Robust Stability for Multiple Model Adaptive Control: Part II - Gain Bounds

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***Abstract*—The axiomatic development of a wide class of Estimation based Multiple Model Switched Adaptive Control (EMMSAC) algorithms considered in the first part of this two part contribution forms the basis for the proof of the gain bounds given in this paper. The bounds are determined in terms of a cover of the uncertainty set, and in particular, in many instances, are independent of the number of candidate plant models under consideration.The full interpretation, implications and usage of these bounds within design synthesis are discussed in part I. Here in part II, key features of the bounds are also discussed and a simulation example is considered. It is shown that a dynamic EMMSAC design can be universal and hence non-conservative and hence outperforms static EMMSAC and other conservative designs. A wide range of possible dynamic algorithms are outlined, motivated by both performance and implementation considerations.**

1. Introduction

The establishment of closed loop gain bounds lie at the centre of any robustness analysis based on small gain approaches. The primary contribution of this paper is to establish gain bounds for the class of estimation based multiple model adaptive controllers (EMMSAC) introduced in [2]. The nominal bounds are given in a form which is independent of the choice of the candidate plant model set, and which allows the development of bounds dependent only on the underlying complexity of the plant uncertainty. Whilst the bounds are complex in form, the elements of the bound are simple; a detailed interpretation is given. The fully modularised nature of the EMMSAC approach permits systematic (and standard) design procedures to optimise these terms independently. Such bounds lead naturally to design processes, such as those described in [2]. In particular, these bounds lead to simple approaches to determine:

1) The required number of candidate plant models.
2) Suitable geometric distributions of the plant models over the uncertainty set.
3) Robust stability certificates.
4) Non-conservative designs.

The contribution in part I [2] addressed the first three issues – and the primary contribution of this paper is to complete the analysis of part I [2] by the provision of the underpinning gain bound analysis. The algorithms of EMMSAC have been cast in an abstract axiomatic setting, this approach gives generality to the analysis. The derivation of the gain bounds derived here are thus applicable to many different algorithmic variants.

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Manuscript received March 2011, revised September 2012.

The secondary contribution of this paper is to address the fourth issue by consideration of a dynamic version of EMMSAC. This has not been addressed previously for MMAC and we argue for it's importance as follows. It is known that LTI controllers, in contrast to classical adaptive controllers, cannot stabilize systems in the presence of unbounded parametric uncertainties. Furthermore, even if the parameters lie within an a-priori known compact set, the performance of an LTI control design typically degrades with the size of this set. On the other hand, controllers which can stabilize systems with unbounded parametric uncertainties are said to be universal. They possess the feature that even if the uncertainty is known to lie inside a known compact set, the performance for a fixed plant is necessarily independent of the size of this set, and hence is non-conservative. In this situation, universal controllers (for example, classical adaptive controllers) necessarily outperform conservative designs (for example, LTI controllers). Unfortunately, existing MMAC designs are all constrained to a priori fixed and finite candidate plant model sets; hence are not universal, and cannot be applied to many uncertainty sets which contain unbounded parametric variations, which is the classical domain for adaptive control. Further, all the known bounds scale with the size of the uncertainty set and we show that the true gain also scales poorly for classes of MMAC: thus basic MMAC is also conservative. However, versions of dynamic EMMSAC, whereby the candidate plant set is varied in size online, are shown to be universal and hence non-conservative, thus satisfying a key rationale for adaptive control. There are many other algorithmic possibilities with dynamic EMMSAC, and this paper also briefly discuss the possibilities to manage computational complexity in the dynamic setting.

The paper is structured as follows. The notation is developed in Section 2 and the structure and the required properties of an EMMSAC controller are described in Section 3. A full discussion of these requirements and examples of estimators can be found in part I [2]. In Section 4 we present the main result and it's proof. In Section 5 we discuss dynamic EMMSAC algorithms, and show that dynamic EMMSAC is universal. We then characterise a situation in which it outperforms static EMMSAC and other designs such as robust LTI controllers. In Section 6, and we outline other possibilities with dynamic EMMSAC. A simulation example is given in Section 7. The paper is self-contained, but is intended to be read in conjunction with [2].

2. Preliminaries

For $0 \leq a \leq b,\ a,b \in \mathbb{Z}$ let $[a,b] := \{x \in \mathbb{Z} \mid a \leq x \leq b\},\ [a,b) := \{x \in \mathbb{Z} \mid a \leq x < b\}$. Let the size of

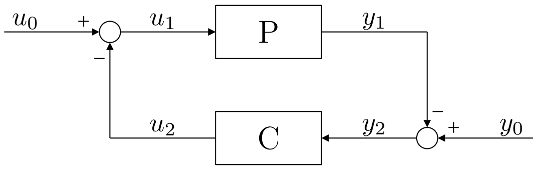

Fig. 1. Closed loop $[P, C]$

$$K : \mathcal{P} \to \mathcal{C} \tag{3.11}$$

$$X : \mathcal{W}_e \to map(\mathbb{N}, map(\mathcal{P}, \mathbb{R}^+)) : w_2 \mapsto [k \to (p \mapsto r_p[k])] \tag{3.12}$$

$$G : \mathcal{W}_e \to map(\mathbb{N}, \mathcal{P}^*) \tag{3.13}$$

$$M : (map(\mathbb{N}, map(\mathcal{P}, \mathbb{R}^+)), map(\mathbb{N}, \mathcal{P}^*)) \to map(\mathbb{N}, \mathcal{P}^*) \tag{3.14}$$

$$\left[k \mapsto (p \mapsto r_p[k]), k \mapsto G(k)\right] \mapsto \left[k \mapsto q_f(k)\right] \tag{3.15}$$

$$q_f(k) := \operatorname*{argmin}_{p \in G(k)} r_p[k], \ \forall k \in \mathbb{N} \tag{3.16}$$

$$\Delta : \mathcal{P} \to \mathbb{N} \tag{3.17}$$

$$D : map(\mathbb{N}, \mathcal{P}) \to map(\mathbb{N}, \mathcal{P}) \tag{3.18}$$

$$[k \mapsto q_f(k)] \mapsto [k \mapsto q(k)] \tag{3.19}$$

$$q(k) := \begin{cases} q_f(k) & \text{if} \quad k - k_s(k) \geq \Delta(q(k_s(k))) \\ q(k_s(k)) & \text{else} \end{cases} \tag{3.20}$$

$$S : \mathcal{W}_e \to map(\mathbb{N}, \mathcal{P}^*) : w_2 \mapsto q \tag{3.21}$$

$$S = DM(X, G) \tag{3.22}$$

$$k_s(k) := \max\{i \in \mathbb{N} \mid 0 \leq i \leq k, \ q(i) \neq q(i-1)\} \tag{3.23}$$

$$C : \mathcal{Y}_e \to \mathcal{U}_e : y_2 \mapsto u_2 \tag{3.24}$$

$$u_2(k) = C_{K(q(k))}(y_2 - \mathscr{T}_{k_s(k)-1} y_2)(k) \tag{3.25}$$

TABLE 1
EQUATIONS SPECIFYING THE ALGORITHM

the given intervals $|\cdot|$ be defined by $|[a, b]| := b - a + 1$ and $|[a, b)| := b - a$. For a signal $v \in \mathcal{S}$ we then define the restriction of v over the interval $I = [c, d]$ by $v|_I := (v(c), \cdots, v(d))$ where $c \leq d, \ c, d \in \mathbb{Z}$, and similarly for $I = [c, d)$. Let $\mathcal{S} := map(\mathbb{Z}, \mathbb{R}^h)$ denote the collection of all maps and let $\mathcal{S}|_{[a,b]} := map([a, b], \mathbb{R}^h)$. Let $\mathscr{T}_t : \mathcal{S} \cup_{b \in \mathbb{Z}} \mathcal{S}|_{[0,b]} \to \mathcal{S}, \ t \in \mathbb{Z}$ denote the truncation operator defined by:

$$(\mathscr{T}_t v)(\tau) = \begin{cases} v(\tau) & \text{if} \quad \tau \in \operatorname{dom}(v), \ \tau \leq t. \\ 0 & \text{otherwise} \end{cases} .$$

For $x \in \mathcal{S}$ define the norms $\|x\| = \|x\|_r = \left(\sum_{i \in \operatorname{dom}(x)} |x(i)|^r\right)^{1/r}$ for $1 \leq r < \infty$, and $\|x\| = \|x\|_\infty = \sup_{i \in \operatorname{dom}(x)} |x(i)|$. We repeatedly use the l_r identity:

$$\| \|x\|_r, \|y\|_r \|_r = \|(x, y)\|_r, \quad x, y \in \mathcal{S}, \ 1 \leq r \leq \infty. \tag{2.1}$$

We consider signal spaces $\mathcal{V} \subset \mathcal{S}$ and extended signal spaces $\mathcal{V}_e \subset \mathcal{S}$:

$$\begin{aligned} \mathcal{V} &:= \{v \in \mathcal{S} \mid v(-t) = 0, \ \forall t \in \mathbb{Z}; \ \|v\| < \infty\} \\ \mathcal{V}_{[a,b]} &:= \left\{v \in \mathcal{S}|_{[a,b]} \,\middle|\, \exists x \in \mathcal{V} \ s.t. \ v = x|_{[a,b]} = v\right\} . \\ \mathcal{V}_e &:= \{v \in \mathcal{S} \mid \forall t \in \mathbb{Z} : \mathscr{T}_t v \in \mathcal{V}\} . \end{aligned} \tag{2.2}$$

We take $\mathcal{V} = l_r$ to be defined by (2.2) with $\|\cdot\| = \|\cdot\|_r$. The input and output signal spaces are defined as: $\mathcal{U} := \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{m} = \mathcal{V}^m, \mathcal{Y} := \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{o} = \mathcal{V}^o$, and let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$. Given a plant $P : \mathcal{U}_e \to \mathcal{Y}_e$ satisfying $P(0) = 0$ and a controller $C : \mathcal{Y}_e \to \mathcal{U}_e$ satisfying $C(0) = 0$, the closed-loop system $[P, C]$ in Figure 1 is defined by equations

$$y_1 = P u_1 \tag{2.3}$$

$$u_0 = u_1 + u_2 \qquad y_0 = y_1 + y_2 \tag{2.4}$$

$$u_2 = C y_2. \tag{2.5}$$

Here $w_i = (u_i, y_i)^\top \in \mathcal{W}_e$ represents the plant input and output (i=1), disturbances (i=0) and observations (i=2). $[P, C]$ is said to be well-posed if for all $w_0 \in \mathcal{W}$ there exists a unique solution $(w_1, w_2) \in \mathcal{W}_e \times \mathcal{W}_e$. Note that linear switched systems are well-posed. Define $\mathcal{P}_{LTI}$ to be the set of all $p = (A, B, C, D) \in \cup_{n \geq 1} \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{o \times n} \times \mathbb{R}^{o \times m}$ such that p is minimal and

$$P_p : \mathcal{U}_e \to \mathcal{Y}_e, \ u_1^p \mapsto y_1^p, \ p = (A, B, C, D) \tag{2.6}$$

$$x_p(k+1) = A x_p(k) + p u_1^p(k) \tag{2.7}$$

$$y_1^p(k) = C x_p(k) + D u_1^p(k) \tag{2.8}$$

$$x_p(-k) = 0, \ k \in \mathbb{N} \tag{2.9}$$

Note that since $x_p(-k) = 0$ for all $k \in \mathbb{N}$ it follows that $y_1^p(-k) = (P_p u_1^p)(-k) = 0$ for all $k \in \mathbb{N}$. Also define $\bar{\mathcal{P}}_{LTI} := \{(A, B, C, D) \in \mathcal{P}_{LTI} \mid D = 0\}$. Analogously, define $\mathcal{C}_{LTI}$ to be the set of all $(A, B, C, D) \in \cup_{n \geq 1} \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times o} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times o}$ such that (A, B, C, D) is minimal, and the control operator

$$C_c : \mathcal{Y}_e \to \mathcal{U}_e : y_2^c \mapsto u_2^c, \ c = (A, B, C, D) \tag{2.10}$$

is defined analogously to equations (2.7) - (2.9). Also let $\bar{\mathcal{C}}_{LTI} := \{(A, B, C, D) \in \mathcal{C}_{LTI} \mid D = 0\}$. Throughout, $\mathcal{P}$ is topologised by the nonlinear gap metric, see [2, Section 2].

3. The EMMSAC Structure

We now recall the structure of the EMMSAC algorithm introduced in [2]. Table 1 summarizes the structural requirements that specify the switching algorithm, where $\mathcal{P}, \mathcal{C}$ denote the parametric space of plants and controllers, e.g. $\mathcal{P} = \bar{\mathcal{P}}_{LTI}$, $\mathcal{C} = \mathcal{C}_{LTI}$ and $\mathcal{P}^*$ denotes the powerset of $\mathcal{P}$. If there are multiple minimising disturbance estimates, an arbitrary ordering on $G(k)$ is imposed a priori, i.e. $G(k) = \{p_1, p_2, \cdots, p_n\}$, and $\operatorname{argmin}_{p \in G(k)} r_p[k]$ is defined to return the parameter $p_i \in G(k)$ with the smallest index i such that $r_p[k]$ is minimal. Recall that G is an example of plant generating operator:

Definition 1: A causal map $Q : \mathcal{W}_e \to map(\mathbb{N}, \mathcal{P}^* \setminus \emptyset)$ is said to be a plant-generating operator. We define $\mathcal{P}^Q$ is the union of all plant model sets possibly represented by Q: $\mathcal{P}^Q := \cup_{w_2 \in \mathcal{W}_e} \cup_{k \in \mathbb{N}} Q(w_2)(k) \subset \mathcal{P}$. Q is said to be finite if $Q(w_2)(k)$ is a finite set for all $w_2 \in \mathcal{W}$, $i \in \mathbb{N}$, constant if $Q(w_2)(i) = Q(w_2)(j)$, for all $w_2 \in \mathcal{W}$, $i, j \in \mathbb{N}$, monotonic if $Q(w_2)(k) \subset Q(w_2)(k+1)$ for all $w_2 \in \mathcal{W}$, $k \in \mathbb{N}$. For notational economy we often write $Q(k) := Q(w_2)(k)$.

To formulate the properties required of the atomic closed loop systems, let

$$y_1^p = P_p u_1^p, \ u_0^p = u_1^p + u_2, \ y_0^p = y_1^p + y_2, \tag{3.26}$$

$$u_2^c \quad = \quad C_c y_2^c \tag{3.27}$$

Furthermore, let $\sigma(c),\ c \in \mathcal{C}$ be defined by

$$\sigma(c) = \min \left\{ k \geq 0 : \begin{array}{c} \forall l \geq 0, \\ u_2^c = C_c y_2^c,\ \hat{u}_2^c = C_c \hat{y}_2^c, \\ (u_2^c, y_2^c)^\top|_{[l,l+k]} = (\hat{u}_2^c, \hat{y}_2^c)^\top|_{[l,l+k]}, \\ y_2^c = \hat{y}_2^c \Rightarrow u_2^c = \hat{u}_2^c \end{array} \right\} \tag{3.28}$$

Similarly define $\sigma(p),\ p \in \mathcal{P}$. Then:

Assumption 2: There exist functions $\alpha, \beta : \mathcal{P} \times \mathcal{C} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the following holds:
1) (Linear growth of $[P_p, C_c]$): Let $p \in \mathcal{P},\ c \in \mathcal{C}$ and the closed-loop system $[P_p, C_c]$ be well-posed. Let $t_1, t_2, t_3, t_4 \in \mathbb{N},\ t_1 < t_2 \leq t_3 < t_4$ and $I_1 = [t_1, t_2), I_2 = [t_2, t_3), I_3 = [t_3, t_4)$. Suppose $w_2, w_2^c, w_1^p \in \mathcal{W}_e,\ w_0^p \in \mathcal{W}$ satisfy equations (3.26),(3.27) on $I_1 \cup I_2 \cup I_3$. Suppose that either

$$\begin{array}{rcll} w_2^c|_{I_1} = 0,\ w_2^c|_{I_2 \cup I_3} & = & w_2|_{I_2 \cup I_3} & \text{or} \\ w_2^c|_{I_1 \cup I_2 \cup I_3} & = & w_2|_{I_1 \cup I_2 \cup I_3} & \end{array}$$

where $|I_1| = t_2 - t_1 \geq \max\{\sigma(p), \sigma(c)\}$. Then, in both cases:

$$\begin{aligned} \|w_2|_{I_3}\| \leq \alpha(p, c, |I_2|, |I_3|)\|w_2|_{I_1}\| \\ + \beta(p, c, |I_2|, |I_3|)\|w_0^p|_{I_1 \cup I_2 \cup I_3}\|. \end{aligned} \tag{3.29}$$

2) (Stability of $[P_p, C_{K(p)}]$): Let $p \in \mathcal{P}$ and $x \in \mathbb{N}$. Then

$$\alpha(p, K(p), a, x) \to 0 \text{ as } a \to \infty \tag{3.30}$$

and α is monotonic in a.
Recall that if $(\mathcal{P}, \mathcal{C}) \in \{(\bar{\mathcal{P}}_{LTI}, \mathcal{C}_{LTI}), (\mathcal{P}_{LTI}, \bar{\mathcal{C}}_{LTI})\}$ it can be shown that Assumption 2(1) holds. If additionally $K : \mathcal{P} \to \mathcal{C}$ is a stabilising design, (i.e. $[P_p, C_{K(p)}]$ is gain stable) it can be shown that Assumption 2(2) holds.

