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N. I. O. Tide Elimination and Prediction Scheme

by

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C O N T E N T S

All the times given are overestimates, as they include some card handling, resetting of labels between runs, etc. on DEUCE. They apply only to DEUCE Mark I and Mercury at the R.A.E. Farnborough at the time of writing.

Introduction

This report contains operating instructions, which must be read in conjunction with the relevant N.I.O. Program descriptions, together with a few notes on the theory behind some of the programs. It is not a complete theoretical account of the scheme.

All details of data, parameters and operation should be entered on the record sheets, and the cards should be labelled as recommended in Sections (2.6), (3.5) and (4.5).

In most cases special plugboards for the Hollerith 902 tabulators at the Royal Aircraft Establishment, Farnborough, have been made to list the results.

Part I. Preparation of basic constants.

See N.I.O. Programs 16 and 33.

The starting point for the whole scheme is the choice of a constant W = 24D where D is an integer > 180, (preferably > 300), and 1·9323D is nearly an integer.

The basic matrices required in N.I.O. Program 13 are produced on tape by N.I.O. Program 33, and then converted to standard Scheme B binary on cards by N.I.O. Program 16.

This calculation is complete for W = 8856 and W = 8520.

If another W is required, note that the output from N.I.O. Program 33 is unlikely to be on one perfect tape. It is advisable to copy all the required matrices on to one new tape, in order of ascending u, replacing the elements which clearly should be 0 or 1 by their true values. The unit matrices are, of course, omitted. Handling tape on DEUCE is not easy, and it is an advantage to have one input tape which can be placed in the reader and then left alone.

Part II. Harmonic constituents : analysis.

See N.I.O. Programs 3, 13, 15, 20 and 37.

Introduction

When observed values of hourly tidal heights for about a year are available the harmonic constituents for the year can be obtained by analysis. This is described in Sections (2.1) to (2.5).

Section (2.1)

Punch the hourly values of height for a year as one series, for increasing time. It is necessary that 0000 hours should start a card. The values are regarded as integers in units of tenths of a foot or hundredths of a metre. There may be more than N + 1 values (N ≡ W is defined in Part I) but not less, so it might be necessary to extend the data at each end of the year. This should be done symmetrically unless otherwise specified.

The punching must be carefully checked, preferably by calling over and by N.I.O. Program 37. A mistake may invalidate over four hours' computing time on DEUCE.

The data is in the usual form of 8 signed 4-digit integers per card in the α-field, starting at the beginning of the first card.

We now suppose the values have been punched correctly, and all this data should be preserved for future occasions.

Section (2.2)

Select the data required for the chosen N; this will consist of N + 1 terms, not necessarily starting at the beginning of a card. Divide the data into two packs with the central value Z_0 at the beginning of a card.

The first pack, S_1 say, contains N/2 terms and the Z_0 term. This pack is then reversed with N.I.O. Program 15 to give a pack known as S^- with n + 1 terms, where N = 2n. Note that the number of terms required on the N.I.O. Program 15 parameter card may be greater than n + 1, because enough (< 8) terms before Z_{-n} must be included in S_1 to make the first data card complete.

The second pack starts with Z_0 and has N/2 terms following Z_0. This pack is S^+ and needs no further preparation.

We now have two series, each with n + 1 terms, denoted by

$$S^- = Z_0, Z_{-1}, Z_{-2}, \ldots\ldots\ldots Z_{-n}$$

$$\text{and } S^+ = Z_0, Z_1, Z_2, \ldots\ldots\ldots Z_{+n}.$$

Note the first, last and central dates of the selected data on the record sheets; also note N, n, Z_0, Z_{-n} and Z_{+n}.

Section (2.3)

Check the restrictions in N.I.O. Program 3, for both S^- and S^+ separately. If necessary, add a negative constant c to each Z_r, using N.I.O. Program 20, but do not multiply Z_r by a constant. Note c on the record sheets.

Section (2.4)

In both S^- and S^+ replace Z_0 by $Z_0/2$. If N.I.O. Program 20 has been used, the first term in each series becomes $(Z_0 + c)/2$ and not $c + Z_0/2$.

Run S^- with N.I.O. Program 3. Here k = 2 and the number of terms is n and not n + 1. The value Z_{-n} is not used but may be left on the card. Cards containing the s-data exist for N = 8856 and N = 8520 and are listed below. The values of s are the same as the u-data listed for input to N.I.O. Program 33. Always s = 0 comes last, just before the end indication.

Run S^+ with N.I.O. Program 3. The parameters are exactly the same as for S^-, except that for Z_{-n} read Z_{+n}.

N	Card no.	s-data							
8520	0000	0001	0002	0012	0013	0024	0026	0304	0305
8520	0001	0306	0317	0318	0319	0320	0330	0332	0343
8520	0002	0344	0345	0346	0353	0354	0355	0356	0357
8520	0003	0358	0367	0368	0369	0380	0382	0647	0649
8520	0004	0660	0661	0662	0673	0674	0675	0676	0684
8520	0005	0686	0688	0697	0698	0699	0700	0709	0710
8520	0006	0711	0712	0722	0723	0724	0725	0734	1016
8520	0007	1029	1040	1042	1066	1359	1371	1372	1383
8520	0008	1396	1397	1398	1420	1422	2045	2057	2058
8520	0009	2069	2082	2083	2084	2106	2108	2744	3430
8520	0010	4116	0000	-					

N	Card no.	s-data							
8856	0000	0001	0002	0013	0014	0025	0027	0316	0318
8856	0001	0329	0330	0331	0332	0343	0345	0356	0357
8856	0002	0358	0359	0367	0368	0369	0370	0371	0372
8856	0003	0381	0382	0383	0384	0395	0397	0672	0673
8856	0004	0675	0686	0688	0699	0700	0701	0702	0711
8856	0005	0713	0715	0724	0725	0726	0727	0737	0738
8856	0006	0739	0740	0751	0753	0754	0763	1056	1069
8856	0007	1070	1081	1083	1108	1412	1413	1426	1437
8856	0008	1438	1451	1453	1476	1478	2125	2126	2139
8856	0009	2150	2151	2164	2166	2189	2191	2852	3565
8856	0010	4278	0000	-					

Section (2.5)

Run N.I.O. Program 13 to give the final harmonic constituents (64 cards)

Section (2.6)

Card labelling for Part II

Column 1 must be left blank.

