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C O N T E N T S

Section 1 - INTRODUCTION

(a) General

It is the intention of this report to calculate the magnitude of the echo received from a fish bladder at a known depth when subjected to a sound wave of known character. The sound wave is the output of the N.I.O. modification to the Edgerton "Boomer" hereafter called the Thumper. For the purposes of the report the bladder is treated as a free spherical bubble.

(b) Method

A mathematical approximation to the Thumper output wave is constructed in a form suitable for analysis by Fourier methods or the Laplace transform. Thus the energy associated with any part of the spectrum crossing unit area parallel to the wavefront at the target can be calculated.

Next the acoustic cross-section of the target is examined as a function of frequency, and using the expression so found the total amount of energy absorbed by the target from the pulse may be found.

This energy is assumed to be re-radiated uniformly in all directions as a damped train of oscillations at the resonant frequency of the target. To find the initial amplitude of this wave the total energy output is calculated and this is equated to the energy absorbed.

Section 2

(a) The Spectrum of the Thumper Pulse

Fig. 1 shows the pressure-time curve of the Thumper pulse. The curve was traced from a photograph of the output of a non-directional hydrophone, the response of which fell off by 6 dB per octave below 200 c/s.

The equation of the acoustic pressure can be written

$$P = P_o\, f(t) \tag{1}$$

where P = acoustic pressure at 1 metre from source at time t. P_o = Peak acoustic pressure at 1 metre and f(t) gives the time variation of P.

The total energy in a small band of angular frequency Δw centred on w which passes through unit area (1 sq.cm.) at 1 metre is

$$\Delta E = \frac{P_o^2}{2\rho c}\, |F(w)|^2 \Delta w \tag{2}$$

where ρc = specific acoustic impedance of the medium and

$$F(w) = \int_{-\infty}^{\infty} e^{-jwt} f(t)\,dt \tag{3}$$

which is the Fourier transform of f(t).

The total energy in the pulse passing through unit area at 1 metre is

$$E = \frac{P_o^2}{2\rho c} \int_{-\infty}^{\infty} |F(w)|^2\, W(w)\, dw \tag{4}$$

and the energy absorbed by a target, whose cross-section W is a function of w, is

$$E_W = \frac{P_o^2}{2\rho c} \int_{-\infty}^{\infty} |F(w)|^2\, W(w)\, dw \tag{5}$$

As a first model of the pulse let

$$\left.\begin{aligned} f(t) &= \cos kt \quad \text{for } {}^{-\pi}/2k < t < {}^{\pi}/2k \\ &= 0 \qquad \text{elsewhere} \end{aligned}\right\} \tag{6}$$

Then

$$F(w) = \int_{-\pi/2k}^{\pi/2k} \cos kt \; e^{-jwt}\, dt$$

$$= \frac{k}{k^2 - w^2}\left(e^{-j\frac{w\pi}{2k}} + e^{j\frac{w\pi}{2k}}\right)$$

$$F(w) = \frac{2k}{k^2 - w^2}\cos\left(\frac{w\pi}{2k}\right) \tag{7}$$

A plot of $F(w)$ (which is in this case real) for values of w from 0 to 2·5k is attached (Fig. 2).

Inspection of the actual pressure-time curve (Fig. 1) shows that this is a poor approximation, and a more realistic model would be

$$\left.\begin{aligned} f(t) &= {}^{t}/t_1 && 0 < t < t_1 \\ &= \frac{2t_1 - t}{t_1} && t_1 < t < 3t_1 \\ &= \frac{t - t_2}{t_2 - 3t_1} && 3t_1 < t < t_2 \\ &= 0 && \text{elsewhere} \end{aligned}\right\} \tag{8}$$

Here it is easier to use Laplace transforms to find $F(w)$. The Laplace transform $F(s)$ of $f(t)$ is found, jw is substituted for s and $|F(w)|^2$ is then the product of $F(jw)$ and its complex conjugate $F(-jw)$. The rules for finding Laplace transforms applicable to this case are as follows:

(1) If $f(t) = t \quad F(s) = \int_0^\infty e^{-st} f(t)\, dt = \frac{1}{s^2}$.

(2) If $f(t)$ is defined only in the interval $t_1 < t < t_2$ then

$Lf(t) = F(s)\,(e^{-st_1} - e^{-st_2})$.

(3) If the transform of $f(t)$ is $F(s)$, that of $f(t-b)$ is $F(s)\, e^{-bs}$.

