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I.O.S.

THE ANALYSIS AND INTERPRETATION

OF RECORDS OF SEA WAVES

PART 2

The spectral analysis

of one dimensional records

NATURAL ENVIRONMENT RESEARCH COUNCIL

INSTITUTE OF OCEANOGRAPHIC SCIENCES

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by
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Institute of Oceanographic Sciences
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Preface to this document

This is intended as one of a series of documents setting out the theory and practice of the analysis and interpretation of wave data. The series will be aimed at engineers or physical oceanographers who are concerned with the use of the data rather than with the niceties of statistical theory. Thus, this report, and probably most of the others, will have a section setting out the principles involved as simply as possible together with the equations which need to be used and how to apply them, and a section giving the derivation of the formula so that the user can, if he wishes, look at the assumptions involved or extend the theory to different cases. Even in this second section some mathematical niceties will be avoided for the sake of clarity. It will be assumed that the reader has a level of mathematical knowledge corresponding to that of a graduate engineer or physicist, together with a general understanding of elementary statistics: that is, that the definition and significance of such terms as mean, variance and standard deviation are understood.

This document, Part 2 of a wider series, has been prepared first because it was required to clarify some questions during the setting up of a routine for spectral analysis of digital wave records at IOS Taunton. Part 1 is intended to be concerned with the simpler non-spectral methods of analysis, and Part 3 with estimation of extreme values. Part 4 will probably be concerned with directional data.

Comments and corrections are welcome.

The main symbols and conventions used

A bar as in $\overline{z^2}$ represents an average over a specific finite sample of the variation of z with time.

The symbol $\langle \quad \rangle$ represents an ensemble average of the quantity within the triangular brackets: that is, the average of a large number of samples from a statistically stationary process.

x , y , z , are the coordinates of a point in the sea-surface, with z increasing upwards along the vertical axis.

$z(t)$ implies that we are considering the variation of z with time at a fixed x and y .

Υ is the length of a conceptual long record of the process, Υ eventually being taken to ∞

p is the harmonic number of a Fourier component of this record.

T is the length of the finite sample of the process which is available for analysis.

n is the harmonic number of a component of this sample record.

m is the difference frequency (in terms of harmonic number of the sample record) between a component of the conceptual long record and the harmonic n of the sample record being considered.

a_n and b_n are the amplitudes of the cosine and sine components of the nth Fourier harmonic of a finite record of length T

$A_n = a_n + i b_n$ is the complex amplitude of this harmonic

There appears to be no accepted convention for spectral densities. In this report the following will be used.

$\phi(f)$ is an estimate of the spectral density computed from a finite record of $\zeta(t)$. It is defined by the following:
that part of the variance of $\zeta(t)$ carried by frequency components in the range $(f - \Delta f/2) < f < (f + \Delta f/2)$ is $\phi(f)\,\Delta f$

$E(f)$ is the ensemble average of $\phi(f)$, as $\Delta f \to 0$. That is, it is the spectral density function of the underlying process of which we have a sample. It is in terms of the variance of the sea-surface elevation, but this is proportional to energy per unit area of the sea-surface. The 'E' can also be thought of as standing for the "Expectation" of $\phi(f)$

PART 2A

General principles and
the application of the formulae

2.1 Basic concepts

2.1.1 The finite Fourier Transform

The concept of the representation of a finite record of a time variable by a series of harmonic components is fundamental not only to the way we analyse wave records nowadays, but also to most of the theory in this paper. This section is therefore devoted to giving the fundamental formulae, explaining their significance, and saying a little about the typical mathematical processes used.

Consider a time variable $\zeta(t)$ over a finite time interval T : then the Fourier theorem says that it may be represented exactly by

$$\zeta(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos 2\pi n t/T + b_n \sin 2\pi n t/T) \qquad 2.1/1$$

where

$$\left.\begin{aligned} a_q &= \frac{2}{T}\int_{-T/2}^{T/2} \zeta(t)\cos 2\pi q t/T \; dt \\ b_q &= \frac{2}{T}\int_{-T/2}^{T/2} \zeta(t)\sin 2\pi q t/T \; dt \end{aligned}\right\} \qquad 2.1/2$$

n and q are integers

$\frac{1}{2}\,a_0$ is the mean value of $\zeta(t)$ during the interval T

For convenience we shall often put $\omega_n = 2\pi n/T$ etc.

The validity of the theorem will be demonstrated below for the complex notation version of the equations.

The standard method of analysis of sea waves into spectra is started by a computer calculating all a_q and b_q using equations 2.1/2.

If the variable $\zeta(t)$ is sampled at regular intervals Δt, then the sum in 2.1/1 is taken to $n = T/2\Delta t$. The values of a_n and b_n are in general different to those obtained when the continuous variable is transformed, but the difference is insignificant if any components in $\zeta(t)$ with frequencies above $1/2\Delta t$ have negligible amplitudes (this is a standard theorem in telecommunications theory).

In the present context there are two important facts to note about the Fourier Transform.

1. It is a purely mathematical device. Any assumptions about the nature of the time series arise in the way we interpret the physical significance of the amplitudes of the components.

2. Another way of putting this is that the Fourier Series is an exact representation of $z(t)$ in the interval $-T/2 < t < T/2$ and the coefficients a_n and b_n contain all the information contained in this record, since it can be reconstructed exactly from a knowledge of them. However, the relationship of these coefficients to the statistical parameters of the process $z(t)$ of which we have a sample is complex, and forms the subject of this paper.

For mathematical analytic work it is often convenient to represent the Fourier Transform in complex notation.

$$z(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} A_n \, e^{i 2\pi n t/T} \qquad 2.1/3$$

$$A_q = \frac{2}{T} \int_{-T/2}^{T/2} z(t) \, e^{-i 2\pi q t/T} \, dt \qquad 2.1/4$$

where $A_n = a_n - i b_n$

$$a_{-n} = a_n$$

$$b_{-n} = -b_n$$

This statement is precisely equivalent to equations 2.1/1 and 2.1/2. For example, remembering that

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

Then

$$A_n e^{i\omega_n t} = (a_n - i b_n)(\cos \omega_n t + i \sin \omega_n t)$$

and

$$A_{-n} e^{i(-\omega_n)t} = (a_n + i b_n)(\cos \omega_n t - i \sin \omega_n t)$$

Thus, adding the positive and negative components in 2.1/3 we get

$$\frac{1}{2}\left[A_n e^{i\omega_n t} + A_{-n} e^{i(-\omega_n)t}\right] = a_n \cos \omega_n t + b_n \sin \omega_n t$$

In the context of the present paper the negative frequencies are a mathematical device and have no physical meaning: when we wish to interpret the results we always convert to real quantities and positive frequencies.

Note that there appears to be no accepted convention for the symbolism in this complex notation: in some books the factors of $\frac{1}{2}$ and 2 in 2.1/3 and 2.1/4 respectively are omitted, but the author feels that the symbolism used here is the best in the present context.

Note also that the amplitude of the n^{th} complex component is $\frac{1}{2} A_n$, but that the amplitude of the n^{th} real component is $(a_n^2 + b_n^2)^{1/2} = |A_n|$

The phases of the components only take on significance when the non-linear properties of waves are important. In the present paper the assumption will be made that the waves in the sea are a linear process (this will be discussed in more detail in the next section), and if this is the case the phases are random and tell us nothing about the wave system.

To finish this section it is worth demonstrating the validity of the complex notation version of the Fourier Transform, because it illustrates in a simple case a type of mathematical process which will be used many times in this paper.

Putting the expression for $\zeta(t)$ from equation 2.1/3 into equation 2.1/4, we should obtain an identity:

$$A_q \equiv \frac{2}{T}\int_{-T/2}^{T/2}\left[\frac{1}{2}\sum_{n=-\infty}^{\infty} A_n\, e^{i2\pi nt/T}\right] e^{-i2\pi qt/T}\, dt$$

There is no convergence problem so that the order of integration and summation may be interchanged and the exponential terms multiplied.

$$\therefore\quad A_q \equiv \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-T/2}^{T/2} A_n\, e^{i2\pi(n-q)t/T}\, dt$$

For $n \neq q$ the general term in the sum

$$= \frac{1}{T} A_n \left[\frac{T}{i2\pi(n-q)}\, e^{i2\pi(n-q)t/T}\right]_{-T/2}^{T/2}$$

$$= \frac{A_n}{i2\pi(n-q)}\left[e^{i\pi(n-q)} - e^{-i\pi(n-q)}\right]$$

Remembering that n and q are integers, this

$$= \frac{A_n}{2\pi(n-q)} \cdot 2\sin\pi(n-q) = 0$$

Thus the only finite term in the sum is when $q = n$, giving

$$A_q \equiv \frac{1}{T}\int_{-T/2}^{T/2} A_q\, e^0\, dt = A_q$$

2.1.2 The representation of a wave system by a frequency spectrum

We shall here consider the output of a single wave height sensor to be a function of time $\zeta(t)$: that is, no account will be taken at this stage of the directional properties of the waves. Two basic assumptions are made.

1. That the statistical properties of the wave system do not vary during the time T for which we sample $\zeta(t)$ on any occasion: that is, for the duration of the record, it is a "statistically stationary system".

2. That the sea behaves as a linear system, so that it may be considered as a superposition of sinusoidal waves all travelling independently. This is not strictly true, and for certain purposes the non-linearity of waves is important. However, for a wide range of practical purposes the assumption is valid to an adequate degree of accuracy. The main practical circumstance in which the non-linearity has to be taken into account is when one is considering the detailed properties of an extreme wave or of a breaking wave.

The statistical parameters of the wave system at any one time and place describe the <u>sea-state</u> at that time. The statistics of the sea-state parameters themselves over a long period of time describe the <u>wave climate</u> at that place. In this paper, we shall be concerned with determining the sea-state parameters by spectral analysis.

The complete derivations of formulae will be given in this section since they are necessary to understand the principles involved. In later sections, the derivations will be omitted and can be found in Part B.

If $\zeta(t)$ is the elevation of a point on the sea surface, then the average energy of a wave system is $\frac{1}{2}\rho g\langle\zeta^2\rangle$ per unit area. The variance of the surface elevation $\langle\zeta^2\rangle$ is often loosely referred to as the energy, particularly in the context of the "energy spectrum" by which is meant the variance/frequency spectrum which is the subject of this section. Here we shall work entirely in terms of the variance of the surface elevation, $\overline{\zeta^2}$ or $\langle\zeta^2\rangle$

In this section we are concerned with the properties of the wave system itself, not with what can be deduced from a finite record of it. It is, however, convenient to start with a conceptual long, but finite, record of the process with a duration Υ . At a convenient point Υ will be taken to infinity. This device will be used frequently in this paper.

Note that when in later sections we come to consider a real finite record of duration T, it will sometimes be considered as a section of this longer conceptual record.

Thus we may consider $\zeta(t)$ in the interval $-\tau/2 < t < \tau/2$ to be represented by its Fourier series

$$\zeta(t) = \frac{1}{2} a_0 + \sum_{p=1}^{\infty} \left(a_p \cos \omega_p t + b_p \sin \omega_p t \right) \qquad 2.1/5$$

The variance of the sea surface (or, loosely speaking, its energy) is given by the average value of $\zeta^2(t)$

$$\zeta^2(t) = \left[\frac{1}{2} a_0 + \sum_{p=1}^{\infty} \left(a_p \cos \omega_p t + b_p \sin \omega_p t \right) \right]^2$$

Expanding the square of the series in the right hand side:

$$\zeta^2(t) = \frac{1}{4} a_0^2 + a_0 \sum_{p=1}^{\infty} \left(a_p \cos \omega_p t + b_p \sin \omega_p t \right)$$

$$+ \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left(a_p \cos \omega_p t + b_p \sin \omega_p t \right)\left(a_q \cos \omega_q t + b_q \sin \omega_q t \right)$$

Consider the following term of the product:

$$a_p a_q \cos \omega_p t \cos \omega_q t = \frac{1}{2} a_p a_q \left[\cos(\omega_p + \omega_q) t + \cos(\omega_p - \omega_q) t \right]$$

The average over the record of this is given by

$$\frac{1}{\tau} \frac{a_p a_q}{2} \int_{-\tau/2}^{\tau/2} \left[\cos 2\pi (p+q) t/\tau + \cos 2\pi (p-q) t/\tau \right] dt$$

Remembering that p and q are integers, it can quickly be shown that this $= 0$ except when $p = q$, when the average equals $\frac{1}{2} a_p^2$

Similarly, the term $b_p b_q \sin \omega_p t \sin \omega_q t$ averages to $\frac{1}{2} b_p^2$, the products of sines and cosines give zero average, as do the products of a_0 and sinusoidal terms. Thus

$$\overline{\zeta^2(t)} = \frac{1}{4} a_0^2 + \sum_{p=1}^{\infty} \frac{1}{2} \left(a_p^2 + b_p^2 \right) \qquad 2.1/6$$

$\frac{1}{2} \left(a_p^2 + b_p^2 \right)$ is the mean-square ordinate, or variance, of the p^{th} component.

