Spin structures of flat manifolds of diagonal type

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Abstract

We give a novel and purely combinatorial description of Stiefel-Whitney classes of closed flat manifolds with diagonal holonomy representation. Using this description, for each integer d at least two, we construct non-spin closed oriented flat manifolds with holonomy group $\mathbb{Z}_2^d$ with the property that all of their finite proper covers have a spin structure. Moreover, all such covers have trivial Stiefel-Whitney classes. In contrast to the case of real Bott manifolds, this shows that for a general closed flat manifold the existence of a spin structure may not be detected by its finite proper covers.

1 Introduction

In this paper, we shall give a characterization of spin structures on closed flat manifolds with a diagonal holonomy representation. In general, it is a difficult problem to classify spin structures on oriented flat manifolds. If one is successful in defining a spin structure, it naturally leads to the definition of spinor fields, a Dirac operator and η-invariants on the manifolds (see e.g. [4, 10]).

Until now, the main direction of research in this area has been on the relation between the existence of a spin structure and properties of the holonomy group and its representation. For example, an oriented flat manifold has a spin structure if and only if its cover corresponding to a 2-Sylow subgroup of

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the holonomy has a spin structure. Hence, from this point of view, more interesting flat manifolds are the ones with 2-group holonomy. From this class of manifolds, the simplest to describe are the flat manifolds with holonomy group isomorphic to an elementary abelian 2-groups with representation of diagonal type. In fact, one of the first examples of oriented flat manifolds without a spin structure is of this type (see [9]). For more information on this, we refer the reader to [12, §6.3].

Let us recall that every closed flat Riemannian manifold M can be realized as a quotient of a Euclidean space by a discrete subgroup of the group of isometries $\Gamma \subseteq \mathrm{Iso}(\mathbb{R}^n)$ called a *Bieberbach group.* More explicitly, considering the isomorphism $\mathrm{Iso}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes \mathrm{O}(n)$, any element of Γ acts on $\mathbb{R}^n$ by a rotation and by a translation in a canonical way.

By the classical Bieberbach theorems (see [2, 3]), $\mathbb{R}^n \cap \Gamma$ is a lattice and the quotient $G = \Gamma/(\mathbb{R}^n \cap \Gamma)$ is a finite group called the *holonomy group* of M. This leads to an exact sequence:

$$0 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1$$

where π is the quotient map. M is said to be of *diagonal (holonomy) type* if the induced representation $\rho : G \to \mathrm{GL}(n, \mathbb{Z})$ is diagonal. The composition $\Gamma \xrightarrow{\pi} G \hookrightarrow \mathrm{O}(n)$ is just the holonomy representation (see [12, Ch. 2, (2.6)]).

It follows that the holonomy group of any finite cover M' of M is a quotient of a subgroup G' of G. If, in addition, G' is a proper subgroup of G, we say that M' is a *proper* cover.

We denote by $\mathrm{Spin}(n)$ the spin (double covering) group of $\mathrm{SO}(n)$. We also write $\lambda_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ for the covering homomorphism. A *spin structure* on a smooth orientable manifold M is an equivariant lift of its orthonormal frame bundle via the covering λ_n. It is well-known that M has a spin structure if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes (see [6, p. 33-34]).

Let us point out that every closed oriented flat manifold with holonomy group $\mathbb{Z}_2$ has a spin structure (see [8, Theorem 3.1(3)], [10, Proposition 4.2]). For any $d \in \mathbb{N}$, set

$$n(d) = \binom{d+1}{2} + \begin{cases} 2 & d = 0 \bmod 2 \\ 1 & d = 1 \bmod 4 \\ 3 & d = 3 \bmod 4 \end{cases}$$

Our main result is the following theorem.

Theorem. *For any integer $d \geq 2$, there exists a closed oriented flat manifold M_d of rank $n(d)$ with holonomy group $\mathbb{Z}_2^d$ and with the second Stiefel-Whitney class $w_2(M_d) \neq 0$ such that every finite proper cover of M_d has all vanishing Stiefel-Whitney classes.*

The key ingredient of the proof is a purely combinatorial description of Stiefel-Whitney classes of flat manifolds of diagonal holonomy type (see Section 3).

This result is in stark contrast to the case of real Bott manifolds which in part motivated our discussion. Real Bott manifolds are a special type of flat manifolds with diagonal holonomy. By a result of A. Gąsior (see [7, Theorem 1.2]), it follows that a real Bott manifold with holonomy group of even $\mathbb{Z}_2$-rank has a spin structure if and only if all its finite covers with holonomy group $\mathbb{Z}_2^2$ have a spin structure. Our examples show that the general case of diagonal flat manifolds is much more complicated.

