$\frac{n}{3}$ -NEIGHBOURLY MOMENT-ANGLE COMPLEXES AND THEIR UNSTABLE SPLITTINGS

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ABSTRACT. Given an $\frac{n}{3}$ -neighbourly simplicial complex K on vertex set $[n]$, we show that the moment-angle complex $\mathcal{Z}_K$ is a *co-H-space* if and only if K satisfies a homotopy analogue of the Golod property.

1. INTRODUCTION

A ring $R = \mathbf{k}[v_1, \dots, v_n]/I$ for I a homogeneous ideal is said to be *Golod* if the multiplication and higher Massey products are trivial in $\mathrm{Tor}_{\mathbf{k}[v_1, \dots, v_n]}^+(R, \mathbf{k})$. The Poincaré series of the ring $\mathrm{Tor}_R(\mathbf{k}, \mathbf{k})$ of R represents a rational function whenever R is Golod [8]. In combinatorics, one says a simplicial complex K on vertex set $[n] = \{1, \dots, n\}$ is *Golod* over $\mathbf{k}$ if its Stanley-Reisner ring $\mathbf{k}[K]$ is Golod. Equivalently, when $\mathbf{k}$ to be a field or $\mathbb{Z}$, K is Golod over $\mathbf{k}$ if all cup products and Massey products (of positive degree elements) are trivial in the cohomology ring $H^*(\mathcal{Z}_K; \mathbf{k})$ of the *moment-angle complex* $\mathcal{Z}_K$. This follows from isomorphisms of graded commutative algebras given in [6, 7, 2, 13]

$$(1) \quad H^*(\mathcal{Z}_K; \mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1, \dots, v_n]}(\mathbf{k}[K], \mathbf{k}) \cong \bigoplus_{I \subseteq [n]} \tilde{H}^*(\Sigma^{|I|+1}|K_I|; \mathbf{k})$$

where K_I is the restriction of K to vertex set $I \subseteq [n]$. The multiplication on the right hand side is induced by maps $\iota_{I,J}: |K_{I \cup J}| \rightarrow |K_I * K_J| \cong |K_I| * |K_J| \simeq \Sigma|K_I| \wedge |K_J|$ that realize the canonical inclusions $K_{I \cup J} \rightarrow K_I * K_J$ whenever I and J are non-empty and disjoint, and are defined zero otherwise. Thus, the Golod condition is equivalent to $\iota_{I,J}$ inducing trivial maps on $\mathbf{k}$ -cohomology for all disjoint non-empty $I, J \subseteq [n]$, as well as Massey products vanishing.

Since the cohomology ring $H^*(\mathcal{Z}_K)$ is in its simplest algebraic form when K is Golod, it is natural to ask what the homotopy type of $\mathcal{Z}_K$ is in this case. Quite a lot of work has been done in this direction [10, 11, 12, 15, 9, 3, 14], and it has been conjectured that K is Golod if and only if $\mathcal{Z}_K$ is a *co-H-space*. One can then determine the homotopy type of $\mathcal{Z}_K$ since, by [15, 1], $\mathcal{Z}_K$ is a *co-H-space* if and only if $\mathcal{Z}_K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+1}|K_I|$. This conjecture has been verified for flag complexes [9, 5], and rationally or localized at large primes [4, 3].

A simplicial complex K on vertex set $[n]$ is said to be *k-neighbourly* if every subset of k or less vertices in $[n]$ is a face of K . That is, K_I is a simplex for each $I \subseteq [n]$, $|I| \leq k$. If K is *k-neighbourly* then it is *k'-neighbourly* for all $k' \leq k$. When K is $\frac{n}{3}$ -neighbourly, we show that a homotopy analogue of Golodness is equivalent to the moment-angle complex $\mathcal{Z}_K$ being a *co-H-space*.

Theorem 1.1. *Let K be $\frac{n}{3}$ -neighbourly. Then $\mathcal{Z}_K$ is a co-H-space (equivalently, $\mathcal{Z}_K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+1}|K_I|$) if and only if*

$$\Sigma^{|I \cup J|} \iota_{I,J}: \Sigma^{|I \cup J|}|K_{I \cup J}| \rightarrow \Sigma^{|I \cup J|}|K_I * K_J|$$

is nullhomotopic for each disjoint non-empty $I, J \subseteq [n]$.

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Remark 1.2. The necessity part of Theorem 1.1 is true in general (see Proposition 2.4 below, or Proposition 5.4 in [3]). Sufficiency generally holds up to a certain coherence condition ([3], Theorem 1.2). Since this coherence condition involves joins $|K_{I_1} * \dots * K_{I_m}|$ over disjoint non-empty $I_1, \dots, I_m \subseteq [n]$ and $m \geq 3$, coherence should be given to us for free when K is $\frac{n}{3}$ -neighbourly, as 3-or-more-fold joins are contractible in this case. This is the heuristic behind Theorem 1.1.

Example 1.3. Suppose K is $\frac{n}{2}$ -neighbourly. Then each $|K_I|$ is contractible when $|I| \leq \frac{n}{2}$, is at least $(\lfloor \frac{n}{2} \rfloor - 1)$ -connected, and has dimension at most $n - 2$ when K is not a simplex. From isomorphism (1), or by Theorem 6.33 [6], one sees $\mathcal{Z}_K$ is contractible when K is a simplex or has the homotopy type of an n -connected CW -complex of at most dimension $2n - 1$. Thus $\mathcal{Z}_K$ is a co - H -space by the Freudenthal suspension theorem. This also follows from distinct arguments found in [3, 14]. On the other hand, it is an easy application of Theorem 1.1. Since for each disjoint non-empty $I, J \subseteq [n]$ at least one of K_I or K_J must be a simplex, each $|K_{I \cup J}| \xrightarrow{\iota_{I,J}} |K_I * K_J|$ must be nullhomotopic since $|K_I * K_J| \simeq \Sigma |K_I| \wedge |K_J|$ is contractible, so $\mathcal{Z}_K$ is a co - H -space by Theorem 1.1.

