assumed it was fixed in matter space, but this requires (6.183) to hold. Hence this contradicts the result in (6.199) above.

6.7 Kalb-Ramond version

We will now try a variation using the Kalb-Ramond gauge field to see if this helps us shed some light on the results of the previous section. As we have seen before, the conservation law

$$\nabla_a n^a = 0, \tag{6.200}$$

is automatically satisfied provided we have

$$n^a = \frac{1}{3!}\epsilon^{abcd} n_{bcd}, \tag{6.201}$$

with the condition

$$\nabla_{[a} n_{bcd]} = 0. \tag{6.202}$$

This last relation implies the existence of a two-form B_{ab} such that

$$n_{abc} = 3\nabla_{[a} B_{bc]}. \tag{6.203}$$

Hence the definition (6.201) can be rewritten in the form

$$n^a = \frac{1}{2}\epsilon^{abcd}\nabla_b B_{cd}. \tag{6.204}$$

Next we consider the vorticity ω_{ab} as an independent, but not entirely 'free', variable.

6.7.1 Version 1

Use Legendre transformation;

$$\mathcal{L} = \Lambda - \frac{1}{2}\omega^{ab} B_{ab} = \Lambda - \frac{1}{4}\epsilon^{abcd} B_{ab}\omega_{cd}. \tag{6.205}$$

Assuming that $\Lambda = \Lambda(n)$ we get

$$\delta\mathcal{L} = -\frac{1}{3!}\mu^{abc}\delta n_{abc} - \frac{1}{2}B_{ab}\delta\omega^{ab} - \frac{1}{2}\omega^{ab}\delta B_{ab}, \tag{6.206}$$

where we have made use of

$$\frac{\partial \Lambda}{\partial n_{abc}} = -\frac{1}{3!}\mu^{abc}. \tag{6.207}$$

However, we now know that

$$\delta n_{abc} = 3\nabla_{[a}\delta B_{bc]}, \tag{6.208}$$

so (6.206) will take the form

$$2\delta\mathcal{L} = -\mu^{abc}\nabla_a \delta B_{bc} - B_{ab}\delta\omega^{ab} - \omega^{ab}\delta B_{ab}. \tag{6.209}$$

Ignoring surface terms, we can rewrite this as

$$\delta\mathcal{L} = \left(\nabla_a \mu^{abc} - \omega^{bc}\right)\delta B_{bc} - \frac{1}{2}B_{ab}\delta\omega^{ab}. \tag{6.210}$$

This suggests that we should have

$$\omega^{bc} = \nabla_a \mu^{abc}, \tag{6.211}$$

which can be rewritten in the form

$$\omega_{ab} = -\frac{1}{2}\epsilon_{abcd}\omega^{cd} = 2\nabla_{[a}\mu_{b]}, \tag{6.212}$$

giving the standard definition of the vorticity tensor. However, with a free variation we would also have $B_{ab} = 0$. To avoid this, we introduce a two dimensional matter space with coordinates χ^A. We know that this gives

$$\Delta\chi^A = 0 \quad \longrightarrow \quad \delta\chi^A = -\mathcal{L}_\xi \chi^A. \tag{6.213}$$

Next, we use $\Delta\omega_{ab} = 0$ to see that

$$\delta\omega^{ab} = \frac{1}{2}\epsilon^{abcd}\delta\omega_{cd} = \epsilon^{abcd}\nabla_c\left(\omega_{de}\xi^e\right). \tag{6.214}$$

This means that the last term in (6.210) leads to, ignoring surface terms,

$$\epsilon^{abcd}\omega_{de}\nabla_c B_{ab} = 0. \tag{6.215}$$

But, from (6.203), it then follows that

$$n^a \omega_{ab} = 0. \tag{6.216}$$

Thus, we recover the usual perfect fluid equations of motion. Of course, we have not achieved much other than that. This result shouldn't be a surprise as we have already seen that the perfect fluid equations of motion can be found using the Kalb-Ramond gauge field in Section 4.5.

