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NATURAL ENVIRONMENT RESEARCH COUNCIL

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ABSTRACT

The errors inherent in the harmonic analysis of short tidal records have been investigated using synthetic and real tidal velocity records. The sources of error were identified as the presence of random noise, tidal constituents not included in the analysis and the length of the record. Of these sources of error the record length was shown to be the greatest contributor to the total error. It is also shown that the use of digital filtering prior to analysis to reduce random noise error or additional constituents can also lead to significant errors.

1. INTRODUCTION

One of the most important parameters in any coastal oceanographic study is the specification of the tidal characteristics of the area. Tides in the sea result from the gravitational pull of the moon, the sun and the planets and from local meteorological effects. The effect of the varying gravitational pull due to the elliptical and inclined orbits of the moon about the earth and the earth about the sun is complex but predictable. Consequently that part of the tide produced by these forces can also be predicted to quite a high degree of accuracy. In general the tide-generating force is expressed as a series of harmonic constituents. The periods and amplitudes of some of these constituents which account for over 80% of the tide generating force are shown in Table 1 using data from Defant (1961) - Volume 2.

Component	Symbol	Period (hours)	Amplitude ratio	
Principal lunar	M_2	12.42	100	semi-diurnal
Principal solar	S_2	12.00	46.6	
Larger lunar elliptic	N_2	12.66	19.2	
Luni-solar semi-diurnal	K_2	11.97	12.7	
Luni-solar diurnal	K_1	23.93	58.4	diurnal
Principal lunar diurnal	O_1	25.82	41.5	
Principal solar diurnal	P_1	24.07	19.4	

Table 1: Principal Constituents of Tide Generating Force

The tidal elevation at a given location is commonly defined in terms of the amplitudes and phase (relative to the equilibrium tide at Greenwich) of a harmonic series whose constituents have the same frequencies as the tide generating force. However, non-linear interactions between the different tidal constituents result in additional constituents which must be included for a complete description of tidal elevations. The major contributions from non-linear interactions are the quarter diurnal constituents M_4, S_4 and MS_4.

One problem that has received considerable attention has been, given a tidal record, what are the amplitudes and phases of its component tidal constituents. The most commonly used method of analysis is to fit a harmonic series consisting of a finite number of specified constituents to the tidal record by the method of least squares. This method has been studied in considerable detail and is now refined to a considerable degree of sophistication (eg Tidal Institute Recursive Analysis program (TIRA) as used by IOS Bidston). One restriction of this technique is the length of the time series required to separate the constributions due to constituents with similar frequencies. In particular, it is the resolution of M_2 and S_2 which governs the minimum length of the series. For these two constituents a fifteen day record is generally required, although in some circumstances seven and a half days can be acceptable. However, in some cases information on the tidal constituents is required at locations where only much shorter time series (less than three days) are available. This is particularly true of velocity data.

The first objective of this report is to investigate the accuracy with which the amplitudes and phases of the tidal constituents can be obtained from short (<3 day) tidal records of elevation or velocity data, using conventional techniques.

The second objective is to provide a method by which the errors in the analysis of short tidal records can be estimated. The major part of this study is concerned with the analysis of an artificially produced 15 day time series representing a tide elevation record. This time series is composed of a small number of tidal constituents with varying degrees of random noise superimposed. In this way, the exact amplitudes and phases of each constituents are known and thus any errors from the analysis of this series can be accurately determined. The effect of record length on the error in amplitude and phase of the component constituents is investigated using this series and also a number of techniques for minimising these errors are considered. On the basis of these results an empirical method is suggested for estimating the error in real records. This method is then tested using an actual 29 day current meter record, the tidal constituents of which have previously been determined using IOS's standard harmonic analysis routine TIRA. This data was collected and processed for the Swansea Bay Project (Heathershaw and Hammond, 1979).

The harmonic analyses of both the synthetic and real time series discussed herein used the least squares species analysis method described by Chamberlain (1975), unless otherwise stated. The full details of this method as implemented by the authors are given in Appendix A.

2. HARMONIC ANALYSIS OF SYNTHETIC TIDAL RECORDS

The errors inherent in the analysis of short tidal records were initially investigated using a synthetic time series containing only M_2, M_4 and S_2 constituents. The amplitudes and phases of the three constituents were taken from a tidal record at Boscastle on the north coast of Cornwall. These values were provided by the Hydrographic Department, Taunton, from a record of 15 days, centred on 2 May 1972 from an automatic tide gauge located at 50° 41.5'N 4° 51.9'W.

A discrete, 15 day, synthetic record was established from the relationship

$$\zeta_n^{(o)} = \underset{M_2}{2.36\cos(28.98n\delta t - 172.0)} + \underset{S_2}{0.89\cos(30.0n\delta t - 231.0)} \qquad (1)$$

$$+ \underset{M_4}{0.13\cos(57.97n\delta t - 90.0)} \quad \text{for} \quad n = 1, 2, 3, \ldots\ldots, 361$$

where $\zeta_n^{(o)}$ is the elevation at time $n\delta t$, and δt was taken to be 1 hour, and the constituent frequencies and phases are in $^o/_{hr}$ and ^{o} respectively.

Effect of Noise

The record represented by equation (1) does not simulate an actual record as it contains no random noise component. In addition, the synthetic record contains only a limited number of constituents, all of which are known. A real record is made up of a large number of constituents, only a proportion of which can be included in the harmonic analysis. The major restriction on the number of constituents included in the analysis is the length of the record, since, as mentioned earlier, constituents with similar frequencies cannot be resolved in the shorter record lengths.

In random noise the total energy is equally distributed throughout all frequencies and hence for a given noise level the energy in any one frequency is very small. Thus, for a record which contains a limited number of known constituents, the addition of random noise should have an insignificant effect on the subsequent harmonic analysis. This was vertified by the addition of a random noise perturbation, $\zeta_n^{(1)}$, to the base elevation $\zeta_n^{(0)}$. The random noise was given by

$$\zeta_n^{(1)} = 0.0236\, R_n \quad \text{with} \quad n = 1, 2, 3, \ldots, 361 \tag{2}$$

where R_n is a random integer in the range $-100 \leq R_n \leq 100$ produced by a standard random number generator. The synthetic record for analysis was then taken as

$$\zeta_n = \zeta_n^{(0)} + \alpha\, \zeta_n^{(1)} \quad \text{with} \quad n = 1, 2, 3, \ldots, 361 \tag{3}$$

where α is the peak amplitude of the noise as a proportion of the M_2 amplitude.

To verify the spread of energy in the random noise, the signals given by $0.1\,\zeta_n^{(1)}$ and $0.5\,\zeta_n^{(1)}$ was harmonically analysed for the constituents M_2, M_4, M_6, M_8 and M_{10}. The results, shown in Table 2, revealed that, as predicted, there is no significant energy in any one frequency and that the energy is evenly spread throughout all frequencies. Thus, in any subsequent analysis of the synthetic record, the 'correct' amplitudes and phases of the record are those used to generate the base record used in equation 1.

