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Long-Term Wave Height Distribution

at Seven Stations around the British Isles

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SUMMARY

Long-term instrumental measurements of significant wave height and mean zero-crossing period at 7 stations are analyzed. The marginal distribution of significant heights is far better described by a Weibull law than by a log-normal law. The long-term distribution of individual wave heights is calculated from the joint distribution of significant wave height and mean wave period. It is found to be nearly exponential.

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1. INTRODUCTION

The N.I.O. has carried out wave measurements at a number of locations. The measurements generally cover a one-year period, and the results may serve as a basis for estimating extreme wave conditions in the respective areas. In making such estimates, the one-year data must be extrapolated. To this end the data are considered to be the result of random sampling from a population, the distribution of which is to be estimated. Once a distribution is found which gives a sufficiently close fit to the data, then extrapolation beyond the original range of the measurements can be made. The confidence which one has in the extrapolations increases with increasing goodness of fit of the distribution on which it is based. Some authors, following Jasper (1956), have stated that the logarithm of the significant wave height would be Gaussian distributed. This distribution function was found not to give a fully satisfactory fit to the N.I.O. data, the measured wave heights in the upper range tending to fall below the line of best fit for a given probability of exceedance. In view of the extrapolation referred to above it is important that the upper tail of the distribution fits the data well. It was therefore desirable to obtain a function which would give a better representation of the data, and this is one purpose of the work reported herein. A second purpose is to compute the long-term distribution of individual wave heights from the data.

Station	Location	Depth (fathoms)	Dates of observations	Total number of observations	Reference
OWS Station India	59°N. 19°W	-	'52 - '64 (intermittently	2400	Draper and Squire
OWS Station Juliett	52°30' N. 20°W	-	'52 - '64 (intermittently)	1440	Draper and Whitaker
Sevenstones	20mi. S.W. of Land's End	33	Jan '62 - '63	2920	Draper and Fricker
Morecambe Bay	15mi. W of Fleetwood	12	Nov '56 - '57	2920	Draper
Mersey Bar	3 mi. W. of buoyed channel to the Mersey	9.6	Sept '65 - '66	2920	Draper and Blakey
Varne	Dover Strait	15	Feb '65 - '66	2920	Draper and Graves
Smith's Knoll	22mi. E.N.E. of Great Yarmouth	27	Mar '59 - '60	2920	Draper

TABLE 1

2. WAVE DATA USED

The majority of long-term wave data presently available is based on visual observations. Instrumental data are far fewer both in number of locations and in time. It was nevertheless decided to use only instrumental data in the present study because of their greater reliability. There exists a systematic difference between the two sets for relatively large wave heights. Draper and Tucker (1970) report that at Ocean Weather Ship station "India" the significant height exceeds 10m in 1.5% of the instrumental measurements, and in only 0.02% of the visual observations. This difference will be discussed again in section 5.1.

The instrumental data chosen for analysis have been obtained by the N.I.O. from measurements with shipborne wave recorders. Table 1 contains pertinent information about the wave data. "India" and "Juliett" are Ocean Weather Ship stations, and the others are Light Vessel stations. The locations are indicated in Figure 1.

The original data generally consist of records of 12 minutes duration, taken every 3 hours during one year, for a total of 2920 records. For purposes of analysis each record is regarded as a (short) sample from a stationary random Gaussian process. The work of Rice (1944), Longuet-Higgins (1952), Cartwright and Longuet-Higgins (1956) and Cartwright (1958) provides the theoretical basis for the subsequent analysis, a convenient procedure for which has been described by Tucker (1961). Each record yields, among others, an estimate of the significant height $H_{1/3}$ and of the mean zero-crossing period T appropriate to the random process of which the record is a sample. The Figures 2 to 8, taken from the publications referred to in Table 1, give the probabilities, expressed in parts per thousand, that $H_{1/3}$ and T simultaneously fall in certain ranges. The original publications give many additional statistics but in this report only these so-called scatter diagrams for $H_{1/3}$ and T will be used. It is to be noted that these diagrams represent the lumped data for one year. Seasonal variations are suppressed.

3. VARIABLES CONSIDERED

It is necessary to distinguish statistics obtained from a single record, and statistics obtained from a collection of records, covering in this report a period of one year. The former are conveniently called short-term statistics, the latter long-term statistics. The short-term probability structure can, in principle at least, and apart from scale factors, be deduced theoretically assuming that one is dealing with a random process which is approximately stationary and Gaussian. The long-term probability structure is a reflection of local and distant climatological features and cannot be dealt with by deductive methods.

The only wave parameters which will be considered herein are:

- Wave height, H , the difference between maximum and minimum water surface elevation between two adjacent zero up-crossings (sometimes referred to as "individual wave height").
- The short-term mean of the highest one third of the wave heights, the significant height $H_{1/3}$
- The short-term mean of zero-crossing periods, i.e. the short-term mean value of the time intervals between adjacent zero up-crossings, denoted by T . (The data to be used herein do not carry information about individual values of zero up-crossing time intervals, although these could of course be extracted from the original records if desired).

The remainder of this report deals mainly with the long-term probability distributions of $H_{1/3}$ and H .

4. PROBABILITY DISTRIBUTIONS

In this chapter several probability functions will be defined, and some relationships between them will be given. Most of these are stated in general terms, without reference to local situations. Some of the relationships will be used in chapter 5 for the analysis of the measurements.

4.1 Notation

In the context of this report the three variables introduced in the previous chapter are considered as stochastic variables, denoted by capital letters. Particular values which each of them may assume will be denoted by the corresponding lower case letter. Probability densities will be written as " p ", and cumulative probabilities as " P ".

4.2 Joint distribution of $H_{1/3}$ and T .

The joint probability density (p.d.) of $H_{1/3}$ and T is $p(h_{1/3}, t)$, estimates of which are given in the Figures 2 to 8. No attempt has been made to find analytical approximations to the measurements.

A conspicuous feature of all the scatter diagrams is the cut-off at some upper limit of $H_{1/3} / T^2$. There seems to be a limiting steepness s_{max} ranging from 1:16 to 1:20, with most values near 1:18, where the steepness s is defined as the ratio of significant height to the deep-water wave length based on mean zero-crossing period:

$$s = \frac{2\pi H_{1/3}}{g T^2} \tag{1}$$

This has been noted by Draper et al (references in Table 1) who furthermore compare the 1 : 18 value with the 1 : 7 value which is the theoretical limiting steepness of irrotational, periodic, progressive, two-dimensional gravity waves in deep water. Sea waves depart too much from waves of these categories for the comparison to be satisfactory. Particularly the assumption that the waves are periodic is unrealistic. This assumption is not made in the calculation which is outlined in the following, and which is believed to provide a more meaningful basis for comparison with the measurements.

The elevation of the sea surface above its mean value is considered as a random, stationary process in time with a variance density spectrum $S(\omega)$. If the moments of $S(\omega)$ are given by

$$m_j = \int_0^\infty \omega^j S(\omega)\, d\omega \tag{2}$$

then

$$H_{1/3} = 4\sqrt{m_0} \tag{3}$$

(Longuet-Higgins, 1952; Hess et al, 1969) and

$$T = 2\pi\sqrt{\frac{m_0}{m_2}} \tag{4}$$

so that

$$s = \frac{2}{\pi g}\frac{m_2}{\sqrt{m_0}} \tag{5}$$

If it is supposed that $S(\omega)$ has the shape of a Pierson-Moskowitz-Bretschneider spectrum, then

$$S(\omega) = \alpha g^2 \omega^{-5} e^{-\beta(\omega/\omega_0)^{-4}} \tag{6}$$

which gives

$$\vartheta = \sqrt{\frac{\alpha}{\pi}} \tag{7}$$

The maximum values of α , determined from equilibrium ranges in the spectra of wind-driven waves, vary from (0.8 to 1.4) 10^{-2} (Phillips, 1966). This gives maximum values of ϑ ranging from 1:20 to 1:15, in very close agreement with the observed range of 1:20 to 1:16.

4.3 Marginal distributions of $H_{1/3}$ and T .

The marginal p.d. of $H_{1/3}$ and T are given by

$$p(h_{1/3}) = \int p(h_{1/3}, t)\, dt \tag{8}$$

and

$$p(t) = \int p(h_{1/3}, t)\, dh_{1/3} \tag{9}$$

Only $p(h_{1/3})$ will be considered further in this report (Chapter 5).

4.4 Conditional distributions of $H_{1/3}$ and T .

The conditional p.d. of $H_{1/3}$ is given by

$$p(h_{1/3} | t) = \frac{p(h_{1/3}, t)}{p(t)} \tag{10}$$

and a similar formula holds for $p(t | h_{1/3})$. Conditional distributions of $H_{1/3}$ or T will not be dealt with here. Reference may be made in this respect to Nordenstrøm (1969) who analysed $P(h_{1/3} | t)$ as obtained from visual observations at a number of stations in the North Atlantic as well as from the instrumental data for station "India" published by Draper and Squire (1967). It appeared that a Weibull distribution fitted the data well. It should be pointed out, however, that the fitted conditional distributions of $H_{1/3}$, unlike the marginal distributions, should not be extrapolated beyond the upper limit discussed in section 4.2. In this respect there is a fundamental difference between the conditional and the marginal distribution of $H_{1/3}$.

4.5 Conditional distribution of H.

The so-called short-term p.d. of individual wave heights H is the conditional p.d. of H for given $H_{1/3}$ and T, formally written as $p(h | h_{1/3}, t)$. It is approximately given by the Rayleigh p.d., with only one parameter, $h_{1/3}$, and which is independent of t:

$$p(h | h_{1/3}, t) = \frac{4h}{h_{1/3}^2} e^{-2(h/h_{1/3})^2} \quad (11)$$

The cumulative probability is

$$P(h | h_{1/3}, t) = 1 - e^{-2(h/h_{1/3})^2} \quad (12)$$

The validity of (11) and (12) will be assumed here without further inquiry. Reference may be made to Hess et al (1969) for a recent survey of empirical evidence in support of the Rayleigh distribution.

4.6 Marginal distribution of H.

The marginal (long-term) p.d. of individual wave heights, $p(h)$, can be derived as a weighted sum of Rayleigh probability densities.

The weight factor should not only include the variability of $H_{1/3}$ but that of T as well, despite the fact that the Rayleigh p.d. does not contain t as a parameter. The reason for this is that probabilities of occurrence of certain $H_{1/3}$ - values, expressed as fractions of time, are transformed into probabilities of occurrence of certain H - values, expressed as fractions of a number of waves. At some stage in the transformation one is converting time intervals into numbers of waves; in other words, one must divide by the wave period. It follows that the marginal (long-term) p.d. of H can be found as a sum of the conditional (short-term) p. densities, weighted with T^{-1} and with the probability that $H_{1/3}$ and T simultaneously fall in certain ranges:

$$p(h) = \frac{\iint p(h \mid h_{1/3}, t)\, t^{-1}\, p(h_{1/3}, t)\, dh_{1/3}\, dt}{\iint t^{-1}\, p(h_{1/3}, t)\, dh_{1/3}\, dt} \qquad (13)$$

The denumerator in this expression is equal to $\overline{T^{-1}}$, the long-term average number of waves per unit time.

A step-by-step derivation of (13) may be given as follows. For brevity, the following abbreviations are used:

Exp.	=	expected;
n.o.w.	=	number of waves;
p.u.t.	=	per unit time;
(I)	=	$[h_{1/3} < H_{1/3} \leq h_{1/3} + dh_{1/3}$ and $t < T \leq t + dt]$

From the definition of the joint p.d. of $H_{1/3}$ and T it follows that

$$\frac{\text{Exp. time during which (I)}}{\text{total time}} = p(h_{1/3}, t)\, dh_{1/3}\, dt \qquad (14)$$

Therefore,

$$\frac{\text{Exp. n.o.w. in time during which (I)}}{\text{total time}} = t^{-1} p(h_{1/3}, t)\, dh_{1/3}\, dt \tag{15}$$

Of these waves, a fraction $p(h \mid h_{1/3}, t)\, dh$ has a height H such that $h < H \leq h + dh$. Thus

$$\frac{\text{Exp. n.o.w. in the time during which (I) and for which } h < H \leq h+dh}{\text{total time}} \tag{16}$$

$$= p(h \mid h_{1/3}, t)\, t^{-1} p(h_{1/3}, t)\, dh_{1/3}\, dt\, dh$$

and

$$\left.\begin{aligned} &\frac{\text{Exp. n.o.w. for which } h < H \leq h+dh}{\text{total time}} \\ &= \text{Exp. n.o.w. p.u.t. for which } h < H \leq h+dh \\ &= dh \iint p(h \mid h_{1/3}, t)\, t^{-1} p(h_{1/3}, t)\, dh_{1/3}\, dt \end{aligned}\right\} \tag{17}$$

from which it follows that

$$\text{Exp. total n.o.w. p.u.t.} = \int dh \iint p(h \mid h_{1/3}, t)\, t^{-1} p(h_{1/3}, t)\, dh_{1/3}\, dt$$

$$= \iint t^{-1} p(h_{1/3}, t) \left[\int p(h \mid h_{1/3}, t)\, dh \right] dh_{1/3}\, dt \tag{18}$$

$$= \overline{T^{-1}}$$

because the expression in brackets equals 1.

Finally,

$$\left.\begin{aligned} &\frac{\text{Exp. n.o.w. p.u.t. for which } h < H \leq h + dh}{\text{Exp. total n.o.w. p.u.t.}} \\ &= \text{fraction of the waves for which } h < H \leq h + dh \\ &= p(h)\, dh \end{aligned}\right\} \tag{19}$$

in which $p(h)$ is the long-term p.d. of wave heights H .

From (17), (18), and (19), (13) results.

Integration of (13) with respect to h gives the cumulative probability of H :

$$\text{Prob}[H \leq h] = P(h) = \frac{\int_0^h dh^* \iint t^{-1} p(h_{1/3}, t)\, p(h^* \mid h_{1/3}, t)\, dh_{1/3}\, dt}{\iint t^{-1} p(h_{1/3}, t)\, dh_{1/3}\, dt} \tag{20}$$

or

$$P(h) = \frac{\iint t^{-1} p(h_{1/3}, t) P(h \mid h_{1/3}, t)\, dh_{1/3}\, dt}{\overline{T^{-1}}} \tag{21}$$

in which $P(h \mid h_{1/3}, t)$ is the cumulative conditional probability of H. Substitution of the Rayleigh law for $P(h \mid h_{1/3}, t)$, given by (12), yields

$$1 - P(h) = \frac{\iint t^{-1} p(h_{1/3}, t)\, e^{-2(h/h_{1/3})^2}\, dh_{1/3}\, dt}{\overline{T^{-1}}} \tag{22}$$

for the probability that H will exceed h. This equation will serve as a basis for the computations to be mentioned in section 5.2. It differs from equivalent expressions usually given (Jasper, 1956; ISSC, 1964; Lewis, 1967; Nordenstrøm, 1969), in which the effect on $p(h)$ of the variability of T is not mentioned at all:

$$1 - P(h) = \int p(h_{1/3})\, e^{-2(h/h_{1/3})^2}\, dh_{1/3} \tag{23}$$

The effect of this omission depends on the degree of correlation which exists between $H_{1/3}$ and T. If these are stochastically independent then both approaches yield identical results, as can be seen by substituting

$$p(h_{1/3}, t) = p(h_{1/3})\, p(t) \tag{24}$$

into (22). Generally, however, there is a positive correlation between $H_{1/3}$ and T, as can be seen by inspection of Figures 2 to 8. This means that neglecting the effects of variations of T results in over-estimating H, because the number of large waves occurring in a given length of time will on the average be less than the number of small waves. In chapter 5.2 a comparison will be made of the results from both methods.