Columns 2, 3 and 4 cannot be punched by the DEUCE punch, so are generally only used on hand punched cards.

Columns 5 to 9 are reserved for job numbers punched on DEUCE output.

Columns 10, 11 and 12 are summarised below.

Column \ Section	(2.1)	(2.2)	(2.3)	(2.4)	(2.5)
10	-	-	1	0,1	0,1
11	-	9	9,1	9,1	8
12	-	-	-	1,2	1,2

Column 10   0 ≡ no constant added to S^- and S^+
1 ≡ constant added to S^- and S^+

Column 11   9 ≡ S^- or associated results
1 ≡ S^+ or associated results
8 ≡ complete harmonic constituents

Column 12   1 ≡ N = 8856
2 ≡ N = 8520

Section (2.7)

Numbered dates

Notation   D = date
h = hour ($0 \leq h \leq 23$)
d = number associated with D.   (See table).

D, h and d are positive integers.
For a leap year, add 1 to d if $d \geq 59$.
February 29 is numbered 59.
Let $D_0.h_0$ be a fixed date, numbered d_0.

Dates before D_0

Let D_1, h_1, d_1 represent a date before D_0.
Let r_1 be the number of $D_1.h_1$, counting in hours from $D_0.h_0$ as origin; that is, $D_0.h_0$ is numbered 0. Note that $r_1 < 0$.

(1) $D_1.h_1$ known, r_1 unknown.
Look up d_1 from the table, taking account of leap years. Then

$$r_1 = 24(d_1 - d_0) + h_1 - h_0.$$

(2) r_1 known, D_1 and h_1 unknown.
Work out α_1 and β_1, where

$$-r_1 = 24\alpha_1 + \beta_1, \quad 0 \leq \beta_1 \leq 23.$$

Then $d_1 = d_0 - \alpha_1$,
and $h_1 = h_0 - \beta_1$.

If it is a leap year, subtract 1 from d_1 if $d_1 \geq 60$ before looking up D_1. If $d_1 = 59$ before subtraction the date is February 29.

Note that subtraction may involve carry from days to hours.

Dates after D_0

Let D_2, h_2, d_2 represent a date after D_0. Define r_2 as above.
Note that $r_2 > 0$.

Proceed exactly as above, but using the following formulae.

(1) $D_2.h_2$ known, r_2 unknown.
Here $r_2 = 24(d_2 - d_0) + h_2 - h_0$.

(2) r_2 known, D_2 and h_2 unknown.
Here $r_2 = 24\alpha_2 + \beta_2$,

$d_2 = d_0 + \alpha_2$,

and $h_2 = h_0 + \beta_2$.

Day	(31) Dec.	(31) Jan.	(28) Feb.	(31) Mar.	(30) Apr.	(31) May	(30) Jun	(31) Jul	(31) Aug.	(30) Sep.	(31) Oct.	(30) Nov.	(31) Dec.	(31) Jan.
1	-31	0	31	59	90	120	151	181	212	243	273	304	334	365
2	-30	1	32	60	91	121	152	182	213	244	274	305	335	366
3	-29	2	33	61	92	122	153	183	214	245	275	306	336	367
4	-28	3	34	62	93	123	154	184	215	246	276	307	337	368
5	-27	4	35	63	94	124	155	185	216	247	277	308	338	369
6	-26	5	36	64	95	125	156	186	217	248	278	309	339	370
7	-25	6	37	65	96	126	157	187	218	249	279	310	340	371
8	-24	7	38	66	97	127	158	188	219	250	280	311	341	372
9	-23	8	39	67	98	128	159	189	220	251	281	312	342	373
10	-22	9	40	68	99	129	160	190	221	252	282	313	343	374
11	-21	10	41	69	100	130	161	191	222	253	283	314	344	375
12	-20	11	42	70	101	131	162	192	223	254	284	315	345	376
13	-19	12	43	71	102	132	163	193	224	255	285	316	346	377
14	-18	13	44	72	103	133	164	194	225	256	286	317	347	378
15	-17	14	45	73	104	134	165	195	226	257	287	318	348	379
16	-16	15	46	74	105	135	166	196	227	258	288	319	349	380
17	-15	16	47	75	106	136	167	197	228	259	289	320	350	381
18	-14	17	48	76	107	137	168	198	229	260	290	321	351	382
19	-13	18	49	77	108	138	169	199	230	261	291	322	352	383
20	-12	19	50	78	109	139	170	200	231	262	292	323	353	384
21	-11	20	51	79	110	140	171	201	232	263	293	324	354	385
22	-10	21	52	80	111	141	172	202	233	264	294	325	355	386
23	-9	22	53	81	112	142	173	203	234	265	295	326	356	387
24	-8	23	54	82	113	143	174	204	235	266	296	327	357	388
25	-7	24	55	83	114	144	175	205	236	267	297	328	358	389
26	-6	25	56	84	115	145	176	206	237	268	298	329	359	390
27	-5	26	57	85	116	146	177	207	238	269	299	330	360	391
28	-4	27	58	86	117	147	178	208	239	270	300	331	361	392
29	-3	28	?	87	118	148	179	209	240	271	301	332	362	393
30	-2	29	-	88	119	149	180	210	241	272	302	333	363	394
31	-1	30	-	89	-	150	-	211	242	-	303	-	364	395

Part III. Harmonic constituents : use of previous analysis.

