A Modeling Framework for Solving Restricted Planar Location Problems Using Phi-Objects

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Abstract

This paper presents a general modeling framework for restricted facility location problems with arbitrarily shaped forbidden regions or barriers, where regions are modeled using phi-objects. Phi-objects are an efficient tool in mathematical modeling of 2D and 3D geometric optimization problems, and are widely used in cutting and packing problems and covering problems. The paper shows that the proposed modeling framework can be applied to both median and center facility location problems, either with barriers or forbidden regions. The resulting models are either mixed-integer linear or nonlinear programming formulations, depending on the shape of the restricted region and the considered distance measure. Using the new framework, all instances from the existing literature for this class of problems are solved to optimality. The paper also introduces and optimally solves a realistic multi-facility problem instance derived from an archipelago vulnerable to earthquakes. This problem instance is significantly more complex than any other instance described in the literature.

Keywords. mathematical modeling, facility location, phi-objects

1 Introduction

Facility location problems are concerned with the optimal placement of facilities in order to minimize a general cost function including that of distance and demand from customer sites. If the underlying space for both the potential facility sites and the customer sites is continuous, such problems are termed continuous facility location problems. These problems are different from their discrete counterparts, where facilities can only be placed on a finite set of candidate sites. Dasci and Verter (2001) state that discrete location models may provide optimal solutions, but when the model becomes more realistic, data and the computational requirements increase significantly, which is likely to result in a decrease in model accuracy.

The most commonly studied continuous facility location problems are the minisum and the minimax problems (Drezner, 1995). The former corresponds to minimizing the total weighted distance from customer sites, while the latter is concerned with minimizing the maximum weighted distance. Minisum problems with p facilities are often referred to as p-median problems; similarly minimax problems with p facilities are called p-center problems. When real life instances are studied, however, these problems might fail to model the problem in a realistic manner, since there can be geographical restrictions on the location of the facilities or on the paths to the facilities. For instance, a huge mountain range can hinder facility location and transportation. These mountain ranges can be seen as a restriction on location and transportation between facilities.

The restriction on locations presents a challenge for such class of problems. According to Canbolat and Wesolowsky (2010), it is possible to classify the restriction types into three categories, namely forbidden regions, barriers and congested regions. The placement of new facilities within a forbidden region is not allowed, but traveling through it is permitted. For barriers, neither placement of facilities within or travel through is permitted. The placement of new facilities is also not permitted in congested regions, but traveling within may be allowed in return for a penalty. In this paper, we will focus on two planar facility location problem restrictions; one with forbidden regions, another with barriers. There are various applications of such types of problems. Assembly of printed circuit boards (Foulds and Hamacher 1994), obnoxious facility planning (Carrizosa and Plastria 1993) and location of emergency facilities are some examples for problems with forbidden regions (Hamacher and Nickel, 1995), whereas urban applications considering lakes, parks, cemeteries and rivers can be listed as examples for problems with barriers.

The problem at hand is to find optimal locations of new facilities on a continuous plane, considering the restrictions mentioned above. These restricted regions on the plane make the feasible region non-convex and arbitrarily shaped. To define a feasible region for this problem type, we use phi-objects, which are an efficient tool in mathematical modeling for 2D and 3D geometric optimization problems. The concept of phi-objects is mainly associated with cutting and packing problems and covering problems. It lends itself to the problems addressed in this paper as they have the capability to model any arbitrary shaped region, including regions that are bounded by arcs, and as a result, naturally form a mathematical model. While they have the advantage of efficiently modeling shapes to a high level of fidelity, they can also be used where a lower level of fidelity is acceptable and take advantage of the computational efficiency offered by approximation of the shape. A model that has general applicability at any chosen level of fidelity is new to the literature. On top of the capability of modeling any arbitrary shaped regions, using phi-objects enables us to model different types of restrictions within a single modeling framework, which would allow formulating and solving real-life instances featuring various types of restrictions and distance metrics. In this paper, we will show that it is possible to adapt this concept to continuous facility location problems with restricted regions and model arbitrarily shaped regions by using phi-objects. The main idea is to use phi-objects to create a general model for this class of problems, which can then be used to solve them to optimality. In this modeling approach, phi-objects can represent either restricted or feasible regions on the plane. We show that the resulting formulations are either mixed-integer linear programs (MILP) or mixed-integer nonlinear programs (MINLP), depending on the shape of the restricted region and the considered distance measure.

In addition to the models, we also present an application of a cutting plane technique to solve models with a significant number of barriers. This technique is used to relax models by decreasing the number of constraints significantly.

The contributions of this paper are listed below:

- A new modeling approach for continuous facility location problems with arbitrarily shaped forbidden regions or barriers is presented by using the concept of phi-objects. This concept enables us to deal with arcs as well as planar edges defining the region. It is shown that problems with different restriction types can be solved within a single modeling framework. The flexibility provided by phi-objects increases the variety of geometric objects that can be considered as restricted regions. Thus, real geographical shapes found in nature can be modeled with negligible approximations.

- It is shown that the proposed general modeling framework can be adapted to multi-facility location problems with forbidden regions and to single facility location problems with barriers with various objective functions.
- We present computational results showing that all instances for this class of problems previously described in the literature can be solved to optimality by using the proposed approach.

The remainder of this paper is organized as follows. In Section 2, we review the relevant literature on continuous location problems with a particular focus on forbidden regions and barriers. Section 3 will present background information on phi-objects and illustrate the application of this concept to restricted continuous facility location problems with forbidden regions or barriers. We introduce the general modeling framework in Section 4. In Section 5, we test our approach by solving instances from the literature, as well as a new instance derived from a real geographical setting in Section 6. Conclusions are given in Section 7.

2 Literature Review

Restricted planar location problems can be classified into three categories, namely those with (i) forbidden regions, (ii) barriers and (iii) congested regions. In this paper, we focus on the restricted planar problems with forbidden regions and barriers. For discussion of the congested region problem, see Butt and Cavalier (1997) and Sarkar et al. (2004).

2.1 Restricted Problems with Forbidden Regions

To our knowledge, the first study on this problems is by Aneja and Parlar (1994), who describe algorithms for the single facility location problem with forbidden regions and barriers separately, with a metric defined on $\mathbb{R}^n$ and with respect to a parameter $1 \leq p \leq \infty$, namely the l_p metric, which has a general form represented as $[|x - x_i|^p + |y - y_i|^p]^{1/p}$. For $p = 1$, the l_p metric is known as the rectilinear distance. Similarly, l_2 is known as the Euclidean distance and l_∞ is the Chebyshev distance. The algorithms proposed by Aneja and Parlar (1994) are applicable to cases where $1 < p \leq 2$. For the forbidden region problem, they present algorithms for polygonal convex and non-convex forbidden regions. The algorithms are based on the premise that if an optimal solution of a problem solved by ignoring the forbidden region is located outside the forbidden region, then this solution would also be the optimal solution for the original problem with the forbidden region. If not, then they use an algorithm that assumes that the optimal solution would be on the boundary of the forbidden region.

Hamacher and Nickel (1994) introduce several algorithms for the 1-median problem on a plane using distance metrics including Manhattan, squared Euclidean and Chebyshev. Their assumption is that the considered forbidden region is a union of pairwise disjoint convex sets. They present some basic examples to test their algorithms. Hamacher and Nickel (1995) extend their previous work to multi-facility median and center problems, where the forbidden region is again a union of pairwise disjoint convex sets. They present a heuristic algorithm which consists of a sequential solution of p single facility problems for the p-median problem with a forbidden region, and an efficient solution algorithm based on level sets and lines for the 1-center problem with Manhattan and Chebyshev distance metrics.