Thus, one can see that the variance of the sea surface is composed of the sum of the variances of its sinusoidal components.

Now the Energy Density Function $E(f)$ is defined as follows. That part of the variance of the signal carried by components in the range of frequencies $(f - \delta f/2)$ to $(f + \delta f/2)$ is $E(f)\,\delta f$

Thus, if the original signal $\zeta(t)$ is passed through a filter of bandwidth δf the variance of the output is by definition $E(f)\,\delta f$, so that

$$E(f)\,\delta f = \sum \tfrac{1}{2}(a_p^2 + b_p^2)$$

where the summation is taken over values of p for which the corresponding frequency f_p lies in the range $(f - \delta f/2) < f_p < (f + \delta f/2)$. This is strictly true when $\tau \to 0$ so that the interval δf contains a large number of components. Thus, assuming that $E(f)$ is constant in the interval δf, so that all the values of $(a^2 + b^2)$ within it come from the same statistical population, and remembering that the frequency separation of the harmonics is $1/\tau$, one can see that

$$\left\langle \tfrac{1}{2}(a_p^2 + b_p^2) \right\rangle = E(f_p) \cdot \frac{1}{\tau} \qquad 2.1/7$$

A plot of the Energy Density Function $E(f)$ against frequency is called a variance/frequency spectrum. Since $\zeta(t)$ is usually given in terms of the elevation of the sea-level, the units of $E(f)$ in the SI system will be $m^2\ Hz^{-1}$

2.2 Estimating the spectrum from a wave record of finite length: introduction

In practice we are only able to record the waves for a finite time, the limit being set either by limited recording capacity or by changing sea conditions: the effects of these will be discussed later. Here we shall assume that we have a record of the waveheight $\zeta(t)$ in the time interval $-T/2 < t < T/2$

The most straightforward way of analysing this is to start by subjecting it to a Fourier Transform (equation 2.1/1).

$$\zeta(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos 2\pi n t/T + b_n \sin 2\pi n t/T \right]$$

where n is an integer

or in complex notation (equation 2.1/3)

$$\zeta(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} A_n e^{i 2\pi n t/T}$$

where

$$A_n = a_n - i b_n$$

$$a_n = a_{-n}$$

$$b_n = -b_{-n}$$

Each component has an exact number of waves in time T, and they are in fact "Harmonics" of the fundamental frequency $1/T$.

The Fourier coefficients a_n and b_n now have to be interpreted. If we put

$$\phi(n) \Delta f = \frac{1}{2} (a_n^2 + b_n^2) \qquad 2.2/1$$

where $\Delta f = 1/T$ is the frequency interval between harmonics, then one would expect $\phi(n)$ to be an estimate of the energy density $E(n)$: this has been shown in effect in section 2.1.2 (equation 2.1/7) which demonstrates that the expected value of $\phi(n)$ defined as its mean value if we were able to determine many values of it from independent records of $\zeta(t)$, is $E(n)$. However a single estimate is subject to enormous sampling errors: in section 2.4.1 it will be shown that the root-mean-square random sampling error equals the mean value. Thus, to get a meaningful estimate of $E(n)$ we must average over a number of estimates $\phi(n)$

A common practice is to average 10 values from adjacent harmonics, but this still leaves a standard sampling error of $1/\sqrt{10}$, or approximately 30%. This problem will be considered in more detail in section 2.5.

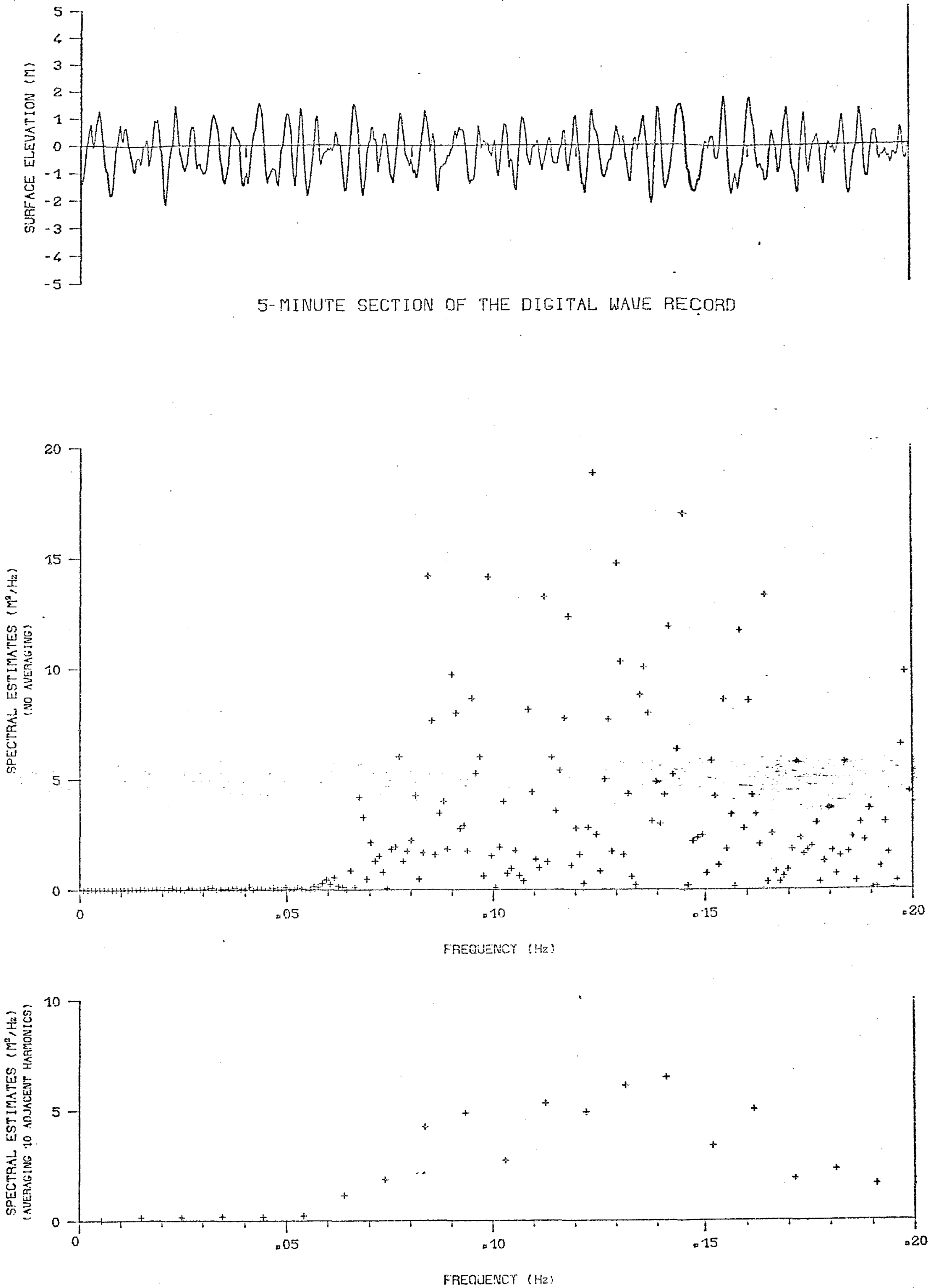


Figure 2.2/1

Top: A 300s length of a wave record of total duration 1024 s

Middle: The spectral estimates derived from squaring the amplitudes of the individual fourier harmonics of the record

Bottom: The spectral estimates obtained by averaging over 10 harmonics at a time

An example of a section of wave record, its individual Fourier harmonics, and the estimate of the energy spectrum derived by smoothing these, is given in figure 2.2/1.

Apart from the random sampling error just discussed, the estimates $\phi(n)$ also contain contributions from a wide range of frequencies in the original process. This problem will now be examined.

2.3 <u>The frequency resolution of a Fourier Analysis</u>

2.3.1 Basic formulae

It has been shown that when analysing for the $n^{\underline{th}}$ Fourier component of a finite record, there is no contribution from any other component of that record. However, when one considers the finite record of length T as only part of a process extending over a much longer time Υ, then we shall show that the energy of the $n^{\underline{th}}$ component of the sample record contains contributions from all frequencies in the long record <u>except</u> those precisely coincident with the

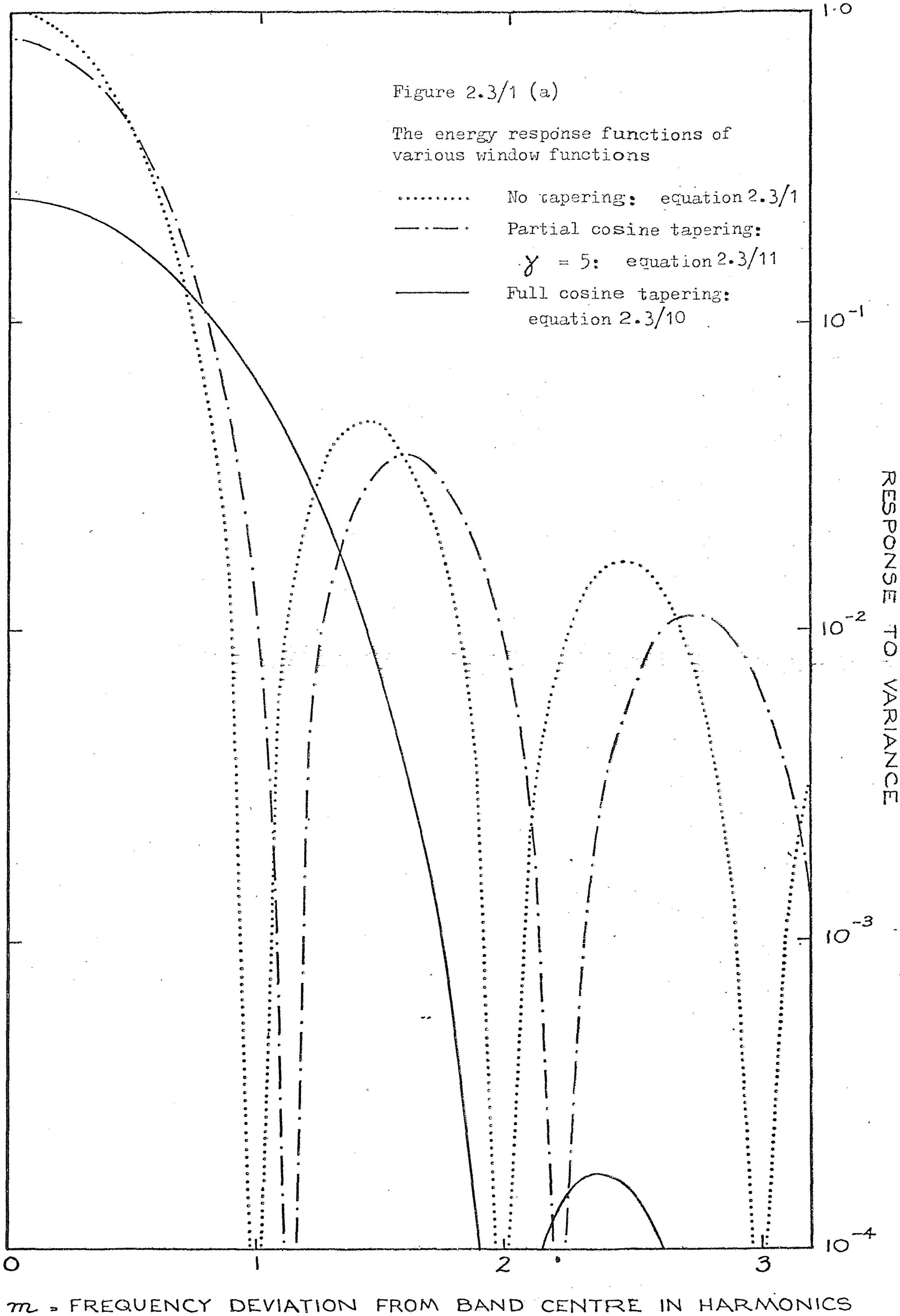


Figure 2.3/1 (a)
The energy response functions of various window functions
No tapering: equation 2.3/1
Partial cosine tapering: $\gamma = 5$: equation 2.3/11
Full cosine tapering: equation 2.3/10
1·0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
RESPONSE TO VARIANCE
0
1
2
3
m = FREQUENCY DEVIATION FROM BAND CENTRE IN HARMONICS