See N.I.O. Programs 18 and 24.

Introduction

Consider a year B, say, for which a tide prediction is required at a certain place; for example, Stornoway. No observed hourly tidal heights are available, so harmonic constituents cannot be obtained by analysis. They are therefore estimated from another year's data. Sections (3.1) to (3.4) describe how 64 cards, corresponding to N.I.O. Program 13 output, are produced.

Section (3.1)

Decide on a central date (0000 hours) for year B. Calculate the parameters s, h, p and N, (see Section (2.7)) and write the results, with the check, on the record sheets. Run N.I.O. Program 18 (no data) to give i, f, u and V for year B (i = 0(1)62). Give the results a Q-number.

Section (3.2)

There are two alternatives here. Both achieve the same result; namely, 63 cards containing i, H and g in the layout of N.I.O. Program 18 output. (i = 0(1)62). H and g are absolute constants for a given place.

(i) Punch H and g directly from tables in the correct layout. i = 0(1)62 must also be punched.

(ii) Select a year A, say, for which full data is available for the place where the prediction is required. In general, but not necessarily, year A is before year B. Carry out Sections (1) to (5) of Part II for year A if this has not already been done.

We now have 64 cards consisting of N.I.O. Program 13 output for year A.

The central date of year A will have been decided already, so calculate s, h, p and N as in Section (3.1). Run N.I.O. Program 18 (with data) to give i, H, g, f, u and V for year A. Give the results a Q-number. Although f, u and V are present on the cards, they are not required for year A.

Section (3.3)

Run N.I.O. Program 24, using the results of (3.1) and (3.2) as data.

We now have 63 cards, corresponding to the first 63 cards of N.I.O. Program 13 output.

Section (3.4)

The 64th card, containing α_0, must now be added. As in Section (3.2) there are two alternatives.

(i) Punch α_0 directly from tables. For the layout see N.I.O. Program 13 output.

(ii) If α_0 for year A is available (for example, if section (3.2)(ii) has been followed), then this value may be taken as α_0 for year B. The card containing α_0 (card no. 63 of N.I.O. Program 13 output) should be copied, taking care to alter the card labelling where necessary.

Section (3.5)

Card labelling for Part III.

Columns 1 to 4 are used as in Part II.
Columns 5 to 8 are reserved for job numbers punched on DEUCE output. No run numbers are necessary.
Columns 9 to 12 are summarised below.

Column \ Section	(3.1)	(3.2)	(3.3)	(3.4)
9, 10	Q	Q	P	P
11	[illegible]	[illegible]	7	7
12	8	8	9	9

Columns 9, 10. Set a 2 digit Q or P number (see record sheets).

Column 11. 7 ≡ harmonic constituents (using previous analysis).

Column 12. 8 ≡ cards contain f, u and V and possibly H, g.
9 ≡ no associated N.

Part IV. Prediction.

See N.I.O. Programs 21 and 23.

Introduction

The prediction follows on from either Part II or Part III, as 64 cards containing the specified harmonic constituents are required as data.

Section (4.1)

Besides the harmonic constituents, the data for N.I.O. Program 23 consists of the two series known as S^- and S^+. If the observed hourly tidal heights are available, the preparation of S^- and S^+ is as described in Sections (2.1) and (2.2). They will, of course, already be in the correct form if Part II has been followed to obtain the harmonic constituents. If hourly tidal heights are not available, artificial series S^- and S^+ must be constructed. This is best done by putting

$$S^- \equiv Z_0 = 0, \text{ and } S^+ \equiv Z_0 = 0,$$

so that [illegible] + 1 = 1. Note that in this case $T_r \equiv \Sigma_r$ for all r.

If prediction is required for a few months only, S^- and S^+ must be specially prepared (if available).

Section (4.2)

Run N.I.O. Program 23.

The calculation itself is simple, but the slow speed, small store and general unreliability of DEUCE make division of the range into smaller sections necessary. Great care must be taken in assembling the final results. The sections should overlap so there is at least one check card in each case. If the prediction is required at 3 hourly intervals, a satisfactory set of range cards is

i) a = 0 , c = 897
ii) a = 888 , c = 1785
iii) a = 1776, c = 2673
iv) a = 2664, c = 3561
v) a = 3552, c = 4449

In all cases b = 3. Note that the results for S^- and S^+ are not consecutive in the final pack unless a = 0. (See Section (2.7) when selecting range cards.)

We now suppose that the suggested range cards have been used. Each range card produces 150 cards of results, and the assembled pack is numbered

600-674, 450-524, 300-374, 150-224, 0-74 for S^-
and 75-149, 225-299, 375-449, 525-599, 675-749 for S^+.

The card count must not be reset to zero between runs. Cards 674, 524, 374, 224, 149, 299, 449 and 599 are check cards, and should compare exactly for S^-, and in all but the last 8 columns for S^+, with the α-field of the first card of the succeeding group. When agreement is obtained the 8 check cards must be scrapped. Note that the last 8 columns on card 749 are meaningless. List the results, omitting the check cards.

It is advisable to inspect the count r = a(b)c occasionally to see that the program is working properly. The relevant integers are in

19_3 r.P_1 12_{10} b.P_1
12_9 a.P_1 12_{11} c.P_1.