The first examples outside the stable range occur when K is $(\frac{n}{2} - 1)$ -neighbourly.

Example 1.4. Suppose K is $(\frac{n}{2} - 1)$ -neighbourly and n is even. Then either (1) $K = \partial \Delta^{\frac{n}{2}-1} * \partial \Delta^{\frac{n}{2}-1}$ and $\mathcal{Z}_K$ is a sphere product; (2) K is $\partial \Delta^{\frac{n}{2}-1} * \partial \Delta^{\frac{n}{2}-1}$ with some faces removed; or else (3) K has only one minimal missing face with $\frac{n}{2}$ vertices. In the third case, each $\iota_{I,J}$ for disjoint I and J is nullhomotopic since at least one of $|K_I|$ or $|K_J|$ is contractible. In the second case, $\iota_{I,J}$ is nullhomotopic since $|K|$ is at most $(n - 4)$ -dimensional and $|K_I * K_J|$ is at least $(n - 4)$ -connected. Then in the last two cases, $\mathcal{Z}_K$ is a co - H -space by Theorem 1.1.

Massey products do not factor into Theorem 1.1. In view of the isomorphism (1) only cup products are relevant. This is as might be expected when localized at the rationals, since k -connected CW -complexes of dimension $\leq 3k + 1$ are rationally formal by [18], therefore they have trivial Massey products in their rational cohomology.

2. SOME BACKGROUND

2.1. Conventions. We take -1 to be the basepoint of $D^1 = [-1, 1]$. The suspension ΣX of a space X is taken to be the reduced suspension $D^1 \times X / (\{-1, 1\} \times X \cup D^1 \times \{*\})$ whenever X is basepointed with basepoint $*$. Otherwise it is the unreduced suspension, in other words, the quotient space of $D^1 \times X$ under identifications $(-1, x) \sim *_{-1}$ and $(1, x) \sim *_1$. In any case, ΣX is always basepointed, in the unreduced case the basepoint taken to be $*_{-1}$.

2.2. Moment-Angle Complexes. Let K be a simplicial complex on n vertices, and (X, A) a CW -pair. The polyhedral product $(X, A)^K$ is the subspace of $X^{\times n}$ defined as the union

$$(X, A)^K = \bigcup_{\sigma \in K} Y_1^\sigma \times \dots \times Y_n^\sigma,$$

where $Y_i^\sigma = X$ if $i \in \sigma$, or else $Y_i^\sigma = A$ if $i \notin \sigma$. The *moment-angle complex* is $\mathcal{Z}_K = (D^2, S^1)^K$ and the *real moment-angle complex* is $\mathbb{R}\mathcal{Z}_K = (D^1, S^0)^K$, where $S^1 = \partial D^2$ and $S^0 = \{-1, 1\} = \partial D^1$.

Given $I \subseteq [n]$, we can think of $\mathcal{Z}_{K_I}$ and $\mathbb{R}\mathcal{Z}_{K_I}$ as the subspace of $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ consisting of those points whose coordinates not indexed by I are basepoints. In this case we define $\mathcal{Z}_K^\ell$ and $\mathbb{R}\mathcal{Z}_K^\ell$ to be the unions of $\mathcal{Z}_{K_I}$ and $\mathbb{R}\mathcal{Z}_{K_I}$ respectively over all $I \subseteq [n]$ such that $|I| = \ell$, i.e. the subspaces of $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ consisting of those points that have at least $n - \ell$ coordinates the basepoint. The *quotiented moment-angle complexes* are defined as $\widehat{\mathcal{Z}}_K = \mathcal{Z}_K / \mathcal{Z}_K^{n-1}$, $\mathbb{R}\widehat{\mathcal{Z}}_K = \mathbb{R}\mathcal{Z}_K / \mathbb{R}\mathcal{Z}_K^{n-1}$, $\widehat{\mathcal{Z}}_{K_I} = \mathcal{Z}_{K_I} / \mathcal{Z}_{K_I}^{|I|-1}$, $\mathbb{R}\widehat{\mathcal{Z}}_{K_I} = \mathbb{R}\mathcal{Z}_{K_I} / \mathbb{R}\mathcal{Z}_{K_I}^{|I|-1}$. Alternatively, they are the images of $\mathcal{Z}_K$, $\mathbb{R}\mathcal{Z}_K$, $\mathcal{Z}_{K_I}$, and $\mathbb{R}\mathcal{Z}_{K_I}$ under the quotient maps $(D^2)^{\times n} \rightarrow (D^2)^{\wedge n}$, $(D^1)^{\times n} \rightarrow (D^1)^{\wedge n}$, $(D^2)^{\times |I|} \rightarrow (D^2)^{\wedge |I|}$, and $(D^1)^{\times |I|} \rightarrow (D^1)^{\wedge |I|}$, respectively. Then $\mathbb{R}\widehat{\mathcal{Z}}_K = \bigcup_{\sigma \in K} Y_1^\sigma \wedge \dots \wedge Y_n^\sigma$, where $Y_i^\sigma = D^1$ if $i \in \sigma$, or $Y_i^\sigma = S^0$ if $i \notin \sigma$.