The required properties of the estimator are given by assumption 4 below, given that the restriction operator $\mathscr{R}_{\sigma,t} : \mathcal{S} \to \mathbb{R}^{h(\sigma+1)}$ is given by $\mathscr{R}_{\sigma,t} v := (v(t-\sigma), \cdots, v(t))$ and we have the definition:

Definition 3: Let $a \leq b,\ a, b \in \mathbb{Z}$. The set of weakly consistent disturbance signals $\mathcal{N}_p^{[a,b]}(w_2)$ for a plant $P_p,\ p \in \mathcal{P}$ and the observation $w_2 = (u_2, y_2)^\top$ is defined by:

$$\mathcal{N}_p^{[a,b]}(w_2) := \left\{ v \in \mathcal{W}|_{[a,b]} \;\middle|\; \begin{array}{l} \exists (u_0^p, y_0^p)^\top \in \mathcal{W}_e \ s.t. \\ \mathscr{R}_{b-a,b} P_p\,(u_0^p - u_2) = \mathscr{R}_{b-a,b}(y_0^p - y_2), \\ v = (\mathscr{R}_{b-a,b} u_0^p, \mathscr{R}_{b-a,b} y_0^p) \end{array} \right\}.$$

Assumption 4: Let $\lambda \in \mathbb{R}$ be given. The residual operator X factorises $X = NE$ where N is the norm operator, E is an estimation operator, and:
1) (Causality): E is causal.
2) (Weak consistency): For all $p \in \mathcal{P}$ there exists a map $\Phi_\lambda \;:\; map(\mathbb{N}, \mathbb{R}^h) \to \mathbb{R}^{m(\lambda+1)} \times \mathbb{R}^{o(\lambda+1)}$, such that for all $w_2 \in \mathcal{W}_e$ and for all $k \in \mathbb{N}$,

$$\begin{array}{rcll} \Phi_\lambda E(w_2)(k)(p) & \in & \mathcal{N}_p^{[k-\lambda,k]}(w_2), & \text{and,} \\ \|\Phi_\lambda E(w_2)(k)(p)\| & \leq & \|\mathscr{R}_{\lambda,k} E(w_2)(k)(p)\|. & \end{array}$$

3) (Monotonicity): For all $p \in \mathcal{P}$, for all $k, l \in \mathbb{N}$ with $0 \leq k \leq l$ and for all $w_2 \in \mathcal{W}_e$,

$$\|E(w_2)(k)(p)\| \leq \|\mathscr{T}_k E(w_2)(l)(p)\|.$$

$$X : \mathcal{W}_e \to map(\mathbb{N}, map(\mathcal{P}, \mathbb{R}^+)) : w_2 \mapsto [k \to (p \mapsto r_p[k])]$$

$$X = NE$$

$$E : \mathcal{W}_e \to map(\mathbb{N}, map(\mathcal{P}, map(\mathbb{N}, \mathbb{R}^h))) \tag{3.31}$$

$$w_2 \mapsto \big[k \mapsto (p \mapsto d_p[k])\big] \tag{3.32}$$

$$d_p[k] : \mathbb{N} \to map(\mathbb{N}, \mathbb{R}^h)$$

$$d_p[k] = (d_p[k](0), d_p[k](1), \ldots, d_p[k](k), 0, \cdots)$$

$$N : map(\mathbb{N}, map(\mathcal{P}, map(\mathbb{N}, \mathbb{R}^h))) \to map(\mathbb{N}, map(\mathcal{P}, \mathbb{R}^+)) \tag{3.33}$$

$$\big[k \mapsto (p \mapsto d_p[k])\big] \mapsto \big[k \mapsto (p \mapsto \|d_p[k]\| = r_p[k])\big]. \tag{3.34}$$

TABLE 2
FACTORISATION OF THE RESIDUAL OPERATOR

4) (Continuity): There exists a function $\chi : \mathcal{P} \times \mathcal{P} \to \mathbb{R}^+$, $\chi(p,p) = 0$ for all $p \in \mathcal{P}$, such that for all $k \in \mathbb{N}$, $p_1, p_2 \in \mathcal{P}$ and $w_2 \in \mathcal{W}_e$,

$$\|E(w_2)(k)(p_1) - E(w_2)(k)(p_2)\| \leq \chi(p_1, p_2)\|\mathscr{T}_k w_2\|.$$

5) (Minimality): There exists $\mu > 0$ such that for all $k \geq 0$, for $p \in \mathcal{P}$ and for all $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfying equations (2.3)–(2.4) for $P = P_p$,

$$\|E(w_2)(k)(p)\| \leq \mu\|\mathscr{T}_k w_0\|.$$

See [2] for examples of estimators that satisfy these assumptions. Recall that an implementation of the algorithm requires realisation of X, and and not necessarily a realisation of E, as exemplified by the Kalman Filter realisation of the l^2 infinite horizon estimator [2].

The class of controllers under consideration is then:

Definition 5: An EMMSAC controller $C(U, K, \Delta, G, X)$ is said to be standard if it satisfies:

- $K \;:\; \mathcal{P} \to \mathcal{C}$ is a given control design satisfying Assumption 2(1),(2)
- U satisfies
$$\sigma = \max_{p_1, p_2 \in \mathcal{P}^U} \max\{\sigma(p_1), \sigma(K(p_2))\} < \infty. \tag{3.35}$$
- $\Delta : \mathcal{P} \to \mathbb{N}$ is a delay transition function satisfying
$$\Delta(p) > \sigma,\ \forall p \in \mathcal{P}^U. \tag{3.36}$$
- K, Δ satisfy
$$J(\xi) \sup_{p_1 \in \mathcal{P}^U} \alpha^\xi(p_1, K(p_1), \Delta(p_1) - \sigma, \sigma) < 1 \tag{3.37}$$
where α is defined in Assumption 2 and $J(\cdot)$ is defined by Table 3.
- E satisfies Assumptions 4(1)–(5) where
$$\lambda = \max_{p \in \mathcal{P}^U} (2\Delta(p) + \sigma) \tag{3.38}$$
- The switching operator $S = DM(X, G)$ is given by equations (3.12),(3.13)–(3.16) and (3.18)–(3.23)
- The switching controller C is defined by equations (3.24),(3.25).

4. Nominal stability and gain bound analysis

In this section we will establish $l_r,\ 1 \leq r \leq \infty$ norm bounds on the observation signal $w_2 \in \mathcal{W}_e$ in terms of the external disturbance signal $w_0 \in \mathcal{W}$. A particular feature of the bounds is that they depend on the size and geometry of a 'cover' of the plant uncertainty set, rather than the candidate plant set itself. The notion of the cover is as follows. Let $\chi : \mathcal{P} \times \mathcal{P} \to \mathbb{R}^+$ be as in Assumption 4(4). Let

$$H : \mathcal{W}_e \to map(\mathbb{N}, \mathcal{P}^*) \tag{4.39}$$

be a plant-generating operator and let $\nu : \mathcal{W}_e \to map(\mathbb{N}, map(\mathcal{P}, \mathbb{R}^+))$ be given, where for notational convenience we often write $\nu(k)$ for $\nu(w_2)(k)$. Now define

$$\begin{aligned} B_\chi(p, \nu(k)(p)) := \{p\} \cup \{&p_1 \in \mathcal{P} \mid \\ &\chi(p, p_1) < \nu(k)(p)\} \cap U(k),\ p \in \mathcal{P},\ k \in \mathbb{N}. \end{aligned} \tag{4.40}$$

For an appropriate choice of (H, ν), the union of the corresponding neighbourhoods in U then leads to a cover for U:

Definition 6: (H, ν) is said to be a monotonic cover for a plant-generating operator U if for all $k \in \mathbb{N}$, $w_2 \in \mathcal{W}_e$: 1) H and ν define a cover for U: $U(k) \subset R(k)$ for all $k \in \mathbb{N}$ where $R(k) := \cup_{p \in H(k)} B_\chi(p, \nu(k)(p)))$, and 2) The cover is monotonic: $R(k) \subset R(k+1)$, for all $k \in \mathbb{N}$.

The main result now provides gain bounds for the interconnection of the 'true' plant p_* with an EMMSAC controller.

Theorem 7: Let $1 \leq r \leq \infty$. Let $P = P_{p_*}$, where $p_* \in \mathcal{P}^U \subset \mathcal{P}$. Let U be a monotonic plant generating operator and suppose (H, ν) defines a monotonic finite cover for U. Let $k \in \mathbb{N}$. Suppose the EMMSAC controller $C(U, K, \Delta, G, X)$ is standard, and $G(j) \subset U(j),\ j \leq k$. Suppose $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfy the closed loop equations (2.3)–(2.4). Let $\varepsilon > 0$. Let

$$k_* := \begin{cases} \min\{i \in Q_\infty \mid \exists\, p \in G(i),\ \chi(p, p_*) \leq \varepsilon \chi_\nu(H, \nu)\} \\ \text{if } \exists i \text{ s.t. } \exists\, p \in G(j), \chi(p, p_*) \leq \varepsilon \chi_\nu(H, \nu),\ \forall j \geq i, \\ \infty \text{ if not} \end{cases} \tag{4.41}$$

and suppose $k_* < \infty$. If

$$\pi(U(j), H(j), \nu(j), \varepsilon, p^*) > 0, \quad \forall j \leq k \tag{4.42}$$

then:

$$\begin{aligned} \|\mathcal{T}_k w_2\| \quad \leq \quad & \beta(U(k), H(k), \nu(k), \varepsilon, p_*) \|\mathcal{T}_{k_*-1} w_2\| \\ & + \hat{\gamma}(U(k), H(k), \nu(k), \varepsilon, p_*) \|w_0\| \end{aligned} \tag{4.43}$$

where π, β, $\hat{\gamma}$ are given in Table 3.

Note that the set Q_∞ is defined in the next subsection, and coincides with the definition given in [2]. Before proceeding to the proof of Theorem 7, we provide a discussion and interpretation of all the terms in the bound:

$\alpha_{OP}, \beta_{OP}, \alpha_{OS}, \beta_{OS}, \gamma_3, \gamma_4, \gamma_5$ are all constants associated with the performance of the atomic closed loop systems with correctly matched plants and controllers. They are determined by α and β from Assumption 2. α_{OP} bounds the attenuation gain over the relevant set, and can be set as a design parameter (as in the example in Section 6B of [2], which in turn determines the required delay transition $\Delta(p)$ for a given controller $K(p)$.

For $\mathcal{Q}_1 \subset \mathcal{P}^U$ and $\xi = \begin{cases} r & \text{for} \quad 1 \leq r < \infty \\ 1 & \text{for} \quad r = \infty \end{cases}$ let:

$$J(\xi) = \xi \binom{\xi}{\max\{n \in \mathbb{Z} \mid n \leq \xi/2\}} \quad \text{where} \quad \binom{x}{y} := \frac{x!}{y!(x-y)!},$$

$$\begin{aligned}
\alpha_{OP}(\mathcal{Q}_1) &= J(\xi) \sup_{p_1 \in \mathcal{Q}_1} \alpha^\xi(p_1, K(p_1), \Delta(p_1) - \sigma, \sigma) \\
\beta_{OP}(\mathcal{Q}_1) &= J(\xi) \sup_{\substack{\Delta(p_1) \leq x \leq 2\Delta(p_1) \\ p_1 \in \mathcal{Q}_1}} \beta^\xi(p_1, K(p_1), x - \sigma, \sigma) \\
\alpha_{OS}(\mathcal{Q}_1) &= J(\xi) \sup_{\substack{\Delta(p_1) \leq x \leq 2\Delta(p_1) \\ p_1 \in \mathcal{Q}_1}} \alpha^\xi(p_1, K(p_1), 0, x - \sigma) \\
\beta_{OS}(\mathcal{Q}_1) &= J(\xi) \sup_{\substack{\Delta(p_1) \leq x \leq 2\Delta(p_1) \\ p_1 \in \mathcal{Q}_1}} \beta^\xi(p_1, K(p_1), 0, x - \sigma) \\
\gamma_1(p, p_*) &= 1 + \sup_{\Delta(p) \leq x \leq 2\Delta(p)} \alpha(p_*, K(p), 0, x) \\
\gamma_2(p, p_*) &= \sup_{\Delta(p) \leq x \leq 2\Delta(p)} \beta(p_*, K(p), 0, x), \\
\bar{\gamma}_i(\mathcal{Q}_2, \mathcal{Q}_1) &= \sup_{p_2 \in \mathcal{Q}_2} \sup_{p_1 \in \mathcal{Q}_1} \gamma_1(p_2, p_1), \quad i = 1, 2,
\end{aligned}$$

If $1 \leq r < \infty$ let:

$$\begin{aligned}
\gamma_3(\mathcal{Q}_1) &= (1 + \alpha_{OS}^{1/r}(\mathcal{Q}_1)) \left(\frac{\alpha_{OP}(\mathcal{Q}_1)}{1 - \alpha_{OP}(\mathcal{Q}_1)} \right)^{1/r} + \alpha_{OS}^{1/r}(\mathcal{Q}_1) \\
\gamma_4(\mathcal{Q}_1) &= (1 + \alpha_{OS}^{1/r}(\mathcal{Q}_1)) \left(\frac{\beta_{OP}(\mathcal{Q}_1)}{1 - \alpha_{OP}(\mathcal{Q}_1)} \right)^{1/r} \\
\gamma_5(\mathcal{Q}_1) &= \beta_{OS}^{1/r}(\mathcal{Q}_1),
\end{aligned}$$

and if $r = \infty$ let:

$$\begin{aligned}
\gamma_3(\mathcal{Q}_1) &= \max\{1, \alpha_{OS}(\mathcal{Q}_1))\} \alpha_{OP}(\mathcal{Q}_1) + \alpha_{OS}(\mathcal{Q}_1) \\
\gamma_4(\mathcal{Q}_1) &= \max\{1, \alpha_{OS}(\mathcal{Q}_1)\}) \frac{\beta_{OP}(\mathcal{Q}_1)}{1 - \alpha_{OP}(\mathcal{Q}_1)} \\
\gamma_5(\mathcal{Q}_1) &= \beta_{OS}(\mathcal{Q}_1).
\end{aligned}$$

For $\mathcal{Q}_2 \subset \mathcal{P}^H$, $v : \mathcal{P} \to \mathbb{R}^+$, $\mu > 0$, $\varepsilon > 0$ let:

$$\begin{aligned}
\chi_\nu(\mathcal{Q}_2, v) &= 2 \sup_{p \in \mathcal{Q}_2} v(p) \\
\pi(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p^*) &= 1 - 2^{1/r} \varepsilon \chi_\nu(\mathcal{Q}_2, v)(1 + \bar{\gamma}_1^2(\mathcal{Q}_2, \mathcal{Q}_1)) \cdot \\
&\quad \cdot \big(\gamma_4(\mathcal{Q}_1) + \gamma_5(\mathcal{Q}_1)\big)
\end{aligned}$$

$$\eta(\mathcal{Q}_2, v, \varepsilon, p_*) = 2^{\frac{1}{r}} (\mu + \varepsilon \chi_\nu(\mathcal{O}_2, v) \bar{\gamma}_2(\mathcal{O}_2, \{p_*\})(1 + \bar{\gamma}_1(\mathcal{Q}_2, \{p_*\})))$$

$$\beta(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*) = \left(\frac{1 + \gamma_3(\mathcal{Q}_1)}{\pi(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*)} \right)^{|\mathcal{Q}_2|} \prod_{p \in \mathcal{Q}_2} \gamma_1(p, p_*)$$

$$\begin{aligned}
\hat{\gamma}(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*) &= \beta(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*) \Bigg(\sum_{p \in \mathcal{Q}_2} \gamma_2(p, p_*) + \\
&\quad \eta(\mathcal{Q}_2, v, \varepsilon, p_*) \frac{\gamma_4(\mathcal{Q}_1) + \gamma_5(\mathcal{Q}_1)}{\pi(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*)} \Bigg).
\end{aligned}$$

TABLE 3
FUNCTIONS SPECIFYING THE GAIN BOUND.

