Dissipative switched linear differential systems

Jonathan C. Mayo-Maldonado, Paolo Rapisarda

***Abstract*—We develop a dissipativity theory for switched systems whose dynamical modes are described by systems of higher-order linear differential equations. We give necessary and sufficient conditions for dissipativity based on systems of LMIs, constructed from the coefficient matrices of the differential equations describing the modes. The relationship between dissipativity and stability is also discussed and an application to the stabilisation of power converters is provided.**

***Index Terms*—Switched systems; behaviours; quadratic differential forms; dissipativity; power converters.**

I. Introduction

Usually switched systems are described using a bank of state space- (e.g. [16], [19]) or descriptor form- (e.g. [36], [37]) representations, together with a switching rule that determines which of the modes is active; furthermore, *state reset maps* can be used to act at the switching times. In such approaches, the dynamical modes share a global state space. In [23], [24], we argued that in many real-life situations the dynamical modes of switched systems do not share the same state space. Examples include multi-controller control systems, power converters, power sources with multiple loads, charging stations for electric vehicles, hybrid renewable energy systems, etc. Moreover, since switched systems can be studied in higher-order terms (see [22], [23], [24], [30], [32], [38]), the use of state space representations themselves is not a fundamental requirement.

In [24], we introduced a framework for the study of *switched linear differential systems (SLDS)*, where each dynamical mode is associated with a *mode behaviour*, i.e. the set of trajectories satisfying *higher-order linear differential equations*. A *switching signal* determines which of the modes is active. Additionally, *gluing conditions* are introduced to specify the equilibrium conditions of the trajectories at switching instants, e.g. charge/flux conservation principles, kinematic constraints, reset maps, etc. The mode behaviours do not necessarily share the same state space and their modelling does not require to satisfy a particular mathematical structure; consequently the use of a global state space becomes a special case. Moreover, individual modelling is permitted, i.e. new dynamical modes can be added to the underlying bank without altering the existing ones. This feature greatly simplifies not only the modelling phase of switched systems (see e.g. [23] Sec. V), but also the computations necessary for their study, see e.g. Ex. 5, p. 2043 of [24].

In [24], we presented stability conditions for *closed systems*, i.e. systems without *input variables*. In this paper we study *open systems* and their *dissipativity*, first introduced in a control and systems setting in [44]. This concept is useful in dealing with issues such as stability, stabilisability, control, and other important applications (see e.g. [11], [17], [41], [43], [46]). For this reason, dissipativity and its special case *passivity* have been studied extensively in general settings such as impulsive, discontinuous and hybrid systems (see e.g. [12], [13], [14], [15], [27], [48]), as well as in the switched systems setting (see e.g. [3], [8], [21], [49], [50], [51]). In [9], the role of passivity for stability of switched systems has been also studied considering dynamical modes with Hamiltonian structure. In [51], novel definitions of dissipative switched systems are presented involving the use of *cross-supply rates*. This approach also encompasses important results (e.g. stability, passivity, L_2-gain) associated to dissipative nonlinear systems with infinitely differentiable trajectories. In [15], another definition of dissipativity is presented where the use of *connective supply rates* characterises the energy change of inactive modes. More recently, in [21], [20], the notion of *decomposable dissipativity* is introduced for discrete-time switched systems.

In this paper, we give definitions of dissipativity of switched linear differential systems. In order to do so, we use *quadratic differential forms* (see [45]), since they provide suitable mathematical tools to deal with higher-order differential systems. Furthermore, we provide sufficient conditions for dissipativity based on systems of LMIs for arbitrary switching signals and involving the computation of multiple storage functions. Such systems of LMIs can be set up straightforwardly from the equations of the mode dynamics and the gluing conditions. Following the behavioural setting for linear systems (see [28]), the mode equations and the gluing conditions are represented by one-variable polynomial matrices, and the functionals (e.g. supply rates and storage functions) by two-variable ones. This feature also opens up the possibility to solve parametric design problems by an efficient exploration of design spaces. We also study the relationship between dissipativity and stability of switched systems by studying *passive systems*, and we provide a detailed example to show the potential of our approach in solving stability problems in *power converters* with *constant power loads*, a current pressing research issue in power electronics (see [6]).

We use the following notation. The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and that of $\mathtt{m}\times\mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. $\mathbb{R}^{\bullet\times\mathtt{m}}$ denotes the space of real matrices with $\mathtt{m}$ columns and an unspecified finite number of rows. Given matrices $A, B \in \mathbb{R}^{\bullet\times\mathtt{m}}$, $\mathrm{col}(A, B)$ denotes the matrix obtained by stacking A over B. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi]$ denotes the set of all $\mathtt{r}\times\mathtt{w}$ matrices with entries in ξ, and $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$ that of $\mathtt{n}\times\mathtt{m}$ polynomial matrices in ζ and η. The set of rational $\mathtt{m}\times\mathtt{n}$ matrices is denoted by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}(\xi)$. Given $G = G^\top \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$, $\sigma_+(G)$ denotes the number of positive eigenvalues of G. The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is the subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of compact support functions. For a function $f : [t-\epsilon, t) \to \mathbb{R}^\bullet$ we set the notation $f(t^-) := \lim_{\tau\nearrow t} f(\tau)$; and similarly for $f : (t, t+\epsilon] \to \mathbb{R}^\bullet$ we set $f(t^+) := \lim_{\tau\searrow t} f(\tau)$, provided that these limits exist.

II. Switched linear differential systems

In order to provide a self-contained theoretical exposition, we present some basic concepts of the switched linear differential systems framework introduced in [24]. To illustrate the concepts in our framework from a physical point of view, we use the example of a standard *power converter*.

A. Main definitions

In the *switched linear differential systems (SLDS)* framework, each dynamical mode is associated with a *mode behaviour*, the set of trajectories that satisfy the dynamical laws of that mode. A *switching signal* determines when a transition between dynamical modes occurs. At the switching instants the system trajectories must satisfy certain *gluing conditions*, that represent algebraic constraints enforced by physical principles.

Definition 1 ([24] p.2039, Def. 1)**.** A *switched linear differential system (SLDS)* Σ is a quadruple $\Sigma = \{\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G}\}$ where

- $\mathcal{P} = \{1,\dots,N\} \subset \mathbb{N}$, is the set of *indices*;
- $\mathcal{F} = \{\mathfrak{B}_1,\dots,\mathfrak{B}_N\}$, with $\mathfrak{B}_j$ a linear differential behaviour and $j \in \mathcal{P}$, is the *bank of behaviours*;
- $\mathcal{S} = \{s : \mathbb{R} \to \mathcal{P}\}$, with s piecewise constant and right-continuous, is the set of admissible *switching signals*; and
- $\mathcal{G} = \left\{(G^-_{k\to\ell}(\xi), G^+_{k\to\ell}(\xi)) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]\times\mathbb{R}^{\bullet\times\mathtt{w}}[\xi] \mid 1 \le k,\ell \le N\ ,\ k \neq \ell\right\}$, is the set of *gluing conditions*.

The set of *switching instants* associated with $s \in \mathcal{S}$ is defined by $\mathbb{T}_s := \{t \in \mathbb{R} \mid s(t^-) \neq s(t^+)\} = \{t_1, t_2, \dots\}$, where $t_i < t_{i+1}$.

The set of all admissible trajectories satisfying the laws of the mode behaviours and the gluing conditions is the *switched behaviour*, and is the central object of study in our framework.

Definition 2 ([24] p.2039, Def. 2)**.** Let $\Sigma = \{\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G}\}$ be a SLDS, and let $s \in \mathcal{S}$. The *s-switched linear differential behaviour* $\mathfrak{B}^s$ is the set of trajectories $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ that satisfy the following two conditions:

1) for all $t_i, t_{i+1} \in \mathbb{T}_s$, $w|_{[t_i,t_{i+1})} \in \mathfrak{B}_{s(t_i)}|_{[t_i,t_{i+1})}$;
2) w satisfies the gluing conditions $\mathcal{G}$ at the switching instants for each $t_i \in \mathbb{T}_s$, i.e.

$$\begin{aligned} G^+_{s(t_{i-1})\to s(t_i)}\left(\frac{d}{dt}\right) w(t_i^+) \\ = G^-_{s(t_{i-1})\to s(t_i)}\left(\frac{d}{dt}\right) w(t_i^-)\ . \end{aligned} \tag{1}$$

The *switched linear differential behaviour (SLDB)* $\mathfrak{B}^\Sigma$ of Σ is defined by $\mathfrak{B}^\Sigma := \bigcup_{s\in\mathcal{S}} \mathfrak{B}^s$.

Since $\mathfrak{B}_i \subseteq \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, $i = 1,\dots,N$ (see App. I-A), it follows that the trajectories in $\mathfrak{B}^\Sigma$ are *piecewise infinitely differentiable functions* from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, i.e. smooth when a mode is active and possibly discontinuous at switching instants.

Example 1. Consider the *high-voltage switching power converter* presented in [5] and depicted in Fig. 1 a). For practical purposes such as voltage/current/power regulation, we are particularly interested in the dynamics at the input/output terminals. Consequently we define the external variable (the set of variables of interest) as $w := \mathrm{col}(E, i_L, v_2, i_o)$. By means of a switching signal, we can arbitrarily induce two possible electrical configurations that occur when the transistor is in either *closed* (see Fig. 1 b)) or *open* (see Fig. 1 c)) operation. Considering a standard modelling of input/output impedances (see [34], p. 123) for each case, we can derive the following physical laws describing the dynamics of the power converter. For simplicity we consider $L = 1H$, $C_1 = C_2 = 1F$,

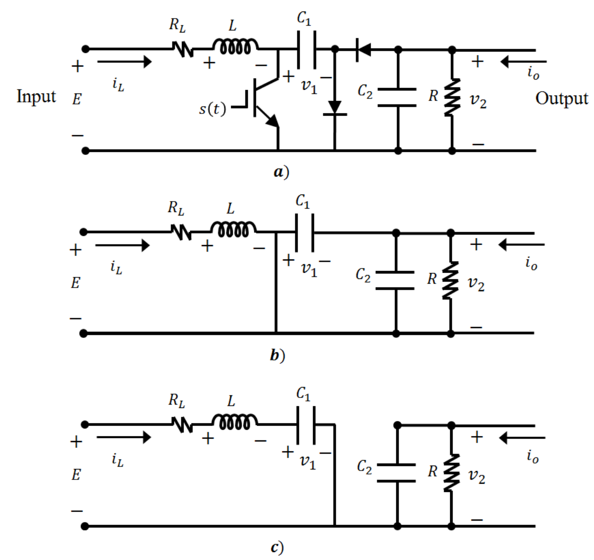

Fig. 1. a) High-voltage switching power converter, b) electrical configuration when the transistor is closed, c) electrical configuration when the transistor is open.

$R_L = 1\Omega$ and $R = 1\Omega$.

$$\text{Mode 1:} \begin{cases} -E + \frac{d}{dt} i_L + i_L = 0 \\ 2\frac{d}{dt} v_2 + v_2 - i_o = 0 \end{cases} . \tag{2}$$

$$\text{Mode 2:} \begin{cases} -\frac{d}{dt} E + \frac{d^2}{dt^2} i_L + \frac{d}{dt} i_L + i_L = 0 \\ \frac{d}{dt} v_2 + v_2 - i_o = 0 \end{cases} . \tag{3}$$

The mode behaviours can be defined as $\mathfrak{B}_j := \ker R_j\left(\frac{d}{dt}\right)$, $j = 1, 2$, (see *kernel representation* in App. I-A) where

$$R_1\left(\frac{d}{dt}\right) := \begin{bmatrix} -1 & \frac{d}{dt} + 1 & 0 & 0 \\ 0 & 0 & 2\frac{d}{dt} + 1 & -1 \end{bmatrix} ;$$
$$R_2\left(\frac{d}{dt}\right) := \begin{bmatrix} -\frac{d}{dt} & \frac{d^2}{dt^2} + \frac{d}{dt} + 1 & 0 & 0 \\ 0 & 0 & \frac{d}{dt} + 1 & -1 \end{bmatrix} .$$

At switching instants the physical laws impose constraints to the system trajectories. By inspecting the circuits in Fig. 1 and using the *principle of conservation of charge* (see [25], sec. 3.3.3), we find the following conditions at switching instants.