We say that a sequence $(I_1, \dots, I_m)$ of subsets of a set I is an *(ordered) partition* of I if they are mutually disjoint, non-empty, and $I_1 \cup \dots \cup I_m = I$. Given a partition $(I_1, \dots, I_m)$ of I with $|I| = k$ and $I_j = \{i_{j1}, \dots, i_{jk_j}\}$ in increasing order, by rearranging coordinates, we can, and usually will, think of the smash product $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_m}}$ as the subspace of $(D^1)^{\wedge k}$ given by

$$(2) \quad \left\{ (x_1, \dots, x_k) \in (D^1)^{\wedge k} \mid (x_{i_{j1}}, \dots, x_{i_{jk_j}}) \in \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_j}} \text{ for each } j \right\}$$

(the order of the smash product therefore being irrelevant). Then $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1 \cup \dots \cup I_m}}$ is a subspace of $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_m}}$, and we let

$$\hat{\iota}_{I_1, \dots, I_m} : \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1 \cup \dots \cup I_m}} \longrightarrow \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_m}}$$

denote the inclusion. This carries over similarly for moment-angle complexes $\mathcal{Z}_K$. By rearranging coordinates there is a homeomorphism $\widehat{\mathcal{Z}}_{K_I} \cong \Sigma^{|I|} \mathbb{R}\widehat{\mathcal{Z}}_{K_I}$, and so we see that the inclusion

$$\hat{\iota}_{I_1, \dots, I_m}^1 : \widehat{\mathcal{Z}}_{K_{I_1 \cup \dots \cup I_m}} \longrightarrow \widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \widehat{\mathcal{Z}}_{K_{I_m}}$$

is homeomorphic to the k -fold suspension of $\hat{\iota}_{I_1, \dots, I_m}$ above.

The homotopy type of a quotiented moment angle complex $\mathbb{R}\widehat{\mathcal{Z}}_K$ is not mysterious, at least not with regards to the combinatorics of K . The existence of a homeomorphism $\mathbb{R}\widehat{\mathcal{Z}}_{K_I} \cong \Sigma|K_I|$ was given in [1, 3], and this is obtainable as a restriction of a homeomorphism $\Sigma|\Delta^{k-1}| \cong (D^1)^{\wedge k}$ where $|I| = k$. More generally, $\Sigma|\Delta^{k-1}| \cong (D^1)^{\wedge k}$ restricts to a homeomorphism

$$\Sigma|K_{I_1} * \dots * K_{I_m}| \cong \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_m}}$$

where $(I_1, \dots, I_m)$ is a partition of I . Consequently, $\hat{\iota}_{I_1, \dots, I_m}$ is homeomorphic to the suspended inclusion $\Sigma\hat{\iota}_{I_1, \dots, I_m} : \Sigma|K_{I_1 \cup \dots \cup I_m}| \longrightarrow \Sigma|K_{I_1} * \dots * K_{I_m}|$.

2.3. A Few Splitting Conditions. If I is a subset of $[n]$ with $k = |I|$ elements, and $\mathcal{R} = (s_1, \dots, s_k)$ is any sequence of real numbers, let i_ℓ denote the ℓ^{th} smallest element in I , s'_j denote the j^{th} smallest element in the set $S = \{s_1, \dots, s_k\}$, and $m = |S|$ be the number of distinct elements in $\mathcal{R}$. Then assign to $\mathcal{R}$ the ordered partition $\mathcal{I}_{\mathcal{R}} = (I_1, \dots, I_m)$ of I where $I_j = \{i_\ell \in I \mid s_\ell = s'_j\}$. For example, if $I = [4]$ and $\mathcal{R} = (-1, \pi, -1, 0)$, then $[4]_{\mathcal{R}} = (\{1, 3\}, \{4\}, \{2\})$. Take the diagonal

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = \dots = x_n\}$$

and consider the following subspace of the smash product $\mathcal{P}_n = (\mathbb{R}^n - \Delta_n) \wedge \Sigma|\Delta^{n-1}|$:

$$\mathcal{Q}_K = \bigcup_{\substack{y \in (\mathbb{R}^n - \Delta_n) \\ (I_1, \dots, I_m) = [n]_y}} \{y\} \wedge \Sigma|K_{I_1} * \dots * K_{I_m}|,$$

where $[n]_y$ is the ordered partition of $[n]$ for the given sequence y , as defined above. Here we took $\mathbb{R}^n - \Delta_n$ to be without a basepoint, so $\mathcal{P}_n$ is the half-smash $((\mathbb{R}^n - \Delta_n) \times \Sigma|\Delta^{n-1}|) / ((\mathbb{R}^n - \Delta_n) \times \{*\})$, and is itself basepointed, as is $\mathcal{Q}_K$. Given any $\omega = (t_1, \dots, t_{n-1}, t, z) \in \Sigma^n|K|$ ($z \in |K|$ and $t_1, \dots, t_{n-1}, t \in [-1, 1]$), let $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$, and define the map

$$\Phi_K : \Sigma^n|K| \longrightarrow \Sigma\mathcal{Q}_K$$

by

$$\Phi_K(\omega) = \begin{cases} * & \text{if } \beta \in \{0, 1\}; \\ (2\beta - 1, (t_1, \dots, t_{n-1}, 0), (t, z)) & \text{if } 0 < \beta < 1. \end{cases}$$

Remark: Since $(t_1, \dots, t_{n-1}, 0)$ cannot be on the diagonal Δ_n , one cannot generally define a nullhomotopy of Φ_K by trying to shift the parameter β to 1.