π lies between 0 and 1 and scales well with the size of the cover: when $|H(k)|$ is large, then typically χ_ν is small and π is close to 1; alternatively the constant ϵ can be made small. In the special case where $U(k)$ is a finite set, then ε can be set to zero and $\pi = 1$. For low dimensional covers, π may be small and contribute significantly to β and $\hat{\gamma}$. A typical application involves determining a value of π close to 1, and then computing the required χ_ν and associated cover.

β is the final bias term. By minimising γ_3 by the atomic control design K, the effect of the exponent $|H(k)|$ can be reduced. In general the control designer has little influence over the term $\prod_{p\in H(k)} \gamma_1(p,p_*)$, with the exception of the terms $\gamma_1(p,p) \geq 1$ (here with $p = p_*$), which are a natural target for minimisation by the control design K. The number of terms in the product is determined by the cover, and hence a cover of small size reduces the effect of this term. $\hat{\gamma}$ is the final gain term. The summation $\sum_{p\in H(k)} \gamma_2(p,p_*)$ may be large and reflects the difficulty of controlling a large model set: the number of terms in the sum is determined directly by the size of the cover. Both terms are directly influenced by the size of the cover. The minimisation constant $\mu \geq 1$ which arises in property 5 of Assumption 4 can be reduced to 1 with optimal estimators, although this term is $O(\mu)$ and has a relatively small effect.

Finally we remark that if (4.43) holds e.g. in l_2, then since $\|w_2\|_\infty \leq \|w_2\|_2$, w_2 is guaranteed to be bounded in both l^2 and l^∞. However, as in [1], if (4.43) holds in l_2 then it does not follow that a similar inequality holds in l^∞. Our viewpoint is very much that the signal space choice is an integral part of the specification of the problem that one is trying to solve. Thus if one was interested in simultaneous l^2 and l^∞ guarantees, it would seem likely that a mixed norm should be chosen, as per [1]; this is outside the scope of the framework at present.

The bounds reflect the modularity of EMMSAC. The control design K can be optimised directly independently of the estimator parameters μ, η and c, and the estimator and atomic controllers performance interact within the bounds via the linkage of the cover. The closed loop $[P,C]$ is independent of the cover (H,ν) which is utilized for analysis terms only. The characterisation and trade-off between χ_ν, ε and $|H(k)|$ is given by the metric entropy, which has been discussed further in [2]. Given K and μ, the optimal cover can in principle be determined directly from the bounds.

We can now begin the construction of the gain bound.

A. Switching times

Let $q_f \in map(\mathbb{N},\mathcal{P})$, and let $q = Dq_f$ (equations (3.18)–(3.23)) denote the switching signal. Let

$$\begin{aligned} L_k &= \{l_0 = 0, l_1, l_2 \cdots\} \\ &= \{l \in \mathbb{N} \mid q(l-1) \neq q(l),\ 0 \leq l \leq k\} \end{aligned} \tag{4.44}$$

be an ordered set, i.e. if $l_i, l_j \in L_k,\ i \leq j$ then $l_i \leq l_j$, where L_k is interpreted as the set of physical switching times up to time $k \in \mathbb{N}$. These are the times where the algorithm switches from one controller to another. To every pair of consecutive physical switching times l_i, l_{i+1} define the set of virtual switching times $V(l_i, l_{i+1})$ by

$$V(l_i,l_{i+1}) = \left\{ a \in \mathbb{N} \;\middle|\; \begin{array}{c} \exists b \in \mathbb{N} \text{ s.t. } a = l_i + b\Delta(q(l_i)), \\ l_i < a \leq l_{i+1} - \Delta(q(l_i)) \end{array} \right\}. \tag{4.45}$$

Virtual switches arise when the algorithm switches to a controller $C_{K(q(l_i))}$ and remain switched to that controller for a period of time longer than the associated transition time $\Delta(q(l_i))$. This is interpreted as a series of consecutive switches to the same controller separated in time by $\Delta(q(l_i))$. A virtual switch differs from a physical switch in that the atomic controller state is not intentionally initialised to zero at the virtual switching time. Note that virtual switching times are defined purely for analytical purposes and do not determine the actual switching algorithm. Now define the ordered set of all switching times, physical and virtual,

$$Q_k = \{k_0 = 0, k_1,\ k_2,\ \cdots\},\ 0 \leq k_i \leq k_{i+1} \leq k \tag{4.46}$$

by

$$Q_k = L_k \cup \bigcup_{i\geq 0} \{V(l_i,l_{i+1}) \mid l_i, l_{i+1} \in L_k\}. \tag{4.47}$$

Let $Q_k(p) = \{i \in Q_k \mid q(i) = p\} \subset Q_k$ be the switching times where the algorithm switches to a plant $p \in \mathcal{P}$. Let $p \in H(k)$ and let

$$Q_k(p,\nu(k)(p)) = \cup_{x\in B_\chi(p,\nu(k)(p))}\{Q_k(x)\} \tag{4.48}$$

be the set of all switching times corresponding to the plants in the neighbourhood $B_\chi(p,\nu(k)(p))$ around a plant $p \in H(k)$. For $p \in H(k)$, let

$$F_k(p,\nu(k)(p)) := \left\{ \begin{array}{c} \{\max(Q_k(p,\nu(k)(p)))\} \\ \text{if } \max(Q_k(p,\nu(k)(p))) \neq \emptyset \\ \emptyset \text{ otherwise} \end{array} \right. \tag{4.49}$$

be the switching time where the algorithm switches to a plant within the neighbourhood $B_\chi(p,\nu(k)(p))$ for the last time in the interval $[0,k]$. Note that $F_k(p,\nu(k)(p))$ is always defined since $\max Q_k(p,\nu(k)(p)) \leq k$. Let

$$F_k = \cup_{p\in H(k)} F_k(p,\nu(k)(p)) \tag{4.50}$$

and note that $F_k(p,\nu(k)(p)) \subset F_k \subset Q_k$. Let

$$O_k(p,\nu(k)(p)) := \left\{ \begin{array}{c} Q(p,\nu(k)(p)) \setminus F_k(p,\nu(k)(p)) \\ \text{if } Q_k(p,\nu(k)(p)) \neq \emptyset \\ \emptyset \text{ otherwise} \end{array} \right. \tag{4.51}$$

be the set of all 'ongoing' switching times corresponding to the plants in the neighbourhood $B_\chi(p,\nu(k)(p))$ around the plant p, i.e. the switching times where the algorithm will switch back to a plant within $B_\chi(p,\nu(k)(p))$ at a subsequent time within the interval $[0,k]$. We let

$$O_k = \cup_{p\in H(k)} O_k(p,\nu(k)(p)) \tag{4.52}$$

and note that $O_k(p,\nu(k)(p)) \subset O_k \subset Q_k$.

For all switching times $k_i \in Q_k$ define the intervals

$$A_i = [k_i - \sigma, k_i), \quad B_i = [k_i, k_{i+1} - \sigma), \tag{4.53}$$

Note that by Lemma 8, $k_{i+1} - k_i \geq \Delta(q(k_i)) > \sigma$ hence $k_{i+1} - \sigma > k_i$, hence A_i, B_i are defined and form a disjoint cover of $\mathbb{N}$. Upper and lower bounds on the switching times are now given as follows:

Lemma 8: Suppose $\Delta : \mathcal{P} \to \mathbb{N}$ is a given delay transition function and suppose the delay operator D is given by equations (3.18)–(3.23). Let $k \in \mathbb{N}$ and let $q_f \in map(\mathbb{N},\mathcal{P}^U)$. Let $q = Dq_f$. Suppose $k_i \leq k_{i+1}$ are consecutive switching times, $k_i, k_{i+1} \in Q_k$, where Q_k is defined by equations (4.44)–(4.47). Let $p = q(k_i)$. Then:

$$\Delta(p) \leq k_{i+1} - k_i < 2\Delta(p). \tag{4.54}$$

Proof: By the definition of the switching delay in equation (3.20) it follows that $\Delta(p) \leq k_{i+1} - k_i$. If k_{i+1} is a virtual switching time, then $k_{i+1} - k_i = \Delta(q(k_i))$ by equation (4.45), and if k_{i+1} is a physical switching time, then

$$k_i := l_i + b\Delta(q(l_i)) \leq k_{i+1} - \Delta(q(l_i)) < l_i + (b+1)\Delta(q(l_i)) = k_i + \Delta(q(l_i)),$$

hence $k_{i+1} - k_i < 2\Delta(q(k_i))$ and inequality (4.54) follows. ■

B. Gain bounds for atomic closed-loop systems

The first result, Proposition 9, establishes bounds on the gain from the disturbance signals w_0 to the internal signals w_2 for the atomic closed-loop interconnection between the true plant and the controller switched into closed-loop at time k_i, i.e. $[P_{p_*}, C_{K(q(k_i))}]$, on the various intervals of type $A_i, B_i,\ k_i \in Q_k$. The two cases $w_2^c|_{A_i} = 0$ and $w_2^c|_{A_i} = w_2|_{A_i}$ correspond to the case whereby the controller is initialised to zero at time k_i i.e. $k_i \in L_k$ (a physical switch) or the case where the controller is not intentionally initialised to zero at time k_i i.e $k_i \in Q_k \setminus L_k$ (a virtual switch).

Proposition 9: Let $1 \leq r \leq \infty$. Suppose $\mathcal{P}^U \subset \mathcal{P}$ satisfies Assumption 3.35. Let $p_* \in \mathcal{P}^U$ and $P = P_{p_*}$. Let $K : \mathcal{P} \to \mathcal{C}$ be a given control design satisfying Assumption 2(1). Suppose Δ is a given delay transition function satisfying Assumption 3.36 and suppose the delay operator D is given by equations (3.18)–(3.23). Let $k \in \mathbb{N}$ and let $q_f \in map(\mathbb{N}, \mathcal{P}^U)$. Let $q = Dq_f$. Suppose $k_i \leq k_{i+1}$ are consecutive switching times, $k_i, k_{i+1} \in Q_k$ where Q_k is defined by equations (4.44)–(4.47) and let the intervals A_i, A_{i+1}, B_i be given by (4.53). Suppose $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e,\ w_2^c \in \mathcal{W}_e$ satisfy equations (2.3)–(2.4),(3.27) on the interval $A_i \cup B_i \cup A_{i+1}$, where $p = q(k_i),\ c = K(p)$ and either

$$w_2^c|_{A_i} = 0, \qquad w_2^c|_{B_i \cup A_{i+1}} = w_2|_{B_i \cup A_{i+1}} \qquad \text{or}$$
$$w_2^c|_{A_i \cup B_i \cup A_{i+1}} = w_2|_{A_i \cup B_i \cup A_{i+1}}.$$

Then, in both cases,

$$\|\mathscr{T}_{k_{i+1}-1} w_2\| \leq \gamma_1(p, p_*)\|\mathscr{T}_{k_i - 1} w_2\| + \gamma_2(p, p_*)\|w_0\|$$

where γ_1 and γ_2 are given by Table 3.

Proof: Let $I_1 = A_i = [k_i - \sigma, k_i),\ I_2 = \emptyset,\ I_3 = B_i \cup A_{i+1} = [k_i, k_{i+1})$. Since $|I_1| = |A_i| = \sigma \geq \max\{\sigma(p_*), \sigma(K(q(k_i)))\}$ by Assumption 2(1), we have for the closed-loop $[P_{p_*}, C_{K(q(k_i))}]$ that

$$\begin{aligned}
\|\mathscr{T}_{k_{i+1}-1} w_2\| &\leq \|\mathscr{T}_{k_i-1} w_2\| + \|w_2|_{I_3}\| \\
&\leq \|\mathscr{T}_{k_i-1} w_2\| \\
&\quad + \alpha(p_*, K(q(k_i)), 0, |I_3|)\|w_2|_{I_1}\| \\
&\quad + \beta(p_*, K(q(k_i)), 0, |I_3|)\|w_0|_{I_1 \cup I_2 \cup I_3}\| \\
&\leq (1 + \alpha(p_*, K(q(k_i)), 0, |I_3|))\|\mathscr{T}_{k_i-1} w_2\| \\
&\quad + \beta(p_*, K(q(k_i)), 0, |I_3|)\|w_0\|.
\end{aligned}$$

By Lemma 8 we now have $\Delta(p) \leq |I_3| = k_{i+1} - k_i \leq 2\Delta(p)$. and obtain:

$$\begin{aligned}
\|\mathscr{T}_{k_{i+1}-1} w_2\| &\leq (1 + \alpha(p_*, K(p), 0, |I_3|))\|\mathscr{T}_{k_i-1} w_2\| \\
&\quad + \beta(p_*, K(p), 0, |I_3|)\|w_0\| \\
&\leq \gamma_1(p, p_*)\|\mathscr{T}_{k_i-1} w_2\| + \gamma_2(p, p_*)\|w_0\|
\end{aligned}$$

as required. ■

The next result establishes bounds on the gain from the disturbance signals w_0^p to the internal signals w_2 for the atomic closed-loop $[P_p, C_{K(p)}],\ p = q(k_i)$ on the various intervals of type $A_i, B_i,\ k_i \in Q_k$. That is the closed-loop interconnection between: the controller to which the algorithm switches to at time k_i, and its corresponding plant.

Proposition 10: Let $1 \leq r \leq \infty$. Suppose $p \in \mathcal{Q} \subset \mathcal{P}^U \subset \mathcal{P},\ c = K(p)$ and $\mathcal{P}^U$ satisfies equation (3.35). Let $K : \mathcal{P} \to \mathcal{C}$ be a given control design satisfying Assumption 2(1),(2). Suppose Δ is a given delay transition function satisfying inequality (3.36) and suppose the delay operator D is given by equations (3.18)–(3.23). Let $k \in \mathbb{N}$ and let $q_f \in map(\mathbb{N}, \mathcal{P}^U)$. Let $q = Dq_f$ and suppose $q(k_{i+1}) = p$. Suppose $k_i \leq k_{i+1}$ are consecutive switching times, $k_i, k_{i+1} \in Q_k$ where Q_k is defined by equations (4.44)–(4.47). Let the intervals A_i, A_{i+1}, B_i be given by (4.53). Suppose $(w_0^p, w_1^p, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e,\ w_2^c \in \mathcal{W}_e$ satisfy equations (3.26),(3.27) on the interval $A_i \cup B_i \cup A_{i+1}$ and either

$$w_2^c|_{A_i} = 0,\ w_2^c|_{B_i \cup A_{i+1}} = w_2|_{B_i \cup A_{i+1}}, \quad \text{or} \quad (4.55)$$
$$w_2^c|_{A_i \cup B_i \cup A_{i+1}} = w_2|_{A_i \cup B_i \cup A_{i+1}}. \quad (4.56)$$

Then, in both cases, for $1 \leq r < \infty$:

$$\begin{aligned}
\|w_2|_{A_{i+1}}\|_r^r &\leq \alpha_{OP}(\mathcal{Q})\|w_2|_{A_i}\|_r^r \\
&\quad + \beta_{OP}(\mathcal{Q})\|w_0^{q(k_{i+1})}|_{A_i \cup B_i \cup A_{i+1}}\|_r^r \\
\|w_2|_{B_i}\|_r^r &\leq \alpha_{OS}(\mathcal{Q})\|w_2|_{A_i}\|_r^r \\
&\quad + \beta_{OS}(\mathcal{Q})\|w_0^{q(k_{i+1})}|_{A_i \cup B_i}\|_r^r
\end{aligned}$$

and similarly for $r = \infty$:

$$\begin{aligned}
\|w_2|_{A_{i+1}}\|_\infty &\leq \alpha_{OP}(\mathcal{Q})\|w_2|_{A_i}\|_\infty \\
&\quad + \beta_{OP}(\mathcal{Q})\|w_0^{q(k_{i+1})}|_{A_i \cup B_i \cup A_{i+1}}\|_\infty, \\
\|w_2|_{B_i}\|_\infty &\leq \alpha_{OS}(\mathcal{Q})\|w_2|_{A_i}\|_\infty \\
&\quad + \beta_{OS}(\mathcal{Q})\|w_0^{q(k_{i+1})}|_{A_i \cup B_i}\|_\infty
\end{aligned}$$

where α_{OP}, β_{OP}, α_{OS} and β_{OS} are given by Table 3.