(4) The transform of a sum is the sum of the separate transforms. Further, t is restricted to values greater than zero. The transform of $f(t)$ can now be written out term by term as

$$\left.\begin{aligned} F(s) = \frac{1}{t_1}\cdot\frac{1}{s^2}\,(1 - e^{-st_1}) \qquad & \text{1st segment} \\ - \frac{(e^{-st_1} - e^{-3st_1})}{t_1\, s^2}\, e^{-2st_1} \qquad & \text{2nd segment} \\ + \frac{1}{s^2}\cdot\frac{1}{t_2 - 3t_1}\,(e^{-3st_1} - e^{-st_2})\, e^{-st_2} \qquad & \text{3rd segment} \end{aligned}\right\} \tag{9}$$

Putting jw for s and $t_2 = nt_1$ where $n > 3$ this becomes

$$F(jw) = \frac{-2j}{w^2 t_1}\left[e^{-j\frac{wt_1}{2}}\sin\left(\frac{wt_1}{2}\right) - e^{-4jwt_1}\sin(wt_1) + e^{\frac{-3(n+1)}{2}jwt_1}\sin\left[\left(\frac{n-3}{2}\right)wt_1\right]\right] \tag{10}$$

The computation is much simplified while retaining a fair approximation by putting $n = 4$, which makes the pulse a symmetrical sawtooth. $F(w)$ then takes the real form

$$F(w) = \frac{2t_1}{(wt_1)^2}\left[\sin wt_1 - 2 \sin \frac{wt_1}{2} \cos \frac{7wt_1}{2}\right] \tag{11}$$

and a plot of this function is attached (Fig. 3).

(b) The Target

This is a fish-bladder, which for the purposes of this report is treated as a free spherical bubble. Much work remains to be done on bubbles and the effects of shape and surrounding tissue, but some figures are available from "Principles and Applications of Underwater Sound" NRDC Div 6 Summary Technical Report (1946) pp 84-86.

Here it is seen that the acoustic cross-section of a spherical bubble is appreciable only in the region of its resonant frequency, and the ratio of the acoustic cross-section σ to the actual projected area $\frac{1}{4}\pi d^2$ at this frequency is about 700. Also the Q of the resonance is about 12-15. Actually σ is approximately $\pi^2 d^2 Q^2$. The resonant frequency $f_0 = \frac{.04P^{\frac{1}{2}}}{d}$ where f_0 is in Kc/s, d is the diameter in inches, and P the pressure in feet of water.

(c) The Energy Absorbed by the Target

Strictly speaking this is given by equation (5) where $W = \sigma$, but in view of the strongly resonant nature of the target equation (2) can be used with little loss of accuracy.

Thus $$E_T = \frac{p_0^2}{2\rho c} |F(w)|^2 \sigma \Delta w \tag{12}$$

where p_0 = peak pressure at target = P_0/r (r = depth in metres)
E_T = energy absorbed by target.

(d) The Echo

E_T as found in 2(c) is re-radiated uniformly in all directions as a damped train of oscillations described by

$$p_e = p_e(o)\, e^{mt} \cos (wt + \phi) \tag{13}$$

where p_e = echo amplitude at hydrophone
$p_e(o)$ = initial amplitude at hydrophone.

Now the rate of energy transfer through unit area by an acoustic disturbance of pressure p is $p^2/\rho c$ ergs/sec, so the total rate of transfer through a sphere of radius r, centred on the source is

$$\frac{\partial E}{\partial t} = \frac{4\pi r^2}{\rho c} p_e^2$$

Thus the total energy is

$$E = \int_0^\infty \frac{4\pi r^2}{\rho c} p_e^2 \, dt \tag{14}$$

which with equation (13) becomes

$$E = \frac{4\pi r^2}{\rho c} p_e^2(o) \int_0^\infty e^{2mt} \cos^2(wt + \phi)\, dt \tag{15}$$

ϕ is the starting phase of the wave at the radius r; and as the Q is large, ϕ has negligible effect on the total energy. Accordingly integration of (15) is carried out with $\phi = o$ to give

$$E = \frac{4\pi r^2}{\rho c} |p_e(o)|^2 \times \frac{-1}{4m}\left[1 + \frac{m^2}{m^2 + w^2}\right] \tag{16}$$

For exponential damping $m = -\frac{f_o \pi}{Q}$ and $w = 2\pi f_o$

thus $\frac{m^2}{m^2 + w^2} = \frac{1}{Q^2[4 + \frac{1}{Q^2}]}$ and if $Q = 15$ $\frac{m^2}{m^2 + w^2} \approx 1/900 \ll 1$

This shows that ϕ has in fact a negligible effect on E, since if $\phi = \pi/2$, $\cos(wt + \phi) = \sin wt$, and equation (16) would be unchanged except for the term in brackets which would be $\left[1 - \frac{m^2}{m^2 + w^2}\right]$. Thus the integral $\int_0^\infty e^{2mt} \cos^2(wt + \phi)dt$ cannot lie outside the limits $\frac{-1}{4m}\left[1 \pm \frac{m^2}{m^2 + w^2}\right]$

so that the energy E is

$$E = -\frac{\pi r^2}{\rho c} |p_e(o)|^2 \cdot \frac{1}{m} \quad \text{with } m = -\frac{f_o \pi}{Q} \tag{17}$$