Five time series, each of 15 days length, were constructed from equation 3 with α values of 0, 0.05, 0.1, 0.25 and 0.5. The two records corresponding to $\alpha = 0$ and $\alpha = 0.5$ are shown in Figure 1 together with their difference which represents $0.5\,\zeta_n^{(1)}$. These five time series were analysed for M_2, S_2 and M_4 harmonics, the results being given in Table 3. The analysis procedure was able to resolve exactly the amplitude and phase of the three constituents for the smooth record $(\alpha = 0)$. As the noise level was increased slight errors in the amplitude and phase of each constituent were introduced although these were generally insignificant. Thus, for long time series, random noise has little effect on the results of a species analysis using the least squares technique.

Another potential source of error in harmonic analysis is the presence in the record of tidal harmonics which are not included in the analysis. If a tidal record containing additional harmonics is analysed, then some of the energy in the additional harmonics contributes to the harmonics included in the analysis. One method of minimising this source of error is to filter the original record with a suitable band-pass digital filter before the time series is analysed. The band-width of the filter is chosen to contain only that range of frequencies used in the harmonic analysis. However, considerable care must be taken in the choice of the digital filter employed.

Consider the filter

$$\hat{\zeta}(t) = \int_{-\infty}^{\infty} \zeta(t)\, G(t,\tau)\, d\tau \tag{4}$$

with

$$G(t,\tau) = \begin{cases} 1 & \text{when } |\tau - t| < \tfrac{1}{2}\Delta t \\ \tfrac{1}{2} & \text{when } |\tau - t| = \tfrac{1}{2}\Delta t \\ 0 & \text{when } |\tau - t| > \tfrac{1}{2}\Delta t \end{cases} \tag{5}$$

which corresponds to a simple square-wave, band-pass filter with a bandwidth of Δt. This filter was applied to the synthetic record given by equation (3) with $\alpha = 0.5$. The record was filtered using an upper cut-off frequency of 0.25 cycles per hour (ie. Δt = 4 hr). The results of the harmonic analysis of this filtered signal, presented in Table 4, showed significant departures from the exact values. Even more severe departures were found when the upper cut-off frequency was reduced to 0.125 cycles per hour (ie. Δt = 8 hr) such that the calculated M_2 and S_2 amplitudes were only half their original values. It should be noted that the Nyquist frequency of the series was 0.5 cycles per hour and this represents the maximum upper cut-off frequency for any filter applied to this record. These results illustrate the potentially drastic effects of pre-filtering a tidal record before harmonic analysis. This is also illustrated graphically in Figure 2 where the original record and the two filtered series are shown.

Unfortunately, computational limitations prevented further investigation of the use of digital filters, particularly on synthetic series with much higher Nyquist frequencies ($\sim$ 5 cycles per hour) typical of the real velocity records discussed in the next section. However, the effect of pre-filtering real records is discussed below.

Analysis of short records

Records of various durations starting at different times through the synthetic record were harmonically analysed. The summary of the results for 2½ and 3 day records are shown in Table 5. These results are surprisingly good for the amplitudes of the dominant constituents, the error increasing to a 26% overestimate in M_2 for α = 0.5 with maximum errors in S_2 and M_4 being 56% and 31% respectively. However, quite large errors in the phases of these constituents did occur. The effect of random noise is much more apparent in the shorter records than in the 15 day records but the variation in the results due to increasing noise is less than the variation due to the position of the short record in the spring-neap cycle.

As an experiment to determine whether any useful information could be obtained from very short records (one tidal cycle) a number of 12 hour samples taken from the 15 synthetic record were analysed for M_2 and M_4 only.

These results are given in Table 6. These amplitudes lie in the expected range between the sum and difference of the real semi-diurnal components (M_2 and S_2) and quarter diurnal components (M_4 and S_4) amplitudes while the phases represent the result of a complex interaction between the energy at semi-diurnal and at quarter-diurnal frequencies. If any useful information is to be obtained from the analysis of very short records, the way in which the least squares fitting procedure combines the energy at a number of semi- or quarter- diurnal frequencies into just one semi- or quarter- diurnal frequency must be determined.

3. ANALYSIS OF A REAL TIDAL VELOCITY RECORD

From the results of the harmonic analysis of a synthetic record it was concluded that, while random noise had some effect on the subsequent harmonic analysis of the series, it was predictable. However, little information was available on the effect of additional tidal constituents to those included in the analysis. To investigate this latter aspect, a velocity record for Station A in Swansea Bay (a Plessey MO21 current meter moored at 10 m in about 20 m of water: Record 66757 of Heathershaw and Hammond, 1979) was used. This record was one of 54 days duration starting at 1150 GMT on 26 September 1977. The data was in the form of speed and direction readings at 9.972 minute intervals. The time series analysed was the Eastings component of the velocity derived from this record.

The first 29 days of this series had already been analysed by the TIRA routine developed by IOS Bidston. The results of this analysis are shown as part of Table 7 and these were taken to be the exact results for this series. The phase of each constituent generated by the TIRA routine is standardised to a time origin of 00:00 GMT on 1 January 1980. To compare the results obtained from the analysis routine used herein with those from TIRA the amplitudes and phases of the present results have to be modified. The amplitude of the ith constituent must be multiplied by a factor f_i while the phase must be altered by an angle $+E_i + u_i$ where $f_i \simeq 1$ and $u_i \simeq 0$ for all constituents considered, except M_1 and K_2.

All these corrections are calculated for 00:00 GMT on the day of the beginning of the series. The full details of the calculation of f_i, E_i and u_i are given in Appendices B and C.

The velocity record was first analysed over its complete length using only every sixth point to make the time interval approximately one hour. The results from this analysis, shown in Table 7, differed appreciably from the TIRA values due to one or more of the following:

1) the use of only 1/6 of the data points;
2) the effective increase in noise level of the sampled series;
or
3) using a program less sophisticated than TIRA.

Again, computational restrictions prevented an analysis using each data point so the filter defined by equations 4 and 5 with Δt = 0.9972 hours was applied to the record. This created a new series with an initial time of 12:20 GMT on the 26 September 1977 and a digitising interval of 0.9972 hours. Records of various durations taken from different parts of the first 29 days of the record were analysed using various numbers of constituents. The results of the long duration records (> 7½ days) are shown in Table 7 while those from the shorter time series (<7½ days) are given in Table 8.

For record lengths greater than 7½ days the inclusion of more constituents has little effect on the results for the major constituents. However, appreciable differences occur when the length of the series is reduced using the same number of constituents. For short records, the amplitude and phase can vary as much ±50% from the true values for records of 36 hours or less.