Proof: By Lemma 8, inequality (4.54) we have

$$\Delta(p) \leq |B_i \cup A_{i+1}| = |B_i| + \sigma = k_{i+1} - k_i \leq 2\Delta(p). \quad (4.57)$$

Let $I_1 = A_i = [k_i - \sigma, k_i),\ I_2 = B_i = [k_i, k_{i+1} - \sigma)$ and $I_3 = A_{i+1} = [k_{i+1} - \sigma, k_{i+1})$. By equation (3.35),

$$|I_1| = |A_i| = \sigma \geq \max\{\sigma(p), \sigma(K(p))\},$$

and it follows from Assumption 2(1) inequality (3.29) that:

$$\begin{aligned}
\|w_2|_{A_{i+1}}\|_r^\xi &\leq \big(\alpha(p, K(p), |B_i|, |A_{i+1}|)\|w_2|_{A_i}\|_r \\
&\quad + \beta(p, K(p), |B_i|, |A_{i+1}|) \\
&\quad \cdot \|w_0^p|_{A_i \cup B_i \cup A_{i+1}}\|_r\big)^\xi \\
&\leq \big(\alpha(p, K(p), \Delta(p) - \sigma, \sigma)\|w_2|_{A_i}\|_r \\
&\quad + \beta(p, K(p), |B_i|, \sigma)\|w_0^p|_{A_i \cup B_i \cup A_{i+1}}\|_r\big)^\xi,
\end{aligned}$$

where the second inequality follows from Assumption 2(2)) and since $|B_i| \geq \Delta(p) - \sigma$ (inequality (4.57)). Hence by inequality (4.57) and since

$$(a+b)^\xi \leq J(\xi)(a^\xi + b^\xi), \qquad a, b \geq 0 \quad (4.58)$$

we obtain

$$
\begin{aligned}
\|w_2|_{A_{i+1}}\|_r^\xi &\le J(\xi)\alpha^\xi(p,K(p),\Delta(p)-\sigma,\sigma)\|w_2|_{A_i}\|_r^\xi \\
&\quad +J(\xi)\beta^\xi(p,K(p),|B_i|,\sigma) \\
&\quad \cdot\|w_0^p|_{A_i\cup B_i\cup A_{i+1}}\|_r^\xi \\
&\le \alpha_{OP}\|w_2|_{A_i}\|_r^\xi+\beta_{OP}\|w_0^p|_{A_i\cup B_i\cup A_{i+1}}\|_r^\xi.
\end{aligned}
$$

Now let $I_1=A_i=[k_i-\sigma,k_i)$, $I_2=\emptyset$, $I_3=B_i=[k_i,k_{i+1}-\sigma)$. By equation (3.35), $|I_1|\ge\sigma\ge\max\{\sigma(p_1),\sigma(K(p_2))\}$ and it follows from Assumption 2(1) (inequality (3.29)), inequality 4.58 and inequality (4.57) that:

$$
\begin{aligned}
\|w_2|_{B_i}\|_r^\xi &\le \big(\alpha(p,K(p),0,|B_i|)\|w_2|_{A_i}\|_r \\
&\quad +\beta(p,K(p),0,|B_i|)\|w_0^p|_{A_i\cup B_i}\|_r\big)^\xi \\
&\le J(\xi)\alpha^\xi(p,K(p),0,|B_i|)\|w_2|_{A_i}\|_r^\xi \\
&\quad +J(\xi)\beta^\xi(p,K(p),0,|B_i|)\|w_0^p|_{A_i\cup B_i}\|_r^\xi \\
&\le \alpha_{OS}\|w_2|_{A_i}\|_r^\xi+\beta_{OS}\|w_0^p|_{A_i\cup B_i}\|_r^\xi
\end{aligned}
$$

as required. ∎

C. Bounds on disturbance estimates

The next proposition follows gives a bound on a series of disturbance estimates corresponding to the switching signal. Repeated application of the minimality property (Assumption 4(2)) allows us to bound the series of disturbance estimates in terms of the disturbance estimate corresponding to the switch at the end of the sequence.

Proposition 11: Let the switching operator $S=DM(NE,G)$ be given by equations (3.14)–(3.16),(3.18)–(3.23), (3.31)–(3.34), where G is a plant-generating operator. E is given by equations (3.31),(3.32) and satisfies Assumptions 4(2)–(3) for $\lambda\in\mathbb{N}$. Suppose $w_2\in\mathcal{W}_e$. Let $k_i\in Q_k$, $i\in\mathbb{N}$ be defined by equations (4.44)–(4.47), $q(k_i)=S(w_2)(k_i)$, let $\tilde{k}_i=k_{i+1}-1$, and let $d_z[k]=E(w_2)(k)(z)$. Then:

$$
\begin{aligned}
&\|\Phi_\lambda d_{q(k_{m+1})}[\tilde{k}_m],\Phi_\lambda d_{q(k_{m+2})}[\tilde{k}_{m+1}],\cdots,\Phi_\lambda d_{q(k_{n+1})}[\tilde{k}_n]\| \\
&\qquad\le 2^{1/r}\|d_{q(k_{n+1})}[k_{n+1}]\| \qquad (4.59)
\end{aligned}
$$

where Φ_λ is defined by Assumption 4(2).

Proof: We first claim that for $1\le j\le i$:

$$
\begin{aligned}
&\|\Phi_\lambda d_{q(k_{m+1})}[\tilde{k}_m],\Phi_\lambda d_{q(k_{m+2})}[\tilde{k}_{m+1}],\cdots,\Phi_\lambda d_{q(k_{j+1})}[\tilde{k}_j]\| \\
&\qquad\le\|d_{q(k_j)}[\tilde{k}_{j-1}],d_{q(k_{j+1})}[\tilde{k}_j]\|. \qquad (4.60)
\end{aligned}
$$

We now prove the claim by induction. Let $i=j=1$. For ease of notation let $\mathscr{R}_\sigma d_z[k]=\mathscr{R}_{\sigma,k}d_z[k]$. Since

$$
\|\mathscr{R}_\lambda d_{q(k_{l+1})}[\tilde{k}_l]\|\le\|d_{q(k_{l+1})}[\tilde{k}_l]\|, \qquad (4.61)
$$

we have

$$
\begin{aligned}
&\|\Phi_\lambda d_{q(k_{m+1})}[\tilde{k}_m],\Phi_\lambda d_{q(k_{m+2})}[\tilde{k}_{m+1}]\| \\
&\overset{4(2),(2.1)}{\le}\|\|\mathscr{R}_\lambda d_{q(k_{m+1})}[\tilde{k}_m]\|,\|\mathscr{R}_\lambda d_{q(k_{m+2})}[\tilde{k}_{m+1}]\|\| \\
&\qquad\overset{(4.61),(2.1)}{\le}\|d_{q(k_{m+1})}[\tilde{k}_m],d_{q(k_{m+2})}[\tilde{k}_{m+1}]\|.
\end{aligned}
$$

Therefore the base step is shown. For the inductive step, assume inequality (4.60) holds for $1\le j\le i-1$. We first show for $l\ge 0$,

$$
\|d_{q(k_{l-1})}[\tilde{k}_{l-2}],\mathscr{R}_\lambda d_{q(k_{l+1})}[\tilde{k}_l]\|\le\|d_{q(k_{l+1})}[\tilde{k}_l]\|. \qquad (4.62)
$$

This follows since:

$$
\begin{aligned}
\|d_{q(k_{l-1})}[\tilde{k}_{l-2}]\| &= \|d_{q(k_{l-1})}[k_{l-1}-1]\| \\
&\overset{\text{Ass. }4(3)}{\le}\|d_{q(k_{l-1})}[k_{l-1}]\| \\
&\le\|d_{q(k_{l+1})}[k_{l-1}]\| \qquad (4.63)
\end{aligned}
$$

where the third inequality follows from the definition of the switch (3.16) and the fact that $k_{l-1}\in Q_k$, hence $q_f(k_{l-1})=q(k_{l-1})$. Then since $k_{l-1}<\tilde{k}_l-\lambda$,

$$
\begin{aligned}
&\|d_{q(k_{l-1})}[\tilde{k}_{l-2}],\mathscr{R}_\lambda d_{q(k_{l+1})}[\tilde{k}_l]\| \\
&\overset{(4.63)}{\le}\|d_{q(k_{l+1})}[k_{l-1}],\mathscr{R}_\lambda d_{q(k_{l+1})}[\tilde{k}_l]\| \\
&\overset{\text{Ass. }4(3)}{\le}\|\mathscr{T}_{k_{l-1}}d_{q(k_{l+1})}[k_l],\mathscr{R}_\lambda d_{q(k_{l+1})}[\tilde{k}_l]\| \\
&\le\|d_{q(k_{l+1})}[\tilde{k}_l]\|.
\end{aligned}
$$

as required. Then by the inductive hypothesis:

$$
\begin{aligned}
&\|\Phi_\lambda d_{q(k_{m+1})}[\tilde{k}_m],\Phi_\lambda d_{q(k_{m+2})}[\tilde{k}_{m+1}],\cdots,\Phi_\lambda d_{q(k_{i+1})}[\tilde{k}_i]\| \\
&\overset{(4.60),(2.1)}{\le}\left\|\begin{matrix}\|d_{q(k_{i-2})}[\tilde{k}_{i-3}]\|,\|d_{q(k_{i-1})}[\tilde{k}_{i-2}]\|,\\ \|\Phi_\lambda d_{q(k_i)}[\tilde{k}_{i-1}]\|,\|\Phi_\lambda d_{q(k_{i+1})}[\tilde{k}_i]\|\end{matrix}\right\| \\
&\overset{\text{Ass. }4(2)}{\le}\left\|\begin{matrix}\|d_{q(k_{i-2})}[\tilde{k}_{i-3}]\|,\|d_{q(k_{i-1})}[\tilde{k}_{i-2}]\|,\\ \|\mathscr{R}_\lambda d_{q(k_i)}[\tilde{k}_{i-1}]\|,\|\mathscr{R}_\lambda d_{q(k_{i+1})}[\tilde{k}_i]\|\end{matrix}\right\| \\
&\overset{(2.1)}{=}\left\|\begin{matrix}\|d_{q(k_{i-2})}[\tilde{k}_{i-3}],\mathscr{R}_\lambda d_{q(k_i)}[\tilde{k}_{i-1}]\|,\\ \|d_{q(k_{i-1})}[\tilde{k}_{i-2}],\mathscr{R}_\lambda d_{q(k_{i+1})}[\tilde{k}_i]\|\end{matrix}\right\| \\
&\overset{(4.62)}{\le}\|d_{q(k_i)}[\tilde{k}_{i-1}],d_{q(k_{i+1})}[\tilde{k}_i]\|
\end{aligned}
$$

This completes the inductive step and establishes the claimed inequality (4.60) for $j=n$ as required. Since (3.16) implies $\|d_{q(k_n)}[k_n]\|\le\|d_{q(k_{n+1})}[k_n]\|$, the result follows:

$$
\begin{aligned}
&\|d_{q(k_n)}[\tilde{k}_{n-1}],d_{q(k_{n+1})}[\tilde{k}_n]\| \\
&\overset{\text{Ass.4(3)}}{\le}\|d_{q(k_n)}[k_n],d_{q(k_{n+1})}[k_{n+1}]\|, \\
&\le\|d_{q(k_{n+1})}[k_n],d_{q(k_{n+1})}[k_{n+1}]\| \\
&\overset{\text{Ass.4(3)}}{\le}\|d_{q(k_{n+1})}[k_{n+1}],d_{q(k_{n+1})}[k_{n+1}]\| \\
&\le\|1,1\|\|d_{q(k_{n+1})}[k_{n+1}]\| \\
&=2^{1/r}\|d_{q(k_{n+1})}[k_{n+1}]\|.
\end{aligned}
$$

∎

The next proposition shows that if the algorithm switches at time x to a plant z then the disturbance estimate at time x, can be bounded by the real disturbance w_0 and a correction factor dependent on w_2 and the χ distance between the true plant p^* (which generates the closed loop system signals) and a plant $p\in G(x)$ (which models p^*).

Proposition 12: Let $1\le r\le\infty$. Suppose $p_*\in\mathcal{P}^U\subset\mathcal{P}$. Suppose Δ is a given delay transition function and suppose the delay operator D is given by equations (3.18)–(3.23). Suppose G is a plant-generating operator. Suppose

E satisfies Assumptions 4(1)–(5) for some $\lambda \in \mathbb{R}$ and the switching operator $S = DM(NE,G)$ is given by equations (3.14)–(3.16),(3.18)–(3.23), (3.31)–(3.34). Let $k \in \mathbb{N}$. Suppose $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfy equations (2.3)–(2.4) for $P = P_{p_*}$. Let $x \in Q_k,\ z = q(x) = S(w_2)(x)$ where Q_k is defined by equations (4.44)–(4.47) and suppose $p \in G(x)$. Then:

$$\|E(w_2)(z)(x)\| = \|d_z[x]\| \le \mu \|\mathscr{T}_x w_0\| + \chi(p,p_*)\|\mathscr{T}_x w_2\|.$$

Proof: By definition, $q_f(t) = M(NE,G)(t),\ t \in \mathbb{N}$, will always point to the plant whose corresponding disturbance estimates are minimal. By the definition of M and since $p \in G(x)$, we have

$$\|d_{q_f(x)}[x]\| = \inf_{p \in G(x)} \|d_p[x]\| \le \|d_p[x]\|.$$

SInce $q_f(x) = q(x)$ by the definition of D, it follows that

$$\|d_z[x]\| = \|d_{q(x)}[x]\| = \|d_{q_f(x))}[x]\| \le \|d_p[x]\|. \qquad (4.64)$$

Then

$$\begin{aligned} \|d_p[x]\| &= \|E(w_2)(x)(p)\| \\ &\le \|E(w_2)(x)(p_*)\| + \chi(p,p_*)\|\mathscr{T}_x w_2\| \\ &\le \mu\|\mathscr{T}_x w_0\| + \chi(p,p_*)\|\mathscr{T}_x w_2\|. \end{aligned}$$

where the first inequality follows by Assumption 4(4) and the second by Assumption 4(5), and the result follows. ■

D. Gain bounds for non-final switching intervals

We first give an intermediate result that is self-contained and purely combinatorial.

Proposition 13: Let $1 \le r \le \infty$ and $\xi = \begin{cases} r & \text{for } 1 \le r < \infty \\ 1 & \text{for } r = \infty \end{cases}$. Let $z, f, \beta, \epsilon : \mathbb{N} \to \mathbb{R}^+$ and $a, b, d, e \in \mathbb{R}^+,\ a < 1$. Let $m, n \in \mathbb{N}$ and suppose for all $m \le i \le n$:

$$\begin{aligned} z_{i+1}^{\xi} &\le a z_i^{\xi} + d\beta_i^{\xi} \qquad (4.65) \\ f_i^{\xi} &\le b z_i^{\xi} + e\epsilon_i^{\xi}. \qquad (4.66) \end{aligned}$$

Then:

$$\begin{aligned} &\left\| z|_{[m+1,n+1]}, f|_{[m,n]} \right\| \\ &\quad \le \tilde{\gamma}_3(\mathcal{G})|z_m| + \tilde{\gamma}_4(\mathcal{G})\|\beta|_{[m,n]}\| + \tilde{\gamma}_5(\mathcal{G})\|\epsilon|_{[m,n]}\| \end{aligned}$$

where $\mathcal{G} = (a,b,d,e)$ and

$$\tilde{\gamma}_3(\mathcal{G}) = \begin{cases} \left(\frac{(1+b^{1/r})^r a}{1-a}\right)^{1/r} + b^{1/r} & \text{if } 1 \le r < \infty, \\ \max\{1,b\}a + b & \text{if } r = \infty, \end{cases}$$

$$\tilde{\gamma}_4(\mathcal{G}) = \begin{cases} \left(\frac{(1+b^{1/r})^r d}{1-a}\right)^{1/r} & \text{if } 1 \le r < \infty, \\ \max\{1,b\}\frac{d}{1-a} & \text{if } r = \infty, \end{cases}$$

$$\tilde{\gamma}_5(\mathcal{G}) = \begin{cases} e^{1/r} & \text{for } 1 \le r < \infty, \\ e & \text{for } r = \infty. \end{cases}$$

Proof: Let $1 \le \xi = r < \infty$. By (4.65) we have

$$\begin{aligned} z_{m+1}^r &\le a z_m^r + d\beta_m^r \\ z_{m+2}^r &\le a^2 z_m^r + d\left(a\beta_m^r + \beta_{m+1}^r\right) \\ z_{m+3}^r &\le a^3 z_m^r + d\left(a^2\beta_m^r + a\beta_{m+1}^r + \beta_{m+2}^r\right) \\ &\vdots \\ z_{n+1}^r &\le a^{n-m+1} z_m^r + d\Big(\beta_m^r a^{n-m} + \cdots \\ &\qquad \cdots + \beta_{m+1}^r a^{n-m-1} + \beta_{n-1}^r a + \beta_n^r\Big). \end{aligned}$$