We do not know whether all the finite proper covers of the manifolds M_d are parallelizable. Therefore, we ask the following question.

Question. *For any integer $d \geq 2$, does there exist a closed oriented non-spin flat manifold M_d with holonomy group $\mathbb{Z}_2^d$ such that every finite proper cover of M_d is parallelizable?*

Of course, the manifolds constructed in the main theorem are potential candidates.

2 Characterizing diagonal flat manifolds

In this section we give a combinatorial description of diagonal flat manifolds. This language will be essential in our analysis of the Stiefel-Whitney classes of such manifolds.

Suppose we have a short exact sequence of groups

$$0 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1. \tag{2.1}$$

We shall call Γ *diagonal* or *diagonal type* if the image of the *holonomy representation* $\rho : G \to \mathrm{GL}(n, \mathbb{Z})$:

$$\rho(g)(z) = \iota^{-1}(\gamma\iota(z)\gamma^{-1}), \ \ \forall g \in G, \ \pi(\gamma) = g, \gamma \in \Gamma, \forall z \in \mathbb{Z}^n,$$

is a subgroup of the group of diagonal matrices $D \cong \mathbb{Z}_2^n \subseteq \mathrm{GL}(n, \mathbb{Z})$ where

$$D = \{A = [a_{ij}] \in \mathrm{GL}(n, \mathbb{Z}) \ | \ a_{ij} = 0, i \neq j; a_{ii} = \pm 1, 1 \leq i, j \leq n\}.$$

It follows that $G = \mathbb{Z}_2^k$ for some $1 \leq k \leq n - 1$.

Let S^1 be the unit circle in $\mathbb{C}$. As in [11], we consider the automorphisms $g_i : S^1 \to S^1$, given by

$$g_0(z) = z, \ g_1(z) = -z, \ g_2(z) = \bar{z}, \ g_3(z) = -\bar{z}, \ \ \forall z \in S^1. \tag{2.2}$$

Equivalently, with the identification $S^1 = \mathbb{R}/\mathbb{Z}$, for any $[t] \in \mathbb{R}/\mathbb{Z}$ we have:

$$(2.3)\quad g_0([t]) = [t],\ \ g_1([t]) = \left[t + \frac{1}{2}\right],\ \ g_2([t]) = [-t],\ \ g_3([t]) = \left[-t + \frac{1}{2}\right].$$

Let $\mathcal{D} = \langle g_i \mid i = 0,1,2,3\rangle$. It is easy to see that $\mathcal{D} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $g_3 = g_1 g_2$. We define an action $\mathcal{D}^n$ on T^n by

$$(t_1,\dots,t_n)(z_1,\dots,z_n) = (t_1 z_1,\dots,t_n z_n), \tag{2.4}$$

for $(t_1,\dots,t_n) \in \mathcal{D}^n$ and $(z_1,\dots,z_n) \in T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$.

Any subgroup $\mathbb{Z}_2^d \subseteq \mathcal{D}^n$ defines a $(d \times n)$-matrix with entries in $\mathcal{D}$ which in turn defines a matrix A with entries in the set $S = \{0,1,2,3\}$ under the identification $i \leftrightarrow g_i$, $0 \le i \le 3$. Note that the group action in S is defined by this correspondence:

$$\forall_{i,j,k \in S} i + j = k \Leftrightarrow g_i g_j = g_k,$$

hence we can add distinct rows of A to obtain a row vector with entries in S.

Remark 2.1. *Note that if Γ is a diagonal Bieberbach group then it can be realized as a subgroup of $\mathrm{GL}(n+1,\mathbb{Q})$ of matrices of the following form*

$$\Gamma = \left\{ \begin{bmatrix} \rho(g) & s(g) + z \\ 0 & 1 \end{bmatrix} \mid g \in G, z \in \mathbb{Z}^n \right\}$$

where $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ is the holonomy representation defined above and $s\colon G \to \{0,\frac{1}{2}\}^n$ is a map called vector system. If $G \cong \mathbb{Z}_2^d$ is generated by $b_1,\dots,b_d$, I_n is the identity matrix of degree n and e_i is the i-th column of I_n, for $i = 1,\dots,n$, then Γ is generated by the following elements

$$\gamma_j = \begin{bmatrix} \rho(b_j) & s(b_j) \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} I_n & e_i \\ 0 & 1 \end{bmatrix}$$

where $j = 1,\dots,d, i = 1,\dots,n$.