When switching from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ at t_i:

$$\begin{aligned} & i_L(t_i^+) = i_L(t_i^-) \ , \\ & E(t_i^+) - i_L(t_i^+) - \frac{d}{dt} i_L(t_i^+) = v_2(t_i^-) \ , \\ & v_2(t_i^+) = v_2(t_i^-) \ . \end{aligned} \tag{4}$$

When switching from $\mathfrak{B}_2$ to $\mathfrak{B}_1$ at t_i:

$$\begin{aligned} & i_L(t_i^+) = i_L(t_i^-) \ , \\ & 2v_2(t_i^+) = E(t_i^-) - i_L(t_i^-) - \frac{d}{dt} i_L(t_i^-) + v_2(t_i^-) \ . \end{aligned} \tag{5}$$

Consequently, the gluing conditions can be defined as

$$G^+_{1\to 2}\left(\frac{d}{dt}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 - \frac{d}{dt} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;$$
$$G^-_{1\to 2}\left(\frac{d}{dt}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;$$
$$G^+_{2\to 1}\left(\frac{d}{dt}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} ;$$
$$G^-_{2\to 1}\left(\frac{d}{dt}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 - \frac{d}{dt} & 1 \end{bmatrix} .$$

Hence equations in (4)-(5) can be compactly written as

$$G^+_{1\to 2}\left(\frac{d}{dt}\right) w(t_i^+) = G^-_{1\to 2}\left(\frac{d}{dt}\right) w(t_i^-) ;$$
$$G^+_{2\to 1}\left(\frac{d}{dt}\right) w(t_i^+) = G^-_{2\to 1}\left(\frac{d}{dt}\right) w(t_i^-) .$$

□

Remark 1. From Ex. 1, we can draw some basic conclusions regarding important features of modelling of physical switched systems that are supported in the SLDS framework. 1) By applying first principles, we usually obtain sets of linear differential equations, possibly of higher-order. 2) The mode dynamics can be associated with different state spaces, i.e. *different minimal state space representations*[1] can be constructed from sets of the linear differential equations, e.g. for (2) we can choose a state vector $x_1 := \begin{bmatrix} i_L & v_2 \end{bmatrix}^\top$, while for (3) we can select $x_2 := \begin{bmatrix} i_L & \frac{d}{dt} i_L & v_2 \end{bmatrix}^\top$. 3) At switching instants the physical laws may impose gluing conditions to the system trajectories, i.e. algebraic constraints such as charge/flux conservation principles, kinematic constraints, reset maps, etc. □

Remark 2. As previously pointed out, a SLDS as in Def.s 1 and 2 admits different state spaces for its dynamical modes. However, the external variables are the same for every dynamical mode: they have been chosen as the variables of interest during the modelling stage according to each particular application, see e.g. Ex. 1. □

[1]The adjective *minimal* refers to the standard notion of *McMillan degree*, see App. I-B

B. Latent variables

Controllable mode behaviours can be described using *observable* image representations $w = M_j\left(\frac{d}{dt}\right)\ell_j$, $j = 1, ...N$, see App. I-A. It follows that every trajectory of the *latent variable* ℓ_j corresponds to a unique trajectory of the external variable w when the j-th mode is active.

Example 2 (Cont'd from Ex. 1**).** Recall that $w := \mathrm{col}(E, i_L, v_2, i_o)$. It can be verified that the mode behaviours $\mathfrak{B}_j$, $i = 1, 2$, are *controllable* and thus can be described by image representations $w = M_j\left(\frac{d}{dt}\right)\ell_j$, $j = 1, 2$ (see App. I-A), where

$$M_1\left(\frac{d}{dt}\right) := \begin{bmatrix} \frac{d}{dt}+1 & 0 \\ 0 & 2\frac{d}{dt}+1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ;$$

$$M_2\left(\frac{d}{dt}\right) := \begin{bmatrix} \frac{d^2}{dt^2}+\frac{d}{dt}+1 & 0 \\ 0 & \frac{d}{dt}+1 \\ \frac{d}{dt} & 0 \\ 0 & 1 \end{bmatrix} ;$$

and $\ell_1 := \mathrm{col}(i_L, v_2)$, $\ell_2 := \mathrm{col}(v_1, v_2)$. Moreover, since $M_j(\lambda)$, $j = 1, 2$, are full column rank for all $\lambda \in \mathbb{C}$ we conclude that the latent variables ℓ_j, $j = 1, 2$ are observable from w (see App. I-A). $\square$

According to Def.s 1 and 2, the gluing conditions are algebraic constraints acting on the external variables at switching instants; however, they can be rewritten in terms of latent variables in the following manner. Define $\overline{G}^+_{s(t_{i-1})\to s(t_i)}\left(\frac{d}{dt}\right) := \left(G^+_{s(t_{i-1})\to s(t_i)}M_{s(t_i)}\right)\left(\frac{d}{dt}\right)$, and $\overline{G}^-_{s(t_{i-1})\to s(t_i)}\left(\frac{d}{dt}\right) := \left(G^-_{s(t_{i-1})\to s(t_i)}M_{s(t_i-1)}\right)\left(\frac{d}{dt}\right)$, with $s \in \mathcal{S}$. Consequently, if w and ℓ_j are related by $w = M_j\left(\frac{d}{dt}\right)\ell_j$, the gluing conditions in (1) can be equivalently written as $\overline{G}^+_{s(t_{i-1})\to s(t_i)}\left(\frac{d}{dt}\right)\ell_{s(t_i)}(t_i^+) = \overline{G}^-_{s(t_{i-1})\to s(t_i)}\left(\frac{d}{dt}\right)\ell_{s(t_i-1)}(t_i^-)$.

Example 3 (Cont'd from Ex. 2**).** Given the gluing conditions in Ex. 1, we can reformulate them in terms of latent variables using $M_1\left(\frac{d}{dt}\right)$ and $M_2\left(\frac{d}{dt}\right)$ as follows.

$$\overline{G}^-_{1\to 2}\left(\frac{d}{dt}\right) := \left(G^-_{1\to 2}M_1\right)\left(\frac{d}{dt}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^\top ,$$

$$\overline{G}^+_{1\to 2}\left(\frac{d}{dt}\right) := \left(G^+_{1\to 2}M_2\right)\left(\frac{d}{dt}\right) = \begin{bmatrix} \frac{d}{dt} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top ,$$

$$\overline{G}^-_{2\to 1}\left(\frac{d}{dt}\right) := \left(G^-_{2\to 1}M_2\right)\left(\frac{d}{dt}\right) = \begin{bmatrix} \frac{d}{dt} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} ,$$

$$\overline{G}^+_{2\to 1}\left(\frac{d}{dt}\right) := \left(G^+_{2\to 1}M_1\right)\left(\frac{d}{dt}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

$\square$

The gluing conditions are in general algebraic constraints that can be freely selected to act at switching instants (see Def. 1), however in order for them to be realistic, they should be *well-defined* and *well-posed*. In order to introduce this concepts, we use the notion of *state maps* recalled in App. I-B.

Definition 3. Let Σ be a SLDS and let $X_j \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_j)\times 1}[\xi]$, induce minimal state maps for $\mathfrak{B}_j := \mathrm{im}\ M_j\left(\frac{d}{dt}\right)$, $j = 1, ..., N$. The gluing conditions are *well-defined* if there exist constant matrices $F^-_{j\to k}$ and $F^+_{j\to k}$, with $j, k = 1, ..., N$, $j \neq k$, such that $\overline{G}^-_{j\to k}(\xi) = F^-_{j\to k}X_j(\xi)$ and $\overline{G}^+_{j\to k}(\xi) = F^+_{j\to k}X_k(\xi)$, with $j, k = 1, ..., N$, $j \neq k$. $\square$

Remark 3. Well-definedness implies that gluing conditions are linear functions of the state of the corresponding modes before and after the switch. Consequently, they do not impose restrictions to the trajectories of the input variables, since the latter must be *free* (see App. I-A). $\square$

Definition 4. Let Σ be a SLDS with $\mathfrak{B}_j := \mathrm{im}\ M_j\left(\frac{d}{dt}\right)$, $j = 1, ..., N$. The well-defined gluing conditions $\mathcal{G} := \{(F^-_{j\to k}X_j(\xi), F^+_{j\to k}X_k(\xi))\}_{j,k=1,...,N,\, j\neq k}$. are *well-posed* if for all $k, j = 1, \ldots, N$ with $k \neq j$, there exists a *re-initialisation* map $L_{j\to k} : \mathbb{R}^{\mathtt{n}(\mathfrak{B}_j)} \to \mathbb{R}^{\mathtt{n}(\mathfrak{B}_k)}$ such that given a switching signal $s \in \mathcal{S}$ such that $s(t_{i-1}) = j$ and $s(t_i) = k$; for all $t_i \in \mathbb{T}_s$ and all admissible $w \in \mathfrak{B}^\Sigma$ with associated latent variable trajectories, it holds that $X_j\left(\frac{d}{dt}\right)\ell_j(t_i^+) = L_{k\to j}X_k\left(\frac{d}{dt}\right)\ell_k(t_i^-)$.

Remark 4. Well-posedness implies that if a transition occurs between $\mathfrak{B}_j$ and $\mathfrak{B}_k$ at t_i, and if an admissible trajectory ends at a "final state" $v_j := X_j\left(\frac{d}{dt}\right)\ell_j(t_i^-)$, then there exists at most one "initial state" for $\mathfrak{B}_k$, defined by $X_k\left(\frac{d}{dt}\right)\ell_k(t_i^+) =: v_k$, compatible with the gluing conditions. In other words, for all $j, k = 1, \ldots, N$, $j \neq k$, $F^+_{j\to k}$ is full column rank, and consequently a *re-initialisation map* can be defined as $L_{j\to k} := F^{+*}_{j\to k}F^-_{j\to k}$, where $F^{+*}_{j\to k}$ is a left inverse of $F^+_{j\to k}$. $\square$

Example 4 (Cont'd Ex. 3**).** We illustrate the modelling of gluing conditions using state maps. Consider

$$X_1\left(\frac{d}{dt}\right) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad X_2\left(\frac{d}{dt}\right) := \begin{bmatrix} \frac{d}{dt} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ;$$

inducing the states $X_1\left(\frac{d}{dt}\right)\ell_1 = \mathrm{col}(i_L, v_2)$ and $X_2\left(\frac{d}{dt}\right)\ell_2 = \mathrm{col}(i_L, v_1, v_2)$. The gluing conditions can be written as

$$\begin{aligned}\mathcal{G} &= \left\{\left(\overline{G}^-_{1\to 2}(\xi), \overline{G}^+_{1\to 2}(\xi)\right), \left(\overline{G}^-_{2\to 1}(\xi), \overline{G}^+_{2\to 1}(\xi)\right)\right\} \\ &= \left\{\left(L_{1\to 2}X_1(\xi), X_2(\xi)\right); \left(L_{2\to 1}X_2(\xi), X_1(\xi)\right)\right\}\end{aligned}$$

where

$$L_{1\to 2} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \;;\quad L_{2\to 1} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} .$$

Note that the gluing conditions are thus well-defined and well-posed according to Def. 3 and Def. 4. □

C. General assumptions

The results presented in this paper are rested on the following standing assumptions.

1) Switching signals are well-defined: We assume that for every $s \in \mathcal{S}$ and for every finite interval of $\mathbb{R}$, there exists only a finite number of switching instants. This is a conventional assumption in switched systems literature (see e.g. [19], [33], [52]) that prevent phenomena such as the *Zeno behaviour*.

2) Compact support trajectories: We often require the integration of functionals acting on $w \in \mathfrak{B}^{\Sigma}$. In order to ensure that such integrals exist, we assume that they involve piecewise infinitely differentiable trajectories of compact support whose set is denoted by $\mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. For this reason we introduce the notation $\mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

3) Inputs and outputs: We consider dynamical modes sharing the same external variable and admitting the same input-output partition $w = \operatorname{col}(u, y)$ (see App. I-A). Note that this consideration includes systems with *ports* and *conjugate variables* (see [26]) e.g. mechanical, electrical, thermodynamical systems, etc.