Summing vertically gives:

$$\begin{aligned} \sum_{i=m+1}^{n+1} z_i^r &= z_m^r \sum_{i=1}^{n-m+1} a^i + d\Big(\beta_m^r \sum_{i=0}^{n-m} a^i + \cdots \\ &\qquad \cdots + \beta_{m+1}^r \sum_{i=0}^{n-m-1} a^i + \beta_{n-1}^r \sum_{i=0}^{1} a^i + \beta_n^r\Big) \\ &\le z_m^r \sum_{i=1}^{n-m+1} a^i + d\sum_{j=m}^{n} \beta_j^r \sum_{i=0}^{n-j} a^i. \end{aligned}$$

Since $a < 1$, it follows that $\sum_{i=0}^{j} a^i \le \frac{1}{1-a}$ for all $j > 0$, and hence

$$\|z|_{[m+1,n+1]}\|_r^r = \sum_{i=m+1}^{n+1} z_i^r \le \frac{1}{1-a}\left(a z_m^r + d\sum_{i=m}^{n} \beta_i^r\right). \qquad (4.67)$$

Therefore

$$\|z|_{[m+1,n+1]}\|_r \le \left(\frac{1}{1-a}\right)^{\frac{1}{r}} \left(a^{\frac{1}{r}}|z_m| + d^{\frac{1}{r}}\left\|\beta|_{[m,n]}\right\|_r\right).$$

By inequality (4.66) we have

$$\begin{aligned} \left\|f|_{[m,n]}\right\|_r &\le \left(b\sum_{i=m}^{n} z_i^r + e\sum_{i=m}^{n} \epsilon_i^r\right)^{\frac{1}{r}} \\ &\le b^{\frac{1}{r}}\left\|z|_{[m,n]}\right\|_r + e^{\frac{1}{r}}\left\|\epsilon|_{[m,n]}\right\|_r. \qquad (4.68) \end{aligned}$$

By inequalities (4.67) and (4.68) we then obtain:

$$\begin{aligned} &\left\|z|_{[m+1,n+1]}, f|_{[m,n]}\right\|_r \\ &\quad \le \left\|z|_{[m+1,n+1]}\right\|_r + b^{\frac{1}{r}}\left\|z|_{[m,n]}\right\|_r + e^{\frac{1}{r}}\left\|\epsilon|_{[m,n]}\right\|_r \\ &\quad \le \left(1 + b^{\frac{1}{r}}\right)\left\|z|_{[m+1,n+1]}\right\|_r + b^{\frac{1}{r}}|z_m| + e^{\frac{1}{r}}\left\|\epsilon|_{[m,n]}\right\|_r \\ &\quad \le \left(\frac{\left(1 + b^{\frac{1}{r}}\right)^r}{1-a}\right)^{\frac{1}{r}} \left(a^{\frac{1}{r}}|z_m| + d^{\frac{1}{r}}\left\|\beta|_{[m,n]}\right\|_r\right) \\ &\quad + b^{\frac{1}{r}}|z_m| + e^{\frac{1}{r}}\left\|\epsilon|_{[m,n]}\right\|_r \\ &\quad \le \tilde{\gamma}_3(\mathcal{G})|z_m| + \tilde{\gamma}_4(\mathcal{G})\|\beta|_{[m,n]}\|_r + \tilde{\gamma}_5(\mathcal{G})\|\epsilon|_{[m,n]}\|_r \end{aligned}$$

as required.

Let $r = \infty$, so $\xi = 1$. By inequality (4.65) we have

$$\begin{aligned} z_{m+1} &\leq a z_m + d\beta_m \\ z_{m+2} &\leq a^2 z_m + d\left(a\beta_m + \beta_{m+1}\right) \\ z_{m+3} &\leq a^3 z_m + d\left(a^2\beta_m + a\beta_{m+1} + \beta_{m+2}\right) \\ \vdots \;&\;\vdots\;\; \vdots \\ z_{n+1} &\leq a^{n-m+1} z_m + d\big(\beta_m a^{n-m} + \beta_{m+1} a^{n-m-1} \\ &\qquad + \cdots + \beta_{n-1} a + \beta_n\big). \end{aligned}$$

Taking norms leads to

$$\begin{aligned} \left\| z|_{[m+1,n+1]} \right\|_\infty &\leq a|z_m| + d\sum_{i=0}^{n-m} a^i \|\beta|_{[m,n]}\|_\infty \\ &\leq a|z_m| + \frac{d}{1-a}\|\beta|_{[m,n]}\|_\infty. \end{aligned}$$

Furthermore by inequality (4.66) we have

$$\left\| f|_{[m,n]} \right\|_\infty \leq b\left\| z|_{[m,n]} \right\|_\infty + e\left\| \epsilon|_{[m,n]} \right\|_\infty .$$

Substitutions lead to:

$$\begin{aligned} &\left\| z|_{[m+1,n+1]}, f|_{[m,n]} \right\|_\infty \\ &\leq \max\{\|z|_{[m+1,n+1]}\|_\infty, b\|z_{[m,n]}\|_\infty + e\|\epsilon_{[m,n]}\|_\infty\} \\ &\leq \max\{1,b\}\|z|_{[m+1,n+1]}\|_\infty + b|z_m| + e\|\epsilon_{[m,n]}\|_\infty \\ &\leq \big(\max\{1,b\}a + b\big)|z_m| + \max\{1,b\}\frac{d}{1-a}\|\beta|_{[m,n]}\|_\infty \\ &+ e\|\epsilon_{[m,n]}\|_\infty \\ &\leq \tilde{\gamma}_3(\mathcal{G})|z_m| + \tilde{\gamma}_4(\mathcal{G})\|\beta|_{[m,n]}\ \ \|_\infty + \tilde{\gamma}_5(\mathcal{G})\|\epsilon_{[m,n]}\|_\infty \end{aligned}$$

as required. ■

In Proposition 10 we established a gain relationship between w_2 and disturbance signals w_0^p which are consistent with the plant $p \in \mathcal{P}$ and the observed signal $w_2 \in \mathcal{W}_e$ over some finite interval. Since it is the overall goal to establish a bound on the gain from the disturbances w_0 to the internal signals w_2 we need to bound the consistent disturbance signals w_0^p by the true disturbances w_0. We do this by considering intervals $[k_m, k_n],\ m, n \in \mathbb{N},\ m \leq n,\ k_m, k_n \in Q_k$ where all intermediate switching times are ongoing, i.e. $k_i \in O_k,\ m \leq i \leq n$ and then use the fact that after a series of ongoing switches there must follow a final switch hence Proposition 12 is applicable. The next result establishes bounds on w_2 in terms of w_0 over certain intervals.

Proposition 14: Let $1 \leq r \leq \infty$. Suppose $p_* \in \mathcal{P}^U \subset \mathcal{P}$ where $\mathcal{P}^U$ satisfies Assumption 3.35. Let $P = P_{p_*}$. Let U be a monotonic plant generating operator and suppose (H, ν) defines a monotonic cover for U. Let $k \in \mathbb{N}$. Suppose the EMMSAC controller $C(U, K, \Delta, G, X)$ is standard, and $G(j) \subset U(j),\ j \leq k$. Suppose $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfy the closed-loop $[P, C]$ equations (2.3)–(2.5) over the interval $[0, k)$. Let $k_i,\ i \in \mathbb{N}$ be defined by equations (4.44)–(4.47) and suppose $k_{n+1} \leq k$. Let $m, n \in \mathbb{N}$, suppose $F_k \cap [k_m - \sigma, k_{n+1}] = \emptyset$. Let $\varepsilon > 0$. If there exists $p \in G(j),\ j \geq k_m$ such that $\chi(p, p_*) \leq \varepsilon\chi_\nu(H, \nu)$,

$$\pi(\mathcal{Q}_1, \mathcal{Q}_2, \nu(k), \varepsilon, p_*) > 0,\ \forall j \leq k \tag{4.69}$$

and $\alpha_{OP}(U(k)) < 1$ then

$$\begin{aligned} \|\mathscr{T}_{k_{n+1}-1} w_2\| &\leq \gamma_6(U(k), H(k), \nu(k), \varepsilon, p_*)\|\mathscr{T}_{k_m-1} w_2\| \\ &\quad + \gamma_7(U(k), H(k), \nu(k), \varepsilon, p_*)\|w_0\|, \end{aligned} \tag{4.70}$$

where α_{OP}, χ_ν, γ_3, γ_4, γ_5, π, η are given by Table 3 and γ_6, γ_7 are given by

$$\gamma_6(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*) = \frac{1 + \gamma_3(\mathcal{Q}_1)}{\pi(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*)}$$

$$\gamma_7(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*) = \frac{\eta(\mathcal{Q}_2, v, \varepsilon, p_*)\big(\gamma_4(\mathcal{Q}_1) + \gamma_5(\mathcal{Q}_1)\big)}{\pi(\mathcal{Q}_1, \mathcal{Q}_2, v, \varepsilon, p_*)}$$

Proof: Let $k \in \mathbb{N}$. Let $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ denote the solution to the closed-loop equations (2.3)–(2.5) with $P = P_{p_*}$ and C as in equations (3.24),(3.25). Let the intervals A_i, B_i be defined by (4.53). In particular $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfy equations (2.3)–(2.5) on the intervals $A_i \cup B_i \cup A_{i+1}$ where

$$A_i \cup B_i \cup A_{i+1} \subseteq [k_m - \sigma, k_{n+1}) \subseteq [0, k)$$

for $m \leq i \leq n$. For $k_i \in Q_k$, let $\bar{k}_i = k_{i+1} - k_i + \sigma - 1$, $\tilde{k}_i = k_{i+1} - 1$ and note that $A_i \cup B_i \cup A_{i+1} = [\tilde{k}_i - \bar{k}_i, \tilde{k}_i]$.

We now intend to apply Proposition 10. By Lemma 8, inequality (3.36) and equations (3.38) we have

$$0 \leq \bar{k}_i = k_{i+1} - k_i + \sigma - 1 \leq 2\Delta(q(k_i)) + \sigma \leq \lambda. \tag{4.71}$$

Let $p = q(k_i)$. Define

$$w_0^p(k) = \begin{cases} \Phi_{\bar{k}_i} d_p[\tilde{k}_i](k) & \text{if } k \in A_i \cup B_i \cup A_{i+1} \\ 0 & \text{otherwise} \end{cases}.$$

By Assumption 4(2) we know that $\Phi_{\bar{k}_i} d_p[\tilde{k}_i] \in \mathcal{N}_p^{[\tilde{k}_i - \bar{k}_i, \tilde{k}_i]}(w_2)$. For every $k_i \in Q_k$ let $w_2^c \in \mathcal{W}_e$ satisfy

$$w_2^c(k) = \begin{cases} w_2(k) \text{ if } k \in B_i \cup A_{i+1} \text{ and } k_i \in L_k \\ w_2(k) \text{ if } k \in A_i \cup B_i \cup A_{i+1} \text{ and } k_i \in Q_k \setminus L_k \\ 0 \quad\quad \text{otherwise.} \end{cases}$$

Note that w_2, w_2^c satisfy equations (4.55),(4.56) of Proposition 10. There exists a $w_1^p \in \mathcal{W}_e$ such that $(w_0^p, w_1^p, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfies equations (2.3)–(2.5) for $P = P_p$ and C as in equation (3.25) on the intervals

$$A_i \cup B_i \cup A_{i+1} = [k_i - \sigma, k_{i+1}) = [\tilde{k}_i - \bar{k}_i, \tilde{k}_i].$$

To see this observe that w_2 is generated by the special structure of C, i.e. from the controller C_c at time k_i which is initialised to zero if $k_i \in L_k$ and inherits an initial value at time k_i determined from $w_2|_{A_i}$ if $k_i \in Q_k \setminus L_k$. Let

$$\begin{aligned} a &= \alpha_{OP}(U(k)) & b &= \alpha_{OS}(U(k)) \\ d &= \beta_{OP}(U(k)) & e &= \beta_{OS}(U(k)) \\ z_i &= \|w_2|_{A_i}\|_r & f_i &= \|w_2|_{B_i}\|_r \end{aligned}$$

$$\beta_i = \|w_0^{q(k_{i+1})}|_{A_i \cup B_i \cup A_{i+1}}\|_r = \|\Phi_{\bar{k}_i} d_{q(k_{i+1})}[\tilde{k}_i]\|_r$$

$$\epsilon_i = \|w_0^{q(k_{i+1})}|_{A_i \cup B_i}\|_r \leq \beta_i = \|\Phi_{\bar{k}_i} d_{q(k_{i+1})}[\tilde{k}_i]\|_r$$

where we note that $z_i = \|w_2|_{A_i}\|_r \geq \|w_2^c|_{A_i}\|_r$. Since U is monotonic, hence $G(k_i) \subset U(k_i) \subset U(k)$, it follows that for all $k_i \in Q_k$ that:

$$\begin{aligned}\alpha_{OP}(G(k_i)) &\leq \alpha_{OP}(U(k_i)) \leq \alpha_{OP}(U(k)) < 1\\ \alpha_{OS}(G(k_i)) &\leq \alpha_{OS}(U(k_i)) \leq \alpha_{OS}(U(k))\\ \beta_{OP}(G(k_i)) &\leq \beta_{OP}(U(k_i)) \leq \beta_{OP}(U(k))\\ \beta_{OS}(G(k_i)) &\leq \beta_{OS}(U(k_i)) \leq \beta_{OS}(U(k)).\end{aligned}$$

Since $\|w_2|_{B_i}\|_r = \|w_2^c|_{B_i}\|_r$ it follows from Proposition 10 that $z_{i+1}^\xi \leq az_i^\xi + d\beta_i^\xi$ and $f_i^\xi \leq bz_i^\xi + e\epsilon_i^\xi$. Since $\epsilon_i \leq \beta_i$ it follows that $\|\epsilon|_{[m,n]}\| \leq \|\beta|_{[m,n]}\|$ and by Proposition 13 we then have for $1 \leq r \leq \infty$ that:

$$\begin{aligned}&\|w_2|_{[k_m,k_{n+1})}\|\\ &\quad= \big\|\|w_2|_{A_{m+1}}\|, \|w_2|_{A_{m+2}}\|, \cdots, \|w_2|_{A_{n+1}}\|,\\ &\qquad\qquad \|w_2|_{B_m}\|, \|w_2|_{B_{m+1}}\|, \cdots, \|w_2|_{B_n}\|\big\|\\ &\quad= \big\|z|_{[m+1,n+1]}, f|_{[m,n]}\big\|\\ &\quad\leq \gamma_3(U(k))|z_m| + \gamma_4(U(k))\|\beta|_{[m,n]}\|\\ &\qquad + \gamma_5(U(k))\|\epsilon|_{[m,n]}\|\\ &\quad\leq \gamma_3(U(k))|z_m|\\ &\qquad + \big(\gamma_4(U(k)) + \gamma_5(U(k))\big)\|\beta|_{[m,n]}\|. \qquad (4.72)\end{aligned}$$

It remains to show that $\big\|\beta|_{[m,n]}\big\|$, $|z_m|$ are bounded by $\|w_0\|$ and $\|w_2\|$. Recall that $\mathscr{R}_i d_p[j] := \mathscr{R}_{i,j} d_p[j]$, $i \leq j$, $p \in \mathcal{P}$. By Assumption 4(2) we have

$$\begin{aligned}&\|w_0^{q(k_{i+1})}|_{A_i \cup B_i \cup A_{i+1}}\|\\ &\qquad= \|\Phi_{\bar{k}_i} d_{q(k_{i+1})}[\tilde{k}_i]\| \leq \|\mathscr{R}_{\bar{k}_i} d_{q(k_{i+1})}[\tilde{k}_i]\|.\end{aligned}$$

Observe that $\tilde{k}_i = k_{i+1} - 1 \leq \tilde{k}_{i+1} = k_{i+2} - 1$ and that $0 \leq \bar{k}_i \leq \lambda$ (equation (4.71)). By Proposition 11, inequality (2.1) and by Assumption 4(2), we obtain:

$$\begin{aligned}\|\beta|_{[m.n]}\| = &\ \|\|\Phi_{\bar{k}_m} d_{q(k_{m+1})}[\tilde{k}_m]\|, \|\Phi_{\bar{k}_{m+1}} d_{q(k_{m+2})}[\tilde{k}_{m+1}]\|,\\ &\qquad \cdots, \|\Phi_{\bar{k}_n} d_{q(k_{n+1})}[\tilde{k}_n]\|\|\\ \leq &\ \|\Phi_\lambda d_{q(k_{m+1})}[\tilde{k}_m], \Phi_\lambda d_{q(k_{m+2})}[\tilde{k}_{m+1}], \cdots\\ &\qquad \cdots, \Phi_\lambda d_{q(k_{n+1})}[\tilde{k}_n]\|\\ \leq &\ 2^{1/r}\|d_{q(k_{n+1})}[k_{n+1}]\| \qquad (4.73)\end{aligned}$$