The rows of the matrix A generate a complete set of transversals of $\mathbb{Z}^n$ in Γ as follows: let $1 \le j \le d$, $\rho(b_j) = \mathrm{diag}(X_1,\dots,X_n)$ and $s(b_j) = [x_1,\dots,x_n]^T$, where $X_i \in \{\pm 1\}$ and $x_i \in \{0,\frac{1}{2}\}$ for $i = 1,\dots,n$. Then the corresponding element of $\mathcal{D}^n$ is a n-tuple $(t_1,\dots,t_n)$ of maps from (2.3) defined by

$$\forall_{1 \le i \le n} \forall_{t \in \mathbb{R}} t_i([t]) = [X_i t + x_i].$$

We get that $(t_1,\dots,t_n) = (g_{i_1},\dots,g_{i_n})$ where $i_1,\dots,i_n \in S$ and hence the j-th row of the matrix A is equal to $(i_1,\dots,i_n)$.

The sum of rows j_1 and j_2 of the matrix A corresponds, by the above construction, to the element

$$\begin{bmatrix} \rho(g_{j_1}g_{j_2}) & s(g_{j_1}g_{j_2}) \\ 0 & 1 \end{bmatrix} \in \Gamma.$$

From the discussion in the Remark 2.1 we obtain the following.

Lemma 2.2. *Using the notation of Remark 2.1 we get that*

$$\forall_{1\leq k\leq n} i_k \in \{0,1\} \Leftrightarrow X_k = 1 \text{ and } i_k \in \{2,3\} \Leftrightarrow X_k = -1.$$

Example 2.3. *Let Γ be a group generated by*

$$\gamma_1 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \gamma_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the corresponding matrix A, given by the construction in Remark 2.1 is equal to

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

The sum of the two rows $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ corresponds to the matrix

$$\gamma = \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \Gamma.$$

Note that

$$\gamma = \gamma_1\gamma_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, as stated in Remark 2.1, γ and $\gamma_1\gamma_2$ differ by a lattice element of Γ.

We have the following characterization of the action of $\mathbb{Z}_2^d$ on T^n and the associated orbit space $T^n/\mathbb{Z}_2^d$ via the matrix A.

Lemma 2.4. *Let $\mathbb{Z}_2^d \subseteq \mathcal{D}^n$ and $A \in S^{d\times n}$. Then,*

(i) the action of $\mathbb{Z}_2^d$ on T^n is free if and only if there is 1 in the sum of any distinct collection of rows of A,

(ii) $\mathbb{Z}_2^d$ is the holonomy group of $T^n/\mathbb{Z}_2^d$ if and only if there is either 2 or 3 in the sum of any distinct collection of rows of A.

Proof. Part(i) follows from the fact that g_1 (which corresponds to 1 in S) is the only element in $\mathcal{D}$ that has no fixed points.

For part (ii), let Γ be a group defined by A that fits into the short exact sequence (2.1). Let $\varphi\colon \mathcal{D} \to \mathrm{GL}(1,\mathbb{Z})$ be a homomorphism defined by Lemma 2.2, i.e. $\varphi(g_2) = \varphi(g_3) = -1$. We have the following diagram

$$\mathcal{D} \supset \mathbb{Z}_2^d \xrightarrow{\varphi^n} \rho(G) \xleftarrow{\rho} G,$$

where $\varphi^n = \varphi \times \ldots \times \varphi$. Since the holonomy representation ρ is faithful, $\mathbb{Z}_2^d$ is the holonomy group of $T^n/\mathbb{Z}_2^d = \mathbb{R}^n/\Gamma$ if and only if every its n-tuple contains g_2 or g_3. This is equivalent to the statement that the sum of any distinct collection of rows of A contains 2 or 3. □

When the action of $\mathbb{Z}_2^d$ on T^n defined by (2.4) is free, we will say that the associated matrix A is *free* and we will call it the *defining matrix* of $T^n/\mathbb{Z}_2^d$. In addition, when $\mathbb{Z}_2^d$ is the holonomy group of $T^n/\mathbb{Z}_2^d$, we will say that A is *effective*.

3 Combinatorial Stiefel-Whitney classes

We use defining matrices of diagonal flat manifolds to express their characteristic algebras and Stiefel-Whitney classes using the language introduced in the previous section.