Definition 2.1. K is *weakly coherently homotopy Golod* if K is a single vertex, or (recursively) the vertex deletion $K \setminus \{i\}$ is weakly coherently homotopy Golod for each $i \in [n]$, and the map $\Phi_K : \Sigma^n|K| \longrightarrow \Sigma\mathcal{Q}_K$ is nullhomotopic.

Remark 2.2. Equivalently, K is *weakly coherently homotopy Golod* if $\Phi_{K_I} : \Sigma^{|I|}|K_I| \rightarrow \Sigma \mathcal{Q}_{K_I}$ is nullhomotopic for each $I \subseteq [n]$.

Proposition 2.3 (Theorem 4.3 in [3]). $\mathcal{Z}_K$ is a co- H -space if K is weakly coherently homotopy Golod. $\square$

On the other hand, a somewhat weaker looking condition is a necessary one.

Proposition 2.4 (Proposition 4.4 in [3]). If $\mathcal{Z}_K$ is a co- H -space, then each of the suspended inclusions

$$\Sigma^{|I \cup J|} \iota_{I,J} : \Sigma^{|I \cup J|} |K_{I \cup J}| \rightarrow \Sigma^{|I \cup J|} |K_I * K_J|$$

is nullhomotopic for every disjoint non-empty $I, J \subseteq [n]$.

We include the proof since it is short and illustrative.

Proof of Proposition 2.4. Recall that Y is a co- H -space with comultiplication $Y \xrightarrow{\psi} Y \vee Y$ if and only if the diagonal map $Y \xrightarrow{\Delta} Y \times Y$ is homotopic to the composite $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\text{include}} Y \times Y$. In particular, this implies that the reduced diagonal map $\bar{\Delta} : Y \xrightarrow{\Delta} Y \times Y \rightarrow (Y \times Y)/(Y \vee Y) \xrightarrow{\cong} Y \wedge Y$ is nullhomotopic whenever Y is a co- H -space.

Suppose $\mathcal{Z}_K$ is a co- H -space. Since $\mathcal{Z}_{K_{I \cup J}}$ is a retract of $\mathcal{Z}_K$, then $\mathcal{Z}_{K_{I \cup J}}$ is a co- H -space for any disjoint non-empty $I, J \subseteq [n]$, and so $\mathcal{Z}_{K_{I \cup J}} \xrightarrow{\bar{\Delta}} \mathcal{Z}_{K_{I \cup J}} \wedge \mathcal{Z}_{K_{I \cup J}}$ is nullhomotopic. Then the composite

$$\mathcal{Z}_{K_{I \cup J}} \xrightarrow{\bar{\Delta}} \mathcal{Z}_{K_{I \cup J}} \wedge \mathcal{Z}_{K_{I \cup J}} \rightarrow \mathcal{Z}_{K_I} \wedge \mathcal{Z}_{K_J} \rightarrow \widehat{\mathcal{Z}}_{K_I} \wedge \widehat{\mathcal{Z}}_{K_J}$$

is nullhomotopic (where the second last map is the smash of the coordinate-wise projection maps onto $\mathcal{Z}_{K_I}$ and $\mathcal{Z}_{K_J}$, and the last map is the smash of quotient maps), and since it is equal to

$$\mathcal{Z}_{K_{I \cup J}} \xrightarrow{\text{quotient}} \widehat{\mathcal{Z}}_{K_{I \cup J}} \xrightarrow{\hat{\iota}_{I,J}^1} \widehat{\mathcal{Z}}_{K_I} \wedge \widehat{\mathcal{Z}}_{K_J},$$

this last composite is nullhomotopic as well. But $\mathcal{Z}_{K_{I \cup J}}$ is a co- H -space, so by Proposition 2.11 in [3], the quotient map $\mathcal{Z}_{K_{I \cup J}} \rightarrow \widehat{\mathcal{Z}}_{K_{I \cup J}}$ has a right homotopy inverse, implying $\hat{\iota}_{I,J}^1$ is nullhomotopic. Since $\hat{\iota}_{I,J}^1$ is up to homeomorphism the $(|I \cup J|)$ -fold suspension of $\hat{\iota}_{I,J} : \mathbb{R} \widehat{\mathcal{Z}}_{K_{I \cup J}} \rightarrow \mathbb{R} \widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R} \widehat{\mathcal{Z}}_{K_J}$, and since $\hat{\iota}_{I,J} : \Sigma^{|I \cup J|} \mathcal{Z}_{K_{I \cup J}} \rightarrow \Sigma^{|I \cup J|} \mathcal{Z}_{K_I} \wedge \Sigma^{|I \cup J|} \mathcal{Z}_{K_J}$ is nullhomotopic, then $\Sigma^{|I \cup J|+1} \iota_{I,J}$ is nullhomotopic.

Now $\Sigma^{|I \cup J|} |K_{I \cup J}|$ is at most $(2(|I| + |J|) - 1)$ -dimensional and $\Sigma^{|I \cup J|} |K_I * K_J| \simeq \Sigma^{|I \cup J|+1} |K_I| \wedge |K_J|$ is at least $(|I| + |J|)$ -connected, so the suspension homomorphism

$$[\Sigma^{|I \cup J|} |K_{I \cup J}|, \Sigma^{|I \cup J|} |K_I * K_J|] \rightarrow [\Sigma^{|I \cup J|+1} |K_{I \cup J}|, \Sigma^{|I \cup J|+1} |K_I * K_J|]$$

is an isomorphism by the Freudenthal suspension theorem ([17], Theorem 4.10). Thus $\Sigma^{|I \cup J|} \iota_{I,J}$ is nullhomotopic since $\Sigma^{|I \cup J|+1} \iota_{I,J}$ is. $\square$