4.7 Formulas for discrete data

The estimates in Figures 2 to 8 are in discrete form, and the formulas used in this chapter therefore need a slight modification. Let the midpoints of the class intervals of $H_{1/3}$ and T be $h_{1/3\,i}$ and t_j resp. for i = 1, 2, ... and j = 1, 2, ... , let the class widths be $\Delta h_{1/3}$ and Δt , and let the numbers given in the Figures be $w_{i,j}$. These numbers represent estimates of the probability element at the point (i,j) :

$$10^{-3} w_{i,j} \approx p(h_{1/3}, t)\, \Delta h_{1/3}\, \Delta t \tag{25}$$

All integrals of $p(h_{1/3}, t)$ should be replaced by summations of $w_{i,j}$. Thus, the cumulative marginal probability of $H_{1/3}$ is

$$\mathrm{Prob}[H_{1/3} \le h_{1/3}] = P(h_{1/3}) = \int_0^{h_{1/3}} dh_{1/3}^{*} \int_0^{\infty} p(h_{1/3}^{*}, t)\, dt \tag{26}$$

which becomes

$$\mathrm{Prob}[H_{1/3} \le h_{1/3,i}] = P(h_{1/3,i}) = \sum_{k=1}^{i} \sum_{\mathrm{all}\,j} w_{k,j} \tag{27}$$

Correction

P 15 should be

n^{-1}

6 lines up

Similarly, (22) is rewritten in the following form:

$$Prob[H > h] = \frac{\sum_{all\ i} \sum_{all\ j} e^{-2(h/h_{1/3,i})^2} t_j^{-1} w_{i,j}}{\sum_{all\ i} \sum_{all\ j} t_j^{-1} w_{i,j}} \tag{28}$$

In the Figures 5, 6 and 7 a number is given for the probability of occurrence of calms. These numbers were considered to apply to a class of zero wave heights. A wave period cannot be associated with them. A consequence of this will be considered in section 5.2.

4.8 Return period and risk.

In engineering applications of probability distributions it is customary to introduce the return period, which is equal to the average (time) interval between occurrences of the event being considered. Let the result of a random experiment be X . Successive trials are assumed independent; in other words, the prob $[X \leq x] = P(x)$ at each trial, independent of the outcome of the other trials. If n^{-1} is the fraction of (a great number of) trials for which $X > x_n$ then n is the dimensionless return period corresponding to exceedances of x_n :

$$n = \frac{1}{Prob[X > x_n]} = \frac{1}{1 - P(x_n)} \tag{29}$$

If the trial is repeated every τ (time) units then the dimensional return period y would be

$$y = n\tau = \frac{\tau}{1 - P(x_n)} \tag{30}$$

in the same units.

If the return period of an event is known, one can calculate: the probabilities that it will occur a given number of occasions within a given interval of time; the expected damage which may result from it; etc. These risk-analyses are of great importance, for the return period concept by itself is of rather limited usefulness, and may in fact be misleading if used superficially. For example, the probability that the event $[X > x_n]$ will not occur in n trials (i.e., in a time interval equal to its return period) is given by $\{P(x_n)\}^n = (1 - \frac{1}{n})^n$, which approaches $e^{-1} \approx 0.37$ for large n. Thus, the probability that an event will occur (at least once) during one return period is 1 - 0.37 = 0.63. If a structure is designed to (just) withstand a wave which has a return period equal to the lifetime of the structure, then the chances of its being destroyed are 63%. If the latter risk must be reduced to 10% then the return period should be almost 10 times the lifetime of the structure! Reference is made to Borgman (1963) for an analysis of these and related subjects.

<u>Applications to individual wave heights.</u>

In section 5.2 the probability of exceedance of individual wave heights will be calculated for selected values of the return period (1 month, 1 year, 20 years, 50 years, 100 years). The long-term expected number of waves per unit time is $\overline{T^{-1}}$, and the expected number of waves during the return period y is therefore $n = y\overline{T^{-1}}$, from which it follows that

$$\text{Prob}\,[H > h_n] = 1 - P(h_n) = \frac{1}{y\overline{T^{-1}}} \tag{31}$$

Nordenstrøm (1969) uses $\tau = \overline{T}$ as the time unit for converting probability of exceedance into return period. This does not seem to be correct, because the return period is based on the expected number of occurrences, which is $y\overline{T^{-1}}$ and not $y(\overline{T})^{-1}$.

<u>Application to significant wave heights</u>

Nordenstrøm (1969) calculates return periods from $P(h_{1/3})$ on the basis of τ = 12 minutes, assuming that 12 min. records are taken continuously (and analysed once every 12 min). Applying the result to an extrapolated log-normal distribution, he finds $H_{1/3}$ = 35 m for a return period of 1 year at station "India", from which he concludes that the log-normal distribution must be rejected because it would predict significant heights of unrealistic magnitude. This argument does not seem to be valid (although the conclusion that the log-normal distribution can be rejected is valid, but on other grounds). An interval τ = 12 min. between observations is too small compared with the time scale of variation of $H_{1/3}$ in the ocean. The return period in this case has no relevance to actual time intervals between geophysical events, such as severe storms, which give rise to the occurrences of very large heights. The following equalities are considered in order to clarify this point:

2 sec/day = 12 min/year = 10 hours/50 years = 20 hours/100 years

Each of these ratios equals the probability of exceedance of a significant wave height, say 20 m, with a return period of 1 year based on one observation every 12 minutes. By the same reasoning, this height has a return period of 1 day based on observations every 2 secs. (such a frequency of observations is possible with remote-sensing equipment scanning an extended area of the sea surface). Clearly, the notion of return period, defined as the average time interval between "events" in the statistical sense (= observations $H_{1/3} > 20m$) has no connection whatsoever with the intervals between geophysical "events" (= storms) during which $H_{1/3} >$ 20m. The latter intervals might typically be 50 to 100 years for an assumed storm duration of 10 to 20 hours, as can be seen from the last two equations above. Equally clearly the probability distribution of $H_{1/3}$ does not provide sufficient information to take the storm-duration effects into account. Knowledge of $P(h_{1/3})$ alone does not enable one to distinguish, for example, one storm of 10 hours' duration from 10 storms of 1 hour's duration. This is evidently an unsatisfactory situation. The point will not be pursued here, but it would seem that an adequate description of wave climates should include information about the probabilities of the duration and intensity of major geophysical events such as storms.

The difficulties in applying the idea of return period to $H_{1/3}$ arise from the fact that $H_{1/3}$ is defined (has a value) at each instant of time. Thus one cannot speak of the number of occurrences that $H_{1/3}$ has a given value. For this reason the probability of exceedance of $H_{1/3}$, dealt with in section 5.1, will not be converted into return periods.

The notation of return periods can perhaps be fruitfully applied to $H_{1/3}$ by considering the maximum value reached each year. This variate should have the Fisher-Tippett double exponential distribution of extremes because the underlying or parent distribution is approximately given by the Weibull distribution (see section 5.1), which is of the exponential type (Gumbel, 1958). However, many years of wave measurements would be required for such an analysis. The data treated in this report cover a one-year period only.

4.9 The log-normal distribution and the Weibull distribution.

The following two distributions will be used in the following chapter. They are here defined in terms of the probability of exceedance:

Log-normal:

$$Prob[X > x] = Prob[\log X > \log x] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\log x} e^{-\frac{\log^2(x^*/m)}{2\sigma^2}} d\log x^* \tag{32}$$

in which log m and σ are the mean and the standard deviation of log X. It plots as a straight line on paper with a Gaussian scale as one co-ordinate and log x as the other.

Weibull:

$$Prob[X > x] = 1 - P(x) = e^{-\left(\frac{x-A}{B}\right)^C} \quad \text{for } x \geq A \qquad \left.\right\} \text{ and } \begin{matrix} B > 0 \\ C > 0 \end{matrix} \tag{33}$$
$$= 1 \quad \text{for } x < A$$

A is a lower limiting value of X

B is a scale parameter

C is a shape parameter. The distribution becomes steeper (the prob. density function narrower) with increasing C. The mean value of X is given by

$$\bar{X} = A + B\Gamma(1 + \tfrac{1}{C}) \tag{34}$$

and the standard deviation is

$$\sigma_x = B\left[\Gamma\left(1+\frac{2}{c}\right) - \Gamma^2\left(1+\frac{1}{c}\right)\right]^{1/2} \tag{35}$$

From (33) it follows that

$$\ln \ln \{1 - P(x)\}^{-1} = C \ln (x - A) - C \ln B \tag{36}$$

so that a plot of the Weibull distribution is a straight line on paper with $\ln \ln \{1 - P(x)\}^{-1}$ as one co-ordinate and $\ln (x - A)$ as the other.

5. ANALYSIS OF THE DATA

5.1 Marginal distribution of $H_{1/3}$.

As stated in the introduction, the distribution of $\log H_{1/3}$ was found to be clearly non-Gaussian in the upper ranges. Examples of the measurements of $\log H_{1/3}$ plotted on Gaussian paper are given in Figs. 9 and 10. The co-ordinates of the plotted data points are the upper limit of the class interval, and the fraction of the observations for which $\log H_{1/3}$ is less than this upper limit. This plotting rule has been used throughout. It is the most convenient one because the basic data in the Figures 2 to 8 give probabilities (of occurrence of $H_{1/3}$ and T falling within certain limits) in parts per thousand. A disadvantage of this rule is that the uppermost observation cannot be plotted.

The examples given in the Figures 9 and 10 are based on the data from stations "India" and "Smith's Knoll". These were chosen because they seemed to represent the best and the worst fit of the Gaussian distribution for log $H_{1/3}$. (The data from "Juliett" are almost identical with those from "India" and could equally well have been chosen for this purpose).

The poor fit of the Gaussian distribution has also been noted by Nordenstrøm (1969), who proposes to use the Weibull distribution for the description of long-term instrumental wave data at "India" and "Juliett" and visual data at these and other stations in the North Atlantic. The application of the Weibull distribution to wind wave problems had previously been suggested by Bretschneider (1965) for a description of the short-term statistics.

In the Figures 11 to 17 the data for the 7 stations have been plotted on Weibull paper, for both A = 0 and, where necessary, for $A \neq 0$ such that the best fit was obtained, as judged by eye. The parameters B and C have been estimated from the best-fitting straight line so obtained. The results are tabulated below:

Station	A (m)	B (m)	C -	Area
India	0.80	2.70	1.22	Atlantic Ocean
Juliett	0.90	2.70	1.24	
Sevenstones	0.60	1.67	1.21	
Morecambe Bay	0.00	0.78	1.05	Irish Sea
Mersey Bar	0.00	0.69	1.01	
Varne	0.00	1.05	1.30	North Sea
Smith's Knoll	0.08	0.89	1.28	

Table 2 - PARAMETERS OF FITTED WEIBULL DISTRIBUTIONS OF $H_{1/3}$

The seven stations where the data were obtained can be broadly grouped into three areas, as indicated in the last column of Table 2. It is noteworthy that the shape parameter C does not vary much between stations from one area, although it varies appreciably between areas. The parameter A , which can be loosely described as an indication of "background noise" (such as might be due to swells) appears to be correlated with the degree of exposure of the locations.

A comparison of Fig. 9 and 10 with Fig. 11 and 17 shows that the Weibull distribution fits the data far better than the Gaussian distribution does. In a few cases the fit is almost perfect (Juliett, Fig. 12; Smith's Knoll, Fig. 17). In some cases it is quite good except for the lowermost point, which is not very important (Mersey Bar, Fig. 15;

Varne, Fig. 16). In the remaining three cases the measurements show a certain sinuosity (particularly in Morecambe Bay, Fig. 14) although the overall-fit seems fair. The significance of the sinuosity is not clear, Due to lack of time no statistical tests of goodness of fit were applied. It seems worthwhile to carry out such tests at a later time.

Hogben (1967) has compared the fit of the log-normal and the Weibull distribution to visual wave height observations at a number of locations in the North Atlantic and to the instrumental data from "India" reported by Draper and Squire (1967). He concludes: "The log-normal distributions seem to give a better overall fit extending down to quite low wave heights. In the important region of large heights, however, the Weibull plottings appear more nearly straight." The first of these conclusions seems largely to be based on the fact that the Weibull distribution gave a poor fit in the lower range. However, Hogben considered a two-parameter distribution only (setting $A = 0$ a priori). Inclusion of the third parameter A greatly improves the fit of the Weibull distribution, as can be seen in Figure 11. The second conclusion by Hogben quoted above is stated in cautious terms which are not suggestive of the differences which can be seen, for instance, between the Figures 9 and 11. The reason for this seems to be that Hogben's conclusion is mainly based on visual data, which do not include observations of $H_{1/3} >$ approx. 10 m. Instrumental data at "India" were also considered by him, but for reasons unknown to the present author the upper tail of the measurements (prob. of exc. $< 0.6\%$ appr.) was not included in the figures. This is precisely the range where the measurements deviate strongly from the log-normal law, while the Weibull law still appears to fit.

5.2 Marginal distribution of H

The cumulative marginal probability distribution of H were calculated on the basis of equation (28) and the $w_{i,j}$ values contained in the Figs. 2 to 8. Values of $P(h)$ were obtained for $h =$ 0 ft., 4 ft., 8 ft., etc; up to a value of twice the maximum significant height measured at the station. This upper limit was chosen because it is fairly representative of the upper range of the measurements, in as much as for these data the most probable maximum wave height in 3 hours, as well as its expected value, is approximately twice the significant height. The results are plotted in the Figures 18 to 24 using a co-ordinate system in which the Weibull distribution is represented by a straight line. The Figures show that a two-parameter Weibull distribution, with $A = 0$, fits the computed values quite well, except for the lower range at "India" and "Juliett". The values of the scale - and shape parameters B and C were estimated from the straight lines drawn through the points by eye. They are given in Table 3 for the respective stations. The shape parameter C is fairly close to 1 in all cases but one (Morecambe Bay), which implies that the long-term distribution of individual wave height is nearly exponential. This type of distribution has previously been found to apply to wave induced stress "heights" in a drilling rig (Bell and Walker, 1970) and in ship's hulls (Nordenstrøm, 1965).

Station	B (ft)	C -	$\bar{T}$ (sec)	$(\overline{T^{-1}})^{-1}$ (sec)	F -	$\{(1-F)\overline{T^{-1}}\}^{-1}$ (sec)
India	5.76	0.97	9.43	9.26	0	9.26
Juliett	5.82	.99	9.53	9.34	0	9.34
Sevenstones	3.76	.97	8.03	7.80	0	7.80
Morecambe Bay	1.53	.85	5.40	4.98	0.159	5.92
Mersey Bar	1.87	1.06	4.98	4.82	0.517	9.98
Varne	2.33	1.03	5.38	5.25	0.065	5.61
Smith's Knoll	1.67	.93	6.15	5.96	0	6.15

Table 3

PARAMETERS OF LONG-TERM DISTRIBUTION OF

The three stations Morecambe Bay, Mersey Bar and Varne require special consideration because calms are reported there during a given fraction of time. The "calm" conditions are not defined explicitly in the reports from which the Figs. 2 to 8 were taken. The values have been accepted at face value.

In the calculation of the distributions shown in the Figs. 18 to 24 the occurrence of calms was completely ignored. In other words, the summations in equation (28) only extended over the values of $w_{i,j}$ for the non-calm conditions. The resulting values of $P(h)$ must therefore be interpreted as the expected ratio between the number of waves for which $H \leq h$, and the total number of waves occurring; by definition, no waves occur during calms.

The occurrence of calms necessitates a slight modification of the relationship between return period y and cumulative probability $P(h)$. The expected number of waves per unit time, given that it is not calm, is $\overline{T^{-1}}$. If the fraction of time during which calms occur is F, then the expected number of waves in the return period is

$$n = y(1-F)\overline{T^{-1}} = \frac{1}{1-P(h_n)} \qquad (37)$$

Values of the (long-term) mean zero crossing period $\overline{T}$, and of the reciprocals of $\overline{T^{-1}}$ and $(1-F)\overline{T^{-1}}$, are given in Table 3. The difference between $(\overline{T^{-1}})^{-1}$ and $\overline{T}$ (which is used by Nordenstrøm to convert y into P) is relatively minor and has in no case been found to have a noticeable effect on the height calculated from a given return period.

For each of the stations, values of $P(h)$ were computed according to (37) for y = 30 days, 1 year, 20 years, 50 years and 100 years. The corresponding values of h_n can be read off the graphs provided the measured distributions are extrapolated beyond y = 1 year.