Finally, a further method of filtering was applied in an attempt to remove the non-tidal part of the record. This was done by applying the Doodson and Warburg Xo filter (Groves,1955) like a running average, to obtain the non-tidal residuals over the series. These values were then subtracted from the original record values, hopefully to obtain only the tidal part. This produces a series with an initial time of 07:20 GMT on 27 September 1977, due to the running average form of the Xo filter, applied over 39 hourly values. The results of this analysis are also shown in Table 7. The amplitudes of the dominant constituents are similar to those obtained from the series including the non-tidal part, but large differences in phase occur. No explanation for this is available. However, these results reinforce the comments in the previous section concerning the care required when pre-filtering a time series before analysing harmonically.

4. DISCUSSION

It is interesting to compare the results from the synthetic time series with those from the real record in terms of the errors induced by analysing short records. Define normalised error, ε, as

$$\varepsilon = \frac{|\text{Analysed value} - \text{True value}|}{\text{True value}} \qquad (6)$$

where for the phases, since Analysed value - true value can have more than one value, the minimum is used (ie if $|$Analysed value - true value$|$ = P, then $\varepsilon = min$ (P, 360-P)/True value). Using equation (6), the variation of ε with record length for the synthetic and real records were compared. The results for M_2 amplitude are shown in Figure 3. The best fit linear regression lines to the results from the synthetic series with α = 0,0.5, 0.1, 0.25 and 0.5 are shown together with the results for the real series. Two lines are shown for the real series,one for those records starting at the beginning of the series and one for those records starting elsewhere in the series. The similarity of these latter two regressions suggests that, on average, the starting time of the short record has significantly less effect on the error than than actual length of the record.

It is also interesting to note that an estimate of the noise level of the real record, based on the peak amplitude of the deviations of the real record from the results of the harmonic analysis, is equivalent to α = 0.3. This fits very well with the results of the synthetic series and suggests that estimates of the error involved in the harmonic analysis of short time series may be possible by considering synthetic time series. It is also interesting to note that even for records as short as 12 hours the maximum error in amplitude is ±50% and can be as small as ±20%. However, to ensure errors in amplitude of less than ±10% at least 100 hours (4 days) of record is required.

Frequently, records of only semi-diurnal length are available. Such a short record can be analysed for only M_2 in the semi-diurnal bandwidth. However it is possible to recover from this amplitude and phase estimates for both M_2 and S_2 components, provided the ratio of the amplitudes and difference in phases are known. The method used here applies a least squares fit.

Let σ, ω be the frequencies of M_2 and S_2 respectively and let A, ϕ denote the amplitude and phase from the analysis for M_2 alone. The aim is to find the best fit of

$$a\cos(\sigma t_i - \nu) + b\cos(\omega t_i - \beta) \qquad (7)$$

to

$$A\cos(\sigma t_i - \phi) \qquad (8)$$

for $t_i : i = 1, N$

and so determine values for a, ν, b and β when

$b/a = R$ and $\beta - \nu = \gamma$ are known.

Replacing b and β by Ra and $\gamma + \nu$ respectively in expression (8) and rearranging it becomes

$$p C_i + q S_i \tag{9}$$

where

$$p = a\cos\nu \quad , \qquad q = a\sin\nu \, ,$$

$$C_i = \cos\sigma t_i + R\cos(\omega t_i - \gamma) \tag{10}$$

and $$S_i = \sin\sigma t_i + R\sin(\omega t_i - \gamma).$$

Equations for p and q are obtained by minimising the function

$$\sum_{i=1}^{N} \left[A\cos(\sigma t_i - \phi) - pC_i - qS_i\right]^2$$

with respect to p and q.

These are

$$p = \frac{\left[\sum_{i=1}^{N} C_i S_i \sum_{i=1}^{N} \mathcal{A}_i S_i - \sum_{i=1}^{N} S_i^2 \sum_{i=1}^{N} \mathcal{A}_i C_i\right]}{\left[\left(\sum_{i=1}^{N} C_i S_i\right)^2 - \sum_{i=1}^{N} C_i^2 \sum_{i=1}^{N} S_i^2\right]} \tag{11}$$

and

$$q = \frac{\left[\sum_{i=1}^{N} C_i S_i \sum_{i=1}^{N} \mathcal{A}_i C_i - \sum_{i=1}^{N} C_i^2 \sum_{i=1}^{N} \mathcal{A}_i S_i \right]}{\left[\left(\sum_{i=1}^{N} C_i S_i \right)^2 - \sum_{i=1}^{N} C_i^2 \sum_{i=1}^{N} S_i^2 \right]} \tag{12}$$

where $\mathcal{A}_i = A \cos(\sigma t_i - \phi)$.

Then a and α can be determined from (10):

$$a = \sqrt{p^2 + q^2}$$

$$\nu = \tan^{-1}(p/q)$$

whence

$$b = Ra$$

and $\beta = \gamma + \nu$.

To illustrate this method it was applied to two sets of A , ϕ , R and γ , obtained from a 12 hour synthetic series ($\alpha = 0.0$) and a 12 hour real record.

Synthetic Series:

The value were taken from Table 6, for the 12 hour series starting at 0.0, ie A = 2.98 and ϕ = 185.6°. From the M_2 and S_2 constituents used in the synthesis R = 0.377 and γ = 59°.

Values of p and q were determined for N = 14 and t_i = 0.0 to 13.0 at 1.0 hour increments, from which the M_2 and S_2 constituents were obtained.

	Amp.	Phase	Synthesis Amp.	Synthesis Phase
M_2	a = 2.35	ν = 1 .1°	2.36	172.0
S	b = 0.89	β = 231.1°	0.89	231.0

These are in excellent agreement with the M_2 and S_2 constituents used in the synthesis. Table 9 shows the residuals of the least square fit and the error in recovering the M_2 and S_2 constituents over the t_i values. The errors are to within 1% of the M_2 amplitude.

Real Record:
Values for A and ϕ were taken from Table 8, ie those for the 12 hour series starting at 0.0:

A = 63.41 and ϕ = 120.0°.

R and γ were deduced from the M_2 and S_2 constituents produced by TIRA, namely

R = 0.348 and γ = 52.3°.

For the same t_i that were used for the synthetic series the results were

	Amp.	Phase	TIRA amp.	TIRA phase
M_2	a = 50.1	ν = 108.6°	47.6	87.6°
S_2	b = 17.4	β = 160.9°	16.6	139.9°

These constituents recovered from A and ϕ are a considerable improvement on the original analysis. The residuals of the least squares fit in

Table 10, are all less than 14% of the M_2 amplitude, while the recovered M_2 and S_2 constituents generate a series which agrees with that from the TIRA constituents to within 60%. Most of this error is due to the difference in phases.

5. CONCLUSIONS

From the analysis of synthetic and real tidal records considered herein the following conclusions can be drawn:

1. The amplitudes and phases of the dominant constituents in a given record are relatively insensitive to any random noise component in the record.

2. The amplitudes and phases of the dominant constituents in a given record are relatively insensitive to the number of constituents included in the analysis provided the dominant constituents are present.

3. The amplitudes and phases of the dominant constituents determined from a harmonic analysis are very sensitive to the length of the record.