Hamacher and Schöbel (1997) study a center problem with a polyhedral convex forbidden region. A polynomial solution algorithm is proposed, using the Euclidean distance metric. With few modifications, the algorithm can be used for a problem with a polyhedral non-convex forbidden region. Later, Woeginger (1998) shows that this problem can be solved with a better time complexity by applying standard computational geometry techniques.

2.2 Restricted Problems with Barriers

Restricted planar location problems with barriers, to the best of our knowledge, are first presented by Katz and Cooper (1981), where the barrier is a single circular region and the distance metric is Euclidean. In this paper, an algorithm is described to compute the shortest distance following, which the modified objective function with the shortest distance is converted into a sequence of unconstrained minimization problems. Klamroth (2004) presents new structural results for the problem described by Katz and Cooper (1981). Larson and Sadiq (1983) solve a p-median problem in a case of rectilinear distance. A cell formation technique is used to reduce the p-median problem to a discrete search problem. The authors state that any of the existing algorithms available for the p-median problem can be applied to solve their problem.

Aneja and Parlar (1994) use the concept of visibility and the Dijkstra algorithm to compute the shortest distance between customer sites and the location of the new facility, using polygons as barriers. Butt and Cavalier (1996) consider the restricted 1-median problem with convex polygonal forbidden regions. They use the Euclidean distance metric and describe an iterative solution procedure.

Hamacher and Klamroth (2000) consider the distances defined by polyhedral norms in the restricted single facility median problem, with the barriers being convex polyhedral subsets. They use a grid construction method to prove a discretization result, which implies a polynomial algorithm to solve location problems with barriers and block norms. Klamroth (2001) presents an exact algorithm and a heuristic solution procedure for the single facility problem with polyhedral barriers based on reducing this non-convex optimization problem to a finite set of convex subproblems. Dearing et al. (2002) present a single facility center problem with polygonal barriers and considered rectilinear distance as a distance metric. A polynomial algorithm is developed to solve the problem to optimality. The same authors later adapt this modification model to a problem considering block distance (Dearing et al., 2005).

Dearing and Segars Jr. (2002a) develop a modification technique for barriers, and prove that the objective function values for both the original problem and the modified problem are the same. This modification technique is based on properties of rectilinear distances. In a companion paper of this work, the authors present a solution algorithm based on partitioning the feasible region into convex subsets by using the mentioned modification technique (Dearing and Segars Jr., 2002b).

McGarvey and Cavalier (2003) develop an iterative solution procedure based on a modified "Big Square Small Square" branch-and-bound method to solve 1-facility median problem with convex polygonal barriers, where the distance metric is Euclidean. Bischoff and Klamroth (2007) reduce the original problem to a finite series of convex subproblems to solve them by using Weiszfeld algorithm to find a heuristic solution in the case of Euclidean distances. The visibility arguments are also taken into consideration to decrease the number of convex subproblems.

Recent research on restricted facility location problems have considered stochastic features, for example, Canbolat and Wesolowsky (2010), Amiri-Aref et al. (2011), Shiripour et al. (2012), Amiri-Aref et al. (2013) and Javadian et al. (2014), where the location of the barriers are uncertain.

2.3 Restricted Problems with Forbidden Regions and Barriers

To the best of our knowledge, the only paper that considers a facility planar location problem with both forbidden regions and barriers is that by Batta et al. (1989). It is a p-median problem in the presence of arbitrarily shaped barriers and convex forbidden regions, with a Manhattan (rectilinear) distance metric. To solve the problem, they extend a cell formation technique introduced by Larson and Sadiq (1983).

2.4 Discussion

As the preceding review shows, it is possible to categorize restricted location problems based on the types of restriction. While each problem has so far been individually addressed, there is a lack of a general approach that is able to collectively address all problems in each category above. Table 1 details the types of problems addressed in the literature and the applicability of the published approaches to these problems.

Table 1: Restricted location problems literature overview

	Restriction Type		Distance Metric			Objective		Restricted Region Shape		
	Forbidden Region	Barrier	Rectilinear	Euclidean	Chebyshev	Median	Center	Polygon	Circular	Arbitrary
Katz and Cooper (1981)		✓		✓		✓			✓	
Larson and Sadiq (1983)		✓	✓			✓				✓
Aneja and Parlar (1994)	✓	✓		✓		✓		✓		
Hamacher and Nickel (1994)	✓		✓	✓	✓	✓		✓		
Hamacher and Nickel (1995)	✓		✓	✓	✓	✓	✓	✓		
Butt and Cavalier (1996)		✓		✓		✓		✓		
Hamacher and Schöbel (1997)	✓			✓			✓	✓		
Woeginger (1998)	✓			✓			✓	✓		
Klamroth (2001)		✓	✓	✓	✓	✓	✓	✓		
Dearing et al. (2002)		✓	✓				✓	✓		
Dearing and Segars Jr. (2002a)		✓	✓			✓	✓			✓
Dearing and Segars Jr. (2002b)		✓	✓			✓	✓			✓
McGarvey and Cavalier (2003)		✓		✓		✓		✓		
Klamroth (2004)		✓		✓		✓			✓	
Bischoff and Klamroth (2007)		✓		✓		✓		✓		
This paper	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

As Table 1 shows, previous studies limit the shape of the restricted regions to a polygon, even though these shapes can vary substantially in real-world instances. Furthermore, previous studies seem to focus only on specific objectives (e.g. median problem, center problem). A general approach that can deal with various objectives is needed. To address the need for such an approach, we present a modeling framework to formulate problems with various restriction categories, objectives and distance metrics, and arbitrarily shaped restricted regions, such that the models can efficiently be solved to optimality using existing software. Before doing so, we first describe how regions are represented using phi-objects in the next section.

3 Phi-objects for Modeling Forbidden Regions

This section describes the concept of phi-objects. Phi-objects are most commonly found in the cutting and packing literature. They are used to model real objects mathematically. These geometric objects are canonically closed sets of points. Point sets that contain isolated points, points that are removed from an object (deleted points) and objects with self-intersection of their frontiers are not classified as phi-objects. Note that in the cutting and packing literature, the concept extends to generating the convolution of objects, while here we simply use the objects in their original form.

The example presented in Figure 1a is a phi-object, formed by a closed set of points. The example shown in Figure 1b is not a phi-object, since it self-intersects with its frontier. Similarly, the example in Figure 1c contains deleted points, and is not classified as a phi-object. Readers are referred to Bennell et al. (2010) for further details on phi-objects. In this paper, phi-objects are used as geometrical tools to model complex geographical formations such as lakes, archipelagos, bays which often give rise to forbidden regions or barriers.

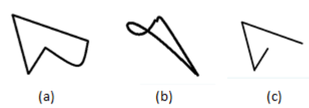

Figure 1: Examples of arbitrarily shaped 2D objects

To model a complex shaped geometric object, it is convenient to divide phi-objects into three categories, namely composed, basic and primitive objects (Bennell et al., 2015). A half-plane, a circle and the complement of a circle (circular hole) are types of primitive objects, as shown in Figure 2. The representation of composed objects is through unions of a finite number of basic objects, whereas a basic object is obtained by intersections of a finite number of primitive objects. Figure 3 shows four examples of basic objects.