6.7.2 Version 2

Now let Λ depend on ω_{ab} as well, such that

$$\delta\Lambda = -\frac{1}{3!}\mu^{abc}\delta n_{abc} - \frac{1}{2}\lambda^{ab}\delta\omega_{ab}. \tag{6.217}$$

Then we find

$$\delta\mathcal{L} = -\frac{1}{2}\mu^{abc}\nabla_a \delta B_{bc} - \frac{1}{2}\lambda^{ab}\delta\omega_{ab} - \frac{1}{4}\epsilon^{abcd}\left(B_{ab}\delta\omega_{cd} + \omega_{cd}\delta B_{ab}\right). \tag{6.218}$$

Using the same argument as we did previously gives us

$$\delta\mathcal{L} = \frac{1}{2}\left(\nabla_a \mu^{abc} - \omega^{ab}\right)\delta B_{ab} - \frac{1}{2}\left(\lambda^{cd} + \frac{1}{2}\epsilon^{abcd}B_{ab}\right)\delta\omega_{cd}, \tag{6.219}$$

and hence (6.211). In addition we get, after introducing the two dimensional matter space and the associated Lagrangian displacement,

$$\omega_{de}\nabla_c\left(\lambda^{cd} + \frac{1}{2}\epsilon^{abcd}B_{ab}\right) = \omega_{de}\left(\nabla_c\lambda^{cd} - n^d\right) = 0. \tag{6.220}$$

The question is: What is this description good for? Suppose we consider an explicit model where $\Lambda = \Lambda(n^2, \mathcal{N}^2)$. Then we need

$$\mathcal{N}^2 = \frac{1}{2}\omega^{ab}\omega_{ab} = \frac{1}{2}g^{ac}g^{bd}\omega_{ab}\omega_{cd}, \tag{6.221}$$

which means that

$$\frac{\partial \Lambda}{\partial \omega_{ab}} = \frac{\partial \Lambda}{\partial \mathcal{N}^2} \omega^{ab} = -\frac{1}{2} \lambda^{ab}, \tag{6.222}$$

and we arrive at

$$n^a \omega_{ab} = \omega_{ab} \nabla_c \lambda^{ca} = 2 \omega_{ab} \nabla_c \left(\frac{\partial \Lambda}{\partial \mathcal{N}^2} \omega^{ac} \right). \tag{6.223}$$

This brings us back to (6.199) from the previous section. This derivation makes use of the two dimensional matter space, similar to the elasticity formulation we created previously and also only requires the variation of ω_{ab}.

In Chapter 5 we produced the following equation for a multifluid system in which all of the components were travelling together

$$u^a_{\rm v} \omega_{ab} = 0. \tag{6.224}$$

Introducing a velocity difference between the vortices and the neutron fluid gave us

$$u^a_{\rm v} = \tilde{\gamma} \left(u^a + v^a \right), \tag{6.225}$$

which in turn produced the Euler equation for the fluid with the Magnus force

$$u^a \omega_{ab} = -v^a \omega_{ab} \equiv f^{\rm M}_b. \tag{6.226}$$

So, comparing this result to (6.223) above, we can see that

$$n^a \omega_{ab} = -n v^a \omega_{ab} = \omega_{ab} \nabla_c \lambda^{ca} = 2 \omega_{ab} \nabla_c \left(\frac{\partial \Lambda}{\partial \mathcal{N}^2} \omega^{ac} \right). \tag{6.227}$$

This enables us to find v^a and hence the vortex velocity $u^a_{\rm v}$. The vortex velocity is given by

$$u^a_{\rm v} = \tilde{\gamma} u^a - \frac{\tilde{\gamma}}{n} \nabla_c \lambda^{ca} = \tilde{\gamma} \left[u^a + \frac{2}{n} \nabla_c \left(\frac{\partial \Lambda}{\partial \mathcal{N}^2} \omega^{ca} \right) \right]. \tag{6.228}$$

From here we can use the results from the mutual friction calculation in Chapter 5 in order to add the vortex mutual friction into this derivation.

Final Remarks

7

Final Remarks

The aim of this thesis was to create a superfluid model in general relativity, closely following that of Newtonian calculations. Hence, we aimed to include quantised vortices and their properties in such a system, the Magnus force, mutual friction and elasticity. A rotating superfluid can be modelled in Newtonian physics using a quantised vorticity vector as we saw in Chapter 3. This vector contains the information required to propel the model from a normal fluid to a superfluid model. Another key feature of superfluids is that they can flow relative to other components due to having zero viscosity. After the integration of the vortices into the system and finding an equation of motion for the superfluid, the introduction of velocity differences gave rise to the Magnus and mutual friction forces. The collection of the equations for the superfluid, the normal fluid and the force balance equation for the vortices tells us how the system behaves. This is the system we wished to produce in general relativity.