The marginal distributions of H were not only calculated according to equation 22, but also according to equation 23, which is the conventional relationship. Table 4 gives the results from both methods for station "India", for h = 0 (8) 96 ft. The effect of not taking the period variability into account is to over-estimate the probabilities of exceedance of individual wave heights. This is to be expected in view of the positive correlation between $H_{1/3}$ and T, as noted in section 4.6. The magnitude of the relative error increases with h. At all the 7 stations it was approximately 50% for the height with a return period of 1 year. For station "India" this can be seen in the last line of Table 4.

h (ft.)	$1 - P(h)$	
	acc. to eq. 22	acc. to eq. 23
0	1.0000	1.0000
8	.2806	.3022
16	$.6526 * 10^{-1}$	$.7520 * 10^{-1}$
24	$.1795 * 10^{-1}$	$.2172 * 10^{-1}$
32	$.5435 * 10^{-2}$	$.6813 * 10^{-2}$
40	$.1679 * 10^{-2}$	$.2159 * 10^{-2}$
48	$.5077 * 10^{-3}$	$.6679 * 10^{-3}$
56	$.1484 * 10^{-3}$	$.1997 * 10^{-3}$
64	$.4194 * 10^{-4}$	$.5779 * 10^{-4}$
72	$.1149 * 10^{-4}$	$.1619 * 10^{-4}$
80	$.3045 * 10^{-5}$	$.4371 * 10^{-5}$
88	$.7723 * 10^{-6}$	$.1124 * 10^{-5}$
96	$.1844 * 10^{-6}$	$.2708 * 10^{-6}$

Table 4.

PROBABILITIES OF EXCEEDANCE OF INDIVIDUAL WAVE HEIGHTS AT STATION "INDIA"

6. CONCLUSIONS

1. The upper envelope bounding the observed joint distribution of significant wave height $H_{1/3}$ and (short-term) mean zero crossing period T, has been shown to be consistent with current knowledge of energy spectra of wind driven waves.
2. The conditional distribution of $H_{1/3}$, for given T, should not be extrapolated beyond the limit mentioned in 1.
3. The statement that the logarithm of $H_{1/3}$ is Gaussian distributed does not apply to the data analysed herein.
4. The measured marginal significant wave height distributions can be well approximated by the Weibull function. This statement is based on visual inspection, rather than statistical tests of goodness-of-fit.
5. Long-term distributions of individual wave heights H have been calculated from the measured joint distributions of $H_{1/3}$ and T. The results are well described by Weibull function with an exponent close to 1.
6. The long-term distribution of H is conventionally calculated from the marginal distribution of $H_{1/3}$, disregarding the effect of period variability. This leads to a considerable overestimate of the probabilities of exceedance of H.
7. The conversion of a probability of exceedance of H into a return period (or vice versa) should strictly speaking be based on the long-term expected number of waves per unit time $\overline{T^{-1}}$, rather than on the mean wave period $\overline{T}$ which is sometimes used. However, the differences were found to be very small.

8. The various distributions referred to are based on the data for a whole year; no distinction between seasons has been made. It would be useful to carry out a more comprehensive analysis, based on more extensive data, in which seasonal variations are not suppressed. The same can be said with regard to intensities and duration of storms.

ACKNOWLEDGEMENTS

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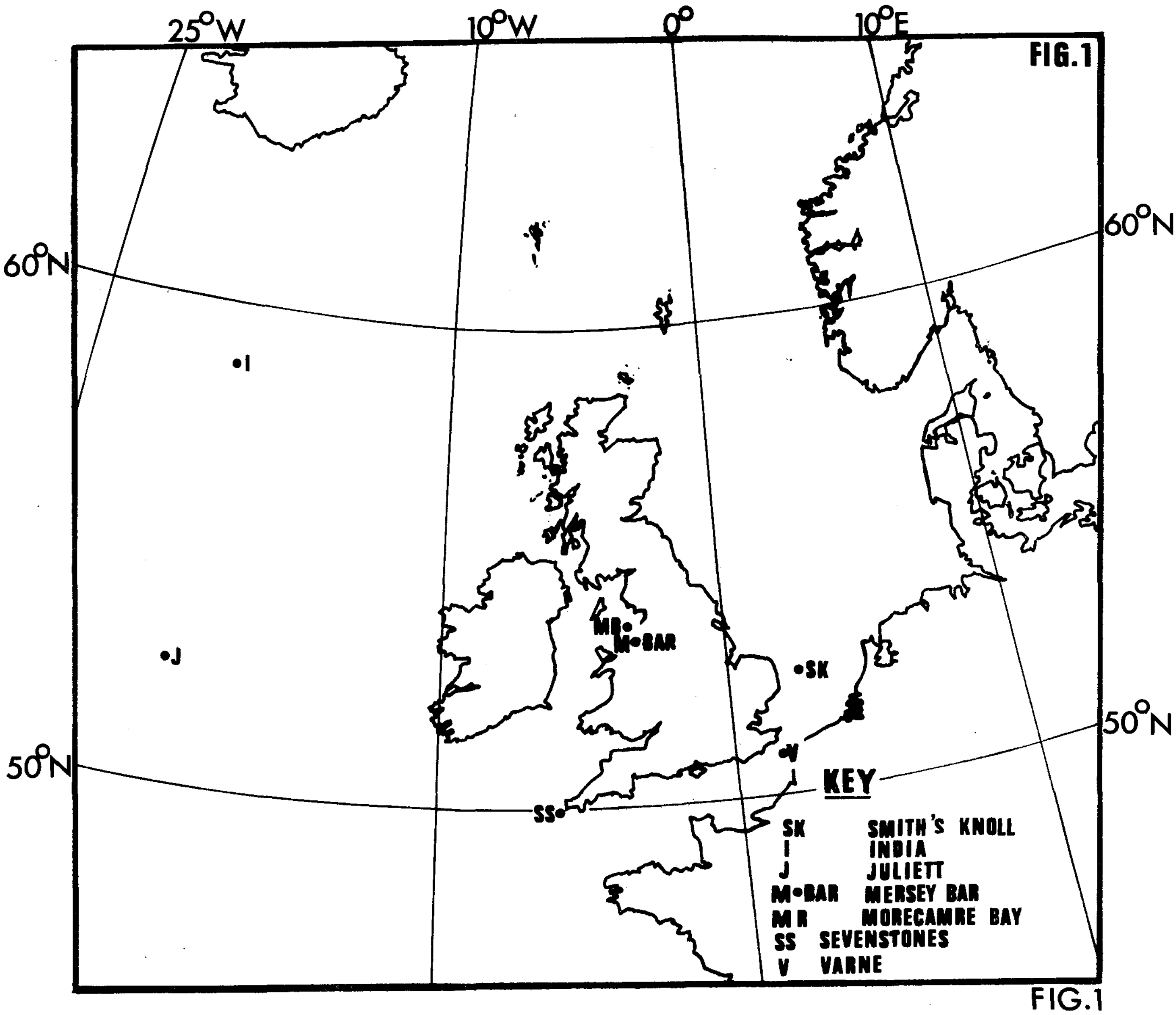
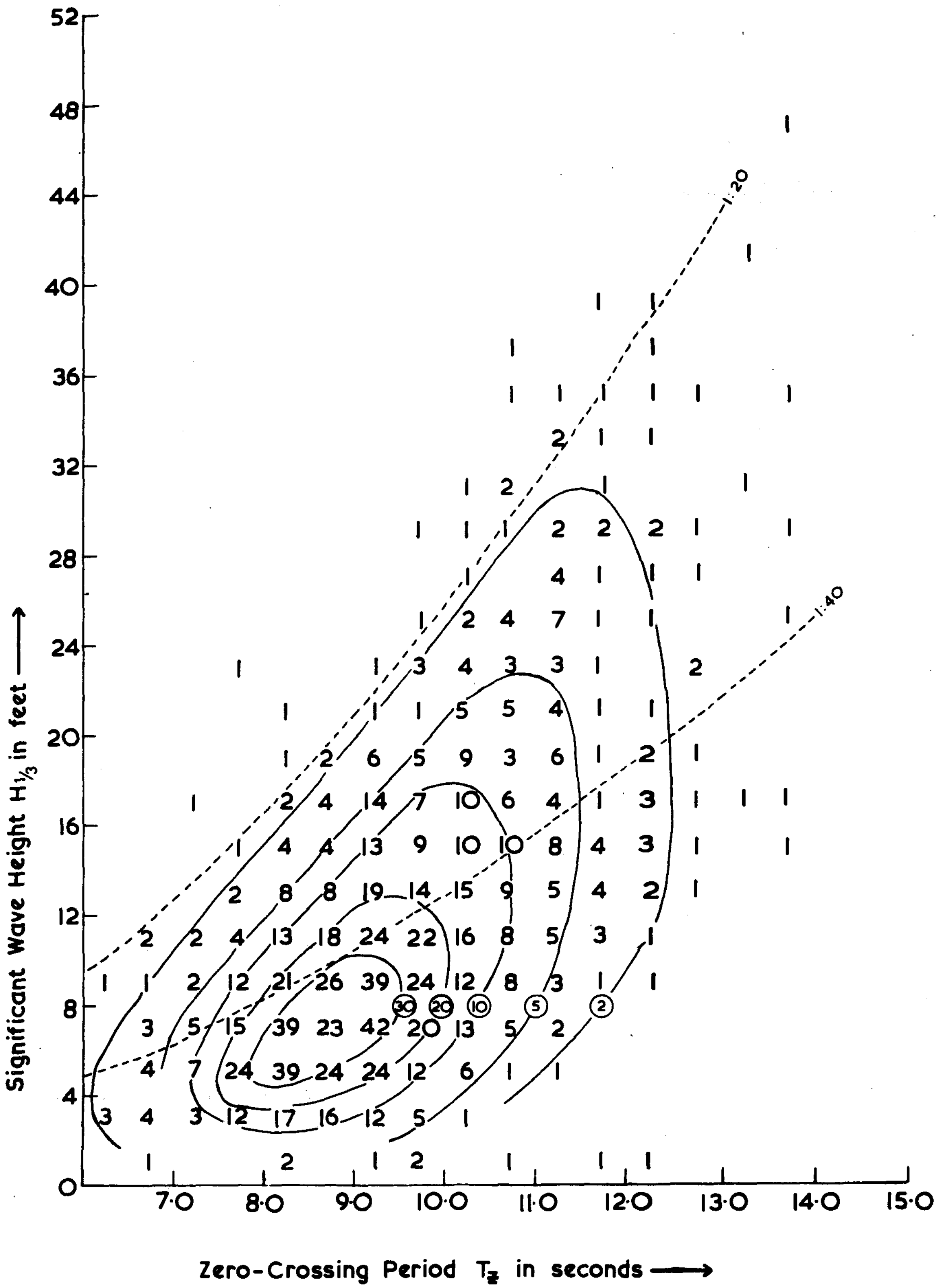

25°W
10°W
0°
10°E
FIG.1
60°N
60°N
50°N
50°N
I
J
MB
M•BAR
SK
V
SS
KEY
SK SMITH'S KNOLL
I INDIA
J JULIETT
M•BAR MERSEY BAR
MR MORECAMBE BAY
SS SEVENSTONES
V VARNE

FIG.1


Significant Wave Height $H_{1/3}$ in feet
Zero-Crossing Period T_z in seconds
1:20
1:40

INDIA

FIG. 2.

Fig. 3

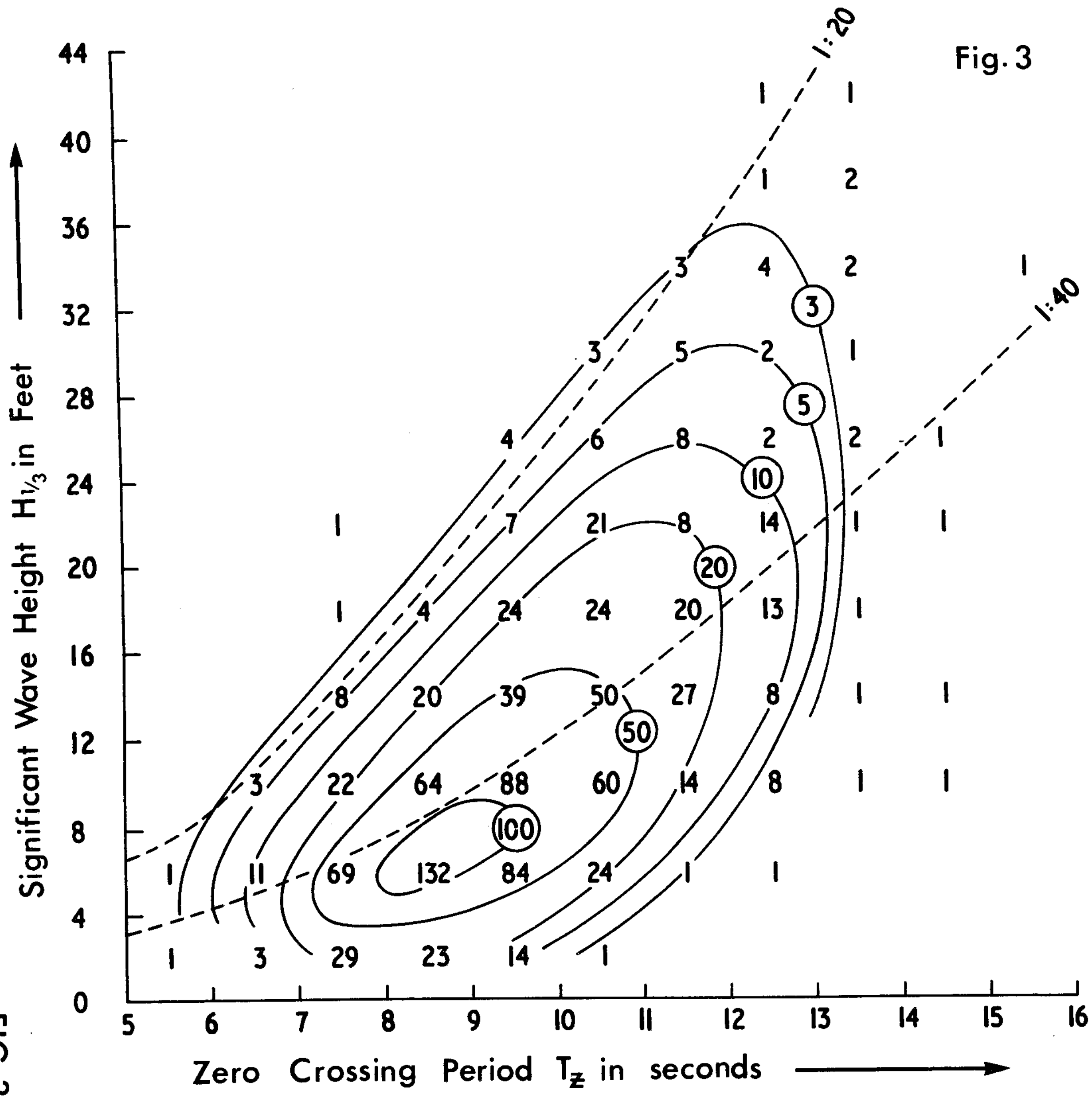
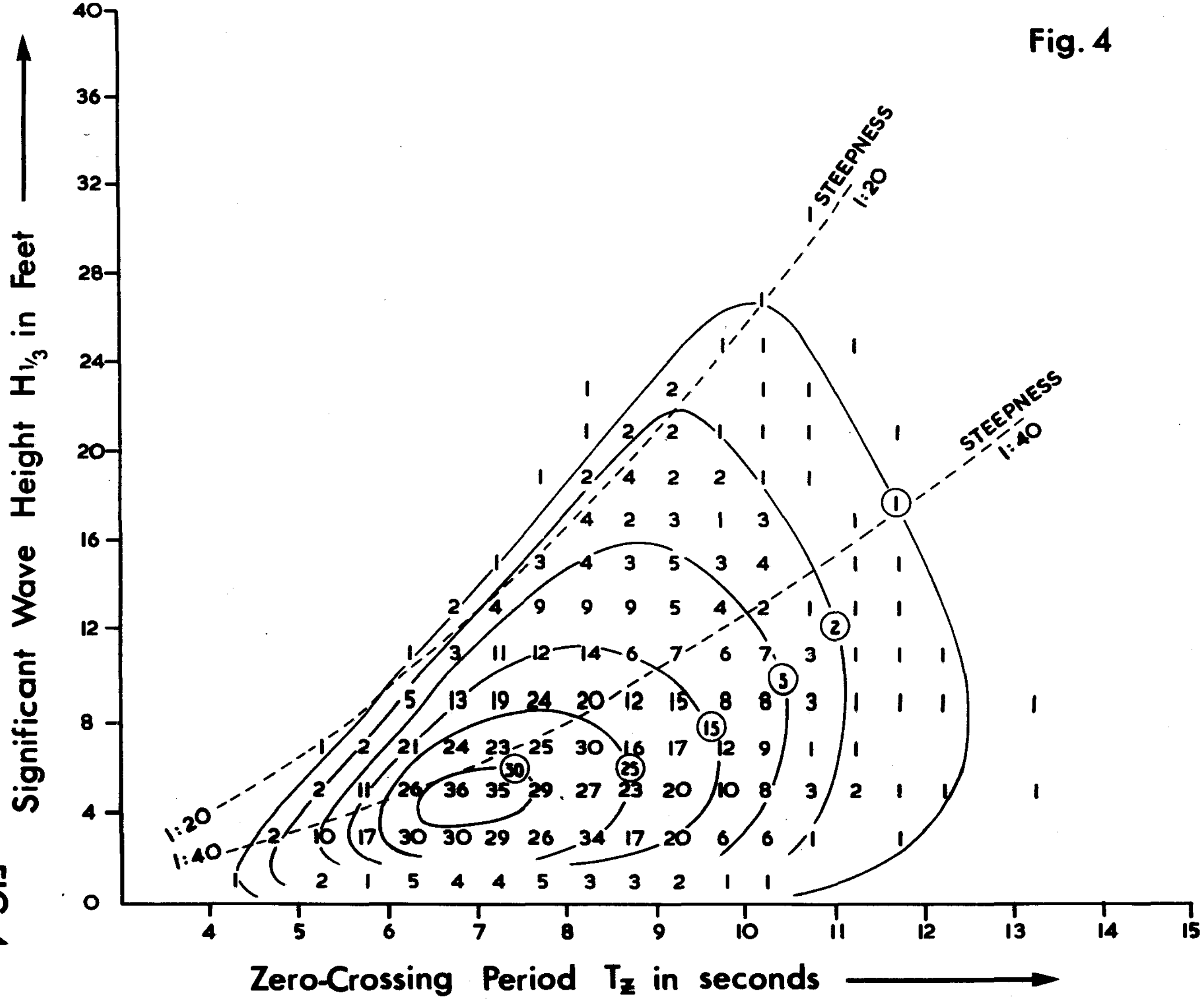