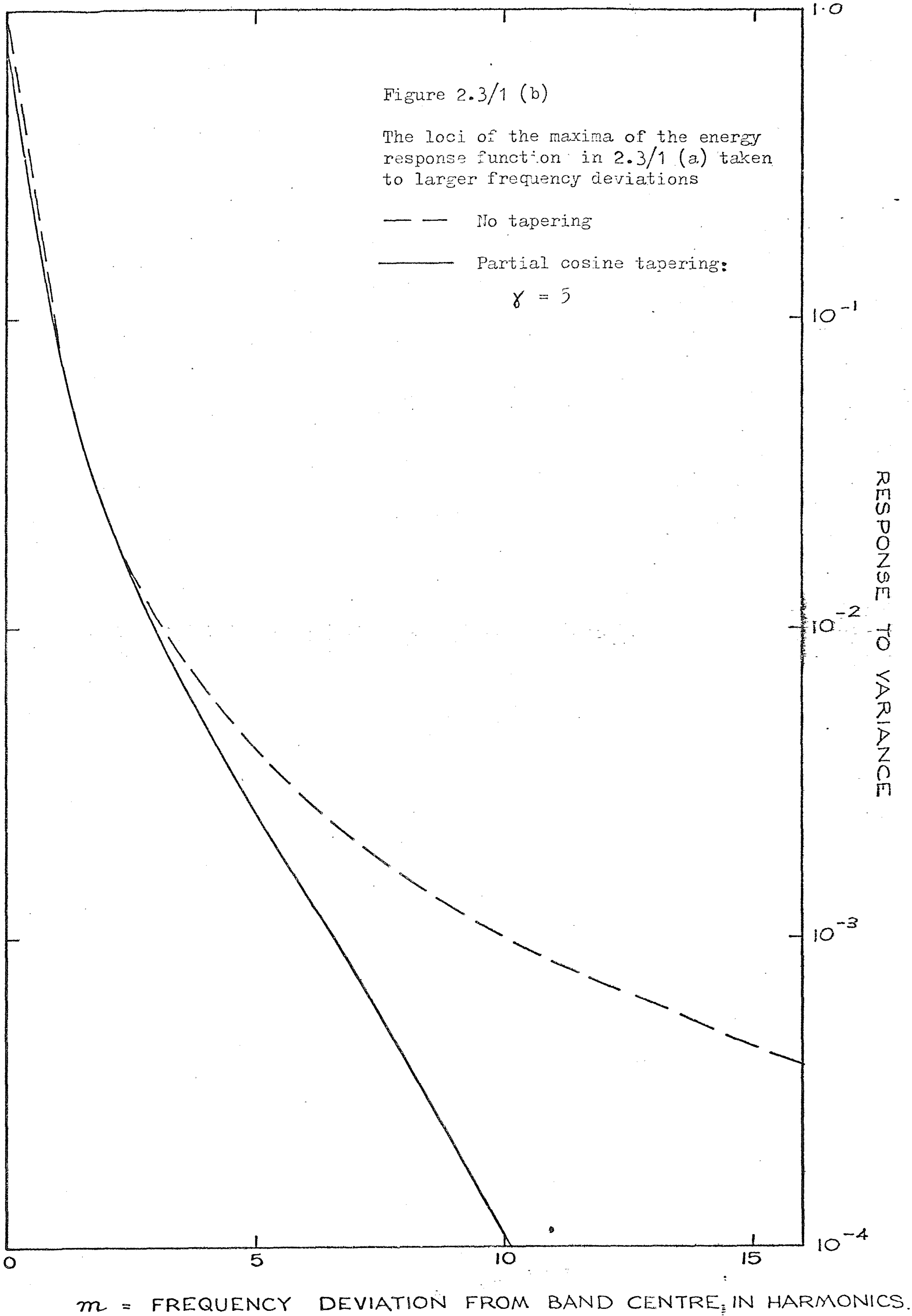


Figure 2.3/1 (b)
The loci of the maxima of the energy response function in 2.3/1 (a) taken to larger frequency deviations
No tapering
Partial cosine tapering:
γ = 5
1·0
10⁻¹
10⁻²
10⁻³
10⁻⁴
RESPONSE TO VARIANCE
0
5
10
15
m = FREQUENCY DEVIATION FROM BAND CENTRE IN HARMONICS.

frequencies of the harmonics of the sample record. These contributions can be described in terms of the filter function of the estimate, as will be seen, and the skirts of this filter function are by no means negligible. Later in this section we shall show how these skirts can be reduced, but first the theory of the simple Fourier analysis will be looked at.

The detailed theory is given in section 2.6.1 where it is shown that for the nth Fourier coefficient, if n is large* and $E(f_m)$ is the energy density at a frequency $f_m = \frac{1}{T}(n+m)$ where m is continuous, then

$$\text{Expectation of } \frac{1}{2}\left(a_n^2 + b_n^2\right) = \frac{1}{T}\int_{m=-n}^{\infty} E(f_m)\left(\frac{\sin \pi m}{\pi m}\right)^2 dm \qquad 2.3/1$$

The analysis of the nth harmonic is thus like putting the original signal through a filter with an amplitude response function of $\dfrac{\sin \pi m}{\pi m}$

The energy response function $\left(\dfrac{\sin \pi m}{\pi m}\right)^2$ is plotted in figure 2.3/1.

For some purposes it is useful to know the expected contribution to the variance of the nth harmonic from the harmonic interval lying about the $(n+M)^{th}$ harmonic. Calling this contribution $E(v_M)$

$$E(v_M) = \frac{1}{T}\int_{M-\frac{1}{2}}^{M+\frac{1}{2}} E(f_m)\left(\frac{\sin \pi m}{\pi m}\right)^2 dm$$

If we assume $E(f_m)$ to be effectively constant over this interval and $= E(f_M)$

$$E(v_M) = \frac{1}{T}E(f_M)\int_{M-\frac{1}{2}}^{M+\frac{1}{2}}\left(\frac{\sin \pi m}{\pi m}\right)^2 dm$$

$$= \frac{1}{T}E(f_M)\, w_M \qquad 2.3/2$$

$$\text{where } w_M = \int_{M-\frac{1}{2}}^{M+\frac{1}{2}}\left(\frac{\sin \pi m}{\pi m}\right)^2 dm$$

This equation can be converted to a form which is tabulated in standard books, but nowadays it is almost as easy to integrate it numerically on a computer, which is what has been done in this case. The weights are given in Table 2.3/1. Note that, on the assumption used here that n is large, $w_M = w_{-M}$

---

*The criterion for n to be adequately large is that $\left(\dfrac{1}{\pi n}\right)^2 \ll 1$

2.3.2 Correction of estimates by a maximum likelihood method

More generally, in the discussion on equation 2.2/1 in section 2.2 it was shown that $\phi(n)$ defined in that equation is an estimate of the true energy density $E(n)$. We can now see that

$$\langle \phi(n) \rangle = \sum_{M=-n}^{\infty} E(n+M)\, \omega_M$$

or

$$E(n) = \frac{1}{\omega_0}\left[\langle \phi(n) \rangle - \sum_{M=-n}^{-1} E(n+M)\,\omega_M - \sum_{M=+1}^{\infty} E(n+M)\,\omega_M \right] \qquad 2.3/3$$

If we replace $\langle \phi(n) \rangle$ by the actual measured $\phi(n)$, and $E(n+M)$ by the estimates $\phi(n+M)$, we get a formula to correct $E(n)$ for its side lobes. However, in principle we can do better by replacing $E(n+M)$ in the formula by its corrected estimate $\phi^*(n+M)$. Then

$$\phi^*(n) = \frac{1}{\omega_0}\left[\phi(n) - \sum_{M=-n}^{-1} \phi^*(n+M)\,\omega_M - \sum_{M=+1}^{\infty} \phi^*(n+M)\,\omega_M \right] \qquad 2.3/4$$

The series of equations for $\phi^*(n)$ are now a series of simultaneous equations giving a matrix which can be inverted by computer to give the values of $\phi^*(n)$. These are now the most likely estimate of $E(n)$ from the actual data.

However, in practice, it will be seen in the next section that a sensible choice of analysis procedures gives values of ω_M which are negligible except for ω_0 and ω_1. In addition, sampling errors are such that the total corrections in most cases are of the same order as, or less than, the rms sampling error. Thus, at the most, one might use the simplified correction (remembering that $\omega_1 = \omega_{-1}$).

$$\phi^*(n) = \frac{1}{1-2\omega_1}\left\{ \phi(n) - \omega_1\left[\phi(n-1) + \phi(n+1)\right] \right\} \qquad 2.3/5$$

Table 2.3/1

M	w_M	$\sum_{q=M+1}^{\infty} w_q$
0	.7737	.1131
1	.0786	.0345
2	.0140	.0205
3	.0060	.01461
4	.00326	.01135
5	.00207	.00928
6	.00143	.00785
7	.00105	.00680
8	.00080	.00600
9	.00063	.00537
10	.00051	.0048

The expected contribution to the value of the $(\text{amplitude})^2$ of a Fourier harmonic from the harmonic interval centred M harmonics away from it is $E(f_M)\, w_M$

This table is for a sample record with no windowing.

The asymptotic formula for w_M is

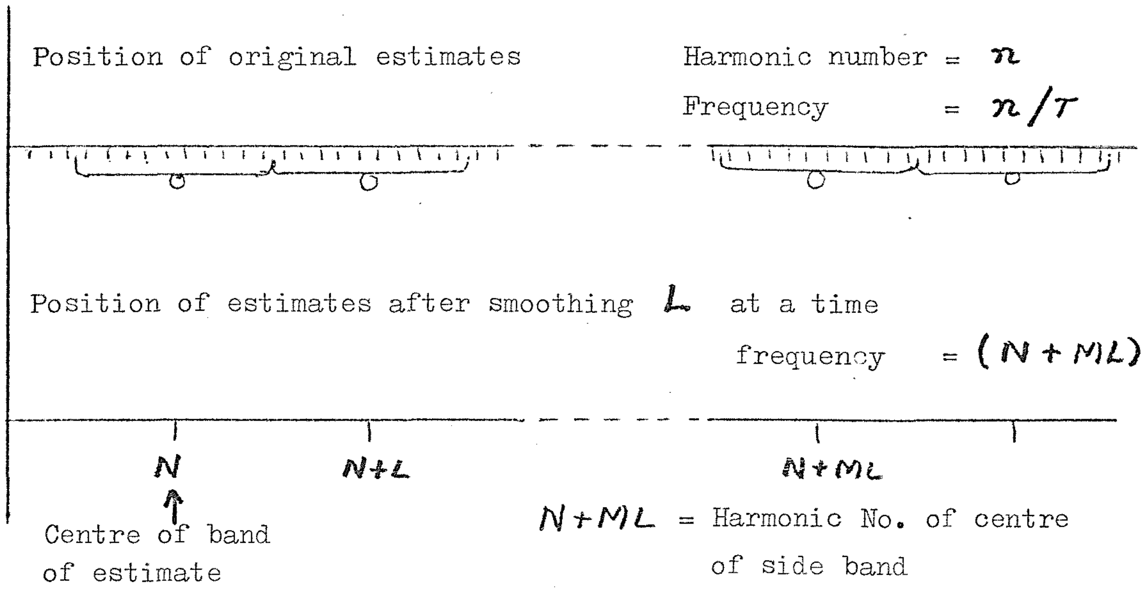
$$w_M = \frac{1}{2(\pi M)^2} = 5.066 \times 10^{-2}/M^2$$

This is correct within 1% for $M > 10$.