Section (4.3)

We now assume that the results of N.I.O. Program 23 have been assembled and checked correctly. The sum of tidal components (Σ_r) and the remainder ($T_r = Z_r - \Sigma_r$) are next separated and the results arranged so that one card corresponds to one day, starting at 0000 hours.

We have 10 groups of 74 or 75 cards each, numbered

600 - 673	(74 cards)
450 - 523	(")
300 - 373	(")
150 - 223	(")
0 - 74	(75 cards)
75 - 148	(74 cards)
225 - 298	(")
375 - 448	(")
525 - 598	(")
675 - 749	(75 cards)

The last two numbers on card 74 correspond to r = 0; that is, the central date with original reading Z_0. Mark the listing at this point.

We now require T_r and Σ_r separated, but only for $-n \leqslant r \leqslant +n$, which may mean setting aside a few cards at the beginning and end. Range card ([illegible]) deliberately covers a wide range, so that all likely values of N are included.

In general $|T_r| << |\Sigma_r|$, so it is easy to pick out T_{-n} and Σ_{-n} by inspection. They are the first pair of values for which $T_r \neq -\Sigma_r$. Mark T_{-n} and Σ_{-n} on the listing. Also mark T_n and Σ_n, which may be found in a similar way. If this method is not possible (that is, if $S^- \equiv S^+ \equiv 0$), the appropriate terms can be found from the table in Section (4.4).

We now come to the sorting, using N.I.O. Program 21. Unfortunately DEUCE is too small to sort the whole lot at once, so for convenience divide the cards into the usual S^- and S^+ sections. The S^- section contains cards up to and including card 74, and the S^+ section starts with card 74. This provides an overlap of one value, namely T_0 or Σ_0.

At the end of this section is a table of cards to be included in the sorting, together with parameters for N.I.O. Program 21. These parameters have been chosen to ensure that results for one day appear on a single card, at 3 hourly intervals from 0000 hours to 2100 hours. Note that the wanted results begin and end at 1200 hours. The parameters for N = 8856 have been chosen so that for a pure prediction ($S^- \equiv S^+ \equiv 0$), where N does not matter, the complete first and last cards containing Σ_r are valid.

Run N.I.O. Program 21 with the S^- section of data, resetting the card count to zero before both T_r and Σ_r. Each output pack may end with a few cards containing zeros, due to the imperfections of the punch subroutine in N.I.O. Program 21, and these should be scrapped. The last card now contains one non-zero number (T_0 or Σ_0) and this is the check card. Note carefully the number (N^+) of this card, as the card count for the S^+ section must begin here.

Run N.I.O. Program 21 with the S^+ section of data, setting the card count to N^+ before both T_r and Σ_r.

Compare the first card in each case with its check card, to ensure that T_0 (or Σ_0) agrees. Then scrap the surplus S^- card.

We now have two packs, T_r and Σ_r, each with the card count running continuously. The first valid result is the fifth number on the first card (T_{-n} or Σ_{-n}), that is, at 1200 hours. Similarly the last valid result (T_n or Σ_n) is the fifth number on the last card (other than cards of zeros).

Notes on N.I.O. Program 21

	Section	N = 8856 First card	N = 8856 Last card		N = 8520 First card	N = 8520 Last card	
Input (N.I.O. 23 output retained)	S^-	600	74		614	74	
	S^+	74	748		74	733	
	Section	n	s	t	n	s	t
Parameters	S^-	2968	6	0	2856	6	0
	S^+	2968	6	0	2848	6	0
	Section	First card	Last card		First card	Last card	
Output	S^-	0	185		0	178	
	S^+	185	369		178	355	

See Section (4.2) for the relevant range cards etc.

Section (4.4)

As a final check, compare the original Z_r with $T_r + \Sigma_r$ for a few dates, remembering that Z_r is listed hourly and T_r and Σ_r are listed every 3 hours. At least the values for $r = 0$, $r = \pm$ n should be checked.

Location of useful values

Source	Value	Section	N = 8856 Card	N = 8856 Term	N = 8520 Card	N = 8520 Term
N.I.O. 23 output	T_{-n}	S^-	601	7	615	7
	T_o	S^-	74	7	74	7
	T_{+n}	S^+	747	7	733	7

Σ_r immediately follows T_r.

Source	Value	Section	N = 8856 Card	N = 8856 Term	N = 8520 Card	N = 8520 Term
N.I.O. 21 output	T_{-n}	-	0	5	0	5
	T_o	-	185	0	178	0
	T_{+n}	-	369	5	355	5

Σ_r is in the same position as T_r.

See Section (4.2) for the relevant range cards etc.

Section (4.5)

Card labelling for Part IV

Columns 1 - 4 are used as in Part II.

Section (4.1) S^+ will always have the original data labels. There are two possibilities for S^-:-

(i) If S^- is a normal half-year's data, it is labelled as in Section (2.2).

(ii) If S^- covers a few months only and is a section of original data reversed specially, then it is labelled as in Section (2.2), with a 3 punched in Column 12.

Sections (4.2) and (4.3) If Part IV was preceded by Part II, Columns 5 - 9 contain the job and run numbers and Column 10 is left blank. If Part IV was preceded by Part III, Columns 5 - 8 contain the job number and Columns 9 and 10 contain the P number.

In either case Columns 11 and 12 are as follows:-

Column \ Section	(4.2)	(4.3)
11	9,1	2,3
12	1,2,9	1,2,9

Column 11 9 ≡ results associated with S^-
1 ≡ results associated with S^+
2 ≡ T_r
3 ≡ Σ_r

Column 12 1 ≡ N = 8856 } These only apply when Part IV follows Part II,
2 ≡ N = 8520 } with the same S^- and S^+ in each part.
9 ≡ Any other prediction.

