

CONFIGURATION SPACES AND POLYHEDRAL PRODUCTS

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ABSTRACT. This paper aims to find the most general combinatorial conditions under which a moment-angle complex $(D^2, S^1)^K$ is a co- H -space, thus splitting unstably in terms of its full subcomplexes. In this way we study to which extent the conjecture holds that a moment-angle complex over a Golod simplicial complex is a co- H -space. Our main tool is a certain generalisation of the theory of labelled configuration spaces.

1. INTRODUCTION

Polyhedral products have been the subject of quite a bit of interest recently, beginning with their appearance as homotopy theoretical generalisations of various objects studied in toric topology. Of particular importance are the polyhedral products $(D^2, S^1)^K$ and $(\mathbb{C}P^\infty, *)^K$, known as *moment-angle complexes* and *Davis-Januszkiewicz spaces* respectively. The homotopy theory of these spaces has many applications - from complex and symplectic geometry (c.f. [8, 27, 26]), to combinatorial and homological algebra (c.f. [32, 2]). For example, moment-angle manifolds appear as intersection of quadrics or as quasitoric manifolds after taking a certain orbit space, *Stanley-Reisner rings* of simplicial complexes are realised by the cohomology of Davis-Januszkiewicz spaces (equivalently, the equivariant cohomology ring of moment-angle complexes), while the cohomology ring of moment-angle complexes is closely related to the study of the cohomology of local rings (c.f. [10, 9]). One would like to know how the combinatorics of the underlying simplicial complex K encodes geometrical and topological properties of polyhedral products, and vice-versa. The case of Golod complexes is especially relevant. A ring $R = \mathbf{k}[v_1, \dots, v_n]/I$ for I a homogeneous ideal is said to be *Golod* if all products and higher Massey products in $\text{Tor}_{\mathbf{k}[v_1, \dots, v_n]}^+(R, \mathbf{k})$ vanish. Golod [17] showed that the Poincaré series of the homology ring $\text{Tor}_R(\mathbf{k}, \mathbf{k})$ of R is a rational function whenever R is Golod. From the context of combinatorics, a simplicial complex K on vertex set $[n] = \{1, \dots, n\}$ is said to be *Golod* over $\mathbf{k}$ if the *Stanley-Reisner ring* $\mathbf{k}[K]$ is Golod, and if this is true for all fields $\mathbf{k}$ and $\mathbf{k} = \mathbb{Z}$, we simply say that K is Golod. Fixing $\mathbf{k}$ to be a field or $\mathbb{Z}$, by [9, 15, 4, 22] there are isomorphisms of graded commutative algebras

$$H^*((D^2, S^1)^K; \mathbf{k}) \cong \text{Tor}_{\mathbf{k}[v_1, \dots, v_n]}(\mathbf{k}[K], \mathbf{k}) \cong \bigoplus_{I \subseteq [n]} \tilde{H}^*(\Sigma^{|I|+1} |K_I|; \mathbf{k})$$

where $\mathbf{k}[K]$ is the Stanley-Reisner ring of K , K_I is the restriction of K to vertex set $I \subseteq [n]$, and the multiplication in the rightmost algebra is realised by maps

$$\iota_{I,J}: |K_{I \cup J}| \longrightarrow |K_I * K_J| \cong |K_I| * |K_J| \simeq \Sigma |K_I| \wedge |K_J|,$$

induced by the canonical inclusions $K_{I \cup J} \longrightarrow K_I * K_J$ whenever I and J are non-empty and disjoint. The Golod condition can then be reinterpreted as $\iota_{I,J}$ inducing trivial maps on $\mathbf{k}$ -cohomology for disjoint non-empty I and J together with Massey products vanishing in $H^*((D^2, S^1)^K; \mathbf{k})$. This topological interpretation of the Golod condition has been a starting point for applying the homotopy theory of moment-angle complexes to the problem of determining which simplicial complexes K are Golod, see [20] for example. In the opposite direction, the cohomology of a moment-angle complex

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$(D^2, S^1)^K$ takes its simplest algebraic form when we restrict to Golod K . Golod complexes are therefore a natural starting point for studying the homotopy types of moment-angle complexes.

Considerable work has been done on the homotopy theory of moment-angle complexes over Golod complexes [19, 20, 21, 24, 23, 18], culminating in the following conjectured topological characterisation of the Golod complexes.

Conjecture 1.1. A moment-angle complex $(D^2, S^1)^K$ is a co- H -space if and only if K is Golod.

The right-hand implication follows from the general fact that cohomology rings of co- H -spaces have trivial cup and Massey products. The left-hand implication, if true (or to whichever extent it is true), immediately gives a nice explicit combinatorial description for the homotopy type of $(D^2, S^1)^K$. Namely, $(D^2, S^1)^K$ is a co- H -space if and only if

$$(D^2, S^1)^K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+1} |K_I|.$$

This was shown by Iriye and Kishimoto [23], and follows from a general suspension splitting due to Bahri, Bendersky, Cohen, and Gitler (the *BBCG splitting*) [3]

$$(1) \quad \Sigma(D^2, S^1)^K \simeq \bigvee_{I \subseteq [n]} \Sigma^{|I|+2} |K_I|.$$

Our goal in this paper is to determine the extent to which Conjecture 1.1 holds. To do this we consider several simplifications, the first of these being localisation. We confirm a *large primes* version of Conjecture 1.1, thus reproducing the rational result of Berglund [6] without making any significant use of rational homotopy theory.

Theorem 3.1. *Localised at any sufficiently large prime p , $(D^2, S^1)^K$ is a co- H -space if and only if K is Golod over $\mathbb{Z}_p$.*

Another simplification is to strengthen the hypothesis on a simplicial complex being Golod. One way of doing so is to require some suspension of each inclusion $\iota_{I,J}$ to be nullhomotopic (instead of only inducing trivial maps on cohomology) for every disjoint and non-empty $I, J \subsetneq [n]$. In Proposition 3.5 we show that after appropriate suspensions of $\iota_{I,J}$, this is a necessary condition for $(D^2, S^1)^K$ to be a co- H -space, and we call this the *homotopy Golod* condition.

Proposition 3.5. *If $(D^2, S^1)^K$ is a co- H -space, then it is homotopy Golod.*

Moreover, we will see that under certain additional coherence conditions on these nullhomotopies, the homotopy Golod condition becomes sufficient for $(D^2, S^1)^K$ to be a co- H -space, though possible no longer necessary. We outline this coherence condition as follows. Take the natural inclusion $\iota_{I,J}: |K_{I \cup J}| \rightarrow |K_I * K_J|$ for disjoint I, J . When I, J_1 , and J_2 are disjoint and non-empty and ι_I is the identity $|K_I| \rightarrow |K_I|$, there is a commutative diagram

$$\begin{array}{ccccc} & & |K_{I \cup J_1} * K_{J_2}| & & \\ & \nearrow \iota_{I \cup J_1, J_2} & & \searrow (\iota_{I, J_1}) * \iota_{J_2} & \\ |K_{I \cup J_1 \cup J_2}| & & & & |K_I * K_{J_1} * K_{J_2}| \\ & \searrow \iota_{I, J_1 \cup J_2} & & \nearrow \iota_I * (\iota_{J_1, J_2}) & \\ & & |K_I * K_{J_1 \cup J_2}| & & \end{array}$$

Then in defining the coherent homotopy Golod condition, we require the composites

$$\begin{aligned} \text{Cone}(\Sigma |K_{I \cup J_1 \cup J_2}|) &\xrightarrow{\hat{\iota}_{I \cup J_1, J_2}} \Sigma |K_{I \cup J_1} * K_{J_2}| \xrightarrow{\Sigma(\iota_{I, J_1}) * \iota_{J_2}} \Sigma |K_I * K_{J_1} * K_{J_2}| \\ \text{Cone}(\Sigma |K_{I \cup J_1 \cup J_2}|) &\xrightarrow{\hat{\iota}_{I, J_1 \cup J_2}} \Sigma |K_I * K_{J_1 \cup J_2}| \xrightarrow{\Sigma \iota_I * (\iota_{J_1, J_2})} \Sigma |K_I * K_{J_1} * K_{J_2}| \end{aligned}$$

to be homotopic to each other via a homotopy that is fixed on the base $\Sigma|K_{I \cup J_1 \cup J_2}|$ of the cone $\text{Cone}(\Sigma|K_{I \cup J_1 \cup J_2}|)$, where $\hat{\iota}_{I \cup J_1, J_2}$ and $\hat{\iota}_{I, J_1 \cup J_2}$ are the extensions given by the nullhomotopies of $\Sigma\iota_{I \cup J_1, J_2}$ and $\Sigma\iota_{I, J_1 \cup J_2}$. One then continues in this manner defining higher coherences for longer joins. This bears some resemblance to the coherence of homotopy in Stasheff's higher homotopy associativity of A_n -spaces [33]. In our case, the *associahedron* is replaced with the simplicial complex associated with ordered partitions of the vertex set $[n]$ (in other words, the dual of an order n *permutohedron*), while H -space multiplication maps are replaced with nullhomotopies of certain reduced diagonal maps, which up to homeomorphism are the above inclusions of cones into joins of full subcomplexes. This is made precise in Section 3, which we call the *coherent homotopy Golod* condition. The left-hand implication of the conjecture then holds for coherently homotopy Golod complexes.

Theorem 3.11. *If K is coherently homotopy Golod, then $(D^2, S^1)^K$ is a co- H -space. In particular, coherently homotopy Golod complexes are Golod.*

In particular, notice that the coherent homotopy Golod condition implies that all Massey products vanish in the cohomology of our moment-angle complex. The *extractible* complexes defined in [23] are currently the most general subclass of Golod complexes K for which $(D^2, S^1)^K$ is known to be a co- H -space. We will show that extractible complexes are coherently homotopy Golod, and give an example of a class of Golod complexes that are coherently homotopy Golod, but which are not extractible. In a follow-up paper [5] we use a weaker form of the coherent homotopy Golod condition to show that the much simpler homotopy Golod condition is both sufficient and necessary for $(D^2, S^1)^K$ to be a co- H -space whenever K is $\frac{n}{3}$ -neighbourly.

The main idea in this paper is the following well-known fact due to Ganea: a space Y is a co- H -space if and only if the evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ has a right homotopy inverse. Our method for constructing right homotopy inverses of $\Sigma\Omega(D^2, S^1)^K \xrightarrow{ev} (D^2, S^1)^K$ depends on a construction of a certain *scanning map*

$$\gamma: \mathcal{C}((D^1, S^0)^K) \longrightarrow \Omega(D^2, S^1)^K$$

together with a suspension splitting $\Sigma\mathcal{C}((D^1, S^0)^K) \simeq \bigvee_i \Sigma\mathcal{D}_i((D^1, S^0)^K)$. Here $\mathcal{C}((D^1, S^0)^K)$ is a configuration space that generalises classical labelled configuration spaces by allowing particles to collide under certain rules, and $\mathcal{D}_i((D^1, S^0)^K)$ is a certain *quotiented* configuration space. We remark that there are various constructions of configuration spaces in the literature that are related to these [11, 25, 29, 12, 13]. The upshot of this one is that the summands $\Sigma\mathcal{D}_i((D^1, S^0)^K)$ in the splitting of $\Sigma\Omega(D^2, S^1)^K$ are in a sense more combinatorial in their behaviour in comparison to polyhedral products, allowing us to obtain combinatorial statements about the homotopy type of $(D^2, S^1)^K$. This is analogous to the summands in the BBCG splitting (1) being combinatorial objects - each of them is known to be homeomorphic to certain *quotiented* moment-angle complexes.

Throughout this paper, -1 is taken to be the basepoint of $D^1 = [-1, 1]$. Points in S^1 are thought of as real numbers in D^1 with -1 and 1 identified as the basepoint. The suspension ΣX of a space X is taken to be the reduced suspension $D^1 \times X / (\{-1, 1\} \times X \cup D^1 \times \{*\})$ whenever X is basepointed with basepoint $*$. Otherwise it is the unreduced suspension, in other words, the quotient space of $D^1 \times X$ under identifications $(-1, x) \sim *_{-1}$ and $(1, x) \sim *_1$. In any case, ΣX is always basepointed, in the unreduced case the basepoint is taken to be $*_{-1}$.

Fix a product of connected basepointed spaces $X = X_1 \times \cdots \times X_n$ and a subspace $W \subseteq X$. For any product $Y = Y_1 \times \cdots \times Y_n$ of basepointed spaces, define the *coordinate smash* $Y \wedge_X W$ of Y and W in X to be the subspace of $Y \wedge_X X = (Y_1 \wedge X_1) \times \cdots \times (Y_n \wedge X_n)$ given by

$$Y \wedge_X W = \{((y_1, x_1), \dots, (y_n, x_n)) \in Y \wedge_X X \mid (x_1, \dots, x_n) \in W\}.$$

In particular, when each Y_i is an ℓ -sphere S^ℓ , we call $Y \wedge_X W$ the ℓ -fold *coordinate suspension* of W in X and denote it by W^ℓ . This is then the subspace of $X^\ell = (\Sigma^\ell X_1) \times \cdots \times (\Sigma^\ell X_n)$ given by

$$W^\ell = \{((t_1, x_1), \dots, (t_n, x_n)) \in X^\ell \mid (x_1, \dots, x_n) \in W\}$$

for pairs (t_i, x_i) in the reduced suspension $\Sigma^\ell X_i = D^\ell \times X_i / (\partial D^\ell \times X_i \cup D^\ell \times \{*\})$, D^ℓ being the unit ℓ -disk. Notice $W^0 = W$ and $(W^{\ell_1})^{\ell_2} = W^{\ell_1 + \ell_2}$.

Example 1.2. Let W be some union of product subspaces $A_1 \times \cdots \times A_n$. Then W^ℓ is a union of the subspaces $\Sigma^\ell A_1 \times \cdots \times \Sigma^\ell A_n$. In particular, thinking of the A_i 's as cells in X_i , if W is a subcomplex of X , then we see that W^ℓ is a subcomplex of X^ℓ .

Example 1.3. If $(\underline{X}, \underline{A}) = ((X_1, A_1), \dots, (X_n, A_n))$ is a sequence of pairs of spaces, K is a simplicial complex on n vertices, and W is the *polyhedral product*

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} Y_1^\sigma \times \cdots \times Y_n^\sigma \subseteq X_1 \times \cdots \times X_n,$$

where

$$Y_i^\sigma = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma, \end{cases}$$

then W^ℓ is the polyhedral product $(\Sigma^\ell \underline{X}, \Sigma^\ell \underline{A})^K \subseteq X^\ell$ of the n -tuple $(\Sigma^\ell \underline{X}, \Sigma^\ell \underline{A})$ of pairs of spaces $((\Sigma^\ell X_1, \Sigma^\ell A_1), \dots, (\Sigma^\ell X_n, \Sigma^\ell A_n))$.

1.1. Cofibrations and Splittings. Given any subset $I = \{i_1, \dots, i_k\} \subseteq [n] = \{1, \dots, n\}$, let W_I^ℓ be the image of W^ℓ under the projection $X^\ell \rightarrow X_I^\ell$, and $\hat{W}_I^\ell$ the image of W_I^ℓ under the quotient map $X_I^\ell \rightarrow \hat{X}_I^\ell$, where

$$\begin{aligned} X_I^\ell &= \Sigma^\ell X_{i_1} \times \cdots \times \Sigma^\ell X_{i_k}, \\ \hat{X}_I^\ell &= \Sigma^\ell X_{i_1} \wedge \cdots \wedge \Sigma^\ell X_{i_k}. \end{aligned}$$

Definition 1.4. We say W satisfies the *simplicial property* if given any point $x \in W$, the point $x' \in X$ obtained from x by replacing some coordinate in x with a basepoint is also a point in W .

Example 1.5. Polyhedral products have the simplicial property. If $\ell \geq 1$, then W^ℓ always has the simplicial property as a subspace of X^ℓ . Also, if each X_i is a basepointed CW -complex (i.e. with a fixed basepoint, built up by attaching cells of dimension ≥ 1 via basepoint preserving attaching maps), and the zero skeleton of X_i contains a single point (the basepoint), then W has the simplicial property.

Suppose that $\ell > 0$, or $\ell = 0$ and W satisfies the simplicial property. Then

$$W_I^\ell \cong \{(x_1, \dots, x_n) \in W^\ell \mid x_i = * \text{ if } i \notin I\}.$$

We will often think of W_I^ℓ as this subspace of W^ℓ . Thus, for any $0 \leq k \leq n$, we can take the union

$$(2) \quad W_k^\ell = \bigcup_{I \subseteq [n], |I|=k} W_I^\ell.$$

In other words,

$$W_k^\ell = \{(x_1, \dots, x_n) \in W^\ell \mid \text{at least } n - k \text{ coordinates } x_i \text{ are the basepoint } * \in X_i\}.$$

The equality $(W^\ell)_k = (W_k^\ell)^\ell$ is clear, so we forget brackets. Take the quotient spaces

$$\hat{W}_k^\ell = \frac{W_k^\ell}{W_{k-1}^\ell}.$$

and $\hat{W}_I^\ell = W_I^\ell / (W_I^\ell)_{|I|-1}$. Applying these constructions to X in place of W , since $X_I^\ell \cap X_J^\ell = X_{I \cap J}^\ell$ as subspaces of X^ℓ , and $X_I^\ell \cap X_{k-1}^\ell = (X_I^\ell)_{|I|-1}$ if $|I| = k$, then

$$(3) \quad \hat{X}_k^\ell = \frac{X_k^\ell}{X_{k-1}^\ell} = \bigvee_{I \subseteq [n], |I|=k} \hat{X}_I^\ell$$

(note not to confuse X_k^ℓ with the factors X_i of X). One can think of $\hat{W}_k^\ell$ as the image of W_k^ℓ under the quotient map $X_k^\ell \rightarrow \hat{X}_k^\ell$, and this restricts to $W_I^\ell \rightarrow \hat{W}_I^\ell$ for any $|I| = k$. Thus

$$(4) \quad \hat{W}_k^\ell = \bigvee_{I \subseteq [n], |I|=k} \hat{W}_I^\ell.$$

Mapping $\hat{W}_I^\ell$ via the homeomorphism $\hat{X}_I^\ell \xrightarrow{\cong} \Sigma^{\ell|I|} \hat{X}_I$ that sends $((t_1, x_1), \dots, (t_{|I|}, x_{|I|}))$ to the point $((t_1, \dots, t_{|I|}), (x_1, \dots, x_{|I|}))$, we see that

$$\begin{aligned} \hat{W}_I^\ell &\cong \Sigma^{\ell|I|} \hat{W}_I \\ \hat{W}_k^\ell &\cong \Sigma^{\ell k} \hat{W}_k. \end{aligned}$$

Note W_I^ℓ and W_k^ℓ are subcomplexes of W^ℓ if each X_i is a CW -complex, and W is a subcomplex of X .

For the remainder of this section we assume that each X_i is a connected basepointed CW -complex, and W is a connected subcomplex of their product X satisfying the simplicial property. The next two propositions generalise the polyhedral product splittings in [3] and [23]. For example, we may recover the BBCG suspension splitting of $W^1 = (D^2, S^1)^K$ by applying the homeomorphism $\hat{W}_I \cong \Sigma|K_I|$ given in [3].

Proposition 1.6. *For $1 \leq k \leq n$ and either $\ell > 0$, or $\ell = 0$ and W satisfying the simplicial property, there exists a suspension splitting*

$$\Sigma W_k^\ell \simeq \bigvee_{1 \leq j \leq k} \Sigma \hat{W}_j^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma \hat{W}_I^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma^{\ell|I|+1} \hat{W}_I.$$

Proof. Since W_k^ℓ has the simplicial property, given a subset $I = \{i_1, \dots, i_s\} \subseteq [n]$, we can define a map $q_I: W_k^\ell \rightarrow W_I^\ell \rightarrow \hat{W}_I^\ell \cong \Sigma^{\ell|I|} \hat{W}_I$ as the restriction of the composite $X^\ell \rightarrow X_I^\ell \rightarrow \hat{X}_I^\ell$ of the projection and quotient map to W_k^ℓ . Notice that q_I is the composite $W_k^\ell \rightarrow \hat{W}_k^\ell \xrightarrow{\hat{q}_I} \hat{W}_I^\ell$ when $|I| = k$, where $\hat{q}_I$ is the quotient onto the summand $\hat{W}_I^\ell$ with respect to homeomorphism (4).