4. Care must be taken when pre-filtering a record before harmonic analysis as the filter can seriously affect both the amplitude and phase of the dominant constituents in the record.

5. Estimates of the error involved in the analysis of short real records may be obtained from the analysis of simple synthetic records. This aspect requires further work, but does indicate that for 12 hour records the errors in amplitude lie between ±20% and ±50%, and to ensure amplitude errors of less than ±10% a record length of at least 100 hours (4 days) is required.

6. For very short records, eg of one semi-diurnal cycle, when the harmonic analysis can produce only an M_2 constituent, which includes M_2 and S_2 components, then estimates of the M_2 and S_2 constituents can be recovered from the combined constituent provided the ratio of the two amptitudes and the difference in the phases is known.

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APPENDIX A

Theory of species analysis

The ordinate Y of a tidal record at time t may be expressed as

$$Y = A_0 + A_1\cos(\omega_1 t-\phi_1) + A_2\cos(\omega_2 t-\phi_2) +\ldots+ A_n\cos(\omega_n t-\phi_n)$$

where A_0 is the mean tide level

ω_n is the frequency of the tidal component n

A_n is the amplitude of the tidal component n

ϕ_n is the phase of the component w.r.t a time datum.

The determination of the constants is facilitated thus:

$$A\cos(\omega t-\phi) = a\cos\omega t + b\sin\omega t \tag{A1}$$

where $a = A\cos\phi$

$b = A\sin\phi$

The analysis evaluates the coefficients which give the best fit to the function

$$F(t) \simeq a_0\cos\omega_0 t + a_1\cos\omega_1 t +\ldots+ a_n\cos\omega_n t + b_0\sin\omega_0 t + b_1\sin\omega_1 t +\ldots+ b_n\sin\omega_n t$$

where F(t) is the time series under analysis

ω_1 to ω_n are the frequencies chosen

ω_0 is zero, giving the constant a_0 as mean level, and $b_0 = 0$.

There are 2n + 1 unknowns; this is therefore the minimum number of values of F(t) needed to solve the equation. A large number of values will be used, in actual fact, to minimise the effect of errors in the data.

A 'best fit' may be obtained by the least-square method: a_n and b_n are found to minimise the sum:

$$\sum_{i=1}^{I} [F_i - \sum_{n=0}^{N} (a_n \cos \, {}_n t_i + b_n \sin \, {}_n t_i)]^2 \tag{A2}$$

where I is the total number of data pairs F_i t_i, N is the total number of frequencies considered.

Differentiating (A2) with respect to a_m where m = 0...N, to minimise sum, derivative is zero:-

$$\sum_{i=1}^{I} [F_i - \sum_{n=0}^{N} (a_n \cos\omega_n t_i + b_n \sin\omega_n t_i)]\cos\omega_m t_i = 0 \tag{A3}$$

Similarly for b_m where m = 1...N

$$\sum_{i=1}^{I} [F_i - \sum_{n=0}^{N} (a_n \cos\omega_n t_i + b_n \sin\omega_n t_i)]\sin\omega_m t_i = 0 \tag{A4}$$

When differentiating with respect to individual values of a and b the other terms (the series $\sum_n a_n \cos\omega_n t_i + b_n \sin\omega_n t_i$) are constant and remain as a sum in the derivative.

Rearranging (A3) and (A4) we have

$$\sum_{i=1}^{I} F_i \cos\omega_m t_i = \sum_{i=1}^{I} [\sum_{n=0}^{N} (a_n \cos\omega_n t_i)\cos\omega_m t_i]$$

$$+ \sum_{i=1}^{I} [\sum_{n=0}^{N} (b_n \sin\omega_n t_i)\cos\omega_m t_i] \tag{A5}$$

$$\sum_{i=1}^{I} F_i \sin\omega_m t_i = \sum_{i=1}^{I} \left[\sum_{n=0}^{N} (a_n \cos\omega_n t_i) \sin\omega_m t_i \right]$$

$$+ \sum_{i=1}^{I} \left[\sum_{n=1}^{N} (b_n \sin\omega_n t_i) \sin\omega_m t_i \right] \qquad \text{(A6)}$$

Equations (A5) and (A6) represent a set of 2N+1 equations as m = 0....N for (A5) and m = 1....N for (A6). This set of equations is then put in matrix form by interchanging the order of summations and introducing the following notation:-

L.H.S. column vectors

$$\underline{FC}_m \equiv \sum_{i=1}^{I} F_i \cos\omega_m t_i \qquad \text{where } m = 0...N \qquad \text{(A7)}$$

$$\underline{FS}_m \equiv \sum_{i=1}^{I} F_i \sin\omega_m t_i \qquad \text{where } m = 1...N \qquad \text{(A8)}$$

R.H.S. matrices, taking out $\sum_{n=0}^{N} a_n$ and $\sum_{n=0}^{N} b_n$

$$\underline{\underline{CC}}_{mn} \equiv \sum_{i=1}^{I} \cos\omega_n t_i \; \cos\omega_m t_i \text{ where } n, m = 0...N \qquad \text{(A9)}$$

$$\underline{\underline{SS}}_{mn} \equiv \sum_{i=1}^{I} \sin\omega_n t_i \; \sin\omega_m t_i \text{ where } n, m = 1...N \qquad \text{(A10)}$$

$$\underline{\underline{SC}}_{mn} \equiv \sum_{i=1}^{I} \sin\omega_n t_i \; \cos\omega_m t_i \text{ where } \begin{matrix} n = 1...N \\ m = 0...N \end{matrix} \qquad \text{(A11)}$$

$$\underline{\underline{CS}}_{mn} \equiv \sum_{i=1}^{I} \cos\omega_n t_i \; \sin\omega_m t_i \quad \text{where } \begin{array}{l} n = 0\ldots N \\ m = 1\ldots N \end{array} \qquad \text{(A12)}$$

Coefficients a_n and b_n can be thought of as components of column vectors ie

$$(\underline{a})_n \equiv a_n \qquad \text{where } n = 0\ldots N$$

$$(\underline{b})_n \equiv b_n \qquad \text{where } n = 1\ldots N$$

Thus equations (A5) and (A6) become

$$\underline{\underline{FC}} = \underline{\underline{CC}} \cdot \underline{a} + \underline{\underline{SC}} \cdot \underline{b} \qquad \text{N+1 equations (rows)} \qquad \text{(A13)}$$

$$\underline{\underline{FS}} = \underline{\underline{CS}} \cdot \underline{a} + \underline{\underline{SS}} \cdot \underline{b} \qquad \text{N equations (rows)} \qquad \text{(A14)}$$

For example

$$(\underline{\underline{CC}} \cdot \underline{a})_m \equiv \sum_{n=0}^{N} \Big[\sum_{i=1}^{I} \cos\omega_n t_i \; \cos\omega_m t_i \; \cdot \; a_n \Big] \text{where} \qquad m = 0\ldots N$$

$$\equiv \sum_{i=1}^{I} \Big[\sum_{n=0}^{N} (a_n \cos\omega_n t_i)\cos\omega_m t_i \Big] \quad \text{where } m = 0\ldots N$$

which is part of equation (A5).