Part V. Notes on the theory.

Introduction

This is not a comprehensive account of the theory of tide elimination and prediction, but merely a collection of comments on some of the Deuce and Mercury programs in the N.I.O. Scheme. In general, the "Method" section is extended, and if possible more notes on the accuracy of results are included. Definitions given in the program descriptions are assumed here.

The two main aspects covered are the basic matrices (Part I) and the harmonic analysis (Part II).

Section (5.1)

The preparation of these matrices, using N.I.O. Program 33, is the first step in tide prediction (see Section (5.6)).

We start with 63 given tidal constants v_j (speeds in degrees/hour) based on astronomical data. Terms of the same order (that is, belonging to the same tidal species) are grouped together and each of the 9 resulting groups is treated separately.

We also require a constant W (or N), which must not differ greatly from the number of hours in a year; and which makes $Wv_j/360$ very nearly integral for as many values of j as possible. The best choice is W = 8520, but W = 8856 is fairly good. As in N.I.O. Program 33 we define

$$\alpha = \alpha_{ij} = Wv_j/360 - u_i \tag{1}$$

and the integers u_i are chosen so that for each i

$$|\alpha| \leqslant A \tag{2}$$

for one value of j. (See Section (5.7)). For W = 8520, A = 0·10, but for W = 8856, A = 0·39. For the last 3 constituents A may be larger than the specified value. If (2) cannot be satisfied, the two nearest integers are included, and this is why M > N in general, where i = 0(1)M-1 and j = 0(1)N-1 for any typical example of the 9 groups mentioned above.

The cosine and sine matrices, $\underset{\sim}{C}^*$ and $\underset{\sim}{D}^*$, have elements

$$a_{ij} = k_{ij} \sin\pi v_j/180 \tag{3}$$

$$\text{and} \quad b_{ij} = k_{ij} \sin 2\pi u_i/W \tag{4}$$

respectively, where

$$k_{ij} = \frac{\sin\pi\alpha}{\pi\alpha} \Bigg/ \frac{\sin\pi\alpha/W}{\pi\alpha/W} \cdot \sin \pi(\alpha + 2u_i)/W. \tag{5}$$

These relations hold for all i and j; for since

$$0 < v_j < 180 \quad \text{for all } j, \tag{6}$$

$$\text{and } 0 < u_i < W/2 \quad \text{for all } i, \tag{7}$$

by definition, we see that

$$\left|\frac{\alpha}{W}\right| < 1 \tag{8}$$

$$\text{and } 0 < \frac{\alpha + 2u_i}{W} < 1, \tag{9}$$

so that the denominator of k_{ij} is never zero.

If $\alpha = 0$, then the appropriate row or column has

$$a_{ij} = b_{ij} = 1 \text{ on the "diagonal"}, \qquad (10)$$

$$\text{and } a_{ij} = b_{ij} = 0 \text{ otherwise.} \qquad (11)$$

This only happens for the pure solar tides S_1, S_2, etc.

Section (5.2)

The maximum error in a_{ij} or b_{ij} can be much larger than the initial error in v_j, and inspection of (1) to (5) shows that $\frac{Wv_j}{360}$ is the only term in which an error initially present in v_j is increased. Thus the maximum error which occurs when calculating a_{ij} and b_{ij} is contained in the term $\frac{\sin\pi\alpha}{\pi\alpha}$.

We consider units of the 6th decimal place, and suppose that the maximum errors in v_j, $\pi\alpha$ and $\frac{\sin\pi\alpha}{\pi\alpha}$ are ϵ_1, ϵ_2 and ϵ_3 respectively.

The v_j are given to 7D, so we may assume that

$$|\epsilon_1| \leq 0\cdot05,$$

whence $$|\epsilon_2| \leq 24\cdot6\ \pi|\epsilon_1| < 3\cdot9,$$

since u_i is exact and $W = 8856$ or 8520.

Now $$|\epsilon_3| = \left|\frac{\sin(\pi\alpha + \epsilon_2)}{\pi\alpha + \epsilon_2} - \frac{\sin\pi\alpha}{\pi\alpha}\right|$$

$$= |\epsilon_2\beta| + 0(\epsilon_2^2),$$

where $$\beta = \frac{\pi\alpha\cos\pi\alpha - \sin\pi\alpha}{(\pi\alpha)^2}.$$

It can be shown that $|\beta|$ reaches its maximum value of about $0\cdot44$ near $\pi\alpha = 2$, so that the largest error is likely to occur in terms just off the diagonal, when $\alpha(\doteqdot 0\cdot6)$ is small but not too small.

So now $$|\epsilon_3| < 0\cdot45\ |\epsilon_2| < 1\cdot8,$$

so that it is safe to assume a maximum error of $\pm$ 2 in the 6th decimal place for all the a_{ij} and b_{ij}.

This is in fact the maximum error in those elements which should a priori be exactly zero or unity. The final tapes and listings for $W = 8856$ and $W = 8520$ have been altered to the true values in these cases.

It is assumed that the least squares inversion (see Section (5.6)) does not increase the error.

Section (5.3)

In the following notes on the harmonic analysis W is replaced by N.

We start with a set of (N + 1) tidal height observations to the nearest $\frac{1}{10}$ foot taken at hourly intervals over a period about one year. Deuce cannot accept such a large amount of data, so it is necessary to divide the series into two parts S^- and S^+, defined as follows:-

$$S^- = Z_0, Z_{-1}, Z_{-2}, \ldots\ldots Z_{-n}, \qquad (1)$$

$$S^+ = Z_0, Z_1, Z_2, \ldots\ldots Z_n. \qquad (2)$$

where $N = 2n$. The two cases considered are $N = 8856$ and $N = 8520$.