To simplify notation, we identify $i \leftrightarrow g_i$ for $i = 0, 1, 2, 3$. Let us consider the epimorphisms:

$$\alpha, \beta : \mathcal{D} \to \mathbb{F}_2 = \{0, 1\}, \tag{3.1}$$

where the values of α and β on $\mathcal{D}$ are given in the following table:

	0	1	2	3
α	0	1	1	0
β	0	1	0	1

Table 1: α and β on $\mathcal{D}$

For $j = 1, \ldots, n$, and $\mathbb{Z}_2^d \subseteq \mathcal{D}^n$ we define the epimorphisms:

$$\alpha_j : \mathbb{Z}_2^d \subseteq \mathcal{D}^n \xrightarrow{pr_j} \mathcal{D} \xrightarrow{\alpha} \mathbb{F}_2, \;\; \beta_j : \mathbb{Z}_2^d \subseteq \mathcal{D}^n \xrightarrow{pr_j} \mathcal{D} \xrightarrow{\beta} \mathbb{F}_2 \tag{3.2}$$

by:

$$\alpha_j(t_1,\dots,t_n)=\alpha(t_j),\ \ \beta_j(t_1,\dots,t_n)=\beta(t_j).$$

Using definitions of α, β and the translations given in the equation (2.3), we obtain the following lemma.

Lemma 3.1. *Suppose a subgroup $\mathbb{Z}_2^d\subseteq\mathcal{D}^n$ acts freely and effectively on T^n. Then a holonomy representation $\rho\colon\mathbb{Z}_2^d\to\mathrm{GL}(n,\mathbb{Z})$ of the flat manifold $T^n/\mathbb{Z}_2^d$ is given by*

$$\forall_{x\in\mathbb{Z}_2^d}\ \rho(x)=\mathrm{diag}\left((-1)^{(\alpha_1+\beta_1)(x)},\dots,(-1)^{(\alpha_n+\beta_n)(x)}\right).$$

Proof. It is enough to note that the map φ defined in the proof of the Lemma 2.4 is given by the formula

$$g\mapsto(-1)^{(\alpha+\beta)(g)}$$

for every $g\in\mathcal{D}$. □

Since $H^1(\mathbb{Z}_2^d;\mathbb{F}_2)=Hom(\mathbb{Z}_2^d,\mathbb{Z}_2)$ we can view α_j and β_j as 1-cocycles and define:

$$\theta_j=\alpha_j\cup\beta_j\in H^2(\mathbb{Z}_2^d;\mathbb{F}_2),\tag{3.3}$$

where $\cup$ denotes the cup product. It is well-known that

$$H^*(\mathbb{Z}_2^d;\mathbb{F}_2)\cong\mathbb{F}_2[x_1,\dots,x_d]$$

where $\{x_1,\dots,x_d\}$ is a basis of $H^1(\mathbb{Z}_2^d,\mathbb{F}_2)$. Hence, the elements α_j and β_j correspond to:

$$\alpha_j=\sum_{i=1}^d\alpha(pr_j(b_i))x_i,\ \ \beta_j=\sum_{i=1}^d\beta(pr_j(b_i))x_i\in\mathbb{F}_2[x_1,\dots,x_d],\tag{3.4}$$

where $\{b_1,\dots,b_d\}$ is the standard basis of $\mathbb{Z}_2^d$ and $j=1,\dots,n$ (cf. [4, Proposition 1.3]).

Moreover, from definition of the matrix $A\in S^{d\times n}$ we have $A_{i,j}=pr_j(b_i)$ and hence we can write (3.4) and (3.3) as:

$$\alpha_j=\sum_{i=1}^d\alpha(A_{i,j})x_i,\ \ \beta_j=\sum_{i=1}^d\beta(A_{i,j})x_i,\ \ \theta_j^A=\alpha_j\cup\beta_j=\alpha_j\beta_j.\tag{3.5}$$

Next, we will make use of the Lyndon-Hochschild-Serre spectral sequence $\{E_r^{p,q},d_r\}$ associated to the group extension of (2.1). Since Γ is of diagonal type, we have:

$$E_2^{p,q}\cong H^p(\mathbb{Z}_2^d;\mathbb{F}_2)\otimes H^q(\mathbb{Z}^n;\mathbb{F}_2).$$

There is an exact sequence:

$$
\begin{aligned}
(3.6) \qquad 0 \to H^1(\mathbb{Z}_2^d;\mathbb{F}_2) &\xrightarrow{\pi^*} H^1(\Gamma;\mathbb{F}_2) \\
&\xrightarrow{\iota^*} H^1(\mathbb{Z}^n;\mathbb{F}_2) \xrightarrow{d_2} H^2(\mathbb{Z}_2^d;\mathbb{F}_2) \xrightarrow{\pi^*} H^2(\Gamma;\mathbb{F}_2),
\end{aligned}
$$

where d_2 is the transgression and π^* is induced by the quotient map $\pi : \Gamma \to \mathbb{Z}_2^d$ (e.g. [5, Corollary 7.2.3]).