These two matrix equations may be combined into one by writing

$$\underline{A} \equiv \frac{\underline{a}}{\underline{b}} \quad \underline{F} \equiv \frac{\underline{FC}}{\underline{FS}} \quad \text{2N+1 vectors}$$

$$\underline{\underline{S}} \equiv \frac{\underline{\underline{CC}}}{\underline{\underline{CS}}} \; \frac{\underline{\underline{SC}}}{\underline{\underline{SS}}} \quad \text{(2N+1) x (2N+1) matrix .}$$

Therefore

$$\underline{\underline{S}} \cdot \underline{A} = \underline{F} \tag{A15}$$

2N+1 equations for 2N+1 unknowns, $a_o \ldots a_n$, $b_1 \ldots b_n$, $\underline{\underline{S}}$ is symmetric.

In the analysis $\underline{\underline{S}}$ and $\underline{F}$ are built up by substituting each pair of values F_i and t_i in turn in equations (A7) to (A12) to give the sum over i. The matrix equation (A15) is then solved for $\underline{A}$ by a standard process of Gaussian elimination. The normal harmonic constants A (amplitude) and ϕ (phase) are obtained from the vector $\underline{A}$, with reference to equation (A1) by

$$A_n = \sqrt{(a_n^2 + b_n^2)}$$

$$\phi_n = \tan^{-1} (b_n/a_n) \quad .$$

APPENDIX B

Equilibrium Phase

The Equilibrium Phase, is the phase of the corresponding idealised constituent at Greenwich at a particular time, based on 00:00 hours GMT on 1 January 1900. The idealised constituent is that which would be generated by the relevant astronomical force acting on a body of water covering the earth, assuming the absence of land masses and inertia in the water.

The Equilibrium Phase is calculated for each constituent from several orbital elements given below, for zero hour on the day of the start of the analysed record.

The orbital elements for zero hour GMT are

Y - the year,

D - the number of days elapsed since January 1 in the year Y (eg for midnight on January 3/4, = 3),

i - the number of leap years between 1900 and the year Y excluding 1900 and Y (ie $i = \frac{\mathrm{INT}(Y-1901)}{4}$).

The mean longitude of the moon

$$s = 277.02^\circ + 129.3848^\circ(Y-1900)-13.1764^\circ(D + i).$$

The mean longitude of the sun

$$h = 280.19^\circ - 0.2387^\circ(Y-1900) + 0.9856^\circ(D + i).$$

The mean longitude of the lunar perigee

$$p = 334.39^\circ + 40.6625^\circ(Y-1900) + 0.1114^\circ(D + i).$$

The mean longitude of the ascending node

$$N = 259.16^\circ - 19.3282^\circ(Y-1900) - 0.0530^\circ(D + i)$$

he the mean longitude of the lunar perigee

$$p' = 282.00^\circ \text{ for the century 1900-2000.}$$

For Y = 1977, D = 268 + d and i = 19 we have :

Constituent	E
M_2	$2h-2s = 46.80 - 24.38d$
M_4	$4h-4s = 93.60 - 48.76d$
S_2	0
M_6	$6h-6s = 140.40 - 73.14d$
MS_4	E of M_2
M_1	$h-s+90 = 293.40 - 12.08d$
μ_2	E of M_4
N_2	$2h-3s+p = 322.89 - 37.43d$
K_2	$2h = 9.36 + 1.96d$

APPENDIX C

Calculation of standard phase g

To compare the analysed phase ϕ , for a particular constituent (frequency ω) with TIRA's results it was necessary to modify them by the Equilibrium Phase, E, calculated for the beginning of the day on which the series begins plus a factor, frequency x time t_0 hrs where t_0 is the time to the nearest minute of the start of the series. Hence the complete equation for the standard phase g given an analysed phase ϕ is

$$g = \phi + E + \omega t_0.$$

In addition to the modifications described above it is also necessary, in some cases, to consider the variation of the moon's orbit which has a period of 18.61 years. This is allowed for by a factor, f, applied to the amplitude of each constituent and an increment, u, to the phase. The factors are calculated from the orbital element, N (described in Appendix B) but are small except in the cases of constituents M_1 and K_2 where

for M_1: $f\cos u = 2\cos p + 0.4\cos(p-N)$

$f\sin u = \sin p + 0.2\sin(p-N)$

which gives f = 1.32 and u = 60.92°

for K_2: $f = 1.024 + 0.286\cos N + 0.008\cos 2N$

$u = -17.7° \sin N + 0.7° \sin 2N$

which gives f = 0.75, u = 5.15°.

		NOISE LEVEL	
		$\alpha = 0.0$	$\alpha = 0.5$
MEAN		-0.010	-0.052
M_2	AMP	0.017	0.086
(28.98)	PHASE	214.067	214.067
S_2	AMP	0.011	0.054
(30.00)	PHASE	80.937	80.937
M_4	AMP	0.001	0.007
(57.47)	PHASE	271.030	271.030
M_6	AMP	0.016	0.080
(86.95)	PHASE	356.786	356.786
M_8	AMP	0.017	0.086
(115.94)	PHASE	333.928	333.928
M_{10}	AMP	0.007	0.034
(144.92)	PHASE	9.492	9.492

Table 2: Analysis of the random noise.

		TIDAL COMPONENTS								
		M_2 (28.98)			M_4 (57.97)			S_2 (30.0)		
		AMP	PHASE		AMP	PHASE		AMP	PHASE	
SYNTHESIS VALUES		2.36	172.00°		0.13	90.00°		0.89	231.00°
α	Mean	-	-		-	-		-	-
0.00	0.000	2.36	172.00°		0.13	90.00°		0.89	231.00°
0.05	-0.005	2.37	172.14°		0.13	90.01°		0.88	230.82°
0.01	-0.011	2.37	172.28°		0.13	90.01°		0.88	230.65°
0.25	-0.026	2.39	172.69°		0.13	89.97°		0.87	230.11°
0.50	-0.053	2.42	173.36°		0.12	89.99°		0.84	229.17°

Table 3: Results of analysis of 15 day synthetic series $\zeta_n = \zeta_n^{(0)} + \alpha\,\zeta_n^{(1)}$ for five values of α.

		TIDAL COMPONENTS							
		M_2 (28.98)			M_4 (57.97)			S_2 (30.0)	
		AMP	PHASE		AMP	PHASE		AMP	PHASE
SYNTHESIS VALUES		2.36	172.00		0.13	90.00		0.89	231.00
Δt	Mean	-	-		-	-		-	-
4 hr	-0.063	1.89	168.09		0.06	66.46		0.65	222.63
8 hr	-0.071	1.19	146.44		0.01	112.36		0.44	200.48

Table 4: Results of analysis of filtered 15 day synthetic series with $\alpha = 0.5$.