4) Controllability and observability: We consider switched linear differential systems with *controllable* mode behaviours, i.e. $\mathfrak{B}_j \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, $j = 1, ...N$ (see App. I-A), described by *observable* image form representations (see App. I-A) $w = M_j\left(\frac{d}{dt}\right)\ell_j$, $j = 1, ...N$, with $M_j \in \mathbb{R}^{\mathtt{w}\times 1}[\xi]$. Controllability ensures that compact support trajectories exist (see assumption 2 above), while observability ensures that every trajectory of the latent variable ℓ_j corresponds to a unique trajectory of the external variable w when the j-th mode is active.

III. Dissipative SLDS

Our concept of dissipativity is fundamentally based on that for linear differential systems which is summarised in App. I-D.

In the theory of dissipative linear differential systems [40], [41], [45], [46], the reformulation of QDFs in terms of the latent variable as in App. I-C is often used since it simplifies certain computations such as positivity tests. Given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, if w and ℓ are related by $w = M\left(\frac{d}{dt}\right)\ell$, defining $\Phi'(\zeta, \eta) := M(\zeta)^{\top}\Phi(\zeta, \eta)M(\eta)$ implies $Q'_{\Phi}(\ell) = Q_{\Phi}(w)$, and consequently it follows e.g. that $Q_{\Phi} \geq 0$ for all $w \in \mathfrak{B}$ (shortly $Q_{\Phi} \overset{\mathfrak{B}}{\geq} 0$) if and only if $Q_{\Phi'} \geq 0$ on $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{1})$. Since in this work we deal with $\mathfrak{B}_i \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, $i = 1, ..., N$, associated to observable image form representations $w = M_i\left(\frac{d}{dt}\right)\ell_i$, $i = 1, ..., N$, we adopt the notation Q_{Φ} to refer to a QDF acting on the external variable and $Q_{\Phi'}$, to denote the associated QDFs acting on the latent variables.

A. Dissipativity

In order to introduce the main definition and results about dissipative SLDS, we define the following notation.

Let $s \in \mathcal{S}$ be a fixed but otherwise arbitrary switching signal, whose associated set of switching instants is $\mathbb{T}_s := \{t_1, t_2, ..., t_n, ...\}$. We denote by $|\mathbb{T}_s|$ the total number of switching instants in $\mathbb{T}_s$.

If $|\mathbb{T}_s| = \infty$, define $\int Q_{\Phi}(w) := \int_{-\infty}^{t_1^-} Q_{\Phi}(w)\,dt + \int_{t_1^+}^{t_2^-} Q_{\Phi}(w)\,dt + ... + \int_{t_n^+}^{t_{n+1}^-} Q_{\Phi}(w)\,dt + ...$; and $\int \|w\|_2^2 := \int_{-\infty}^{t_1^-} \|w\|_2^2\,dt + \int_{t_1^+}^{t_2^-} \|w\|_2^2\,dt + ... + \int_{t_n^+}^{t_{n+1}^-} \|w\|_2^2\,dt + ...$.

If $0 < |\mathbb{T}_s| < \infty$, then define $\int Q_{\Phi}(w) := \int_{-\infty}^{t_1^-} Q_{\Phi}(w)\,dt + \sum_{k=2}^{|\mathbb{T}_s|}\int_{t_{k-1}^+}^{t_k^-} Q_{\Phi}(w)\,dt + \int_{t_{|\mathbb{T}_s|}^+}^{\infty} Q_{\Phi}(w)\,dt$; and $\int \|w\|_2^2 := \int_{-\infty}^{t_1^-} \|w\|_2^2\,dt + \sum_{k=2}^{|\mathbb{T}_s|}\int_{t_{k-1}^+}^{t_k^-} \|w\|_2^2\,dt + \int_{t_{|\mathbb{T}_s|}^+}^{\infty} \|w\|_2^2\,dt$.

If $|\mathbb{T}_s| = 0$, i.e. no switching takes place, then $\int Q_{\Phi}(w) := \int_{-\infty}^{+\infty} Q_{\Phi}(w)\,dt$; and $\int \|w\|_2^2 := \int_{-\infty}^{+\infty} \|w\|_2^2\,dt$.

Moreover, given a trajectory $w \in \mathfrak{B}^{\Sigma}$, we denote the switching signal associated to it (see Def. 2) as s_w.

Definition 5. Let Σ be a SLDS and let Q_{Φ} be a QDF. Σ is Φ*-dissipative* if $\int Q_{\Phi}(w) \geq 0$ for all $w \in \mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$; and *strictly* Φ*-dissipative* if there exists $\epsilon > 0$ such that $\int Q_{\Phi}(w) \geq \epsilon \int \|w\|_2^2$ holds for all $w \in \mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

In the previous definition, the quadratic differential form Q_{Φ} can be interpreted as power, consequently, its integral over the real line measures the energy that is being supplied to, or flows out from the SLDS. If the net flow of energy is nonnegative then we call the SLDS Φ-dissipative.

Remark 5. The definition of dissipativity is not uniform in the literature for switched/hybrid systems. For instance, in [51] *multiple-* and *cross-supply rates* are considered to characterise the energy change of inactive modes for the case when they share the same state space. A similar concept is used in [15], where *connective* *supply rates* are used. These definitions permit the modelling of dynamical modes with different inputs, which is a suitable approach in cases such as multi-controller control systems. In our definition, we consider the use of a *main supply rate* acting on a fixed external variable for modes that do not necessarily share the same state-space. This definition is most suitable for the study of switched systems whose variables of interest are fixed, consequently the modes interchange energy with the environment in the same manner for every mode e.g. by means of ports. $\square$

Proposition 1. *Let Σ be a SLDS. If Σ is (strictly) Φ-dissipative according to Def. 5, then $\mathfrak{B}_i$, $i = 1, ...N$, are (strictly) Φ-dissipative linear differential behaviours.*

Proof: See App. II. $\blacksquare$

In the following result we use the notion of *storage function* for linear differential behaviours (see App. I-D).

Proposition 2. *Let Σ be a (strictly) Φ-dissipative SLDS. For all $i \in \mathcal{P}$ there exists a QDF Q_{Ψ_i} that is a storage function for $\mathfrak{B}_i$. Let $a < b$, then for all $w \in \mathfrak{B}^{\Sigma}$ with $s_w(t) = i$ for $t \in [a, b]$, it holds that $\int_a^b Q_\Phi(w)\ dt \geq Q_{\Psi_i}(w)(b) - Q_{\Psi_i}(w)(a)$.*

Proof: See App. II. $\blacksquare$

Remark 6. QDFs act on $\mathfrak{C}^\infty$-functions, while trajectories of a SLDS are non-differentiable; however, this mismatch in differentiability is irrelevant to the results of this paper. We use the calculus of QDFs as an *algebraic* tool, considering only their value before and after a switch. $\square$

B. Multiple storage functions

As discussed in the literature (see e.g. [15], [51]), the use of a global storage function for all dynamical modes of a dissipative switched system is not only conservative but also not supported by physical considerations. Note for instance that physical switched systems may have different ways to store energy depending on the mode that is active.

Example 5 (Cont'd from Ex. 1)**.** Consider the electrical circuit in Fig. 1 and its associated dynamical modes (2)-(3). Following first principles, the stored energy for each mode is given by the QDFs $Q_{\Psi_1}(w) := \frac{1}{2}Li_L^2 + \frac{1}{2}(C_1+C_2)v_2^2$ and $Q_{\Psi_2}(w) := \frac{1}{2}Li_L^2 + \frac{1}{2}C_1(E + R_Li_L - L\frac{d}{dt}i_L)^2 + \frac{1}{2}C_2v_2^2$. $\square$

When switching between modes, the trajectories of w are in general subject to algebraic constraints modelled via the gluing conditions. Consequently, the transition between storage functions becomes of interest in dissipative systems. The second law of thermodynamics prevents stored energy in a dissipative system to increase at switching instants, since the process of dissipation cannot be reversed and energy is strictly provided by external sources characterised by the supply rate. Consequently any change in the physical stored energy must be accounted necessarily as energy losses. This point of view has been elaborated in [25] where the analysis of a wide variety of physical systems exhibiting discontinuities is presented; the same principle is also discussed in [7], [10], [35]. This energy condition is also used for a definition of passivity for hybrid systems in [48], Prop. 1, where the nonincreasing condition for *multiple Lyapunov functions* introduced in [2] is used for multiple storage functions. Here we illustrate such condition for dissipative systems from a physical point of view using the power converter in Fig. 1.

Example 6 (Cont'd from Ex. 5)**.** Let us compute the changes in stored energy of the circuit at a switching instant t_i. Taking into account the gluing conditions in Ex. 1 and after some straightforward computations, the change in stored energy when switching respectively from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ and vice versa can be computed as $Q_{\Psi_1}(w)(t_i^-) - Q_{\Psi_2}(w)(t_i^+) = 0$, i.e. there is no loss; and $Q_{\Psi_2}(w)(t_i^-) - Q_{\Psi_1}(w)(t_i^+) = \frac{1}{4}\left(E(t_i^-) + i_L(t_i^-) - \frac{d}{dt}i_L(t_i^-) - v_2(t_i^-)\right)^2$; evidently the latter quantity is nonnegative implying that the circuit loses energy. $\square$

Definition 6. Let Σ be a SLDS and let $s \in \mathcal{S}$. An N-tuple $(Q_{\Psi_1}, ..., Q_{\Psi_N})$ is a *multiple storage function* for Σ with respect to Q_Φ if

1) $\frac{d}{dt}Q_{\Psi_i} \overset{\mathfrak{B}_i}{\leq} Q_\Phi$, $i = 1, ..., N$.
2) $\forall\ w \in \mathfrak{B}^{\Sigma}$ s.t. $s = s_w$ and $\forall\ t_k \in \mathbb{T}_s$, it holds $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0$.

Remark 7. In condition 1) of Def. 6 we require each mode behaviour to be Φ-dissipative (see App. I-D) which is equivalent to Q_{Ψ_i} satisfying the dissipation inequality for the i-th mode. In condition 2) we require that the storage function does not increase when we switch from one mode to another: switching cannot increase the amount of stored energy in the system. $\square$

Theorem 1. *Let Σ be a SLDS and let Q_Φ be a QDF. Assume that there exists a multiple storage function as in Def. 6. Then Σ is Φ-dissipative.*

Proof: See App. II. $\blacksquare$

A multiple storage function is not necessarily unique, morever the set of all possible multiple storage functions is a convex set.

Proposition 3. *Let Σ be a Φ-dissipative SLDS. Let the N-tuples $Q_{\mathbf{\Psi}} := (Q_{\Psi_1}, ..., Q_{\Psi_N})$ and $Q_{\mathbf{\Psi}'} := \left(Q_{\Psi'_1}, ..., Q_{\Psi'_N}\right)$ be multiple storage functions for Σ. Then, for all $0 \geq \alpha \geq 1$, the N-tuple $\alpha Q_{\mathbf{\Psi}} + (1-\alpha) Q_{\mathbf{\Psi}'}$, is a multiple storage function for Σ.*

Proof: See App. II. ■

In Th. 1, we proved that the existence of a multiple storage function as in Def. 6 is a sufficient condition for dissipativity. In the classical theory for linear differential behaviours, dissipativity is actually equivalent to the existence of a storage function (see Prop. 8, in App. I-D). In the following we show that if Φ is a constant matrix, then strict Φ-dissipativity implies the existence of a multiple storage function for SLDS.

Theorem 2. *Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ and let Σ be a strictly Φ-dissipative SLDS with $\mathcal{G}$ well-defined and well-posed, and with mode behaviours $\mathfrak{B}_k$, $k = 1, ..., N$. There exist storage functions Q_{Ψ_i}, $i = 1, ...N$, for the linear differential behaviours $\mathfrak{B}_k$, $i = 1, ..., N$, with respect to Q_Φ, such that for all $t_k \in \mathbb{T}_s$ and for all $i, j \in \mathcal{P}$, $i \neq j$, it holds that $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0$. Consequently, $(Q_{\Psi_1}, ..., Q_{\Psi_N})$ is a multiple storage function for Σ.*

Proof: See App. II. ■

Derived from strict dissipativity and the fact that for constant supply rates, storage functions are quadratic functions of the state (see Prop. 9 in App. I-D), we can construct an LMI equivalent with condition 2) in Def. 6.