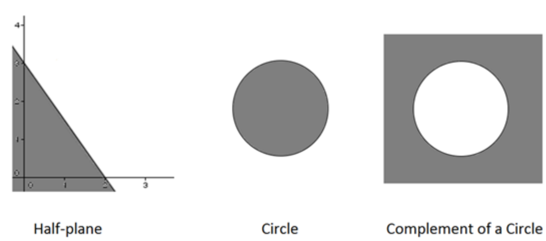

Figure 2: Primitive object types

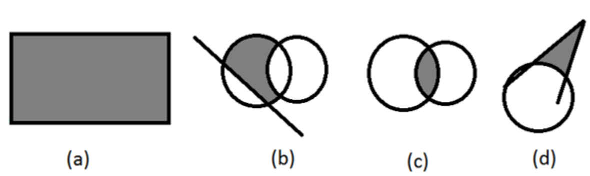

Figure 3: Examples of basic objects

In Figure 3a, the basic object is obtained by the intersection of four half-planes. The non-convex object shown in Figure 3b is the intersection of a circle, the complement of another circle and a half plane. In Figure 3c, the basic object is an intersection of two circles. The basic object presented in Figure 3d is the intersection of two half-planes and the complement of a circle.

Figure 4 shows the union of the four basic objects that are presented in Figure 3. This arbitrarily shaped non-convex shape is a composed object.

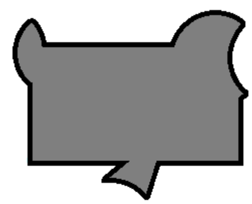

Figure 4: A composed object

More formally, let **B** be the index set of basic objects and **P** be the index set of primitive objects. Each basic object B_t, $t \in \mathbf{B}$, is defined with respect to an index set $\mathsf{P}_t \subseteq \mathbf{P}$ of primitive objects, where each primitive object is denoted by P_k. Thus, B_t can be written as

$$B_t = \bigcap_{k \in \mathsf{P}_t} P_k \;\; \forall t \in \mathbf{B}. \tag{3.1}$$

Let **R** be the index set of composed objects. For each composed object R_i, $i \in \mathbf{R}$, let $\mathsf{B}_i \subseteq \mathbf{B}$ denote the index set of basic objects defining this region. A composed object R_i is then represented by

$$R_i = \bigcup_{k \in \mathsf{B}_i} B_k \;\; \forall i \in \mathbf{R}. \tag{3.2}$$

Functions can be defined for phi-objects to describe the interaction between a point and phi-object. These functions allow the arrangement of points, in our case facility locations, with respect to phi-objects, in our case forbidden regions or barriers, to optimize a given objective to be formulated as a mathematical model. Each primitive object P_k is associated with a half-plane, circle or the complement of a circle. Let $f_k(X) \leq 0$, where X is a point with a coordinate (x, y) and $f_k(X)$ is a function defining the boundary of the primitive object in such a way that it is less than zero if X is inside the phi-object. Since basic object B_t is defined by an intersection of primitive objects, the maximum of $f_k(X)$, $k \in P_t$ is taken. Therefore, the function $\gamma_t(X, B_t)$ representing the relationship between point X and the basic object B_t is

$$\gamma_t(X, B_t) = \max_{k \in \mathsf{P}_t} \{f_k(X)\}. \tag{3.3}$$

If $\gamma_t(X, B_t) = 0$, then the point X is on the boundary of B_t. Alternatively, if $\gamma_t(X, B_t) > 0$, then the point X is outside the basic object B_t.

Similarly, since a composed object R_i is a union of basic objects B_k, $k \in \mathbf{B}_i$, it is defined by the minimum of $\gamma_k(X, B_t)$. Therefore, the function $\Gamma_i(X, R_i)$ representing the relationship between point X and the composed object R_i is

$$\Gamma_i(X, R_i) = \min_{k \in \mathsf{B}_i} \{\gamma_k(X, B_t)\}. \tag{3.4}$$

If $\Gamma_i(X, R_i) = 0$, then the point X is on the boundary of R_i. Alternatively, if $\Gamma_i(X, R_i) > 0$, then the point

X is outside the composed object R_i.

The above mathematical model is based on the set of primitive and basic phi-objects. These objects are known and described in Bennell et al. (2010) and Bennell et al. (2015). In order to model an arbitrary object, the object needs to be decomposed into basic and primitive objects. Chernov et al. (2012) provides a decomposition algorithm for this purpose. They define four types of basic objects. These are (i) a convex polygon (Figure 5a), formed by an intersection of three or more half-planes, (ii) a circular segment (Figure 5b), as an intersection of a circle and a half-plane, (iii) a hat (Figure 5c), formed by an intersection of a complement of a circle and two half-planes that corresponding boundaries are tangent to the circle, and (iv) a horn (Figure 5d), an intersection of a circle, a complement of a circle and a half-plane. Chernov et al. (2012) showed that any arbitrarily shaped composed object bounded by circular arcs and line segments can automatically be decomposed into a set of defined basic objects by the given algorithm.

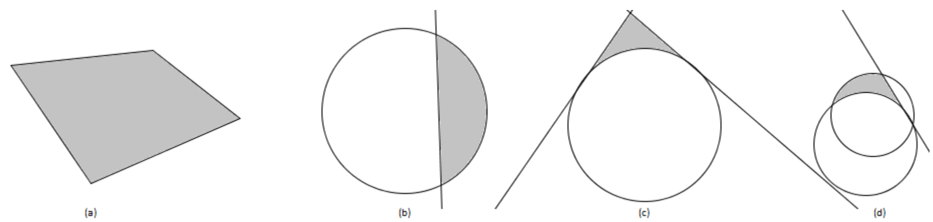

Figure 5: Four types of basic objects (Chernov et al., 2012)

These functions will be used in the proposed modeling framework to define the feasible regions for the possible locations of new facilities. The next section describes the framework in greater detail.

4 A General Modeling Framework

In this section, we present a general modeling framework to formulate continuous facility location problems with restricted regions where regions are modeled using the concept of phi-objects.

4.1 Restricted Problems with Forbidden Regions

By using the functions defined above, it is possible to develop a general model for problems with forbidden regions, in which locating a facility is forbidden but traveling through is permitted. Let **M** be the index set of customer sites and **N** be the index set of new facilities, where $X_1, ..., X_n$ are the locations for new facilities with coordinates $(x_1, y_1), ..., (x_n, y_n)$ respectively. Similarly, $\bar{X}_1, ..., \bar{X}_m$ are locations for customer sites with coordinates $(\bar{x}_1, \bar{y}_1), ..., (\bar{x}_m, \bar{y}_m)$ respectively, and $d(X_n, \bar{X}_m)$ is the distance between nth new facility and mth customer site. Let $\Gamma_i(X_n, R_i)$ denote the function defining the relationship between a new facility X_n and a region R_i. A binary variable z_{mn} is defined, which is equal to 1 if point $\bar{X}_m$ is assigned to a new facility X_n, and 0 otherwise. The general modeling framework of the continuous facility location problem with a forbidden region is shown below.

$$\text{minimize } F\left(z_{mn}, \left(d(X_n, \bar{X}_m)\right)_{n\in\mathbf{N}, m\in\mathbf{M}}\right) \tag{4.1}$$

subject to

$$\Gamma_i(X_n, R_i) \geq 0 \qquad \forall i \in \mathbf{R}, n \in \mathbf{N} \tag{4.2}$$

$$\sum_{n \in \mathbf{N}} z_{mn} = 1 \qquad \forall m \in \mathbf{M} \tag{4.3}$$

$$z_{mn} \in \{0, 1\} \qquad \forall m \in \mathbf{M}, n \in \mathbf{N}. \tag{4.4}$$

In constraints (4.2), " $\geq$ " is used because locating a new facility is not permitted in region R_i. If phi-objects represent the feasible region on a plane, then constraints (4.2) should read $\Gamma_i(X_n, R_i) \leq 0$.