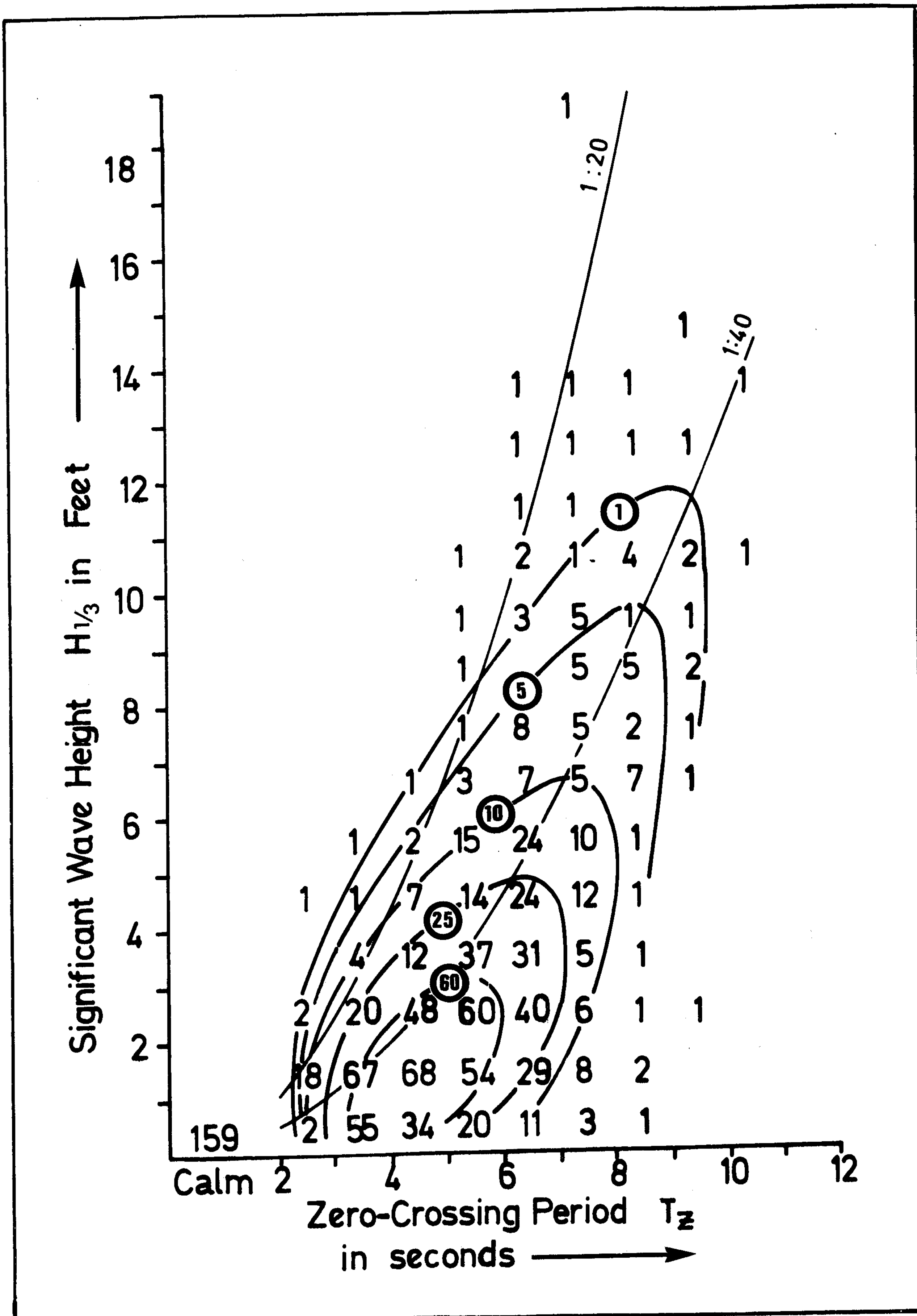
FIG. 3.

JULIETT

Fig. 4

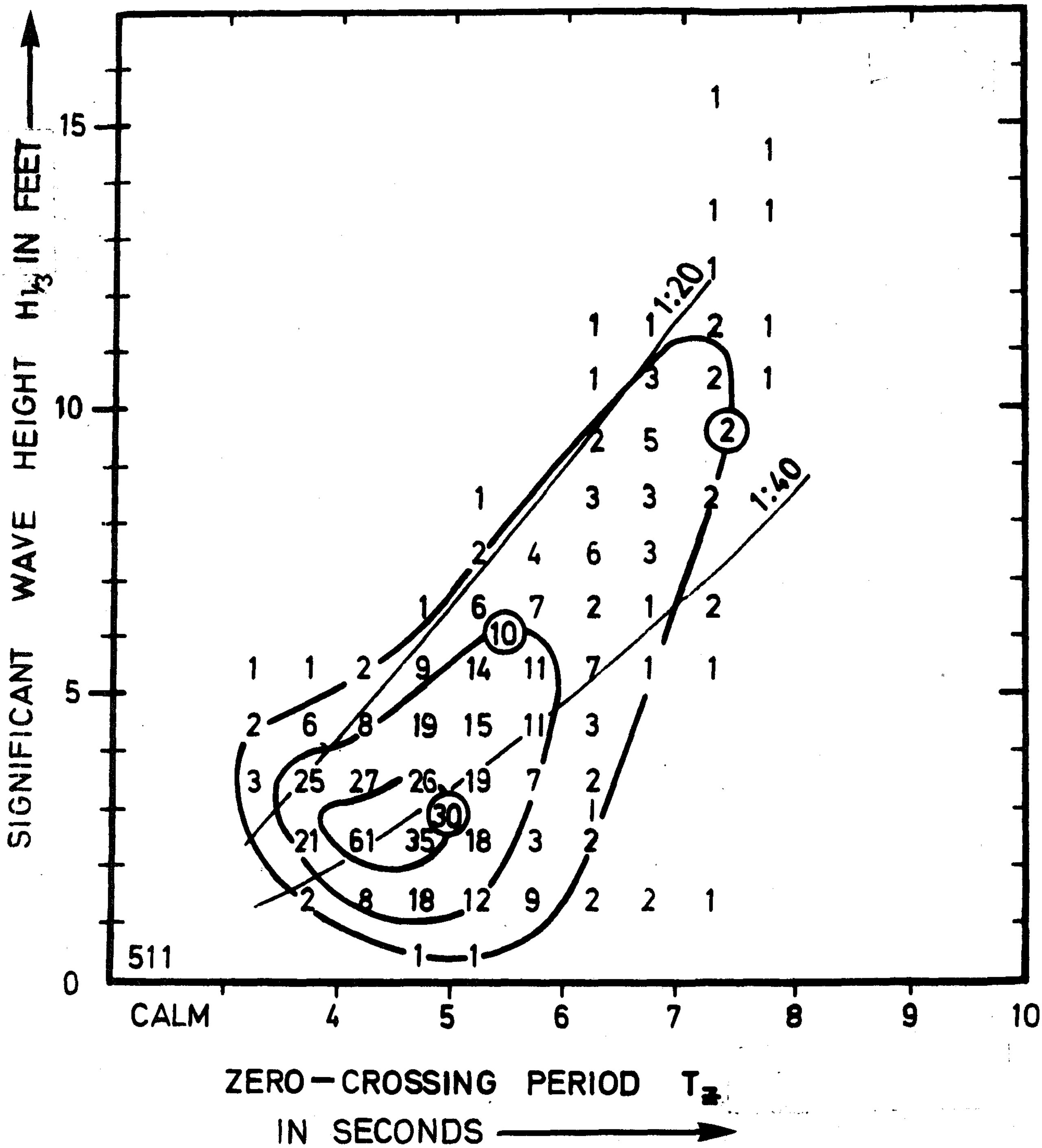


SEVENSTONES



MORECAMBE BAY

FIG. 5.



MERSEY BAR **FIG. 6.**

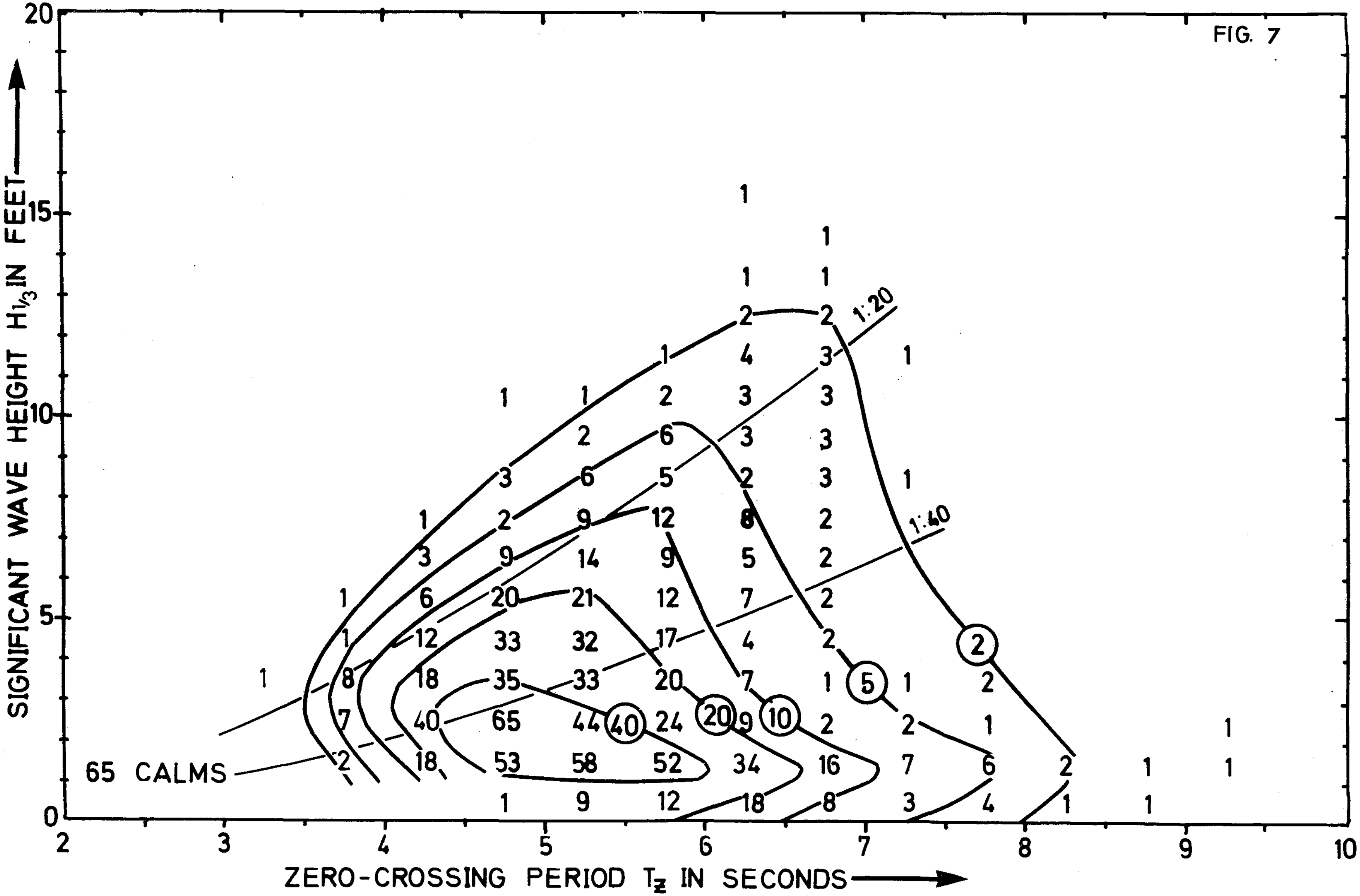
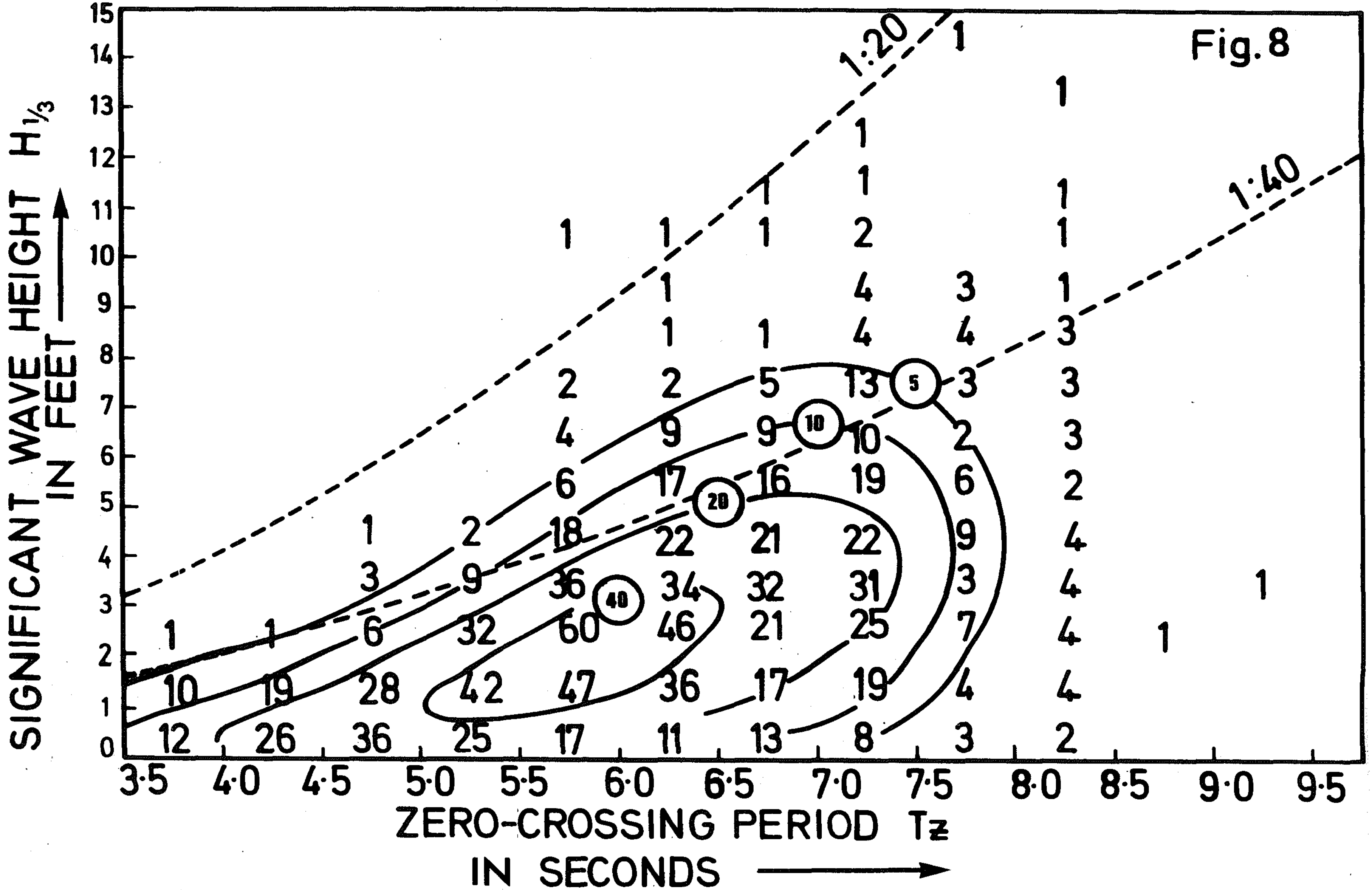


FIG. 7
SIGNIFICANT WAVE HEIGHT H1/3 IN FEET
ZERO-CROSSING PERIOD Tz IN SECONDS
65 CALMS
1:20
1:40

FIG.7

VARNE



SMITH'S KNOLL

FIG. 8.