Lemma 1. *Under the assumptions of Th. 2, let ℓ_i, $i = 1, ..., N$, be unique latent variable trajectories associated with the external variable, i.e. $w = M_i\left(\frac{d}{dt}\right)\ell_i$, $i = 1, ..., N$. Let $X_i \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_i) \times \mathtt{l}}[\xi]$ induce minimal state maps for $\mathfrak{B}_i$, $i = 1, ..., N$, and let $Q_{\Psi'_i}(\ell_i) = Q_{\Psi_i}(w)$, $i = 1, ..., N$. Let $L_{i \to j} \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_j) \times \mathtt{n}(\mathfrak{B}_i)}$ for all $i, j \in \mathcal{P}$, $i \neq j$, be the re-initialisations maps. There exist $K_i = K_i^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_i) \times \mathtt{n}(\mathfrak{B}_i)}$, such that $\Psi'_i(\zeta, \eta) = X_i(\zeta)^\top K_i X_i(\eta)$, $i = 1, ..., N$.*

Moreover, the following conditions are equivalent: for all $w \in \mathfrak{B}^\Sigma$, $t_k \in \mathbb{T}_s$ and $i, j \in \mathcal{P}$, $i \neq j$,

1) $Q_{\Psi_i}(w)(t_k^-) \geq Q_{\Psi_j}(w)(t_k^+)$.
2) $Q_{\Psi'_i}(\ell_i)(t_k^-) \geq Q_{\Psi'_j}(\ell_j)(t_k^+)$.
3) $K_i \geq L_{i \to j}^\top K_j L_{i \to j}$.

Proof: See App. II. ■

An important consequence of Lemma 1 is the following result.

Proposition 4. *Under the assumptions of Th. 2 and Lemma 1, if the re-initialisation maps $L_{i \to j}$ associated to the switching between $\mathfrak{B}_i$ to $\mathfrak{B}_j$, for all $i, j \in \mathcal{P}$ and $i \neq j$, are the identity, there exists Q_Ψ such that $(Q_\Psi, Q_\Psi, ..., Q_\Psi)$ is a multiple storage function for Σ with respect to Q_Φ.*

Proof: See App. II. ■

As a special case of Th. 2, Prop. 4 can be interpreted in the following way. If the mode behaviours share the same state space and the state trajectories are continuous at switching instants, strict dissipativity implies the existence of a *common storage function* Q_Ψ for *open* systems. Consequently, note that this result is analogous to the *converse Lyapunov theorem* (see Th. 2.2 of [19], p. 25), where asymptotic stability implies the existence of a *common Lyapunov function* for *closed* systems under analogous conditions.

C. Half-line dissipativity

When the energy absorbed by a SLDS is positive in any arbitrary interval of time, we call such SLDS *half-line dissipative*.

In order to introduce the definition and results regarding half-line dissipativity, we use the following notation. Let $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and $\tau \in \mathbb{R}$. Let $s = s_w \in \mathcal{S}$ whose associated set of switching instants is $\mathbb{T}_s := \{t_1, t_2, ..., t_n, ...\}$, we define

$$\int^\tau Q_\Phi(w)\, dt := \int_{-\infty}^{t_1^-} Q_\Phi(w)\, dt + \sum_{k=2}^{n} \int_{t_{k-1}^+}^{t_k^-} Q_\Phi(w)\, dt + \int_{t_n^+}^{\tau} Q_\Phi(w)\, dt\ ; \tag{6}$$

where $n = \max\{k \mid t_k \in \mathbb{T}_s,\ \text{and}\ t_k \leq \tau\}$.

Definition 7. Let Q_Φ be a QDF. A SLDS Σ is *half-line Φ-dissipative* if for every $\tau \in \mathbb{R}$ and for all $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, it holds that $\int^\tau Q_\Phi(w) \geq 0$.

Half-line dissipativity appears very frequently in physical systems. For instance, in n-port driven electrical circuits we can select a external variable $w := \mathrm{col}(V, I)$ consisting of a vector of voltages $V := \mathrm{col}(V_1, ..., V_n)$ and currents $I := \mathrm{col}(I_1, ..., I_n)$ across and through the ports. We thus say that the circuit is passive if for the supply rate defined as $Q_\Phi(w) := V^\top I$, it follows that for all τ and for all the trajectories of w with compact support $\int_{-\infty}^\tau Q_\Phi(w)\, dt \geq 0$ (cf. the classical definitions in [1], [26]).

Proposition 5. *Let Σ be a SLDS. If Σ is half-line Φ-dissipative, then $\mathfrak{B}_i$, $i = 1, ...N$, are half-line Φ-dissipative linear differential behaviours.*

Proof: The proof of the proposition follows readily from the same argument used in Prop. 1. ■

Consider now the following proposition regarding half-line dissipativity of SLDS. We consider the case when the *liveness condition* is satisfied (see [46], sec. IV-B), namely, given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $w = \mathrm{col}(u, y) \in \mathfrak{B}^{\Sigma}$, the number of components in the input u, denoted by $\mathtt{m}(\mathfrak{B}^{\Sigma})$, equals the number of positive eigenvalues of Φ, denoted by $\sigma_+(\Phi)$.

Theorem 3. *Let Σ be a SLDS and let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume that $\sigma_+(\Phi) = \mathtt{m}(\mathfrak{B}^{\Sigma})$. If there exists a multiple storage function as in Def. 6, then Σ is half-line Φ-dissipative.*

Proof: See App. II. ■

IV. Computation of multiple storage functions

In this section, we develop procedures based on LMIs to compute multiple storage functions. We first introduce results that provide conditions based on LMIs for the existence of a storage function for linear differential behaviours. For practical purposes, we consider the two-variable polynomial matrix version $\Phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta) + \Delta(\zeta, \eta)$ of the dissipation equality $Q_\Phi = \frac{d}{dt}Q_\Psi + Q_\Delta$ (see App. I-D, Prop. 8).

Proposition 6. *Let $M \in \mathbb{R}^{\mathtt{w}\times 1}[\xi]$ be defined as $M = col(U, Y)$, such that YU^{-1} is strictly proper. Let $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times 1}[\xi]$ be a minimal state map for* $\mathrm{im}\, M\left(\frac{d}{dt}\right)$. *Write $M(\xi) = \sum_{i=0}^{L} M_i \xi^i$, with $M_i \in \mathbb{R}^{\mathtt{w}\times 1}$, $i = 0, \ldots, L$; then there exist $X_i \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times 1}$, $i = 0, 1, ..., L-1$, such that $X(\xi) = \sum_{i=0}^{L-1} X_i \xi^i$.*

Proof: See App. II. ■

Proposition 7. *Under the assumptions of Prop. 6, let $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Define $\widetilde{M} := \begin{bmatrix} M_0 & \ldots & M_L \end{bmatrix}$ and $\widetilde{X} := \begin{bmatrix} X_0 & \ldots & X_{L-1} \end{bmatrix}$. Let $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$. The following statements are equivalent:*

1) *$\Psi(\zeta, \eta) := X(\zeta)^\top K X(\eta)$ and $\Delta \in \mathbb{R}^{1\times 1}[\zeta, \eta]$ satisfy $\Delta(\zeta, \eta) = M(\zeta)^\top \Phi M(\eta) - (\zeta + \eta)\Psi(\zeta, \eta)$;*
2) $\widetilde{\Delta} := \widetilde{M}^\top \Phi \widetilde{M} - \begin{bmatrix} 0_{1\times\mathtt{n}(\mathfrak{B})} \\ \widetilde{X}^\top \end{bmatrix} K \begin{bmatrix} \widetilde{X} & 0_{\mathtt{n}(\mathfrak{B})\times 1} \end{bmatrix} - \begin{bmatrix} \widetilde{X}^\top \\ 0_{1\times\mathtt{n}(\mathfrak{B})} \end{bmatrix} K \begin{bmatrix} 0_{\mathtt{n}(\mathfrak{B})\times 1} & \widetilde{X} \end{bmatrix}$.

Proof: See App. II. ■

Lemma 2. *Under the assumptions of Prop. 6, let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and define $\mathfrak{B} := \mathrm{im}\, M\left(\frac{d}{dt}\right)$. Assume that $\mathfrak{B}$ is Φ-dissipative. Then there exists $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ such that any of the statements 1) and 2) in Prop. 7 holds, and moreover $Q_\Delta \geq 0$ or equivalently $\widetilde{\Delta} \geq 0$.*

Proof: See App. II. ■

The results in Lemma 2 permit to transform the computation of storage functions into solving the expression 2) in Prop. 7 as an LMI, i.e. $\widetilde{\Delta} \geq 0$, involving coefficient matrices that can be straightforwardly set up from the equations describing the laws of the system. We now provide an analogous result for multiple storage functions.

Theorem 4. *Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and let Σ be a SLDS with $\mathcal{G}$ well-defined and well-posed. Let $\mathfrak{B}_k := \mathrm{im}\, M_k\left(\frac{d}{dt}\right)$, with $M_k \in \mathbb{R}^{\mathtt{w}\times 1}[\xi]$, $k = 1, ...N$, be strictly Φ-dissipative. Let $X_k \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_k)\times 1}[\xi]$ be a minimal state map for $\mathfrak{B}_k$, $i = 1, ..., N$, and let $L_{i\to j} \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_j)\times\mathtt{n}(\mathfrak{B}_i)}$ for all $i, j \in \mathcal{P}$, $i \neq j$, be the re-initialisations maps of Σ. Denote the coefficient matrix of $M_k(\xi)$ by $\widetilde{M}_k := \begin{bmatrix} M_{k,0} & \ldots & M_{k,L_k} \end{bmatrix}$; then that of $X_k(\xi)$ can be written as $\widetilde{X}_k := \begin{bmatrix} X_{k,0} & \ldots & X_{k,L_k-1} \end{bmatrix}$.*

There exist $K_k = K_k^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_k)\times\mathtt{n}(\mathfrak{B}_k)}$, $k = 1\ldots, N$, such that

$$\widetilde{M}_k^\top \Phi \widetilde{M}_k - \begin{bmatrix} 0_{1\times\mathtt{n}(\mathfrak{B}_k)} \\ \widetilde{X}_k^\top \end{bmatrix} K_k \begin{bmatrix} \widetilde{X}_k & 0_{\mathtt{n}(\mathfrak{B}_k)\times 1} \end{bmatrix} - \begin{bmatrix} \widetilde{X}_k^\top \\ 0_{1\times\mathtt{n}(\mathfrak{B}_k)} \end{bmatrix} K_k \begin{bmatrix} 0_{\mathtt{n}(\mathfrak{B}_k)\times 1} & \widetilde{X}_k \end{bmatrix} \geq 0\,. \tag{7}$$

Moreover, if for $k, j = 1, \ldots, N$, $k \neq j$, it holds that

$$K_k - L_{k\to j}^\top K_j L_{k\to j} \geq 0\,, \tag{8}$$

then $(\Psi_k(\zeta, \eta) := X_k(\zeta)^\top K_k X_k(\eta))_{k=1,\ldots,N}$ induces a multiple storage function for Σ, and Σ is Φ-dissipative.

Proof: See App. II. ■

Theorem 4 reduces the computation of multiple storage functions to the solution of a system of structured LMIs (7)-(8), a straightforward matter for standard LMI solvers.

Example 7. Consider the switched electrical circuit in Fig. 2. The switching occurs when at an arbitrary instant of time, the inductor L_2 is connected. We select $w := \mathrm{col}(V, i_1)$ as the external variables. For simplicity we consider $C_1 = 1\,F$, $L_1 = 1\,H$, and $L_2 = 1\,H$.

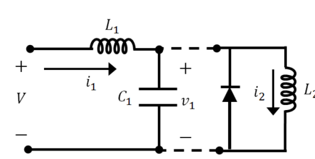

Fig. 2. Switched electrical circuit.