Remark 2.5. Proposition 2.3 is a consequence of the following. Given any map $f : \Sigma X \rightarrow Y$, f factors as $\Sigma X \xrightarrow{\Sigma f^{ad}} \Sigma \Omega Y \xrightarrow{ev} Y$ where f^{ad} is the adjoint of f and ev the adjoint of the identity $\mathbb{1}_{\Omega Y}$. Taking f to be an attaching map $\Sigma^n |K| \rightarrow \mathcal{Z}_K^{n-1}$ whose cofiber is $\mathcal{Z}_K$, it was shown in [3] that Σf^{ad} factors through $\Sigma^n |K| \xrightarrow{\Phi_K} \Sigma \mathcal{Q}_K$. Then Proposition 2.3 follows from the fact that f is nullhomotopic if and only if Σf^{ad} is.

2.4. A Reformulation of $\mathcal{Q}_K$. It will be convenient to redefine $\mathcal{Q}_K$ and Φ_K in terms of quotiented moment-angle complexes. Take the following subspace of $(\mathbb{R}^n - \Delta_n) \wedge (D^1)^{\wedge n}$:

$$\mathcal{Q}'_K \cong \bigcup_{\substack{y \in (\mathbb{R}^n - \Delta_n) \\ (I_1, \dots, I_m) = [n]_y}} \{y\} \wedge \mathbb{R} \widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \mathbb{R} \widehat{\mathcal{Z}}_{K_{I_m}}.$$

As remarked before, there is a homeomorphism $\Sigma|\Delta^{n-1}| \cong (D^1)^{\wedge n}$ restricting for each partition $(I_1, \dots, I_m)$ of $[n]$ to a homeomorphism $\Sigma|K_{I_1} * \dots * K_{I_m}| \cong \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_1}} \wedge \dots \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_{I_m}}$, such the inclusion $\hat{\iota}_{I_1, \dots, I_m}$ is up to homeomorphism the suspended inclusion $\Sigma\iota_{I_1, \dots, I_m}$. Thus, $\mathbb{R}\widehat{\mathcal{Z}}_K \cong \Sigma|K|$ and there is a homeomorphism $(\mathbb{R}^n - \Delta_n) \wedge \Sigma|\Delta^{n-1}| \cong (\mathbb{R}^n - \Delta_n) \wedge (D^1)^{\wedge n}$ restricting to a homeomorphism

$$\mathcal{Q}_K \cong \mathcal{Q}'_K,$$

with respect to which Φ_K is the map

$$\Phi'_K: \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \longrightarrow \Sigma\mathcal{Q}'_K$$

defined by: $\Phi'_K(t_1, \dots, t_{n-1}, x) = (2\beta - 1, (t_1, \dots, t_{n-1}, 0), x)$ if $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$ is between 0 and 1, otherwise Φ'_K maps to the basepoint. We will work with $\mathcal{Q}'_K$ and Φ'_K instead of $\mathcal{Q}_K$ and Φ_K from now on.

2.5. Clusters and Comultiplication. Given integers $m \leq n$, a metric space M , a fixed point $x_n \in M$, and a function $f: M^{\times(n-1)} \longrightarrow \mathbb{R}$, consider the configuration space $\mathcal{A} \subseteq M^{\times(n-1)}$ consisting of all points $y = (x_1 \dots x_{n-1}) \in M^{\times(n-1)}$ such that each x_i (including x_n) has a subset of m of the x_j 's ($j \neq i$) clustered around it within distance $f(y)$. We describe the topology of $\mathcal{A}$ when M is the interval $(0, 1)$, $m = \lfloor \frac{n}{3} \rfloor$, $x_n = 0$, and $f(y)$ is proportional to the maximal distance between the x_i 's.

As a convention, $\min \emptyset = 0$ and $\max \emptyset = 0$. If $y = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$, let $\hat{y} = (t_1, \dots, t_{n-1}, 0) \in \mathbb{R}^n$. Given $I \subseteq [n]$, $n \geq 2$, and $z = (t_1, \dots, t_n) \in \mathbb{R}^n$, define the following non-negative values

$$\nu_{I,i}^z = \min \left\{ \max \{ |t_i - t_j| \mid j \in S \} \mid S \subseteq I - \{i\} \text{ and } |S| = \left\lfloor \frac{n}{3} \right\rfloor \right\},$$

$$\nu_I^z = \max \{ \nu_{I,i}^z \mid i \in I \},$$

$$\delta^z = \frac{1}{n} (\max \{ t_1, \dots, t_n \} - \min \{ t_1, \dots, t_n \}).$$

In other words, $\nu_{I,i}^z$ is the smallest value such that the interval $[t_i - \nu_{I,i}^z, t_i + \nu_{I,i}^z]$ contains at least $\lfloor \frac{n}{3} \rfloor$ of the t_j 's over $j \in I - \{i\}$, and is zero if no such value exists. Consider the subspace $\mathcal{A}$ of the interior $\text{int}(D^{n-1})$ of the $(n-1)$ -disk $D^{n-1} = [-1, 1]^{\times(n-1)}$ given by

$$\mathcal{A} = \left\{ y \in \text{int}(D^{n-1}) \mid \nu_{[n]}^{\hat{y}} < \delta^{\hat{y}} \right\}.$$