CHARTWELL GRAPH SHEET No. 5571 ARITHMETICAL PROBABILITY

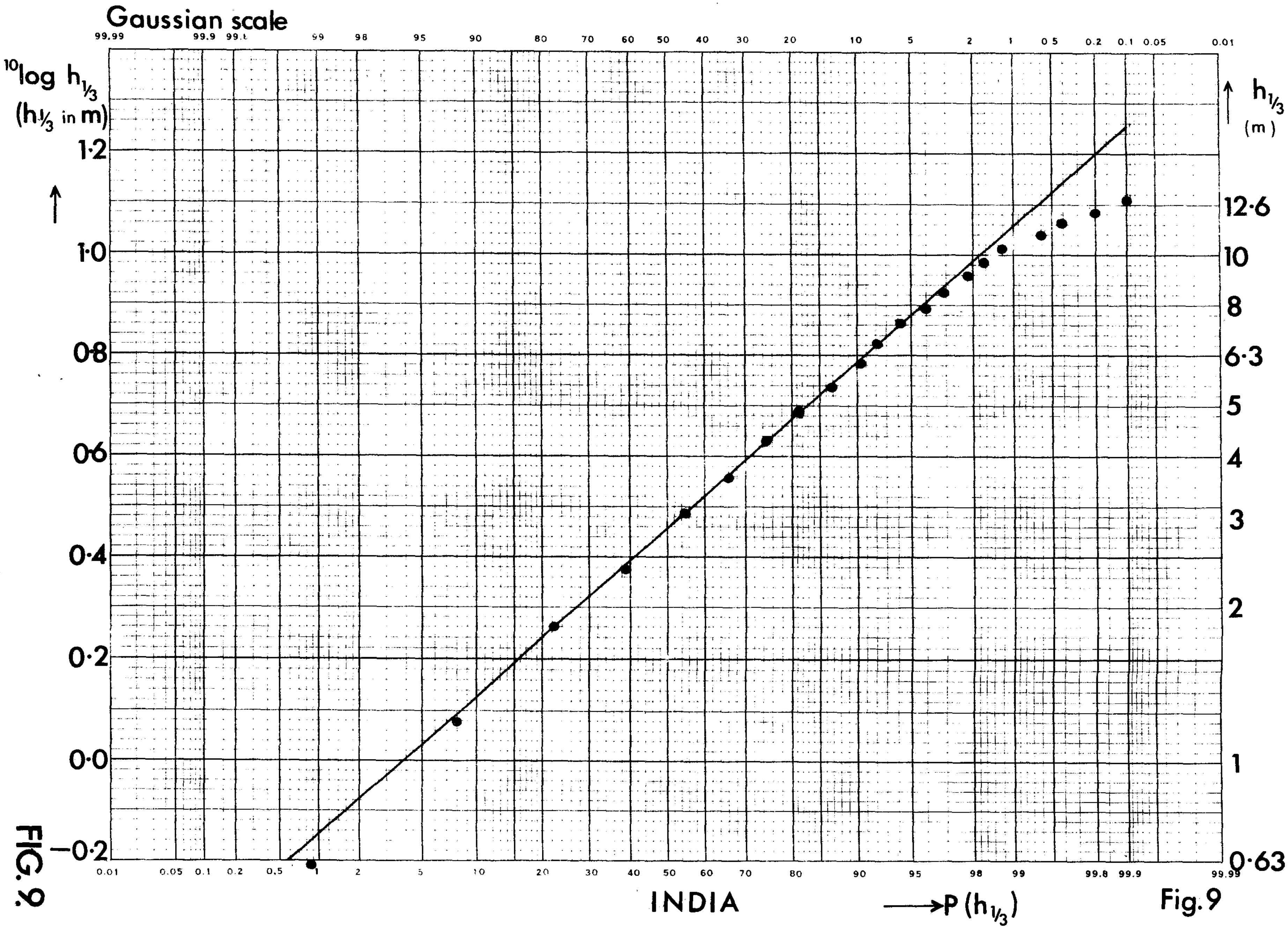


FIG. 9

Fig. 9

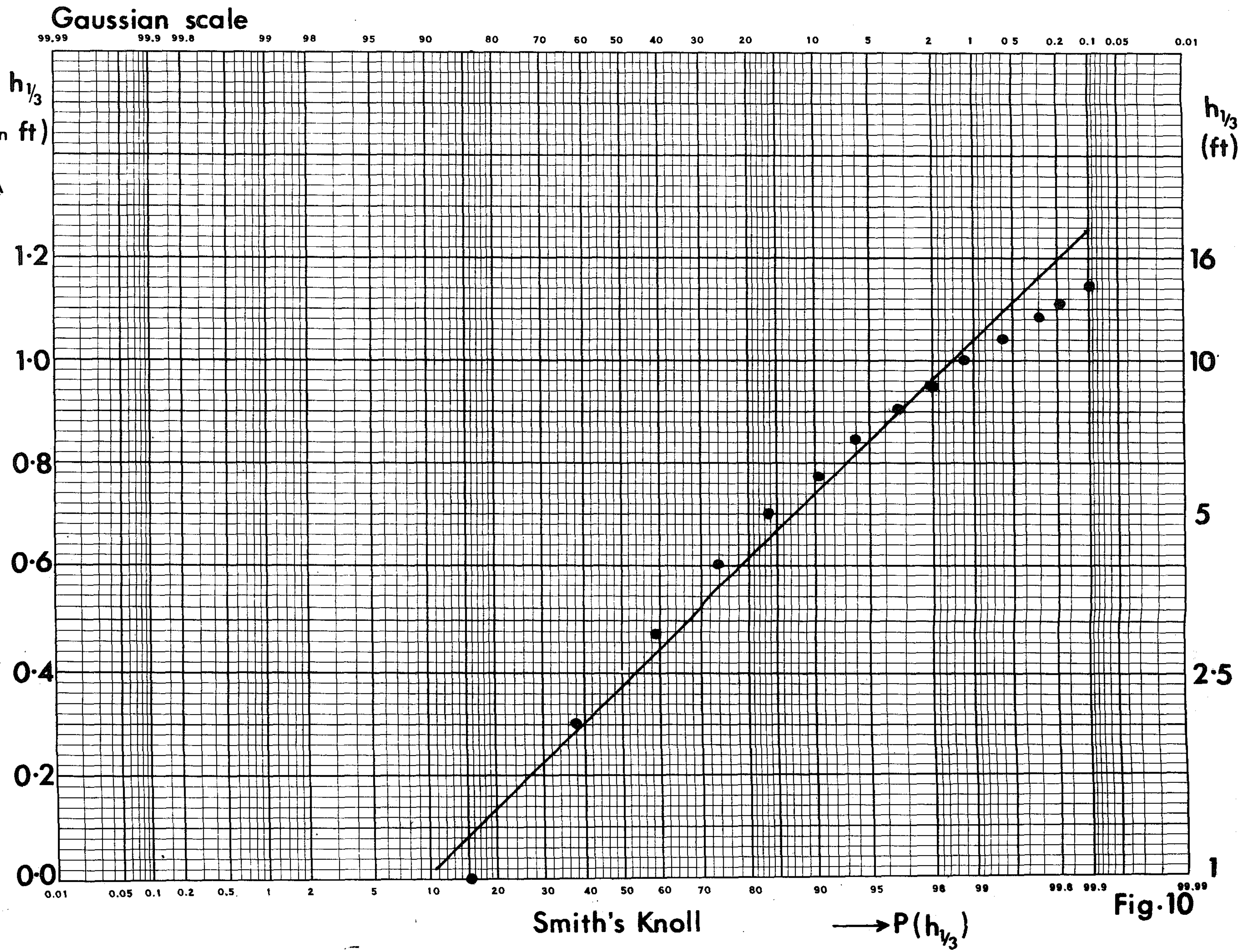

Gaussian scale
$^{10}\log h_{1/3}$
($h_{1/3}$ in ft)
$h_{1/3}$
(ft)
Smith's Knoll
$\longrightarrow P(h_{1/3})$
Fig·10

FIG. 10.

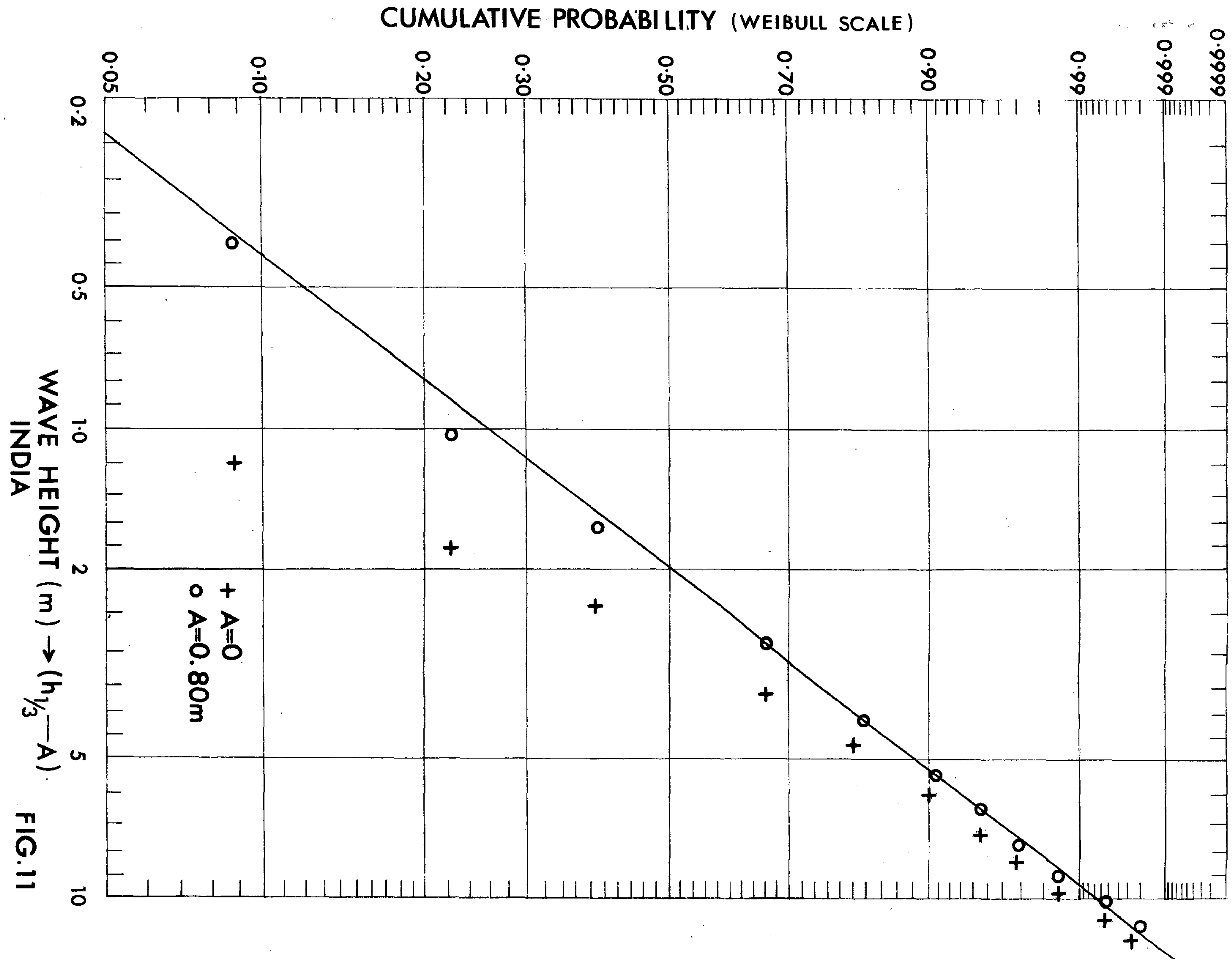
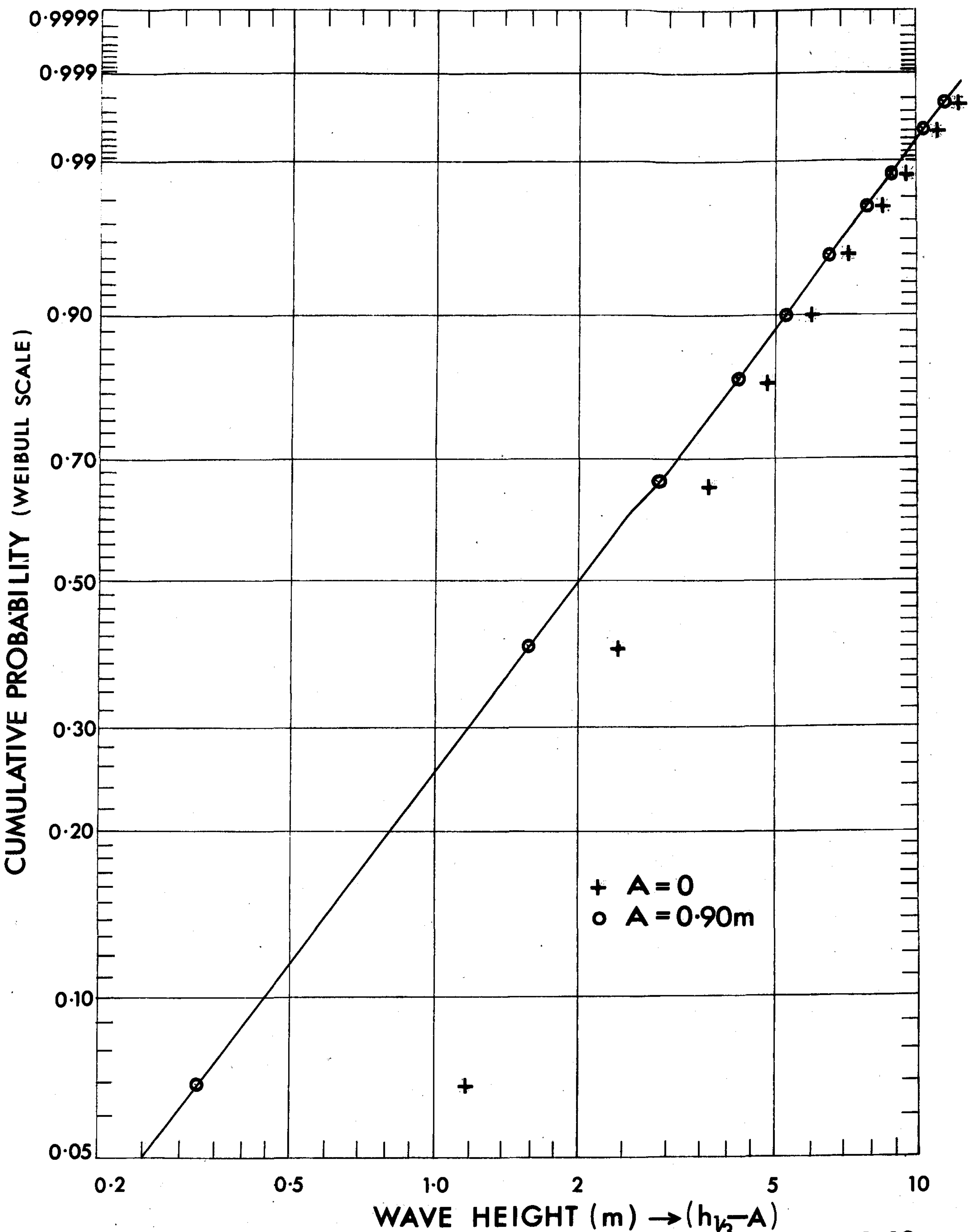
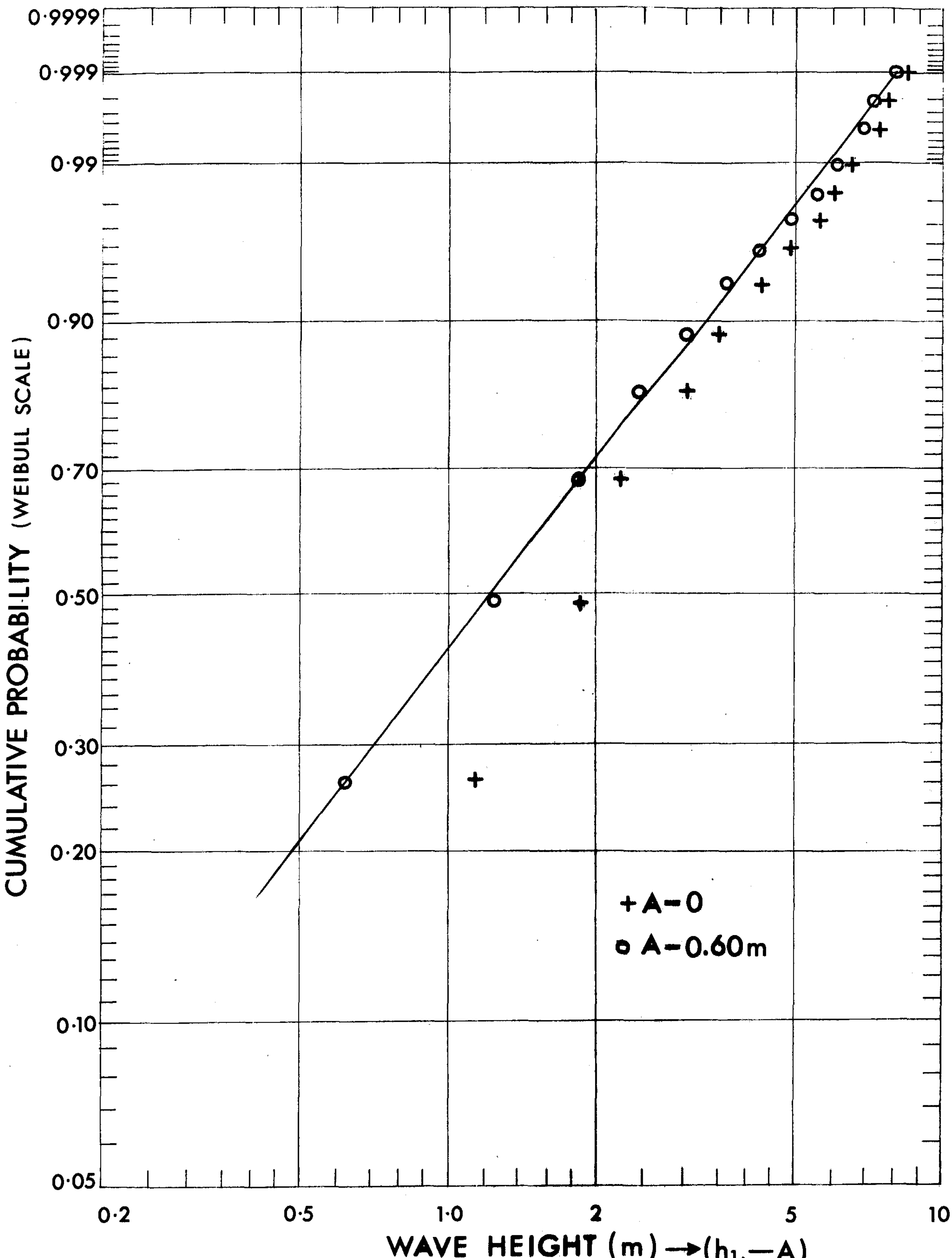