• *Mode behaviours:* By applying first principles we can model the mode behaviours $\mathfrak{B}_i$, $i = 1, 2$, which can be easily verified to be controllable. Moreover, they can be described by observable image representations $w =$

$M_i\left(\frac{d}{dt}\right)\ell_i$, $i=1,2$, with $M_1\left(\frac{d}{dt}\right):=\begin{bmatrix}\frac{d^2}{dt^2}+1\\ \frac{d}{dt}\end{bmatrix}$, $\ell_1:=v_1$, for the first mode; and $M_2\left(\frac{d}{dt}\right):=\begin{bmatrix}\frac{d^3}{dt^3}+2\frac{d}{dt}\\ \frac{d^2}{dt^2}+1\end{bmatrix}$, $\ell_2:=i_2$, when the inductor is connected.

• *Gluing conditions:* We consider the state maps acting on the latent variables induced by $X_1(\xi):=\begin{bmatrix}1 & \xi\end{bmatrix}^\top$ and $X_2(\xi):=\begin{bmatrix}\xi & \xi^2 & 1\end{bmatrix}^\top$. The physics of the circuit imposes that for every $t_k\in\mathbb{T}_s$, the gluing conditions can be expressed as $X_2\left(\frac{d}{dt}\right)i_2(t_k^+)=L_{1\to 2}X_1\left(\frac{d}{dt}\right)v_1(t_k^-)$ and $X_1\left(\frac{d}{dt}\right)v_1(t_k^+)=L_{2\to 1}X_2\left(\frac{d}{dt}\right)i_2(t_k^-)$, where $L_{1\to 2}^\top:=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}=:L_{2\to 1}$.

• *LMI conditions:* Define $\Phi:=\frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, corresponding to the supply rate $Q_\Phi(w)=Vi$. Based on Th. 4, we construct the LMIs (7) and (8) for this case. Then using standard LMI solvers we obtain

$$K_1:=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix}\ ;\quad K_2:=\begin{bmatrix}0.500 & 0 & 0\\ 0 & 0.500 & 0\\ 0 & 0 & 0.577\end{bmatrix}.$$

Thus, $(X_i(\zeta)^\top K_i X_i(\eta))_{i=1,2}$ induces a multiple storage function for the SLDS. Note that the system is thus dissipative according to Prop. 1 and in fact half-line dissipative according to Th. 3, since $\sigma_+(\Phi)=\mathtt{m}(\mathfrak{B}^\Sigma)=1$. □

V. Passivity

In this section we study passive SLDS, i.e. dissipative with respect to the positive-real supply rate, i.e. $\Phi:=\frac{1}{2}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$. In particular, we are interested in determining under which circumstances a multiple storage function is also a *multiple Lyapunov function* in the sense of Sec. III of [24].

We call a SLDB $\mathfrak{B}^\Sigma$ *asymptotically stable* if $w\in\mathfrak{B}^\Sigma \implies \lim_{t\to\infty}w(t)=0$. Since constant switching signals are admissible, it follows from this definition that for a SLDS to be asymptotically stable, all mode behaviours $\mathfrak{B}_i$ must be asymptotically stable and consequently autonomous (see [24], sec. III). Consider the following theorem obtained from [24] (Th. 1, Sec. III, p.2042).

Theorem 5. *Let Σ be a SLDS with autonomous mode behaviours $\mathfrak{B}_i\in\mathfrak{L}^{\mathtt{w}}$, $i=1,...,N$. Let $s=s_w\in\mathcal{S}$ be an admissible switching signal. Assume that there exist QDFs Q_{Ψ_i}, $i=1,...,N$, such that*

1. $Q_{\Psi_i}\overset{\mathfrak{B}_i}{\geq}0$, $i=1,...,N$;
2. $\frac{d}{dt}Q_{\Psi_i}\overset{\mathfrak{B}_i}{<}0$, $i=1,...,N$;
3. $\forall\ w\in\mathfrak{B}^\Sigma$ *and* $\forall\ t_j\in\mathbb{T}_s$, $Q_{\Psi_{s(t_{j-1})}}(w)(t_j^-)\geq Q_{\Psi_{s(t_j)}}(w)(t_j^+)$.

Then Σ is asymptotically stable .

Proof: See [24], Th. 1, Sec. III, p.2042. ■

An N-tuple $(Q_{\Psi_1},...,Q_{\Psi_N})$ in Th. 5 is called a *multiple Lyapunov function*, and its computation has been presented in [24].

In a dissipative SLDS the external variables include inputs, consequently the modes are not autonomous. However, we can associate to Σ an autonomous SLDS as follows.

Definition 8. Let $\Sigma:=\{\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G}\}$ be a SLDS with switched behaviour $\mathfrak{B}^\Sigma$ and $w=\mathrm{col}(u,y)$. The *unforced* SLDS Σ_{aut} associated to Σ is defined as $\Sigma_{\text{aut}}:=\{\mathcal{P},\mathcal{F},\mathcal{S},\mathcal{G}\}$, with switched behaviour $\mathfrak{B}^\Sigma_{\text{aut}}:=\{w=\mathrm{col}(u,y)\in\mathfrak{B}^\Sigma\mid u=0\}$.

Note that $\mathfrak{B}^\Sigma_{\text{aut}}$ is not empty, since it contains at least the zero trajectory $w=0$. The following proposition deals with asymptotic stability of unforced SLDS as in Def. 8.

Theorem 6. *Let Σ and Σ_{aut} be as in Def. 8 and let $\Phi:=\frac{1}{2}\begin{bmatrix}0 & I_1\\ I_1 & 0\end{bmatrix}$. Assume that Σ is strictly Φ-dissipative, then Σ_{aut} is asymptotically stable.*

Proof: See App. II. ■

VI. Stabilisation of power converters with constant power loads

We illustrate an application to our results to the problem of stabilisation of switching power converters (see e.g. [6]). We study networks consisting of a *source power converter* feeding a *constant power load*, which is a potential destabiliser in energy distribution networks since it is not a passive element and its interconnection with power converters may lead to an unstable operation (see [47]).

We now show that this problem can be solved using the dissipativity concepts studied in this paper. Consider the power converter in Fig. 3 which corresponds to a *boost converter* and the local approximation of a constant power load consisting of a negative resistor and a constant current source (see [29], Sec. II). In order to prevent instability, we consider the realisation of a *passive filter* (a circuit consisting of inductors, capacitors and resistors) that when connected to the switched network as depicted in Fig. 4, the overall circuit results in a passive system.

We model the filter using a 1-port admittance function $Y(\xi)=\frac{q(\xi)}{p(\xi)}$, with $p,q\in\mathbb{R}[\xi]$, or equivalently an image

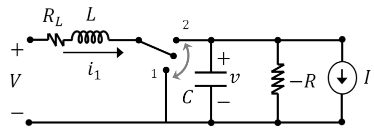

Fig. 3. Boost converter with a constant power load

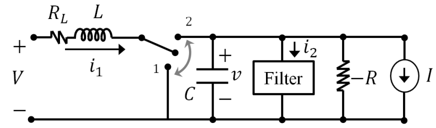

Fig. 4. Filter interconnection

representation

$$\begin{bmatrix} v \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} p\left(\frac{d}{dt}\right) \\ q\left(\frac{d}{dt}\right) \end{bmatrix}}_{=:M'\left(\frac{d}{dt}\right)} \ell' \ ,$$

where ℓ' is a latent variable associated to the internal dynamics of the filter and whose physical meaning can be determined after the realisation. Applying fundamental current and voltage laws, the mode dynamics when the switch is in position 1 are described by

$$\begin{aligned} L\frac{d}{dt}i_1 &= V - R_L i_1 \ , \\ C\frac{d}{dt}v &= -q\left(\frac{d}{dt}\right)\ell' + \frac{1}{R}v + I \ . \end{aligned}$$

When the switch is in position 2, the mode dynamics are described by

$$\begin{aligned} L\frac{d}{dt}i_1 &= V - R_L i_1 - v \ , \\ C\frac{d}{dt}v &= i_1 - q\left(\frac{d}{dt}\right)\ell' + \frac{1}{R}v + I \ . \end{aligned}$$

By selecting the external and latent variables as $w := \begin{bmatrix} V & I & i_1 & v \end{bmatrix}^\top$ and $\ell := \begin{bmatrix} i_1 & \ell' \end{bmatrix}^\top$, the mode dynamics can be modelled using image form representations $w = M_k\left(\frac{d}{dt}\right)\ell$, $k = 1, 2$, where

$$M_1\left(\frac{d}{dt}\right) := \begin{bmatrix} L\frac{d}{dt} + R_L & 0 \\ 0 & -\left(\frac{1}{R} - C\frac{d}{dt}\right)p\left(\frac{d}{dt}\right) + q\left(\frac{d}{dt}\right) \\ 1 & 0 \\ 0 & p\left(\frac{d}{dt}\right) \end{bmatrix}$$

$$M_2\left(\frac{d}{dt}\right) := \begin{bmatrix} L\frac{d}{dt} + R_L & p\left(\frac{d}{dt}\right) \\ -1 & -\left(\frac{1}{R} - C\frac{d}{dt}\right)p\left(\frac{d}{dt}\right) + q\left(\frac{d}{dt}\right) \\ 1 & 0 \\ 0 & p\left(\frac{d}{dt}\right) \end{bmatrix}$$

Based on Th. 4 we can construct the following set of matrix inequalities:

$$\begin{aligned} &\widetilde{M}_i^\top \Phi \widetilde{M}_i - \begin{bmatrix} 0_{1\times \mathtt{n}(\mathfrak{B}_i)} \\ \widetilde{X}_i^\top \end{bmatrix} K_i \begin{bmatrix} \widetilde{X}_i & 0_{\mathtt{n}(\mathfrak{B}_i)\times 1} \end{bmatrix} \\ &- \begin{bmatrix} \widetilde{X}_i^\top \\ 0_{1\times \mathtt{n}(\mathfrak{B}_i)} \end{bmatrix} K_i \begin{bmatrix} 0_{\mathtt{n}(\mathfrak{B}_i)\times 1} & \widetilde{X}_i \end{bmatrix} \geq 0 \ ; \ \ i = 1, 2 \ , \end{aligned} \tag{9}$$

with

$$\Phi := \frac{1}{2}\begin{bmatrix} 0_{2\times 2} & I_2 \\ I_2 & 0_{2\times 2} \end{bmatrix} \ ,$$

Note that the matrices X_i, $i = 1, 2$, correspond to the coefficient matrices of state maps for the dynamical modes and $\widetilde{M}_i$, $i = 1, 2$, depend on the unknown coefficients associated to the filter. Moreover classic results in circuit theory show that a transfer function can be realisable as a passive circuit if and only if it is positive-real [26], i.e. the filter must be dissipative with respect to

$$\Phi' = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ ,$$

hence we add the following inequality

$$\begin{aligned} &\widetilde{M}'^\top \Phi' \widetilde{M}' - \begin{bmatrix} 0_{1\times \deg(p)} \\ \widetilde{X}'^\top \end{bmatrix} K' \begin{bmatrix} \widetilde{X}' & 0_{\deg(p)\times 1} \end{bmatrix} \\ &- \begin{bmatrix} \widetilde{X}'^\top \\ 0_{1\times \deg(p)} \end{bmatrix} K' \begin{bmatrix} 0_{\deg(p)\times 1} & \widetilde{X}' \end{bmatrix} \geq 0 \ . \end{aligned} \tag{10}$$

The existence of solutions for (9)-(10) guarantees that the SLDS is Φ-dissipative under arbitrary switching signals. Note that (9)-(10) is not a set of LMIs, but a set of *bilinear matrix inequalities (BMIs)*, for which sub-optimal solutions can be obtained using standard softwares such as `Yalmip`. However for simplicity of exposition, we fix the coefficients of the polynomial p so as to achieve LMIs. Such choice can be justified observing that often we can set poles/zeros for the admittance function whose patterns are related to important features of the filter dynamics such as time response, characteristic frequencies and the structure of the electrical circuit to be obtained, which are essential filter design considerations (see sec. 5-8 of [34]).