2.3.3 The effect of averaging over adjacent harmonics

In order to reduce the random sampling error, it is common practice to average over a number L of adjacent harmonics (Figure 2.3/2). We shall assume below that the spectral density function $E(f)$ of the wave system may be considered to be constant over such an interval.

Figure 2.3/2



The estimate $\phi(f)$ of $E(f)$ then becomes

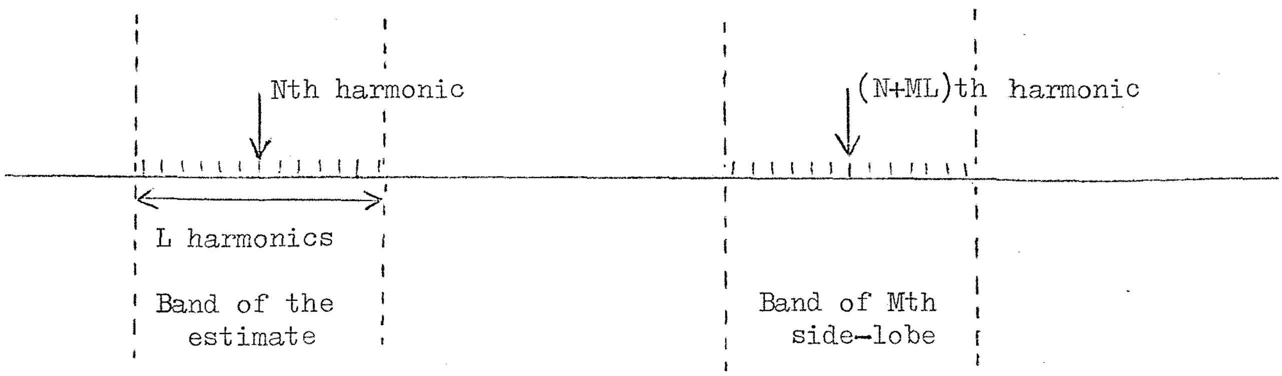
$$\phi(f)\,\Delta f = \frac{1}{L} \sum \left(a_n^2 + b_n^2\right)$$

where the sum is taken over L adjacent harmonics centred on the Nth harmonic. If L is even, N will be an integer $+\frac{1}{2}$. The estimates will be separated by intervals of L harmonics and we wish to know the weighting function W_M such that the expected contribution to the Nth estimate $\phi(f_N)$ from the energy in the $(N+ML)^{\text{th}}$ interval is

$$W_M \, E\left(f_{N+ML}\right)$$

We use the same assumptions and approximations as before. The equations are less cumbersome if we work with exceedances: that is, the total energy contained in all side-bands including the one of interest and beyond.

Figure 2.3/3



Using Figure 2.3/3 consider first the lefthand harmonic in the estimate band. Remembering that it is only one of L harmonics contributing to the estimate, its side-lobes will contribute to the M^{th} side-lobe and beyond by a proportion

$$\frac{1}{L} \sum_{q=ML}^{\infty} w_q$$

Similarly the next will contribute by a proportion

$$\frac{1}{L} \sum_{q=ML-1}^{\infty} w_q$$

So the total proportional contribution will be

$$\sum_{Q=M}^{\infty} W_Q = \frac{1}{L} \sum_{m=(M-1)L+1}^{ML} \; \sum_{q=m}^{\infty} w_q \qquad 2.3/6$$

This can be computed from a table such as 2.3/1 if it is extended to higher values of m . Note that W_0 is a special case. Values of W_Q can be obtained by differencing.

As an example, values of W_M have been computed for $L = 10$ and are given in table 2.3/2.

Table 2.3/2

Values of the weighting function W_m for estimates of spectral density obtained by averaging over 10 Fourier harmonics (no windowing)

M	W_M	$\sum_{Q=M+1}^{\infty} W_Q$
0	.9542	.02290
1	.0193	.0036
2	.00144	.00216
3	.00062	.00154
4	.00034	.00120

The contribution to the estimate centred on the Nth harmonic from the spectral density $E\left(f_{N+10m}\right)$ centred on the $(N+10m)^{th}$ harmonic is

$W_m \, E\left(f_{N+10m}\right)$

2.3.4 Reduction of the skirts of the filter function by windowing

It has been shown that the variance of a given Fourier harmonic of a finite record receives contributions from a wide range of frequencies present in the wave pattern. It has also been shown that these skirts can be reduced to acceptable proportions for most engineering purposes by transforming a time series which is long enough to allow averaging over adjacent harmonics at, say, 10 at a time. However, there are some applications where such averaging is impracticable because of limitations in computing power or cost, and others where the size of the skirts is still unacceptable. An example of the latter is where one is examining the activity at medium frequencies (3×10^{-4} to 3×10^{-2} Hz). The spectral density in parts of this band can be in the region of 10^{-6} of that in the main wave spectrum. Thus, more powerful methods for reduction of the skirts are required.

One way of looking at the Fourier analysis of a finite record is to join many copies of it end to end to produce a repetitive waveform of fundamental frequency $1/T$. The harmonics of this fundamental frequency are then continuous sinusoidal waves. However, any sinusoidal component of the original signal with a frequency which is not one of these harmonics has a discontinuity at the join, and this discontinuity clearly contains a wide range of other frequencies. Thus, in order to restrict the range of these spurious frequencies one must smooth the ends of the record in some way. This is done by multiplying the record by a "Window Function" which tapers the ends of the record to zero. That is, we produce a new record $Z(t) = z(t)\,F(t)$ where $F(t)$ is the window function.

It is shown in section 2.6.3 that the expectation of the variance of the nth harmonic after windowing* is

$$\frac{1}{2}\langle a_n^2 + b_n^2 \rangle = \frac{1}{4}\int_0^\infty E(f_k)\left[\frac{g(f_n - f_k)}{T}\right]^2 df_k \qquad 2.3/7$$

where

$$g(f) = 4\int_0^\infty F(t)\cos 2\pi f t \; dt \qquad 2.3/8$$

To confirm that this gives the correct answer in the simple case already calculated in the previous section, put $F(t) = 1$ for $-T/2 < t < +T/2$

$= 0$ elsewhere

---

*It has been assumed that $g(f)$ is negligible for $f > f_n$ and that the window function is symmetrical about $t = 0$

Then $g(f) = 4 \int_0^T \cos 2\pi f t \; dt$

$$= \frac{4}{2\pi f} \sin 2\pi f T/2$$

$$= 2 \; \frac{\sin \pi f T}{\pi f}$$

Inserting this in 2.3/7 gives

$$\frac{1}{2} \langle a_n^2 + b_n^2 \rangle = \int_0^\infty E(f_k) \left[\frac{\sin \pi (f_n - f_k) T}{\pi (f_n - f_k) T} \right]^2 df_k$$

which is identical with equation 2.3/1 if f is converted into the equivalent harmonic number.

The wider significance of equations 2.3/7 and 2.3/8 is that the shape of the filter function of the analysis is the shape of the spectrum of the window function. Thus, in order to reduce the size of the skirts, a window function must be chosen with a spectrum which falls off quickly at higher frequencies. A function commonly used is:

$$F(t) = \frac{1}{2} + \frac{1}{2} \cos 2\pi t/T \qquad \text{for } -T/2 < t < +T/2 \qquad 2.3/9$$

$= 0$ elsewhere

The Fourier transform of this is

$$g(f) = 4 \int_0^{T/2} \left(\frac{1}{2} + \frac{1}{2} \cos 2\pi t/T \right) e^{-i 2\pi f t} \; dt$$

which after some manipulation, gives

$$g(f) = T \cdot \frac{\sin \pi f T}{\pi f T} \cdot \frac{1}{1 - (fT)^2} \qquad 2.3/10$$

The shapes of $\left[\frac{g(f)}{T} \right]^2$ for this and the square window are given in figure 2.3/1 It will be seen that the windowed function falls off very quickly at high frequencies. The corresponding weighting functions are given in Table 2.3/3.

It will be shown in section 2.4 that this window function has the major disadvantage that it increases the random sampling error by a factor close to $\sqrt{2}$ This is because it in effect throws away roughly half the data. Thus, people have searched for window functions which reduce the skirts of the filter function while retaining more of the data.

A commonly used class of functions is the partial cosine tapers, defined as follows.

Table 2.3/3

Weighting factors for a full cosine taper

The spectral analysis is performed on $Z(t) = z(t)\left(\frac{1}{2} + \frac{1}{2}\cos 2\pi t/T\right)$

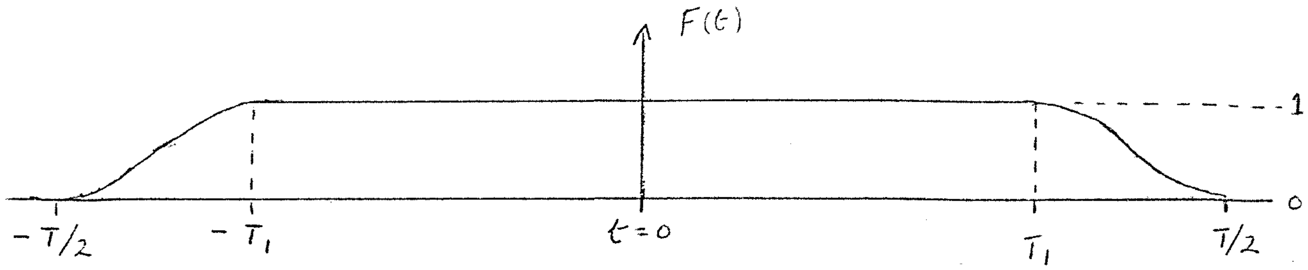
(a) With no averaging over adjacent harmonics

M	w_M	$\sum_{q=M+1}^{\infty} w_q$
0	.60061	.19968
1	.19697	.00271
2	.00262	.00009
3	.00008	.00001
4	.00001	0

(b) Adjacent harmonics averaged 10 at a time

M	W_M
0	.9595
1	.02025
2	10^{-6}

Figure 2.3/4



$$F(t) = 1 \qquad \text{for } 0 < t < T_1$$

$$F(t) = \tfrac{1}{2}\left[1 + \cos(\alpha t + \beta)\right] \qquad \text{for } T_1 < t < T/2$$

$$F(t) = F(-t)$$

$$\alpha = 2\pi\gamma/T \qquad \text{where } \gamma \text{ is an integer}$$

$$\beta = \pi \qquad \text{for } \gamma \text{ even}$$

$$\beta = 0 \qquad \text{for } \gamma \text{ odd}$$

It is shown in section 2.6.5 that the Fourier Transform of this function is (for $\gamma \neq 0$)

$$g(f) = \frac{1}{\pi f} \cdot \frac{\gamma^2}{\gamma^2 - (fT)^2}\left[\sin\left(\pi f T\,\frac{\gamma - 1}{\gamma}\right) + \sin \pi f T\right] \qquad 2.3/11$$

This function is plotted in figure 2.3/1 for $\gamma = 5$ and the weighting functions are given in Table 2.3/4. It will be seen that when the estimates are derived from smoothing harmonics 10 at a time, this is very effective at reducing the skirts of the filter function, while losing comparatively little data.