CUMULATIVE PROBABILITY (WEIBULL SCALE)
0·05
0·10
0·20
0·30
0·50
0·70
0·90
0·99
0·999
0·9999
0·2
0·5
1·0
2
5
10
+ A=0
o A=0.80m
WAVE HEIGHT (m) → ($h_{1/3}$ — A)
INDIA

FIG.11



JULIETT

FIG.12



SEVEN STONES

FIG.13

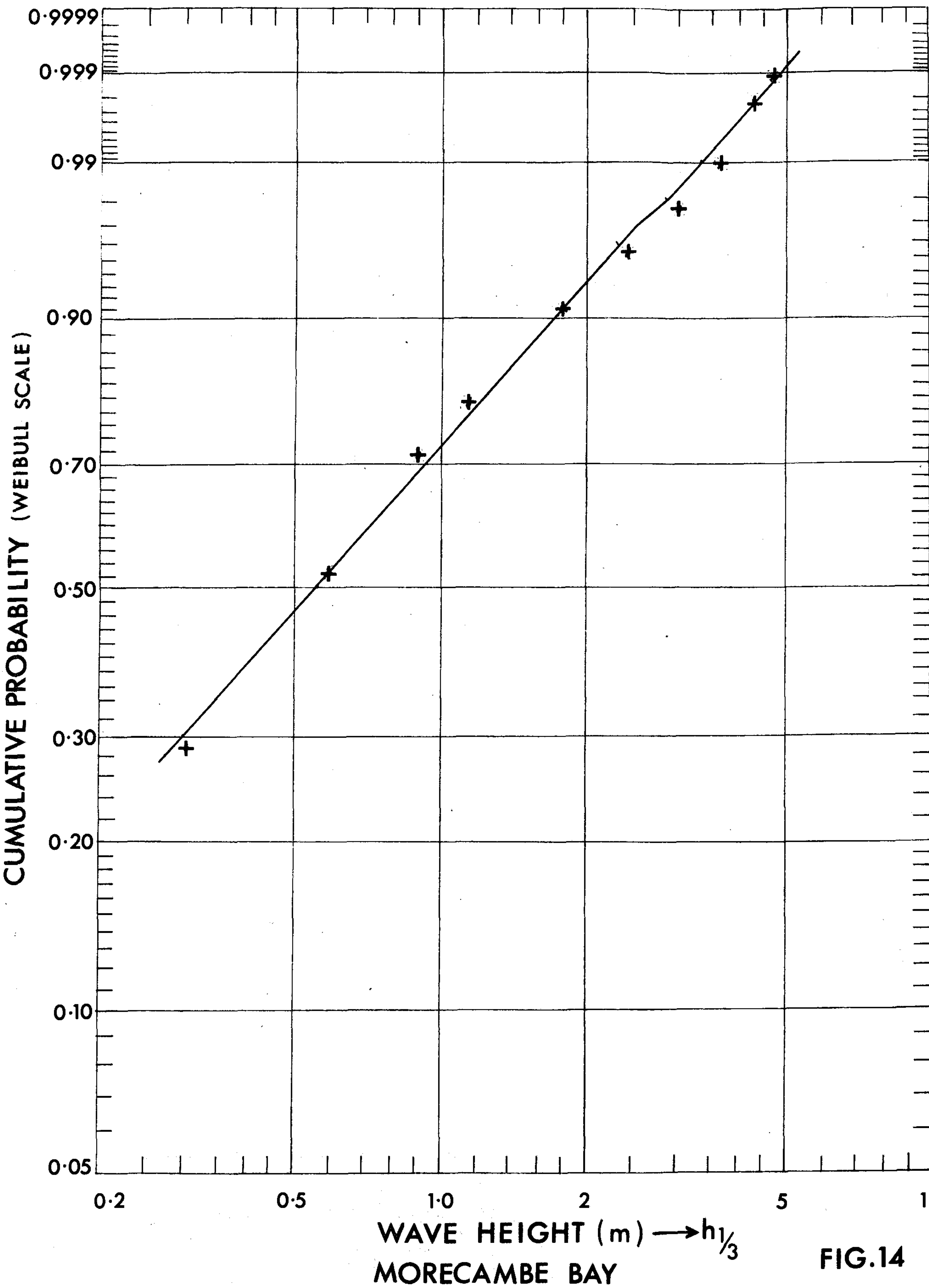
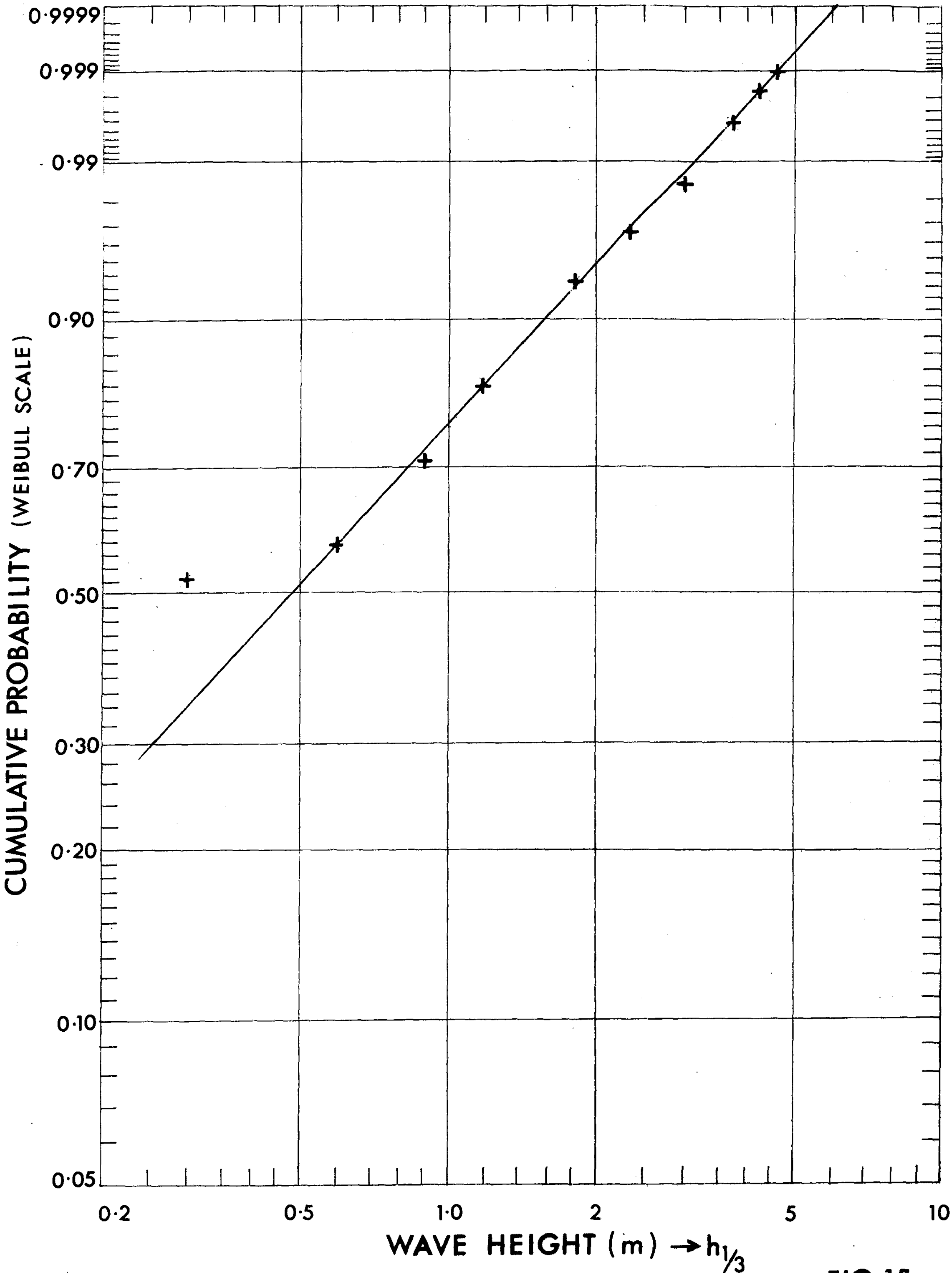
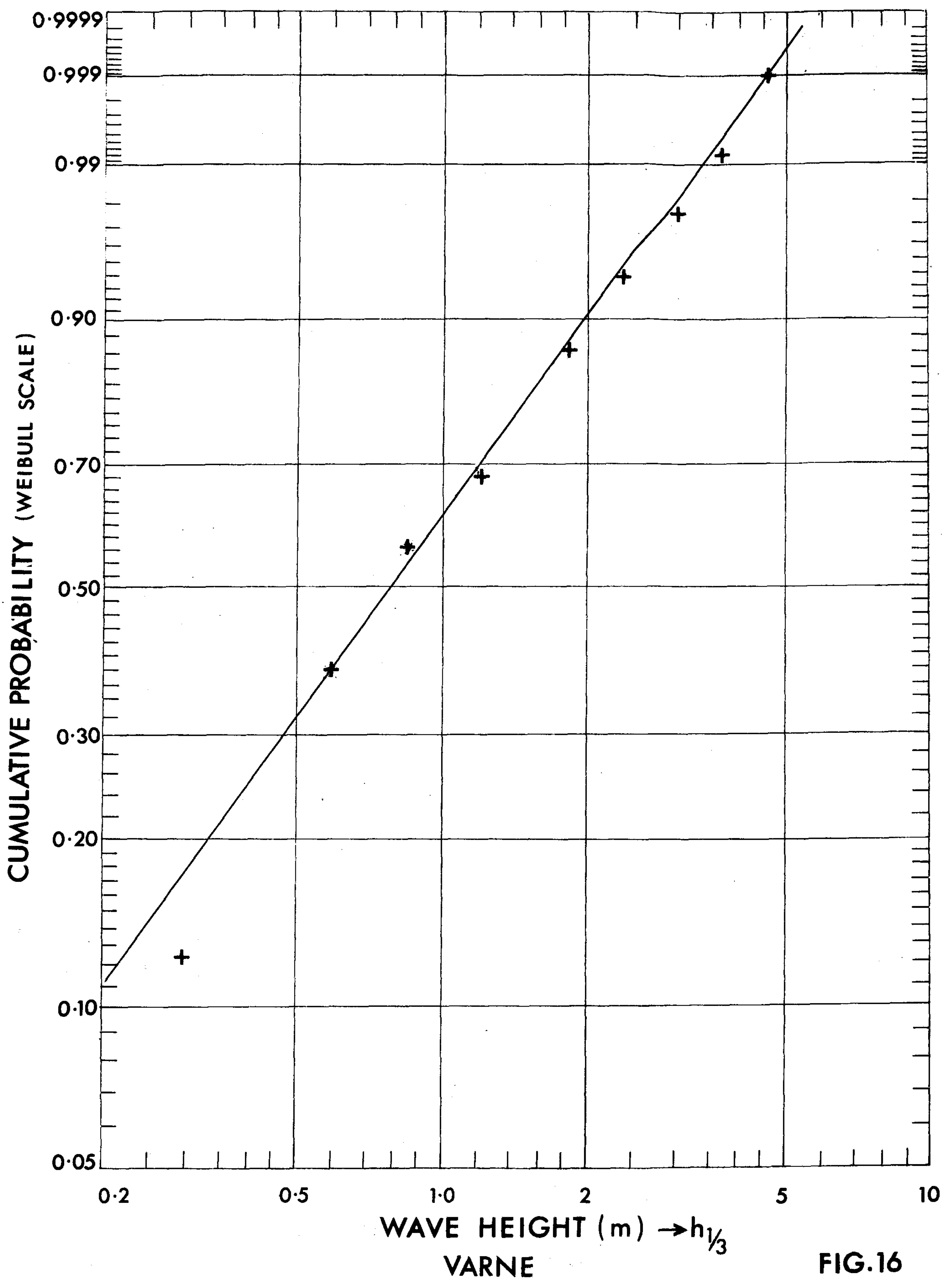