Take the subset of ordered partitions of $[n]$

$$\mathcal{O} = \left\{ (I, J) \mid I \cup J = [n], I \cap J = \emptyset, \text{ and } |I|, |J| > \frac{n}{3} \right\},$$

and for each $(I, J) \in \mathcal{O}$, consider the subspace of $\text{int}(D^{n-1})$

$$\mathcal{A}_{I,J} = \left\{ y = (t_1, \dots, t_{n-1}) \in \text{int}(D^{n-1}) \mid t_j - t_i > \delta^{\hat{y}}, \nu_{I,i}^{\hat{y}} < \delta^{\hat{y}}, \nu_{J,j}^{\hat{y}} < \delta^{\hat{y}} \text{ for } i \in I, j \in J, \text{ where } t_n = 0 \right\}.$$

Let

$$\hat{\mathcal{A}}_{I,J} = \text{cl}(\mathcal{A}_{I,J}) / \partial \mathcal{A}_{I,J}$$

where $\text{cl}(\mathcal{A}_{I,J})$ is the closure of $\mathcal{A}_{I,J}$, and $\partial \mathcal{A}_{I,J}$ is its boundary.

Lemma 2.6. (i) The $\mathcal{A}_{I,J}$'s are mutually disjoint, and $\mathcal{A}$ is a union of $\mathcal{A}_{I,J}$ over all partitions $(I, J) \in \mathcal{O}$; and (ii) each $\mathcal{A}_{I,J}$ is an open subspace of $\text{int}(D^{n-1})$ homeomorphic to the interior of D^{n-1} . Hence

$$D^{n-1} / (D^{n-1} - \mathcal{A}) \cong \bigvee_{(I,J) \in \mathcal{O}} \hat{\mathcal{A}}_{I,J} \cong \bigvee_{(I,J) \in \mathcal{O}} S^{n-1}$$

where $\hat{\mathcal{A}}_{I,J} \cong S^{n-1}$ for each $(I, J) \in \mathcal{O}$.

Proof of (i). Take any distinct $(I, J), (I', J') \in \mathcal{O}$, and let $t_n = 0$ and $y = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$. Since we have $t_i < t_j$ when $y \in \mathcal{A}_{I,J}$, $i \in I$ and $j \in J$; or when $y \in \mathcal{A}_{I',J'}$, $i \in I'$ and $j \in J'$; then $\mathcal{A}_{I,J} \cap \mathcal{A}_{I',J'} = \emptyset$ whenever $I \cap J' \neq \emptyset$ and $J \cap I' \neq \emptyset$. Suppose $I \cap J' = \emptyset$ or $J \cap I' = \emptyset$. Since (I, J) and (I', J') are distinct partitions of $[n]$, then one of $I' \not\subseteq I$, $I \not\subseteq I'$, $J' \not\subseteq J$, or $J \not\subseteq J'$ holds. Without loss of generality, suppose $I' \not\subseteq I$. If $y \in \mathcal{A}_{I,J}$, we have $\nu_{I,i}^{\dot{y}} < \delta^{\dot{y}}$ for any $i \in I$, so given that $i \in I - I' \subseteq J'$, there must be an $S \subseteq I$ such that $|S| = \lfloor \frac{n}{3} \rfloor$ and $\max\{|t_i - t_j| \mid j \in S\} < \delta^{\dot{y}}$. Since $|I|, |J|, |I'| > \frac{n}{3}$ and $I' \not\subseteq I$, then $|I - I'| < \frac{n}{3}$, so S must contain at least one element i' in I' . But if $y \in \mathcal{A}_{I',J'}$, then $\max\{|t_i - t_j| \mid j \in S\} \geq |t_i - t_{i'}| \geq \delta^{\dot{y}}$ since $i \in J'$ and $i' \in I'$, a contradiction. Thus $\mathcal{A}_{I,J} \cap \mathcal{A}_{I',J'} = \emptyset$ here as well. In summary, $\mathcal{A}_{I,J}$ and $\mathcal{A}_{I',J'}$ are disjoint for any distinct $(I, J), (I', J') \in \mathcal{O}$.

Suppose that $y \in \mathcal{A}$. Let $a = \min\{t_1, \dots, t_n\}$ and $b = \max\{t_1, \dots, t_n\}$. Since there are at most $n - 2$ t_i 's in the interval (a, b) , there is an l and an r in $[n]$ such that there are no t_i 's in the interval (t_l, t_r) , and $t_r - t_l > \frac{1}{n}(b - a) = \delta^{\dot{y}}$. Then we have disjoint non-empty sets $I = \{i \in [n] \mid t_i \leq t_l\}$ and $J = \{j \in [n] \mid t_j \geq t_r\}$ partitioning $[n]$ with $t_j - t_i > \delta^{\dot{y}}$ for every $i \in I$ and $j \in J$. Notice that this last fact implies for every $k \in I$ there cannot be an $S \subseteq [n] - \{k\}$ such that S contains an element in J and: (*) $|S| = \lfloor \frac{n}{3} \rfloor$ and $\max\{|t_k - t_j| \mid j \in S\} < \delta^{\dot{y}}$. But since $\nu_{[n]}^{\dot{y}} < \delta^{\dot{y}}$ (as $y \in \mathcal{A}$), there is for each $k \in I$ at least one such $S \subseteq [n] - \{k\}$ such that (*) holds, and we must have $S \subseteq I$ since S cannot contain elements in J . Therefore $|I| > \frac{n}{3}$ and $\nu_{I,k}^{\dot{y}} < \delta^{\dot{y}}$ for every $k \in I$. Similarly, $|J| > \frac{n}{3}$ and $\nu_{J,k}^{\dot{y}} < \delta^{\dot{y}}$ for every $k \in J$. It follows that (I, J) is in $\mathcal{O}$, and $y \in \mathcal{A}_{I,J}$. Thus $\mathcal{A} \subseteq \bigcup_{(I,J) \in \mathcal{O}} \mathcal{A}_{I,J}$.