We proceed by induction on k . Assume that for some $m < k$ a co- H -space sum

$$h_m: \Sigma W_m^\ell \xrightarrow{\text{pinch}} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma W_m^\ell \xrightarrow{\vee \Sigma q_I} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma \hat{W}_I^\ell$$

is a homotopy equivalence. This is trivial when $m = 1$. We assume the order in which these co- H -space sums h_j are taken is such that the maps Σq_I are summed before the maps Σq_J when $|I| < |J|$. Then there is a homotopy commutative diagram of cofibration sequences

$$(5) \quad \begin{array}{ccccc} \Sigma W_m^\ell & \longrightarrow & \Sigma W_{m+1}^\ell & \longrightarrow & \Sigma \hat{W}_{m+1}^\ell \\ \downarrow h_m & & \downarrow h_{m+1} & & \downarrow \hat{h}_{m+1} \\ \bigvee_{I \subseteq [n], |I| \leq m} \Sigma \hat{W}_I^\ell & \longrightarrow & \bigvee_{I \subseteq [n], |I| \leq m+1} \Sigma \hat{W}_I^\ell & \longrightarrow & \bigvee_{I \subseteq [n], |I| = m+1} \Sigma \hat{W}_I^\ell \end{array}$$

where the bottom horizontal maps are respectively the summand-wise inclusion and the quotient map onto the appropriate summands. The top maps are the suspended inclusion and quotient. Since for $|I| = m+1$ the composite of the first two maps in the sequence $X_m^\ell \rightarrow X_{m+1}^\ell \rightarrow \hat{X}_{m+1}^\ell \rightarrow \hat{X}_I^\ell$ is the constant map (where the last map is the quotient map with respect to splitting (3)), its restriction $W_m^\ell \rightarrow W_{m+1}^\ell \xrightarrow{q_I} \hat{W}_I^\ell$ is the constant map. Then since the suspended inclusion $\Sigma W_m^\ell \rightarrow \Sigma W_{m+1}^\ell$ is a co- H -map, and h_m and h_{m+1} are the co- H -space sums of the maps Σq_I as defined above, we see that the left-hand square commutes up to homotopy. The right-hand map $\hat{h}_{m+1}$ is taken to be a co- H -space sum of the composites $\Sigma \hat{q}_I: \Sigma \hat{W}_{m+1}^\ell \xrightarrow{\cong} \bigvee_{|I|=m+1} \Sigma \hat{W}_I^\ell \rightarrow \Sigma \hat{W}_I^\ell$ over all $|I| = m+1$, where the first map is the suspension of the homeomorphism (4), and the last map the suspended quotient map. The first map being a homeomorphism, we see $\hat{h}_{m+1}$ induces an isomorphism on homology, thus is a homotopy equivalence. We assume the co- H -space sum $\hat{h}_{m+1}$ is taken in the same order

as the last maps Σq_I for $|I| = m + 1$ are taken in the co- H -space sum h_{m+1} . Then since Σq_I is the composite of the suspended quotient map $\Sigma W_{m+1}^\ell \rightarrow \Sigma \hat{W}_{m+1}^\ell$ and $\Sigma \hat{q}_I$, and the suspended quotient map being a co- H -map, we see that the right-hand square commutes up to homotopy.

The left-hand and right-hand vertical maps being homotopy equivalences, h_{m+1} induces an isomorphism of homology groups. All spaces being simply connected CW -complexes, h_{m+1} is a homotopy equivalence. This completes the induction. $\square$

Lemma 1.7. *W_k^ℓ is simply connected for $\ell \geq 2$ and each $k \leq n$. Moreover, if W is connected, then W_k^1 is also simply connected.*

Proof. Since X^ℓ has no 1-cells when $\ell \geq 2$, neither does its subcomplex W_k^ℓ , so it is simply-connected.

Suppose W is connected. Then each W_I is connected, and since W has the simplicial property, the W_I 's intersect at the basepoint, so W_k is also connected. Notice the 1-skeleton of ΣX_i consists of points (t, y) such that y is in the 0-skeleton of X_i . Since cells in X^1 are products of cells in each ΣX_i , the 1-skeleton of the subcomplex $W_k^1 \subseteq X^1$ is contained in the subspace $V \subseteq W_k^1 = (W_k)^1$ consisting of points $z = ((t_1, x_1), \dots, (t_n, x_n))$ such that $x = (x_1, \dots, x_n)$ is a point in the 0-skeleton of W_k . Notice that V is a wedge of subspaces $V_x = \{((t_1, x_1), \dots, (t_n, x_n)) \mid 0 \leq t_i \leq 1\}$ for each of the disjoint points x in the 0-skeleton of W_k . Since W_k is connected, we can define a nullhomotopy of V_x in W_k^1 by picking a path in W_k^1 from x to the basepoint $(*, \dots, *)$ to define a homotopy of $z = ((t_1, x_1), \dots, (t_n, x_n))$ to the basepoint $((t_1, *), \dots, (t_n, *)) \sim *$. This therefore defines a nullhomotopy of V in W_k^1 , and since the 1-skeleton of W_k^1 is contained in V , a nullhomotopy of its 1-skeleton inside W_k^1 . Thus W_k^1 is simply connected. $\square$

The following folk statement will be useful, and for completeness we include a proof.

Lemma 1.8. *If Y is a co- H -space, and $\psi: Y \rightarrow Y \vee Y$ its comultiplication, then $\Sigma \psi$ is homotopic to the pinch map $\Sigma Y \xrightarrow{\text{pinch}} \Sigma Y \vee \Sigma Y$.*

Proof. Let H and H' be the homotopies to the identity of the composites $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{1 \vee * } Y$ and $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{* \vee 1} Y$. The composite $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\iota} Y \times Y$, where ι is the canonical inclusion, is homotopic to the diagonal map $Y \xrightarrow{\Delta} Y \times Y$ via the homotopy $G_t(y) = (H_t(y), H'_t(y))$. Take the composite

$$f: \Sigma(Y \times Y) \xrightarrow{\text{pinch}} \Sigma(Y \times Y) \vee \Sigma(Y \times Y) \rightarrow \Sigma Y \vee \Sigma Y$$

where the last map is the wedge sum of the suspended projection maps $\Sigma(Y \times Y) \xrightarrow{\Sigma \pi_i} \Sigma Y$, $i = 1, 2$, projecting onto the first factor and the second factor respectively. The composite f is a left homotopy inverse of $\Sigma \iota$. Therefore $\Sigma Y \xrightarrow{\Sigma \psi} \Sigma(Y \vee Y) \xrightarrow{\Sigma \iota} \Sigma(Y \times Y) \xrightarrow{f} \Sigma Y \vee \Sigma Y$ is homotopic to $\Sigma \psi$, and $\Sigma Y \xrightarrow{\Sigma \Delta} \Sigma(Y \times Y) \xrightarrow{f} \Sigma Y \vee \Sigma Y$ is the pinch map $\Sigma Y \xrightarrow{\text{pinch}} \Sigma Y \vee \Sigma Y$. These two composites are homotopic since $\Sigma \iota \circ \Sigma \psi = \Sigma(\iota \circ \psi)$ is homotopic to $\Sigma \Delta$. $\square$

Proposition 1.9. *Let $\ell \geq 0$ and $1 \leq k \leq n$, and consider the following:*

- (a) W_k^ℓ is a co- H -space;
- (b) there is a splitting

$$W_k^\ell \simeq \bigvee_{0 \leq j \leq k} \hat{W}_j^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \hat{W}_I^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma^{|I|} \hat{W}_I;$$

- (c) the quotient map $W_m^\ell \rightarrow \hat{W}_m^\ell$ has a right homotopy inverse for each $m \leq k$;
- (d) W_I^ℓ is a simply connected co- H -space for each $I \subseteq [n]$ with $|I| \leq k$;
- (e) the quotient map $W_I^\ell \rightarrow \hat{W}_I^\ell$ has a right homotopy inverse for each $I \subseteq [n]$ with $|I| \leq k$.

Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) when $\ell \geq 2$, or when $\ell = 1$ and W is connected. Generally, (a) $\Rightarrow$ (b) provided W_j^ℓ is simply connected for $j \leq k$.

Proof. When $\ell \geq 2$, or $\ell = 1$ and W is connected, Lemma 1.7 ensures that all our spaces are simply connected. The implication $(b) \Rightarrow (a)$ is clear when $\ell \geq 1$. To see that $(a) \Rightarrow (b)$ holds, for $m \leq k$, consider the composite

$$h'_m : W_m^\ell \longrightarrow W_k^\ell \xrightarrow{\psi'} \bigvee_{I \subseteq [n], |I| \leq m} W_k^\ell \xrightarrow{\vee q_I} \bigvee_{I \subseteq [n], |I| \leq m} \hat{W}_I^\ell,$$

where the first map is the inclusion and ψ' is given by iterating the comultiplication $W_k^\ell \xrightarrow{\psi} W_k^\ell \vee W_k^\ell$ in some order. By Lemma 1.8, $\Sigma\psi'$ is homotopic to the iterated pinch map, so by naturality the composite of $\Sigma\psi'$ with the suspended inclusion $\Sigma W_m^\ell \rightarrow \Sigma W_k^\ell$ is an iterated pinch map. The map $\Sigma h'_m$ is then homotopic to the homotopy equivalence h_m in the proof of Proposition 1.6, meaning h'_m induces isomorphisms of homology groups, and since this is a map between simply-connected CW -complexes, h'_m is a homotopy equivalence.

Suppose (b) holds. From the commutativity of the right square in diagram (5), we may take as a right homotopy inverse of the suspended quotient map $\Sigma W_m^\ell \rightarrow \Sigma \hat{W}_m^\ell$ the composite

$$f : \Sigma \hat{W}_m^\ell \xrightarrow{\cong} \bigvee_{I \subseteq [n], |I|=m} \Sigma \hat{W}_I^\ell \longrightarrow \bigvee_{I \subseteq [n], |I| \leq m} \Sigma \hat{W}_I^\ell \xrightarrow{h_m^{-1}} \Sigma W_m^\ell,$$

where the second map is the inclusion. Since $h_m \simeq \Sigma h'_m$ and h'_m is a homotopy equivalence, we may take $h_m^{-1} = \Sigma(h'_m)^{-1}$. All the maps in this composite being suspensions, f desuspends to a map f' such that the composite $\hat{W}_m^\ell \xrightarrow{f'} W_m^\ell \rightarrow \hat{W}_m^\ell$ induces isomorphisms on homology groups. Since each of the summands in the splitting of $\hat{W}_m^\ell$ are summands in the splitting of ΣW_m^ℓ , this last composite must be homotopic to the identity. Thus $(b) \Rightarrow (c)$ holds.

Since each of the cofibrations $W_{m-1}^\ell \rightarrow W_m^\ell \rightarrow \hat{W}_m^\ell$ for $m \leq k$ trivializes when there is right homotopy inverse of $W_m^\ell \rightarrow \hat{W}_m^\ell$, $(c) \Rightarrow (b)$ holds. Thus, we have shown $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

The restriction $W_k^\ell \rightarrow W_I^\ell$ of the projection map $X^\ell \rightarrow X_I^\ell$ to W_k^ℓ has a right inverse $W_I^\ell \rightarrow W_k^\ell$ whenever $|I| \leq k$, given by restricting the inclusion $X_I^\ell \rightarrow X^\ell$ to W_I^ℓ . Thus $(a) \Rightarrow (d)$.

Suppose (d) holds. By $(a) \Rightarrow (c)$ with W_I^ℓ in place of W_k^ℓ , the quotient map $W_I^\ell \rightarrow \hat{W}_I^\ell$ has a right homotopy inverse. Thus $(d) \Rightarrow (e)$.

Suppose (e) holds. For any $|I| = m$ we have a commutative square

$$\begin{array}{ccc} W_I^\ell & \longrightarrow & \hat{W}_I^\ell \\ \downarrow & & \downarrow \\ W_m^\ell & \longrightarrow & \hat{W}_m^\ell \end{array} \xrightarrow{\cong} \bigvee_{J \subseteq [n], |J|=m} \hat{W}_J^\ell$$

with the vertical maps the inclusions, the right-hand one being the inclusion into the summand $\hat{W}_I^\ell$. Therefore the wedge sum of maps $\hat{W}_I^\ell \xrightarrow{s_I} W_I^\ell \rightarrow W_m^\ell$ over all $|I| = m$, where s_I is our given right homotopy inverse, is a right homotopy inverse of $W_m^\ell \rightarrow \hat{W}_m^\ell$. Thus $(e) \Rightarrow (c)$. $\square$

Proposition 1.9 has an interesting interpretation for polyhedral products $(\underline{X}, \underline{A})^K$.

Corollary 1.10. *If $(\underline{X}, \underline{A})^K$ is a co- H -space, then $(\underline{X}, \underline{A})^K \simeq \bigvee_{I \subseteq [n]} \widehat{(\underline{X}, \underline{A})}^{K_I}$.*

Note that by [3], when each X_i is contractible, then $(\underline{X}, \underline{A})^K \simeq \bigvee_{I \nsubseteq K} |K_I| * \hat{A}^I$, and when each A_i is contractible, $\widehat{(\underline{X}, \underline{A})}^{K_I} \simeq \bigwedge_{i \in I} X_i$ when K_I is a simplex, otherwise homotopy equivalent to a point.

2. CONFIGURATION SPACES

Let M be any path connected space, $N \subseteq M$ a subspace, and Y a basepointed space with basepoint $*$. Let $D_0(M, N; Y) = *$ and take the quotient space

$$D_k(M, N; Y) = \coprod_{i=0}^k M^{\times i} \times Y^{\times i} / \sim$$

where the equivalence relation $\sim$ is given by

$$(z_1, \dots, z_i; x_1, \dots, x_i) \sim (z_{\sigma(1)}, \dots, z_{\sigma(i)}; x_{\sigma(1)}, \dots, x_{\sigma(i)})$$

for any permutation σ in the symmetric group Σ_i , and

$$(z_1, \dots, z_i; x_1, \dots, x_i) \sim (z_1, \dots, z_{i-1}; x_1, \dots, x_{i-1})$$

whenever x_i is the basepoint $*$ or $z_i \in N$. Then

$$D(M, N; Y) = \bigcup_{k=0}^{\infty} D_k(M, N; Y).$$

Notice $D(M, N; Y)$ is just the infinite symmetric product $SP((M/N) \wedge Y)$. We may think of $D(M, N; Y)$ as the space of (possibly colliding) particles in M with labels in Y that are annihilated in N . We thus refer to points in $D(M, N; Y)$ as *configurations of particles* in M with labels in Y , and by this we usually mean, without loss of generality, that the particles in such a configuration are *non-degenerate*, or in other words, that they are outside N and their labels are not the basepoint. The empty configuration is vacuously non-degenerate. It will sometimes be convenient to think of a configuration $(z_1, \dots, z_i; x_1, \dots, x_i)$ as a set $\{(z_1, x_1), \dots, (z_i, x_i)\}$ where each of the pairs (z_i, x_i) are considered distinct regardless of the values of z_i or x_i .

Fix W to be a subspace of a product $X = X_1 \times \dots \times X_n$.

Definition 2.1. A multiset of points $\mathcal{S} = \{\{x_1, \dots, x_s\}\} \subset \coprod_i X_i$ is *represented* by a point in W if there exists a $(\bar{x}_1, \dots, \bar{x}_n) \in W$ and some injective function $f_{\mathcal{S}}: [s] \rightarrow [n]$ such that $x_i = \bar{x}_{f_{\mathcal{S}}(i)}$.

Equivalently, when $s > 0$, $\mathcal{S}$ is represented by a point in W if $(x_{\sigma(1)}, \dots, x_{\sigma(s)})$ is in W_I for some distinct permutation $\sigma \in \text{Aut}([s])$ and $I \subseteq [n]$ with $|I| = s$.

Remark: $\mathcal{S}$ is not represented by a point if it contains two or more elements in the same X_i for some i .

Definition 2.2. Let $X^\vee = X_1 \vee \dots \vee X_n$ be the wedge at basepoints $*$ $\in X_i$. Define $\mathcal{C}(M, N; W)$ to be the subspace of $D(M, N; X^\vee)$ given by the following rule: a configuration of non-degenerate particles $(z_1, \dots, z_k; x_1, \dots, x_k) \in D(M, N; X^\vee)$ is in $\mathcal{C}(M, N; W)$ if and only if

- for each increasing integer sequence $1 \leq i_1 < \dots < i_s \leq k$, if $z_{i_1} = \dots = z_{i_s}$, then the multiset of labels $\{\{x_{i_1}, \dots, x_{i_s}\}\}$ is represented by some point in W .

The condition $z_{i_1} = \dots = z_{i_s}$ is vacuously true when $s = 1$, thus, the label x of each particle is an element of $W_{\{i\}} \subseteq X_i$ for some i . We filter $\mathcal{C}(M, N; W)$ by the subspaces

$$\mathcal{C}_k(M, N; W) = \mathcal{C}(M, N; W) \cap D_k(M, N; X^\vee)$$

having no more than k non-degenerate particles.

These configuration spaces generalise McDuff's and Bødgeheimer's [28, 7], originally studied by Boardman, Vogt, May, Giffen, and Segal. Several other generalisations have appeared since then, some of these bearing certain similarities to these (see [11, 25, 29, 12, 13] for example).

Example 2.3. If $W = X_1 \times \dots \times X_n$, then the space $\mathcal{C}(M, N; W)$ is homeomorphic to the product of classical labelled configuration spaces $C(M, N; X_1) \times \dots \times C(M, N; X_n)$. If $W = X_1 \vee \dots \vee X_n$, then $\mathcal{C}(M, N; W) = C(M, N; X_1 \vee \dots \vee X_n)$.

When referring to *collided particles* in a labelled configuration of particles in M , we mean a subset of all particles in that configuration sharing the same point in M . We can think of these configuration spaces as being like classical configuration spaces with labels in $X^\vee$, but with certain collisions allowed governed by the way W sits inside its ambient space X . Here a single particle is considered to collide with itself, and at most n particles can collide.

2.1. Faithful Neighbourhoods. Recall that a CW -complex Z is said to be *regular* if its attaching maps are embeddings. A *mapping cylinder neighbourhood* of a subspace $A \subseteq Z$ is a neighbourhood of A in Z homeomorphic to a mapping cylinder $M_f = A \sqcup ([0, 1] \times B) / \sim$, where $(0, b) \sim f(b)$, for some map $f: B \rightarrow A$. An *open mapping cylinder neighbourhood* is a subspace of M_f of the form $A \sqcup ([0, 1] \times B) / \sim$. A *sub-mapping cylinder* of M_f is a subspace of M_f of the form $A \sqcup ([0, 1] \times B') / \sim$ for some subspace $B' \subseteq B$, where f' restricts f to B' . We call the subspaces $(0, 1] \times \{b\} \subseteq M_f$ for $b \in B$ the *fibers* of M_f .

It is easy to see via a cell-by-cell induction that any subcomplex of a (geometric) simplicial complex has a mapping cylinder neighbourhood. Moreover, each of its fibers lie entirely in the interior of a single simplex. Since the cells of a regular CW -complex can be subdivided to give it the structure of a simplicial complex, subcomplexes of regular CW -complexes also have mapping cylinder neighbourhoods, with each of their fibers lying entirely in a single cell.

The following is a coordinate-wise version of the neighbourhood deformation retraction property of a pair.

Definition 2.4. Any proper subcomplex W has a *faithful neighbourhood* if it has an open neighbourhood $U \subseteq X$ with associated basepoint preserving homotopies $h_{i,t}: X_i \rightarrow X_i$ for $i = 1, \dots, n$, such that: (1) $h_{i,0}$ is the identity and $h_{i,t}(\ast) = \ast$; (2) there is an open neighbourhood U_i of the basepoint $\ast \in X_i$ such that $h_{i,1}(U_i) = \ast$, and $h_{i,t}(U_i)$ is an open neighbourhood of $\ast$ for each $t \in [0, 1]$; and (3) the homotopy $h_t = h_{1,t} \times \dots \times h_{n,t}: X \rightarrow X$ satisfies $h_t(W) \subseteq W$ for each $t \in [0, 1]$, $h_1(U) = W$, and $h_t(U)$ is an open neighbourhood of W for each $t \in [0, 1]$.

Since any open subset of X is a union of products of open subsets of each X_i , the projection $X \rightarrow X_I = \prod_{i \in I} X_i$ maps open subsets to open subsets for any $I \subseteq [n]$.

Lemma 2.5. *The image of our faithful neighbourhood U of W under the projection $X \rightarrow X_I$ is a faithful neighbourhood of W_I in X_I , inheriting from U the associated homotopies $h_{i,t}$ where $i \in I$. $\square$*

Example 2.6. Given a simplicial complex K on vertex set $[n]$, the real moment-angle complex $W = (D^1, S^0)^K$ with basepoint 0 has as a faithful neighbourhood the polyhedral product $U = (D^1, [0, \frac{1}{4}] \cup (\frac{3}{4}, 1])^K$ with each $h_{i,t}: D^1 \rightarrow D^1$ defined explicitly by $h_{i,t}(s) = s(1-t)$ for $s \in [0, \frac{1}{4}]$, $h_{i,t}(s) = s + t(s - \frac{1}{2})$ for $s \in (\frac{1}{4}, \frac{3}{4})$, and $h_{i,t}(s) = s + t(1-s)$ for $s \in [\frac{3}{4}, 1]$. Note U and the $h_{i,t}$'s also form a faithful neighbourhood for each subcomplex W_k .