A final caveat should perhaps be added. All the theory in this section strictly speaking applies to analogue processing. For the typical circumstances of digital analysis of wave records for engineering purposes the results can be applied in a straightforward manner. However in unusual circumstances, such as if one were trying to look at the very low activity round about 1 minute period in the presence of sizeable ordinary waves, it would be necessary to look closely at a number of other factors before the results could be applied in a straightforward manner. Such factors are the noise due to the finite digitising amplitude intervals, aliassing due to the finite digitising time interval, and possibly the departure of the windowing formulae from the analogue values when applied to a finite number of samples.

Table 2.3/4

Weighting factors for a partial cosine taper

The figures below correspond to the case where the tapering is carried out over the first and last 10%'s of the record, the middle 80% being at full amplitude, (that is, $\gamma = 5$).

(a) No smoothing over adjacent harmonics

M	w_M	$\sum_{q=M+1}^{\infty} w_q$
0	.74833	.12583
1	.09609	.02974
2	.01681	.01293
3	.00654	.00639
4	.00313	.00326
5	.00160	.00166
6	.00082	.00084
7	.00042	.00042
8	.00021	.00021
9	.00011	.00010
10	.00006	.00004

(b) Smoothing over adjacent harmonics 10 at a time

M	W_M
0	.963724
1	.018133
2	5×10^{-6}
3	$< 10^{-6}$

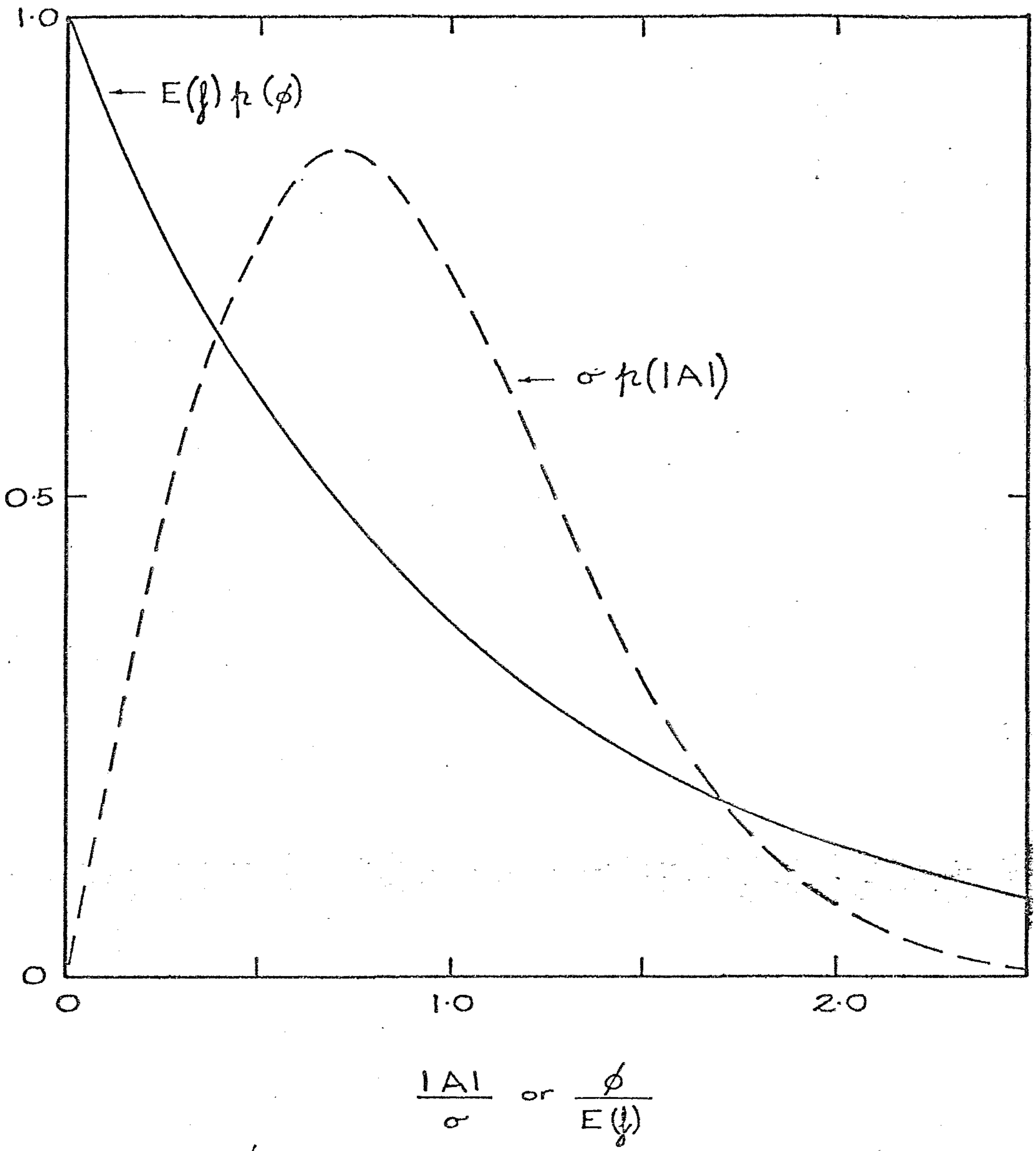


Figure 2.4/1

The probability distributions of the amplitude $|A|$ of a Fourier harmonic ($p(|A|)$: the Rayleigh distribution) and of the estimate ϕ of spectral density derived from it ($p(\phi)$: the negative exponential distribution). Note the curious fact that the most likely value of ϕ is always zero. This is well illustrated by the middle diagram of figure 2.2/1.

σ^2 = ensemble average of $|A|^2$

$E(\phi)$ = " " " ϕ

2.4 Random Sampling Errors

2.4.1 The random sampling error when averaging over uncorrelated harmonics

It has been mentioned once or twice in passing that the estimates $\phi(f)$ of the spectral density $E(f)$ have a large sampling error. In this section this statement will be quantified.

It has already been demonstrated that the complex amplitude of a given harmonic is made up of the superposition of a large number of contributions from components in the original record. It is in the nature of a random gaussian process that the phases of these contributions are random: the amplitudes also have a random variation, but it is not necessary to assume this here. Thus, if these components are added together on a phase diagram the classical Random Walk situation is obtained and a standard statistical theorem says that the probability that the amplitude of the resultant vector will fall between $|A|$ and $|A| + \delta|A|$ is $p(|A|)\,\delta A$ where

$$p(|A|) = \frac{2}{\sigma^2}|A| \exp\left(-\frac{|A|^2}{\sigma^2}\right) \qquad 2.4/1$$

where σ^2 is the mean value of $|A|^2$

This is the well-known Rayleigh Distribution.

For use in the frequency/variance spectrum which is being discussed here, it is the square of the amplitude $|A|^2 = a^2 + b^2$ which is of interest. This happens to be an easy transformation (see section 2.6.5) and gives

$$p(|A|^2) = \frac{1}{\sigma^2} \exp\left(-\frac{|A|^2}{\sigma^2}\right) \qquad 2.4/2$$

Now $\quad a_n^2 + b_n^2 = |A_n|^2 = 2\,\phi_n\,\Delta f$

and $\quad \sigma^2$ = mean value of $|A_n|^2 = 2\,E(f_n)\,\Delta f$

Thus, this equation can be written (see section 2.6.5)

$$p(\phi_n) = \frac{1}{E(f_n)} \exp\left[-\frac{\phi_n}{E(f_n)}\right] \qquad 2.4/3$$

The Rayleigh distribution of 2.4/1 and this negative exponential distribution are plotted in Figure 2.4/1. It will be seen that any particular value of ϕ_n is likely to be a very poor measure of the corresponding spectral density of variance $E(f_n)$ In fact, the standard error, defined as $\sqrt{\langle[\phi_n - E(f_n)]^2\rangle}$ is $E(f_n)$ or the proportional standard error is 1.

Another standard statistical theorem states that if random uncorrelated samples of a variable are taken and then averaged L at a time, the variance of these averages is proportional to $1/L$, or the standard deviation of the averages is proportional to $(1/L)^{\frac{1}{2}}$. Thus, the standard error of an estimate of $E(\phi_n)$ obtained by averaging over L uncorrelated harmonics is given by

$$\text{Proportional standard error of } \phi_n = 1/L^{\frac{1}{2}} \qquad 2.4/4$$

This is a result of great significance. It shows, for example, that even averaging 10 harmonics at a time, the standard error of the resulting estimate is still $1/\sqrt{10} \simeq 32\%$

The probability distribution of the estimates is a chi-squared distribution with $2L$ degrees of freedom. This will not be derived in this paper, but the distributions and their confidence limits can be looked up in standard books of tables. The factor of 2 arises because the average is in fact being taken over harmonic phase components (that is, L values of a^2 plus L values of b^2). a and b have Normal probability distributions, so that the average is being taken of $2L$ normally distributed components.

2.4.2 The effect of windowing on the random sampling error

A little thought will show that the arguments leading to the Rayleigh probability distribution for $|A_n|$ (equation 2.4/1) are not affected by windowing. The effect of windowing on the random sampling error arises because adjacent values of $|A_n|$ from one analysis are no longer effectively independent. This is clearly seen from table 2.3/3, for example, which shows that when using a full cosine taper, nearly 40% of the expectation of ϕ_n is due to the energy density in the intervals on either side of it. This means that when averaging over adjacent harmonics, the standard error of the estimate is reduced by a factor smaller than $\sqrt{L}$

The precise theory of the effect of this correlation is cumbersome, but a reasonable approximation can be obtained comparatively simply. It is shown in section 2.6.6 that the standard error of the measurement of the total wave energy is proportional to

$$\overline{F^4(t)}^{\frac{1}{2}} \Big/ \overline{F^2(t)} \qquad 2.4/5$$

where $F(t)$ is the weighting function applied to the original record. For a full cosine taper for example, this factor is $(35/18)^{\frac{1}{2}} = 1.4$. If the estimates after averaging become effectively independent, then clearly this ratio must also apply to them.

Table 2.3/3b shows that using a full cosine taper and averaging over 10 harmonics at a time, the contribution to an estimate from the intervals on either side is only approximately 4% (compared with the 40% of the un-averaged estimates). Thus, assuming that the estimates are uncorrelated is a reasonable approximation and equation 2.4/4 can be applied to give an approximate measure of the effect of windowing.

Thus, when spectral density estimates are obtained by averaging over a number of adjacent harmonics, the effect of windowing by a full cosine taper, for example, is to increase the standard sampling error by a factor of approximately 1.4.

2.5 Practical sampling and analysis procedures

2.5.1 General

In practice, sampling and analysis procedures have to be a compromise. The parameters which have to be decided when the samples are taken are

(a) Digitising time interval Δt

(b) Digitising amplitude interval $\Delta \zeta$

(c) Sample record length T

(d) Interval between sample records

The parameters which have to be decided when the sample records are analysed are

(e) The frequency resolution required

(f) Whether to use segmented Fourier Transforms

(g) Whether to use windowing

These two sets of parameters interact to some extent, but it is convenient to consider them separately.

2.5.2 The sampling scheme

(a) Digitising time interval

Standard communication theory shows that the real spectrum is folded back on itself at a frequency $1/2\Delta t$. Thus, one must choose Δt such that the spectral energy folded back is small compared with the real spectral density in the frequency range of interest. In practice, wave sensors often have an instrumental cut-off at high frequencies, and Δt can be chosen so that $1/2\Delta t$ is above this cut-off frequency. Δt should not, however, be made smaller than necessary since many of the other important parameters are limited by the number of individual readings that can be stored and processed economically.

(b) Digitising amplitude interval

The process produces a white noise over the whole spectrum with an integrated variance of $1/4\,(\Delta \zeta)^2$. Thus, the noise in any frequency interval can readily by calculated.

(c) Sample record length

This is where one reaches the difficult decisions. As has been seen, the sampling error is unfortunately large with any practical analysis scheme. For a given frequency resolution, it is proportional to $(1/T)^{1/2}$

Thus, there is pressure to increase the record length. However, particularly when dealing with routine analysis, the cost and even the availability of computer time is limited, and the computer time required for an analysis is roughly proportional to T^2.