The required harmonic components for the whole series are

$$\begin{matrix}\alpha_u \\ \beta_u\end{matrix} = \frac{2}{N} \sum_{-n}^{+n} y_r \begin{matrix}\cos \\ \sin\end{matrix} \frac{2\pi ru}{N}, \quad u > 0 \tag{3}$$

$$\alpha_o = \frac{1}{N} \sum_{-n}^{+n} y_r, \quad \beta_o = 0 \tag{4}$$

where $y_r = Z_r$ for $-n < r < +n$

and $y_{-n} = \frac{1}{2}Z_{-n}$, $y_n = \frac{1}{2}Z_{+n}$. (5)

Splitting these in accordance with (1) and (2) we find that

$$\alpha_u = \frac{1}{2}\left[\frac{2}{n}\sum_{o}^{n} y_r \cos\frac{2\pi ru}{N} + \frac{2}{n}\sum_{o}^{n} y_{-r}\cos\frac{2\pi ru}{N}\right], u > 0 \tag{6}$$

$$\beta_u = \frac{1}{2}\left[\frac{2}{n}\sum_{o}^{n} y_r \sin\frac{2\pi ru}{N} - \frac{2}{n}\sum_{o}^{n} y_{-r}\sin\frac{2\pi ru}{N}\right], u > 0 \tag{7}$$

$$\alpha_o = \frac{1}{2}\left[\frac{1}{n}\sum_{o}^{n} y_r + \frac{1}{n}\sum_{o}^{n} y_{-r}\right], \beta_o = 0 \tag{8}$$

where now $y_r = Z_r$ for $0 < r < n$

and $y_o = \frac{1}{2}Z_o$, $y_n = \frac{1}{2}Z_n$. (9)

Denoting the harmonic components for S^- by a_u, b_u and those for S^+ by A_u, B_u, as calculated by N.I.O. Program 3, with $k = 2$, we have, for example

$$\begin{matrix}A_u \\ B_u\end{matrix} = \frac{2}{n} \sum_{o}^{n-1} y_r \begin{matrix}\cos \\ \sin\end{matrix} \frac{2\pi ru}{2n}, \quad u > 0 \tag{10}$$

$$A_o = \frac{1}{n} \sum_{o}^{n-1} y_r, \quad B_o = 0 \tag{11}$$

where $y_r = Z_r$ for $0 < r \leq n - 1$

and $y_o = \frac{1}{2}Z_o$ (12)

Thus from (6), (7) and (10) we find that

$$\alpha_u = \frac{1}{2}\left[A_u + a_u + \frac{2}{n}\left\{\frac{1}{2} Z_{-n} + \frac{1}{2} Z_{+n}\right\}\cos \pi u\right]$$

$$\text{or} \quad \alpha_u = \frac{1}{2}\left[A_u + a_u\right] + \frac{(-1)^u}{N}\left[Z_{-n} + Z_{+n}\right], \quad u > 0 \tag{13}$$

$$\text{and} \quad \beta_u = \frac{1}{2}\left[B_u - b_u\right], \quad u > 0 \tag{14}$$

Also from (8) and (11) we have

$$\alpha_o = \tfrac{1}{2}\left[A_o + a_o + \frac{1}{n}\left\{\tfrac{1}{2}Z_{-n} + \tfrac{1}{2}Z_{+n}\right\}\right]$$

$$\text{or} \quad \alpha_o = \tfrac{1}{2}\left[A_o + a_o\right] + \frac{1}{2N}\left[Z_{-n} + Z_{+n}\right] \tag{15}$$

and $\beta_o = 0$.

These harmonic components A_u, B_u, a_u and b_u depend on c (see Section (5.4)) and have no physical meaning, and must be regarded as intermediate values which will combine to give unique values to α_u and β_u.

Section (5.4)

It may happen that the restrictions in N.I.O. Program 3 are not satisfied unless Z_r is replaced by $Z_r' = Z_r + c$ (see N.I.O. Program 20), where c is a constant. To see what effect this has on α_u and β_u, we first consider A_u and B_u. Similar expressions hold for a_u and b_u.

If we replace y_r and Z_r by y_r' and Z_r' in (10), (11) and (12), we can define A_u' and B_u' in exactly the same way as A_u and B_u. Making use of

$$Z_r' = Z_r + c \tag{16}$$

we can show that

$$A_u + i\,B_u = A_u' + i\,B_u' + P_u + i\,Q_u \tag{17}$$

where P_u and Q_u are real and not necessarily zero. By straightforward algebra we find that

$$P_u + i\,Q_u = -\frac{c}{n} - \frac{2ic}{n}\cot\frac{\pi u}{2n}, \quad u > 0 \text{ and odd,} \tag{18}$$

$$P_u + i\,Q_u = +\frac{c}{n}, \quad u > 0 \text{ and even,} \tag{19}$$

$$\text{and} \quad P_o + i\,Q_o = -c + \frac{c}{2n}. \tag{20}$$

Substituting (18) and (19) in (17) in turn, and assuming similar formulae for a_u and b_u, we find from (13) and (14) that

$$\alpha_u = \tfrac{1}{2}[A_u' + a_u'] + \frac{(-1)^u}{N}\,[Z_{-n}' + Z_{+n}'], \quad u > 0 \tag{21}$$

$$\text{and} \quad \beta_u = \tfrac{1}{2}[B_u' - b_u']. \tag{22}$$

Thus α_u and β_u are independent of c, since all terms involving c cancel out.