Example 2.7. More generally, let Y be a regular CW -complex and A a subcomplex. The polyhedral product $W = (Y, A)^K$ has as a faithful neighbourhood the polyhedral product $U = (Y, M_f)^K$, where M_f is an open mapping cylinder neighbourhood $A \cup_f ([0, \frac{1}{2}] \times B)$ of A for some map $f: B \rightarrow A$. Each homotopy $h_{i,t}: Y \rightarrow Y$ is given as follows. Define $g_{i,t}: Y \rightarrow Y$ by extending to Y the deformation retraction of M_f onto A to give our homotopy $g_{i,t}$. Then define $g'_{i,t}: Y \rightarrow Y$ by taking an open mapping cylinder neighbourhood $M'_\ast$ of the basepoint $\ast$ in A , extending it to an open mapping cylinder $U_i = M'_\ast$ of $\ast$ in Y , and extending the deformation retraction of $M'_\ast$ onto $\ast$ to give our homotopy $g'_{i,t}$. Note since $M'_\ast$ is a sub-mapping cylinder of $M_\ast$, the deformation retraction of $M_\ast$ onto $\ast$ restricts to a deformation retraction in A of $M'_\ast$ onto $\ast$, so we can take $g'_{i,t}$ so that it restricts to $g'_{i,t}: A \rightarrow A$ on A , which we do. Then let $h_{i,t} = g'_{i,t} \circ g_{i,t}$.

If A is a disjoint union $A' \sqcup \{\ast\}$, so that M_f is a disjoint union $M_{f'} \sqcup M_\ast$, then each $W_k \subseteq Z_K(Y, A)$ has the subspace

$$U_k = \{(x_1, \dots, x_n) \in Z_K(Y, M_f) \mid x_i \in M_\ast \text{ for at least } n-k \text{ coordinates } x_i\}$$

as a faithful neighbourhood, with the same associated homotopies $h_{i,t}$.

2.2. The Case of a Disk and its Boundary. From now on we assume that each X_i is a connected basepointed regular CW -complex, W is a connected subcomplex of their product X containing the basepoint, and W has a faithful neighbourhood in X .

Consider the inclusion

$$\iota: W^\ell \longrightarrow \mathcal{C}(D^\ell, \partial D^\ell; W)$$

given by sending $((z_1, x_1), \dots, (z_n, x_n))$ to the configuration $(z_1, \dots, z_n; x_1, \dots, x_n)$. Notice ι is the restriction of the likewise defined map $X^\ell \xrightarrow{\iota} \mathcal{C}(D^\ell, \partial D^\ell; X)$. This map in turn is just the product of inclusions $\Sigma^\ell X_i \longrightarrow \mathcal{C}(D^\ell, \partial D^\ell; X_i)$ into the classical configuration spaces, each of which is a homeomorphism of $\Sigma^\ell X_i$ onto its image [28, 7]. Therefore ι is homeomorphism of W^ℓ onto its image, and we think of W^ℓ as a subspace of $\mathcal{C}(D^\ell, \partial D^\ell; W)$ under this inclusion. Notice that a configuration of non-degenerate particles is in the image of ι if and only if the multiset of labels of all its non-degenerate particles is represented by a point in W .

Lemma 2.8. *There is a deformation retraction of $\mathcal{C}(D^\ell, \partial D^\ell; W)$ onto the subspace W^ℓ given by the inclusion ι .*

Proof. We use a two-step radial expansion argument, generalizing the deformation retraction for the classical labelled configuration space $\mathcal{C}(D^\ell, \partial D^\ell; Y)$ onto $\Sigma^\ell Y$. We think of D^ℓ as the closed unit-disk centered at the origin in $\mathbb{R}^\ell$, and the radial expansion $R_t: D^\ell \longrightarrow D^\ell$ here is given by $R_t(x) = tx$ if $|tx| < 1$ and $R_t(x) = \frac{tx}{|tx|}$ if $|tx| \geq 1$.

Let $\mathcal{C}$ denote $\mathcal{C}(D^\ell, \partial D^\ell; W)$, and $\mathcal{C}'$ be the subspace of configurations in $\mathcal{C}$ whose non-degenerate particles have labels in distinct X_i 's (thus, there can be no more than n of such particles). To define a deformation retraction of $\mathcal{C}$ onto $\mathcal{C}'$ we define a homotopy $H_t: \mathcal{C} \longrightarrow \mathcal{C}$, for $t \in [0, \infty)$, by applying the radial expansion to non-degenerate particles in a given configuration $\omega \in \mathcal{C}$, while at the same time we homotope labels 'near' the basepoint towards the basepoint. We do this by picking a mapping cylinder neighbourhood M_i of the basepoint in each X_i (these exist since X_i is a regular CW -complex), keeping a label fixed if it is the basepoint or outside the interior of some M_i , or else homotoping it towards the basepoint along the corresponding fibre $(0, 1] \times \{b\} \subseteq M_i$ of the mapping cylinder, this being done at a speed that tends to zero the closer the label is to the boundary of M_i . We may assume each of the fibers $(0, 1] \times \{b\}$ of M_i lie in a single cell in X_i . Then since W as a subcomplex of X is a union of products of cells in the X_i 's, under this homotopy of labels, collided particles still have their set of labels represented by a point in W . This defines H .

Notice only particles that are collided at the centre of D^ℓ are unaffected by the radial expansion, but then their multiset of labels is represented by a point in W , so in particular their labels must already be in distinct X_i 's. On the other hand, particles away from the centre may eventually enter the boundary ∂D^ℓ under the radial expansion, or have their labels homotoped to the basepoint, in either case becoming degenerate, until we are left with at most n non-degenerate particles with labels in distinct X_i 's. Thus, there is a smallest sufficiently large time t_ω depending on ω such that $H_{t_\omega}(\omega)$ is a configuration in $\mathcal{C}'$, at which point we stop the homotopy H . To see that the map $\mathcal{C} \longrightarrow [0, \infty)$, $\omega \mapsto t_\omega$, is continuous, notice our homotopy of labels near the basepoint towards it ensures t_ω varies continuously as we vary the labels by making them approach a basepoint. Then given an open neighbourhood $V \subseteq [0, \infty)$ of t_ω , for sufficiently small open neighbourhoods $V' \subseteq \mathcal{C}$ of ω , the only possible configurations $\mathfrak{x}$ that could be in V' , and for which $t_{\mathfrak{x}}$ could not be in V , would have to have at least two particles collided at the origin with labels in the same X_i . But such configurations cannot exist in $\mathcal{C}$, since they would not be represented by a point in W . This defines our first deformation retraction.

Finally, to define a deformation retraction of $\mathcal{C}'$ onto W^ℓ , notice the set of labels of non-degenerate particles in a configuration in $\omega \in \mathcal{C}'$ corresponds to a distinct point x in X whose coordinates are each either a basepoint or one of the labels. So we can think of the set of labels of ω as a point $x \in X$. We define a homotopy $G_t: \mathcal{C}' \longrightarrow \mathcal{C}'$, $t \in [0, \infty)$, by again apply the radial expansion R_t to ω starting at $t = 0$, meanwhile, homotoping the set of labels towards W when it is 'near' W . We

must be careful how we perform this homotopy of labels, since we must ensure labels of particles that are collided at a point remain represented by a point in W_n , and that the homotopy does not become discontinuous as a particle enters the boundary and becomes degenerate (in which case the coordinate in x corresponding to the label of this particle becomes the basepoint). We therefore use a faithful neighbourhood U of W with associated homotopy $h_t: X \rightarrow X$, and homotop labels near W towards W by mapping x to $h_{f(t)}(x)$ at each time t , where $f(t) = \frac{t}{1+t}$. This takes care of both of these issues since h_t maps W to W and is defined coordinate-wise as a product of basepoint preserving maps $h_{i,t}: X_i \rightarrow X_i$. In particular, for each $I \subseteq [n]$, the product $h_{I,t}: X_I \rightarrow X_I$ of the $h_{i,t}$'s over $i \in I$ maps W_I to W_I . So if a subset of $|I|$ non-degenerate particles in ω , each with a label in X_i for each $i \in I$, are represented by a point in W , that is, their set of labels are the coordinates of a distinct point in W_I , they remain so under the homotopy $h_{f(t)}$. This defines G_t .

As those particles unaffected by radial expansion are collided at the origin and already represented by a point in W , we see that there is a smallest sufficiently large time t'_ω depending on ω when enough particles are annihilated at the boundary, or have the set of labels homotoped to a point in W , so that the labels of the non-degenerate particles are represented by a point in W . That is, $G_{t'_\omega}(\omega)$ becomes a point in the image of ι , at which point we stop the homotopy G . As before, $\mathcal{C}' \rightarrow [0, \infty)$ mapping $\omega \mapsto t'_\omega$ is continuous. Here, since ω is in $\mathcal{C}'$, the labels of any subset of particles in ω are the coordinates of a point $p \in X_I$ for some $I \subseteq [n]$, and by Lemma 2.5, the homotopy h_t , restricting to $h_{I,t}$ on the labels p makes p approach W_I near W_I . Moreover, since each $h_{i,t}$ by definition homotopes points close to the basepoint in X_i towards it, any label near a basepoint is homotoped towards it by $h_{f(t)}$. So t'_ω varies continuously as we homotope a subset of labels of ω towards a point in W_I , or each towards a basepoint. Then given an open neighbourhood $V \subseteq [0, \infty)$, for sufficiently small open neighbourhoods $V' \subseteq \mathcal{C}'$ of ω , the only possible configurations $\mathfrak{x}$ that could be in V' , and for which $t'_\mathfrak{x}$ could not be in V , would have to have particles collided at the origin that are not represented by a point in W , but these do not exist in $\mathcal{C}'$. This defines our second deformation retraction. $\square$

2.3. The Scanning Map. Let $\mathcal{C}(W)$ denote $\mathcal{C}(\mathbb{R}, \emptyset; W)$, and $\mathcal{C}_i(W)$ the subspace $\mathcal{C}_i(\mathbb{R}, \emptyset; W)$. We define a *scanning map*

$$\gamma: \mathcal{C}(W) \xrightarrow{\cong} \mathcal{C}(D^1; W) \xrightarrow{\gamma} \text{map}(D^1, \partial D^1; W^1) \cong \Omega W^1$$

analogously to the one for classical labelled configuration spaces [30] as follows. The first homotopy equivalence in this composite is induced by the map $\mathbb{R} \rightarrow D^1$ sending $t \mapsto \frac{t}{1+|t|}$ (with a choice of homotopy inverse induced by the inclusion $D^1 \rightarrow \mathbb{R}$). Let $D_{\varepsilon,z}$ denote the closed interval $[z-\varepsilon, z+\varepsilon] \subseteq \mathbb{R}$. For any configuration $\omega \in \mathcal{C}(W)$, $\gamma(\omega)$ is the map $D^1 \rightarrow W^1$ defined for any $z \in D^1$ by taking $\gamma(\omega)(z)$ to be the image of ω under the composite

$$(6) \quad \mathcal{C}(W) \rightarrow \mathcal{C}(D_{\varepsilon,z}, \partial D_{\varepsilon,z}; W) \xrightarrow{\cong} \mathcal{C}(D^1, \partial D^1; W) \xrightarrow{r} W^1$$

where the first map forgets those particles from ω that are in $\mathbb{R} - D_{\varepsilon,z}$, the second homeomorphism induced by the homeomorphism $D^1 \cong D_{\varepsilon,z}$, and r is the retraction from Lemma 2.8.

With regards to labels, $\mathcal{C}()$ is a functor from the category of subspaces of length n products of basepointed CW -complexes, and coordinate-wise maps between them. More precisely, given $Y = Y_1 \times \cdots \times Y_n$ a product of basepointed CW -complexes, basepoint preserving maps $f_i: X_i \rightarrow Y_i$, and a subspace $V \subseteq Y$, suppose the coordinate-wise map $\bar{f} = (f_1 \times \cdots \times f_n): X \rightarrow Y$ restricts to a map $f: W \rightarrow V$. This induces a map

$$\mathcal{C}(f): \mathcal{C}(W) \rightarrow \mathcal{C}(V)$$

sending $(z_1, \dots, z_n; x_1, \dots, x_n) \mapsto (z_1, \dots, z_n; f_1(x_1), \dots, f_n(x_n))$, and a map

$$f^1: W^1 \rightarrow V^1$$

sending $((z_1, x_1), \dots, (z_n, x_n)) \mapsto ((z_1, f_1(x_1)), \dots, (z_n, f_n(x_n)))$. The scanning map γ is then a natural transformation in the homotopy category:

Lemma 2.9. *There is a homotopy commutative square*

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{\mathcal{C}(f)} & \mathcal{C}(V) \\ \downarrow \gamma & & \downarrow \gamma \\ \Omega W^1 & \xrightarrow{\Omega f^1} & \Omega V^1. \end{array}$$

Proof. Since the following square commutes

$$\begin{array}{ccc} W^1 & \xrightarrow{f^1} & V^1 \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{C}(D^1, \partial D^1; W) & \xrightarrow{\mathcal{C}(f)} & \mathcal{C}(D^1, \partial D^1; V), \end{array}$$

we may replace the homotopy equivalences ι with their homotopy inverse r to obtain a homotopy commutative square. Thus we have a diagram

$$\begin{array}{ccccccc} \mathcal{C}(W) & \longrightarrow & \mathcal{C}(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W) & \xrightarrow{\cong} & \mathcal{C}(D^1, \partial D^1; W) & \xrightarrow{r} & W^1 \\ \downarrow \mathcal{C}(f) & & & & \downarrow \mathcal{C}(f) & & \downarrow f^1 \\ \mathcal{C}(V) & \longrightarrow & \mathcal{C}(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; V) & \xrightarrow{\cong} & \mathcal{C}(D^1, \partial D^1; V) & \xrightarrow{r} & V^1, \end{array}$$

the left square commuting strictly, and the homotopy $f^1 \circ r \simeq r \circ \mathcal{C}(f)$ in the right square being independent of $z \in \mathbb{R}$. This then gives a homotopy $\gamma(\mathcal{C}(f)(\omega)) \simeq \Omega f^1 \circ \gamma(\omega)$ of maps $D^1 \rightarrow W^1$ independently of z and ω , so in turn, it defines a homotopy $\gamma \circ \mathcal{C}(f) \simeq \Omega f^1 \circ \gamma$. $\square$

2.4. Suspension Splittings. Our goal is to obtain a (single) suspension splitting of $\mathcal{C}(W)$. Take the quotient space

$$\mathcal{D}_m(W) = \frac{\mathcal{C}_m(W)}{\mathcal{C}_{m-1}(W)}$$

Lemma 2.10. *The pair $(\mathcal{C}_m(W), \mathcal{C}_{m-1}(W))$ is an NDR-pair. Thus, the sequence*

$$\mathcal{C}_{m-1}(W) \rightarrow \mathcal{C}_m(W) \rightarrow \mathcal{D}_m(W)$$

is a cofibration sequence.

Proof. Since X_i is a regular CW -complex, as in the proof of Lemma 2.8, we can take a mapping cylinder neighbourhood M_i of the basepoint $\ast \in X_i$ such that each of the fibers $(0, 1] \times \{b\} \subseteq M_i$ lie entirely in a single cell in X_i . Let $f_i: X_i \rightarrow [0, 1]$ be the map given by $f_i(\ast) = 0$, $f_i(x) = 1$ if $x \notin M_i$, and $f_i(x) = s$ if x is at the point (s, b) along a fiber $(0, 1] \times \{b\}$. Notice that f_i is continuous as the projection to the parameter s . Define $f: X_1 \vee \dots \vee X_n \rightarrow [0, 1]$ by restricting to f_i on X_i , and let $u: \mathcal{C}_m(W) \rightarrow [0, 1]$ be given for any $\omega = \{(t_1, x_1), \dots, (t_m, x_m)\}$ by $u(\omega) = 0$ if $\omega \in \mathcal{C}_{m-1}(W)$, otherwise $u(\omega) = \prod_{i=1}^m f(x_i)$. Notice $u^{-1}(0) = \mathcal{C}_{m-1}(W)$. Define a homotopy $H: \mathcal{C}_m(W) \times \mathbb{R} \rightarrow \mathcal{C}_m(W)$ by homotoping the labels of a configuration $\omega \in \mathcal{C}_m(W)$ simultaneously as follows. For each particle $(t, x) \in \omega$, we either keep the label x fixed if x is a basepoint or x is not in the interior of any mapping cylinder M_i , or else we homotop x towards the basepoint along the corresponding fibre of $(0, 1] \times \{b\}$ of the mapping cylinder. This is done at a speed that tends to zero the closer x is to the boundary of corresponding mapping cylinder. Since each of the fibers of M_i lies in a single cell of X_i , and since W as a subcomplex of X is a union of products of cells in the X_i 's, then under this homotopy of labels collided particles still have their set of labels represented by a point in W as necessary. We stop homotoping the labels of ω as soon as there are less than m particles (among our homotoped particles in ω) that have non-basepoint labels. Thus H_t restricts to the identity on $\mathcal{C}_{m-1}(W)$. Notice at least one of the labels is in the interior of a mapping cylinder M_i when $u(\omega) < 1$, and $u(\omega)$ approaches 1 if the labels of ω all approach points outside these mapping cylinders. Then for any $(s, b) \in (0, 1] \times \{b\}$, there is a sufficiently large time t such

that $H_t(u^{-1}([0, s)))$ is contained in $\mathcal{C}_{m-1}(W)$. This therefore defines a neighbourhood deformation retraction for $\mathcal{C}_{m-1}(W)$ in $\mathcal{C}_m(W)$. $\square$

Let

$$V_m(W) = \bigvee_{i=1}^m \mathcal{D}_i(W).$$

We now define a map

$$\zeta : \mathcal{C}(W) \longrightarrow C(\mathbb{R}; V_\infty(W)),$$

where the right-hand space is the classical configuration space of non-overlapping particles in $\mathbb{R}$ with labels in $V_\infty(W)$. Since $X_1 \vee \dots \vee X_n$ is a CW -complex, we may pick a map

$$f : X_1 \vee \dots \vee X_n \longrightarrow [0, 1]$$

such that $f(*) = 0$ and $f(x) > 0$ when $x \neq *$. Let $\zeta_0 : \mathcal{C}_0(W) \longrightarrow C(\mathbb{R}; V_\infty(W))$ be given by mapping the empty configuration to the empty configuration, and define

$$\zeta_m : \mathcal{C}_m(W) - \mathcal{C}_{m-1}(W) \longrightarrow C(\mathbb{R}; V_\infty(W))$$

for $m > 0$ as follows. Notice that any configuration $\omega = \{(t_1, x_1), \dots, (t_m, x_m)\} \in \mathcal{C}_m(W) - \mathcal{C}_{m-1}(W)$ has only non-degenerate particles, that is, the label x_i of each particle in ω is not the basepoint $*$. Since any two non-degenerate particles in a configuration in $\mathcal{C}(W)$ cannot collide when their labels are in the same summand X_l (so $t_i \neq t_j$ when $x_i, x_j \in X_l - \{*\}$ by definition of $\mathcal{C}(\mathbb{R}, \emptyset; W)$), there is a strict total order on ω given for any $y = (t_i, x_i), y' = (t_j, x_j) \in \omega$ by the relation

- $y < y'$ if and only if either $x_i \in X_{i'} - \{*\}$ and $x_j \in X_{j'} - \{*\}$ such that $i' < j'$, or else both $x_i, x_j \in X_l - \{*\}$ for some l and $t_i < t_j$.

Given $y \in \omega$, let $\sigma(y)$ be the integer such that y is the $\sigma(y)^{th}$ smallest element in ω with respect to this ordering. Then take the strict total order on the set 2^ω of subsets of ω , given for any $S, S' \subseteq \omega$ by the relation

- $S < S'$ if and only if $\sum_{y \in S} 2^{\sigma(y)} < \sum_{y' \in S'} 2^{\sigma(y')}$.