				TIDAL COMPONENTS					
				M_2 (28.98)		M_4 (57.97)		S_2 (30.0)	
				AMP	PHASE	AMP	PHASE	AMP	PHASE
SYNTHESIS VALUES				2.36	172.00	0.13	90.00	0.89	231.00
Noise level α	Length (hrs)	Initial Time (hrs)	Mean	-	-	-	-	-	-
0.0	60.0	0.0	0.000	2.36	172.00	0.13	90.01	0.89	230.99
		150.0	0.000	2.36	172.00	0.13	89.99	0.89	231.00
		300.0	0.000	2.36	172.00	0.13	90.01	0.89	231.01
0.05	60.0	0.0	-0.006	2.39	172.47	0.13	87.10	0.83	231.27
		150.0	-0.006	2.36	172.06	0.12	92.94	0.88	230.53
		300.0	-0.002	2.37	170.71	0.12	89.53	0.96	230.84
0.10	72.0	0.0	-0.012	2.47	172.23	0.12	84.43	0.76	235.49
		150.0	-0.017	2.41	143.80	0.12	41.16	0.87	46.64
		280.0	-0.007	2.48	338.09	0.11	72.08	0.83	117.43
0.25	72.0	0.0	-0.031	2.64	172.54	0.10	73.96	0.59	245.78
		150.0	-0.042	2.48	142.93	0.11	52.28	0.86	39.89
		280.0	-0.016	2.66	340.22	0.11	96.31	0.77	128.63
0.50	72.0	0.0	-0.061	2.93	172.98	0.09	50.65	0.39	281.47
		150.0	-0.084	2.60	141.60	0.10	74.26	0.87	28.48
		280.0	-0.032	2.98	343.20	0.14	131.61	0.74	150.13

Table 5: Results of the analysis of short periods taken from the synthetic tidal records for different values of α.

					TIDAL COMPONENTS					
					M_2 (28.98)			M_4 (57.97)	
					AMP	PHASE		AMP	PHASE
	SYNTHESIS VALUES				2.36	172.00		0.13	90.00
Noise Level α	Length (Hrs)	Initial Time (Hrs)	Mean		–	–		–	–
0.0	12.0	0.0	-0.013		2.98	185.56		0.16	78.45
		180.0	-0.013		1.91	151.50		0.15	77.34
		340.0	-0.018		2.80	189.14		0.15	103.11
0.05	12.0	0.0	-0.002		3.00	185.09		0.17	77.27
		180.0	-0.001		1.89	151.84		0.17	74.59
		340.0	-0.022		2.79	189.40		0.14	117.92
0.10	12.0	0.0	0.008		3.02	184.63		0.18	76.34
		180.0	0.011		1.86	152.19		0.19	72.54
		340.0	-0.075		2.79	189.65		0.14	134.02
0.25	12.0	0.0	0.040		3.09	183.27		0.21	74.00
		180.0	0.045		1.78	153.30		0.24	67.85
		340.0	-0.036		2.76	190.41		0.19	171.11
0.50	12.0	0.0	0.093		3.21	181.64		0.26	71.27
		180.0	0.103		1.66	155.39		0.33	63.53
		340.0	-0.054		2.72	191.71		0.35	194.02

Table 6: Results of analyses of 12 hr records selected from 15 day synthetic records for different values of α.

Series		Original		Smoothed						Filtered (X_o)
		TIRA	Full							
Length (days)		29	54	29	15	15	15	15	7½	13½
Time intervals used in analysis (hours)		0.166200	1	0.997203	1	1	1	0.997203	0.997203	0.997203
Mean		1.9373	1.782	1.930	1.167	1.167	1.167	1.151	1.699	0.006
M_2	Amp	47.56	40.95	48.82	47.70	45.87	45.93	45.52	40.75	45.38
(28.98)	Phase	87.61	147.48	90.07	94.00	93.05	93.11	85.23	96.01	21.28
M_4	Amp	3.34	1.61	3.36	3.33	3.33	3.33	3.46	3.70	3.28
(57.97)	Phase	299.66	69.78	302.42	311.80	311.89	312.31	295.45	306.13	163.29
M_6	Amp	1.41	0.72	1.44	1.25	1.25	1.26	1.16	1.01	1.25
(89.95)	Phase	218.12	211.15	219.89	240.33	250.69	250.75	221.09	251.72	22.42
S_2	Amp	16.57	13.90	19.78	27.52	20.74	20.80	19.85	26.67	20.47
(30.0)	Phase	139.86	179.70	145.61	121.30	136.46	136.34	137.77	132.36	76.29
M_1 *	Amp	0.52	0.58	0.55	0.32	0.34	0.45	0.62	-	0.46
(14.49)	Phase	318.10	194.57	122.04	101.52	111.67	124.44	126.69	-	231.00
MS_4	Amp	1.60	0.65	1.58	2.06	2.06	-	-	-	-
(58.98)	Phase	335.59	94.49	340.09	358.60	358.37	-	-	-	-
μ_2	Amp	3.06	1.94	1.64	2.47	3.73	-	-	-	-
(27.968)	Phase	154.47	97.91	168.46	143.84	145.73	-	-	-	-
N_2	Amp	9.55	9.73	9.27	2.26	0.10	-	-	-	-
(28.44)	Phase	86.31	177.97	99.14	106.75	344.21	-	-	-	-
K_2 *	Amp	4.51	-	3.27	6.72	-	-	-	-	-
(30.08)	Phase	139.86	-	83.32	295.64	-	-	-	-	-

* These components have been modified by f and u which are significant for these constituents.

Table 7: Results of the analyses of the long period records taken from the real tidal record.

SERIES	Length (Hrs)	Time Interval used in analysis (Hrs)	Initial Time (Hrs)	Mean	TIDAL COMPONENTS M_2 (28.98)		M_4 (57.97)		M_6 (89.95)		S_2 (30.00)	
					AMP	PHASE	AMP	PHASE	AMP	PHASE	AMP	PHASE
(TIRA) ORIGINAL					47.56	87.61	3.34	299.06	1.41	218.12	16.57	139.86
	24	0.1662	0.0	1.84	63.93	115.17	4.38	331.01	2.72	294.88	-	-
SMOOTHED	72	1	35.90	1.414	44.45	99.27	6.05	311.62	1.94	264.93	23.47	118.79
	72	1	131.63	1.360	41.42	86.52	2.81	244.78	2.77	145.47	25.31	130.74
	72	1	215.40	0.376	47.65	64.11	2.36	253.57	1.43	170.35	23.19	103.23
	36	0.9972	0.00	2.065	42.09	113.08	3.77	339.43	1.93	293.65	22.14	103.44
	36	0.9972	129.64	2.271	26.50	100.78	2.83	271.81	2.57	173.26	36.55	155.88
	36	0.9972	209.41	0.390	29.88	3.25	3.25	250.45	2.28	174.31	6.69	139.97
	36	0.9972	0.00	1.811	63.49	115.61	4.49	332.71	2.27	296.28	-	-
	36	0.9972	129.64	1.538	53.88	65.29	2.44	298.42	2.99	179.56	-	-
	36	0.9972	209.41	0.606	27.30	50.69	3.08	250.30	2.23	173.36	-	-
	15	0.9972	0.00	2.408	64.08	119.04	3.97	340.96	2.46	319.93	-	-
	12	0.9972	0.00	1.720	63.41	120.02	4.82	333.60	2.95	329.94	-	-
	12	1	0.00	1.762	63.50	120.41	4.62	335.76	-	-	-	-
	12	1	179.50	0.103	39.50	54.17	5.12	256.51	-	-	-	-
	12	1	339.05	3.357	55.57	94.00	3.62	320.40	-	-	-	-

Table 8: Results of the analyses of short period records taken from the real tidal record.