In order to show a numeric solution for the present example, set the values $R_L = 0.01\Omega$, $L = 1000\mu H$, $C = 50\mu F$ and $-R = -500\Omega$. Define $p(\xi) := \xi^2 + 295\xi + 13200$, and let $q(\xi) := a_2\xi^2 + a_1\xi + a_0$ be the numerator of the admittance function. The selected purely real poles correspond to the realisation of an RC low-pass circuit (see [34], sec. 3). Compute state maps for $\mathfrak{B}_i$, $i = 1, 2$, e.g. $X_1(\xi) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \xi & \xi^2 \end{bmatrix}^\top =: X_2(\xi)$. According to the physics of the switched circuit, the re-initialisation maps are given by $L_{1\to 2} = L_{2\to 1} = I_4$. Similarly, a state map associated to the passive filter can be computed as $X'\left(\frac{d}{dt}\right) := \begin{bmatrix}1 & \xi\end{bmatrix}^\top$. Note that since the re-initialisation maps are equal to the identity, according to Prop. 4 the storage function is unique and the set of LMIs (8) can been omitted. Using standard LMI solvers for this choice of p, we can compute the coefficients of the polynomial q, consequently $q(\xi) = 102\xi^2 + 40677\xi + 3030456$. Note that in the present example, the polynomials p and q are coprime, consequently the controllability and observability assumptions used in this paper are satisfied. In order to ensure that this condition is satisfied in general, an additional easy-to-construct condition based on the *Sylvester resultant* can be used (see e.g. [28], p. 191).

Finally, the filter with admittance

$$Y(\xi) = \frac{102\xi^2 + 40677\xi + 3030456}{\xi^2 + 295\xi + 13200} ,$$

can be realised using any suitable method of circuit synthesis, e.g. using the *Cauer method* (see Sec. 6 of [34]) we obtain the circuit in Fig. 5

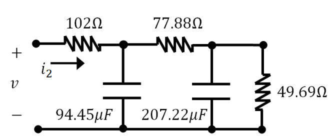

Fig. 5. Filter realisation.

Remark 8. We can also determine the physical realisation of the auxiliary variable ℓ', associated to the filter and used for the modelling of the dynamical modes. In order to do so note that according to Cor. 2.5.12 of [28], since p and q are coprime, then there exists $a, b \in \mathbb{R}[\xi]$ such that $p(\xi)a(\xi) + q(\xi)b(\xi) = 1$. Since the coefficients of p and q are known, we can easily compute $a(\xi) = 1.524 \times 10^{-6} + 112.148 \times 10^{-6}\xi$ and $b(\xi) = -274.224 \times 10^{-6} - 1.143 \times 10^{-6}\xi$. Finally, recalling Remark 6.4.11 of [28], we can conclude that since $v = p\left(\frac{d}{dt}\right)\ell'$ and $i_2 = q\left(\frac{d}{dt}\right)\ell'$, then $\ell' = a\left(\frac{d}{dt}\right)v + b\left(\frac{d}{dt}\right)i_2$. $\square$

Remark 9. Our method can be regarded as a modification of the *output impedance* of the power converter (see Sec. 7 of [53]), in our case achieved by adding a passive filter to the output stage of the converter. Note also that our method concides with the "physical interpretation" of the output impedance control based on a feedback scheme provided in Sec. 8 of [53], where an admittance-like function is associated with controller gains. $\square$

VII. Conclusions

We presented a theory of dissipativity for switched systems in which the dynamical modes are not described in state space form, and do not necessarily share a common state space. We provided necessary and sufficient conditions for the existence of multiple storage functions, and a method to compute them using sets of LMIs. We studied the notion of passivity as a special case as well as its relationship with stability. We also showed an application of our approach to the stabilisation problem of a power converter with a constant power load.

Appendix I
Background material

A. Linear differential behaviours

$\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is a *linear time-invariant differential behaviour* if it is the set of solutions of a finite system of constant-coefficient linear differential equations, i.e. if there exists $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\} =: \ker\ R(\frac{d}{dt})$. If $\mathfrak{B} = \ker R(\frac{d}{dt})$, then we call R a *kernel representation* of $\mathfrak{B}$. We denote with $\mathfrak{L}^{\mathtt{w}}$ the set of all linear time-invariant differential behaviours whose trajectories take their values in the signal space $\mathbb{R}^{\mathtt{w}}$. The behaviour $\mathfrak{B} := \ker\ R\left(\frac{d}{dt}\right)$ is *controllable* (see [28], sec. 5.2) if and only if $R(\lambda)$ is full row rank for all $\lambda \in \mathbb{C}$.

When $\mathfrak{B}$ is controllable, it can be also represented in *image form*

$$w = M\left(\frac{d}{dt}\right)\ell , \tag{11}$$

where $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$ and ℓ is an auxiliary variable also called a *latent variable*; i.e., $\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ such that (11) holds$\} =:$ im $M\left(\frac{d}{dt}\right)$. The latent variable ℓ in (11) is called *observable* from w if $[w = M(\frac{d}{dt})\ell = 0] \Longrightarrow [\ell = 0]$ (this is the case if and only if $M(\lambda) \in \mathbb{C}^{\mathtt{w}\times\mathtt{l}}$ has full column rank for all $\lambda \in \mathbb{C}$, see [28], Th. 5.3.3). A controllable behaviour always admits an observable image representation. The set of linear differential controllable behaviours whose trajectories take their values in $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$.

Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, it may be possible to choose some components of the external variable w freely. If such components are maximally free (in the sense of Def. 3.3.1 of [28]), they are called *input variables*. It can be shown that the number of input variables is an invariant, denoted by $\mathtt{m}(\mathfrak{B})$. Once $\mathtt{m}(\mathfrak{B})$ free variables have been chosen, the remaining components of w are *output variables*; evidently, the number $\mathtt{p}(\mathfrak{B}) := \mathtt{w} - \mathtt{m}(\mathfrak{B})$ of output variables is also an invariant.

If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is associated to an image representation (11), there exists a permutation matrix P, such that $PM = \text{col}(U, Y)$, with YU^{-1} a matrix of proper rational functions (see [28], Sec. 3.3). This corresponds to the permutation of the elements of the external variable as $w = (u, y)$ where $u = U\left(\frac{d}{dt}\right)\ell$ is an input variable, and $y = Y\left(\frac{d}{dt}\right)\ell$ is an output variable. Moreover, it can be shown that $\mathtt{m}(\mathfrak{B}) = \mathtt{l}$, i.e. the number of input variables is equal to the dimension of ℓ, see ([46], Sec. VI-A).

B. State maps

A latent variable ℓ is a *state variable* for $\mathfrak{B}$ iff there exist $E, F \in \mathbb{R}^{\bullet\times\bullet}$, $G \in \mathbb{R}^{\bullet\times\mathtt{w}}$ such that $\mathfrak{B} = \left\{w \mid \exists\ \ell \text{ s.t. } E\frac{d\ell}{dt} + F\ell + Gw = 0\right\}$, i.e. if $\mathfrak{B}$ has a representation of first order in ℓ and zeroth order in w. The minimal number of state variables needed to represent $\mathfrak{B}$ in this way is called the *McMillan degree* of $\mathfrak{B}$, denoted by $\mathtt{n}(\mathfrak{B})$.

A state variable for $\mathfrak{B}$ can be computed as the image of a polynomial differential operator called a *state map* (see [31],[42]). Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be a state map for $\mathfrak{B}$. A polynomial differential operator $d\left(\frac{d}{dt}\right)$, where $d \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$, is called a (linear) *function of the state* of $\mathfrak{B}$ if there exists a constant vector $f \in \mathbb{R}^{1\times\mathtt{w}}$ such that $d\left(\frac{d}{dt}\right)w = fX\left(\frac{d}{dt}\right)w$ for all $w \in \mathfrak{B}$.

To construct state maps for $\ker R\left(\frac{d}{dt}\right)$, with $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ nonsingular, consider the set

$$\mathfrak{X}(\ker\ R) := \{f \in \mathbb{R}^{1\times\mathtt{w}}[\xi]\ \mid\ fR^{-1} \text{ is strictly proper}\}\ . \tag{12}$$

$\mathfrak{X}(\ker\ R)$ is a finite-dimensional subspace of $\mathbb{R}^{1\times\mathtt{w}}[\xi]$ over $\mathbb{R}$, (see [31], Prop. 8.4), of dimension $n := \deg(\det(R))$ (see [31], Cor. 6.7). To compute a state map for $\mathfrak{B}$, choose a set of generators $x_i \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$, $i = 1, \dots, N$ of $\mathfrak{X}(\ker\ R)$, and define $X := \text{col}(x_i)_{i=1,\dots,N}$; to obtain a *minimal state map*, choose $\{x_i\}_{i=1,\dots,N}$ so that they form a basis of $\mathfrak{X}(\ker\ R)$.

Let (11) be an observable image representation of $\mathfrak{B}$; we now summarise the main results concerning state maps acting on the latent variable ℓ. If necessary, permute the components of w so that $M = \text{col}(U, Y)$ with $U \in \mathbb{R}^{1\times 1}[\xi]$, $\det(U) \neq 0$, and YU^{-1} is a proper rational matrix. Consider the finite-dimensional vector space over $\mathbb{R}$ defined as

$$\mathfrak{X}(\text{im}\ M) := \{r \in \mathbb{R}^{1\times 1}[\xi]\ \mid\ rU^{-1} \text{ is strictly proper}\}\ . \tag{13}$$

X is a state map for (11) if and only if its rows span the vector space (13), and a *minimal* one if and only if its rows form a basis for (13) (see [31], Sec. 8). It can be shown that if (11) is observable, then $\mathtt{n}(\mathfrak{B}) = \deg(\det(U))$ (see Prop. 3.5.5 of [31]).

C. Quadratic differential forms

Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$; then $\Phi(\zeta, \eta) = \sum_{h,k} \Phi_{h,k}\zeta^h\eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is called *coefficient matrix*, and the sum extends over a finite set of nonnegative indices. $\Phi(\zeta, \eta)$ induces the *quadratic differential form* (QDF) acting on $\mathfrak{C}^\infty$-trajectories defined by $Q_\Phi(w) := \sum_{h,k}\left(\frac{d^h w}{dt^h}\right)^\top \Phi_{h,k}\frac{d^k w}{dt^k}$. Without loss of generality a QDF is induced by a *symmetric* two-variable polynomial matrix $\Phi(\zeta, \eta)$, i.e. one such that $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$; we denote the set of such matrices by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}_s[\zeta, \eta]$.

Given Q_Ψ, its *derivative* is the QDF Q_Φ defined by $Q_\Phi(w) := \frac{d}{dt}(Q_\Psi(w))$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$; this holds if and only if $\Phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta)$ (see [45], p. 1710).

Q_Φ is *nonnegative along* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$; and *positive along* $\mathfrak{B}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{>} 0$, if $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $[Q_\Phi(w) = 0] \implies [w = 0]$. If $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, then we call Q_Φ simply *nonnegative*, respectively *positive*. For algebraic characterisations of these properties see [45], pp. 1712-1713.

D. Dissipative linear differential behaviours

Denote $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ as the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of compact support functions.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. $\mathfrak{B}$ is called Φ-*dissipative* if for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ it holds that $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0$; and *strictly* Φ-*dissipative* if there exists $\epsilon > 0$ such that $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq \epsilon\int_{-\infty}^{\infty} \|w\|_2^2\, dt$. The QDF Q_Φ is called *supply rate*. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is *half-line* Φ-*dissipative* if for every $\tau \in \mathbb{R}$ and for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ it holds that $\int_{-\infty}^{\tau} Q_\Phi(w)dt \geq 0$.

A QDF Q_Ψ is a *storage function* for $\mathfrak{B}$ with respect to Q_Φ if $\frac{d}{dt}Q_\Psi \overset{\mathfrak{B}}{\leq} Q_\Phi$. Moreover, a QDF Q_Δ is a *dissipation function* for $\mathfrak{B}$ with respect to Q_Φ if $Q_\Delta \overset{\mathfrak{B}}{\geq} 0$ and $\int_{-\infty}^{\infty} Q_\Phi(w)dt = \int_{-\infty}^{\infty} Q_\Phi(w)dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Storage functions, supply rates and dissipation functions are associated as follows.