This total order does not change as we move these particles and vary their labels in such a way that ω moves along any path in $\mathcal{C}_m(W) - \mathcal{C}_{m-1}(W)$, and if we take a subconfiguration $\omega' \subset \omega$ in $\mathcal{C}_{m'}(W) - \mathcal{C}_{m'-1}(W)$, $m' < m$, then the ordering on $2^{\omega'} \subset 2^\omega$ is the one inherited from 2^ω . Write $S \leq S'$ if either $S < S'$ or $S = S'$, and let

$$\eta_{\omega, S'} = \sum_{\substack{S \subseteq \omega \\ S \leq S'}} \left(\prod_{(t_i, x_i) \in S} f(x_i) \right).$$

Since each $x_i \neq *$, we have $f(x_i) > 0$, so $\eta_{\omega, S} < \eta_{\omega, S'}$ whenever $S < S'$. Then we may define

$$\zeta_m(\omega) = \{(\eta_{\omega, S}, \chi_S) \mid S \subseteq \omega\} \in C(\mathbb{R}; V_\infty(W))$$

where the label $\chi_S \in V_\infty(W)$ is the image of the configuration $S \in \mathcal{C}_{|S|}(W)$ under the composite of the quotient map and inclusion

$$(7) \quad \mathcal{C}_{|S|}(W) \longrightarrow \mathcal{D}_{|S|}(W) \xrightarrow{\text{include}} V_\infty(W)$$

($S \subseteq \omega$ is indeed a configuration in $\mathcal{C}_{|S|}(W)$, since every subset of a represented multiset of labels is itself represented by a point in W). Notice that when ω approaches some configuration $\omega' \in \mathcal{C}_{m'}(W) - \mathcal{C}_{m'-1}(W)$ for some $m' < m$, that is, when some subset L of at least $m - m'$ labels x_i from ω approaches the basepoint, then $\prod_{y=(x_i, t_i) \in S} f(x_i)$ approaches 0 and χ_S approaches $*$ in $V_\infty(W)$ for all $S \subseteq \omega$ that contain a particle with a label in L . Therefore, we see that $\zeta_m(\omega)$ approaches $\zeta_{m'}(\omega')$. Thus, we can define ζ by restricting to ζ_m on $\mathcal{C}_m(W) - \mathcal{C}_{m-1}(W)$, for any $m > 0$, and ζ_0 on the empty configuration.

Notice ζ_m maps to the subspace $C(\mathbb{R}; V_m(W)) \subseteq C(\mathbb{R}; V_\infty(W))$. Restricting ζ to a subspace $\mathcal{C}_i(W)$, there is a commutative diagram

$$(8) \quad \begin{array}{ccccc} \mathcal{C}_{i-1}(W) & \longrightarrow & \mathcal{C}_i(W) & \longrightarrow & \mathcal{D}_i(W) \\ \downarrow \zeta & & \downarrow \zeta & & \downarrow \\ C(\mathbb{R}; V_{i-1}(W)) & \longrightarrow & C(\mathbb{R}; V_i(W)) & \longrightarrow & C(\mathbb{R}; \mathcal{D}_i(W)) \end{array}$$

with the left-hand and right-hand horizontal maps the inclusions and quotient maps respectively, and the bottom maps induced by the inclusion $V_{i-1}(W) \rightarrow V_i(W)$ and the quotient map $V_i(W) \rightarrow \mathcal{D}_i(W)$. Since the composite of these two maps is the constant map, the composite of the maps in the bottom sequence is the constant map, which gives the right-hand vertical extension. Notice this extension sends a (quotiented) configuration ω to a configuration with a single particle, whose label is ω . Recall from the theory of classical labelled configuration spaces [30] there is a *scanning map*

$$\lambda: C(\mathbb{R}; V_\infty(W)) \rightarrow \Omega \Sigma V_\infty(W),$$

which in fact is a homotopy equivalence. This is also just a special case of our scanning map constructed above. Here one takes $\Sigma V_\infty(W)$ as the coordinate suspension of $V_\infty(W)$ on a single coordinate, meaning $\mathcal{C}(\mathbb{R}; V_\infty(W)) = C(\mathbb{R}; V_\infty(W))$. Composing the vertical maps in Diagram (8) with the corresponding scanning maps λ , and taking the adjoints of these, we obtain a commutative diagram of cofibration sequences

$$(9) \quad \begin{array}{ccccc} \Sigma \mathcal{C}_{i-1}(W) & \longrightarrow & \Sigma \mathcal{C}_i(W) & \longrightarrow & \Sigma \mathcal{D}_i(W) \\ \downarrow \zeta' & & \downarrow \zeta' & & \downarrow \simeq \\ \Sigma V_{i-1}(W) & \longrightarrow & \Sigma V_i(W) & \longrightarrow & \Sigma \mathcal{D}_i(W), \end{array}$$

the left and middle vertical maps being the restrictions of the adjoint $\zeta': \Sigma \mathcal{C}(W) \rightarrow \Sigma V_\infty(W)$ of the composite $\lambda \circ \zeta$, and the bottom cofibration sequence the composite of the inclusion and quotient onto the summand $\Sigma(\mathcal{D}_i(W))$. The right-hand vertical extension can be homotoped to the identity using a homotopy to the identity of the composite $\Sigma V_\infty(W) \rightarrow \mathcal{C}(D^1, \partial D^1; V_\infty(W)) \rightarrow \Sigma V_\infty(W)$ of the inclusion from Lemma 2.8 and its left homotopy inverse used to define the scanning map. Then the middle vertical ζ' is a homotopy equivalence whenever the left-hand ζ' is, so by induction, we see that $\Sigma \mathcal{C}(W) \xrightarrow{\zeta'} \Sigma V_\infty(W)$ is a homotopy equivalence. Thus

Proposition 2.11. *There is a splitting*

$$\Sigma \mathcal{C}(W) \simeq \bigvee_{i \geq 1} \Sigma \mathcal{D}_i(W).$$

□

Remark: If each $X_i = Y$ is a regular CW -complex, and W is a polyhedral product of the form $(Y, *)^K$, then by [11], $\mathcal{C}(W)$ is homotopy equivalent to $\Omega(\Sigma Y, *)^K$. This improves on the stable splitting of $\Omega(\Sigma Y, *)^K$ in [11].

2.5. A Refinement. Suppose W has the simplicial property. Recall our filtration $* = W_0 \subseteq W_1 \subseteq \dots \subseteq W_n = W$, where $W_k = \bigcup_{I \subseteq [n], |I|=k} W_I$. Again, $\mathcal{C}_m(W_k)$ denotes $\mathcal{C}_m(\mathbb{R}, \emptyset; W_k)$, these configuration spaces being constructed by thinking of W_k as a subspace of X . We still refer to elements in $\mathcal{D}_m(W_k) = \mathcal{C}_m(W_k)/\mathcal{C}_{m-1}(W_k)$ as configurations of labelled particles, in the sense that they are under the image of the quotient map $\mathcal{C}_m(W_k) \rightarrow \mathcal{D}_m(W_k)$. They must have exactly m particles, and are identified with the basepoint when one of them is a degenerate particle, that is, when its label is a basepoint. The empty configuration is the basepoint. Since $\mathcal{C}_m(W_k)$ is a subspace of $SP_m(Y)$, $Y = (\mathbb{R} \times X^\vee)/(\mathbb{R} \times \{*\})$, then $\mathcal{D}_m(W_k)$ is a subspace of $\dot{S}P_m(Y) = SP_m(Y)/SP_{m-1}(Y)$.

We call a sequence $\mathcal{S} = (s_1, \dots, s_n)$ an (m, n) -*partition* of a non-negative integer m if $s_1 + \dots + s_n = m$ and each $s_i \geq 0$. Given such a sequence, let $\mathcal{D}_{\mathcal{S}}(W_k)$ be the subspace of $\mathcal{D}_m(W_k)$ consisting of

configurations ω such that either exactly s_i non-degenerate particles in ω have labels in X_i for each $1 \leq i \leq n$, or else ω is the basepoint. Notice $\mathcal{D}_m(W_k)$ is the union of $\mathcal{D}_{\mathcal{S}}(W_k)$ over all (m, n) -partitions. To homotope the label of a particle in a configuration in $\mathcal{D}_m(W_k)$ from one summand X_j to another summand X_i we must make at least one label go through the basepoint, in which case the entire configuration must go through the basepoint. Therefore $\mathcal{D}_{\mathcal{S}}(W_k) \cap \mathcal{D}_{\mathcal{S}'}(W_k) = \{*\}$ for any (m, n) -partitions $\mathcal{S}$ and $\mathcal{S}'$, and

$$(10) \quad \mathcal{D}_m(W_k) = \bigvee_{(m,n)\text{-partitions } \mathcal{S}} \mathcal{D}_{\mathcal{S}}(W_k)$$

wedged at the basepoint, and the inclusion

$$\iota_{m,k}: \mathcal{D}_m(W_{k-1}) \longrightarrow \mathcal{D}_m(W_k)$$

induced by $W_{k-1} \longrightarrow W_k$, and extending $\bar{\iota}_{m,k}: \mathcal{C}_m(W_{k-1}) \longrightarrow \mathcal{C}_m(W_k)$, restricts to an inclusion

$$\iota_{\mathcal{S},k}: \mathcal{D}_{\mathcal{S}}(W_{k-1}) \longrightarrow \mathcal{D}_{\mathcal{S}}(W_k)$$

for each (m, n) -partition $\mathcal{S}$. Given a basepoint preserving map $X \xrightarrow{f} Y$ of non-degenerately basepointed spaces, let $M(f) = M_f/([0, 1] \times \{*\}) = (Y \cup_f [0, 1] \times X)/([0, 1] \times \{*\})$ be the reduced mapping cylinder of f , and $\mathbb{C}(f) = M(f)/(\{1\} \times X)$ the reduced mapping cone of f (or homotopy cofiber of f). We write points in $\mathbb{C}(f)$ as pairs (t, y) , where $t \in [0, 1]$, $y \in Y$ if $t = 0$, $y \in X$ if $t > 0$, and (t, y) is the basepoint $*$ when $t = 1$ or y is the basepoint. Since $\iota_{m,k}$ is as a wedge sum of $\iota_{\mathcal{S},k}$ over (m, n) -partitions $\mathcal{S}$,

$$(11) \quad \mathbb{C}(\iota_{m,k}) \cong \bigvee_{(m,n)\text{-partitions } \mathcal{S}} \mathbb{C}(\iota_{\mathcal{S},k}).$$

Notice that the following property holds when W has the simplicial property.

- (A) For each $m < k$ a multiset $\{\{x_1, \dots, x_m\}\}$ is represented by a point in W_k if and only if it is represented by a point in W_m .

Lemma 2.12. *Assume W has the simplicial property. If at most $k-1$ elements in $\mathcal{S}$ are nonzero, then $\mathcal{D}_{\mathcal{S}}(W_{k-1}) = \mathcal{D}_{\mathcal{S}}(W_k)$ as subspaces of $\hat{S}P_m(Y)$, and $\iota_{\mathcal{S},k}$ is the identity. In particular, $\mathcal{D}_m(W_{k-1}) = \mathcal{D}_m(W_k)$, $\iota_{m,k}$ is the identity, when $m \leq k-1$.*

Proof. Any non-degenerate configuration $\omega \in \mathcal{D}_{\mathcal{S}}(W_k)$ has m particles when $\mathcal{S}$ is an (m, n) -partition, but since collisions are not allowed between particles with labels in the same X_i , no more than $k-1$ can collide at a point when at most $k-1$ elements in $\mathcal{S}$ are nonzero. Since the labels of any $j \leq k-1$ particles in ω that are collided at a point are represented by a point in W_k , by Property (A) they are represented by a point in $W_j \subseteq W_{k-1}$. Therefore $\omega \in \mathcal{D}_{\mathcal{S}}(W_{k-1})$. $\square$

2.6. Comparing Filtrations. We again assume each X_i is a connected basepointed regular CW -complex, W is a connected subcomplex of their product X containing the basepoint and having a faithful neighbourhood. Moreover, we assume W has the simplicial property, and that its subcomplex W_{n-1} also has a faithful neighbourhood. Note W_k^1 is simply connected by Lemma 1.7.

Take the (n, n) -partition $\mathcal{A} = (1, \dots, 1)$. Then $\mathcal{D}_{\mathcal{A}}(W_k)$ is the subspace of $\mathcal{D}_n(W_k)$ containing those configurations of n particles whose labels are each in a distinct X_i . Our goal will be to prove the following.

Proposition 2.13. *There exists a homotopy commutative diagram*

$$\begin{array}{ccccccc}
 \mathbb{C}(\iota_{\mathcal{A},n}) & \xrightarrow{\partial} & \Sigma \mathcal{D}_{\mathcal{A}}(W_{n-1}) & \xrightarrow{\Sigma \iota_{\mathcal{A},n}} & \Sigma \mathcal{D}_{\mathcal{A}}(W_n) & \longrightarrow & \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \\
 \uparrow \phi \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\
 & & \Sigma \mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma \bar{\iota}_{n,n}} & \Sigma \mathcal{C}_n(W_n) & \longrightarrow & \Sigma \mathbb{C}(\bar{\iota}_{n,n}) \\
 & & \downarrow ev & & \downarrow ev & & \downarrow \simeq \\
 \Sigma^{n-1} \hat{W}_n & \longrightarrow & W_{n-1}^1 & \longrightarrow & W_n^1 & \longrightarrow & \hat{W}_n^1
 \end{array}$$

where consecutive maps in each row form homotopy cofibration sequences, the right-hand vertical homotopy equivalences are extensions induced by the homotopy commutativity of the middle squares. The top second and third vertical maps are homotopy inverses of the quotient maps coming from the splitting in Proposition 2.11, and the bottom maps ev are the adjoints $\Sigma \mathcal{C}_n(W_i) \xrightarrow{\Sigma \gamma} \Sigma \Omega W_i^1 \xrightarrow{eval} W_i^1$ of the corresponding scanning map γ . The map ∂ is the connecting map, given by collapsing the subspace $\mathcal{D}_{\mathcal{A}}(W_n)$ of $\mathbb{C}(\iota_{\mathcal{A},n})$.

To some extent, this reduces the study of the filtration of $W^1 = W_n^1$ by the spaces W_k^1 to the corresponding filtration for $\mathcal{D}_{\mathcal{A}}(W_n)$.

Corollary 2.14. *Considering the following:*

- (a) $W_n^1 \longrightarrow \hat{W}_n^1$ has a right homotopy inverse;
- (b) $\Sigma \mathcal{D}_{\mathcal{A}}(W_n) \longrightarrow \Sigma \mathbb{C}(\iota_{\mathcal{A},n})$ has a right homotopy inverse;
- (c) $\mathbb{C}(\iota_{\mathcal{A},n}) \xrightarrow{\partial} \Sigma \mathcal{D}_{\mathcal{A}}(W_{n-1})$ is nullhomotopic.

Then $(c) \Leftrightarrow (b) \Rightarrow (a)$.

Proof. The implication $(b) \Rightarrow (a)$ holds by Proposition 2.13, while $(c) \Rightarrow (b)$ is a standard property of homotopy cofibration sequences. Suppose (b) holds. Then the homotopy cofibration sequence

$$(12) \quad \Sigma \mathcal{D}_{\mathcal{A}}(W_{n-1}) \xrightarrow{\Sigma \iota_{\mathcal{A},n}} \Sigma \mathcal{D}_{\mathcal{A}}(W_n) \longrightarrow \Sigma \mathbb{C}(\iota_{\mathcal{A},n})$$

trivializes. In particular $\Sigma \iota_{\mathcal{A},n}$ has a left homotopy inverse $\hat{\iota}$. Since $\mathbb{C}(\iota_{\mathcal{A},n}) \xrightarrow{\partial} \Sigma \mathcal{D}_{\mathcal{A}}(W_{n-1}) \xrightarrow{\Sigma \iota_{\mathcal{A},n}} \Sigma \mathcal{D}_{\mathcal{A}}(W_n)$ is a homotopy cofibration sequence, $\Sigma \iota_{\mathcal{A},n} \circ \partial$ is nullhomotopic, so $(\hat{\iota} \circ \Sigma \iota_{\mathcal{A},n}) \circ \partial \simeq \partial$ is nullhomotopic. Thus $(b) \Rightarrow (c)$. $\square$

Corollary 2.15. W_n^1 is a co- H -space if one of the conditions in Corollary 2.14 hold for W_I in place of W for each $I \subseteq [n]$.

Proof. Combine Corollary 2.14 with Proposition 1.9. $\square$

2.7. Proof of Proposition 2.13. Take a configuration $\omega = (t_1, \dots, t_n; x_1, \dots, x_n) \in \mathcal{D}_{\mathcal{A}}(W_n)$. Configurations in $\mathcal{D}_{\mathcal{A}}(W_n)$ have particles whose labels are each in a distinct X_i , so we can and will assume without loss of generality that $x_i \in X_i$. Given $x = (x_1, \dots, x_n) \in \hat{W}_n$, then $\{x_1, \dots, x_n\}$ is represented by $(x_1, \dots, x_n) \in W_n$, so there is an inclusion $\hat{W}_n \longrightarrow \mathcal{D}_{\mathcal{A}}(W_n)$ given by mapping $x \mapsto (0, \dots, 0; x_1, \dots, x_n)$. We can extend this to an inclusion

$$\phi: \Sigma^{n-1} \hat{W}_n \longrightarrow \mathbb{C}(\iota_{\mathcal{A},n})$$

by mapping the basepoint to the basepoint, otherwise

$$((t_1, \dots, t_{n-1}), (x_1, \dots, x_n)) \mapsto (\max\{|t_1|, \dots, |t_{n-1}|, 0\}, (t_1, \dots, t_{n-1}, 0; x_1, \dots, x_n)),$$

where each parameter t_i is in the unit 1-disk $D^1 = [-1, 1]$. Notice ϕ is a homeomorphism of $\Sigma^{n-1} \hat{W}_n$ onto its image, so we regard $\Sigma^{n-1} \hat{W}_n$ as a subspace of $\mathbb{C}(\iota_{\mathcal{A},n})$ under this inclusion.

Remark: Since points in W_{n-1} have at least one coordinate a basepoint, no more than $n - 1$ non-degenerate particles in a configuration in $D_{\mathcal{A}}(W_{n-1})$ can collide at a point. Therefore any non-degenerate point $(t, (t_1, \dots, t_n; x_1, \dots, x_n)) \in \mathbb{C}(\iota_{\mathcal{A},n})$ must have $t = 0$ when $t_1 = \dots = t_n$, i.e., we must be at the base of the mapping cone $D_{\mathcal{A}}(W_n) \subseteq \mathbb{C}(\iota_{\mathcal{A},n})$.

Lemma 2.16. *There is a deformation retraction of $\mathbb{C}(\iota_{\mathcal{A},n})$ onto the subspace $\Sigma^{n-1}\hat{W}_n$ given by the inclusion ϕ .*

Proof. Let A be the subspace of $\mathbb{C}(\iota_{\mathcal{A},n})$ consisting of the points $(t, (t_1, \dots, t_n; x_1, \dots, x_n))$ such that $t_n = 0$ and $t = \max\{|t_1|, \dots, |t_{n-1}|, 0\} \leq 1$, as well as the basepoint. Define a deformation retraction of $\mathbb{C}(\iota_{\mathcal{A},n})$ onto A by keeping the basepoint fixed, otherwise homotoping a point $(t, (t_1, \dots, t_n; x_1, \dots, x_n)) \in \mathbb{C}(\iota_{\mathcal{A},n})$ to a point in A , first by shifting the t_i 's, by linearly homotoping each t_i to $\bar{t}_i = t_i - t_n$ via $t_{i,s} = (1-s)t_i + s\bar{t}_i$. Then, depending on t and $\beta = \max\{|\bar{t}_1|, \dots, |\bar{t}_{n-1}|, 0\}$, we do one of three things: (i) if $t = \beta$, do nothing since we are done; (ii) if $t > \beta$, radially expand the $\bar{t}_i$'s to $\frac{t}{\beta}\bar{t}_i$; (iii) if $t < \beta$, linearly homotope t towards β , stopping either when $t = \beta$, or when $t = 1$ (we are at the basepoint). Notice β cannot be 0 in case (ii), otherwise the $\bar{t}_i$'s are 0, and by our preceding remark $t = 0$. The homotopy in case (iii) is possible since $\beta > t \geq 0$, so our configuration of particles $(\bar{t}_i, x_i)$ is necessarily in $D_{\mathcal{A}}(W_{n-1})$. Note no new collisions between particles are created during these homotopies, and this homotopy is continuous in the neighbourhood of the basepoint.