Time t_i (hrs)	Estimated $M_2 + S_2$ Contribution [1]	$A_i = A\cos(\sigma t_i - \phi)$	Residuals of least squares fit[2]	True $M_2 + S_2$ Contribution [3]	Error[4]
0.0	-2.88369	-2.96578	0.0821	-2.89713	0.0134
1.0	-2.70541	-2.73524	0.0298	-2.71606	0.0107
2.0	-1.83552	-1.81955	-0.0160	-1.84015	0.0046
3.0	-0.49058	-0.44808	-0.0425	-0.48794	-0.0026
4.0	0.98928	1.03562	-0.0463	0.99856	-0.0093
5.0	2.22691	2.25992	-0.0330	2.24050	-0.0136
6.0	2.90432	2.91813	-0.0138	2.91879	-0.0145
7.0	2.84452	2.84537	-0.0009	2.85617	-0.0117
8.0	2.05777	2.05988	-0.0021	2.06361	-0.0058
9.0	0.74029	0.75841	-0.0181	0.73878	0.0015
10.0	-0.77466	-0.73303	-0.0416	-0.78318	0.0085
11.0	-2.10113	-2.04086	-0.0603	-2.11453	0.0134
12.0	-2.89890	-2.83747	-0.0614	-2.91378	0.0149
13.0	-2.96081	-2.92333	-0.0375	-2.97338	0.0126

1 Calculated from $a\cos(\sigma t_i - \nu) + b\cos(\omega t_i - \beta)$
2 Residual: $a\cos(\sigma t_i - \nu) + b\cos(\omega t_i - \phi) - A\cos(\sigma t_i - \phi)$

with $a = 2.349$, $\nu = 172.1°$; $b = 0.886$, $\beta = 231.1°$; $A = 2.98$, $\phi = 185.6°$

3 Calculated from $2.36\cos(\sigma t_i - 172°) + 0.89\cos(\omega t_i - 231°)$ (values from Table 6).
4 Error: 1 minus 3.

Table 9: Residuals of least squares fit and error after recovering M_2 and S_2 from the results of analysis of 12 hour synthetic series ($\alpha = 0.0$) for M_2 only.

Time t_i (hrs)	Estimated $M_2 + S_2$ Contribution[1]	$\mathcal{A}_i$ = $A\cos(\sigma t_i - \phi)$	Residuals of least squares fit[2]	True $M_2 + S_2$ Contribution[3]	Error[4]
0.0	-32.422	-31.705	-0.717	-3.657	-28.765
1.0	-2.362	-1.124	-1.238	19.138	-21.500
2.0	27.807	29.738	-1.931	44.286	-16.679
3.0	52.260	53.151	-0.891	58.289	-6.029
4.0	62.842	63.251	-0.409	57.505	5.337
5.0	57.453	57.506	-0.053	42.044	15.409
6.0	37.374	37.357	0.017	15.766	21.609
7.0	7.647	7.851	-0.204	-14.683	22.330
8.0	-24.195	-23.622	-0.573	-41.553	17.358
9.0	-50.034	-49.178	-0.856	-57.963	7.929
10.0	-63.240	-62.416	-0.824	-59.668	-3.572
11.0	-60.376	-60.018	-0.358	-46.154	-14.222
12.0	-42.089	-48.587	6.498	-20.793	-21.296
13.0	-12.973	-14.488	1.515	9.997	-22.970

1 Calculated from $a\cos(\sigma t_i - \nu) + b\cos(\omega t_i - \beta)$
2 Residual: $a\cos(\sigma t_i - \nu) + b\cos(\omega t_i - \beta) - A\cos(\sigma t_i - \phi)$

with $\begin{cases} a = 50.1 & \nu = 108.6° \\ b = 17.4 & \beta = 160.9° \\ A = 63.41 & \phi = 120.0° \end{cases}$

3 Calculated from $47.6\cos(\sigma t_i - 87.6) + 16.6\cos(\omega t_i - 139.9)$: values from Table 8.
4 Error: 1 minus 3.

Table 10: Residuals of least squares fit and error after recovering M_2 and S_2 constituents from the results of the analysis on 12 hour real record for M_2 only.

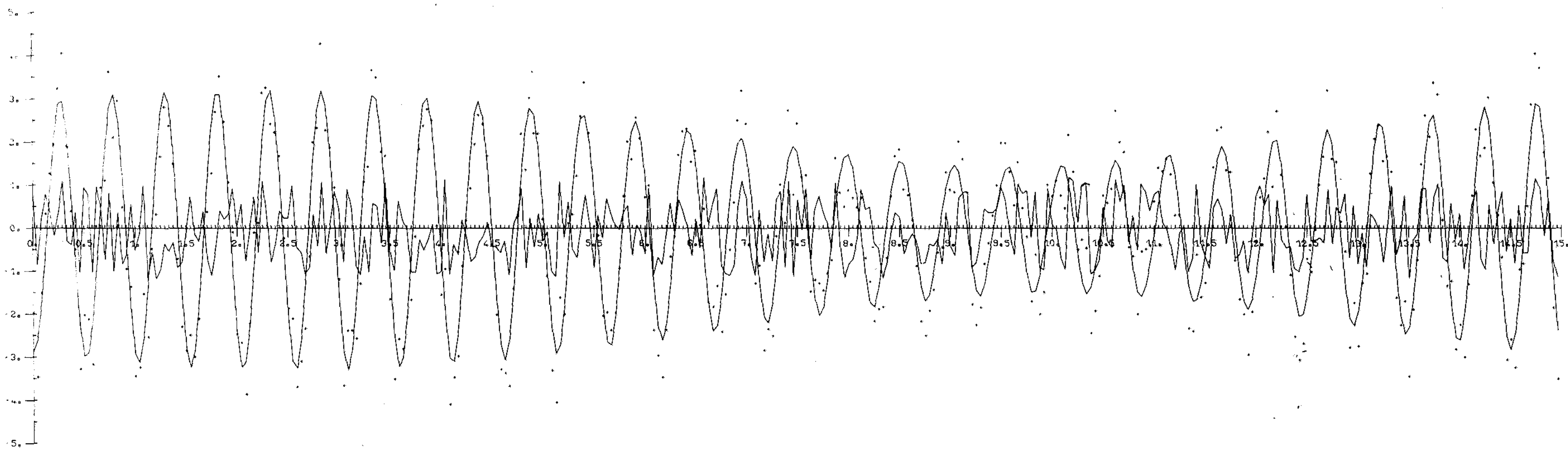


Figure 1: 15 day synthetic records with its components:

— represents $\xi_n^{(0)}$ (ie $\alpha = 0.0$)

+ represents $\xi_n = \xi_n^{(0)} + 0.5\,\xi_n^{(1)}$

represents $0.5\,\xi_n^{(1)}$.