(d) Interval between sample records

The question here is "over what period do the sea-state parameters change significantly?". A significant change means one larger than the sampling error. In some circumstances the sea can change quite quickly.

Taking (c) and (d) together, clearly the ideal would be to record continuously and divide this continuous record into lengths such that the maximum change between records was approximately equal to the random sampling error. In practice, it is still not economically possible to do this for routine recording, though it may now be possible to do it using on-line microprocessors. However, where data vetting is required before frequency analysis (and this is usually the case if the data is telemetered over a radio link before recording), the computing time is still a limiting factor, as is also the storage capacity of magnetic tape digital recorders suitable for use in remote locations.

For the time being, a typical sampling routine is to sample for 1024 seconds at half-second intervals every 3 hours.

2.5.3 The analysis scheme

(e) The frequency resolution required

Since the practical considerations discussed above limit the length of the sample record and hence the frequency separation of the Fourier harmonics, there is a simple compromise between frequency resolution and sampling error.

(f) The use of segmented FFT's

One way of reducing computation time is to divide the record into a number of equal lengths, carry out a Fourier transform of each length, and then average over the corresponding harmonics of each analysis. The number of multiplications required for a Fourier transform is proportional to T^2 : dividing the record into L equal lengths therefore reduces computation time by a factor of approximately L . The price paid is a major widening of the skirts of the equivalent filter function: compare tables 2.3/1 and 2.3/2.

In fact, with modern FFT algorithms, the penalty for analysing the longer records is not so great as indicated above. Thus, for the 1024 s records, most operators consider it worth transforming the complete record.

(g) Should one use windowing?

If segmented FFT's <u>are</u> used, then there is a case for using windowing to reduce the skirts of the equivalent filter function. However, the width of the central lobe of the filter function is increased considerably. (Compare tables 2.3/1 and 2.3/3a.)

If the complete record is transformed and averaging performed over adjacent harmonics, then the price paid for the reduction of the skirts is an increase in the sampling error. To get a feel for the balance of advantage, consider three schemes for analysing complete records.

(i) Simple Fourier transform followed by averaging the variances of the harmonics 10 at a time.

(ii) As (i) but using a partial cosine window function with $\gamma = 5$.

(iii) As (i) but using a full cosine window function.

The frequency spreading can be compared using tables 2.3/2, 2.3/4 and 2.3/3b. It will be seen that, surprisingly, W_1 is close to 0.02 in all three cases. The differences arise in W_2 and more distant weights. W_2 is 0.0014 for the first case, 5×10^{-6} for the second case and less than 10^{-6} for the third.

The standard sampling errors are respectively approximately 32%, 35% and 45%.

For all ordinary engineering purposes a value of W_2 of 0.0014 is negligible and it is clear that the use of windowing is not justified. However, if for special reasons the side-lobes have to be reduced, then the second scheme gives a major reduction with only a small penalty in the increase in sampling error.

Perhaps it should be said that a number of variations on the themes described above have been suggested and/or used from time to time, but in the author's view they offer no significant advantages in the present state of technology and it would only cause confusion to consider them here.

At the present time, the standard scheme used for routine analysis by IOS (sampling twice per second for 1024 s every 3 hours) uses only approximately 10% of the data available from a wave sensor. The resulting spectra have a frequency resolution of approximately 0.01 Hz and a standard sampling error of 30%. None of these parameters can be considered to be satisfactory, and it is clear that the way ahead must be to decrease the labour of editing the records and the cost of analysis to the point where a greater proportion of the potentially available data can be analysed.

<u>PART 2B</u>

Theoretical notes and the derivations of some of the formulae used in Part 2A

2.6.1 Equation 2.3/1. The equivalent filter function of a Fourier analysis without windowing.

It is again convenient to consider a long conceptual record of the process extending over the period $-\tau/2 < t < \tau/2$, where τ will eventually be taken to be infinite. This may be represented by its Fourier series (equation 2.1/3).

$$z(t) = \frac{1}{2} \sum_{p=-\infty}^{\infty} C_p \, e^{i\omega_p t}$$

where $\omega_p = 2\pi p/\tau$

We shall look at what happens to the general p th component when we take a sample of $z(t)$ of duration T where $-T/2 < t < T/2$

Using equation 2.1/4, the amplitude of the nth Fourier component of the finite sample is:

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} z(t) \, e^{-i\omega_n t} \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left[\sum_{p=-\infty}^{\infty} C_p \, e^{i\omega_p t} \right] e^{-i\omega_n t} \, dt$$

$$= \frac{1}{T} \sum_{p=-\infty}^{\infty} C_p \int_{-T/2}^{T/2} e^{i(\omega_p - \omega_n)t} \, dt$$

$$= \frac{1}{T} \sum_{p=-\infty}^{\infty} C_p \frac{1}{i(\omega_p - \omega_n)} \left[e^{i(\omega_p - \omega_n)T/2} - e^{-i(\omega_p - \omega_n)T/2} \right]$$

$$= \frac{1}{T} \sum_{p=-\infty}^{\infty} C_p \frac{1}{i(\omega_p - \omega_n)} \cdot 2i \sin\left[(\omega_p - \omega_n) T/2\right]$$

$$= \sum_{p=-\infty}^{\infty} C_p \frac{\sin\left[(\omega_p - \omega_n)T/2\right]}{(\omega_p - \omega_n) T/2.}$$

This amplitude response function must now be converted to an energy (or variance) response function, since we are trying to estimate the power spectrum of the process $\zeta(t)$.

Remembering the definitions (see equation 2.1/4)

$$A_n = a_n - i\, b_n$$

$$A_{-n} = a_n + i\, b_n$$

$$C_p = c_p - i\, d_p$$

$$C_{-p} = c_p + i\, d_p$$

Then $\frac{1}{2}(a_n^2 + b_n^2) = \frac{1}{2} A_n A_{-n}$

$$= \frac{1}{2} \sum_{p=-\infty}^{\infty} C_p \frac{\sin(\omega_p - \omega_n)T/2}{(\omega_p - \omega_n)T/2} \sum_{q=-\infty}^{\infty} C_q \frac{\sin(\omega_q + \omega_n)T/2}{(\omega_q + \omega_n)T/2}$$

Since the values of c_p and d_p are randomly positive and negative, when an ensemble is taken the only consistently positive or negative terms in the product series are when $q = p$ and $q = -p$. Thus

$$\left\langle \tfrac{1}{2}(a_n^2 + b_n^2) \right\rangle = \left\langle \frac{1}{2} \sum_{p=-\infty}^{\infty} C_p^2 \left\{ \frac{\sin(\omega_p - \omega_n)T/2}{(\omega_p - \omega_n)T/2} \cdot \frac{\sin(\omega_p + \omega_n)T/2}{(\omega_p + \omega_n)T/2} \right\} \right.$$

$$\left. + \frac{1}{2} \sum_{p=-\infty}^{\infty} C_p C_{-p} \left\{ \frac{\sin^2(\omega_p - \omega_n)T/2}{[(\omega_p - \omega_n)T/2]^2} \right\} \right\rangle$$

Considering the first term, the factor inside the curly brackets is the same for p and $-p$, and zero for $p = 0$ since $\omega_n = 2\pi n/T$

Also $\langle C_p^2 + C_{-p}^2 \rangle = \langle c_p^2 - 2ic_p d_p - d_p^2 + c_p^2 + 2ic_p d_p - d_p^2 \rangle$

$$= \langle 2(c_p^2 - d_p^2) \rangle$$

$$= 0$$

Considering the second term, the factor inside the curly brackets is also 0 for $p = 0$.

$$\therefore \left\langle \tfrac{1}{2}(a_n^2+b_n^2)\right\rangle = \left\langle \tfrac{1}{2}\sum_{p=-\infty}^{\infty}(c_p^2+d_p^2)\ \frac{\sin^2(\omega_p-\omega_n)T/2}{[(\omega_p-\omega_n)T/2]^2}\right\rangle$$

$$= \sum_{p=1}^{\infty}\left\langle \tfrac{1}{2}(c_p^2+d_p^2)\right\rangle \left\{\frac{\sin^2(\omega_p-\omega_n)T/2}{[(\omega_p-\omega_n)T/2]^2} + \frac{\sin^2(\omega_p+\omega_n)T/2}{[(\omega_p+\omega_n)T/2]^2}\right\}$$

Putting $1/\tau = \delta f$ then taking τ to infinity

$$\sum_{p=1}^{\infty}\left\langle \tfrac{1}{2}(c_p^2+d_p^2)\right\rangle\cdot X = \sum_{p=1}^{\infty} E(f_p)\,\delta f\cdot X \rightarrow \int_0^{\infty} E(f_p)\,X\,df_p$$

$$\therefore \left\langle \tfrac{1}{2}(a_n^2+b_n^2)\right\rangle = \int_0^{\infty} E(f_p)\left\{\frac{\sin^2(\omega_p-\omega_n)T/2}{[(\omega_p-\omega_n)T/2]^2} + \frac{\sin^2(\omega_p+\omega_n)T/2}{[(\omega_p+\omega_n)T/2]^2}\right\} df_p$$

Now in many practical cases we are dealing with values of n from, say, 30 upwards, so that the denominator of the second term varies from $(30\pi)^2 \simeq 10^4$ upwards, and the second term can be neglected. In this case it is also convenient to put

$$\omega_p - \omega_n = 2\pi m/T \quad \text{or} \quad f_p - f_n = m/T$$

That is, we express the difference frequency in terms of harmonic intervals m.

Then $$\left\langle \tfrac{1}{2}(a_n^2+b_n^2)\right\rangle \simeq \int_{m=-n}^{\infty} E(f)\ \frac{\sin^2\pi m}{(\pi m)^2}\cdot\frac{1}{T}\,dm.$$

which is equation 2.3/1

Note that

$$\int_{-\infty}^{\infty}\frac{\sin^2\pi m}{(\pi m)^2}\,dm = 1$$

2.6.2 The infinite Fourier transform of a window function

The window function $F(t)$ is finite in the region $-T/2 < t < T/2$ and zero elsewhere. It will be assumed to be symmetrical about $t = 0$ (see, for example, figure 2.3/4).

Consider it in the longer interval $-\Upsilon/2 < t < \Upsilon/2$

Then the finite Fourier transform is (see equation 2.1/4)

$$G'_q = \frac{2}{\Upsilon}\int_{-\Upsilon/2}^{\Upsilon/2} F(t)\, e^{-i 2\pi q t/\Upsilon}\, dt$$

$G'_q = g'_q + i h'_q$ is the complex amplitude of the q^{th} harmonic of the interval Υ. The reason for the dashes will appear later.

However, since $F(t)$ is zero outside the T interval,

$$G'_q \Upsilon = 2\int_{-\Upsilon/2}^{\Upsilon/2} F(t)\, e^{-i 2\pi q t/\Upsilon}\, dt = 2\int_{-T/2}^{T/2} F(t)\, e^{-i 2\pi q t/\Upsilon}\, dt$$

This is a function of q/Υ which is the frequency f. Thus, for a given frequency $G'\Upsilon$ is independent of Υ, and therefore remains finite if Υ is taken to infinity.

We may therefore define a new function

$$G(f) = 2\int_{-\infty}^{\infty} F(t)\, e^{-i 2\pi f t}\, dt$$

$G(f)$ is called the infinite Fourier transform of $F(t)$. Such a transform applies to pulses which have a finite duration. Note that it is a continuous function of frequency, not discrete as in the case of finite transforms.

If $F(t)$ is symmetrical about $t = 0$, then $F(t) = F(-t)$ and

$$\int_{-\infty}^{\infty} F(t)\, e^{-i 2\pi f t}\, dt = \int_0^{\infty} F(t)\left(e^{-i 2\pi f t} + e^{i 2\pi f t}\right) dt$$

$$= \int_0^{\infty} F(t)\, 2\cos(2\pi f t)\, dt$$

Which shows that $G(f)$ is real, that is, in our usual convention.

$$G(f) = g(f)$$

and

$$g(f) = 4\int_0^{\infty} F(t) \cos 2\pi f t\, dt$$

2.6.3 Derivation of equations 2.3/7 and 2.3/8
The filter function of a general windowing process

The algebra of this section gets rather involved and can obscure the conceptual sequence, so that this will be explained first.