There can be various objectives in facility location problems such as minimizing distance, time, operating cost or maximizing responsiveness. The objective of the proposed modeling framework assumes minimization of a weighted distance, and not the distance itself. A weighted distance is a general term that can represent any of the measures listed above. As an example, if one wishes to minimize time, and time is proportional to distance, then the objective can be modified with suitable weights to reflect this relationship. The following are two of the most commonly studied global objective functions, both of which can easily be included in the proposed modeling framework:

1. F can be decomposed into a sum of one-dimensional functions, i.e. $F\left(z_{mn}, \left(d(X_n, \bar{X}_m)\right)_{n \in \mathbf{N}, m \in \mathbf{M}}\right) = \sum_{n \in \mathbf{N}} \sum_{m \in \mathbf{M}} z_{mn} w_m d(X_n, \bar{X}_m)$, as in the case of the standard median problem, where w_m are non-negative weights of each customer site $m \in \mathbf{M}$ (Akyüz et al., 2010). Median problems often arise in the private sector, since the objective is to minimize the total cost. Several applications for this objective are in industrial transportation and telecommunications.
2. F is the maximum of one-dimensional functions, i.e. $F\left(z_{mn}, \left(d(X_n, \bar{X}_m)\right)_{n \in \mathbf{N}, m \in \mathbf{M}}\right) = \max_{\substack{m \in \mathbf{M} \\ n \in \mathbf{N}}} z_{mn} w_m d(X_n, \bar{X}_m)$, as is the objective function of the well-known center problem (Drezner, 1984). Center problems are more suitable for modeling public sector problems, since the objective is to minimize the maximum weighted distance from demand points to new facilities. An example would be locating an emergency aid center, from where relief items would be sent to all demand points as soon as possible following a possible natural disaster.

In our modeling framework, the function (3.4) that appears in the constraint set includes min and max operators, which introduces a non-linearity. The way to deal with this non-linearity is to represent the function by a series of individual constraints using an either-or representation. While this representation in turn may include linear or non-linear expressions, it is convenient to use this transformation when using off-the-shelf solvers that cannot deal with min and max operators. Let us first look at the case where phi-objects represent forbidden regions and where the new facilities can be located on the boundary of or outside phi-objects. In this case the constraint set (4.2) is transformed into (Schouwenaars et al., 2001):

$$f_k(X_n) \geq 0 - M a_{knt} \qquad \forall k \in \mathbf{P}_t, n \in \mathbf{N}, t \in \mathbf{B} \tag{4.5}$$

$$\sum_{k=1}^{K_t} a_{knt} \leq K_t - 1 \qquad \forall n \in \mathbf{N}, t \in \mathbf{B}, \tag{4.6}$$

where K_t is the total number of elements in set $\mathbf{P}_t$. Here a_{knt} is a binary variable equal to 0 if the constraint associated with a $k \in \mathbf{P}_t, n \in \mathbf{N}$ and $t \in \mathbf{B}$ is binding, and is equal to 1 otherwise. M is a sufficiently large positive number. In this manner at least one corresponding binary variable will take the value 0 for each new facility and basic object. In other words, through constraint (4.5), at least one of the functions forming the basic object B_t will be non-negative, which means the maximum of these functions will be non-negative. Thus, for each basic object forming the forbidden region, at least one corresponding function is non-negative, which guarantees that the minimum of these functions is non-negative.

We now look at the case where the phi-object represents a feasible region and outside of the region is forbidden. In this case, the new facilities can be located on the boundary or inside of the phi-object, and the constraint set

$$\Gamma_{ni}(X_n, R_i) \leq 0 \;\; \forall i \in \mathbf{R}, n \in \mathbf{N}, \tag{4.7}$$

can be converted into

$$f_k(X_n) \leq 0 + M a_{nt} \qquad\qquad \forall k \in \mathbf{P}_t, n \in \mathbf{N}, t \in \mathbf{B} \tag{4.8}$$

$$\sum_{t=1}^{T} a_{nt} \leq T - 1 \qquad \forall n \in \mathbf{N}, \tag{4.9}$$

where T is the total number of elements in set $\mathbf{B}$. Here a_{nt} is a binary variable defined for a basic object B_t and a new facility X_n, and is equal to 1 if the new facility is not contained within B_t, and 0 otherwise. Constraints (4.8) ensure that at least each member of a group of functions forming one of the basic objects B_t will be nonpositive. Some of the other members of other function groups that are forming other basic objects may be positive. Therefore, at least one of the maximums will be non-positive, which implies that the minimum of the maximums will be non-positive.

As it is mentioned above, M is a sufficiently large positive number. It is important to define a lower bound for this number to be able to decrease the value of M, since large values for M can have a negative effect on algorithm performance. In the following two cases, a lower bound value of M can be determined. In the first case where phi-objects represent forbidden regions, we define the smallest rectangle Q that contains all forbidden regions and customer sites within it or on its boundary. Note that the locations of the new facilities must also be inside or on the boundary of Q. In this case,

$$M \geq \max_{X_n \in Q \forall n \in \mathbf{N}, k \in \mathbf{P}_t} \left\{ |f_k(X_n)| \right\}. \tag{4.10}$$

In the second case where phi-objects represent feasible regions, a rectangle is not necessary as in the previous case since the locations of the new facilities will be inside or on the boundary of the phi-objects $R_i, \forall i$. Therefore, a lower bound of M can be defined as,

$$M \geq \max_{X_n \in \bigcup\limits_{i \in \mathbf{R}} R_i \forall n \in \mathbf{N}, k \in \mathbf{P}_t} \left\{ |f_k(X_n)| \right\}. \tag{4.11}$$

After these modifications in the constraint set, it is possible to produce models for both multi-facility median and center problems with an arbitrarily shaped forbidden region and the relevant objective function. These models can be in the form of mixed integer linear programming (MILP) or mixed integer non-linear programming (MINLP) formulations, depending on the objective function and the functions used to model the forbidden region. For instance, if the distance metric is rectilinear and the forbidden region is a convex polygon, then the resulting model is a MILP, whereas if the distance metric is Euclidean, then the model will be in the form of a MINLP due to the non-linearity in the objective function.

4.2 Restricted Single Facility Problems with Barriers

The modeling framework for single facility location problems with barriers is very similar to that of forbidden regions. The main challenge in modeling these problems is the requirement that the paths between customer sites and the new facility must lie outside the barriers. To overcome this difficulty, a large number of points have to be defined for each path connecting each customer site with the new facility, which must all lie outside the barrier, therefore ensuring that the paths themselves will be outside the barriers. To illustrate, we provide an example in Figure 6, which shows the shortest path between points A and B in the existence of a rectangular barrier. Here, the path between A and B is formed by 100 points, which are all restricted to be outside of the rectangle. This idea can clearly be adapted to a facility location problem with barriers as explained below.