To create our model we first contemplated current research in fluids, superfluids and relativity, giving examples of how these are modelled. We then began our quest in the same manner that the Newtonian models are formed. We created a single fluid system using a quantised vorticity vector and produced equations of motion for the vortices as well as the vorticity equation. We compared our single superfluid system to a two form calculation produced by Carter, as well as taking the Newtonian limit to show that our system matches the Newtonian case.

We then continued in Chapter 5, following the Newtonian calculation again, by introducing a velocity difference indicating that the vortices and superfluid do not travel together. This assumption introduced the Magnus force to our system, as one should expect to happen. Further to this, we introduced velocity differences indicating the mutual friction between the vortices and the second fluid. Once more we followed the Newtonian calculation and rearranged the equations to eliminate the vortex velocity. This produced the mutual friction force in a familiar form to that of the Newtonian models. At this point we had created a relativistic multifluid model including quantised vortices and mutual friction.

Introducing elasticity to the system involved an investigation into how elastic stars are modelled, in Chapter 6. From this we attempted to create a three dimensional elasticity term and split it into a vortex tension piece and a vortex elasticity term. In doing this we were left with cross terms which do not fall under the category of vortex elasticity nor vortex tension. In the Newtonian calculation these are easily ignored but in relativity they do not vanish. Hence we decided to focus on creating only the vortex elasticity, by using a two dimensional version of the matter space argument. We achieved this by specifying that the vortex array direction should be aligned with one of the matter space directions. Using this we could use the remaining two dimensional plane to create a two dimensional anisotropic term, from which we found the vortex elasticity. The Newtonian limit of this term matched the general form of elasticity used in the Newtonian framework. From this formulation we have a relativistic Euler equation with elasticity. Combining this with our Magnus force and mutual friction model gives the multifluid system including quantised vortices, mutual friction and elasticity in general relativity.

Finally, we reconsidered the variational approach in which we let the Lagrangian depend on the matter space metric. We discovered in earlier derivations that if one is to consider vortex elasticity, then the matter space metric is a vital component to include. We showed what one obtains when considering variations with respect to this metric. We then considered variations with respect to the vorticity tensor and finally the vorticity tensor along with the number density. These two variations produced the same equation with a term on the right hand side. The next task would be to understand the outcome of these variations and find the link between them and the derivation of the matter space elasticity.

Our model has been extended to the level at which it can be used in applications that require detailed microphysics. Applications such as the long term variability associated with Tkachenko waves, the mechanism responsible for causing pulsar glitches and also the recovery of pulsars after they experience a glitch.

Appendix

A

Additional calculations

We note here the calculations which show the equations and identities mentioned and used in Chapter 6.

$h_{ab}\Delta g^{ab}$

$$\Delta g^{ab} = \Delta(g_{cd}g^{ac}g^{bd}), \tag{A.1}$$

$$\Delta g^{ab} = g^{ac}g^{bd}\Delta g_{cd} + g_{cd}g^{bd}\Delta g^{ac} + g_{cd}g^{ac}\Delta g^{bd}, \tag{A.2}$$

$$h_{ab}\Delta g^{ab} = h^{cd}\Delta g_{cd} + h_{ac}\Delta g^{ac} + h_{bd}\Delta g^{bd}, \tag{A.3}$$

$$\Rightarrow h_{ab}\Delta g^{ab} = -h^{ab}\Delta g_{ab} \tag{A.4}$$

$h_{ac}h_{bd}\Delta g^{cd}$

$$\Delta g^{cd} = \Delta(g_{ef}g^{ce}g^{df}), \tag{A.5}$$

$$\Delta g^{cd} = g^{ce}g^{df}\Delta g_{ef} + g_{ef}g^{df}\Delta g^{ce} + g_{ef}g^{ce}\Delta g^{df}, \tag{A.6}$$

$$h_{ac}h_{bd}\Delta g^{cd} = h_a{}^e h_b{}^f \Delta g_{ef} + h_{ac}h_{be}\Delta g^{ce} + h_{af}h_{bd}\Delta g^{df}, \tag{A.7}$$

$$\Rightarrow h_{ac}h_{bd}\Delta g^{cd} = -h_a{}^c h_b{}^d \Delta g_{cd} \tag{A.8}$$