Proposition 3.2. *Suppose $\mathbb{Z}_2^d$ acts freely and diagonally on T^n. Let $M = T^n/\mathbb{Z}_2^d$, $\Gamma = \pi_1(M)$ and consider the associated to the group extension of (2.1). Then*

(i) $\theta_l = d_2(\varepsilon_l)$, $\forall 1 \le l \le n$, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the basis of $H^1(\mathbb{Z}^n, \mathbb{F}_2)$ dual to the standard basis of $\mathbb{Z}^n \otimes \mathbb{F}_2$.

(ii) The total Stiefel-Whitney class of M is

$$w(M) = \pi^*\Big(\prod_{j=1}^{n}(1 + \alpha_j + \beta_j)\Big) \in H^*(\Gamma;\mathbb{F}_2) = H^*(M;\mathbb{F}_2).$$

Proof. By Theorem 2.5(ii) and Proposition 1.3 of [4] and using (2.3), it follows that

$$d_2(\varepsilon_l) = \sum_{A_{il}=1} x_i^2 + \sum_{i\neq j} x_i x_j,$$

where the second sum is taken for such i, j that

$$(A_{il}, A_{jl}) \in \{(1,2),(2,1),(1,3),(3,1),(3,2),(2,3)\}.$$

On the other hand

$$\theta_l = \alpha_l\beta_l = \sum_{i=1}^{d} \alpha(A_{il})\beta(A_{il})x_i^2 + \sum_{1\le i<j\le d} \big(\alpha(A_{il})\beta(A_{jl}) + \alpha(A_{jl})\beta(A_{il})\big)x_i x_j.$$

Comparing coefficients of the above two polynomials finishes the proof of (i).

For the second part of the proposition, note that the image of the holonomy representation φ of M, defined in Lemma 3.1, is a subgroup of the group D of diagonal matrices of $\mathrm{GL}(n,\mathbb{Z})$. Now, let $\{x_1', \dots, x_n'\}$ be the standard basis of $H^1(D, \mathbb{Z}_2)$ (i.e. x_j' checks whether the j-th entry of the diagonal is ± 1). Using Proposition 3.2 of [4] (see also (2.1) of [9]), we have:

$$w(M) = \pi^*\Big(\prod_{j=1}^{n}(1 + \varphi^*(x_j'))\Big).$$

Furthermore, for every $1 \leq l \leq d$ and $1 \leq j \leq n$, we have

$$\varphi^*(x_j')(b_l) = x_j'(\varphi(b_l)) = (\alpha_j + \beta_j)(b_l)$$

and the result follows. $\square$

We observe that by part (i) of Proposition 3.2, the image of the differential d_2 is the ideal generated by θ_j-s:

$$\langle Im(d_2) \rangle = \langle \theta_1, \dots, \theta_n \rangle \subseteq \mathbb{F}_2[x_1, x_2, ..., x_d].$$

Given $A \in S^{d\times n}$, using (3.5), we will set $I_A = \langle \theta_1^A, \dots, \theta_n^A \rangle$ and call it the *characteristic ideal* of A. The quotient $\mathcal{C}_A = \mathbb{F}_2[x_1, ..., x_d]/I_A$ will be the *characteristic algebra* of A. Whenever there is no confusion, we will suppress the subscripts.

Corollary 3.3. *Suppose $\mathbb{Z}_2^d$ acts freely and diagonally on T^n. There is a canonical homomorphism of graded algebras $\phi : \mathcal{C} \to H^*(T^n/\mathbb{Z}_2^d; \mathbb{F}_2)$ such that $\phi([w]) = w(T^n/\mathbb{Z}_2^d)$ where $[w]$ is the class of*

$$w = \prod_{j=1}^{n} (1 + \alpha_j + \beta_j) \in \mathbb{F}_2[x_1, x_2, ..., x_d]. \tag{3.7}$$

Moreover, ϕ is a monomorphism in degree less than or equal to two.

Proof. This follows directly from the exact sequence (3.6), with ϕ induced by the algebra homomorphism $\pi^* : H^*(\mathbb{Z}_2^d; \mathbb{F}_2) \to H^*(\Gamma; \mathbb{F}_2)$. $\square$

Definition 3.4. Given a matrix $A \in S^{d\times n}$, using (3.5), we define the *(combinatorial) Stiefel-Whitney class* of A, denoted $w(A)$, to be the class $[w] \in \mathcal{C}_A$ defined by (3.7).

Corollary 3.5. *Suppose $A \in S^{d\times n}$ is free and $T^n/\mathbb{Z}_2^d$ is the corresponding flat manifold. Then $\phi(w(A)) = w(T^n/\mathbb{Z}_2^d)$.*

Next, we derive several properties of the Stiefel-Whitney classes and of characteristic ideals which will be key to our discussion later on.