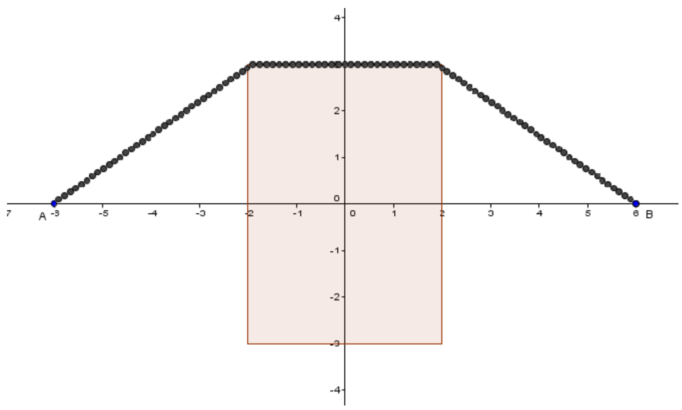

Figure 6: Shortest path between points A and B

Let $\mathbf{J} = \{1, 2, ..., J_{max}\}$ be the index set of the points generating the paths, and $\bar{X}_{mj}$ be the location of the jth point on the path starting from the mth customer site and ending at the location of the new facility. The general modeling framework of the single facility median problem with barriers is shown below.

$$\text{minimize } F\left(\left(d(\bar{X}_{mj-1}, \bar{X}_{mj})\right)_{m\in\mathbf{M}, j\in\mathbf{J}\setminus\{0\}}\right) \tag{4.12}$$

subject to

$$d(\bar{X}_{mj-1}, \bar{X}_{mj}) \leq Vt_{max} \quad \forall m \in \mathbf{M}, j \in \mathbf{J} \setminus \{0\} \tag{4.13}$$

$$\Gamma_{mji}(\bar{X}_{mj}, R_i) \geq 0 \quad \forall i \in \mathbf{R}, m \in \mathbf{M}, j \in \mathbf{J} \tag{4.14}$$

$$X = \bar{X}_{mj} \quad \forall m \in \mathbf{M}, j = J_{max}. \tag{4.15}$$

The objective function (4.12) minimizes a function of distances between each point on each path until the last point, which is the location of the new facility. $F\left(\left(d(\bar{X}_{mj-1}, \bar{X}_{mj})\right)_{m\in\mathbf{M}, j\in\mathbf{J}\setminus\{0\}}\right)$ will be equal to $\sum_{m\in\mathbf{M}} \sum_{j\in\mathbf{J}\setminus\{0\}} w_m d(\bar{X}_{mj-1}, \bar{X}_{mj})$ for single facility median problems, and equal to $\max_{m\in\mathbf{M}} \sum_{j\in\mathbf{J}\setminus\{0\}} w_m d(\bar{X}_{mj-1}, \bar{X}_{mj})$ for single facility center problems. Constraints (4.13) guarantee that the travel time from a point j on a path to point $j + 1$ with a constant velocity V cannot be more time than a specified limit t_{max}. This constraint therefore imposes an upper bound on the distance between any two consecutive points on a path. Setting an upper bound for each distance reduces the chance that the path with pass through the barrier. Constraints (4.14) state that each point on each path between the customer sites and the new facility must not lie in an object that defines a barrier. Constraints (4.15) show that the last point of each path must be the same as the location of the new facility. In our notation, we denote by $\bar{X}_{mj},\ j = 0$ and $\forall m \in \mathbf{M}$, as the locations of each customer site.

5 Results on Instances from the Literature

In this section, instances of restricted continuous facility location problems previously described in the literature are modeled with the proposed general modeling framework. Table 2 shows the characteristics of these instances and where they were published. All instances are solved on a computer with Intel(R) Core(TM) 2.60 GHz processor, and 4.00 GB of RAM, using CPLEX 12.5.1.0 for MILP, BONMIN 1.7 for MINLP, and IPOPT 3.11 for DNLP.

Table 2: Characteristics of the literature instances

Source	Problem Type	Restriction Type	Restricted Region Type	Distance Measure
Katz and Cooper (1981)	1-median	Barrier	Circle	Euclidean
Katz and Cooper (1981)	1-median	Barrier	Circle	Euclidean
Aneja and Parlar (1994)	1-median	Forbidden Region	Polygonal	Euclidean
Aneja and Parlar (1994)	1-median	Barrier	Polygonal	Euclidean
Hamacher and Nickel (1994)	1-median	Forbidden Region	Polygonal	Euclidean
Hamacher and Nickel (1994)	1-median	Forbidden Region	Polygonal	Rectilinear
Hamacher and Nickel (1994)	1-median	Forbidden Region	Polygonal	Chebyshev
Archipelago Instance	1-center	Forbidden Region	Arbitrary	Euclidean
Archipelago Instance	2-center	Forbidden Region	Arbitrary	Euclidean

5.1 Forbidden Region Instances

There exists four instances with forbidden regions in the literature, one described by Aneja and Parlar (1994), and three by Hamacher and Nickel (1994) which we model using the proposed framework, and subsequently solve to optimality. In the remaining sections, we introduce these instances and discuss in detail how they were solved.

5.1.1 The Aneja and Parlar (1994) Instance

The instance introduced by Aneja and Parlar (1994) is shown in Figure 7. This is a non-convex polygonal forbidden region for a single facility median problem. First, we will show that the non-convex forbidden region can be modeled as a phi-object. Then, we will construct a model by using the function defining the relationship between specified forbidden region and the new facility, and solve the model to obtain an optimal solution to the problem.

Originally, this problem is posed as a 1-median problem with a non-convex forbidden region R shown in Figure 7. The coordinates of the customer sites are $e_1 = (0, -10), e_2 = (11, -10), e_3 = (0, 11.6)$ and $e_4 = (11, 11.6)$, each of which has a weight equal to 1. The distance metric is Euclidean.

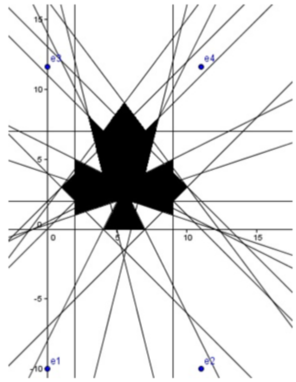

Figure 7: The numerical example of Aneja and Parlar (1994)

The non-convex composed object in Figure 7 is the union of nine basic objects, as shown in Figure 8, each of which is a convex polygon. Each basic object in turn is formed by intersecting primitive objects, which in this case are half planes.

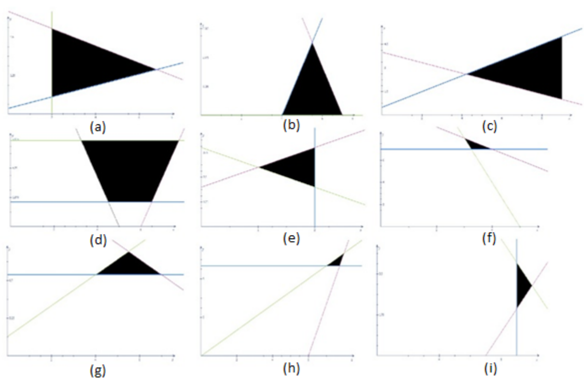

Figure 8: Nine basic objects of the numerical example

For example, the basic object B_1 in Figure 8a is the intersection $B_1 = \bigcap_{k \in \mathsf{P}_1} P_k$ where $\mathsf{P}_1 = \{l_1, l_2, l_3\}$ and $l_1 - l_3$ denote the three half-planes, whose bounds are defined by the following three functions $f_1 : -x + 2 = 0$, $f_2 : -y + 0.333x + 0.333 = 0$ and $f_3 : y + 0.5x - 6 = 0$, respectively. The rest of the basic objects $B_2 - B_9$ shown in Fig8b – Fig8i respectively can be defined in a similar way. The forbidden region R in Figure 7 is then modeled as $\bigcup_{k \in \mathsf{B}} B_k$.