As before, we consider the output $\zeta(t)$ of a waveheight sensor which we shall assume to be measuring a statistically stationary process. We assume that $\zeta(t)$ can be considered as the superposition of a large number of sinusoidal components, and for mathematical convenience these are taken as the Fourier harmonics of a long conceptual record with a duration Υ which will eventually be taken to be infinite. This long record is sampled by multiplying it by a function $F(t)$ which is non-zero only in the range $-T/2 < t < T/2$. For a simple sample $F(t)$ is 1 within this range, but in general can be tapered in some way. $F(t)$ is assumed to be symmetrical about $t = 0$

We first show (equation 2.6/2) that the effect of this process is to spread each component of the original signal over a band whose shape (in amplitude) is that of the amplitude spectrum of the window function (which is in effect its infinite Fourier transform). The original spectrum has in fact been convoluted with the Fourier transform of the window function.

We then show (equation 2.6/4) that the Fourier series transform of the sample picks out just those frequencies of the convoluted spectrum which correspond to harmonics of the sample length T without further spreading.

At first sight it is surprising that the Fourier series transform does not introduce further spreading, but since the sampling has been done before this stage, so that the signal is zero outside the range $-T/2 < t < T/2$, the product of the signal and of the sinusoidal function can be integrated from $-\infty$ to $+\infty$ without changing the answer. This means that it picks out a single frequency component from the windowed signal.

Finally, we show that the expected value of the resulting spectral estimate (that it, its ensemble average) is the integrated product of the true power spectrum and of the square of the Fourier transform of the window function shifted to be centred on the frequency of the estimate. For each Fourier harmonic it is as though the original signal had been passed through a filter centred on the harmonic frequency and with the shape of the Fourier transform of the window function, and the resulting power measured. The tail of the equivalent filter function which would be below zero frequency is reflected back to positive frequencies, but can usually be neglected.

The mathematics will now be given.

As stated above, we consider a conceptual long record of $z(t)$ extending over the interval $-\Upsilon/2 < t < \Upsilon/2$, where Υ will eventually be taken to infinity. The sample of this extends over $-T/2 < t < T/2$, but now this will be multiplied by a "window function" which smooths off the ends (see later). It will be convenient to consider this sampling and windowing process as the one operation of multiplying $z(t)$ by a function $F(t)$ which extends over the long interval $-\Upsilon/2 < t < \Upsilon/2$ but which is 0 except in the sample range. Thus, the windowed sample becomes $Z(t)$ where

$$Z(t) = z(t)\,F(t) \qquad \text{for } -\Upsilon/2 < t < \Upsilon/2$$

and we can put each function in terms of its Fourier Series as follows

$$z(t) = \frac{1}{2}\sum_{p=-\infty}^{\infty} K_p\, e^{i\omega_p t}$$

$$F(t) = \frac{1}{2}\sum_{q=-\infty}^{\infty} G'_q\, e^{i\omega_q t}$$

$$Z(t) = \frac{1}{2}\sum_{r=-\infty}^{\infty} C_r\, e^{i\omega_r t} \qquad 2.6/1$$

K_p G'_q and C_r are complex amplitudes (the fact that G'_q is real, as has been shown in section 2.6.2, will be introduced later).

$$\omega_p = 2\pi p/\Upsilon \qquad \text{etc}$$

The first step is to calculate the values of C_r : that is, to determine how the windowing and sampling process has modified the spectrum of the function before the sample FFT is carried out.

$$C'_r = \frac{2}{\Upsilon}\int_{-\Upsilon/2}^{\Upsilon/2} Z(t)\, e^{-i\omega_r t}\, dt$$

$$= \frac{2}{\Upsilon}\int_{-\Upsilon/2}^{\Upsilon/2} g(t)\, F(t)\, e^{-i\omega_r t}\, dt$$

$$= \frac{2}{\Upsilon}\int_{-\Upsilon/2}^{\Upsilon/2} \frac{1}{4}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} K_p\, G'_q\, e^{i(\omega_p+\omega_q-\omega_r)t}\, dt$$

$$= \frac{1}{2\Upsilon}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} K_p\, G'_q \int_{-\Upsilon/2}^{\Upsilon/2} e^{i2\pi(p+q-r)t/\Upsilon}\, dt$$

Remembering that p , q and r are integers, the integral is zero except when $p+q-r=0$, when it equals . Thus,

$$C'_r = \frac{1}{2}\sum_{p=-\infty}^{\infty} K_p\, G'_{r-p} \qquad 2.6/2$$

The usual Fourier analysis of the sampled function can now be performed (equation 2.1/4)

$$A_n = \frac{2}{T}\int_{-T/2}^{+T/2} Z(t)\, e^{-i2\pi nt/T}\, dt$$

But since $Z(t)$ is zero outside the range $-T/2 < t < T/2$, the limits of integration may be extended to $-\Upsilon/2 < t < \Upsilon/2$

$$\therefore A_n = \frac{2}{T}\int_{-\Upsilon/2}^{\Upsilon/2} Z(t)\, e^{-i2\pi nt/T}\, dt$$

To simplify the mathematics we now postulate that $\Upsilon = \alpha T$ where α is an integer. Since α will eventually be taken to infinity, this in fact loses no generality.

Using equation 2.6/1

$$A_n = \frac{2}{T}\int_{-\gamma/2}^{+\gamma/2} \frac{1}{2}\sum_{r=-\infty}^{\infty} C_r\, e^{i2\pi rt/\gamma}\, e^{-i2\pi\alpha nt/\gamma}\, dt$$

$$= \frac{1}{T}\sum_{r=-\infty}^{\infty} C_r \int_{-\gamma/2}^{\gamma/2} e^{i2\pi(r-\alpha n)t/\gamma}\, dt$$

The integral is zero except when $r = \alpha n$, when it equals

$$\therefore A_n = (\gamma/T)\, C_{\alpha n} \qquad 2.6/4$$

Using equation 2.6/2

$$A_n = \frac{\gamma}{2T}\sum_{p=-\infty}^{\infty} K_p\, G'_{\alpha n-p}$$

It will considerably simplify the maths if it is assumed as in section 2.6.2 that the window function is symmetrical so that $G'_{\alpha n-p}$ is real and therefore equal to $g'_{\alpha n-p}$

$$\frac{1}{2}(a_n^2 + b_n^2) = \frac{1}{2} A_n A_{-n}$$

$$= \frac{\gamma^2}{8T^2}\sum_{p=-\infty}^{\infty} K_p\, g'_{\alpha n-p} \sum_{q=-\infty}^{\infty} K_q\, g'_{-\alpha n-q}$$

The values of K are randomly positive and negative, so that when an ensemble average is taken the only consistently positive or negative terms in the product series are those with $q = p$ and $q = -p$. Thus, remembering that

$$g'_x = g'_{-x}$$

$$\frac{1}{2}\langle a_n^2 + b_n^2\rangle = \frac{\gamma^2}{8T^2}\left\langle \sum_{p=-\infty}^{\infty}\left[K_p^2\, g'_{\alpha n-p}\, g'_{\alpha n+p} + K_p K_{-p}\, g'_{\alpha n-p}\, g'_{\alpha n-p}\right]\right\rangle$$

Consider the first series inside the square bracket, and add the terms for p and $-p$ (remembering that g' depends only on the shape of the window function and is constant in an ensemble average).

$$\langle K_p^2\, g'_{\alpha n-p}\, g'_{\alpha n+p} + K_{-p}^2\, g'_{\alpha n+p}\, g'_{\alpha n-p}\rangle$$
$$= \langle K_p^2 + K_{-p}^2\rangle\, g'_{\alpha n-p}\, g'_{\alpha n+p}$$

It has been shown in section 2.6.1 that

$$\langle K_p^2 + K_{-p}^2\rangle = 0$$

The term for $p = 0$ does not disappear, but tends to zero as $\Upsilon \to \infty$ so will be neglected here for the sake of simplicity.

Thus $$\frac{1}{2}\langle a_n^2 + b_n^2 \rangle = \frac{\Upsilon^2}{8T^2}\left\langle \sum_{p=-\infty}^{\infty} K_p K_{-p} (g'_{\alpha n-p})^2 \right\rangle$$

But $$\langle K_p K_{-p} \rangle = \langle (k_p - i l_p)(k_p + i l_p) \rangle$$
$$= \langle k_p^2 + l_p^2 \rangle = 2E(f_p)\Delta f$$

where $$\Delta f = 1/\Upsilon$$

$$\therefore \frac{1}{2}\langle a_n^2 + b_n^2 \rangle = \frac{\Upsilon^2}{8T^2}\left\langle \sum_{p=-\infty}^{\infty} 2E(f_p)(g'_{\alpha n-p})^2 \Delta f \right\rangle$$

Now it was shown in section 2.6.2 that $\Upsilon g'_{\alpha n-p}$ remains constant as $\Upsilon \to \infty$ so long as the frequency to which g' refers is held constant. In the present case g' refers to the difference in frequency between the αn^{th} harmonic and the pth harmonic of the long record of length Υ. The αn^{th} harmonic of this is just the nth harmonic of the sample length T and thus has the frequency f_n for which we are analysing, whereas the pth harmonic has the general frequency f_p over which we are summing. Thus, as we take $\Upsilon \to \infty$ we can write

$$\Upsilon g'_{\alpha n-p} = g(f_n - f_p)$$

where g is the infinite Fourier transform of the window function as defined in section 2.6.2.

$$\therefore \frac{1}{2}\langle a_n^2 + b_n^2 \rangle = \frac{1}{4T^2}\int_{-\infty}^{\infty} E(f_p)\, g^2(f_n - f_p)\, df_p$$
$$= \frac{1}{4T^2}\int_{0}^{\infty} E(f_p)\left[g^2(f_n - f_p) + g^2(f_n + f_p)\right] df_p$$

since $$E(f_p) = E(-f_p)$$

Since f_p is now only positive in this equation, the criterion for the second term in the square brackets to be negligible is that $g^2(f) \ll 1$ for $f > f_n$. For many practical circumstances this will be the case, and then

$$\frac{1}{2}\langle a_n^2 + b_n^2 \rangle \simeq \frac{1}{4T^2}\int_{0}^{\infty} E(f_p)\, g^2(f_n - f_p)\, df_p \qquad 2.3/7$$

2.6.4 Derivation of equation 2.3/11

The infinite Fourier transform of a partial cosine window function

Refer to figure 2.3/4 for definitions of the window function parameters. To simplify the mathematics it will be assumed that the cosine wave used in the taper is an exact harmonic of the sample record length T : that is, that γ is an integer. In practice, the restriction represented by this condition is unimportant.

In section 2.6.2 it was shown that the infinite Fourier transform of a symmetrical function is given by

$$g(f) = 4\int_0^\infty F(t)\cos 2\pi f t\, dt$$

Putting $\omega = 2\pi f$ for convenience, and inserting the partial cosine function for $F(t)$, this becomes

$$g(f) = 4\int_0^{T_1}\cos\omega t\, dt + 2\int_{T_1}^{T/2}[1+\cos(\alpha t+\beta)]\cos\omega t\, dt$$

$$= \frac{4}{\omega}\sin\omega T_1 + \frac{2}{\omega}(\sin\omega T/2 - \sin\omega T_1) + 2\int_{T_1}^{T/2}\cos(\alpha t+\beta)\cos\omega t\, dt$$

$$= \frac{2}{\omega}(\sin\omega T_1 + \sin\omega T/2) + 2\int_{T_1}^{T/2}\cos\omega t\ \cos(\alpha t+\beta)\, dt \qquad 2.6/5$$

The last term in this equation can be converted to

$$\int_{T_1}^{T/2}\left[\cos(\overline{\omega+\alpha}\ t+\beta) + \cos(\overline{\omega-\alpha}\ t\ \ \beta)\right]dt$$

$$= \frac{1}{\omega+\alpha}\left[\sin(\overline{\omega+\alpha}\ T/2+\beta) - \sin(\overline{\omega+\alpha}\ T_1+\beta)\right]$$

$$+ \frac{1}{\omega-\alpha}\left[\sin(\overline{\omega-\alpha}\ T/2-\beta) - \sin(\overline{\omega-\alpha}\ T_1-\beta)\right] \qquad 2.6/6$$

But by definition

$$\alpha = 2\pi\gamma/T$$

$$\alpha T/2 = \pi\gamma$$

Also, since there are γ wavelengths of the cosine taper sinusoid in the length T and there are $\gamma - 1$ wavelengths between $-T_1$ and $+T_1$

$$\therefore \frac{2T_1}{T} = \frac{\gamma - 1}{\gamma}$$

and

$$\alpha T_1 = 2\pi \gamma T_1 / T = \pi(\gamma - 1)$$

We now have to consider separately the cases for γ even and γ odd.