Δu_a

$$\Delta u^a = \frac{1}{2}u^a u^b u^c \Delta g_{bc} \quad \text{(from [61])}, \tag{A.9}$$

$$\Delta u_a = \Delta(g_{ab}u^b), \tag{A.10}$$

$$\Delta u_a = u^b \Delta g_{ab} + g_{ab}\Delta u^b, \tag{A.11}$$

$$\Rightarrow \Delta u_a = u^b \Delta g_{ab} + \frac{1}{2}u_a u^c u^d \Delta g_{cd} \tag{A.12}$$

$\Delta \eta_{ab}$

$$\Delta\eta_{ab} = \Delta(h_a{}^c h_b{}^d \eta_{cd}) \quad (\text{as } u^a\eta_{ab} = u^b\eta_{ab} = 0), \tag{A.13}$$

$$\Delta\eta_{ab} = h_a{}^c h_b{}^d \Delta\eta_{cd} + h_a^c \eta_{cd}\Delta h_b^d + h_b^d \eta_{cd}\Delta h_a^c, \tag{A.14}$$

$$\Delta\eta_{ab} = h_a{}^c h_b{}^d \Delta\eta_{cd} + h_a^c h_{cd}\Delta(\delta_b^d + u_b u^d) + h_b^d h_{cd}\Delta(\delta_a^c + u_a u^c), \tag{A.15}$$

$$\Delta\eta_{ab} = h_a{}^c h_b{}^d \Delta\eta_{cd} + h_{ad}\Delta(u_b u^d) + h_{bc}\Delta(u_a u^c), \tag{A.16}$$

$$\Delta\eta_{ab} = h_a{}^c h_b{}^d \Delta\eta_{cd} + h_{ad}u_b\Delta u^d + h_{bc}u_a\Delta u^c, \tag{A.17}$$

$$\Delta\eta_{ab} = h_a{}^c h_b{}^d \Delta\eta_{cd} + h_{ad}u_b\left(\frac{1}{2}u^d u^e u^f \Delta g_{ef}\right) + h_{bc}u_a\left(\frac{1}{2}u^c u^e u^f \Delta g_{ef}\right), \tag{A.18}$$

$$\Rightarrow \Delta\eta_{ab} = h_a{}^c h_b{}^d \Delta\eta_{cd} \tag{A.19}$$

$\Delta\eta_{ab}$ when $\Delta k_{ab} = 0$

$$\Delta\eta_{ab} = n^{-2/3}\Delta k_{ab} + k_{ab}\Delta n^{-2/3}, \tag{A.20}$$

$$\Delta\eta_{ab} = n^{2/3}\eta_{ab}\left(-\frac{2}{3}n^{-5/3}\Delta n\right), \tag{A.21}$$

$$\Delta\eta_{ab} = -\frac{2}{3n}h_{ab}\left(-\frac{n}{2}h^{cd}\Delta g_{cd}\right), \tag{A.22}$$

$$\Delta\eta_{ab} = \frac{1}{3}h_{ab}h^{cd}\Delta g_{cd}, \tag{A.23}$$

Newtonian limit of (6.90)

$$\Delta\pi_{ab} = -2n\left(f_1 + 4f_2\right)\left(h_a{}^c h_b{}^d\Delta g_{cd} - \frac{1}{3}h_{ab}h^{cd}\Delta g_{cd}\right), \tag{A.24}$$

$$\begin{aligned}\Delta\pi_{ab} =& -2n\left(f_1 + 4f_2\right)\Bigg(\left(\delta_a{}^c + u_a u^c\right)\left(\delta_b{}^d + u_b u^d\right)\Delta g_{cd} \\ & - \frac{1}{3}\left(g_{ab} + u_a u_b\right)\left(g^{cd} + u^c u^d\right)\Delta g_{cd}\Bigg),\end{aligned} \tag{A.25}$$

$$\begin{aligned}\Delta\pi_{ab} =& -2n\left(f_1 + 4f_2\right)\Bigg(\left(\delta_a{}^c\delta_b{}^d + u_a u^c\delta_b{}^d + \delta_a{}^c u_b u^d + u_a u^c u_b u^d\right)\Delta g_{cd} \\ & - \frac{1}{3}\left(g_{ab}g^{cd} + u_a u_b g^{cd} + g_{ab}u^c u^d + u_a u_b u^c u^d\right)\Delta g_{cd}\Bigg),\end{aligned} \tag{A.26}$$