Lemma 3.6. *Let $A \in S^{d\times m}$, $B \in S^{d\times n}$ and $[A, B] \in S^{d\times(m+n)}$. Then,*

(i) $w([A, B]) = w(A)w(B)$;

(ii) $I_{[A,B]} = I_A + I_B$;

(iii) if j-column of A has only elements $\{0, 2\}$ or $\{0, 3\}$, then $\theta_j^A = 0$.

Proof. By Definition 3.4, we have:

$$w([A,B]) = \prod_{j=1}^{m}(1+\alpha_j+\beta_j)\prod_{j=m+1}^{m+n}(1+\alpha_j+\beta_j) \in \mathbb{F}_2[x_1,x_2,...,x_d]$$

where

$$\alpha_j = \sum_{i=1}^{d}\alpha(A_{i,j})x_i, \quad \beta_j = \sum_{i=1}^{d}\beta(A_{i,j})x_i, \quad \forall\, 1 \le j \le m$$

and

$$\alpha_j = \sum_{i=1}^{d}\alpha(B_{i,j})x_i, \quad \beta_j = \sum_{i=1}^{d}\beta(B_{i,j})x_i, \quad \forall\, m+1 \le j \le m+n.$$

Therefore, $w([A,B]) = w(A)w(B)$.

To prove (ii), recall that $I_{[A,B]} = \langle\theta_1^{[A,B]},\dots,\theta_n^{[A,B]}\rangle$ with $\theta_j^{[A,B]} = \alpha_j\beta_j$. Note that, $\theta_j^{[A,B]} = \theta_j^A$ when $1 \le j \le m$ and $\theta_j^{[A,B]} = \theta_{j-m}^B$ when $m+1 \le j \le m+n$. Hence. $I_{[A,B]} = I_A + I_B$.

Part (iii) follows from that fact that $\theta_j^A = \alpha_j\beta_j$ and that $\alpha_j = 0$ on $\{0,2\}$ and $\beta_j = 0$ on $\{0,3\}$.

□

4 Proof of main theorem

To define minimal non-spin manifolds we will make use of the following matrices:

1. $A_0 = \begin{bmatrix} I_{(d-1)} \\ r \end{bmatrix} \in S^{d\times(d-1)}$, where $I_{(d-1)}$ is the identity matrix and $r = (1,\dots,1)$.

2. $A_1 = \begin{bmatrix} c_1,\dots,c_{d(d-1)/2}\end{bmatrix} \in S^{d\times d(d-1)/2}$ with columns $c_k = 2e_i + 3e_j$ for all $i<j$ ordered in lexicographical order. Here, e_i denotes the column vector with 1 in the i-th coordinate and 0 everywhere else.

3. Let $A = [A_0, A_1]$, $B = 2(e_1+e_2+...+e_d) \in S^{d\times 1}$, and $C = 2e_1 \in S^{d\times 1}$.

4. Let E be the free matrix

$$E = \begin{cases} [A,B,C,C] & d = 0 \bmod 2 \\ [A,B,B] & d = 1 \bmod 4 \\ A & d = 3 \bmod 4 \end{cases}$$

5. Finally, let $F \in S^{d \times n(d)}$ be the free and effective matrix defined by

$$F = \begin{cases} E & d \neq 3 \bmod 4 \\ [E, C, C, C, C] & d = 3 \bmod 4 \end{cases}$$

Note that

$$n(d) = \binom{d+1}{2} + \begin{cases} 2 & d = 0 \bmod 2 \\ 1 & d = 1 \bmod 4 \\ 3 & d = 3 \bmod 4 \end{cases}$$

Example 4.1. *For $d = 2$ and $d = 3$ the matrix F is equal to*

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 3 & 2 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 & 2 & 2 & 2 & 2 \\ 0 & 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

respectively.

Let σ_i be the i-th elementary symmetric polynomial on variable $\{x_1, \ldots, x_d\}$. Consider the ideal $J \subseteq \mathbb{F}_2[x_1, \ldots, x_d]$ generated by the polynomials

$$x_i^2 + x_j^2 \text{ and } x_i x_j$$

where $i \neq j$.