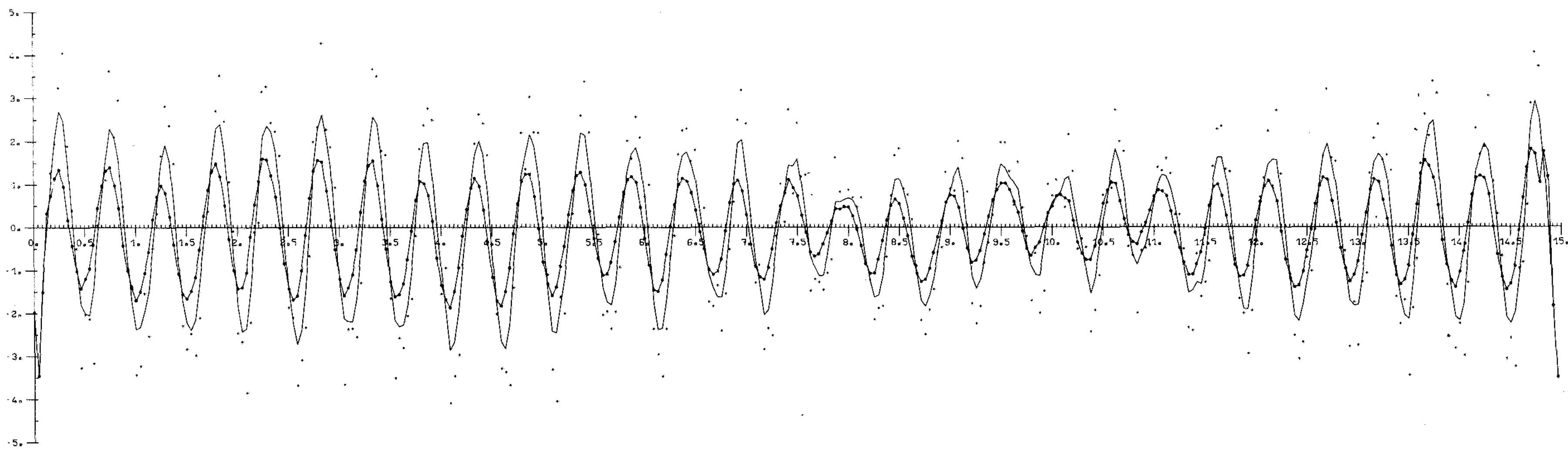
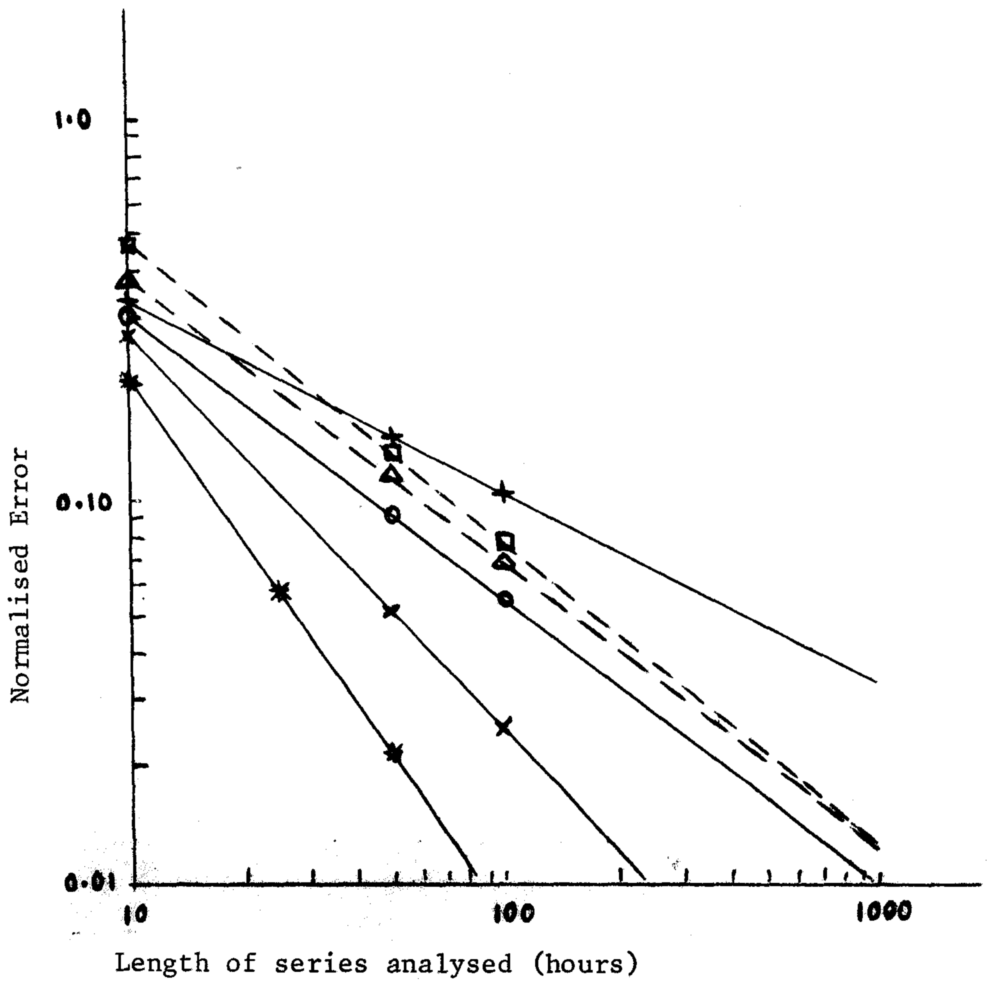


Figure 2: Plot of filtered series compared with unfiltered.

+ are the values $\zeta_n = \zeta_n^{(0)} + 0.50\zeta_n^{(1)}$ for n = 1 to 361

- connects points of the series filtered with Δt = 4 hr

⟜ are points of the series filtered with Δt = 8 hr.



Symbols:

- * Synthetic Series: noise level α = 0.05
- × Synthetic Series: noise level α = 0.10
- ⊙ Synthetic Series: noise level α = 0.25
- + Synthetic Series: noise level α = 0.50
- □ Real series, only those with zero initial time.
- Δ All real series.

(The real tidal record has an estimated noise level of $\alpha \simeq 0.03$.)

Figure 3: Graph of normalised error v length of series, from results of M_2 amplitude.