0·9999
0·999
0·99
0·90
0·70
0·50
0·30
0·20
0·10
0·05
CUMULATIVE PROBABILITY (WEIBULL SCALE)
0·2
0·5
1·0
2
5
10
WAVE HEIGHT (m) ⟶ $h_{1/3}$
MORECAMBE BAY

FIG.14



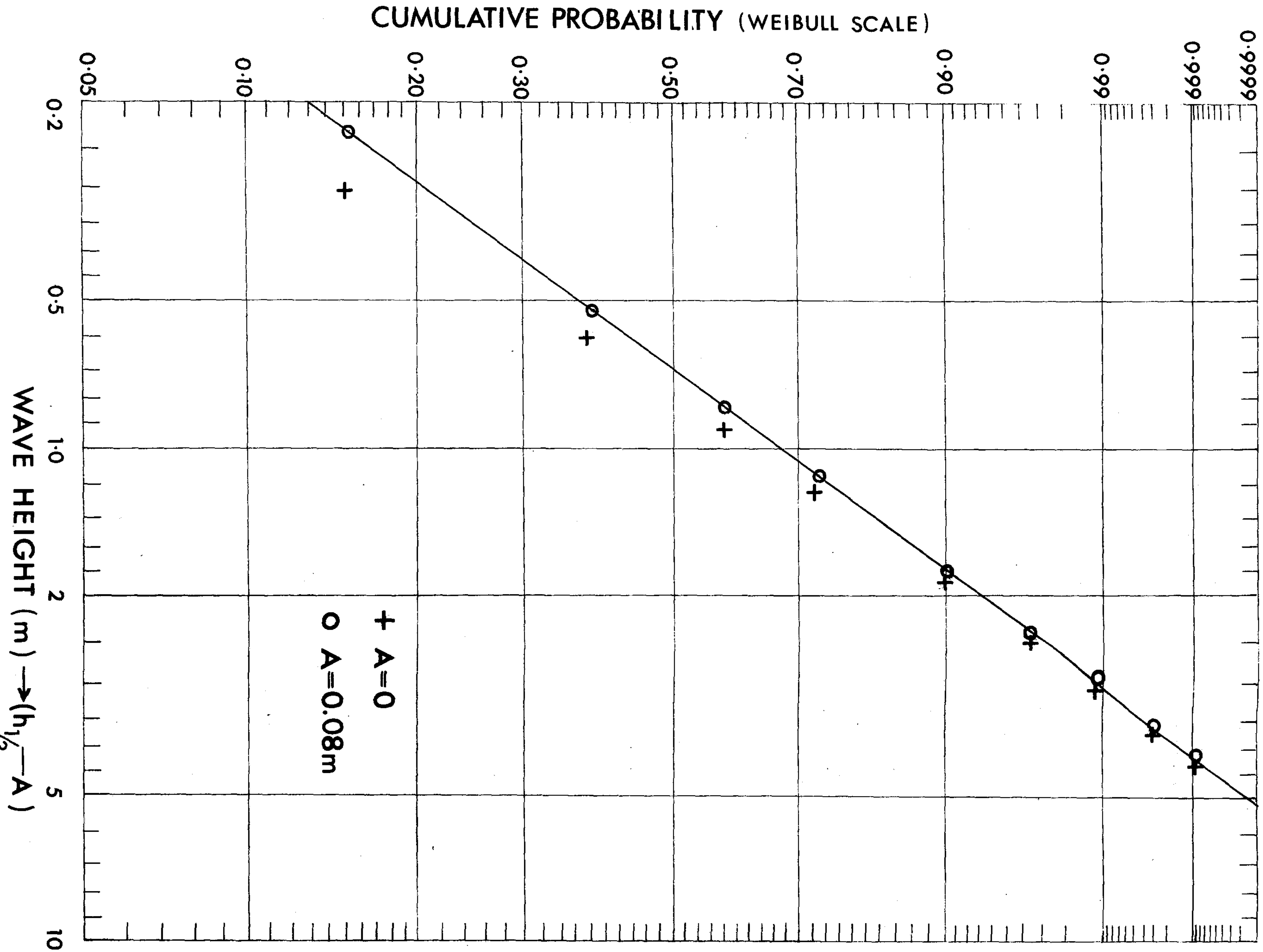
MERSEY BAR

FIG.15



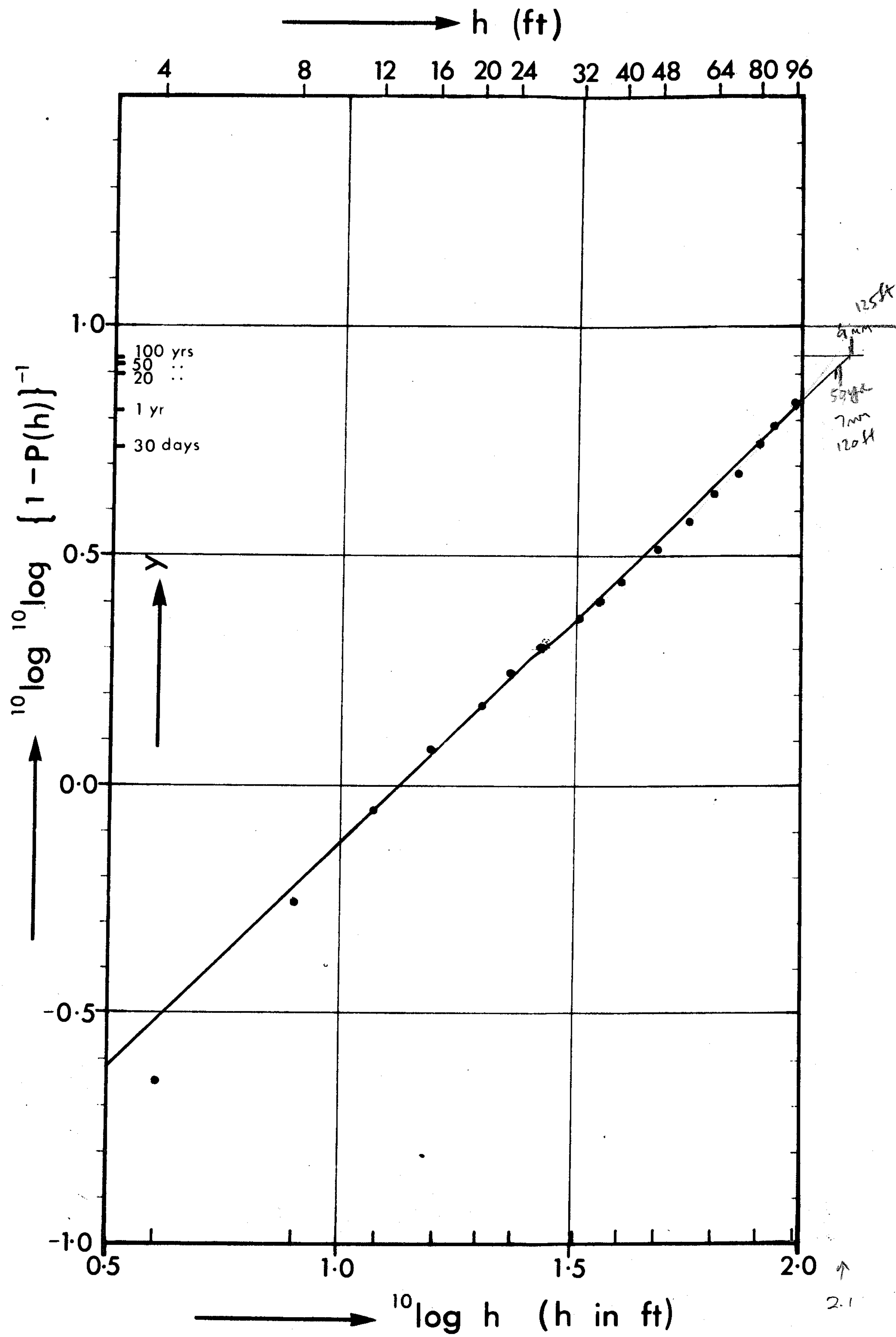
CUMULATIVE PROBABILITY (WEIBULL SCALE)
0·9999
0·999
0·99
0·90
0·70
0·50
0·30
0·20
0·10
0·05
0·2
0·5
1·0
2
5
10
WAVE HEIGHT (m) → $h_{1/3}$
VARNE

FIG.16



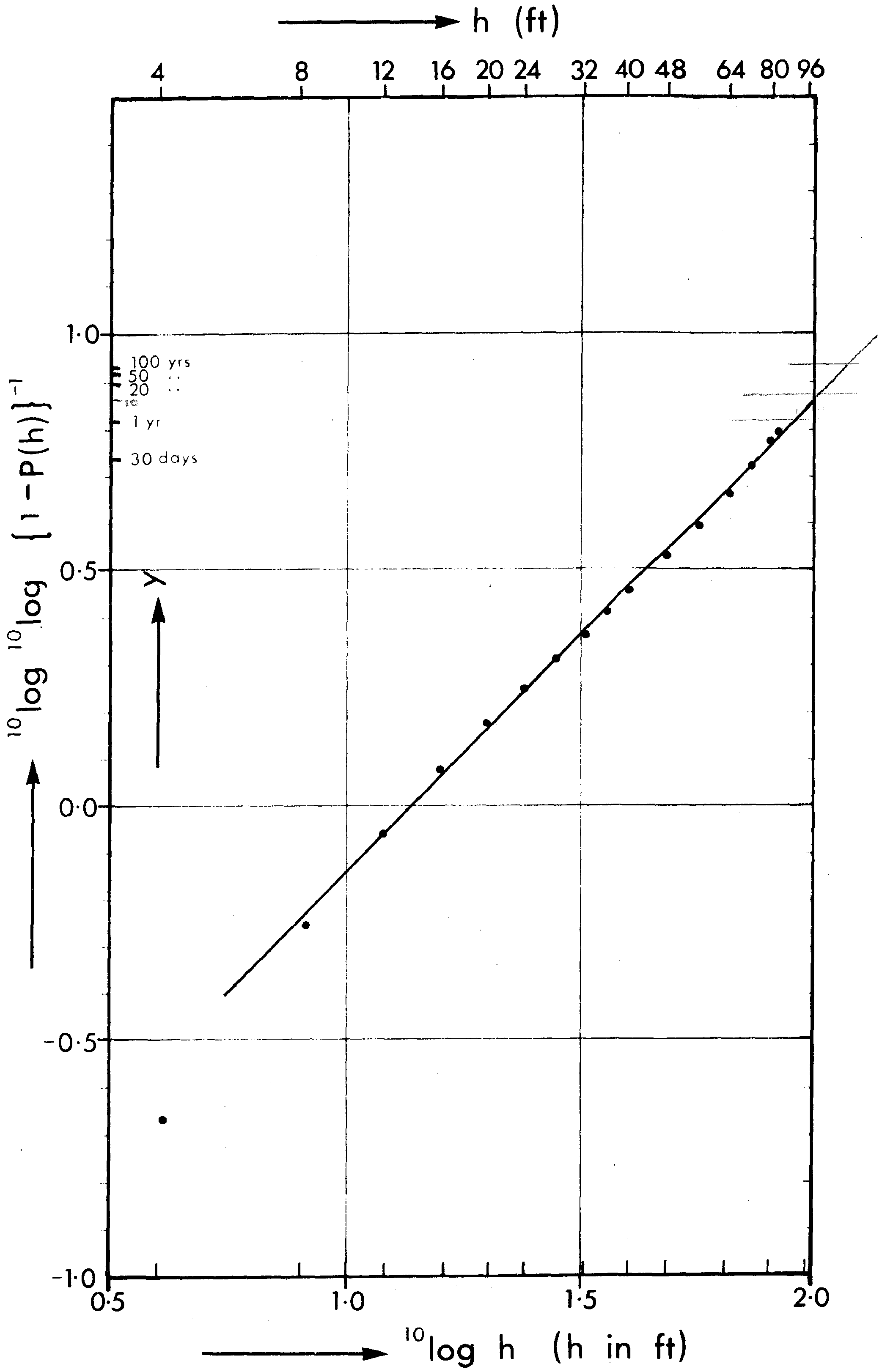
SMITH'S KNOLL

FIG.17

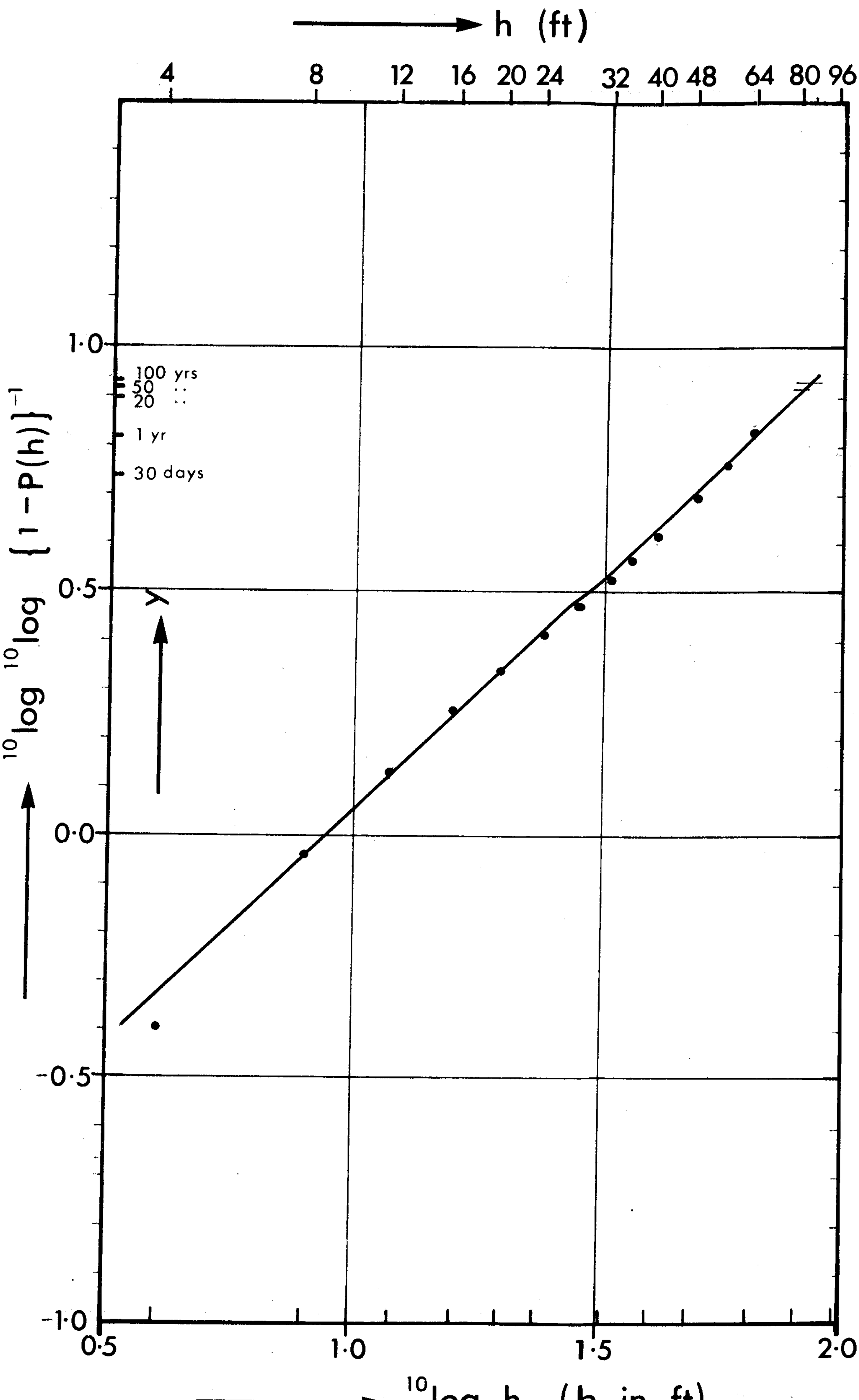


h (ft)
4
8
12
16
20
24
32
40
48
64
80
96
1·0
0·5
0·0
-0·5
-1·0
100 yrs
50
20
1 yr
30 days
y
$^{10}\log\ ^{10}\log\ \{1-P(h)\}^{-1}$
0·5
1·0
1·5
2·0
$^{10}\log h$ (h in ft)