On the other hand, if $y \in \mathcal{A}_{I,J}$, since either $\nu_{I,i}^{\dot{y}} < \delta^{\dot{y}}$ or $\nu_{J,i}^{\dot{y}} < \delta^{\dot{y}}$ for any $i \in [n]$, depending on whether $i \in I$ or $i \in J$, then $\nu_{[n],i}^{\dot{y}} < \delta^{\dot{y}}$ for each i , implying that $\nu_{[n]}^{\dot{y}} < \delta^{\dot{y}}$. Therefore $y \in \mathcal{A}$. Thus $\bigcup_{(I,J) \in \mathcal{O}} \mathcal{A}_{I,J} \subseteq \mathcal{A}$.

Proof of (ii). Notice that $\mathcal{A}$ is an open subspace of $\text{int}(D^{n-1})$ since it does not contain any points in its boundary $\{y \in \text{int}(D^{n-1}) \mid \nu_{[n]}^{\dot{y}} = \delta^{\dot{y}}\}$. Thus $\mathcal{A}_{I,J}$ is also open by part (i).

Given $(I, J) \in \mathcal{O}$, consider the point $b = (b_1, \dots, b_{n-1})$ in $\mathcal{A}_{I,J}$ whose coordinates are defined for indices $i \in I$ and $j \in J$ by either: (1) $b_i = -\frac{1}{2}$ and $b_j = 0$ if $n \in J$; or else (2) $b_i = 0$ and $b_j = \frac{1}{2}$ if $n \in I$. Given $y \in \mathcal{A}_{I,J}$, let $y_t = (1 - t)y + tb$. Notice that $\delta^{\dot{y}_t} = (1 - t)\delta^{\dot{y}} + t(\frac{1}{2n})$, $\nu_{I,i}^{\dot{y}_t} = (1 - t)\nu_{I,i}^{\dot{y}}$, $\nu_{J,j}^{\dot{y}_t} = (1 - t)\nu_{J,j}^{\dot{y}}$ and $((1 - t)t_j + tb_j) - ((1 - t)t_i + tb_i) = (1 - t)(t_j - t_i) + t(\frac{1}{2})$ for any $i \in I$ and $j \in J$, where $t_n = 0$. It follows that y_t is in $\mathcal{A}_{I,J}$ for each $t \in [0, 1]$ whenever y is, and $\mathcal{A}_{I,J}$ deformation retracts onto the point b via the homotopy sending $y \mapsto y_t$ at each time t . Moreover, this is a linear homotopy with y_t lying on the line through b and y , and $\mathcal{A}_{I,J}$ is an open subspace of $\text{int}(D^{n-1})$. We conclude that $\mathcal{A}_{I,J}$ is homeomorphic to $\text{int}(D^{n-1})$. $\square$

Since $\mathcal{A}$ is in the interior of D^{n-1} , $\partial D^{n-1} \subseteq D^{n-1} - \mathcal{A}$. Of use to us will be the comultiplication

$$\psi: S^{n-1} \cong D^{n-1}/\partial D^{n-1} \longrightarrow D^{n-1}/(D^{n-1} - \mathcal{A}) \xrightarrow{\cong} \bigvee_{(I,J) \in \mathcal{O}} S^{n-1}$$

the first map being the quotient map, the last map the wedge of homeomorphisms $\hat{\mathcal{A}}_{I,J} \longrightarrow S^{n-1}$ restricting away from the basepoint to the homeomorphism

$$h: \mathcal{A}_{I,J} \longrightarrow \text{int}(D^{n-1})$$

from Lemma 2.6. On $S^{n-1} \wedge \mathbb{R}\widehat{\mathcal{Z}}_K = \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K$, this gives the comultiplication

$$\psi_K = \psi \wedge \mathbb{1}: \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \longrightarrow \bigvee_{(I,J) \in \mathcal{O}} \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K$$

where $\mathbb{1}: \mathbb{R}\widehat{\mathcal{Z}}_K \longrightarrow \mathbb{R}\widehat{\mathcal{Z}}_K$ is the identity.

3. PROOF OF THEOREM 1.1

Suppose K is $\frac{n}{3}$ -neighbourly. Notice that $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I'}} = (D^1)^{\times |I'|}$ whenever $|I'| \leq \frac{n}{3}$, in which case we have $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I'}} = (D^1)^{\wedge |I'|}$. Thinking of $\mathbb{R}\widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_J}$ as the subspace of $(D^1)^{\wedge n}$ with coordinates rearranged as in (2), we can define for any $(I, J) \in \mathcal{O}$ a map

$$\lambda_{I,J}: \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_J} \longrightarrow \Sigma\mathcal{Q}'_K$$

that is given for any $\omega = (t_1, \dots, t_{n-1}, (x_1, \dots, x_n))$ by

$$\lambda_{I,J}(\omega) = \begin{cases} * & \text{if } \beta \in \{0, 1\}; \\ (2\beta - 1, (t'_1, \dots, t'_{n-1}, 0), (\varphi_1(x_1), \dots, \varphi_n(x_n))) & \text{if } 0 < \beta < 1, \end{cases}$$

where $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$, $y' = (t'_1, \dots, t'_{n-1}) = h^{-1}(t_1, \dots, t_{n-1})$ for $h^{-1}: \text{int}(D^{n-1}) \xrightarrow{\cong} \mathcal{A}_{I,J}$ the inverse of the homeomorphism h in the definition of ψ_K , and

$$\varphi_i(x_i) = (1 + x_i) \left(\max \left\{ 0, \frac{\delta^{y'} - \nu'_{[n],i}}{\delta^{y'}} \right\} \right) - 1.$$