Since $k_{n+1} \in Q_k$, it follows from Proposition 12 that $\|d_{q(k_{n+1})}[k_{n+1}]\| \leq \mu\|\mathscr{T}_{k_{n+1}} w_0\| + \chi(p, p_*)\|\mathscr{T}_{k_{n+1}} w_2\|$. It follows from inequality (4.73) that

$$\begin{aligned}&\|\beta|_{[m.n]}\|\\ &\leq 2^{1/r}\left(\mu\|\mathscr{T}_{k_{n+1}} w_0\| + \chi(p, p_*)\|\mathscr{T}_{k_{n+1}} w_2\|\right)\\ &\leq 2^{1/r}\Big(\mu\|w_0\| + \varepsilon\chi_\nu(H(k), \nu(k))(\|\mathscr{T}_{\tilde{k}_n} w_2\| + |w_2(k_{n+1})|)\Big),\end{aligned}$$

By a double application of Proposition 9 we obtain:

$$\begin{aligned}&|w_2(k_{n+1})|\\ &\quad\leq \gamma_1(q(k_{n+1}), p_*)\|\mathscr{T}_{\tilde{k}_{n+2}} w_2\| + \gamma_2(q(k_{n+1}), p_*)\|w_0\|\\ &\quad\leq \gamma_1(q(k_{n+1}), p_*)\gamma_1(q(k_n), p_*)\|\mathscr{T}_{\tilde{k}_{n+1}} w_2\|\\ &\qquad + (\gamma_1(q(k_n), p_*)\gamma_2(q(k_{n+1}), p_*) + \gamma_2(q(k_n), p_*))\|w_0\|\end{aligned}$$

hence:

$$\begin{aligned}&\|\beta|_{[m.n]}\| \leq\\ &\quad 2^{1/r}\varepsilon\chi_\nu(H(k), \nu(k))((1 + \bar{\gamma}_1^2(H(k), \{p_*\}))\|\mathscr{T}_{\tilde{k}_n} w_2\|\\ &\quad + 2^{1/r}(\mu + \varepsilon\chi_\nu(H(k), \nu(k))\bar{\gamma}_2(H(k), \{p_*\})\cdot\\ &\qquad\qquad \cdot (1 + \bar{\gamma}_1(H(k), \{p_*\}))\|w_0\| \quad (4.74)\end{aligned}$$

Since $|z_m| = \|w_2|_{A_m}\| \leq \|\mathscr{T}_{k_m-1} w_2\|$, by inequalites (2.1), (4.74), (4.73), (4.72) we have

$$\begin{aligned}&\|\mathscr{T}_{\tilde{k}_n} w_2\| \leq \|\mathscr{T}_{k_m-1} w_2\| + \|w_2|_{[k_m,k_{n+1})}\|\\ &\quad\leq \|\mathscr{T}_{k_m-1} w_2\| + \gamma_3(U(k))|z_m|\\ &\qquad + \big(\gamma_4(U(k)) + \gamma_5(U(k))\big)\|\beta|_{[m,n]}\|\\ &\quad\leq (1 + \gamma_3(U(k)))\|\mathscr{T}_{k_m-1} w_2\|\\ &\qquad + \big(\gamma_4(U(k)) + \gamma_5(U(k)\big)\Big(\eta(H(k), \nu(k), \varepsilon, p_*)\|w_0\|\\ &\qquad + 2^{1/r}\varepsilon\chi_\nu(H(k), \nu(k))((1 + \bar{\gamma}_1^2(H(k), \{p_*\}))\|\mathscr{T}_{\tilde{k}_n} w_2\|\Big),\\ &\quad\leq (1 + \gamma_3(U(k)))\|\mathscr{T}_{k_m-1} w_2\|\\ &\qquad + \big(\gamma_4(U(k)) + \gamma_5(U(k)\big)\eta(H(k), \nu(k), \varepsilon, p_*)\|w_0\|\\ &\qquad\qquad + (1 - \pi(U(k), H(k), \nu(k), \varepsilon, p_*)\|\mathscr{T}_{\tilde{k}_n} w_2\|.\end{aligned}$$

Since inequality (4.69) holds, we can now rearrange to obtain (4.70) as required. ■

E. Main result

In Proposition 14 we have established gain bounds for sequences of intervals (ongoing intervals) relating to ongoing switches, i.e. to times $k_i \in O_k$. In Proposition 9 we have established gain bounds which can be applied to intervals (final intervals) relating to final switches, i.e. $k_i \in F_k$. Now observe the following: to every $p \in H(k)$, provided that $Q_k(p, \nu(k)(p))$ is not empty, there exists a plant z in the neighbourhood $B_\chi(p, \nu(k)(p))$, such that the algorithm switches to that plant for the last time on the interval $[0, k]$, i.e. $z = q(F_k(p))$, $z \in B_\chi(p, \nu(k)(p))$. This implies that none, one, or a sequences of ongoing intervals is always followed by a final interval. This progression may repeat itself a maximum of $|H(k)|$ times since there can be only a maximum of $|F_k| = |H(k)|$ final switches. These facts will be used to prove the main result, thus establishing gain bounds on w_2 in terms of w_0.

Proof: Theorem 7. Let γ_6, γ_7 be as in Proposition 14. Suppose $0 \leq k \leq k_* - 1$. For $j \leq k$, observe that since the gain $\gamma_3(U(j)) \geq 0$, and $\pi(U(j), H(j), \nu(j), \varepsilon, p_*) > 0$ by assumption, it follows that $\gamma_6(U(j), H(j), \nu(j), \varepsilon, p_*) \geq 1$. Also observe that since $\alpha(p_*, K(p), 0, x) \geq 0$, for $p \in \mathcal{P}^U$, $x \in \mathbb{N}$, it follows that $\gamma_1(p, p_*) \geq 1$ for all $p \in \mathcal{P}^U$, therefore $\beta(U(j), H(j), \nu(j), \varepsilon, p_*) \geq$ for $j \leq k$, and inequality (4.43) holds as required.

Now suppose $k \geq k_*$. Let $\{k_{f_0} = k_*, k_{f_1}, \cdots, k_{f_m}\} = \cup_{p \in H(k)}\{\max(O_k(p))\} \cup \{k_*\} \cup F_k$ be an ordered set of switching times, i.e. $k_{f_i} \leq k_{f_{i+1}}$, $0 \leq i < m$. Observe that the algorithm might not switch to some neighbourhood $B_\chi(p, \nu(k)(p))$, $p \in H(k)$ at all, i.e. there might exist a $p \in H(k)$ such that $F_k(p) = O_k(p) = \emptyset$, and indeed

$O_k(p_i) \cap F_k(p_j)$ may not be empty for all $i, j \leq k$, however $m = |F_k| + |\cup_{p \in H(k)} \{\max(O_k(p))\}| \leq 2|H(k)|$. Let

$$a_{f_i} = \begin{cases} \gamma_6(U(k), H(k), \nu(k), \varepsilon, p_*) & \text{if} \quad k_{f_i} \in O_k \\ \gamma_1(q(k_{f_i}), p_*) & \text{if} \quad k_{f_i} \in F_k \end{cases}$$

$$b_{f_i} = \begin{cases} \gamma_7(U(k), H(k), \nu(k), \varepsilon, p_*) & \text{if} \quad k_{f_i} \in O_k \\ \gamma_2(q(k_{f_i}), p_*) & \text{if} \quad k_{f_i} \in F_k \end{cases}$$

where $a_{f_i} \geq 0$ since $\gamma_1, \gamma_6 \geq 1$, as previously. Now define $k_{f_{m+1}} = \min\{a > k_{f_m} \mid a \in Q_a\}$ and observe that $k_{f_m} \leq k < k_{f_{m+1}}$ where $k_{f_i} \in Q_k \subset Q_{k_{f_{m+1}}}$, $0 \leq i \leq m$ and $k_{f_m}, k_{f_{m+1}} \in Q_{k_{f_{m+1}}}$. Then, with $z_{f_i} = \|\mathscr{T}_{k_{f_i}-1} w_2\|_r$ for $0 \leq i \leq m+1$, it follows from Propositions 9, 14 that:

$$z_{f_{i+1}} \leq a_{f_i} z_{f_i} + b_{f_i} \|w_0\|, \ 0 \leq i \leq m,$$

and so,

$$\begin{aligned} z_{f_{m+1}} &\leq \prod_{i=0}^{m} a_{f_i} z_{f_0} + \Big(\prod_{i=1}^{m} a_{f_i} b_{f_0} + \prod_{i=2}^{m} a_{f_i} b_{f_1} + \cdots \\ &\qquad \cdots + \prod_{i=m}^{m} a_{f_i} b_{f_{m-1}} + b_{f_m} \Big) \|w_0\| \\ &\leq \prod_{i=0}^{m} a_{f_i} \left(z_{f_0} + \sum_{i=0}^{m} b_{f_i} \|w_0\| \right) \\ &\leq \gamma_6^{|H(k)|}(U(k), H(k), \nu(k), \varepsilon, p_*) \prod_{p \in H(k)} \gamma_1(p, p_*) \\ &\quad \cdot \Big(z_{f_0} + \Big(\mu |H(k)| \gamma_7(U(k), H(k), \nu(k), \varepsilon, p_*) \\ &\qquad + \sum_{p \in H(k)} \gamma_2(p, p_*) \Big) \|w_0\| \Big). \end{aligned}$$

Since $k_{f_0} = k_*$, it follows that $z_{f_0} = \|\mathscr{T}_{k_*-1} w_2\|$, hence,

$$\begin{aligned} \|\mathscr{T}_k w_2\| &\leq z_{f_{m+1}} \\ &\leq \beta(U(k), H(k), \nu(k), \varepsilon, p_*) \|\mathscr{T}_{k_*-1} w_2\| \\ &\quad + \hat{\gamma}(U(k), H(k), \nu(k), \varepsilon, p_*) \|w_0\| \end{aligned}$$

as required. ■

5. Conservatism, Universality and EMMSAC

The following material is based on [4, 5]. One of the key motivating rationales for adaptive control is the ability to overcome conservativeness of alternative control designs for large (structured) uncertainty sets. This follows from the property of universality introduced below. Following [5] we define the notion of a conservative design as follows. Suppose $\{\Delta(\beta)\}_{\beta \geq 0}$ is a parameterised collection of nested subsets of $\mathcal{P}$. Here, the parameter β represents the *(structured) uncertainty level* of the uncertainty set $\Delta(\beta)$. For example, we might be interested in controlling a plant of the form $(a, 1, 1, 0) \in \bar{\mathcal{P}}_{\text{LTI}}$, where a is an uncertain parameter, and $\Delta(\beta)$ could be taken to be:

$$\Delta(\beta) = \{(1 + a, 1, 1, 0) \in \mathbb{R}^4 \ : \ a \in [-\beta, \beta]\}. \tag{5.75}$$

The notion of conservativeness of a control design is the property that nominal performance degrades as the uncertainty set on which the design is based becomes larger. Given a controller C, a bounded set $D \subset \mathcal{W}$ and an uncertainty set $\Omega \subset \mathcal{P}$, we define the worst case cost :

$$\mathcal{J}_R(\Omega, C) = \sup_{p \in \Omega} \sup_{\|w_0\| < R} \left\| \Pi_{P_p // C} w_0 \right\|, \tag{5.76}$$

and make the definition:

Definition 15: Let $R > 0$, and suppose $\{\Delta(\beta)\}_{\beta \geq 0}$ is a parameterised collection of nested subsets of $\mathcal{P}$. A control design $\Gamma \colon \mathbb{R}_+ \to \mathcal{C}$ is said to be:

1) $\mathcal{J}_R(\Delta)$-stable if for all $\beta \geq 0$ and for all $\beta^* \geq \beta$,
$$\mathcal{J}_R(\Delta(\beta), \Gamma(\beta^*)) < \infty.$$
2) $\mathcal{J}_R(\Delta)$-conservative if for all $\beta \geq 0$,
$$\lim_{\beta^* \to \infty} \mathcal{J}_R(\Delta(\beta), \Gamma(\beta^*)) = \infty,$$
3) $\mathcal{J}_R(\Delta)$-semi-universal if for all $\beta \geq 0$, there exists $\bar{\mathcal{J}} > 0$ such that for all $\beta^* \geq \beta$,
$$\mathcal{J}_R(\Delta(\beta), \Gamma(\beta^*)) < \bar{\mathcal{J}}.$$

It is a clear requirement of any control design Γ that it is $\mathcal{J}_R(\Delta)$-stable, this is simply the requirement that the controller $\Gamma(\beta)$ designed for uncertainty level β does indeed stabilize all plants P_p, $p \in \Delta(\beta)$. But, many control designs are also conservative, i.e. have the property that as the uncertainty level β^* used in the control design becomes an increasingly high over-bound of the 'true' uncertainty β, the performance degrades unboundedly. For example, the fact that both LTI and memoryless control designs are conservative for our exemplar uncertainty set given in equation (5.75), is established in continuous time in [5, Proposition 7.5] and [5, Proposition 7.4] respectively. Analogous results holds for alternative model uncertainties and in discrete time.

Clearly a semi-universal design Γ_1 will outperform a conservative design Γ_2 as β^* becomes large w.r.t. β, since for all $\beta \geq 0$ there exists $\beta^{**} \geq \beta$ such that, for all $\beta^* \geq \beta^{**}$,

$$\mathcal{J}_R(\Delta(\beta), \Gamma_1(\beta^*)) < \mathcal{J}_R(\Delta(\beta), \Gamma_2(\beta^*)).$$

Adaptive designs can often be shown to be semi-universal and hence non-conservative via a universality property:

Definition 16: Let $R > 0$, and suppose $\{\Delta(\beta)\}_{\beta \geq 0}$ is a parameterised collection of nested subsets of $\mathcal{P}$. A controller $C \in \mathcal{C}$ is said to be universal if for all $\beta > 0$, $\mathcal{J}_R(\Delta(\beta), C) < \infty$. A $\mathcal{J}_R(\Delta)$-stable mapping $\Gamma \colon \mathbb{R}_+ \to \mathcal{C}$, which is constant is said to be a universal control design.

It is simple to see that a universal control design is automatically $\mathcal{J}_D$-semi-universal. This universality property is a key feature of classical adaptive control designs (see [5] for further discussion). This supplies a clear rationale for adaptive controllers: for large uncertainty sets, nominal performance will degrade for LTI control designs; but it does not degrade for universal adaptive designs.

By establishing an lower bound on performance, Theorem 17 below shows that static EMMSAC is also conservative. It is anticipated that this property holds also for other variants of MMAC by similar constructions, note that the MMAC literature to date restricts attention either to structured uncertainty sets defined by finite candidate model sets or to compact continua [6]; and in all cases the derived upper gain bounds diverge as the uncertainty set becomes larger. This motivates the development of dynamic EMMSAC which is shown to be $\mathscr{J}_R$ semi-universal and hence not conservative in Theorem 18.

A. Static MMAC is conservative

We now consider the construction of universal EMMSAC designs which yield non-conservative performance. Such designs are dynamic, and we show how they outperform static EMMSAC and LTI controllers.

Consider the following example. For $\beta > 0$, let U be a constant plant-generating operator defined by

$$U(j) = \Delta(\beta),\ j \in \mathbb{N}. \tag{5.77}$$

where $\Delta(\beta)$ is given by equation (5.75). A possible sampling of U is then given as follows. Let the refinement level $m > 0$ and the parameter bound $\beta > 0$ define the plant model set

$$\mathcal{P}_m(\beta) = \{(1+\beta - i/m, 1, 1, 0) \in \bar{\mathcal{P}}_{LTI} \mid i \in \mathbb{N},\ |\beta - i/m| \leq \beta\} \tag{5.78}$$

Let the (dead-beat) controller design satisfy:

$$K((a,1,1,0)) = (0,0,0,-a) \in \mathcal{C}_{LTI}. \tag{5.79}$$

Under the conditions of the theorem below, it is straightforward to verify that Γ is $\mathscr{J}_R(\Delta)$-stable provided $m > 0$ is sufficiently small. However, the algorithm is necessarily conservative:

Theorem 17: Let $\mathcal{U} = \mathcal{Y} = l_r,\ 1 \leq r \leq \infty$, and suppose $R = \infty$. Let $m > 0$ and let the plant set $\mathcal{P}_m(\beta)$, constant plant generating operator U and control design K be given by equations (5.77),(5.78), (5.79). Let $X = X_A$ or X_B where $\lambda \geq 3$. Let the constant plant generating operator G_β be given by $G_\beta = \mathcal{P}_{\beta,m}$. Let the switching control design $\Gamma(\beta^*) = C(U, K, 1, G_{\beta^*}, X)$ be determined by equations (3.24),(3.25). Then Γ is $\mathscr{J}_R(\mathcal{P}_m)$-conservative.