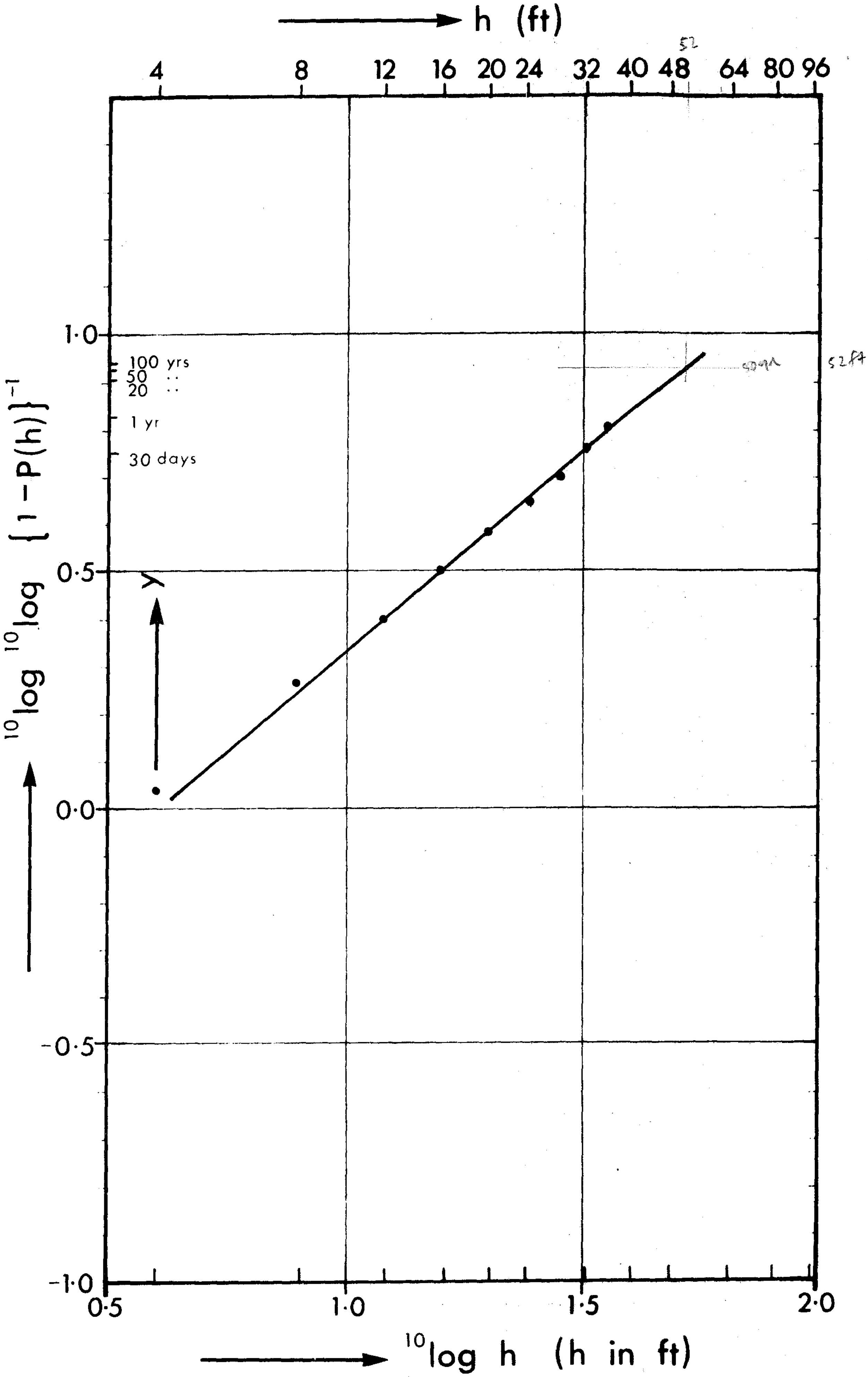
INDIA Fig. 18



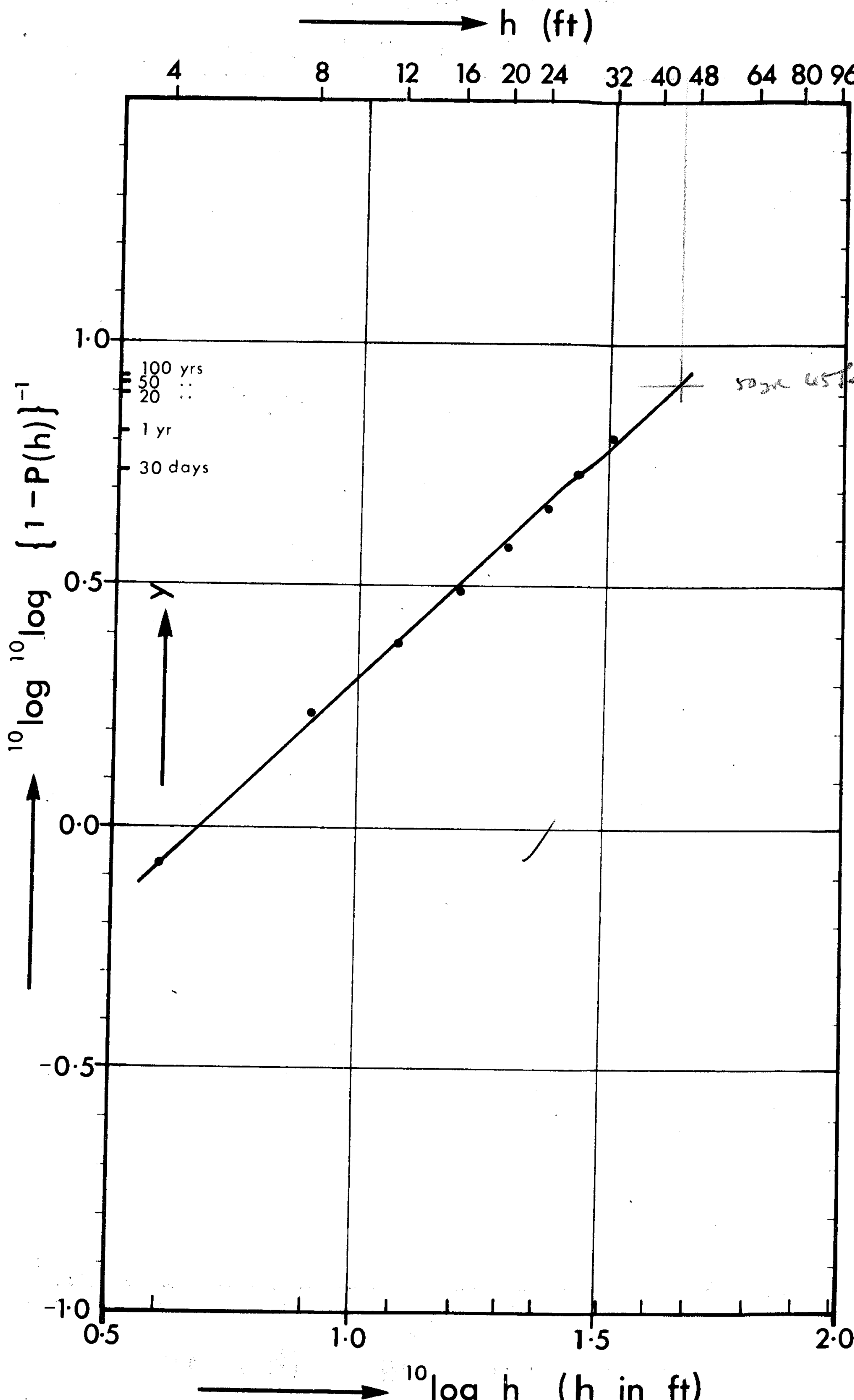
JULIETT Fig. 19



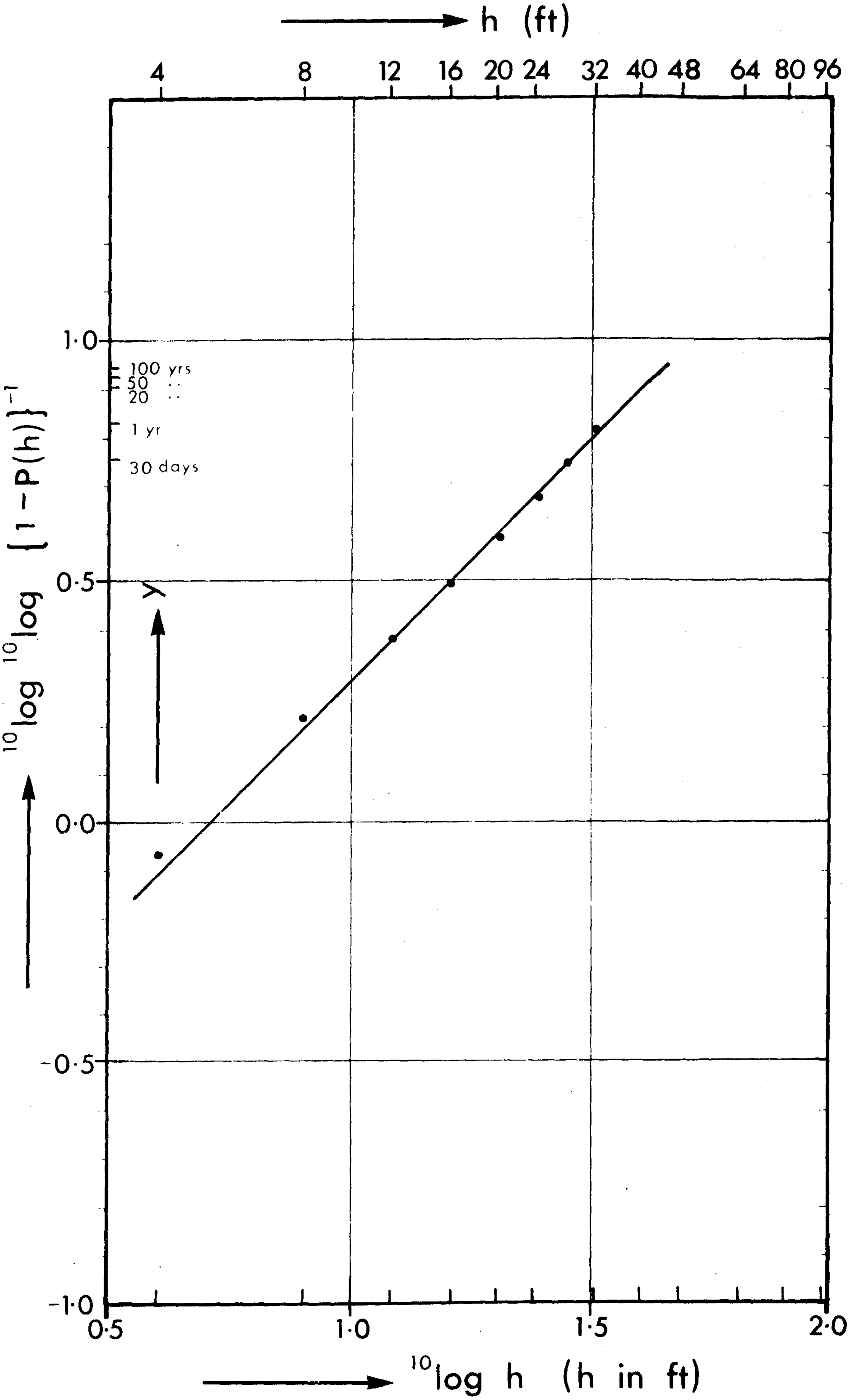
SEVENSTONES Fig. 20



MORECAMBE BAY Fig. 21



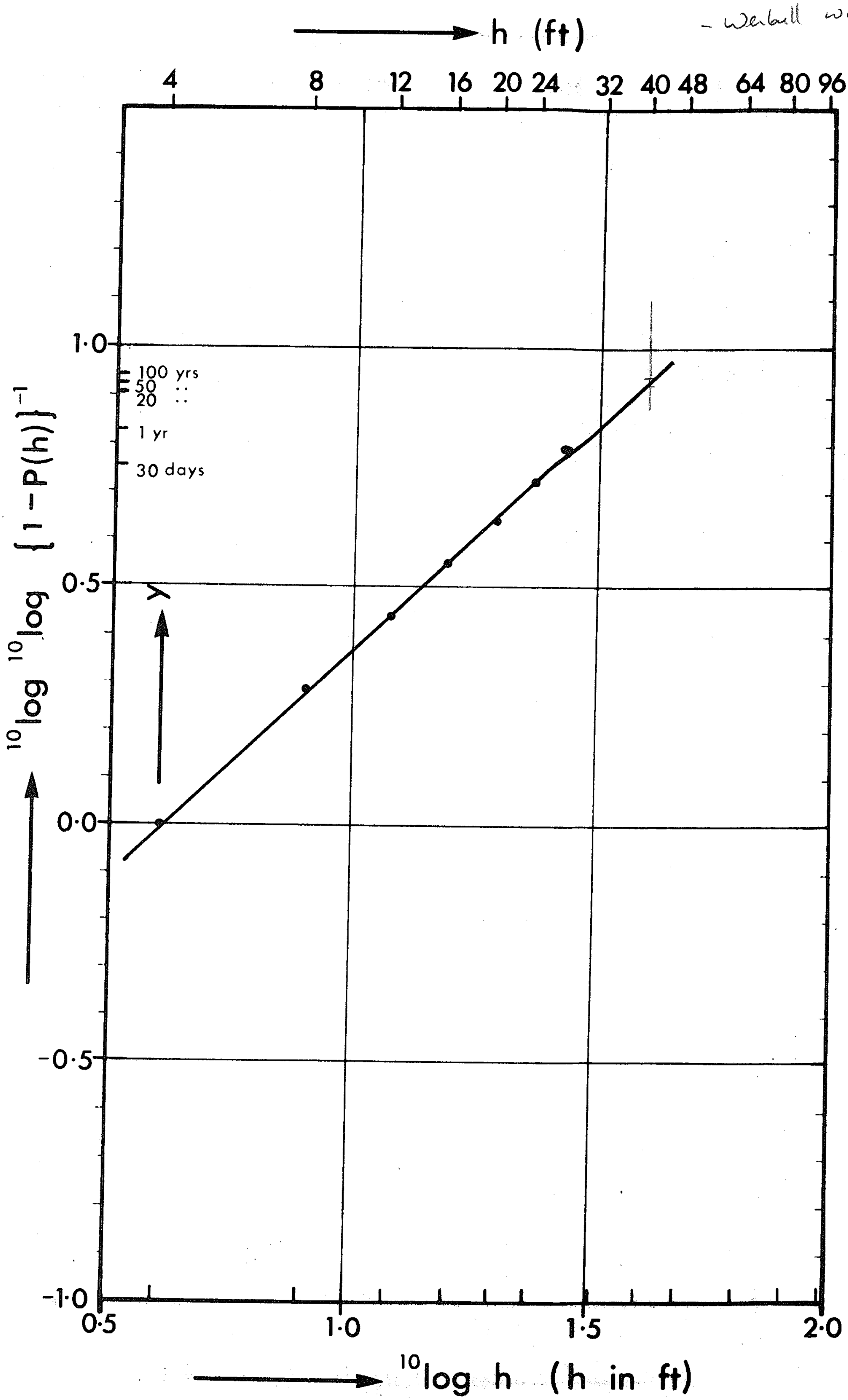
MERSEY BAR Fig. 22


h (ft)
4
8
12
16
20
24
32
40
48
64
80
96
1·0
0·5
0·0
-0·5
-1·0
100 yrs
50
20
1 yr
30 days
y
$^{10}\log\ ^{10}\log\ \{1-P(h)\}^{-1}$
0·5
1·0
1·5
2·0
$^{10}\log h$ (h in ft)

VARNE

Fig. 23

Logarithmic Dist^n
- Weibull with C=1



SMITH'S KNOLL Fig. 24