Lemma 4.2. *The matrix A is free, $I_A = J$ and*

$$w(A) = \left[(1 + \sigma_1)^{d-1}\right] \in \mathcal{C}_A = \mathbb{F}_2[x_1, \ldots, x_d]/J.$$

Proof. The matrix A is clearly free by definition. To see that $J = I_A$, note that, by Lemma 3.6, we have $I_A = I_{A_0} + I_{A_1}$. Recall that $I_{A_0} = \langle \theta_1^{A_0}, \ldots, \theta_{d-1}^{A_0} \rangle$ with $\theta_l^{A_0} = \alpha_l \beta_l$ for $1 \leq l \leq d - 1$. Now, we have:

$$\begin{aligned} \theta_l^{A_0} &= \alpha_l \beta_l \\ &= \sum_{i=1}^{d} \alpha(A_{0il})\beta(A_{0il})x_i^2 + \sum_{1 \leq i < j \leq d} \big(\alpha(A_{0il})\beta(A_{0jl}) + \alpha(A_{0jl})\beta(A_{0il})\big)x_i x_j \\ &= x_l^2 + x_d^2. \end{aligned}$$

Similarly, $I_{A_1} = \langle \theta_1^{A_1}, \dots, \theta_{d(d-1)/2}^{A_1} \rangle$ and $\theta_l^{A_1} = x_i x_j$ for all $1 \leq i < j \leq d$. It is easy to see now that $J = I_{A_0} + I_{A_1}$.

To prove the last claim, we write:

$$\begin{aligned} w(A) =& w(A_0)w(A_1) \\ =& w(A_1) \\ =& [\prod_{i<j}(1 + x_i + x_j)] \\ =& \big[1 + (d-1)\sigma_1 + d(d-2)\sigma_2 + \binom{d-1}{2}\sigma_1^2\big]. \end{aligned}$$

Since $\sigma_2 \in J$, it follows that

$$\begin{aligned} w(A) =& \big[1 + (d-1)\sigma_1 + \binom{d-1}{2}\sigma_1^2\big] \\ =& \big[(1+\sigma_1)^{d-1}\big]. \end{aligned}$$

□

Let $\varphi : P_2 \to \mathbb{F}_2$ be the linear extension of the map given by $\varphi(x_i^2) = 1$ and $\varphi(x_i x_j) = 0$, for $i \neq j$, where P_2 denotes the space of homogeneous polynomials of degree two. We make the following observations.

Lemma 4.3. *Let $J_2 = \{x \in J \mid x$ is an element of degree $2\}$. Then $J_2 = Ker(\varphi)$.*

Proof. The spaces J_2 and $Ker(\varphi)$ have the same basis. □

Lemma 4.4. *We have*

(i) $I_B = 0$ and $w(B) = [1 + \sigma_1]$.

(ii) $I_C = 0$ and $w(C) = [1 + x_1]$.

(iii) The matrix E is free and and $I_E = J$.

Proof. The first two claims follow from Lemma 3.6 (iii) and the formula (3.7) applied to the matrices B and C. For the proof of the last claim, note that, by parts (i), (ii) and Lemma 3.6 (ii), we have that $I_E = I_A$. This finishes the claim, since $I_A = J$. □

Proposition 4.5. *The flat manifold M defined by the matrix E has $w(M) = [1 + x_1^2] \in \mathcal{C}$. In particular, M is oriented, it does not have a spin structure and $w_i(M) = 0$ for all $i > 2$.*

Proof. First, let us observe that $\dim \mathcal{C}_2^E = 1$ and $\mathcal{C}_i^E = 0$ for $i > 2$. In fact, the first formula can be seen from the definition $\mathcal{C}_2 = P_2/J_2$. The second formula follows from noting that any homogeneous polynomial in $\mathbb{F}_2[x_1, \dots, x_d]$ of degree greater than two is in the ideal J.

Let us now calculate the Stiefel-Whitney class $w(M) = w(E)$ of M. We shall consider the following cases.

Case 1 (d is even). We have:

$$\begin{aligned} w(E) =& w(A)w(B)w(C) \\ =& [(1+\sigma_1)^{d-1}(1+\sigma_1)(1+x_1)^2] \\ =& [(1+\sigma_1)^d(1+x_1^2)] \\ =& [(1+\sigma_1^2)^{d/2}(1+x_1^2)]. \end{aligned}$$

Since d is even, σ_1^2 is a sum of even number of squares. Hence, $\sigma_1^2 \in J$ and $w(E) = [1+x_1^2]$. Therefore, $w_i(M) = 0$ for $i \neq 2$ and $w_2(M) = [x_1^2]$. But $x_1^2 \notin J$ because $\varphi(x_1^2) \neq 0$.

Case 2 ($d = 1 \bmod 4$). We have:

$$\begin{aligned} w(E) =& w(A)w(B)^2 \\ =& [(1+\sigma_1)^{d+1}] \\ =& [(1+\sigma_1)^2] \\ =& [1+\sigma_1^2]. \end{aligned}$$

As above, M is orientable and has no spin structure since $\varphi(\sigma_1^2) = d = 1$.