Substituting (20) in (15) gives

$$\alpha_o = \tfrac{1}{2}[A_o' + a_o'] + \frac{1}{2N}\,[Z_{-n}' + Z_{+n}'] - c \tag{23}$$

and $\beta_o = 0$.

The discrepancy between (23) and (15) reflects the change in the mean value of the data, since Z_r is now $Z_r + c$.

Section (5.5)

The most serious restriction in N.I.O. Program 3 is

$$n^2M < 2{\cdot}14 \times 10^9 \qquad (1)$$

The particular method [1] used to calculate A_s and B_s gives rise to this restriction.

We define

$$A = A_s = \sum_{r=o}^{n-1} y_r \cos r\theta \qquad (2)$$

$$B = B_s = \sum_{r=o}^{n-1} y_r \sin r\theta \qquad (3)$$

Then

$$A = t_o - t_1 \cos \theta \qquad (4)$$

$$B = t_1 \sin \theta \qquad (5)$$

where t_o and t_1 are derived from the recurrence relation

$$t_{r+2} - 2t_{r+1} \cos \theta + t_r = y_r, \quad r = 0(1)n-1, \qquad (6)$$

with $t_n = 0$, $t_{n-1} = y_{n-1}$.

The data is y_r, $r = 0(1)n-1$, and the recurrence relation method is faster than calculating A and B directly from (2) and (3). The danger of rounding errors building up is negligible.

To deduce (1), (4) and (5) we first write

$$(E^2 - 2E \cos \theta + 1)\, t_r = y_r \qquad (7)$$

where the operator E is defined in the usual way.
Thus

$$t_r = F(E)y_r, \qquad (8)$$

where $F(E) = (1 - \alpha E)^{-1}(1 - \beta E)^{-1}$ (9)

and $\alpha = e^{i\theta}$, $\beta = e^{-i\theta}$.

$$\text{Now} \quad F(E) = \left[\sum_{s=o}^{\infty} \alpha^s E^s\right]\left[\sum_{t=o}^{\infty} \beta^t E^t\right]$$

$$= \sum_{q=o}^{\infty} E^q \sum_{p=o}^{q} \alpha^p \beta^{q-p}$$

$$= \sum_{q=o}^{\infty} e^{-iq\theta}\, E^q \sum_{p=o}^{q} e^{2pi\theta} \qquad (10)$$

assuming all expansions to be valid.

From (10) we can deduce (4) and (5) directly.

From (8) and (10),

$$t_r = \sum_{q=o}^{\infty} e^{-iq\theta} \left(\sum_{p=o}^{q} e^{2pi\theta}\right) E^q y_r, \qquad (11)$$

$$\text{or } t_r = \sum_{q=0}^{\infty} \frac{\sin (q + 1)\theta}{\sin \theta} \; (E^q y_r). \tag{12}$$

In particular,

$$t_0 = \sum_{q=0}^{n-1} y_q \frac{\sin (q + 1)\theta}{\sin \theta} \tag{13}$$

$$t_1 = \sum_{q=0}^{n-1} y_q \frac{\sin q\theta}{\sin \theta} \tag{14}$$

So that

$$t_0 - t_1 \cos \theta = \sum_{q=0}^{n-1} y_q \cos \theta = A \tag{15}$$

$$\text{and} \quad t_1 \sin \theta = \sum_{q=0}^{n-1} y_q \sin \theta = B \quad \text{as required.} \tag{16}$$

To deduce (1), we note that from (10)

$$|F(E)| \leqslant \sum_{q=0}^{\infty} (q + 1)\, E^q \tag{17}$$

since $|e^{-iq\theta}| \leqslant 1$ and $|e^{2pi\theta}| \leqslant 1$.

But $E^q y_r \equiv 0$ for $q \geqslant n - r$, since y_r only exists for $r = 0(1)n-1$, so that

$$|t_r| = |F(E)y_r| \leqslant \left| \sum_{q=0}^{n-r-1} (q + 1)\, E^q y_r \right|$$

$$< M \sum_{q=0}^{n-r-1} (q + 1) \text{ since } |y_r| < M,$$

$$\leqslant Mn\,(n - 1)/2.$$

Finally $|t_r| < Mn^2/2$.

All numbers must be DEUCE single length, and so

$$2|t_r| < 2^{31} - 1$$

$$\text{or} \quad |t_r| < 2^{30} - \tfrac{1}{2}$$

since $2t_r \cos \theta$ is the largest possible number arising.

Thus we must have

$$Mn^2/2 < 2^{30} - \tfrac{1}{2}$$

$$\text{or} \quad n^2 M < 2{\cdot}14 \times 10^9 \quad \text{as required.}$$

The references are:-

[1] J.M. Watt A note on the evaluation of trigonometric series. The Computer Journal 1, No.4, Jan 1959, p.162.

[2] C.W. Clenshaw A note on the summation of Chebyshev series. M.T.A.C. IX, 1955, p.118.

Section (5.6)

For both N = 8856 and N = 8520 there are 82 values of u, including u = 0. The 81 non-zero values, taken in ascending order of magnitude, are divided into 9 sets with w_i terms in each set. Note that

$$\sum_{0}^{8} w_i = 81, \tag{1}$$

and that the w_i are not necessarily equal. (See Section (5.7).

Two constant matrices correspond to each of these sets, for one N. The "cosine inverse matrix" $\underset{\sim}{C}$ is associated with the α_u and the "sine inverse matrix" $\underset{\sim}{D}$ with the β_u. (See Section (5.3)).