Next, we define a deformation retraction of A onto $\Sigma^{n-1}\hat{W}_n$, homotoping a given point $\omega = (t, (t_1, \dots, t_{n-1}, 0; x_1, \dots, x_n))$ in A as follows. If ω is the basepoint, we are done, so assume ω is not the basepoint. Then $t < 1$ and each x_i is not the basepoint. Since we are in $\mathbb{C}(\iota_{\mathcal{A},n})$, we may assume without loss of generality that $x_i \in X_i$, so $\hat{x} = (x_1, \dots, x_n)$ is a point in $\hat{X} = X_1 \wedge \dots \wedge X_n$, and we regard the set of labels x_i as the point $\hat{x} \in \hat{X}$. Since $\hat{x}$ is not the basepoint, there is a unique $x \in X$ such that $q(x) = \hat{x}$, where $X \xrightarrow{q} \hat{X}$ is the quotient map. To start off the homotopy, if $t = 0$, we do nothing. If $t > 0$, we homotope $t = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$ linearly towards 1 via $t_s = (1-s)t + s$, and homotope each t_i for $i \leq n-1$ by radial expansion via $t_{i,s} = \frac{t_s}{t}t_i$ (so that $t_s = \max\{|t_{1,s}|, \dots, |t_{n-1,s}|, 0\}$), while at the same time, homotoping the set of labels $\hat{x}$ by homotoping x towards W_n when it is 'near' W_n to obtain x_t at each time t , meanwhile letting $\hat{x}_t = q(x_t)$ to obtain our homotopy $\hat{x} \mapsto \hat{x}_t$. We perform this homotopy of x using a faithful neighbourhood of W_n and applying its associated homotopy $h_t: X \rightarrow X$ to x , letting $x_t = h_t(x)$, exactly as in the proof of Lemma 2.8, thereby ensuring that labels of collided particles remain represented by a point in W_n during the homotopy of labels $\hat{x}$. Since h_t is defined coordinate-wise as a product of maps $X_i \rightarrow X_i$ that are basepoint preserving, our homotopy for $\hat{x}$ is continuous for $\hat{x}$ in the neighbourhood of the basepoint. The homotopy stops as soon as x_t becomes a point in W_n , in which case $\hat{x}_t \in \hat{W}_n$, or else when t becomes 1, in which case ω becomes the basepoint. Notice there is no issue of continuity, or the homotopy not stopping, when $t = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$ is at or near 0. This is because all particles are collided when $t = 0$. Thus (as a set) $\hat{x}$ must be represented by a point in W_n , and since there are n labels, $\hat{x}$ must be a point $\hat{W}_n$ and x a point in W_n , and the homotopy stops immediately. At the same time, points ω approaching some point with $t = 0$ must have their label set $\hat{x}$ approach a point in $\hat{W}_n$, so x approaches a point in W_n , and our homotopy of x near W_n towards W_n ensures continuity. We let t_ω be the smallest time at which this homotopy stops for ω . For the same reasons as in the second step of the proof of Lemma 2.8, t_ω depends continuously on $\omega \in A$. This then defines our second deformation retraction. □

We will want to describe a right homotopy inverse of $\Sigma\phi$ that it is compatible with the scanning map. To do so we will need to define a version of the scanning map for quotiented configuration spaces. Fix $\mathcal{C}^\partial = \mathcal{C}(D^1, \partial D^1; W_n)$, $\mathcal{C}_i^\partial = \mathcal{C}_i(D^1, \partial D^1; W_n)$, and $\mathcal{D}_i^\partial = \mathcal{C}_i^\partial / \mathcal{C}_{i-1}^\partial$. Consider the map

$$\hat{\iota}: \hat{W}_n^1 \longrightarrow \mathcal{D}_n^\partial$$

given by sending $((t_1, x_1), \dots, (t_n, x_n))$ to $(t_1, \dots, t_n; x_1, \dots, x_n)$.

Since the inclusion $W_n^1 \xrightarrow{\iota} \mathcal{C}^\partial$ from Lemma 2.8 includes into the subspace $\mathcal{C}_n^\partial$, we have an inclusion $\iota_n: W_n^1 \rightarrow \mathcal{C}_n^\partial$. In turn, ι_n restricts to a map $\iota_{n-1}: W_{n-1}^1 \rightarrow \mathcal{C}_{n-1}^\partial$. Notice then our maps fit into a commutative square

$$(13) \quad \begin{array}{ccc} W_n^1 & \xrightarrow{q} & \hat{W}_n^1 \\ \downarrow \iota_n & & \downarrow \hat{\iota} \\ \mathcal{C}_n^\partial & \xrightarrow{p} & \mathcal{D}_n^\partial, \end{array}$$

where p and q are the quotient maps. Like ι , both ι_n and $\hat{\iota}$ are homeomorphisms onto their images, and we think of W_n^1 and $\hat{W}_n^1$ as subspaces of $\mathcal{C}_n^\partial$ and $\mathcal{D}_n^\partial$ under these inclusions.

Lemma 2.17. *There is a deformation retraction of $\mathcal{C}_n^\partial$ onto the subspace W_n^1 given by the inclusion ι_n . Moreover, it extends to a deformation retraction of $\mathcal{D}_n^\partial$ onto the subspace $\hat{W}_n^1$ given by the inclusion $\hat{\iota}$. Therefore, ι_n and $\hat{\iota}$ are homotopy equivalences.*

Proof. Recall that the deformation retraction $H: \mathcal{C}^\partial \times [0, 1] \rightarrow \mathcal{C}^\partial$ of $\mathcal{C}^\partial$ onto W_n^1 in the proof of Lemma 2.8. This is defined via a two-step radial expansion. A configuration ω is homotoped by radial expanding particles towards the boundary, meanwhile, in the first step, labels near the basepoint are homotoped towards it until there are no more than n particles left, and in the second step, the labels of the remaining particles are homotoped towards W_n when they are near W_n , this being done via a coordinate-wise homotopy $h: X \times [0, 1] \rightarrow X$ that is basepoint preserving on each coordinate. From this, we see that for any $\omega = (t_1, \dots, t_m; x_1, \dots, x_m)$ and each $t \in [0, 1]$, H satisfies $H_t(\omega) = (t'_1, \dots, t'_m; x'_1, \dots, x'_m)$ such that $t_i \in \partial D^1 \Rightarrow t'_i \in \partial D^1$ and $x_i = * \Rightarrow x'_i = *$. Degenerate particles thereby remain degenerate during this homotopy. Thus H restricts to a deformation retraction H^i of $\mathcal{C}_i^\partial$ onto W_n^1 when $i \geq n$, and a deformation retraction $\bar{H}^i$ of $\mathcal{C}_i^\partial$ onto W_i^1 when $i \leq n-1$. Since $H^n: \mathcal{C}_n^\partial \times [0, 1] \rightarrow \mathcal{C}_n^\partial$ restricts to $\bar{H}^{n-1}: \mathcal{C}_{n-1}^\partial \times [0, 1] \rightarrow \mathcal{C}_{n-1}^\partial$, there is a well-defined extension $\hat{H}$ of H^n , as in the composite

$$\hat{H}: \mathcal{D}_n^\partial \times [0, 1] \rightarrow (\mathcal{C}_n^\partial \times [0, 1]) / (\mathcal{C}_{n-1}^\partial \times [0, 1]) \xrightarrow{\hat{H}} \mathcal{D}_n^\partial,$$

with the first map being the quotient map collapsing $* \times [0, 1]$ onto the basepoint. Then $\hat{H}$ is a deformation retraction of $\mathcal{D}_n^\partial$ onto $\hat{W}_n^1$. □

The quotiented scanning map

$$\hat{\gamma}: \mathcal{D}_n(W_n) \rightarrow \Omega \hat{W}_n^1$$

is defined as follows. Let $\mathcal{D}_n(M, N; W_n) = \mathcal{C}_n(M, N; W_n) / \mathcal{C}_{n-1}(M, N; W_n)$. Fix any $\varepsilon > 0$ (this will be the radius of our scanning interval). For any configuration $\omega' \in \mathcal{D}_n(W_n)$, we define $\hat{\gamma}(\omega'): \mathbb{R} \rightarrow \hat{W}_n^1$ as the map given for any $z \in \mathbb{R}$ as the image of ω' under the composite

$$\mathcal{D}_n(W_n) \rightarrow \mathcal{D}_n(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W_n) \xrightarrow{\cong} \mathcal{D}_n^\partial \xrightarrow{\hat{r}} \hat{W}_n^1$$

where $D_{\varepsilon, z} = [z - \varepsilon, z + \varepsilon] \subseteq \mathbb{R}$, and where the first map forgets those particles from ω' that are in $\mathbb{R} - D_{\varepsilon, z}$ (thus maps to the basepoint if at least one particle is in $\mathbb{R} - D_{\varepsilon, z}$), the second homeomorphism is induced by the homeomorphism $D_{\varepsilon, z} \cong D^1$, and $\hat{r}$ is the homotopy inverse of $\hat{\iota}$ taken as the retraction from Lemma 2.17. Recall that the regular scanning map γ is defined similarly. In the case of $\mathcal{C}(W_n)$, the restriction of $\mathcal{C}(W_n) \xrightarrow{\gamma} \Omega W_n^1$ to $\mathcal{C}_n(W_n)$ is defined by sending $\omega \in \mathcal{C}_n(W_n)$ to the map $\gamma(\omega): \mathbb{R} \rightarrow W_n^1$, given for any $z \in \mathbb{R}$ as the image of ω under

$$\mathcal{C}_n(W_n) \rightarrow \mathcal{C}_n(D_{\varepsilon, z}, \partial D_{\varepsilon, z}; W_n) \xrightarrow{\cong} \mathcal{C}_n^\partial \xrightarrow{r} W_n^1$$

where the first map forgets particles in $\mathbb{R} - D_{\varepsilon, z}$ as before, and (since the deformation retraction H in the proof of Lemma 2.17 restricts as it does) r is the homotopy inverse of ι_n taken as the retraction from Lemma 2.17. Replacing the homotopy equivalences ι_n and $\hat{\iota}$ in Diagram (13) with

their homotopy inverses r and $\hat{r}$ gives a homotopy commutative diagram, so we obtain a homotopy $q \circ r \simeq \hat{r} \circ p$. Combining this homotopy with the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}_n(W_n) & \longrightarrow & \mathcal{C}_n(D_{\varepsilon,z}, \partial D_{\varepsilon,z}; W_n) & \xrightarrow{\cong} & \mathcal{C}_n^\partial \\ \downarrow p' & & & & \downarrow p \\ \mathcal{D}_n(W_n) & \longrightarrow & \mathcal{D}_n(D_{\varepsilon,z}, \partial D_{\varepsilon,z}; W_n) & \xrightarrow{\cong} & \mathcal{D}_n^\partial, \end{array}$$

where the vertical maps are the quotient maps, gives a homotopy of $\hat{\gamma}(p'(\omega)) \simeq q(\gamma(\omega))$ that is independent of ω and z . Therefore:

Lemma 2.18. *There is a homotopy commutative square*

$$\begin{array}{ccc} \mathcal{C}_n(W_n) & \longrightarrow & \mathcal{D}_n(W_n) \\ \downarrow \gamma & & \downarrow \hat{\gamma} \\ \Omega W_n^1 & \xrightarrow{\Omega q} & \Omega \hat{W}_n^1. \end{array}$$

□

Let

$$\hat{e}v: \Sigma \mathcal{D}_n(W_n) \longrightarrow \hat{W}_n^1$$

be the adjoint of $\mathcal{D}_n(W_n) \xrightarrow{\hat{\gamma}} \Omega \hat{W}_n^1$. The composite $\Sigma \mathcal{D}_n(W_{n-1}) \xrightarrow{\Sigma \iota_{n,n}} \Sigma \mathcal{D}_n(W_n) \xrightarrow{\hat{e}v} \hat{W}_n^1$ is nullhomotopic as follows. Since $\hat{\gamma}$ is given by scanning an ε -neighbourhood (for some fixed $\varepsilon > 0$) through a configuration ω of labelled particles in $\mathbb{R}$, while applying a radial expansion to particles in this neighbourhood to obtain a point along a based loop in $\hat{W}_n^1$ (as defined in the proof of Lemmas 2.8 and 2.17). If no ε -neighbourhood contains all n particles, then every point in the loop is the basepoint in $\hat{W}_n^1$, in which case $\hat{e}v$ will map (t, ω) to the basepoint in $\hat{W}_n^1$ for all $t \in [-1, 1]$. Then since no more than $n-1$ particles in a configuration $\omega \in \mathcal{D}_n(W_{n-1})$ can collide at a point, we can homotope $\hat{e}v \circ \Sigma \iota_{n,n}(t, \omega)$ to the basepoint by homotoping (t, ω) via radial expansion of the n particles in ω away from $0 \in \mathbb{R}$, until no ε -neighbourhood in $\mathbb{R}$ contains them all (alternatively, one can scan ever smaller neighbourhoods). This defines a nullhomotopy of $\hat{e}v \circ \Sigma \iota_{n,n}$, and in turn, this nullhomotopy defines an extension

$$\rho: \Sigma \mathbb{C}(\iota_{n,n}) \longrightarrow \hat{W}_n^1$$

of $\hat{e}v$. Take the composite

$$\rho_{\mathcal{A}}: \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \longrightarrow \Sigma \mathbb{C}(\iota_{n,n}) \xrightarrow{\rho} \hat{W}_n^1$$

where the first map is the inclusion. This is not the same map as the suspension of the retraction map onto $\hat{W}_n^1 \cong \Sigma^n W_n$ in Lemma 2.16, but it will be of use to us since it fits naturally with the maps ev and $\hat{e}v$.

Corollary 2.19. *The composite $\rho_{\mathcal{A}} \circ \Sigma \phi$ is a homotopy equivalence. Thus, $\rho_{\mathcal{A}}$ is a homotopy equivalence.*

Proof. Take an element $\omega = ((t_1, \dots, t_n), (x_1, \dots, x_n)) \in \hat{W}_n^1$. Since $x = (x_1, \dots, x_n)$ is in W_n , any subset of $\{x_1, \dots, x_n\}$ is represented the point x in W_n . Then since the homeomorphic image of the inclusion $\hat{W}_n \xrightarrow{\hat{\iota}} \mathcal{D}_n^\partial$ in Lemma 2.17 is the subspace of configurations whose particles have labels represented by a point in W_n , and since $\hat{\gamma}$ is defined in terms of a deformation retraction of $\mathcal{D}_n^\partial$ onto this image, then the mapping $\omega \mapsto \rho_{\mathcal{A}} \circ \Sigma \phi(\omega)$ does not change the labels x_i of ω , that is, $\rho_{\mathcal{A}} \circ \Sigma \phi(\omega) = ((\bar{t}_1, \dots, \bar{t}_n), (x_1, \dots, x_n))$ for some $\bar{t}_i$. Moreover, one can check directly from the construction of $\rho_{\mathcal{A}}$ and $\phi(\omega)$ that each $\bar{t}_i$ satisfies the following for each $c = \pm 1$ as $(t_1, \dots, t_n)$ approaches the boundary of $D^n = [-1, 1]^{\times n}$: (1) $\bar{t}_n$ approaches c as t_n approaches c (since the scanning maps scans a neighbourhood from right to left in $\mathbb{R}$); and (2) for $i \leq n-1$, if t_i approaches c , then either $\bar{t}_i$ approaches c , or else t_n approaches $-c$ (as does $\bar{t}_n$). So if we define a linear

homotopy from $\bar{t}_i$ to t_i via $t_{i,s} = (1-s)\bar{t}_i + st_i$, then (1) and (2) hold with $t_{i,s}$ in place of $\bar{t}_i$, and we see that $(t_{1,s}, \dots, t_{n,s})$ approaches the boundary of D^n for each s whenever $(t_1, \dots, t_n)$ does. There are therefore no issues of continuity in defining a homotopy from $\rho_A \circ \Sigma\phi$ to the identity by mapping ω to $((t_{1,s}, \dots, t_{n,s}), (x_1, \dots, x_n))$ at each time s . Then using Lemma 2.16, ρ_A is a homotopy equivalence since both $\rho_A \circ \Sigma\phi$ and $\Sigma\phi$ are. $\square$

Lemma 2.20. *Suppose we are given a homotopy commutative diagram of homotopy cofibration sequences*

$$\begin{array}{ccccccc} Z & \xrightarrow{\partial} & X & \xrightarrow{f} & Y & \longrightarrow & \Sigma Z \\ & & \downarrow \bar{g} & & \downarrow g & & \downarrow \hat{g} \simeq \\ & & A & \xrightarrow{h} & B & \longrightarrow & C, \end{array}$$

where the homotopy equivalence $\hat{g}$ is an induced map of homotopy cofibers, and all spaces are homotopy equivalent to simply connected CW-complexes. Then the bottom homotopy cofibration sequence extends to the left to a homotopy cofibration sequence

$$Z \xrightarrow{\bar{g} \circ \partial} A \xrightarrow{h} B.$$

Proof. Since the induced map $\hat{g}$ is a homotopy equivalence, the left-most square $\mathcal{Q}$ in the above diagram is a homotopy pushout, that is, $\mathcal{Q}$ is equivalent in the homotopy category to a pushout square whose horizontal maps are cofibrations (the dual of Proposition 7.6.1 in [31]). Likewise, if we take B' to be the homotopy cofibre of $\bar{g} \circ \partial$, we obtain a homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccccc} Z & \xrightarrow{\partial} & X & \xrightarrow{f} & Y & \longrightarrow & \Sigma Z \\ \parallel & & \downarrow \bar{g} & & \downarrow g' & & \parallel \\ Z & \xrightarrow{\bar{g} \circ \partial} & A & \xrightarrow{h'} & B' & \longrightarrow & \Sigma Z \end{array}$$

where the last two vertical maps are iteratively induced maps of homotopy cofibers, and so we see the middle square $\mathcal{Q}'$ here is a homotopy pushout as well. We obtain a map of homotopy pushouts $\mathcal{Q} \xrightarrow{q} \mathcal{Q}'$, and thus a map $B' \xrightarrow{p} B$, by mapping X , Y , and A to themselves via the identity, and since these three are homotopy equivalences, so is p . From the cube diagram given by q we see that $p \circ h'$ is homotopic to h , and we are done. $\square$

Proof of Proposition 2.13. Since the quotient map $\mathcal{C}_i(W_n) \rightarrow \mathcal{D}_i(W_n)$ restricts to the quotient map $\mathcal{C}_i(W_k) \rightarrow \mathcal{D}_i(W_k)$ for $k < n$, by definition of labels in the image of $\mathcal{C}_n(W_n) \xrightarrow{\zeta} C(\mathbb{R}; V_\infty(W_n))$ in terms of (7) in Section 2.4, it follows that ζ restricts to $\mathcal{C}_n(W_{n-1}) \xrightarrow{\zeta} C(\mathbb{R}; V_\infty(W_{n-1}))$. Then composing ζ and its restriction with the corresponding scanning maps, using naturality of the scanning map, and taking the adjoints ζ' of their composites (the homotopy equivalences ζ' from (9)), we obtain the left commutative square in the diagram of homotopy cofibration sequences

$$\begin{array}{ccccc} \Sigma \mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma \bar{\iota}_{n,n}} & \Sigma \mathcal{C}_n(W_n) & \longrightarrow & \Sigma \mathbb{C}(\bar{\iota}_{n,n}) \\ \downarrow \zeta' & & \downarrow \zeta' & & \downarrow \hat{\zeta}' \\ \Sigma V_n(W_{n-1}) & \xrightarrow{\bigvee_i \Sigma \iota_{i,n}} & \Sigma V_n(W_n) & \longrightarrow & \bigvee_{1 \leq i \leq n} \Sigma \mathbb{C}(\iota_{i,n}). \end{array}$$

Here, the bottom sequence is the wedge sum of homotopy cofibration sequences $\Sigma \mathcal{D}_i(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma \mathcal{D}_i(W_n) \rightarrow \Sigma \mathbb{C}(\iota_{i,n})$ for $1 \leq i \leq n$, thus giving the right vertical extension $\hat{\zeta}'$. Since the left and middle vertical maps are homotopy equivalences, so is $\hat{\zeta}'$. But since $\iota_{m,k}$ is a homeomorphism when

$m < k$ by Lemma 2.12, the only non-contractible homotopy cofibre $\Sigma\mathbb{C}(\iota_{i,n})$ between $1 \leq i \leq n$ is $\Sigma\mathbb{C}(\iota_{n,n})$, implying that the composite

$$\hat{\nu}: \Sigma\mathbb{C}(\iota_{n,n}) \xrightarrow{\text{include}} \bigvee_{1 \leq i \leq n} \Sigma\mathbb{C}(\iota_{i,n}) \xrightarrow{(\hat{\zeta}')^{-1}} \Sigma\mathbb{C}(\bar{\iota}_{n,n})$$

is a homotopy equivalence, where $(\hat{\zeta}')^{-1}$ is a homotopy inverse of $\hat{\zeta}'$. Replacing the vertical homotopy equivalences ζ' , $\hat{\zeta}'$ with their inverses, and restricting to the homotopy cofibration sequence $\Sigma\mathcal{D}_n(W_{n-1}) \xrightarrow{\Sigma\iota_{n,n}} \Sigma\mathcal{D}_n(W_n) \rightarrow \Sigma\mathbb{C}(\iota_{n,n})$, gives a homotopy commutative diagram of homotopy cofibrations

$$(14) \quad \begin{array}{ccccc} \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n,n}} & \Sigma\mathcal{D}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\iota_{n,n}) \\ \downarrow \nu & & \downarrow \nu & & \downarrow \hat{\nu} \\ \Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\bar{\iota}_{n,n}} & \Sigma\mathcal{C}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\bar{\iota}_{n,n}). \end{array}$$

In turn, the top homotopy cofibration sequence splits as the wedge sum of homotopy cofibrations sequences

$$\Sigma\mathcal{D}_{\mathcal{S}}(W_{n-1}) \xrightarrow{\Sigma\iota_{\mathcal{S},n}} \Sigma\mathcal{D}_{\mathcal{S}}(W_n) \rightarrow \Sigma\mathbb{C}(\iota_{\mathcal{S},n})$$

over all (n,n) -partitions $\mathcal{S}$, and since $\iota_{\mathcal{S},k}$ is a homeomorphism when at most $k-1$ elements in an (m,n) -partition $\mathcal{S}$ are non-zero, the only non-contractible homotopy cofibre in this splitting is $\Sigma\mathbb{C}(\iota_{\mathcal{A},n})$, so the inclusion

$$\Sigma\mathbb{C}(\iota_{\mathcal{A},n}) \rightarrow \Sigma\mathbb{C}(\iota_{n,n})$$

is a homotopy equivalence. Restricting to the sequence $\mathcal{S} = \mathcal{A}$, we obtain the top homotopy commutative diagram of homotopy cofibrations in the statement of the proposition.