Proof: Let $m = 1$. First we show that we can always make the switching algorithm switch to the controller corresponding to the plant with the largest possible $v \in \mathbb{N},\ (0,0,0,v) \in \mathcal{C}_{\beta^*,m}$, that is $v = \beta^* \in \mathbb{N}$. Secondly we show that this switch leads to an unbounded increase in closed loop cost as β^* increases.

Let $p_b = (b,1,1,0), p_{\beta^*} = (\beta^*,1,1,0) \in \mathcal{P}_{\beta^*,1},\ 1 \leq b < \beta^*$. Let $B > 0$ and consider the closed-loop system $[P, C[\mathcal{P}_{\beta^*,1}]]$ with

$$\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} = \left(\begin{pmatrix} 0 \\ B \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B - \beta^* B \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \cdots \right).$$

We now claim that these disturbances make the algorithm switch to the controller $C_{p_{\beta^*}}$ in two time steps, i.e. $q(2) = p_{\beta^*} = q_f(2) = S(w_2)(2)$, and that the signals in Table 4 are consistent with

$$\begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = \Pi_{P//\Gamma(\beta^*)} \begin{pmatrix} u_0 \\ y_0 \end{pmatrix},\ u_0 = u_1 + u_2,\ y_0 = y_1 + y_2.$$

Note that in Table 4 and throughout this proof, an entry marked $\times$ indicates that the entry is irrelevant to the calculation that follows. To establish the claim as follows. Let:

k	$\begin{pmatrix} u_0 \\ y_0 \end{pmatrix}$	$\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}$	$\begin{pmatrix} u_2 \\ y_2 \end{pmatrix}$
0	$\begin{pmatrix} 0 \\ B \end{pmatrix}$	$\begin{pmatrix} B \\ 0 \end{pmatrix}$	$\begin{pmatrix} -B \\ B \end{pmatrix}$
1	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -B \\ B \end{pmatrix}$	$\begin{pmatrix} B \\ -B \end{pmatrix}$
2	$\begin{pmatrix} 0 \\ B - \beta^* B \end{pmatrix}$	$\begin{pmatrix} \beta^*(B - \beta^* B) \\ 0 \end{pmatrix}$	$\begin{pmatrix} \beta^*(\beta^* B - B) \\ B - \beta^* B \end{pmatrix}$
3	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \times \\ \beta^*(B - \beta^* B) \end{pmatrix}$	$\begin{pmatrix} \times \\ \times \end{pmatrix}$

TABLE 4
SIGNALS FOR THE TRUE PLANT $P = P_{p_*}$ UP TO TIME $k = 3$

$$\begin{aligned} P = P_{p_*} : y_1(k+1) &= y_1(k) + u_1(k) \\ P_{p_b} : y_1^b(k+1) &= b y_1^b(k) + u_1^b(k) \\ P_{p_l} : y_1^l(k+1) &= l y_1^l(k) + u_1^l(k), \end{aligned}$$

where $y_1(0) = y_1^b(0) = y_1^l(0) = 0$. Since the zero initial conditions are zero and $y_0(0) = B$, the consistency property forces $d_p[0](0),\ p \in \{p_b, p_\beta^*\}$ to satisfy:

$$d_p[0](0) = \begin{pmatrix} u_0^p(0) \\ y_0^p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ B \end{pmatrix},\ p \in \{p_b, p_\beta^*\}.$$

Hence $\|d_p[0]\| = B,\ p \in \{p_b, p_{\beta^*}\}$. Note that since the disturbance estimates are of identical size, we impose an ordering on G such that $p_{\beta^*} = (1,1,1,0) \in \mathcal{P}$ is the plant model with the smallest index, hence $q(0) = p_* = p_1$. [1] With $u_2(0) = -y_2(0) = -B$ and $u_0(0) = 0$ we have $u_1(0) = B$.

At time $k = 1$ we have $y_1(1) = B$ and since $y_0(1) = 0$ it follows that $y_2(1) = -B$. The smallest disturbance $d_p[1],\ p \in \{p_b, p_\beta^*\}$ consistent with $(\mathscr{T}_0 u_2, \mathscr{T}_1 y_2)$ and $P_{p_b}, P_{p_{\beta^*}}$ can, by the general property $\|d_p[k]\| \leq \|d_p[k+1]\|,\ p \in \mathcal{P},\ k \in \mathbb{N}$, be found to be

$$(d_{p_{\beta^*}}[1](0), \|d_{p_{\beta^*}}[1](1)) = \left(\begin{pmatrix} 0 \\ B \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right),\ p \in \{p_b, p_{\beta^*}\}.$$

Since $\|d_{p_{\beta^*}}[1]\| = \|d_{p_b}[1]\|$, $q(1) = p_*$ and no switch occurs. Furthermore with $u_2(1) = -y_2(1) = B$ and $u_0(0) = 0$ we have $u_1(1) = -B$.

At $k = 2$ we have $y_1(2) = 0$ and since $y_0(2) = B - \beta^* B$ it follows that $y_2(2) = B - \beta^* B$. The smallest disturbance estimate for $d_{p_{\beta^*}}[2]$ consistent with $(\mathscr{T}_1 u_2, \mathscr{T}_2 y_2)$ and $P_{p_{\beta^*}}$ satisfies

$$(d_{p_{\beta^*}}[2](0), d_{p_{\beta^*}}[2](1), d_{p_{\beta^*}}[2](2)) = \left(\begin{pmatrix} 0 \\ B \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

since similarly minimality is ensured by consistency and $\|d_{p_{\beta^*}}[2]\| = \|d_{p_{\beta^*}}[1]\|$. Since $y_0^{p_b}(0) = B$, $\|d_{p_b}[2]\| \geq \|d_{p_{\beta^*}}[2]\|$, however the choice $d_{p_b}[2] = d_{p_{\beta^*}}[2],\ p_b \neq p_{\beta^*}$ is not possible since the trajectories would have the property that $\Pi_{\Gamma(\beta^*)//P_{p_b}} d_{p_b}[2] = \Pi_{\Gamma(\beta^*)//P_{p_b}} d_{p_{\beta^*}}[2] \neq (\mathscr{T}_1 u_2, \mathscr{T}_1 y_2)$. This can be seen by choosing

$$(d_{p_{\beta^*}}[2](0), d_{p_{\beta^*}}[2](1), d_{p_{\beta^*}}[2](2)) = \left(\begin{pmatrix} 0 \\ B \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ y_0^{p_b}(2) \end{pmatrix} \right).$$

[1] Observe that if the smallest index is assigned such that $p_1 = (\beta^*, 1, 1, 0)$, then $q(0) = p_\beta^*$, and the proof can be considerably shortened).

In this case we have $y_1^{p_b}(2) = bB - B$. With $y_2^{p_b}(2) = B - \beta^* B$ from above we would have to choose $y_0^{p_b}(2) = bB - \beta^* B \neq 0$, for all $b \neq \beta^*$ to be consistent with $(\mathscr{T}_1 u_2, \mathscr{T}_2 y_2)$ and P_{p_b}. Hence $\|d_{p_{\beta^*}}[2]\| = B < \|d_{p_b}[2]\|$. Hence we have $q(2) = \beta^*$ and obtain $u_2(2) = \beta^*(\beta^* B - B)$. Furthermore with $u_0(2) = 0$ it follows $u_1(2) = \beta^*(B - \beta^* B)$. A direct calculation shows $y_1(3) = \beta^*(B - \beta^* B)$. This establishes the first claim.

Let $w_0 = (0, B, 0, 0, 0, B(1-\beta^*), 0, 0, \dots)$, and let $\lambda = \frac{R}{B\|1, 1-\beta^*\|}$, so $\|\lambda w_0\| \leq R$. Then since $\Pi_{P//\Gamma(\beta^*)}$ is homogeneous,

$$\begin{aligned} \mathscr{J}_R(\Delta(\beta), \Gamma(\beta^*)) &= \sup_{p \in \Delta(\beta)} \sup_{\|\tilde{w}_0\| \leq R} \left\| \Pi_{P_p//\Gamma(\beta^*)} \tilde{w}_0 \right\|, \\ &\geq \left\| \Pi_{P//\Gamma(\beta^*)} \lambda w_0 \right\|, \\ &\geq \lambda |y_1(3)| \\ &= \frac{R|\beta^* - \beta^{*2}|}{\|1, 1-\beta^*\|} \geq R\beta^* \end{aligned}$$

hence $\mathscr{J}_R(\Delta(\beta), \Gamma(\beta^*)) \to \infty$ as $\beta^* \to \infty$. The analysis is analogous for all $m > 0$ hence the proof is complete. ∎

B. Dynamic EMMSAC as a Universal Algorithm

The conservativeness of static EMMSAC, as illustrated above in Theorem 17, can be overcome by a dynamic version of EMMSAC, which we now develop. The idea is to progressively enlarge the model set, based on monitoring the performance. If the performance is inconsistent with the EMMSAC bound at the current model set size, the model set is enlarged, until the performance is consistent with the EMMSAC bound with the true plant present in the model set.

For simplicity, assume that the uncertainty set, as specified by the plant-generating operator U, is finite. We can therefore let $U = G = H$ and achieve a finite dimensional EMMSAC design. This leads to the following construction of a dynamic EMMSAC algorithm. Let a plant level set be given by

$$\mathcal{P}_i \in \mathcal{P}^*, \; \emptyset \neq \mathcal{P}_j \subset \mathcal{P}_{j+1}, \; 1 \leq j < i, \; i \in \mathbb{N} \tag{5.80}$$

where we assume that all $\mathcal{P}_i, \; i \in \mathbb{N}$ are finite and that there exists an index $i \in \mathbb{N}$ such that $p_* \in \mathcal{P}_l, \; \forall l \geq i$. Let

$$\tilde{\gamma}(\mathcal{Q}) = \max_{p \in \mathcal{Q}} \left(\hat{\gamma}(\mathcal{Q}, \mathcal{Q}, 0, p) + \beta(\mathcal{Q}, \mathcal{Q}, 0, p) \right), \; \mathcal{Q} \subset \mathcal{P}^G$$

where $\hat{\gamma}$ and β are from Theorem 7. Let $v > 2$ and take the expansion rule to be given by

$$G(k) = \mathcal{P}_{i(k)}, \; k \in \mathbb{N}, \tag{5.81}$$

$$i(k) := \begin{cases} \max\{a \in \mathbb{N} \mid \tilde{\gamma}^v(\mathcal{P}_a) - \tilde{\gamma}^v(\mathcal{P}_1) \leq \|\mathscr{T}_k w_2\|\} \\ \qquad \text{if } 0 \leq k < \infty, \\ \infty \qquad \text{if } k = \infty \end{cases}. \tag{5.82}$$

This expansion rule can be interpreted as a soft model falsification procedure (compare to [7, 8]): the model set is only expanded if the previous model set has been falsified at that hypothesised performance level. The form of the update ensures that eventually this performance is necessarily met, for any permissible plant and disturbance level. Theorem 7 applies with the choice $G(k) = U(k) = H(k), \; \nu = 0$. This brings us to our next result:

Theorem 18: Let $k \in \mathbb{N}$. Let $\mathcal{P}_i$ be given by equations (5.80) and suppose that there exists $i \in \mathbb{N}$ such that $p_* \in \mathcal{P}_l, \; l \geq i$. Let the expansion rule be given by equation (5.82) which gives the plant-generating operator G via equation (5.81). Suppose the EMMSAC algorithm is standard. Suppose $(w_0, w_1, w_2) \in \mathcal{W} \times \mathcal{W}_e \times \mathcal{W}_e$ satisfy the closed loop equations (2.3)–(2.5). Then for all $w_0 \in \mathcal{W}$:

$$\|w_2\| \leq \beta_1 + \beta_2 \|w_0\| + \beta_3 \|w_0\|^2$$

where β and $\hat{\gamma}$ are from Theorem 7, $N := \min\{i \geq 1 \mid p_* \in \mathcal{P}_i\}$, and

$$\begin{aligned} \tilde{\gamma}(\mathcal{Q}) &= \max_{p \in \mathcal{Q}} \left(\hat{\gamma}(\mathcal{Q}, \mathcal{Q}, 0, p) + \beta(\mathcal{Q}, \mathcal{Q}, 0, p) \right) \\ \beta_1 &= \tilde{\gamma}^{v+2}(\mathcal{P}_N) + \tilde{\gamma}(\mathcal{P}_N) \tilde{\gamma}^v(\mathcal{P}_1) \\ \beta_2 &= 2\tilde{\gamma}^2(\mathcal{P}_N) + \tilde{\gamma}^{1-v}(\mathcal{P}_N) \tilde{\gamma}^v(\mathcal{P}_1) \\ \beta_3 &= \tilde{\gamma}^{2-v}(\mathcal{P}_N). \end{aligned}$$

Proof: Let $w_0 \in \mathcal{W}$ and let k_* be given by equation (4.41). By equation (5.82)

$$\|\mathscr{T}_k w_2\| \leq \tilde{\gamma}^v(\mathcal{P}_{i(k)+1})) - \tilde{\gamma}^v(\mathcal{P}_1) \leq \tilde{\gamma}^v(\mathcal{P}_{i(k)+1}), \; \forall k \in \mathbb{N}. \tag{5.83}$$

From the definition of k^* it follows that $i(k_*) \geq N \geq i(k_* - 1) + 1$. Hence since $\tilde{\gamma}(\mathcal{P}_i)$ is monotonically increasing with i, we can write equation (5.83) with $k = k_* - 1$ as

$$\|\mathscr{T}_{k_*-1} w_2\| \leq \tilde{\gamma}^v(\mathcal{P}_{i(k_*-1)+1}) \leq \tilde{\gamma}^v(\mathcal{P}_N). \tag{5.84}$$

We now have to consider the two possibilities that either $k_* = \infty$ or $k_* < \infty$. For $k_* = \infty$ we have by equation (5.82) that no plants can be introduced to G hence there does not exits a $k_* \in \mathbb{N}$ such that $p_* \in G(k_*)$. Hence $\beta_1 \geq \tilde{\gamma}^{v+2}(\mathcal{P}_N) \geq \tilde{\gamma}^v(\mathcal{P}_N)$ and $\|w_2\| = \|\mathscr{T}_{k_*-1} w_2\| \leq \tilde{\gamma}^v(\mathcal{P}_N) \leq \beta_1$. For $k \leq k_* - 1$ it follows similarly that $\|\mathscr{T}_k w_2\| \leq \beta_1$. For $k > k_* - 1$ we have by equations (5.82), Theorem 7 and inequality (5.84) that

$$\begin{aligned} \tilde{\gamma}^v(\mathcal{P}_{i(k)}) &\leq \|\mathscr{T}_k w_2\| + \tilde{\gamma}^v(\mathcal{P}_1) \\ &\leq \tilde{\gamma}(\mathcal{P}_{i(k)})(\|\mathscr{T}_{k_*-1} w_2\| + \|w_0\|) + \tilde{\gamma}^v(\mathcal{P}_1) \\ &\leq \tilde{\gamma}(\mathcal{P}_{i(k)})(\tilde{\gamma}^v(\mathcal{P}_N) + \|w_0\|) + \tilde{\gamma}^v(\mathcal{P}_1). \end{aligned}$$

Multiplication with $\tilde{\gamma}^{1-v}(\mathcal{P}_{i(k)}) > 0$ yields

$$\tilde{\gamma}(\mathcal{P}_{i(k)}) \leq \tilde{\gamma}^{2-v}(\mathcal{P}_{i(k)})(\|w_0\| + \tilde{\gamma}^v(\mathcal{P}_N)) + \tilde{\gamma}^{1-v}(\mathcal{P}_{i(k)}) \tilde{\gamma}^v(\mathcal{P}_1).$$

Furthermore, since $\tilde{\gamma}(\mathcal{P}_i)$ is monotonically increasing with i and $i(k_*) \geq N \geq i(k_* - 1) + 1$ we have that $\tilde{\gamma}(\mathcal{P}_N) \leq \tilde{\gamma}(\mathcal{P}_{i(k)})$. Hence $\tilde{\gamma}^{q-v}(\mathcal{P}_{i(k)}) \leq \tilde{\gamma}^{q-v}(\mathcal{P}_N)$ for all $q < v$ and we obtain

$$\tilde{\gamma}(\mathcal{P}_{i(k)}) \leq \tilde{\gamma}^2(\mathcal{P}_N) + \tilde{\gamma}^{2-v}(\mathcal{P}_N)\|w_0\| + \tilde{\gamma}^{1-v}(\mathcal{P}_N) \tilde{\gamma}^v(\mathcal{P}_1). \tag{5.85}$$