Since this instance is of a 1-median problem, the function of point X representing the relationship between location of the new facility and the forbidden region R is shown below, and since R is a forbidden region, the function, $\Gamma(X, R)$, must be greater or equal to zero in the model.

$$\Gamma(X,R) = \min\{\max_{k=1,2,3}\{f_k\}, \max_{k=4,5,6}\{f_k\}, \max_{k=7,8,9}\{f_k\}, \max_{k=10,11,12,13}\{f_k\}, \max_{k=14,15,16}\{f_k\}, \max_{k=17,18,19}\{f_k\}, \max_{k=20,21,22}\{f_k\}, \max_{k=23,24,25}\{f_k\}, \max_{k=26,27,28}\{f_k\}\}, \quad (5.1)$$

where f_k, $k = 1, ..., 28$, is a linear function associated with each primitive object.

As mentioned in Section 4, we use either-or constraints to linearize the terms max and min appearing in the constraints. For instance, $\max_{k=1,2,3}\{f_k\}$ in (5.1) can be converted into the constraints,

$$-x + 2 \geq 0 - Ma_{111} \quad (5.2)$$

$$-y + 0.333x + 0.333 \geq 0 - Ma_{211} \quad (5.3)$$

$$y + 0.5x - 6 \geq 0 - Ma_{311} \quad (5.4)$$

$$a_{111} + a_{211} + a_{311} \leq 2 \quad (5.5)$$

$$a_{111}, a_{211}, a_{311} \in \{0, 1\}. \quad (5.6)$$

The resulting model is a MINLP. BONMIN 1.7 is used to solve the formulation to optimality, yielding the optimal solution $d = (5.5, 0)$, which is same with the solution given in Aneja and Parlar (1994). The execution time is 0.008 seconds.

5.1.2 The Hamacher and Nickel (1994) Instances

Hamacher and Nickel (1994) introduce some single facility median instances with a convex forbidden region represented by a rectangle $R = [3, 11] \times [9, 15]$, which is a basic object in our framework. The forbidden region and the customer sites are shown in Figure 9. The coordinates of the customer sites are $e_1 = (5, 13)$, $e_2 = (7, 11)$, and $e_3 = (5, 11)$, all with unit weights. This instance has been solved with three different distance metrics, namely rectilinear, squared Euclidean and Chebyshev.

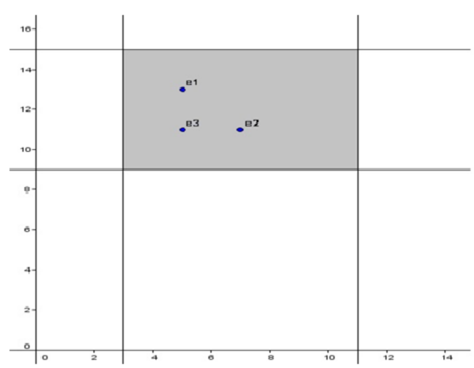

Figure 9: Example in Hamacher and Nickel (1994)

Since the forbidden region in this problem is a basic object, the function defining the relation between the point X and basic object B is shown as follows;

$$\gamma(X, B) = \max_k (f_k), \tag{5.7}$$

where f_k, $k = 1, 2, 3, 4$, is a linear function associated with the half planes that bound each edge of the rectangle. Our proposed solution technique applies to all three distance functions mentioned above, by only modifying the objective function of the model but keeping the constraint set the same.

In these models the either-or constraints are used to define the maximum constraints. In the second and third model, a linearization method is applied, which has resulted in a mixed integer programming formulation, using the linearization technique for rectilinear and Chebyshev distance functions in Hamacher and Nickel (1995). The first model remains a mixed integer non-linear program, since the distance metric is squared Euclidean. In the third model, Chebyshev distance function is transformed to rectilinear distance function (Hamacher and Nickel, 1995).

The first model is solved by BONMIN 1.7 solver. The optimal location for the new facility is $(5.667, 9)$ with an objective value 26.667. The execution time is 0.004 seconds. The second and third models are solved by CPLEX $12.5.1.0$. The optimal location of the new facility for the second model is $(3, 11)$ with an objective value 10. For the third model, $(3, 11)$ is the optimal location for the third model with an objective value, 8.

On a related note, we have observed that these instances exhibit alternative optima. In order to find these solutions, cuts can be added to prune the optimal solutions found. Using such cuts, we were able to obtain $(3, 11.667)$, $(5, 9)$ and $(5, 9)$ as alternative optima for the first, second and third model respectively, which coincide with the solutions reported in Hamacher and Nickel (1994).

5.2 Barrier Instances

There are three barrier instances that have been described in the existing literature, all of which are modeled through the proposed framework. The first two instances are first presented in Katz and Cooper (1981), for which the parameters use the following values: $J_{max} = 100$, $V = 0.01$ m/s and $t_{max} = 0.2$ seconds. The third instance is first presented in Aneja and Parlar (1994), for which the parameters are $J_{max} = 100$, $V = 0.01$ m/s and $t_{max} = 0.2$ seconds. These values are decided by considering the performance of the solvers at hand and the instances that are solved. We now present the solutions of these instances in more detail.

5.2.1 Katz and Cooper (1981) Instances

In the first instance, the barrier is a circle with a radius of two, with its center located at $(0, 0)$. There are five customer sites, with coordinates $(-8, -6)$, $(-7, 13)$, $(-1, -5)$, $(6.6, -0.5)$ and $(4.4, 10)$, each with a weight equal to one. The distance metric is Euclidean. This instance is also studied in various papers including Butt and Cavalier (1996), Klamroth (2001), Bischoff and Klamroth (2007) and Klamroth (2004).

Since the barrier in this problem is a circle, the corresponding phi-object is simply a primitive object, which makes the model a NLP. IPOPT 3.11 is used to solve the formulation to optimality, yielding the solution $(-1.186, 2.060)$ with an objective value of 48.257 as shown in Figure 10. This solution is better than the one found in Katz and Cooper (1981), and same with the solutions given in Butt and Cavalier (1996), Klamroth (2001), Bischoff and Klamroth (2007) and Klamroth (2004). The execution time is 0.234 seconds.

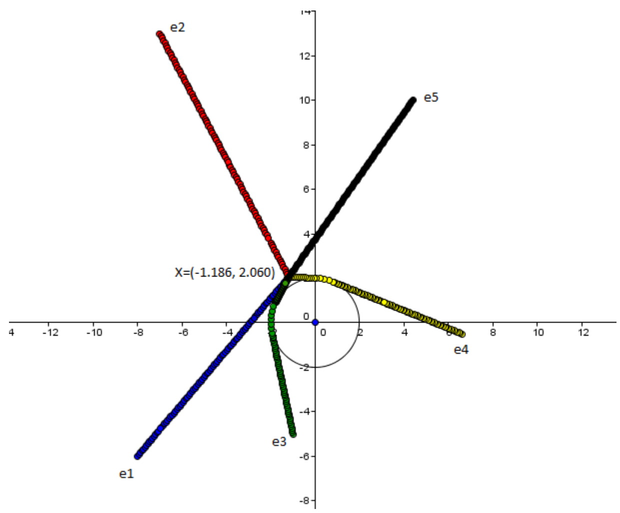

Figure 10: Katz and Cooper (1981) Instance No. 1

The second instance in Katz and Cooper (1981) is very similar to the first one with the exception that the radius is now three. The number of customer sites is increased to 10, which are located on $(8, 8)$, $(5, 7)$, $(6, 4)$, $(-3, 5)$, $(-6, 6)$, $(-3, -4)$, $(-5, -6)$, $(-8, -8)$, $(5, -5)$ and $(8, -8)$. Bischoff and Klamroth (2007) also applied their solution algorithm to this problem.