Proposition 8. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. The following statements are equivalent.*

- *$\mathfrak{B}$ is Φ-dissipative.*
- *There exists a storage function Q_Ψ for $\mathfrak{B}$ with respect to Q_Φ.*
- *There exists a dissipation function Q_Δ for $\mathfrak{B}$ with respect to Q_Φ.*

Moreover, there exists a one-to-one relation between Q_Φ, Q_Ψ and Q_Δ, defined by the dissipation equality $\frac{d}{dt}Q_\Phi = Q_\Phi - Q_\Delta$. *If $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, this equality holds true if and only if $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta)$.*

Proof: See [39], Th. 4.3. ∎

According to [45], Prop. 5.2, the inequality $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0$ is equivalent with the condition $\Phi(-j\omega, j\omega) \geq 0 \; \forall \; \omega \in \mathbb{R}$, consequently a dissipation function can be computed by factorising $\Phi(-\xi,\xi) = D(-\xi)^\top D(\xi)$ with $D \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$; note that such factorisation is not unique. The set of all possible storage functions is bounded from above by the *required supply* Q_{Ψ_+} and from below by the Q_{Φ_-} *available storage*, which can be computed using polynomial methods, as we now show. Consider $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\Phi(-j\omega, j\omega) > 0 \; \forall \; \omega \in \mathbb{R}$. Factorize $\Phi(-\xi,\xi) = A(-\xi)^\top A(\xi)$ corresponding to the *anti-Hurwitz*, and $\Phi(-\xi,\xi) = H(-\xi)^\top H(\xi)$ to the *Hurwitz* spectral factorization, respectively (see [4]). Then $\Psi_+(\zeta,\eta) := \dfrac{\Phi(\zeta,\eta) - A(\zeta)^\top A(\eta)}{\zeta+\eta}$ and $\Psi_-(\zeta,\eta) := \dfrac{\Phi(\zeta,\eta) - H(\zeta)^\top H(\eta)}{\zeta+\eta}$.

The set of storage functions is convex (see [44], Th. 3), i.e. if Ψ_1 and Ψ_2 are storage functions, so is $\Psi_\alpha := \alpha\Psi_1 + (1-\alpha)\Psi_1$ with $0 \leq \alpha \leq 1$.

When the supply rate is constant, the following result holds.

Proposition 9. *Let* $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *and let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. *Define* $\mathfrak{B} = \operatorname{im} M\left(\frac{d}{dt}\right)$, *where* $M \in \mathbb{R}^{\mathtt{w}\times 1}[\xi]$ *and* $M = col(U,Y)$, *corresponds to an input-output parition. Fix a state map* $X \in \mathbb{R}^{\bullet\times 1}[\xi]$ *for* $\mathfrak{B}$. *Assume that* $\mathfrak{B}$ *is* Φ*-dissipative. Let* $\Psi \in \mathbb{R}^{1\times 1}[\zeta,\eta]$ *and* $\Delta \in \mathbb{R}^{1\times 1}[\zeta,\eta]$ *be as in Prop. 8. There exist real symmetric matrices* K *and* Q *of suitable sizes such that* $\Psi(\zeta,\eta) = X(\zeta)^\top K X(\eta)$ *and* $\Delta(\zeta,\eta) = \operatorname{col}(X(\zeta), M(\zeta))^\top Q \operatorname{col}(X(\eta), M(\eta))$.

Proof: The proof follows from Th. 5.5 in [45]. ∎

In the following lemma, an important property of half-line dissipative linear differential behaviours with constant supply rates is shown.

Lemma 3. *Let* $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *be a supply rate and* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. *Assume that* $\sigma_+(\Phi) = \mathtt{m}(\mathfrak{B})$. *If* $\mathfrak{B}$ *is half-line* Φ*-dissipative, then every storage function* Q_Ψ *for* $\mathfrak{B}$ *is such that* $Q_\Psi \overset{\mathfrak{B}}{\geq} 0$.

Proof: See Theorem 6.4 in [45]. ∎

Appendix II
Proofs

Proof of Prop. 1: Let $i \in \{1,...,N\}$; since Σ is (strictly) Φ-dissipative and a constant switching signal $s(t) = i$ for all t is admissible in $\mathcal{S}$, then it necessarily follows that $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0$ (respectively $\exists\, \epsilon > 0$ s.t. $\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq \|w\|_2^2 dt$) for all $w \in \mathfrak{B}_i$ of compact support, i.e. $\mathfrak{B}_i$ is (strictly) Φ-dissipative. ∎

Proof of Prop. 2: Since $\mathfrak{B}_i$ is (strictly) Φ-dissipative, according to Prop. 1, the existence of Q_{Ψ_i}, $i = 1,...,N$, is guaranteed (see Prop. 8 in App. I-D). Now integrate the inequality $\frac{d}{dt}Q_{\Psi_i} \leq Q_\Phi$ between a and b, for all $w \in \mathfrak{B}_i \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. ∎

Proof of Th. 1: We consider the three possible cases, i.e. **A)** $|\mathbb{T}_s| = \infty$, **B)** $0 < |\mathbb{T}_s| < \infty$ and **C)** $|\mathbb{T}_s| = 0$. Let $t_0 := -\infty$. Use Prop. 2 and the fact that $\lim_{t\to\pm\infty} w(t) = 0$ for all $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to obtain the following expressions for cases **A)** and **B)**, where $s = s_w$:

A) $\int Q_\Phi(w) \geq (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+)) + ... + (Q_{\Psi_{s(t_{n-1})}}(w)(t_n^-) - Q_{\Psi_{s(t_n)}}(w)(t_n^+)) + \dots$.

B) $\int Q_\Phi(w) \geq (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+)) + \Sigma_{k=2}^{|\mathbb{T}_s|-1}(Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+)) + (Q_{\Psi_{s(|\mathbb{T}_s|-1)}}(w)(t_{|\mathbb{T}_s|}^-) - Q_{\Psi_{s(|\mathbb{T}_s|)}}(w)(t_{|\mathbb{T}_s|}^+))$.

Since $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0$, $\forall\; t_k \in \mathbb{T}_s$, we conclude that in both cases $\int Q_\Phi(w) \geq 0$.

Finally the claim for **C)** when no switching takes place, i.e. $s(t) = i$ for all t, follows readily from the existence of a storage function Q_{Ψ_i} (see Prop. 2) and the standard result quoted in App. I-D, Prop. 8. ∎

Proof of Prop. 3: Since $Q_\Phi \overset{\mathfrak{B}_i}{\geq} \frac{d}{dt}Q_{\Psi_i}$ and $Q_\Phi \overset{\mathfrak{B}_i}{\geq} \frac{d}{dt}Q_{\Psi'_i}$, $i = 1,...,N$, it follows from standard results regarding dissipative systems (see App. I-D) that $Q_\Phi \overset{\mathfrak{B}_i}{\geq} \frac{d}{dt}\left(\alpha Q_{\Psi_i} + (1-\alpha)Q_{\Psi'_i}\right)$, $i = 1,...,N$. Moreover, to show that condition 2) in Def. 6 is satisfied, let $s \in \mathcal{S}$ and note that since $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) \geq Q_{\Psi_{s(t_k)}}(w)(t_k^+)$ and $Q_{\Psi'_{s(t_{k-1})}}(w)(t_k^-) \geq Q_{\Psi'_{s(t_k)}}(w)(t_k^+)$ for every $t_k \in \mathbb{T}$, it follows that $\alpha Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) + (1-\alpha)Q_{\Psi'_{s(t_{k-1})}}(w)(t_k^-) - \alpha Q_{\Psi_{s(t_k)}}(w)(t_k^+) - (1-\alpha)Q_{\Psi'_{s(t_k)}}(w)(t_k^+) \geq 0$. ∎

Proof of Th. 2: The existence of storage functions Q_{Ψ_i}, $i = 1,...N$, follows from Prop. 1 and Prop. 8 in App. I-D. To prove the rest of the claim let us introduce first the following lemma where concepts such as the construction of state maps in App. I-B and the computation of storage functions in App. I-D are used.

Lemma 4. *Let* $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *and let* Σ *be a strictly* Φ*-dissipative SLDS with* $\mathcal{G}$ *well-posed. Consider two behaviours* $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{F}$, *described by the observable image representations* $w = M_i\left(\frac{d}{dt}\right)\ell_i$, $i = 1,2$, *respectively. Consider the switching signal*

$$s(t) := \begin{cases} 1, & t \leq 0\,, \\ 2, & t > 0\,. \end{cases}$$

Let $X_i \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_i)\times 1}[\xi]$, $i = 1,2$, *be minimal state maps for* $\mathfrak{B}_i$, $i = 1,2$; *and let* $L_{1\to 2} \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_2)\times\mathtt{n}(\mathfrak{B}_1)}$ *be the corresponding re-initialisation map when switching from*

$\mathfrak{B}_1$ to $\mathfrak{B}_2$ at zero. Select a fixed but otherwise arbitrary final state v_1, corresponding to the unique initial state $v_2 := L_{1\to 2}v_1$.

There exists $A_1, H_2 \in \mathbb{R}^{1\times 1}[\xi]$ such that $\det(A_1)$ and $\det(H_2)$ are respectively anti-Hurwitz and Hurwitz polynomials; and $M_1(-\xi)^\top \Phi M_1(\xi) = A_1(-\xi)^\top A_1(\xi)$ and $M_2(-\xi)^\top \Phi M_2(\xi) = H_2(-\xi)^\top H_2(\xi)$.

There exist unique latent variable trajectories $\ell_1, \ell_2 : \mathbb{R} \to \mathbb{R}^1$ such that $A_1\left(\frac{d}{dt}\right)\ell_1 = 0$, $X_1\left(\frac{d}{dt}\right)\ell_1(0^-) = v_1$; and $H_2\left(\frac{d}{dt}\right)\ell_2 = 0$, $X_2\left(\frac{d}{dt}\right)\ell_2(0^+) = L_{1\to 2}v_1$. Consequently, the external variable trajectory defined by

$$w(t) := \begin{cases} M_1\left(\frac{d}{dt}\right)\ell_1, & t \le 0\ , \\ M_2\left(\frac{d}{dt}\right)\ell_2, & t > 0\ ; \end{cases}$$

belongs to $\mathfrak{B}^\Sigma$. Moreover, the final/initial state of w at zero is v_1 and $L_{1\to 2}v_1$ respectively.

Proof: The existence of $A_1, H_2 \in \mathbb{R}^{1\times 1}[\xi]$ satisfying the conditions in the first claim follows directly from standard results in polynomial spectral factorization (see App. I-D).