from the definition of $u^a = (1 - \Phi/c^2 + v^2/2c^2, v^i/c)$ and remembering that coordinate $x^0 = ct$, we simplify this by taking out $\mathcal{O}(1/c)$ terms. We also consider the spatial part of the equation after taking the divergence, which gives

$$\nabla^j\Delta\pi_{ij} = -2n\left(f_1 + 4f_2\right)\nabla^j\left(\nabla_i\xi_j + \nabla_j\xi_i - \frac{2}{3}g_{ij}\nabla_c\xi^c\right), \tag{A.27}$$

$$\nabla^j\Delta\pi_{ij} = -2n\left(f_1 + 4f_2\right)\left(\frac{1}{3}\nabla_i\nabla^j\xi_j + \nabla^2\xi_i\right). \tag{A.28}$$

Derivative $\frac{\partial N}{\partial \bar{g}^{AB}}$

$$N^2 = -N^a N_a = -\frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\hat{\kappa}^e\frac{1}{2}\epsilon_{afgh}\hat{\kappa}^h N^{fgm}\hat{\kappa}_m, \tag{A.29}$$

$$N^2 = -\frac{1}{4}\hat{\kappa}_d\hat{\kappa}^h N_{bce}\hat{\kappa}^e N^{fgm}\hat{\kappa}_m(\epsilon^{abcd}\epsilon_{afgh}), \tag{A.30}$$

$$N^2 = -\frac{1}{4}\hat{\kappa}_d\hat{\kappa}^h N_{bce}\hat{\kappa}^e N^{fgm}\hat{\kappa}_m(-[\delta^b_f(\delta^c_g\delta^d_h - 0) + \delta^b_g(0 - \delta^c_f\delta^d_h) + 0(0-0)]). \tag{A.31}$$

The 0s come from simplifying the contractions of $\hat{\kappa}^a$ with N_{abc}, as they will vanish. Hence

$$N^2 = \frac{1}{4}[\delta^b_f\delta^c_g\delta^d_h\hat{\kappa}_d\hat{\kappa}^h N_{bce}\hat{\kappa}^e N^{fgm}\hat{\kappa}_m - \delta^b_g\delta^c_f\delta^d_h\hat{\kappa}_d\hat{\kappa}^h N_{bce}\hat{\kappa}^e N^{fgm}\hat{\kappa}_m], \tag{A.32}$$

$$N^2 = \frac{1}{4}[\hat{\kappa}_d\hat{\kappa}^d N_{bce}\hat{\kappa}^e N^{bcm}\hat{\kappa}_m - \hat{\kappa}_d\hat{\kappa}^d N_{bce}\hat{\kappa}^e N^{cbm}\hat{\kappa}_m], \tag{A.33}$$

$$N^2 = \frac{1}{4}[2N_{bce}\hat{\kappa}^e N^{bcm}\hat{\kappa}_m], \tag{A.34}$$

$$N^2 = \frac{1}{2}N_{bce}N^{bcm}\hat{\kappa}^e\hat{\kappa}_m = \frac{1}{2}N_{bc}N^{bc}. \tag{A.35}$$

Then we take the derivative

$$N^2 = \frac{1}{2}N_{bce}N^{bcm}\hat{\kappa}^e\hat{\kappa}_m = \frac{1}{2}N_{BCE}N_{DFM}\bar{g}^{BD}\bar{g}^{CF}\hat{\kappa}^E\hat{\kappa}^M, \tag{A.36}$$