The model for this problem is also a NLP which is solved by IPOPT 3.11. The location of the new facility is $(3.306, -0.068)$ with an objective value of 88.326, which is similar to the results given in Bischoff and Klamroth (2007), but contradict the results in Katz and Cooper (1981). The execution time is 0.452 seconds. The solution is shown in Figure 11.

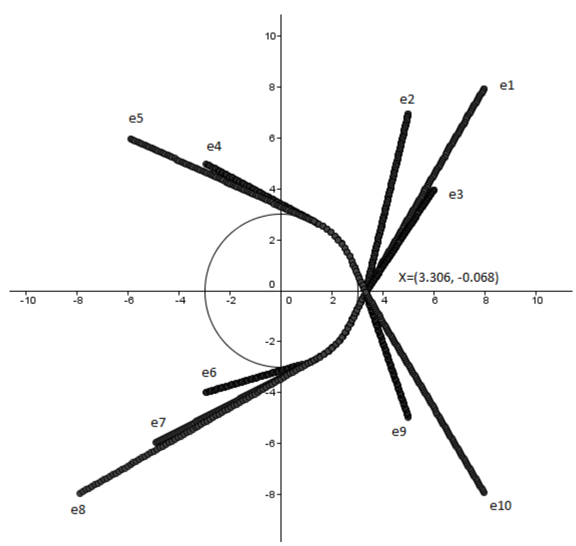

Figure 11: Katz and Cooper (1981) Instance No. 2

5.2.2 Aneja and Parlar (1994) Barrier Instance

In this instance there are 10 convex and two non-convex polygon barriers. There are 18 facilities, each with a unit weight, placed on points $(1, 2)$, $(6, 1)$, $(9, 1)$, $(14, 2)$, $(5, 5)$, $(7, 4)$, $(9, 5)$, $(14, 4)$, $(17, 4)$, $(2, 8)$, $(8, 8)$, $(16, 8)$, $(3, 12)$, $(6, 11)$, $(9, 10)$, $(17, 10)$, $(10, 12)$ and $(19, 13)$. This is a 1-median instance where the aim is to find the location of the new facility that minimizes the total weighted distance.

Since there are numerous barriers in this problem, there will be a large number of constraints in the model, which can be detrimental to the performance of a solver. To overcome this drawback, the model is solved using a Cutting Plane technique, whereby a relaxed version of the model is solved. As a first step, by assuming barriers are forbidden regions. Figure 12 shows the solution of the relaxed problem, where the new facility is placed on the coordinate $(8.913, 6.355)$ and is shown by a red dot.

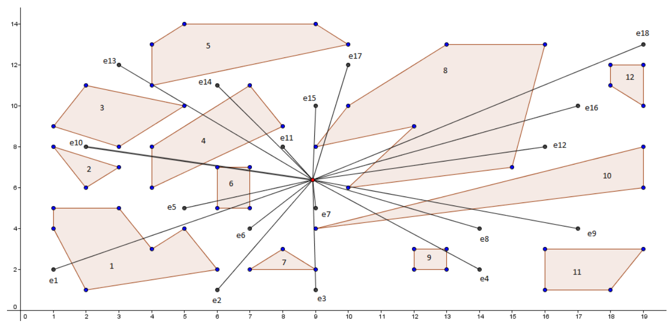

Figure 12: Cutting plane Step 1

In Figure 12, it can be seen that certain barrier constraints are violated. For instance, a path from the first facility e1 to the location of a new facility intersects with barriers 1 and 6. In this case, we add the constraint sets that define regions 1 and 6 as barriers. Other infeasibilities are shown in Table 3, for which the corresponding constraints are also added and the model is resolved. The resulting solution, shown in Figure 13, shows other barriers being crossed, namely 10, 8 and 12. We iteratively add the relevant cuts and resolve the model until a feasible solution is obtained, as shown in Figure 14, after four iterations. By applying cutting plane approach, the total number of single equations in the model is decreased from $27,035$ to $4,635$.

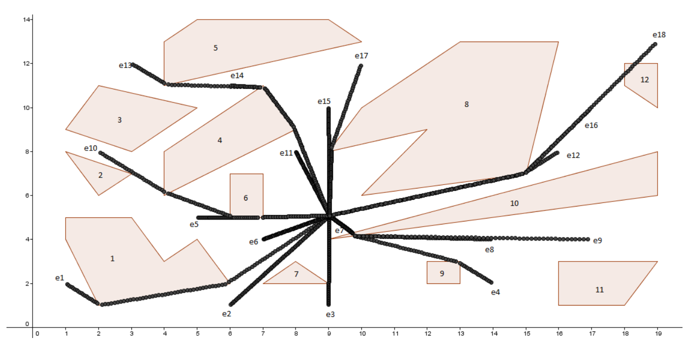

Figure 13: Cutting plane Step 2

Once all the violated constraints are added, there is one other potential issue that needs to be addressed to reach optimality for this instance. Even though constraints are added to force the points that form the paths

Table 3: Barriers for which constraints are added in Step 2

Customer Site	Barrier
1	1, 6
2	-
3	7
4	9, 10
5	6
6	-
7	-
8	10
9	8, 10
10	4, 6
11	-
12	8
13	4, 5
14	4
15	-
16	8
17	8
18	8

Table 4: Barriers for which constraints are added in Step 3

Customer Site	Barrier
3	10
15	8
18	12

to be outside of the barriers, a path may well contain a line segment that crosses the corners of a barrier. This infeasibility can be addressed by adding some constraints to the model, which dictate that one of the two points at either end of such a line segment should be placed on the vertex. For example, the 13^{th} point of Path 1 shown in Table 5 will be placed on the vertex coordinate $(2, 1)$ of barrier 1, and will therefore prohibit the formation of a path that passes through it.

For this instance, optimality is reached after adding these constraints. After applying cutting plane technique, moving specific point to barrier vertecies can be called as post-processing to reach feasible optimal solution. An optimal solution is presented in Figure 15.

The final model for this problem is a NLP with max and min operators in constraints. It may be transformed into a MINLP, but to keep the size as small as possible, we opted to keep it as a NLP and used IPOPT 3.11 as a solver using its internal functions to model the max and min operators. The optimal solution reported by IPOPT 3.11 has a value 119.176 where the new facility is placed at (8.752, 4.979). The execution time is 0.936 seconds. The values are slightly different than the results found in Aneja and Parlar (1994) and Bischoff and Klamroth (2007), both of which report $(8.7667, 4.9797)$ for the location of the new facility and 119.1387 for the objective value.