As for the bottom diagram of homotopy cofibrations, note by adjoining the square in Lemma 2.18, we have a commutative diagram

$$\begin{array}{ccccc} \Sigma\mathcal{C}_n(W_n) & \longrightarrow & \Sigma\mathcal{D}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\iota_{n,n}) \\ \downarrow ev & & \downarrow \hat{ev} & \swarrow \rho & \\ W_n^1 & \longrightarrow & \hat{W}_n^1 & & \end{array}$$

The top left horizontal quotient map fits into a commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc} \Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\bar{\iota}_{n,n}} & \Sigma\mathcal{C}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\bar{\iota}_{n,n}) \\ \downarrow & & \downarrow & & \downarrow \hat{q} \\ \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n,n}} & \Sigma\mathcal{D}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\iota_{n,n}) \end{array}$$

induced by the left square, where the vertical maps are the quotient maps. Combining the right square with the first diagram, we obtain a commutative diagram of homotopy cofibration sequences

$$(15) \quad \begin{array}{ccccc} \Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\bar{\iota}_{n,n}} & \Sigma\mathcal{C}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\bar{\iota}_{n,n}) \\ \downarrow ev & & \downarrow ev & & \downarrow \bar{\rho} \\ W_{n-1}^1 & \longrightarrow & W_n^1 & \longrightarrow & \hat{W}_n^1, \end{array}$$

where $\bar{\rho}$ is the composite

$$\bar{\rho}: \Sigma\mathbb{C}(\bar{\iota}_{n,n}) \xrightarrow{\hat{q}} \Sigma\mathbb{C}(\iota_{n,n}) \xrightarrow{\rho} \hat{W}_n^1.$$

To see that $\bar{\rho}$ is a homotopy equivalence, note by Corollary 2.19 the composite

$$\rho_{\mathcal{A}}: \Sigma\mathbb{C}(\iota_{\mathcal{A},n}) \xrightarrow{\simeq} \Sigma\mathbb{C}(\iota_{n,n}) \xrightarrow{\rho} \hat{W}_n^1$$

is a homotopy equivalence. Since we saw above that the first inclusion is a homotopy equivalence, then so is ρ . Now the composite of $\Sigma\mathcal{C}_n(W_n) \xrightarrow{\zeta'} \Sigma V_n(W_n)$ with the quotient map onto the summand $\Sigma\mathcal{D}_n(W_n)$ is the left homotopy inverse of the map ν in diagram (14), and by the right-hand commutative square in diagram (9) in Section 2.4, this composite is equal to the composite

$$\Sigma\mathcal{C}_n(W_n) \xrightarrow{q} \Sigma\mathcal{D}_n(W_n) \xrightarrow{\simeq} \Sigma\mathcal{D}_n(W_n)$$

of q and some self homotopy equivalence. Thus, q is also a left homotopy inverse of ν , and similarly for W_{n-1} in place of W_n . Then composing (14) with the commutative diagram

$$\begin{array}{ccccc} \Sigma\mathcal{C}_n(W_{n-1}) & \xrightarrow{\Sigma\bar{\iota}_{n,n}} & \Sigma\mathcal{C}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\bar{\iota}_{n,n}) \\ \downarrow q & & \downarrow q & & \downarrow \hat{q} \\ \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n,n}} & \Sigma\mathcal{D}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\iota_{n,n}) \end{array}$$

gives a commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc} \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\bar{\iota}_{n,n}} & \Sigma\mathcal{D}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\bar{\iota}_{n,n}) \\ \downarrow q \circ \nu & & \downarrow q \circ \nu & & \downarrow \hat{q} \circ \hat{\nu} \\ \Sigma\mathcal{D}_n(W_{n-1}) & \xrightarrow{\Sigma\iota_{n,n}} & \Sigma\mathcal{D}_n(W_n) & \longrightarrow & \Sigma\mathbb{C}(\iota_{n,n}). \end{array}$$

Since both composites $q \circ \nu$ are homotopic to the identity, $\hat{q} \circ \hat{\nu}$ is a homotopy equivalence. But $\hat{\nu}$ is a homotopy equivalence, so $\hat{q}$ is homotopy equivalence, and therefore $\bar{\rho} = \rho \circ \hat{q}$ is a homotopy equivalence since ρ is. Thus, we have obtain the right-side of the bottom homotopy commutative diagram of homotopy cofibrations in the statement of the proposition.

It remains to show that the bottom cofibration sequence in the proposition extends to the left, and to obtain the left-hand square involving ϕ . This is clear when $n = 1$, since $W_{n-1}^1 = *$ in this case. Since $\Sigma^{n-1}\hat{W}_n$ is simply connected when $n \geq 2$, and ϕ is a homotopy equivalence by Lemma 2.16, the $n \geq 2$ case follows by Lemma 2.20. $\square$

3. THE GOLOD PROPERTY

From now on we fix $W = (D^1, S^0)^K$ as a subspace of $X = (D^1)^{\times n}$, where K on vertex set $[n]$ has no ghost vertices. Thus, $W^1 = (D^2, S^1)^K$. Since W is connected, has the simplicial property, and each W_k has a faithful neighbourhood by Examples 2.7 and 2.6, all of our previous results apply.

The following confirms a large primes version of Conjecture 1.1, generalising Berglund's rational result in [6].

Theorem 3.1. *Localised at any sufficiently large prime p , $(D^2, S^1)^K$ is homotopy equivalent to a wedge of spheres if and only if K is Golod over $\mathbb{Z}_p$.*

Proof. The right-hand implication is clear. For the left-hand, recall that a space Y is a co- H -space if and only if the evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ has a right homotopy inverse [16]. A suspension ΣY of a simply connected finite CW -complex Y is homotopy equivalent to a wedge of spheres when localised at sufficiently large primes (see for example [14, 1]).

The moment-angle complex W^1 is a 2-connected finite CW -complex. Then ΩW^1 has finite type and the skeleta of ΩW^1 are finite CW -complexes, and suspending, the skeleta of $\Sigma\Omega W^1$ are homotopy equivalent to wedges of spheres when localised at sufficiently large primes. Also, the evaluation map $\Sigma\Omega W^1 \xrightarrow{ev} W^1$ induces a surjection on $\mathbb{Z}_p$ -homology whenever $H^+(W^1; \mathbb{Z}_p)$ has trivial cup products and Massey products, i.e. when K is Golod over $\mathbb{Z}_p$ (see Theorem 3.2 and the proof of Theorem 3.1 in [18]). Then $H_*(W^1; \mathbb{Z}_p)$ being a finite dimensional vector space, for sufficiently large k and prime p , we can construct a right homotopy inverse for ev localised at p by taking the inclusion of a skeleton $(\Sigma\Omega W^1)^{(k)} \rightarrow \Sigma\Omega W^1$ and selecting the appropriate spheres from a p -local splitting of $(\Sigma\Omega W^1)^{(k)}$. In particular, W^1 is homotopy equivalent to a wedge of spheres localised at p . $\square$

We now focus on the integral case. An *ordered partition* of a finite set I is a sequence of disjoint non-empty subsets $\mathcal{I} = (I_1, \dots, I_m)$ of I such that $I = I_1 \cup \dots \cup I_m$. If I is a subset of the integers $[n]$ with $k = |I|$ elements, and $\mathcal{S} = (s_1, \dots, s_k)$ is any sequence of real numbers, let i_ℓ denote the ℓ^{th} smallest element in I , s'_j denote the j^{th} smallest element in the set $S = \{s_1, \dots, s_k\}$, and $m = |\mathcal{S}|$ be the number of distinct elements in $\mathcal{S}$. Then assign to $\mathcal{S}$ the ordered partition $I_{\mathcal{S}} = (I_1, \dots, I_m)$ of I where $I_j = \{i_\ell \in I \mid s_\ell = s'_j\}$. When $I = [n]$ we have $k = n$ and $I_j = \{\ell \mid s_\ell = s'_j\}$, so for example, if $I = [4]$ and $\mathcal{S} = (-1, \pi, -1, 0)$, then $[4]_{\mathcal{S}} = (\{1, 3\}, \{4\}, \{2\})$.

We think of the topological $(n-1)$ -simplex $|\Delta^{n-1}|$ and its subcomplexes $|K|$ as being without basepoint. The first suspensions $\Sigma|\Delta^{n-1}|$ and $\Sigma|K|$ are therefore unreduced. Again, the suspensions $\Sigma|\Delta^{n-1}|$ and $\Sigma|K|$ are themselves basepointed, with the basepoint $\ast_{-1}$ the tip of the double cone corresponding to the basepoint $-1 \in D^1 = [-1, 1]$. Higher suspensions are therefore reduced. This means that any point $(t_1, \dots, t_n, z) \in \Sigma^n|K|$ is identified with the basepoint $\ast$ if and only if $t_n = -1$ or $t_i = \pm 1$ for some $i < n$.

Take the diagonal

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = \dots = x_n\}$$

and consider the following subspace of the smash product $\mathcal{P}_n = (\mathbb{R}^n - \Delta_n) \wedge \Sigma|\Delta^{n-1}|$:

$$\mathcal{Q}_K = \bigcup_{\substack{y \in (\mathbb{R}^n - \Delta_n) \\ (I_1, \dots, I_m) = [n]_y}} \{y\} \wedge \Sigma|K_{I_1} \ast \dots \ast K_{I_m}|,$$

where $[n]_y$ is a partition $[n]_{(y_1, \dots, y_n)}$ of $[n]$ as defined above. Here we took $\mathbb{R}^n - \Delta_n$ to be without a basepoint, so $\mathcal{P}_n$ is the half-smash product $((\mathbb{R}^n - \Delta_n) \times \Sigma|\Delta^{n-1}|) / ((\mathbb{R}^n - \Delta_n) \times \{\ast\})$, and is basepointed.

Definition 3.2. A simplicial complex K on vertex set $[n]$ is *weakly coherently homotopy Golod* if K is a single vertex, or (recursively) $K \setminus \{i\}$ is weakly coherently homotopy Golod for each $i \in [n]$, and the map

$$\Phi_K : \Sigma^n|K| \longrightarrow \Sigma\mathcal{Q}_K$$

given for any $z \in |K|$, $t_1, \dots, t_{n-1}, t \in [-1, 1]$, and $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$ by

$$\Phi_K(t_1, \dots, t_{n-1}, t, z) = (2\beta - 1, (t_1, \dots, t_{n-1}, 0), (t, z))$$

is nullhomotopic.

Theorem 3.3. *If K is weakly coherently homotopy Golod, then $(D^2, S^1)^K$ is a co- H -space.*

We will need a bit of setup before we can prove this. Recall that $\hat{W}$ is the image of W under the quotient map $(D^1)^{\times n} \longrightarrow (D^1)^{\wedge n}$. To prove this theorem we will need to use a simplicial description of $\hat{W}$. In [3] the theory of *diagrams of spaces* was used to show the existence of a homeomorphism $\Sigma|K| \cong \hat{W}$, but for our purposes, we give an explicit description of this homeomorphism as follows. Take the simplex Δ^{n-1} to be on vertex set $[n]$, and think of its geometric realisation as the subspace

$$|\Delta^{n-1}| = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1 \text{ and } t_1 + \dots + t_n = 1\}$$

with each face $\sigma \in \Delta^{n-1}$ corresponding to the subspace $|\sigma| = \{(t_1, \dots, t_n) \in |\Delta^{n-1}| \mid t_i = 0 \text{ if } i \notin \sigma\}$. Take the subspaces of $|\Delta^{n-1}|$ and $(D^1)^{\wedge n}$:

$$U_i = \{(t_1, \dots, t_n) \in |\Delta^{n-1}| \mid t_i = \max\{t_1, \dots, t_n\}\},$$

$$V_{i,t} = \{(t'_1, \dots, t'_n) \in (D^1)^{\wedge n} \mid t'_i = t = \min\{t'_1, \dots, t'_n\}\}.$$

Consider a homeomorphism

$$(16) \quad h : \Sigma|\Delta^{n-1}| \xrightarrow{\cong} (D^1)^{\wedge n}$$

given as follows. For $-1 < t < 1$, take the homeomorphism $h_{i,t} : U_i \longrightarrow V_{i,t}$ given by mapping

$$(t_1, \dots, t_n) \mapsto \left(t + (1-t) \frac{t_i - t_1}{t_i}, \dots, t + (1-t) \frac{t_i - t_n}{t_i} \right),$$

and its inverse homeomorphism $h_{i,t}^{-1}: V_{i,t} \rightarrow U_i$ is given by mapping

$$(t'_1, \dots, t'_n) \mapsto \left(\frac{1 - t'_1}{n - \sum_{k=1}^n t'_k}, \dots, \frac{1 - t'_n}{n - \sum_{k=1}^n t'_k} \right)$$

(to see that $h_{i,t} \circ h_{i,t}^{-1}$ and $h_{i,t}^{-1} \circ h_{i,t}$ are the identity, use the fact that here $t'_i = t$ and $1 = t_1 + \dots + t_n$). Notice that $h_{i,t}$ agrees with $h_{j,t}$ on $U_i \cap U_j$, and $h_{i,t}^{-1}$ agrees with $h_{j,t}^{-1}$ on $V_{i,t} \cap V_{j,t}$. Thus, for each $t \in (-1, 1)$, we can piece together the homeomorphisms $h_{i,t}$ over all $i \in [n]$ to give a homeomorphism

$$h_t: |\Delta^{n-1}| = \left(\bigcup_{i=1}^n U_i \right) \xrightarrow{\cong} \left(\bigcup_{i=1}^n V_{i,t} \right) = V_t$$

defined unambiguously by mapping $z \mapsto h_{i,t}(z)$ if $z \in U_i$, with inverse h_t^{-1} defined by mapping $z' \mapsto h_{i,t}^{-1}(z')$ if $z' \in V_{i,t}$. Notice that $(D^1)^{\wedge n}$ is the disjoint union of V_t over all $t \in [-1, 1]$, with V_1 being a single point $(1, \dots, 1)$, and V_{-1} being the basepoint, and the maps h_t vary continuously with respect to t . Thus, we define the homeomorphism h for any $(t, z) \in \Sigma|\Delta^{n-1}|$ by

$$h(t, z) = \begin{cases} h_t(z) & \text{if } -1 < t < 1; \\ (1, \dots, 1) & \text{if } t = 1; \\ * & \text{if } t = -1. \end{cases}$$

The crucial property of h is that it restricts on an m -dimensional face σ of Δ^{n-1} to a homeomorphism

$$h_\sigma: \Sigma|\Delta^m| \cong \Sigma|\sigma| \xrightarrow{\cong} (Y_1^\sigma \wedge \dots \wedge Y_n^\sigma) \cong (D^1)^{\wedge(m+1)},$$

where

$$Y_i^\sigma = \begin{cases} D^1 & \text{if } i \in \sigma \\ \{1\} & \text{if } i \notin \sigma. \end{cases}$$

Then piecing these together, for any subcomplex L of Δ^{n-1} , h restricts to a homeomorphism

$$\Sigma|L| = \bigcup_{\sigma \in L} \Sigma|\sigma| \xrightarrow{\cong} \bigcup_{\sigma \in L} Y_1^\sigma \wedge \dots \wedge Y_n^\sigma.$$

In particular, when $W = (D^1, S^0)^K$,

$$\hat{W} = \bigcup_{\sigma \in K} Y_1^\sigma \wedge \dots \wedge Y_n^\sigma,$$

so h restricts to a homeomorphism $\Sigma|K| \cong \hat{W}$. More generally, for any partition $(I_1, \dots, I_m)$ of $[n]$, we have an inclusion $\hat{W}_{I_1} \wedge \dots \wedge \hat{W}_{I_m}$ into $(D^1)^{\wedge n}$ given by mapping elements by permuting coordinates, so that if a j^{th} coordinate is the coordinate in a factor $\hat{W}_{I_k}$ that corresponds to $i \in I_k$, upon mapping it becomes the i^{th} coordinate. This then gives a homeomorphism onto its image

$$\hat{W}_{I_1} \wedge \dots \wedge \hat{W}_{I_m} \cong \bigcup_{\sigma_1 \in K_{I_1}, \dots, \sigma_m \in K_{I_m}} Y_1^{(\sigma_1 \cup \dots \cup \sigma_m)} \wedge \dots \wedge Y_n^{(\sigma_1 \cup \dots \cup \sigma_m)},$$

so more generally h restricts to a homeomorphism

$$(17) \quad \Sigma|K_{I_1} * \dots * K_{I_m}| = \bigcup_{\sigma_1 \in K_{I_1}, \dots, \sigma_m \in K_{I_m}} \Sigma|\sigma_1 \cup \dots \cup \sigma_m| \xrightarrow{\cong} \hat{W}_{I_1} \wedge \dots \wedge \hat{W}_{I_m}.$$

In this manner, we will think of $\hat{W}_{I_1} \wedge \dots \wedge \hat{W}_{I_m}$ as a subspace of $(D^1)^{\wedge n}$.

Proof of Theorem 3.3. Let $W = (D^1, S^0)^K$. At most $n - 1$ particles collide in a configuration in $\mathcal{D}_A(W_{n-1})$, and if we look at the subspace of all configurations of n particles fixed at some $t_1, \dots, t_n \in \mathbb{R}$ (that is, all possible labels for particles fixed in this way), then this subspace is homeomorphic to $\{y\} \wedge \hat{W}_{I_1} \wedge \dots \wedge \hat{W}_{I_m}$, where $y = (t_1, \dots, t_n)$, and $(I_1, \dots, I_m) = [n]_y$. This in turn is homeomorphic to $\{y\} \wedge \Sigma|K_{I_1} * \dots * K_{I_m}|$ via the restriction of the homeomorphism $\Sigma|\Delta^{n-1}| \xrightarrow{h} (D^1)^{\wedge n}$ in (16). Thus, we see that the map

$$\tilde{h}: \mathcal{Q}_K \rightarrow \mathcal{D}_A(W_{n-1})$$

sending $((t_1, \dots, t_n), x) \mapsto (t_1, \dots, t_n; x_1, \dots, x_n)$, where $(x_1, \dots, x_n) = h(x)$, is a homeomorphism. Notice this map fits into a commutative square

$$\begin{array}{ccc} \Sigma^n |K| & \xrightarrow{\Phi_K} & \Sigma \mathcal{Q}_K \\ \cong \downarrow \Sigma^{n-1} h & & \cong \downarrow \Sigma \tilde{h} \\ \Sigma^{n-1} \hat{W} & \xrightarrow[\simeq]{\phi} \mathbb{C}(\iota_{\mathcal{A}, n}) \xrightarrow{\partial} & \Sigma \mathcal{D}_{\mathcal{A}}(W_{n-1}), \end{array}$$

where ∂ is the connecting map given by collapsing the subspace $\mathcal{D}_{\mathcal{A}}(W_n)$, and ϕ the homotopy equivalence from Lemma 2.16.