Case 3 ($d = 3 \bmod 4$). We have:

$$\begin{aligned} w(E) =& [(1+\sigma_1)^{d-1}] \\ =& [(1+\sigma_1)^2] \\ =& [1+\sigma_1^2]. \end{aligned}$$

Hence, as above, M is orientable and has no spin structure.

□

Proposition 4.6. *Let $M = T^n/\mathbb{Z}_2^d$ be the flat manifold defined by the matrix E. Let M' be a finite proper cover of M, $\Gamma = \pi_1(M)$, $\Gamma' = \pi_1(M')$ and $i : \Gamma' \to \Gamma$ be the inclusion corresponding to the covering. Suppose $\Gamma'/(\pi_1(T^n) \cap \Gamma') \cong \mathbb{Z}_2^k$ with $k < d$. Then M' has trivial Stiefel-Whitney classes.*

Proof. Let $\mathcal{C} = \mathbb{F}_2[x_1, \dots, x_d]/J$ be the characteristic algebra of M (equivalently, of E) and $\mathcal{C}'$ be the characteristic algebra of M' with characteristic ideal $I_{M'}$. We claim that $\mathcal{C}'_l = 0$ for $l \geq 2$.

To see this, we note that there is a commutative diagram with exact rows:

$$\begin{array}{ccccc} \pi_1(T^n) \cap \Gamma' & \xrightarrow{\iota'} & \Gamma' & \xrightarrow{\pi'} & \mathbb{Z}_2^k \\ \big\downarrow & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ \pi_1(T^n) & \xrightarrow{\iota} & \Gamma & \xrightarrow{\pi} & \mathbb{Z}_2^d \end{array}$$

Combining this with the equation (3.6), yields the commutative diagram:

$$\begin{array}{ccc} H^1(\Gamma; \mathbb{F}_2) & \xrightarrow{d_2} & H^2(\mathbb{Z}_2^d; \mathbb{F}_2) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow{\scriptstyle j^*} \\ H^1(\Gamma'; \mathbb{F}_2) & \xrightarrow{d'_2} & H^2(\mathbb{Z}_2^k; \mathbb{F}_2) \end{array}$$

This shows that

$$j^*(J) = j^*(\langle Im(d_2)\rangle) \subseteq \langle Im(d'_2)\rangle = I_{M'} \subseteq H^*(\mathbb{Z}_2^k; \mathbb{F}_2).$$

Therefore, we get an induced epimorphism of algebras $j^* : \mathcal{C} \to \mathcal{C}'$.

Recall that by Proposition 4.5, $\mathcal{C}_2 = \{0, [x_1^2]\}$. For any $y \in \mathcal{C}_1 \setminus \{0\}$ there is $z \in \mathcal{C}_1$ such that $yz = [x_1^2]$. Suppose otherwise and let $y = [a]$, $a \in \mathbb{F}_2[x_1, \dots, x_d]_1$. If $y\mathcal{C}_1 = \{0\}$, then for any $1 \leq m \leq d$, $ax_m \in J_2 = Ker(\varphi)$. This is impossible since φ corresponds to a non-degenerated symmetric two linear map.

Since $\dim \mathcal{C}_1 = d > k \geq \dim \mathcal{C}'_1$, there exists $y \in \mathcal{C}_1$ such that $j^*(y) = 0$. We can find $z \in \mathcal{C}_1$ so that $yz = [x_1^2] \in \mathcal{C}_2$. Because j^* is an epimorphism and $\mathcal{C}_2$ is one-dimensional, $j^*(yz)$ generates $\mathcal{C}'_2$. But $j^*(yz) = j^*(y)j^*(z) = 0$ and therefore, $\mathcal{C}'_2 = 0$.

Finally, since $\mathcal{C}_l = 0$ for $l > 2$ and i^* is surjection, we obtain the triviality of $\mathcal{C}'_l$ for $l > 2$. This proves our claim and together with Proposition 4.5 finishes the proof. □

We are now ready to prove our main result.

Theorem 4.7. *Suppose M is the flat manifold defined by the matrix F. Then, M is orientable with holonomy group $\mathbb{Z}_2^d$, $w_2(M) \neq 0$ and every finite proper cover of M has all vanishing Stiefel-Whitney classes.*

Proof. Since the matrix F is effective, by Lemma 2.4, we know that the holonomy group is $\mathbb{Z}_2^d$. By Lemmas 3.6 and 4.4, it follows that $I_F = J$,

$\mathcal{C}_F = \mathcal{C}_E$, and $w(F) = w(E) = [1 + x_1^2]$. Hence, M is orientable, but non-spin. The last claim follows from applying the proof of Proposition 4.6 to the manifold M defined by the matrix F in place of E. $\square$

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