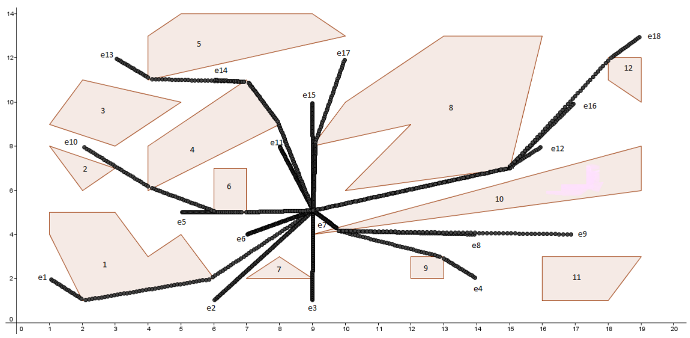

Figure 14: Cutting plane Step 3

Table 5: Vertex Points

Customer Site (Path)	Point (J)	Barrier	Vertex Coordinate
1	13$^{\text{th}}$	1	(2,1)
1	54$^{\text{th}}$	1	(6,2)
3	45$^{\text{th}}$	7	(9,2)
4	18$^{\text{th}}$	9	(13,3)
4	60$^{\text{th}}$	10	(9,4)
8	58$^{\text{th}}$	10	(9,4)
9	60$^{\text{th}}$	10	(9,4)
10	28$^{\text{th}}$	4	(4,6)
10	46$^{\text{th}}$	6	(6,5)
12	26$^{\text{th}}$	8	(15,7)
13	15$^{\text{th}}$	5	(4,11)
13	47$^{\text{th}}$	4	(7,11)
13	65$^{\text{th}}$	4	(8,9)
14	8$^{\text{th}}$	4	(7,11)
14	27$^{\text{th}}$	4	(8,9)
16	38$^{\text{th}}$	8	(15,7)
17	56$^{\text{th}}$	8	(9,8)
18	10$^{\text{th}}$	12	(18,12)
18	52$^{\text{nd}}$	8	(15,7)

6 Case Study: New Single and Multifacility Instances

In this section, we introduce, model and solve new instances based on real geography, for both single and multiple facilities. The new set of instances are derived from an archipelago called Prince Islands in the Marmara Sea near Istanbul, Turkey. These islands have historical and touristic importance. There are nine of these islands, namely Tavsanadasi, Sedefadasi, Buyukada, Heybeliada, Kasikadasi, Burgazada, Kinaliada, Sivriada and Yassiada, but only six are populated. A satellite view of Prince Islands by Google Earth software is shown in Figure 16. Prince Islands are in close proximity to an active fault line, and facing the danger of earthquakes. For this archipelago, we focused on the problem of locating a relief item depot with a helicopter field. From Istanbul, a large city in terms of population that is vulnerable to possible earthquake damage, it may take considerable time to send relief items to Prince Islands. Therefore, it is vital for the residents and the visitors of these islands that they have their own autonomous emergency response system. After a possible earthquake, relief items from the new depot can be sent to each of the

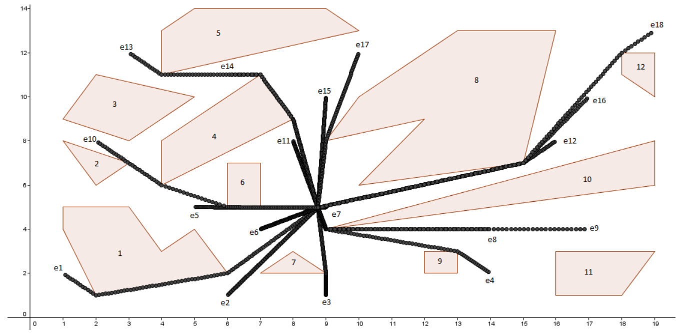

Figure 15: Final solution

populated islands using helicopters. Given the nature of the problem, an appropriate objective here would be to minimize the maximum distance between each populated island and its closest new relief depot.

Figure 16: Prince Islands

For this instance, we solve 1-center and 2-center problems using the modeling framework where the Marmara Sea is modeled as the forbidden region, and the Prince Islands as the feasible regions. In contrast to previous instances, the phi-objects in this case model the feasible regions implying that the forbidden region is non-convex shaped and unbounded. To approximate these phi-objects, 105 half planes, 10 circles and 14 circular holes are used as primitive objects, as shown in Figure 17.

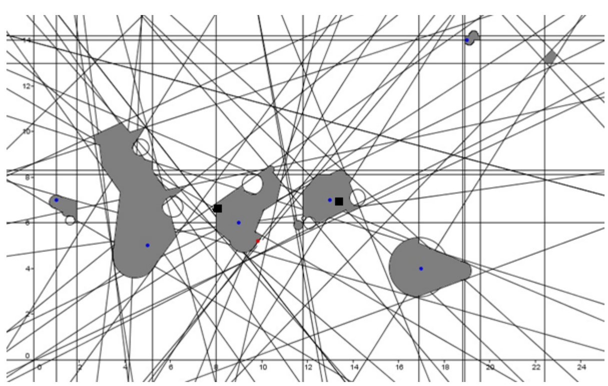

Figure 17: Phi-objects representing the Prince Islands and the solutions of the problems

The helicopter fields in each populated island can be considered as customer sites on the islands. The weights are calculated as follows: The highly populated islands in this archipelago are Buyukada and

Heybeliada, the less populated islands are Sedefadasi and Yassiada. The population of Burgazada and Kinaliada is at a medium level compared to the other islands. It is therefore assumed that the weights of Heybeliada and Buyukada are both three, Burgazada and Kinaliada are both two and Sedefadasi and Yassiada are both one. The coordinates of the customer sites are assumed to be $(1,7)$, $(5,5)$, $(9,6)$, $(13,7)$, $(17,4)$ and $(19,14)$, shown by the blue dots in Figure 17. Since the transportation of relief items will be done by helicopters, Euclidean distance is considered as the metric. The representation of Prince Islands as phi-objects, the locations of existing helicopter fields and the solutions for 1-center and 2-center problems are shown in Figure 17. The 1-center problem for these islands is solved by BONMIN 1.7, which has yielded the location of the depot as $(9.837, 5.192)$, which is shown by a red dot on Heybeliada. The objective value is 14.513, and the execution time for this model is 0.215 seconds. The 2-center problem is again solved by BONMIN 1.7, according to which the new depots are to be placed on $(13.599, 6.978)$ and $(8, 6.813)$, shown by black squares in the same figure, with an objective value 10.517. These new facilities are on Burgazada and Heybeliada. The execution time for this model is 0.437 seconds.

7 Conclusions

This paper has described a general framework for modeling continuous facility location problems with either forbidden regions or barriers by using the concept of phi-objects. The use of phi-objects allows one to model a restricted region or a feasible region, thus an unbounded restricted region can also be taken into consideration. Another advantage is the flexibility of modeling various geometric shapes, which is the case for a real life problems, since complicated geographical shapes (e.g., archipelago, bays, lakes) can be modeled with negligible approximations. Within this framework, it is also possible to formulate both median and center problems with various types of distance metrics such as Euclidean, rectilinear and Chebyshev. Using the models, we have optimally solved all literature instances that we are aware of for this class of problems, using state-of-the-art linear and nonlinear integer programming technology.

Building from the framework we present in this paper, there are two clear research challenges. To the best of our knowledge, there is limited research on multi-facility location problems with barriers. We believe that a solution approach on these problems is needed. Another topic that warrants further research is a similar general modeling framework for the stochastic version of the restricted location problems. There are many real-life examples of this problem, where random obstacles caused by natural disasters, especially floods may occur. Many countries around the globe, including the United Kingdom, face high risks of floods which can occur during heavy storms. Such restricted regions can be treated as 2D obstacles on a plane with a stochastic formation pattern. Although some research on this variant has appeared, there is still a need for a general approach.

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