Induct on number of vertices. Assume the statement holds for simplicial complexes on less than n vertices. Suppose K is weakly coherently homotopy Golod. Directly from definition, each K_I is weakly coherently homotopy Golod, so by our inductive assumption W_I^1 is a co- H -space for $|I| < n$. Then using Proposition 1.9, the quotient map $W_I^1 \rightarrow \hat{W}_I^1$ has a right homotopy inverse for $|I| < n$, and to show W^1 is a co- H -space, it remains to show that $W^1 \rightarrow \hat{W}^1$ has a right homotopy inverse. Since Φ_K is nullhomotopic and ϕ is a homotopy equivalence, ∂ is nullhomotopic by the above commutative square, and so $W^1 \rightarrow \hat{W}^1$ has a right homotopy inverse by Corollary 2.14. $\square$

Example 3.4. Let K be two disjoint vertices. In this case the right-hand factor in the definition of $\mathcal{Q}_K$ is in all cases homeomorphic to D^2 , while left-hand factor is $\mathbb{R}^2$ minus the diagonal. Then $\mathcal{Q}_K$ is homotopy equivalent to the left half smash product $(* \sqcup *) \times D^2$, and as such, it is contractible, and Φ_K is nullhomotopic. Therefore K is weakly coherently homotopy Golod and $(D^2, S^1)^K$ is a co- H -space - as expected since it is homeomorphic to a 3-sphere.

The weakly coherent homotopy Golod condition - as it stands - does not look near enough to the Golod condition for our liking. With this in mind, we start by considering complexes K for which each inclusion $|K_{I \cup J}| \xrightarrow{\iota_{I,J}} |K_I * K_J|$ is nullhomotopic after appropriate suspension. This is just a homotopy version of the Golod property without consideration for Massey products, so appropriately, we call it the *homotopy Golod* condition. It turns out it is a necessary condition for $(D^2, S^1)^K$ to be a co- H -space (and so it might be a good avenue for finding a counter-example to Conjecture 1.1).

Proposition 3.5. *If $(D^2, S^1)^K$ is a co- H -space, then each one of the suspended inclusions*

$$\Sigma^{|I \cup J|+1} \iota_{I,J}: \Sigma^{|I \cup J|+1} |K_{I \cup J}| \rightarrow \Sigma^{|I \cup J|+1} |K_I * K_J|$$

is nullhomotopic for every disjoint non-empty $I, J \not\subseteq [n]$.

Proof. Recall that if Y is a co- H -space with comultiplication $Y \xrightarrow{\psi} Y \vee Y$ if and only if the diagonal map $Y \xrightarrow{\Delta} Y \times Y$ is homotopic to the composite $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\text{include}} Y \times Y$. In particular, this implies the reduced diagonal map

$$\bar{\Delta}: Y \xrightarrow{\Delta} Y \times Y \rightarrow (Y \times Y)/(Y \vee Y) \xrightarrow{\cong} Y \wedge Y$$

is nullhomotopic whenever Y is a co- H -space. As before, let $W = W_n = (D^1, S^0)^K$, in which case $W^1 = W_n^1 = (D^2, S^1)^K$, and suppose W^1 is a co- H -space. By Proposition 1.9, $W_{I \cup J}^1$ is a co- H -space for each disjoint non-empty $I, J \not\subseteq [n]$, and so $W_{I \cup J}^1 \xrightarrow{\bar{\Delta}} W_{I \cup J}^1 \wedge W_{I \cup J}^1$ is nullhomotopic. Then the composite

$$W_{I \cup J}^1 \xrightarrow{\bar{\Delta}} W_{I \cup J}^1 \wedge W_{I \cup J}^1 \rightarrow W_I^1 \wedge W_J^1 \rightarrow \hat{W}_I^1 \wedge \hat{W}_J^1$$

is nullhomotopic (where the second last map is the smash of the coordinate-wise projection maps onto W_I^1 and W_J^1 , and the last map is the smash of quotient maps) and moreover, is equal to the composite

$$W_{I \cup J}^1 \xrightarrow{\text{quotient}} \hat{W}_{I \cup J}^1 \xrightarrow{\iota^1} \hat{W}_I^1 \wedge \hat{W}_J^1$$

where the last map ι^1 is the coordinate-wise inclusion (given by mapping $(x_1, \dots, x_n)$ to the pair $((y_1, \dots, y_{|I|}), (z_1, \dots, z_{|J|}))$ such that $y_i = x_k$ if $k \in I$ is the i^{th} largest element, or else $z_j = x_k$ if $k \in J$ is the j^{th} largest element), so this composite is nullhomotopic as well. But since $W_{I \cup J}^1$ is a co- H -space, by Proposition 1.9 the quotient map $W_{I \cup J}^1 \rightarrow \hat{W}_{I \cup J}^1$ has a right homotopy inverse, so ι^1 must be nullhomotopic. Notice that with respect to the homeomorphisms $\hat{W}_{I'}^1 \cong \Sigma^{|I'|} \hat{W}_{I'}$ given in Section 2.4, ι^1 is the $(|I \cup J|)$ -fold suspension of the coordinate-wise inclusion $\iota^0: \hat{W}_{I \cup J} \rightarrow \hat{W}_I \wedge \hat{W}_J$, and with respect to the homeomorphisms (17), ι^0 is just the suspended inclusion $\Sigma|K_{I \cup J}| \xrightarrow{\Sigma \iota_{I,J}} \Sigma|K_I * K_J|$. Thus, we see that $\Sigma^{|I \cup J|+1} \iota_{I,J}$ is nullhomotopic. $\square$

In a follow-up paper [5], we show that moment-angle complexes over $\frac{n}{3}$ -neighbourly simplicial complexes are homotopy Golod if and only if they are weakly coherently homotopy Golod, and as such, if and only if they are co- H -spaces.

Next, we strengthen a desuspension of the homotopy Golod condition so that it is sufficient for $(D^2, S^1)^K$ to be a co- H -space, though possibly no longer necessary. This is the *coherent homotopy Golod* condition defined below. It will be implicit in the proof of Theorem 3.11 - using Corollary 2.14 - that the coherent homotopy Golod condition implies the weak coherent homotopy Golod condition. Also, the following generalisation of the Golod and homotopy Golod conditions - and in turn the coherent homotopy Golod condition - to a relative context will be useful: if K is a subcomplex of L , and $I, J \not\subseteq [n]$ are disjoint, we may ask if the inclusions $\iota_{I,J}: |K_{I \cup J}| \rightarrow |L_I * L_J|$ induce a trivial map on cohomology or are nullhomotopic after some suspension.

In defining the coherent homotopy Golod condition, recall that we want the inclusions $\Sigma \iota_{I,J}$ to be nullhomotopic for disjoint non-empty $I, J \not\subseteq [n]$, and at the same time we want the coherence conditions between these nullhomotopies to hold as outlined in the introduction. These properties are encoded as follows.

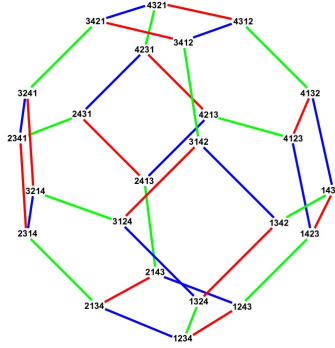


FIGURE 1. 1-skeleton of an order 4 permutohedron

Consider the delta set $\mathcal{K}_n = \{\mathcal{F}_0, \dots, \mathcal{F}_{n-1}\}$, whose set of $(m-2)$ -faces $\mathcal{F}_{m-2}$ is the set of all length m ordered partitions $(I_1, \dots, I_m)$ of $[n]$, with each $I_j \neq \emptyset$, and whose face maps $d_i: \mathcal{F}_{m-2} \rightarrow \mathcal{F}_{m-3}$ are given by $d_i((I_1, \dots, I_m)) = (I_1, \dots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \dots, I_m)$ for $1 \leq i \leq m-1 < n$. These satisfy the required identity $d_i \circ d_j = d_{j-1} \circ d_i$ when $i < j$. Since each ordered partition $\mathcal{S} \in \mathcal{F}_{m-2}$ is uniquely determined by its $(m-3)$ -faces $d_1(\mathcal{S}), \dots, d_{m-1}(\mathcal{S})$, then $\mathcal{K}_n$ is a simplicial complex of dimension $n-2$. We will see that it is a triangulation of S^{n-2} . In fact, $\mathcal{K}_n$ is the dual of the order n permutohedron.

We will from now on assume K is a subcomplex of L on vertex set $[n]$, L has no ghost vertices, though K possibly has less than n vertices if $K \neq L$. If $K = L$, then K also has no ghost vertices. We fix $W' = Z_L(D^1, S^0)$, in which case $\hat{W}' \cong \Sigma|L|$. If $K = L$, then $W' = W$.

Let $\text{Cone}(\Sigma|K|) = [0, 1] \times \Sigma|K| / \sim$ under the identifications $(1, x) \sim (t, *_{-1}) \sim *$.

Definition 3.6. The pair (L, K) on vertex set $[n]$ is *coherently homotopy Golod* if L is a single vertex or $K = \emptyset$, or (recursively) $(L \setminus \{i\}, K \setminus \{i\})$ is coherently homotopy Golod on vertex set $[n] - \{i\}$ for each $i \in [n]$, and there is a map

$$\bar{\Psi}_{L,K} : |\mathcal{K}_n| \times \text{Cone}(\Sigma|K|) \longrightarrow \Sigma|\Delta^{n-1}|$$

such that for any $\gamma \in |\mathcal{K}_n|$, $\bar{\Psi}_{L,K}(\gamma, *)$ is the basepoint $*_{-1} \in \Sigma|\Delta^{n-1}|$, and:

- (1) the restriction of $\bar{\Psi}_{L,K}$ to $\{\gamma\} \times (\{0\} \times \Sigma|K|)$ is the suspended inclusion $\Sigma|K| \longrightarrow \Sigma|L| \subseteq \Sigma|\Delta^{n-1}|$;
- (2) if $\gamma \in |S|$ for some $S = (I_1, \dots, I_m) \in \mathcal{F}_{m-2}$, then $\bar{\Psi}_{L,K}$ maps $\{\gamma\} \times \text{Cone}(\Sigma|K|)$ to a subspace of $\Sigma|L_{I_1} * \dots * L_{I_m}|$ in $\Sigma|\Delta^{n-1}|$.

If $K = L$, we simply say K is *coherently homotopy Golod*, and denote $\bar{\Psi}_K = \bar{\Psi}_{K,K}$.

We will give two more equivalent definitions of coherent homotopy Golodness, which are a bit less directly related to the Golod condition, but which from our standpoint will be more workable. Recall that if $S = (s_1, \dots, s_n)$ is any sequence of real numbers, then $[n]_S$ is the ordered partition $(I_1, \dots, I_m)$ of $[n]$, where $I_j = \{\ell \in [n] \mid s_\ell = s'_j\}$, s'_j is the j^{th} smallest element in the set $\{s_1, \dots, s_n\}$, and $m = |\{s_1, \dots, s_n\}|$ is the number of distinct elements in S .

Definition 3.7 (Second definition of coherent homotopy Golodness). The pair (L, K) on vertex set $[n]$ is *coherently homotopy Golod* if L is a single vertex or $K = \emptyset$, or (recursively) $(L \setminus \{i\}, K \setminus \{i\})$ is coherently homotopy Golod on vertex set $[n] - \{i\}$ for each $i \in [n]$, and for some choice of vertex ordering there is a basepoint preserving map

$$\Psi_{L,K} : \Sigma^n|K| \longrightarrow \Sigma|\Delta^{n-1}|$$

such that for any $y = (t_1, \dots, t_{n-1}) \in \frac{[-1,1]^{\times(n-1)}}{\partial([-1,1]^{\times(n-1)})}$ that is not the basepoint, $\Psi_{L,K}$ maps the subspace $\{y\} \wedge \Sigma|K|$ of $\Sigma^n|K|$ as follows:

- (1) when $t_1 = \dots = t_{n-1} = 0$, the restriction of $\Psi_{L,K}$ to $\{y\} \wedge \Sigma|K|$ is the suspended inclusion $\Sigma|K| \longrightarrow \Sigma|L| \subseteq \Sigma|\Delta^{n-1}|$;
- (2) letting $S_y = (t_1, \dots, t_{n-1}, 0)$ and $(I_1, \dots, I_m) = [n]_{S_y}$, $\Psi_{L,K}$ maps $\{y\} \wedge \Sigma|K|$ to a subspace of $\Sigma|L_{I_1} * \dots * L_{I_m}|$ in $\Sigma|\Delta^{n-1}|$.

For the base case where K is a single vertex, $\Psi_{L,K}$ can be taken as the constant map. If $K = L$, we simply say K is *coherently homotopy Golod*, and denote $\Psi_K = \Psi_{K,K}$.

Remark: If K is on vertex set $J \subseteq [n]$, $|J| = m$, recall the inclusion $\Sigma|K| \longrightarrow \Sigma|\Delta^{n-1}|$ is up to homeomorphism the inclusion $\hat{W} \longrightarrow (D^1)^{\wedge n}$, given as the restriction of the inclusion $(D^1)^{\wedge m} \longrightarrow (D^1)^{\wedge n}$ sending $(x_1, \dots, x_m) \mapsto (x'_1, \dots, x'_n)$, where $x'_i = x_j$ when i is the j^{th} largest element in J , otherwise $x'_i = 1$.

Definition 3.8 (Third definition of coherent homotopy Golodness). Same as the second definition, but with $\Sigma|\Delta^{n-1}|$ replaced with $(D^1)^{\wedge n}$, the subspaces $\Sigma|L_{I_1} * \dots * L_{I_m}|$, $\Sigma|L|$, and $\Sigma|K|$ of $\Sigma|\Delta^{n-1}|$ replaced with the subspaces $\hat{W}'_{I_1} \wedge \dots \wedge \hat{W}'_{I_m}$, $\hat{W}'$, and $\hat{W}$ of $(D^1)^{\wedge n}$, and $\Psi_{L,K}$ becoming a map

$$\Psi_{L,K} : \Sigma^{n-1}\hat{W} \longrightarrow (D^1)^{\wedge n}.$$

If $K = L$, then $\hat{W}'_I = \hat{W}_I \cong \Sigma|K|$, and we denote $\Psi_K = \Psi_{K,K}$.

Proposition 3.9. All three definitions of coherent homotopy Golodness are equivalent.

Proof. Since the homeomorphisms $\Sigma|K| \cong \hat{W}$ and $\Sigma|L_{I_1} * \dots * L_{I_m}| \cong \hat{W}'_{I_1} \wedge \dots \wedge \hat{W}'_{I_m}$ that we defined before are restrictions of the homeomorphism $\Sigma|\Delta^{n-1}| \cong (D^1)^{\wedge n}$, the equivalence of the second and third definitions follows.

Think of S^{n-1} as the unit $(n-1)$ -cube quotient boundary $[-1,1]^{\times(n-1)} / \partial([-1,1]^{\times(n-1)})$. We can write $S^{n-1} = \bigcup_{0 \leq t \leq 1} U_t$, where $U_t = \{(t_1, \dots, t_{n-1}) \in S^{n-1} \mid \max\{|t_1|, \dots, |t_{n-1}|\} = t\}$. When $0 < t < 1$,

U_t is homeomorphic to S^{n-2} , while $U_1 = \{*\}$ and $U_0 = \{(0, \dots, 0)\}$, where $*$ $\in S^{n-1}$ is the basepoint. We construct homeomorphisms

$$\tau_t: |\mathcal{K}_n| \longrightarrow U_t \cong S^{n-2}$$

for $0 < t < 1$ as follows. Given any face $\mathcal{S} = (I_1, \dots, I_m) \in \mathcal{F}_{m-2}$ of $\mathcal{K}_n$, its vertices are the length 2 partitions $\mathcal{S}_i = (I_1 \cup \dots \cup I_i, I_{i+1} \cup \dots \cup I_m)$ of $[n]$ for $i = 1, \dots, m-1$, so we can think of any point $\gamma \in |\mathcal{S}|$ as a formal sum

$$\gamma = s_1 \mathcal{S}_1 + \dots + s_{m-1} \mathcal{S}_{m-1}$$

such that each $s_i \geq 0$ and $s_1 + \dots + s_{m-1} = 1$, and γ is on a boundary face $|d_i(\mathcal{S})| \subseteq |\mathcal{S}|$ if and only if $s_i = 0$, in which case the summand $s_i \mathcal{S}_i$ vanishes. Given $j \in [n]$, let $j_{\mathcal{S}}$ be the integer such that $j \in I_{j_{\mathcal{S}}}$, and let $t_{j,\gamma} = s_0 + s_1 + \dots + s_{j_{\mathcal{S}}-1}$ where $s_0 = 0$. Notice at least two of these must be distinct, so $\beta_{\gamma} = \max\{|t_{1,\gamma} - t_{n,\gamma}|, \dots, |t_{n-1,\gamma} - t_{n,\gamma}|\} > 0$. Then define τ_t by

$$\tau_t(\gamma) = \left(\frac{t}{\beta_{\gamma}}(t_{1,\gamma} - t_{n,\gamma}), \dots, \frac{t}{\beta_{\gamma}}(t_{n-1,\gamma} - t_{n,\gamma}) \right).$$

One can see that τ_t is well-defined by checking this definition agrees on each boundary face $d_i(\mathcal{S})$. If γ is in the interior of some k -face $|\mathcal{S}'| \subseteq |\mathcal{S}|$ for $k \leq m$, then $[n]_{\tau_t(\gamma) \times \{0\}} = \mathcal{S}'$. Then one can check that τ_t is a homeomorphism with inverse $\tau_t^{-1}: U_t \longrightarrow |\mathcal{K}_n|$ given by

$$\tau_t^{-1}((t_1, \dots, t_{n-1})) = \left(\frac{t'_2 - t'_1}{t'_k - t'_1} \right) \mathcal{S}'_1 + \dots + \left(\frac{t'_k - t'_{k-1}}{t'_k - t'_1} \right) \mathcal{S}'_{k-1}$$

where t'_ℓ is the ℓ^{th} smallest integer in the set $S = \{t_1, \dots, t_{n-1}, 0\}$, $k = |S|$, and $\mathcal{S}'_i$ is the length 2 partition $(I'_1 \cup \dots \cup I'_i, I'_{i+1} \cup \dots \cup I'_k)$, given that $\mathcal{S}' = (I'_1, \dots, I'_k)$ denotes the partition $[n]_{(t_1, \dots, t_{n-1}, 0)}$.

For $t = 0$ and $t = 1$, we define $\tau_0: |\mathcal{K}_n| \longrightarrow U_0$ and $\tau_1: |\mathcal{K}_n| \longrightarrow U_1$ to be the constant maps sending all points to $(0, \dots, 0)$ and $*$ respectively.

Take the quotient space

$$Q = |\mathcal{K}_n| \times \text{Cone}(\Sigma|K|) / \sim$$

under the identifications $(\gamma, (0, x)) \sim (\gamma', (0, x)) \sim (0, x)$ and $(\gamma, *) \sim *$ for any $\gamma, \gamma' \in |\mathcal{K}_n|$ and $x \in \Sigma|K|$. Let

$$\tau_K: Q \longrightarrow S^{n-1} \wedge \Sigma|K| \cong \Sigma^n|K|$$

be given by mapping $(\gamma, (t, x)) \mapsto (\tau_t(\gamma), x)$. This is a homeomorphism, with inverse

$$\tau_K^{-1}: S^{n-1} \wedge \Sigma|K| \longrightarrow Q$$

given by mapping $(0, \dots, 0, x) \mapsto (0, x)$, $*$ $\sim$ $(*, x) \mapsto *$, and $(t_1, \dots, t_{n-1}, x) \mapsto (\tau_t^{-1}(t_1, \dots, t_{n-1}), (t, x))$ when $(t_1, \dots, t_{n-1}) \in U_t$ for some $0 < t < 1$. From the first definition of coherent homotopy Golodness, we see $\bar{\Psi}_{L,K}$ factors as

$$\bar{\Psi}_{L,K}: |\mathcal{K}_n| \times \text{Cone}(\Sigma|K|) \xrightarrow{\text{quotient}} Q \xrightarrow{\hat{\Psi}_{L,K}} \Sigma|\Delta^{n-1}|.$$

Thus, given $\bar{\Psi}_{L,K}$ exists as in the first definition, we can construct $\Psi_{L,K}$ as in the second definition by taking it as the composite $\hat{\Psi}_{L,K} \circ \tau_K^{-1}$. Conversely, given $\Psi_{L,K}$ exists, we can define $\hat{\Psi}_{L,K} = \Psi_{L,K} \circ \tau_K$, and then take $\bar{\Psi}_{L,K}$ to be the above composite of the quotient map and $\hat{\Psi}_{L,K}$. Therefore, the first and second definitions are equivalent. $\square$

Notice each parameter t_i in the second and third definition of coherent homotopy Golodness corresponds to the vertex $\{i\}$ in L . But there are n vertices and only $n-1$ parameters t_i , the vertex $\{n\}$ being treated exceptionally here. Then both the second and third definitions depend on the choice of ordering of vertices in K and L by elements in $[n]$. The first definition of coherent homotopy Golodness is however independent of vertex ordering, meaning it holds for all orderings if it holds for one. Moreover, in the proof of Proposition 3.9 the argument that the first definition implies the second and third works for any choice of ordering of vertices. We can thus go from the

second or third definitions to the first definition, change the ordering, and then come back again with a new map $\Psi_{L,K}$ corresponding to the new ordering. Therefore:

Corollary 3.10. *If the second or third definitions of coherent homotopy Golodness hold for one vertex ordering, then they hold for all vertex orderings.* $\square$

We will use the second and third definitions of coherent homotopy Golodness from now on.