$$2N\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{2}N^2\epsilon_{BCE}\epsilon_{DFM}\hat{\kappa}^E\hat{\kappa}^M\frac{\partial}{\partial \bar{g}^{PQ}}(\bar{g}^{BD}\bar{g}^{CF}), \tag{A.37}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{4}N\epsilon_{BCE}\epsilon_{DFM}\hat{\kappa}^E\hat{\kappa}^M(\bar{g}^{BD}\frac{\partial}{\partial \bar{g}^{PQ}}\bar{g}^{CF} + \bar{g}^{CF}\frac{\partial}{\partial \bar{g}^{PQ}}\bar{g}^{BD}), \tag{A.38}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{4}N\epsilon_{BCE}\epsilon_{DFM}\hat{\kappa}^E\hat{\kappa}^M(2\bar{g}^{BD}\frac{\partial}{\partial \bar{g}^{PQ}}\bar{g}^{CF}), \tag{A.39}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{2}N\epsilon_{BCE}\epsilon_{DFM}\hat{\kappa}^E\hat{\kappa}^M\bar{g}^{BD}(\frac{1}{2}(\bar{\delta}^C_P\bar{\delta}^F_Q + \bar{\delta}^F_P\bar{\delta}^C_Q)), \tag{A.40}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{4}N(\epsilon_{BPE}\epsilon_{DQM}\hat{\kappa}^E\hat{\kappa}^M\bar{g}^{BD} + \epsilon_{BQE}\epsilon_{DPM}\hat{\kappa}^E\hat{\kappa}^M\bar{g}^{BD}), \tag{A.41}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{4}N2\epsilon_{BPE}\epsilon_{DQM}\hat{\kappa}^E\hat{\kappa}^M\bar{g}^{BD}, \tag{A.42}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{2}N\epsilon_{BPE}\epsilon^{BGM}\bar{g}_{GQ}\hat{\kappa}^E\hat{\kappa}_M, \tag{A.43}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{2}N(\delta^G_P\delta^M_E - \delta^M_P\delta^G_E)\bar{g}_{GQ}\hat{\kappa}^E\hat{\kappa}_M, \tag{A.44}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{2}N(\bar{g}_{PQ}\hat{\kappa}^E\hat{\kappa}_E - \bar{g}_{EQ}\hat{\kappa}^E\hat{\kappa}_P), \tag{A.45}$$

$$\frac{\partial N}{\partial \bar{g}^{PQ}} = \frac{1}{2}N\bar{g}_{PQ}. \tag{A.46}$$

$$\bar{\eta}_{ab} = \bar{k}^{-1/2}\bar{k}_{ab}$$

Relating the two with a constant C,

$$\bar{\eta}_{ab} = C\bar{k}_{ab}, \tag{A.47}$$

$$\det(\bar{\eta}_{ab}) = 1 = \det(C\bar{k}_{ab}), \tag{A.48}$$

$$1 = C^2 \det(\bar{k}_{ab}), \tag{A.49}$$

$$C^2 = \frac{1}{\det(\bar{k}_{ab})}, \tag{A.50}$$

$$C = \frac{1}{\bar{k}^{1/2}}. \tag{A.51}$$

Hence

$$\bar{\eta}_{ab} = \bar{k}^{-1/2}\bar{k}_{ab}. \tag{A.52}$$

New N^a

Start with the three index number density and split into the 3rd direction

$$N_{bcd} = \psi_b^B \psi_c^C \psi_d^D N_{BCD} = \psi_b^B \psi_c^C \psi_d^3 N_{BC3} + \psi_b^B \psi_c^C \bar{\psi}_d^D N_{BCD}, \tag{A.53}$$

$$N_{bcd} = \bar{\psi}_b^B \bar{\psi}_c^C \psi_d^3 N_{BC3} + \psi_b^B \psi_c^3 \bar{\psi}_d^D N_{B3D} + \psi_b^B \bar{\psi}_c^C \bar{\psi}_d^D N_{BCD}, \tag{A.54}$$

$$N_{bcd} = \bar{\psi}_b^B \bar{\psi}_c^C \psi_d^3 N_{BC3} + \bar{\psi}_b^B \psi_c^3 \bar{\psi}_d^D N_{B3D} + \psi_b^3 \bar{\psi}_c^C \bar{\psi}_d^D N_{3CD}, \tag{A.55}$$

$$N_{bcd} = \left(\bar{\psi}_b^B \bar{\psi}_c^C \psi_d^3 - \bar{\psi}_b^B \psi_c^3 \bar{\psi}_d^C + \psi_b^3 \bar{\psi}_c^B \bar{\psi}_d^C\right) N_{BC3}. \tag{A.56}$$

From the definition of N^a

$$N^a = \frac{1}{3!}\epsilon^{abcd} N_{bcd} = \frac{1}{3!}\left(\epsilon^{abcd}\bar{\psi}_b^B \bar{\psi}_c^C \psi_d^3 - \epsilon^{abcd}\bar{\psi}_b^B \psi_c^3 \bar{\psi}_d^C + \epsilon^{abcd}\psi_b^3 \bar{\psi}_c^B \bar{\psi}_d^C\right) N_{BC3}, \tag{A.57}$$

$$N^a = \frac{1}{2}\epsilon^{abcd}\bar{\psi}_b^B \bar{\psi}_c^C \psi_d^3 N_{BC3}, \tag{A.58}$$

$$N^a = \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\hat{\kappa}^e = \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d \bar{N}_{bc}, \tag{A.59}$$

as $\psi_a^3 = \hat{\kappa}_a$ and say that $\bar{N}_{ab} = N_{abc}\hat{\kappa}^c$.