Consider a typical set w_i, with cosine matrix $\underset{\sim}{C}^*$ and sine matrix $\underset{\sim}{D}^*$. We require $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ defined by

$$\begin{aligned} \underset{\sim}{C}^*\underset{\sim}{X} &= \underset{\sim}{\alpha}, \\ \text{and} \quad \underset{\sim}{D}^*\underset{\sim}{Y} &= \underset{\sim}{\beta}. \end{aligned} \tag{2}$$

The $\underset{\sim}{C}^*$ and $\underset{\sim}{D}^*$ need not be square. By the method of least squares we find that

$$\begin{aligned} \underset{\sim}{X} &= \underset{\sim}{C}\,\underset{\sim}{\alpha}, \\ \text{and} \quad \underset{\sim}{Y} &= \underset{\sim}{D}\,\underset{\sim}{\beta}. \end{aligned} \tag{3}$$

$$\begin{aligned} \text{where} \quad \underset{\sim}{C} &= (\underset{\sim}{C}^{*\prime}\,\underset{\sim}{C}^*)^{-1}\,\underset{\sim}{C}^{*\prime}, \\ \text{and} \quad \underset{\sim}{D} &= (\underset{\sim}{D}^{*\prime}\,\underset{\sim}{D}^*)^{-1}\,\underset{\sim}{D}^{*\prime}. \end{aligned} \tag{4}$$

Both $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are calculated by N.I.O. Program 33 (see Part I), and are used in N.I.O. Program 13 (see Part II).

Section (5.7). Table of v_j, u_i and α_{ij}.
For definitions see Section (5.1)

Constituent Number	Constituent Symbol	v	N = 8856 u	N = 8856 100α	N = 8520 u	N = 8520 100α
0	Sa	0.0410686	1	1	1	-3
1	SSa	0.0821373	2	2	2	-6
2	Mm	0.5443747	13	39	12	88
			14	-61	13	-12
3	MSf	1.0158958	25	-1	24	4
4	Mf	1.0980331	27	1	26	-1
5	$2Q_1$	12.8542862	316	22	304	22
					305	-78
6	σ_1	12.9271398	318	1	306	-6
7	Q_1	13.3986609	329	61	317	10
			330	-39	318	-90
8	ρ_1	13.4715145	331	40	319	-17
			332	-60	320	-117
9	O_1	13.9430356	343	0	330	-1
10	MP_1	14.0251729	345	2	332	-7
11	M_1	14.4920521	356	50	343	-2
			357	-50	344	81
12	χ_1	14.5695476	358	41	345	-19
			359	-59	346	-119
13	π_1	14.9178647	367	-2	353	6
14	P_1	14.9589314	368	-1	354	3
15	S_1	15.0000000	369	0	355	0
16	K_1	15.0410686	370	1	356	-3
17	ψ_1	15.0821353	371	2	357	-6
18	ϕ_1	15.1232059	372	3	358	-8
19	θ_1	15.5125897	381	61	367	13
			382	-39		
20	J_1	15.5854433	383	40	368	86
			384	-60	369	-14
21	SO_1	16.0569644	395	0	380	1
22	OO_1	16.1391017	397	2	382	-4
23	OQ_2	27.3416964	672	61	647	9
			673	-39		
24	MNS_2	27.4238337	675	-37	649	3
25	$2N_2$	27.8953548	686	23	660	19
					661	-81
26	μ_2	27.9682084	688	2	662	-9
27	N_2	28.4397295	699	62	673	7
			700	-38		
28	ν_2	28.5125831	701	41	674	80
			702	-59	675	-20
					676	-120
29	OP_2	28.9019669	711	-1	684	1
30	M_2	28.9841042	713	1	686	-4
31	MKS_2	29.0662415	715	3	688	-10
32	λ_2	29.4556253	724	61	697	12
			725	-39	698	-88
33	L_2	29.5284789	726	40	699	-16
			727	-60	700	-116
34	T_2	29.9589333	737	-1	709	3
35	S_2	30.0000000	738	0	710	0
36	R_2	30.0410667	739	1	711	-3
37	K_2	30.0821373	740	2	712	-6
38	MSN_2	30.5443747	751	39	722	88
					723	-12
39	KJ_2	30.6265120	753	41	724	83
			754	-59	725	-17
40	$2SM_2$	31.0158958	763	-1	734	4

Constituent Number	Symbol	v	N = 8856 u	N = 8856 100α	N = 8520 u	N = 8520 100α
41	MO_3	42.9271398	1056	1	1016	-6
42	M_3	43.4761563	1069	51	1029	-6
			1070	-49		
43	SO_3	43.9430356	1081	0	1040	-1
44	MK_3	44.0251729	1083	2	1042	-7
45	SK_3	45.0410686	1108	1	1066	-3
46	MN_4	57.4238337	1412	63	1359	3
			1413	-37		
47	M_4	57.9682084	1426	2	1371	91
					1372	-9
48	SN_4	58.4397295	1437	62	1383	7
			1438	-38		
49	MS_4	58.9841042	1451	1	1396	-4
50	MK_4	59.0662415	1453	3	1397	90
					1398	-10
51	S_4	60.0000000	1476	0	1420	0
52	SK_4	60.0821373	1478	2	1422	-6
53	$2MN_6$	86.4079380	2125	64	2045	-1
			2126	-36		
54	M_6	86.9523127	2139	3	2057	87
					2058	-13
55	MSN_6	87.4238337	2150	63	2069	3
			2151	-37		
56	$2MS_6$	87.9682084	2164	2	2082	-9
57	$2MK_6$	88.0503457	2166	4	2083	86
					2084	-14
58	$2SM_6$	88.9841042	2189	1	2106	-4
59	MSK_6	89.0662415	2191	3	2108	-10
60	M_8	115.9364169	2852	4	2744	-17
61	M_{10}	144.9205211	3565	4	3430	-21
62	M_{12}	173.9046254	4278	5	4116	-26