By Theorem 7, inequality (5.85) and inequality (5.84) we now have that:

$$\begin{aligned} \|\mathscr{T}_k w_2\| &\leq \tilde{\gamma}(\mathcal{P}_{i(k)})(\|\mathscr{T}_{k_*-1} w_2\| + \|w_0\|) \\ &\leq \Big(\tilde{\gamma}^2(\mathcal{P}_N) + \tilde{\gamma}^{2-v}(\mathcal{P}_N)\|w_0\| \\ &\quad + \tilde{\gamma}^{1-v}(\mathcal{P}_N) \tilde{\gamma}^v(\mathcal{P}_1) \Big) (\|\mathscr{T}_{k_*-1} w_2\| + \|w_0\|) \\ &\leq \beta_1 + \beta_2 \|w_0\| + \beta_3 \|w_0\|^2. \end{aligned}$$

Since this holds for all $k \in \mathbb{N}$, the proof is complete. ∎

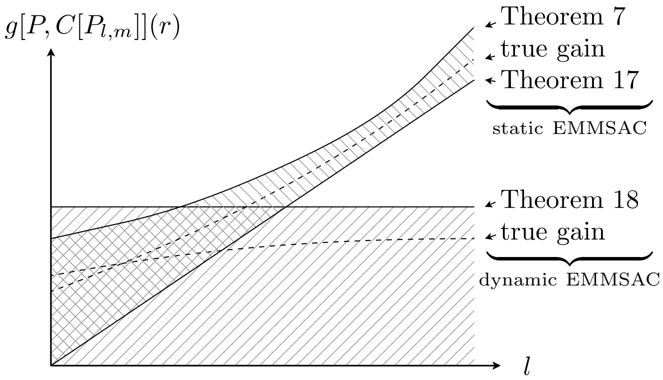

Fig. 2. Gain comparison of static and dynamic EMMSAC for a parametric uncertainty of level β^*

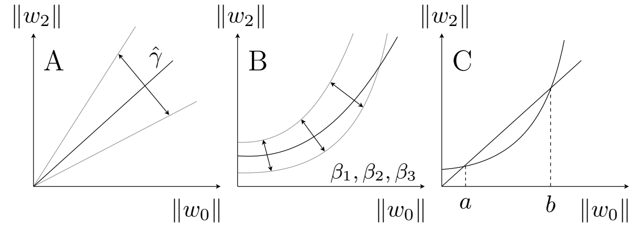

Fig. 3. Gain bound comparison of static and dynamic EMMSAC

Theorem 18 shows that the given dynamic EMMSAC algorithm $\Gamma(\beta^*) = C$ is universal. The constants $\beta_1, \beta_2, \beta_3$ are invariant to any uncertainty level information and only depend on $\mathcal{P}_i$ and N where N defines the smallest 'learning level' i such that the true plant p_* is included in $\mathcal{P}_N$. We are now in the position to compare these result for dynamic EMMSAC to the ones obtained in Theorem 7 for static EMMSAC (and other conservative designs, such as any LTI controller). Consider Figure 2. Earlier in this section we have discussed how the algorithm behaves in the presence of an increasingly large parametric uncertainty governed by the parameter $\beta^* > 0$ and represented by the plant model set $G = \mathcal{P}_{\beta^*,1}$, where it can be seem from the proof of Theorem 17 that the actual closed loop gain $\|\Pi_{P_{p_*}//\Gamma(\beta^*)}\|$ scales at least linearly with the uncertainty level $\beta^* > 0$. This gives a lower bound on the closed loop gain in Figure 2 at a disturbance level $R \in \mathbb{R}$, as a function of $\beta^* > \beta$. Now observe that an increasingly large β^* in $G = \mathcal{P}_{\beta^*,1}$ corresponds to an increasingly large constant U since $G \subset U$. This however means that the upper bound $\hat{\gamma}$ on the closed-loop gain from Theorem 7 scales with β^*. In contrast we have shown in Theorem 18 that for a special (dynamic) choice of G we obtain a gain (function) bound which is invariant to β^*. Hence, for large parametric uncertainties, dynamic EMMSAC outperforms static EMMSAC.

The gain bounds for dynamic/static EMMSAC also have differing scaling characteristics with respect to the size of the disturbance. Recall that for a constant, compact plant-generating operator U and a corresponding constant cover (H, ν), assuming $p_* \in G \subset U$, there follows $k_* = 0$ hence $\|\mathscr{T}_{k_*-1} w_2\| = 0$. By Theorem 7 we then obtain a (linear) gain bound (Figure 3 (A)) of the form $\|w_2\| \leq \hat{\gamma}(U, H, \nu, p_*)\|w_0\|$, where the gain $\hat{\gamma}$ depends on the uncertainty set specified by U and the corresponding cover (H, ν). From Theorem 18, we have for a dynamic construction of $U = G = H, \nu = 0$, assuming that there exists a $k_* < \infty$ such that $p_* \in G(k_*)$, a gain function bound of the form $\|w_2\| \leq \beta_1 + \beta_2\|w_0\| + \beta_3\|w_0\|^2$ where $\beta_1, \beta_2, \beta_3$ are constant and depend on $v > 2$, the design of the level set $\mathcal{P}_i$ and the true plant $P = P_{p_*}$ (Figure 3 (B)). Since our goal is to optimise the bound on the signal amplification from the disturbances $\|w_0\|$ to the internal signals $\|w_2\|$, we can now intersect these two curves and argue by Figure 3 (C) that for disturbances $\|w_0\| < a$ or $\|w_0\| > b$ the gain bound obtained for static EMMSAC is better than that for dynamic EMMSAC whereas for $a < \|w_0\| < b$ the converse holds. Note that the intersection points a, b depend on $\hat{\gamma}$ and $\beta_1, \beta_2, \beta_3$ where in some scenarios they do not intersect at all, i.e. for $\hat{\gamma} < \beta_2$, and a constant plant set should be preferred over a time-varying one. In all other cases the two curves will intersect for sufficiently large $\|w_0\|$ since the (quadratic) gain function grows faster then the (linear) gain. Hence when large disturbances are very likely, a constant plant model set should be preferred over a time-varying one. Observe that increasingly large v (as appears in the expressions for β_1, β_2 and β_3) will effectively straighten the curve since β_3 will become increasingly small and the influence of the quadratic term is diminished. However the offset β_1 will increase. Alternatively, small v will lead to small offsets and a faster quadratic growth. The choice of $v > 2$ is therefore dominated by the available information on the size of $\|w_0\|$, i.e. if $\|w_0\|$ is expected to be large it is advantageous to choose v large since then the gain function curve is more linear, which leads to smaller signal amplification. However if $\|w_0\|$ is expected to be small, v should be small since the constant G case has a zero offset in the gain bound.

6. Dynamic EMMSAC – Managing Computational Complexity

In this last section we tentatively outline further classes of dynamic EMMSAC algorithms, wherein the candidate plant set is adapted on-line in response to the closed loop measurements, in order that the computational complexity of the algorithm is moderated (i.e. the number of candidate plant models), whilst at the same time allowing the highly tuned models into the candidate plant set. We do not prove results or even give concrete algorithms; this section is intended to be of a more speculative and open ended nature: the purpose of which is to illustrate the utility and flexibility of dynamic EMMSAC over and above the rationale established in Section 5.

Although Theorem 18 of [2] shows there is no loss of performance from high plant densities in the candidate plant model set, there are clear implementation constraints which arise from the computational requirements of realising a large number of estimators. One possibility is to control this computational complexity by adaptively refining the plant model set in the regions of model space close to models with low residuals, thus reducing the numbers of models considered (it is only necessary to have a high density of plant models in areas where the estimators are reporting low residuals).

Computational resource can also be released by selectively discontinuing the estimators associated to plant models with high residuals. Although implementations which 'discard' plants with high residuals do not produce monotonic plant operators G, they do behave identically to implementations maintaining these plants within the plant set, providing the switch never points to them. For example, if $p_* \in G(k)$ and $\|w_0\| < W$, where $W > 0$ is known, then at any time $k > 0$ where $X(w_2)(k)(p) \geq \mu W$, the plant P_p has been falsified (hence $q(s) \neq p$ for all $s \geq k$) and can be safely removed from the candidate plant set from time k onward. Hence, in practice, discarding plants with high residuals is safe.

More complex dynamic refinement schemes could include a local search for the smallest disturbance estimate, for example by computing a local gradient from the plants closest to the plant with the smallest residual and then to consecutively add plant models along this gradient (gradient descent): note that the problem of local minima is not an issue as the search is only conducted locally, the estimator ranking can cause a switch to any plant in $G(k)$. Yet another possibility would be to run an on-line parameter identification algorithm, and 'seed' new candidate plants from this. Essentially any performance-driven search scheme which generates a monotonic G can be incorporated in EMMSAC. Such schemes only need to ensure that there exists a time k such that the static algorithm associated to $G(k)$ is stabilising to ensure stability.

Finally we remark that the introduction of a new plant at time k does in principle carry the requirement that the residual is back computed on the interval $[0, k]$, thus the computational cost scales with k. However, it is more pragmatic to take the 'closest' plant in the previous plant model set to define the residual up to time k, and then to begin the recursive update of the residual based on the new model from time step k onward. This is equivalent to thinking of the new model as a switched model, switching from a previously considered model to the new form at time step k. This is straightforward to implement with both the Kalman Filter and finite horizon estimators, and achieves a computational cost which is independent of k, and simply scales with the number of models.

7. Illustrative Example

To illustrate both static and dynamic EMMSAC algorithms, we consider the following example. The continuous time system matrices describing an (inverted or non-inverted) pendulum on a cart are given by

$$p_l = \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(i+ml^2)b}{v} & \frac{m^2gl^2}{v} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mlb}{v} & \frac{mgl(M+m)}{v} & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{i+ml^2}{v} \\ 0 \\ \frac{ml}{v} \end{bmatrix}, I, 0 \right\},$$

where $v = (i + ml^2)(m + M)$ and where $M = 0.6$kg, $m = 0.3$kg, $b = 0.1$N/ms, $i = 0.005\text{m}^2$kg, $g = 9.8\text{m/s}^2$ are the cart mass, pendulum mass, cart-friction, pendulum inertia and gravitational acceleration, respectively. The state vector is given by $\begin{bmatrix} x & \dot{x} & \Phi & \dot{\Phi} \end{bmatrix}^\top$. Φ is the angle between the positive y-axis and the pendulum in the upward configuration and the angle between the negative y-axis and the pendulum in the downward configuration; x is the distance of the cart center to the origin. The control task is to stabilise the pendulum around the vertical axis, i.e. $\Phi = 0$, by applying a force F to the cart, however the pendulum length l and orientation (upwards or downwards) is uncertain. The corresponding uncertainty set is given by $U = \{p_l \;:\; l \in [-0.4, -0.2] \cup [0.2, 0.4]\}$. The corresponding discrete-time models p_l are then constructed via zero-order-hold sampling with sampling period $\tau = 10^{-3}$s. To each $p \in U$, we let $K(p)$ represent the LQR controllers with discrete-time weights $Q = \text{diag}(500, 1, 500, 1)$, $R = 1$. We utilize infinite horizon estimators in l_2 with a constant switching delay of $\Delta(p) = 25$. The uncertainty set is sampled to give $G_+ = \{p_l \;:\; l \in \{0.2, 0.25, 0.3, 0.35, 0.4\}\}$, $G_- = -G_+$ and for static EMMSAC we choose a constant G operator: $G = G_- \cup G_+ \subset U$. We take:

$$(u_0(k), y_0(k)) = \begin{cases} (10 + n_u, n_y) & \text{if } 100 < k < 105 \\ (n_u, n_y) & \text{else} \end{cases},$$

where n_u and n_y are uniformly distributed disturbances in the range $[-1 \cdot 10^{-2}, +1 \cdot 10^{-2}]$. This corresponds to an input disturbance of 10N for the duration of 3 samples (a push to the cart) at time $k = 101$ with additional actuator and sensor noise. Furthermore, suppose that the input of the true plant is perturbed by multiplicative unmodeled dynamics of the form $\frac{500}{s+500} \cdot \frac{500-s}{500+s} \cdot e^{-0.01s}$, i.e. a first order lag, an all-pass factor and an actuator delay of 10ms. Figure 4 illustrates typical trajectories when the true (unknown) plant P is given by $P = P_p$, $p = p_{0.32\text{m}} \notin G$. Note that a laboratory implementation of a similar scheme is documented in [3, Chapter 8].

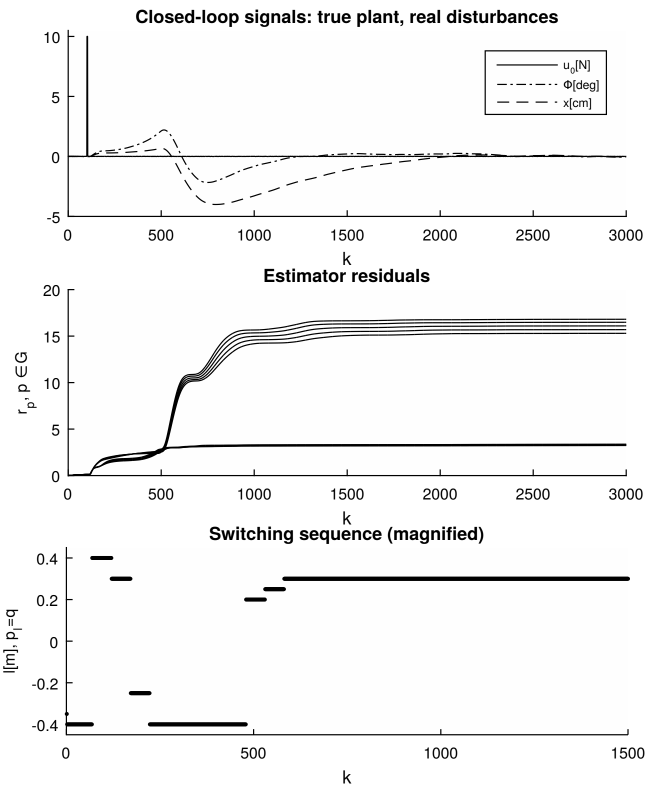

Fig. 4. Closed-loop signals, residuals and switching sequence (magnified) for EMMSAC operating on the pendulum example.

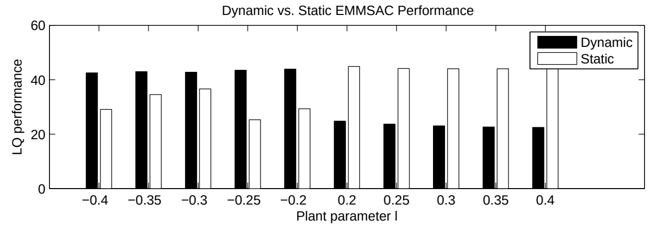

Fig. 5. Performance comparison between static and dynamic EMMSAC.

We observe that the EMMSAC algorithm does not initially identify the sign of the true plant and applies controllers that are designed for a downward configuration of the pendulum, hence providing a transient destabilising response. However, the very act of switching destabilising controllers in closed-loop further exposes the dynamics of P which helps the estimators to identify better plant models. This allows the algorithm to switch to stabilising controllers for $k > 500$. On the other hand, a dynamic version of EMMSAC can trade off the performance on G_- with G_+ as illustrated in Figure 5 which clearly demonstrates the dynamic algorithm outperforming the static approach on G_+ and vice-versa on G_-. Here the dynamic update rule is taken to be:

$$G(k) = \begin{cases} G_+ & \text{if } \|\mathscr{T}_k(u_2, y_2)\| \leq 30 \\ G_+ \cup G_- & \text{if } \|\mathscr{T}_k(u_2, y_2)\| > 30. \end{cases}$$

and the LQ cost with $Q = I$, $R = 1$, averaged over 10 trials, is plotted against the nominal plant value of l. The plant generating operator is constructed to ensure that the unstable plants are rapidly stabilized (the de-stabilising controllers are not available in the first 'learning level'), whilst the stabilisation of the stable plants is delayed (as in this case, the stabilising controllers are in the second 'learning level'), hence the larger transient occurs on the stable plants which can better tolerate recovery from such transients. Hence the trade-off between performance on the unstable plants in G_+ and the stable plants in G_- is entirely appropriate.

8. Conclusion

This paper has established the key gain bound underpinning the axiomatic EMMSAC framework. A key technical feature is the introduction of covers of the uncertainty set which lead to performance bounds dependent explicitly on the cover size rather than on the size of the candidate model set. In turn this leads to the principled design approaches described in part I of this contribution [2]. The secondary focus of this paper concerns dynamic versions of EMMSAC. This enables non-conservative designs to be constructed and for which an analysis and a detailed comparison with the qualitative features of static EMMSAC was provided. The dynamic case opens the door to many algorithmic variants to manage computational resource and adaptively refine the candidate model set and this was informally discussed. Exploring these algorithmic possibilities is a rich area for future research.

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