To verify that the second claim holds true, it is enough to prove that the state space of each mode behaviour equals the state space associated to its supply rate spectral factor, i.e. to prove that $\mathfrak{X}(\text{im}\ M_1\left(\frac{d}{dt}\right)) = \mathfrak{X}(\ker\ A_1\left(\frac{d}{dt}\right))$ and $\mathfrak{X}(\text{im}\ M_2\left(\frac{d}{dt}\right)) = \mathfrak{X}(\ker\ H_2\left(\frac{d}{dt}\right))$. In order to do so, we recall from Prop. 1 that if Σ is strictly Φ-dissipative, it follows that every behaviour in the bank is also strictly Φ-dissipative. Since there exists $\epsilon > 0$ such that $\int Q_\Phi(w) \geq \epsilon\left(\int_{-\infty}^{0^-}\|w\|_2^2\,dt + \int_{0^+}^{\infty}\|w\|_2^2\,dt\right)$; by a Fourier-transform argument we conclude that $\Phi_i'(-j\omega, j\omega) := M_i(-j\omega)^\top \Phi M_i(j\omega) > \epsilon M_i(-j\omega)^\top M_i(j\omega)\ \ \forall\,\omega \in \mathbb{R}$, $i = 1, 2$. Select a submatrix $U_i \in \mathbb{R}^{1\times 1}[\xi]$ of M_i of maximal determinantal degree, then $M_i U_i^{-1}$, $i = 1, 2$ is a proper rational matrix. Consequently $\lim_{\omega\to\infty} U_i(-j\omega)^{-\top}\Phi_i(-j\omega, j\omega)U_i(j\omega)^{-1} = M_{i_\infty}^\top \Phi M_{i_\infty} > \epsilon M_{i_\infty}^\top M_{i_\infty}$, $i = 1, 2$, with $M_{i_\infty} := \lim_{\omega\to\infty} M_i(j\omega)U_i(j\omega)^{-1}$. It is easy to check that M_{i_∞} contains I_1 as a submatrix, and consequently $M_{i_\infty}^\top M_{i_\infty} > 0$, implying that $\lim_{w\to\infty} U_i(-j\omega)^{-\top}\Phi_i(-j\omega, j\omega)U_i(j\omega)^{-1}$, $i = 1, 2$, is invertible and therefore $U_1(-\xi)^{-\top}\Phi_1(-\xi, \xi)U_1(\xi)^{-1} = U_1(-\xi)^{-\top}A_1(-\xi)^\top A_1(\xi)U_1(\xi)^{-1}$, as well as $U_2(-\xi)^{-\top}\Phi_2(-\xi, \xi)U_2(\xi)^{-1} = U_2(-\xi)^{-\top}H_2(-\xi)^\top H_2(\xi)U_2(\xi)^{-1}$, have a proper inverse. Considering (12) and (13) in App. I-B, we conclude that $\mathfrak{X}(\text{im}\ M_1\left(\frac{d}{dt}\right)) = \mathfrak{X}(\ker\ A_1\left(\frac{d}{dt}\right))$ and $\mathfrak{X}(\text{im}\ M_2\left(\frac{d}{dt}\right)) = \mathfrak{X}(\ker\ H_2\left(\frac{d}{dt}\right))$.

Consequently, the trajectories[2] $\ell_1 \in \ker\ A_1\left(\frac{d}{dt}\right)$ and $\ell_2 \in \ker\ H_2\left(\frac{d}{dt}\right)$ are such that $X_1\left(\frac{d}{dt}\right)\ell_1(0^-) = v_1$ and $X_2\left(\frac{d}{dt}\right)\ell_2(0^+) = L_{1\to 2}v_1$. Finally, since the latent variables ℓ_1 and ℓ_2 are observable, they correspond to a unique trajectory $w \in \mathfrak{B}^\Sigma$ defined as in the Lemma with final/initial state v_1 and $L_{1\to 2}v_1$ respectively. The lemma is proved. ■

We now prove the claim of Th. 2 by contradiction. Let $X_i^{\mathtt{n}(\mathfrak{B}_i)\times 1}[\xi]$ be minimal state maps for $\mathfrak{B}_i$, $i = 1, ..., N$ and $L_{j\to k} \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_k)\times\mathtt{n}(\mathfrak{B}_j)}$ with $j, k = 1, ..., N$ the re-initialisation maps. Let w.l.o.g. $i = 1$, $j = 2$ and assume that there exists a final/initial state v_1 and $L_{1\to 2}v_1$ for $w \in \mathfrak{B}^\Sigma$ respectively, such that $Q_{\Psi_1}(w)(0^-) < Q_{\Psi_2}(w)(0^+)$.

Construct latent variable trajectories $\ell_1, \ell_2 : \mathbb{R} \to \mathbb{R}^1$ as in Lemma 4 corresponding to an admissible switched trajectory $w \in \mathfrak{B}^\Sigma$. For this trajectory it holds that $\int Q_\Phi(w) = \int_{-\infty}^{0^-} Q_{\Phi_1'}(\ell_1)dt + \int_{0^+}^{\infty} Q_{\Phi_2'}(\ell_2)dt = Q_{\Psi_1'}(\ell_1)(0^-) - Q_{\Psi_2'}(\ell_2)(0^+) < 0$; which contradicts the fact that Σ is strictly Φ-dissipative.

Note that it follows automatically from the latter results that there exists an N-tuple $(Q_{\Psi_1}, ..., Q_{\Psi_N})$ that satisfies the conditions 1) and 2) in Def 6. The theorem is proved. ■

Proof of Lemma 1: The fact that $\Psi_i'(\zeta, \eta)$, $i = 1, ..., N$, can be factorised as $X_i(\zeta)K_iX_i(\eta)$, $i = 1, ..., N$, follows from Prop. 9 in App. I-D.

The equivalence of conditions 1) and 2) follows from the fact that $w = M_i\left(\frac{d}{dt}\right)\ell_i$, $i = 1, ..., N$, and the standard reformulation of QDFs in terms of latent variables, see App. I-C. We now prove the equivalence of conditions 2) and 3). Use Lemma 4 in the proof of Th. 2 to conclude that since Σ is strictly Φ-dissipative, then the final/initial states at switching instants corresponding to ℓ_i and ℓ_j are arbitrary. Use the factorisations $\Psi_i'(\zeta, \eta) = X_i(\zeta)K_iX_i(\eta)$, $i = 1, ..., N$, to conclude that $v_i^\top K_i v_i \geq v_j^\top K_j v_j$ for all $i, j \in \mathcal{P}$, $i \neq j$. Then use the re-initialisation map to conclude that $v_i^\top K_i v_i \geq v_i^\top L_{i\to j}^\top K_j L_{i\to j} v_i$, which is equivalent to condition 3). ■

Proof of Prop. 4: The proof follows from the fact that the re-initialisation maps are also the identity, and since $K_i \geq K_j$ and $K_j \geq K_i$ for all $i, j \in \mathcal{P}$. Consequently, $K_i = K_j$ and $Q_{\Psi_i} = Q_{\Psi_j}$ for all $i, j \in \mathcal{P}$. ■

Proof of Th. 3: Define $t_0 := -\infty$. Using Prop. 2 and equation (6), it follows that since $\lim_{t\to-\infty}(w)(t) = 0$, we obtain $\int^\tau Q_\Phi(w)\,dt \geq (Q_{\Psi_{s(t_0)}}(w)(t_1^-) -$

[2]Note that ℓ_i, $i = 1, 2$ are not trajectories with compact support, however an approximation argument can be used to complete the proof of the claim.

$Q_{\Psi_{s(t_1)}}(w)(t_1^+)) + \sum_{k=2}^{n}(Q_{\Psi_{s(t_k-1)}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+)) + Q_{\Psi_{s(t_n)}}(w)(\tau)$. Note that $Q_{\Psi_{s(t_{j-1})}}(w)(t_j^-) - Q_{\Psi_{s(t_j)}}(w)(t_j^+) \geq 0$, for every $t_j \in \mathbb{T}_s$. Moreover, since every mode has the same partition $w = \text{col}(u, y)$, it follows that $\sigma_+(\Phi) = \mathtt{m}(\mathfrak{B}_i)$, $i = 1, ..., N$. Use Th. 6.4 in [45] to conclude that $Q_{\Psi_{s(t_n)}}(w)(\tau) \geq 0$, consequently $\int^{\tau} Q_\Phi(w) \geq 0$. ■

Proof of Prop. 6: To prove that the degree of $X(\xi)$ is less than the degree of $M(\xi)$, note that since YU^{-1} and XU^{-1} are strictly proper (see App. I-B), we can apply Lemma 6.3-10 of [18] and conclude that the highest degree of each entry in X is less than the highest degree present in M. ■

Proof of Prop. 7: To prove the equivalence of statements 1) and 2) let us define $S_L(\xi) := \begin{bmatrix} I_{\mathtt{1}} & \xi I_{\mathtt{1}} & \cdots & \xi^L I_{\mathtt{1}} \end{bmatrix}$. The equivalence follows from the equalities $X(\xi) = \begin{bmatrix} \widetilde{X} & 0_{n\times\mathtt{1}} \end{bmatrix} S_L(\xi)$, $\xi X(\xi) = \begin{bmatrix} 0_{n\times\mathtt{1}} & \widetilde{X} \end{bmatrix} S_L(\xi)$, and $M(\xi) = \widetilde{M} S_L(\xi)$. ■

Proof of Lemma 2: Since $\mathfrak{B}$ is Φ-dissipative, then there exists a storage function $\Psi(\zeta, \eta)$, moreover according to Prop. 9, there exists K such that $\Psi(\zeta, \eta) = X(\zeta)^\top K X(\eta)$.

To prove that statements 1) and 2) in Prop. 6 hold, it is enough to recall from Prop. 8 that there exists a dissipation function $\Delta(\zeta, \eta)$ such that $\Delta(\zeta, \eta) = M(\zeta)^\top \Phi M(\eta) - (\zeta + \eta)\Psi(\zeta, \eta)$.

To prove the final claim define $S_L(\xi) := \begin{bmatrix} I_{\mathtt{1}} & \xi I_{\mathtt{1}} & \cdots & \xi^L I_{\mathtt{1}} \end{bmatrix}$. Factorise $\Delta(\zeta, \eta) = S_L(\zeta)^\top \widetilde{\Delta} S_L(\eta)$, with $\widetilde{\Delta} = \widetilde{\Delta}^\top \in \mathbb{R}^{(L+1)\mathtt{1}\times(L+1)\mathtt{1}}$. Then it follows from the definition of a dissipation function that $Q_\Delta \geq 0$ and consequently $\widetilde{\Delta} \geq 0$. ■

Proof of Th. 4: To prove the first part of the claim note that the degree of X_k cannot exceed that of M_k, $k = 1, ..., N$, because of the same argument used in Prop. 6. Moreover solutions K_k, $k = 1, ...N$, for the LMIs (7) exist because of the fact that $\mathfrak{B}_k$, $i = 1, ..., N$, is strictly Φ-dissipative and Lemma 2. Moreover, according to Lemma 2 and Prop. 7 if the LMIs (7) hold, $\Psi_k(\zeta, \eta)$ induces a storage function for $\mathfrak{B}_k$, $k = 1, ..., N$. Finally, note that due to Lemma 1, the LMIs (8) imply condition 2) in Def. 6, then using Th. 1 we conclude that Σ is Φ-dissipative. ■

Proof of Th. 6: Since Σ is strictly Φ-dissipative, it follows from Th. 2 that there exists a multiple storage function $Q_{\mathbf{\Psi}} := (Q_{\Psi_1}, ..., Q_{\Psi_N})$ for Σ. Note that since only the trajectories $\mathfrak{B}^{\Sigma}_{\text{aut}} \subseteq \mathfrak{B}^{\Sigma}$ are permitted for Σ_{aut} according to Def. 8, it necessarily follows that the trajectories of its mode behaviours are also restricted as $\mathfrak{B}'_i := \{w = \text{col}(u, y) \in \mathfrak{B}_i \mid u = 0\}$, $i = 1, ..., N$. We now show that Σ_{aut} is asymptotically stable by showing that $Q_{\mathbf{\Psi}}$ satisfies the conditions 1)-3) in Th. 5. **C1.** The fact that $Q_{\Psi_i} \overset{\mathfrak{B}'_i}{\geq} 0$, $i = 1, ..., N$, follows directly from Lemma 3 in App. I-D, i.e. since $\mathfrak{B}'_i \subseteq \mathfrak{B}_i$ and $Q_{\Psi_i} \overset{\mathfrak{B}_i}{\geq} 0$, $i = 1, ..., N$, then $Q_{\Psi_i} \overset{\mathfrak{B}'_i}{\geq} 0$, $i = 1, ..., N$. **C2.** In order to prove that $\frac{d}{dt}Q_{\Psi_i}$ decreases along $\mathfrak{B}'_i$, $i = 1, ..., N$, use Prop. 1 to show that $\mathfrak{B}_i$, $i = 1, ..., N$, is strictly Φ-dissipative and consequently there exists $\epsilon_i > 0$ such that $Q_\Phi(w) \geq \frac{d}{dt}Q_{\Psi_i}(w) + \epsilon_i \|w\|_2^2$, $i = 1, ..., N$. Since, for every trajectory $\text{col}(0, y) \in \mathfrak{B}'_i$ it follows that $Q_\Phi(w) = 0$, then $\frac{d}{dt}Q_{\Psi_i}(w) \overset{\mathfrak{B}'_i}{\leq} -\epsilon_i \|w\|_2^2 < 0$, for every $w \neq 0$. **C3.** Finally, note that the non increasing condition at switching instants 3) in Th. 5 is equivalent to condition 2) in Def. 6. ■

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