Theorem 3.11. *If K is coherently homotopy Golod, then $(D^2, S^1)^K$ is a co- H -space.*

Proof. Induct on number of vertices. The base case is trivial. Assume the statement holds for simplicial complexes on less than n vertices. Since by definition, each K_I is coherently homotopy Golod when K is, by our inductive assumption W_I^1 is a co- H -space for $|I| < n$. Then using Proposition 1.9, the quotient map $W_I^1 \rightarrow \hat{W}_I^1$ has a right homotopy inverse for $|I| < n$, and to show W^1 is a co- H -space, it remains to show that $W^1 \rightarrow \hat{W}^1$ has a right homotopy inverse.

Given our map $\Psi_K: \Sigma^{n-1} \hat{W} \rightarrow (D^1)^{\wedge n}$, define a basepoint preserving map $g: \Sigma^{n-1} \hat{W} \rightarrow \mathcal{D}_A(W)$ by mapping a point $\sigma = (t_1, \dots, t_{n-1}, x)$ by

$$g(\sigma) = (t_1, \dots, t_{n-1}, 0; \Psi_K(\sigma)).$$

The composite of g and the inclusion $q: \mathcal{D}_A(W) \xrightarrow{\iota} \mathbb{C}(\iota_{A,n})$ is homotopic to the homotopy equivalence $\Sigma^{n-1} \hat{W} \xrightarrow{\phi} \mathbb{C}(\iota_{A,n})$ from Lemma 2.16 as follows. Let $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$. Since no more than $n-1$ particles in $g(\sigma)$ collide at a point when $\beta > 0$, $g(\sigma) \subseteq \mathcal{D}_A(W_{n-1})$ when $\beta > 0$, so we can homotope $q \circ g(\sigma) = (0, g(\sigma))$ to $(\beta, g(\sigma))$ inside $\mathbb{C}(\iota_{A,n})$ preserving the basepoint by homotoping 0 to β linearly. Then we homotope $(\beta, g(\sigma))$ to $\phi(\sigma) = (\beta, (t_1, \dots, t_{n-1}, 0; x))$ via the basepoint preserving homotopy

$$H_t(\sigma) = \begin{cases} * & \text{if } \beta = 1; \\ (\beta, (t_1, \dots, t_{n-1}, 0; \Psi_K((1-t)t_1, \dots, (1-t)t_{n-1}, x))) & \text{if } \beta < 1. \end{cases}$$

Notice $H_t(\sigma)$ is well-defined when $(t_1, \dots, t_{n-1})$ is on the boundary of $[-1, 1]^{\times(n-1)}$, since then $\beta = 1$, and is continuous as $(t_1, \dots, t_{n-1})$ approaches the boundary of $[-1, 1]^{\times(n-1)}$, since then β approaches 1, so $(\beta, (t_1, \dots, t_{n-1}, 0; \Psi_K((1-t)t_1, \dots, (1-t)t_{n-1}, x)))$ approaches the basepoint in $\mathbb{C}(\iota_{A,n})$. Thus, since ϕ is a homotopy equivalence, the composite of g and $\mathcal{D}_A(W) \xrightarrow{q} \mathbb{C}(\iota_{A,n})$ is a homotopy equivalence, so q has a right homotopy inverse, and then by Corollary 2.14, so does $W^1 \rightarrow \hat{W}^1$. $\square$

Next we show that two general classes of Golod complexes - whose moment angle complexes are co- H -spaces - are in fact coherently homotopy Golod. Before getting to these, the next example will be useful.

Example 3.12. Suppose K with no ghost vertices is coherently homotopy Golod on vertex set $[n]$, and let $L = K \cup \{n+1\}$ be K with a disjoint vertex $\{n+1\}$ added. Then the pair (L, K) is coherently homotopy Golod on vertex set $[n+1]$ as follows.

Assume this is true for complexes K with less than n vertices. The base case $K = \emptyset$ holds by definition. Since $L \setminus \{i\} = K \setminus \{i\} \cup \{n+1\}$ when $i \neq n+1$, then by induction each $(L \setminus \{i\}, K \setminus \{i\})$ is coherently homotopy Golod on vertex set $[n+1] - \{i\}$, while K being coherently homotopy Golod implies by definition that $(L \setminus \{n+1\}, K \setminus \{n+1\}) = (K, K)$ is. Thus we are left to show the existence of the map $\Psi_{L,K}$.

As before, $W' = Z_L(D^1, S^0)$ and $W = Z_K(D^1, S^0)$. Given our map $\Psi_K: \Sigma^{n-1} \hat{W} \rightarrow (D^1)^{\wedge n}$ from the third definition of coherent homotopy Golodness of K , define $\Psi_{L,K}: \Sigma^n \hat{W} \rightarrow (D^1)^{\wedge(n+1)}$ by mapping the basepoint to the basepoint, and a non-basepoint $\sigma = (t_1, \dots, t_{n-1}, s, (x_1, \dots, x_n)) \in \Sigma^n \hat{W}$ to

$$\Psi_{L,K}(\sigma) = (x'_1, \dots, x'_n, y),$$

where $(x'_1, \dots, x'_n) = \Psi_K(t_1, \dots, t_{n-1}, (x_1, \dots, x_n))$,

$$y = \begin{cases} 1 & \text{if } \alpha \leq s \leq \beta; \\ 2\left(\frac{1-s}{1-\beta}\right) - 1 & \text{if } \beta < s < 1; \\ 2\left(\frac{1+s}{1+\alpha}\right) - 1 & \text{if } -1 < s < \alpha; \\ -1 & \text{if } |s| = 1, \end{cases}$$

and $\alpha = \min\{t_1, \dots, t_{n-1}, 0\}$ and $\beta = \max\{t_1, \dots, t_{n-1}, 0\}$. Notice there is a discontinuity here for y as both s and β approach 1, or s and α approach -1 . But this does not affect the continuity of $\Psi_{L,K}$ since $(\Psi_K$ being basepoint preserving) $(x'_1, \dots, x'_n)$ approaches the basepoint as α approaches -1 or β approaches 1, so $\Psi_{L,K}(\sigma)$ approaches the basepoint here. Likewise, $\Psi_{L,K}(\sigma)$ approaches the basepoint if (1) some x_i approaches -1 (since some x'_j approaches -1 , since Ψ_K is basepoint preserving); or (2) s approaches 1 or -1 and β or α respectively do not (since y approaches -1). In any case, $\Psi_{L,K}(\sigma)$ approaches the basepoint as σ approaches the basepoint.

Since Ψ_K satisfies the first condition of coherent homotopy Golodness, $\Psi_{L,K}(\sigma) = (x'_1, \dots, x'_n, y) = (x_1, \dots, x_n, 1)$ when the t_i 's and s are zero, so $\Psi_{L,K}$ restricts to the inclusion $\hat{W} \rightarrow \hat{W}'$ here, thus also satisfies the first condition.

Note $\hat{W}'_I = \hat{W}_I$ if $\{n+1\} \notin I$. Let $\mathcal{S} = (t_1, \dots, t_{n-1}, 0)$ and $(I_1, \dots, I_m) = [n]_{\mathcal{S}}$. Since Ψ_K satisfies the second condition of coherent homotopy Golodness, $(x'_1, \dots, x'_n)$ is in $A = \hat{W}_{I_1} \wedge \dots \wedge \hat{W}_{I_m}$. Let $\mathcal{S}' = (t_1, \dots, t_{n-1}, s, 0)$ and $(I'_1, \dots, I'_{m'}) = [n+1]_{\mathcal{S}'}$. If $s = 0$ or $s = t_i$ for some i , then $m = m'$, $I'_k = I_k \cup \{n+1\}$ for some k , $I'_i = I_i$ for $i \neq k$, and $y = 1$ (since then $\alpha \leq s \leq \beta$). In any case, each $I_j \subseteq I'_j$. Thus $\Psi_{L,K}(\sigma) = (x'_1, \dots, x'_n, 1)$ is in the subspace A of $A' = \hat{W}'_{I'_1} \wedge \dots \wedge \hat{W}'_{I'_{m'}}$ given on each factor by the inclusions $\hat{W}_{I_i} \rightarrow \hat{W}'_{I'_i}$ as described in Remark 3, these being equalities when $i \neq k$. On the other hand, if $s \neq 0$ and $s \neq t_i$ for each i , then $m' = m+1$, $I'_k = \{n+1\}$ for some k , and $I'_i = I_i$ and $I'_j = I_{j-1}$ for $i < k < j$, so $\hat{W}_{I_k} = D^1$, $\hat{W}_{I_i} = \hat{W}_{I'_i}$, and $\hat{W}_{I_{j-1}} = \hat{W}_{I'_j}$, for $i < k < j$. Then $\Psi_{L,K}(\sigma)$ is in the (homeomorphic) subspace $A \wedge D^1$ of A' . Thus, we see $\Psi_{L,K}$ satisfies the second condition of coherent homotopy Golodness.

Example 3.13. Recall from [23] that a complex K on vertex set $[n]$ (without ghost vertices) is *extractible* if either $K \setminus \{i\}$ is a simplex for some $i \in [n]$, or else (recursively) $K \setminus \{i\}$ is extractible for each $i \in [n]$ and the wedge sum of inclusions $\bigvee_{i \in [n]} \Sigma|K \setminus \{i\}| \rightarrow \Sigma|K|$ has a right homotopy inverse

$$s: \Sigma|K| \rightarrow \bigvee_{i \in [n]} \Sigma|K \setminus \{i\}|.$$

We show that K is coherently homotopy Golod by inducting on number of vertices. Suppose any extractible complex on less than n vertices is coherently homotopy Golod, the base case being trivial. Then since each $K \setminus \{i\}$ is extractible when K is, each $K \setminus \{i\}$ is coherently homotopy Golod on vertex set $[n] - \{i\}$ by induction. Thus, to show K is coherently homotopy Golod, we are left only to construct the map $\Psi_K: \Sigma^{n-1}\hat{W} \rightarrow (D^1)^{\wedge n}$ (or $\Psi_K: \Sigma^n|K| \rightarrow \Sigma|\Delta^{n-1}|$ from the second definition).

First consider the case where $K \setminus \{i\}$ is a simplex for some $i \in [n]$. Without loss of generality, assume $i = n$. Let $W = (D^1, S^0)^K$ and define $\Psi_K: \Sigma^{n-1}\hat{W} \rightarrow (D^1)^{\wedge n}$ by mapping $\sigma = (t_1, \dots, t_{n-1}, (x_1, \dots, x_n))$ to

$$\Psi_K(\sigma) = (x'_1, \dots, x'_{n-1}, x_n).$$

where $x'_i = (1 - |t_i|)x_i - |t_i|$. The last coordinate x_n that is unchanged corresponds to the vertex $\{n\}$. This clearly satisfies the first condition of coherent homotopy Golodness. One can see that it satisfies the second condition precisely because K_I is a simplex for any $I \subseteq [n] - \{n\}$, that is, $\hat{W}_I = (D^1)^{\wedge |I|}$. So for the second condition to hold, each of the first $n-1$ coordinates x'_i of $\Psi_K(\sigma)$ still has the freedom to take on any value in D^1 , as long as the corresponding t_i is non-zero.

For the general case, let $K^i = K \setminus \{i\}$ and take the disjoint union $L^i = (K^i) \cup \{i\}$. Since by induction each K^i is coherently homotopy Golod on vertex set $[n] - \{i\}$, by Example 3.12 the pair

(L^i, K^i) is coherently homotopy Golod on vertex set $[n]$. Let $\Psi_{L^i, K^i}: \Sigma^n |K^i| \rightarrow \Sigma |\Delta^{n-1}|$ be the associated map. By Corollary 3.10 we can assume each Ψ_{L^i, K^i} has been taken so that $\{n\}$ is the exceptional vertex, and each of the first $n-1$ vertices $\{i\} \in L^i$ correspond to the parameter t_i in the second definition of coherent homotopy Golodness. Consider the map

$$\Psi'_K: \Sigma^n |K| \xrightarrow{\Sigma^{n-1}s} \bigvee_{i \in [n]} \Sigma^n |K^i| \xrightarrow{\vee \Psi_{L^i, K^i}} \bigvee_{i \in [n]} \Sigma |\Delta^{n-1}| \rightarrow \Sigma |\Delta^{n-1}|$$

where the last map is the fold map. Notice that Ψ'_K satisfies the second condition of coherent homotopy Golodness since each Ψ_{L^i, K^i} does, since the first map $\Sigma^{n-1}s$ and second map are $(n-1)$ -fold suspensions (thus, do not change the first $n-1$ coordinates t_i), and since each $|L_{I_1}^i * \dots * L_{I_m}^i|$ is a subspace of $|K_{I_1} * \dots * K_{I_m}|$ in $|\Delta^{n-1}|$ (since K has no ghost vertices). However, Ψ'_K does not necessarily satisfy the first condition. Instead, as each Ψ_{L^i, K^i} satisfies the first condition of coherent homotopy Golodness and $\Sigma^{n-1}s$ is an $(n-1)$ -fold suspension, Ψ'_K maps the subspace $\{(0, \dots, 0)\} \wedge \Sigma |K|$ to $\Sigma |K|$ in $\Sigma |\Delta^{n-1}|$ via the composite $\Sigma |K| \xrightarrow{s} \bigvee_i \Sigma |K \setminus \{i\}| \rightarrow \Sigma |K|$, thus the restriction of Ψ'_K to this subspace is homotopic to the identity. Denote this homotopy by $H: \Sigma |K| \times [0, 1] \rightarrow \Sigma |K|$. Pick some $0 < \varepsilon < 1$. Let

$$\chi(t) = \begin{cases} 0 & \text{if } |t| \leq \varepsilon; \\ \frac{t-\varepsilon}{1-\varepsilon} & \text{if } t > \varepsilon; \\ \frac{t+\varepsilon}{1-\varepsilon} & \text{if } t < -\varepsilon, \end{cases}$$

and for any $\sigma = (t_1, \dots, t_{n-1}, x) \in \Sigma^n |K|$, where $x \in \Sigma |K|$, let $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$, and

$$t_\sigma = \begin{cases} \frac{\varepsilon-\beta}{\varepsilon} & \text{if } \beta \leq \varepsilon; \\ 0 & \text{if } \beta > \varepsilon. \end{cases}$$

Note $|\chi(t)|$ approaches 1 as $|t|$ approaches 1. Define $\Psi_K: \Sigma^n |K| \rightarrow \Sigma |\Delta^{n-1}|$ by

$$\Psi_K(\sigma) = \Psi'_K(\chi(t_1), \dots, \chi(t_n), H_{t_\sigma}(x)).$$

Now Ψ_K satisfies the first condition of coherent homotopy Golodness. Since $\chi(t_i) \leq \chi(t_j)$ when $t_i < t_j$, $\chi(t_i) = \chi(t_j)$ when $t_i = t_j$, and Ψ_K is defined in terms of H_{t_σ} (which maps to $\Sigma |K| \subseteq \Sigma |\Delta^{n-1}|$) and Ψ'_K (which satisfies the second condition of coherent homotopy Golodness), we see Ψ_K also satisfies the second condition.

Example 3.14. Define a simplicial complex K on vertex set $[n]$ to be *m-neighbourly* if every subset of at most m vertices in $[n]$ is a face in K . We say K is *neighbourly* if it is $\lfloor \frac{n}{2} \rfloor$ -neighbourly.

Suppose K is neighbourly. Since $|I| \leq \lfloor \frac{n}{2} \rfloor$ or $|J| \leq \lfloor \frac{n}{2} \rfloor$ for any disjoint non-empty $I, J \subsetneq [n]$, then at least one of $|K_I|$ or $|K_J|$ is a simplex, implying $|K_I * K_J| \simeq \Sigma |K_I| \wedge |K_J|$ is contractible. Therefore each inclusion $\Sigma |K| \xrightarrow{\Sigma \iota_{I,J}} \Sigma |K_I * K_J|$ is nullhomotopic, implying K is Golod. We show K is coherently homotopy Golod by inducting on number of vertices as follows.

Suppose any neighbourly complex on less than n vertices is coherently homotopy Golod. Since each $K \setminus \{i\}$ is neighbourly when K is, $K \setminus \{i\}$ is coherently homotopy Golod by induction. Now define

$$\Psi_K: \Sigma^{n-1} \hat{W} \rightarrow (D^1)^{\wedge n}$$

by mapping the basepoint to the basepoint, and a non-basepoint $\sigma = (t_1, \dots, t_{n-1}, (x_1, \dots, x_n))$ to

$$\Psi_K(\sigma) = (f_1(x_1), \dots, f_n(x_n)),$$

where $\beta = \max\{|t_1|, \dots, |t_{n-1}|, 0\}$,

$$f_i(x_i) = \begin{cases} (1 - 2\alpha_i\beta)(x_i + 1) - 1 & \text{if } 0 \leq \alpha_i < \frac{1}{2}; \\ (1 - \beta)(x_i + 1) - 1 & \text{if } \alpha_i \geq \frac{1}{2}, \end{cases}$$

and (letting $t_n = 0$)

$$\alpha_i = \min \left\{ \sum_{j \in S} |t_i - t_j| \mid S \subseteq [n] - \{i\} \text{ and } |S| = \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

We must show Ψ_K is continuous. Any possible discontinuity happens when β approaches 1, in other words, when $(t_1, \dots, t_{n-1})$ approaches the boundary of $[-1, 1]^{*(n-1)}$, or when $x = (x_1, \dots, x_n)$ approaches the basepoint. To see that there is no issue here, notice $S \cap S'$ is non-empty for any $S \subseteq [n] - \{i\}$ and $S' \subseteq [n] - \{n\}$ such that $i \neq n$ and $|S| = |S'| = \left\lfloor \frac{n}{2} \right\rfloor$, implying $\max\{\alpha_n, \alpha_i\} \geq \frac{|t_i - t_n|}{2}$. So if $|t_i|$ approaches 1 for some $i \neq n$, since t_n is always 0, at least one of α_n or α_i approaches some value $L \geq \frac{1}{2}$, so one of $f_i(x_i)$ or $f_n(x_n)$ approaches -1 . Also, since $f(x_i)$ approaches -1 as x_i does, $\Psi_K(\sigma)$ approaches the basepoint when x does. In any case, $\Psi_K(\sigma)$ approaches the basepoint as σ approaches the basepoint.

Clearly Ψ_K satisfies the first condition of coherent homotopy Golodness. Since $\hat{W}_I = (D^1)^{\wedge |I|}$ whenever $|I| \leq \left\lfloor \frac{n}{2} \right\rfloor$, in order for Ψ_K satisfy the second coherent homotopy Golod condition, for each i , $f_i(x_i)$ has freedom to taken on any value in $[-1, 1]$ whenever there are no more than $\left\lfloor \frac{n}{2} \right\rfloor$ values of j for which $t_j = t_i$, so defining f_i as we did presents no difficulties here. On the other hand, when there are strictly more than $\left\lfloor \frac{n}{2} \right\rfloor$ values of j for which $t_j = t_i$, we have $f_i(x_i) = x_i$ and each $f_j(x_j) = x_j$ (since $\alpha_i = \alpha_j = 0$). So because $x = (x_1, \dots, x_n) \in \hat{W}$, the restriction of x to the coordinates corresponding to x_i and each x_j is an element in some $\hat{W}_I$ (this does not give a continuous map $\hat{W} \rightarrow \hat{W}_I$), so there is no issue here either. Thus Ψ_K satisfies the second condition, and K is coherently homotopy Golod, therefore a co- H -space.

We note that another argument showing $\left\lfloor \frac{n}{2} \right\rfloor$ -neighbourly moment-angle complexes are co- H -spaces appears in [5], this time using the Freudenthal suspension theorem.

Example 3.15. Let $K = (\partial\Delta^2 * \partial\Delta^1) \cup_{\partial\Delta^1} \Delta^1$ on vertex set $[5]$. Since every pair of distinct vertices in K is connected by an edge, K is neighbourly, and therefore coherently homotopy Golod. But $H_2(|K|) \cong \mathbb{Z}$, while $H_2(|K \setminus \{i\}|) = 0$ for each $i \in [5]$, so K cannot be extractible.

Question 3.16. Do any of the following implications hold: (1) K Golod $\Rightarrow K$ homotopy Golod; (2) K homotopy Golod $\Rightarrow K$ weakly coherently homotopy Golod; (3) K weakly coherently homotopy Golod $\Rightarrow K$ coherently homotopy Golod;

Conjecture 1.1 is false if (1) is false, and true if both (1) and (2) are true.

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