The t'_i 's cannot all be 0 = t'_n since by definition of $\mathcal{A}_{I,J}$, we have $t'_i < t'_j$ when $i \in I$ and $j \in J$, thus $\lambda_{I,J}(\omega)$ is in $\Sigma(\mathbb{R}^n - \Delta_n) \wedge (D^1)^{\wedge n}$ at the very least. By this last fact, and since we have $\varphi_i(x_i) = x_i$ for any i such that there is an $I' \subseteq [n]$ with $i \in I'$, $|I'| > \frac{n}{3}$, and $t'_i = t'_j$ for any $j \in I'$, it follows that $\lambda_{I,J}(\omega) \in \Sigma\mathcal{Q}'_K$. This is necessary since $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I'}}$ is not necessarily $(D^1)^{\wedge |I'|}$ when $|I'| > \frac{n}{3}$. Otherwise, when $|I'| \leq \frac{n}{3}$, we have $\mathbb{R}\widehat{\mathcal{Z}}_{K_{I'}} = (D^1)^{\wedge |I'|}$, meaning $\varphi_i(x_i)$ is free to take on any value in $D^1 = [-1, 1]$.

Lemma 3.1. *Suppose K is $\frac{n}{3}$ -neighbourly, $n \geq 2$. Then $\Phi'_K: \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \longrightarrow \Sigma\mathcal{Q}'_K$ is homotopic to the composite*

$$\zeta: \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \xrightarrow{\psi_K} \bigvee_{(I,J) \in \mathcal{O}} \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \longrightarrow \bigvee_{(I,J) \in \mathcal{O}} \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_J} \longrightarrow \Sigma\mathcal{Q}'_K$$

where the second map is the wedge of the suspended inclusions $\Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \xrightarrow{\Sigma^{n-1}\hat{\iota}_{I,J}} \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_J}$ over $(I, J) \in \mathcal{O}$, and the last map restricts to $\lambda_{I,J}$ on a summand $\Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_J}$.

Proof. Define the homotopy $H_t: \Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K \longrightarrow \Sigma\mathcal{Q}'_K$ for each $\omega = (t_1, \dots, t_{n-1}, (x_1, \dots, x_n))$ in $\Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K$, and at each time $t \in [0, 1]$, by mapping to the basepoint if $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$ is 0 or 1, otherwise

$$H_t(\omega) = (2\beta - 1, (t_1, \dots, t_{n-1}, 0), (\varphi_{1,t}(x_1), \dots, \varphi_{n,t}(x_n))),$$

where $\varphi_{i,t}(x_i) = (1 - t)x_i + t\varphi_i(x_i)$ and φ_i is as defined before for $y = (t_1, \dots, t_{n-1})$ in place of y' . We see that $H_t(\omega) \in \Sigma(\mathbb{R}^n - \Delta_n) \wedge (D^1)^{\wedge n}$ is in $\Sigma\mathcal{Q}'_K$ for similar reasons as outlined for $\lambda_{I,J}$.

Clearly $H_0 = \Phi'_K$. By definition of $\mathcal{A}$, ψ_K , and φ_i , we have $\psi_K(\omega) = *$ and $\varphi_{i,1}(x_i) = \varphi_i(x_i) = -1$ (the basepoint in D^1) for some i when $y \notin \mathcal{A}$, thus $\zeta(\omega) = H_1(\omega) = *$ in this case. On the other hand $\mathcal{A}$ is the union of disjoint sets $\bigcup_{(I,J) \in \mathcal{O}} \mathcal{A}_{I,J}$, and if $y \in \mathcal{A}_{I,J}$ for some $(I, J) \in \mathcal{O}$, then by definition of ψ_K , $\psi_K(\omega) = (h(y), (x_1, \dots, x_n))$ in the summand $\Sigma^{n-1}\mathbb{R}\widehat{\mathcal{Z}}_K$ corresponding to (I, J) , in which case

$$\begin{aligned} \zeta(\omega) &= \lambda_{I,J} \circ \Sigma^{n-1}\hat{\iota}_{I,J}(\psi_K(\omega)) \\ &= \lambda_{I,J}(h(y), (x_1, \dots, x_n)) \\ &= (2\beta - 1, (h^{-1}(h(y)), 0), (\varphi_1(x_1), \dots, \varphi_n(x_n))) \\ &= (2\beta - 1, (t_1, \dots, t_{n-1}, 0), (\varphi_1(x_1), \dots, \varphi_n(x_n))) \\ &= H_1(\omega). \end{aligned}$$

Thus, H_1 is the composite ζ .

□

Proof of Theorem 1.1. Necessity follows from Proposition 2.4. The theorem is trivial to check when $n = 1$ since $\mathcal{Z}_K = D^2$ here, so assume $n \geq 2$. Since K is $\frac{n}{3}$ -neighbourly, then so is K_I , or $\frac{|I|}{3}$ -neighbourly in particular. Since the maps Φ'_K and $\hat{\iota}_{I,J}: \mathbb{R}\widehat{\mathcal{Z}}_K \rightarrow \mathbb{R}\widehat{\mathcal{Z}}_{K_I} \wedge \mathbb{R}\widehat{\mathcal{Z}}_{K_J}$ are homeomorphic to Φ_K and $\Sigma|K_I| \times \Sigma|K_J| \rightarrow \Sigma|K|$ respectively, sufficiency follows from Proposition 2.3 and applying Lemma 3.1 for each K_I , $|I| \geq 2$, in place of K . $\square$

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