N Conservation

We have the definition

$$N^a = Nu^a = \frac{1}{3!}\epsilon^{abcd}N_{bcd}, \tag{A.60}$$

which obeys the conservation equation

$$\nabla_a(Nu^a) = 0. \tag{A.61}$$

Next we rewrite N^a, as in Chapter 6, in the form

$$N^a = \frac{1}{3!}\epsilon^{abcd}N_{bcd} = \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\hat{\kappa}^e = \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_b \bar{N}_{cd}, \tag{A.62}$$

by specifying that $\hat{\kappa}_a = \psi_a^A \delta_A^3$. From this we see that

$$\nabla_a(\frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\hat{\kappa}^e) = 0, \tag{A.63}$$

$$\frac{1}{2}\epsilon^{abcd}\nabla_a(\hat{\kappa}_d N_{bce}\hat{\kappa}^e) = 0, \tag{A.64}$$

$$\frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d\hat{\kappa}^e\nabla_a N_{bce} + \frac{1}{2}\epsilon^{abcd}N_{bce}\hat{\kappa}^e\nabla_a\hat{\kappa}_d + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0, \tag{A.65}$$

$$\frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d\hat{\kappa}^e\nabla_a N_{bce} + \frac{1}{2}\epsilon^{abcd}N_{bce}\hat{\kappa}^e\delta_D^3\nabla_a\psi_d^D + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0. \tag{A.66}$$

The partial derivatives on the second term commute and vanish, giving

$$\frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d\hat{\kappa}^e\nabla_a N_{bce} + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0. \tag{A.67}$$

We have that

$$N^a = \frac{1}{3!}\epsilon^{abcd}N_{bcd} \iff N_{abc} = \epsilon_{abcd}N^d \tag{A.68}$$

Which we plug into first term of the equation so see

$$\frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d\hat{\kappa}^e\nabla_a(\epsilon_{bcef}N^f) + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0, \tag{A.69}$$

$$\frac{1}{2}\epsilon_{bcef}\epsilon^{bcad}\hat{\kappa}_d\hat{\kappa}^e\nabla_a N^f + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0, \tag{A.70}$$

$$-\frac{1}{2}2!(\delta^a_e\delta^d_f - \delta^d_e\delta^a_f)\hat{\kappa}_d\hat{\kappa}^e\nabla_a N^f + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0, \tag{A.71}$$

$$-\hat{\kappa}_f\hat{\kappa}^a\nabla_a N^f + \hat{\kappa}_e\hat{\kappa}^e\nabla_a N^a + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0, \tag{A.72}$$

$$-\nabla^{\parallel}_f N^f + \nabla_a N^a + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0, \tag{A.73}$$

as $\hat{\kappa}^a\hat{\kappa}_a = 1$ and $\hat{\kappa}_a\hat{\kappa}^b\nabla_b = \nabla^{\parallel}_a$. Also, as $\nabla_a = \nabla^{\perp}_a + \nabla^{\parallel}_a$ we see

$$\nabla^{\perp}_f N^f + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d N_{bce}\nabla_a\hat{\kappa}^e = 0. \tag{A.74}$$

Using the same definition of N_{abc} on second term implies that

$$\nabla^{\perp}_f N^f + \frac{1}{2}\epsilon^{abcd}\hat{\kappa}_d\epsilon_{bcef}N^f\nabla_a\hat{\kappa}^e = 0, \tag{A.75}$$

hence

$$\nabla^{\perp}_f N^f - \hat{\kappa}_f N^f\nabla_a\hat{\kappa}^a + \hat{\kappa}_d N^a\nabla_a\hat{\kappa}^d = 0. \tag{A.76}$$

As $\hat{\kappa}_a u^a = 0$ and $\nabla_a(\hat{\kappa}_b\hat{\kappa}^b) = 0$ we get

$$\nabla^{\perp}_f N^f = 0. \tag{A.77}$$

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