For γ even $\beta = \pi$ by definition

$$\therefore \beta + \alpha T/2 = \pi + \text{integer} \times 2\pi$$

$$\beta + \alpha T_1 = 0 + \text{integer} \times 2\pi$$

For γ odd $\beta = 0$ by definition

$$\therefore \beta + \alpha T/2 = \pi + \text{integer} \times 2\pi$$

$$\beta + \alpha T_1 = 0 + \text{integer} \times 2\pi$$

In both cases, 2.6/6 becomes

$$\frac{1}{\omega + \alpha}\left[-\sin \omega T/2 - \sin \omega T_1\right] + \frac{1}{\omega - \alpha}\left[-\sin \omega T/2 - \sin \omega T_1\right]$$

$$= -\left[\frac{1}{\omega + \alpha} + \frac{1}{\omega - \alpha}\right]\left[\sin \omega T/2 + \sin \omega T_1\right]$$

$$= -\frac{2\omega}{\omega^2 - \alpha^2}\left[\sin \omega T/2 + \sin \omega T_1\right]$$

Inserting this in 2.6/5

$$g(f) = \frac{2}{\omega}\left[\sin \omega T_1 + \sin \omega T/2\right] - \frac{2\omega}{\omega^2 - \alpha^2}\left[\sin \omega T_1 + \sin \omega T/2\right]$$

$$= \frac{2}{\omega}\left[1 - \frac{\omega^2}{\omega^2 - \alpha^2}\right]\left[\sin \omega T_1 + \sin \omega T/2\right]$$

$$= \frac{2}{\omega} \cdot \frac{\alpha^2}{\alpha^2 - \omega^2}\left[\sin \omega T_1 + \sin \omega T/2\right]$$

But $\alpha = 2\pi\gamma/T$

$$T_1 = \frac{\gamma-1}{\gamma}\cdot\frac{T}{2}$$

$$\omega = 2\pi f$$

$$\therefore \quad g(f) = \frac{1}{\pi f}\cdot\frac{\gamma^2}{\gamma^2-(fT)^2}\left[\sin \pi f T\,\frac{\gamma-1}{\gamma} + \sin\pi f T\right] \qquad 2.3/11$$

$\gamma \neq 0$

Checking that this gives correct results for the full cosine window and for no tapering:

For full cosine tapering $\gamma = 1$

so $$g(f) = \frac{1}{\pi f}\cdot\frac{1}{1-(fT)^2}\sin\pi f T$$

which is equation 2.3/10

For no tapering $\gamma \to \infty$

$$\therefore \quad g(f) = \frac{1}{\pi f}\cdot 2\sin(\pi f T)$$

$$= 2\,\frac{\sin\pi f T}{\pi f}$$

Inserting this in equation 2.3/7 we get equation 2.3/1, remembering that m in the latter equation is defined by: the difference frequency $f_r - f_p = m/T$

2.6.5 Derivation of equations 2.4/2 and 2.4/3

The probability distributions of the square of the amplitude of a harmonic and of the spectral estimate derived from it.

From equation 2.4/1, the probability that the amplitude of a Fourier harmonic will fall between $|A|$ and $|A| + \delta|A|$ is

$$p(|A|)\,\delta|A| = \frac{2}{\sigma^2}|A| \exp\left(-\frac{|A|^2}{\sigma^2}\right)\delta|A|$$

This is also the probability that the square of the amplitude will fall between $|A|^2$ and $(|A| + \delta|A|)^2 \simeq |A|^2 + 2|A|\,\delta|A|$

Thus
$$p(|A|^2).\,2|A|\,\delta|A| = \frac{2}{\sigma^2}|A| \exp\left(-\frac{|A|^2}{\sigma^2}\right)\delta|A|$$

or
$$p(|A|^2) = \frac{1}{\sigma^2}\exp\left(-\frac{|A|^2}{\sigma^2}\right) \qquad 2.4/2$$

The definition of the spectral estimate ϕ is

$$|A_n|^2 = 2\phi_n\,\Delta f$$

where
$$\Delta f = 1/T$$

From 2.4/2 the probability that the square of the amplitude will fall between $|A_n|^2$ and $|A_n|^2 + \delta|A_n|^2$ is

$$p(|A_n|^2)\,\delta|A_n|^2 = \frac{1}{\sigma^2}\exp\left(-\frac{|A_n|^2}{\sigma^2}\right)\delta|A_n|^2$$

This is also the probability that the spectral estimate ϕ_n will fall between

$$|A_n|^2/2\Delta f \quad \text{and} \quad (|A_n|^2 + \delta|A_n|^2)/2\Delta f$$

Thus
$$p(\phi_n)\,\delta|A_n|^2/2\Delta f = \frac{1}{\sigma^2}\exp\left(-\frac{|A_n|^2}{\sigma^2}\right)\delta|A_n|^2$$

$$\therefore\ p(\phi_n) = \frac{2\Delta f}{\sigma^2}\exp\left(-\frac{|A_n|^2}{\sigma^2}\right)$$

$$= \frac{2\Delta f}{\sigma^2}\exp\left(-\frac{2\phi_n\Delta f}{\sigma^2}\right)$$

But σ^2 = the ensemble average of $|A_n|^2 = 2E(\phi)\Delta f$

$$\therefore\ p(\phi_n) = \frac{1}{E(\phi_n)}\exp\left(-\frac{\phi_n}{E(\phi_n)}\right) \qquad 2.4/3$$

2.6.6 Derivation of equation 2.4/5

The random sampling error of the total energy of a sample record which has been windowed.

To approach this problem with reasonably simple mathematics we must start by considering a record $z(t)$ which has been sampled at intervals Δt to produce a series z_1 z_2 -- z_n--- z_N (these are before windowing). Δt is assumed to be large enough to ensure no correlation between adjacent samples, though this condition is in fact not essential in the present context, since we are concerned with the change in the sampling error due to windowing compared with an unwindowed function, and it will be shown that the effect of correlation is the same for both.

It is again assumed that the process being sampled is statistically stationary: that is, that the probability distribution of z_n is independent of n. The variance of z (which is proportional to the energy of the system) will be denoted by $E = \langle z^2 \rangle$. The estimate ϕ of E from our unwindowed sample is

$$\phi = \frac{1}{N} \sum_{n=1}^{N} z_n^2$$

$$\langle \phi \rangle = E$$

We are concerned with the sampling error of ϕ, and this is quantified by its variance V

$$V = \langle (\phi - E)^2 \rangle$$

The process of windowing is to multiply each sample by a corresponding weight u_n so that the series becomes:

$$u_1 z_1 \quad u_2 z_2 \quad \text{------} \quad u_n z_n \quad \text{----} \quad u_N z_N$$

Consider $\displaystyle\sum_{n=1}^{N} \langle (u_n z_n)^2 \rangle = \sum_{n=1}^{N} u_n^2 \langle z_n^2 \rangle$

$$= \sum_{n=1}^{N} u_n^2 \, E$$

$$= E \sum_{n=1}^{N} u_n^2$$

$$\therefore \quad E = \sum_{n=1}^{N} \langle (u_n^2 z_n^2) \rangle \Big/ \sum_{n=1}^{N} u_n^2$$

Thus, for a single sample record the estimate ϕ of E is

$$\phi = \sum_{n=1}^{N} (u_n z_n)^2 \Big/ \sum_{n=1}^{N} u_n^2 \qquad 2.6/6$$

For convenience put

$$C = \sum_{n=1}^{N} u_n^2 \quad \text{(which is constant for an ensemble of samples)}$$

$$Z_n = \langle (u_n z_n)^2 \rangle$$

$$\Delta_n = (u_n z_n)^2 - Z_n$$

Then from 2.6/6

$$C\phi = \sum_{n=1}^{N} (Z_n + \Delta_n)$$

$$C\langle\phi\rangle = CE = \sum_{n=1}^{N} Z_n$$

$$\therefore C(\phi - E) = \sum_{n=1}^{N} \Delta_n$$

$$C^2(\phi - E)^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} \Delta_n \Delta_m$$

$$\therefore C^2 V = C^2 \langle (\phi - E)^2 \rangle$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \Delta_n \Delta_m \rangle$$

But since Δ_n and Δ_m are randomly positive and negative, $\langle \Delta_n \Delta_m \rangle = 0$ except when $n = m$

$$\therefore C^2 V = \sum_{n=1}^{N} \langle \Delta_n^2 \rangle$$

$$= \sum_{n=1}^{N} \langle [(u_n z_n)^2 - Z_n]^2 \rangle$$

$$= \sum_{n=1}^{N} \langle (u_n z_n)^4 - 2(u_n z_n)^2 Z_n + Z_n^2 \rangle$$

Remembering that $\langle (u_n z_n)^2 \rangle = Z_n$

$$c^2 V = \sum_{n=1}^{N} \left[\langle (u_n z_n)^4 \rangle - \langle (u_n z_n)^2 \rangle^2 \right]$$

Remembering that u_n is constant for an ensemble of samples and that $\langle z_n^2 \rangle$ and $\langle z_n^4 \rangle$ are independent of n

$$c^2 V = \sum_{n=1}^{N} \left[u_n^4 \langle z^4 \rangle - u_n^4 \langle z^2 \rangle^2 \right]$$

$$= \left[\langle z^4 \rangle - E^2 \right] \sum_{n=1}^{N} u_n^4$$

The proportional variance of the sample error is V/E^2, so, substituting back for c

$$\frac{V}{E^2} = \frac{\langle z^4 \rangle - E^2}{E^2} \cdot \frac{\sum_{n=1}^{N} u_n^4}{\left(\sum_{n=1}^{N} u_n^2 \right)^2}$$

If $$\overline{u_n^4} = \frac{1}{N} \sum_{n=1}^{N} u_n^4$$

and $$\overline{u_n^2} = \frac{1}{N} \sum_{n=1}^{N} u_n^2$$

Then $$\frac{V}{E^2} = \frac{1}{N} \cdot \frac{\langle z^4 \rangle - E^2}{E^2} \cdot \frac{\overline{u_n^4}}{\overline{u_n^2}^{\,2}}$$

This reveals the three factors governing the sampling error.

1. The number of samples N. In a continuous record, this becomes the effective number of independent samples, the precise definition of which need not concern us here. It is enough to note that N is clearly proportional to the length of the record.

2. $\dfrac{\langle \zeta^4 \rangle - E^2}{E^2}$ is a function of the shape of the probability distribution of the wave elevations ζ. Waves approximate to a random gaussian process for which the distribution is Normal (or "Gaussian") and this factor = 2.

3. $\dfrac{\overline{u_n^4}}{\overline{u_n^2}^{\,2}}$ is dependent only on the shape of the window function. If u_n is constant, then this = 1

In terms of a continuous window function $F(t)$ such as we have been considering in section 2.3.4, this factor becomes

$$\overline{F^4(t)} \Big/ \overline{F^2(t)}^{\,2}$$

The standard error is defined as the square root of the proportional variance, and is therefore proportional to

$$\overline{F^4(t)}^{\,1/2} \Big/ \overline{F^2(t)} \qquad 2.4/4$$