Having assumed that the energy absorbed is re-radiated without loss the value of p (o) can be obtained by equating the right hand sides of equations (12) and (17).

$$|p_e(o)|^2 \frac{r^2}{\rho c} \frac{Q}{f_o} = \frac{p_o^2}{2\rho c} |F(w)|^2 \sigma \, \Delta w \tag{18}$$

where the symbols have all been previously defined. And solved for $p_e(o)$, this reads

$$p_e(o) = \frac{p_o}{2r} |F(w)| \frac{w}{Q} \sqrt{\frac{\sigma}{\pi}} \tag{19}$$

Section 3

(a) Typical Values

As an illustration of the echo likely to be received the following figures would be typical. A fish of 2-3 lb weight would have an air bladder about 4·5 cms in diameter. At a depth of 100 metres (about 50 fm) this would resonate at 500 c/s. Thus

$r = 10^4$ cms, $d = 4{\cdot}5$ cms, $f_o = 500$, $w_o = 2\pi \times 500$

$p_o = \frac{P_o}{r \times 10^{-2}}$, where the best estimate of P_o is $2{\cdot}5 \times 10^5$ dynes/cm² at 1 metre.

$t_1 = 0{\cdot}3 \times 10^{-3}$ from Fig. 1 and

$|F(w)| = 2 \times 2t_1 = 1{\cdot}2 \times 10^{-3}$ using Fig. 3

$Q = 12$ and $\sigma/\frac{1}{4}\pi d^2 = 700$ $\therefore \sigma = \frac{700\pi\, d^2}{4}$

$$\text{Thus } p_e(o) = \frac{2{\cdot}5 \times 10^5 \times 1{\cdot}2 \times 10^{-3}}{2 \times 10^4 \times 10^2} \times \frac{10^3\pi}{12} \times \sqrt{\frac{700\pi \times 4{\cdot}5^2}{4\pi}}$$

$$= \frac{1{\cdot}5\pi}{12} \times 10^{-1} \times \frac{4{\cdot}5}{4} \times 10\sqrt{7}$$

$$\underline{p_e(o) = 1{\cdot}2 \text{ dynes/cm}^2} \tag{20}$$

(b) Discussion

Equation (20) shows that an appreciable echo is returned from a typical fish bladder provided that it behaves as a free spherical bubble.
1·2 dynes/cm² is roughly the RMS noise to be expected in Thumper work due to sea noise, ship noise and towing noise of the hydrophone and transducer. But owing to the large number of cycles in the echo it should be easily detectable. However, the figure of 1·2 dynes/cm² may be an optimistic

estimate for the following reasons:

(a) The peak pressure at 1 metre may be less than $2{\cdot}5 \times 10^5$ dynes/cm^2.

(b) The effective cross-section of a bubble is not well known at frequencies below 1 or 2 Kc/s.

(c) The body of the fish may reduce the efficiency of the absorption and re-radiation processes quite appreciably. And these are in any case unlikely to be perfectly efficient even for a bubble. Experimental work remains to be done on these last two points.

It is interesting to see how $p_e(o)$ can be expected to vary with the parameters. Equation (19) can be re-written

$$p_e(o) \propto p_o |F(w)| \frac{1}{r} 2\pi f_o \frac{1}{Q} \pi^{3/2} d\ Q \qquad (21)$$

since $\sigma = \pi^2 d^2 Q^2$

But $f_o \propto \frac{P^{\frac{1}{2}}}{d}$ $\qquad P$ = static pressure $\propto r$

Hence

$$p_e(o) \propto p_o |F(w)| \frac{1}{r} \frac{r^{\frac{1}{2}}}{d} \frac{1}{Q} d\ Q$$

$$\propto \frac{p_o |F(w)|}{r^{\frac{1}{2}}} = \frac{p_o}{r^{3/2}} |F(w)|$$

This shows that provided $|F(w)|$ varies little in the frequency range of interest (as indeed it does), then $p_e(o)$ is not critically dependent on frequency. Furthermore it is independent of Q, though the detectability of the echo is not, if Q is less than about 6-10.

(c) <u>Conclusion</u>

It seems possible that resonant reflection of sound impulses by fish bladders may be detectable in a wide band system, if several fish occur in a group, and reverberation and noise can be kept within fairly low limits. However, much quantitative work remains to be done.

APPENDIX

Extension of the Complex Fourier Series to the Fourier Transform

This is intended to bring out the physical significance of the amplitude transform of a single pulse by extending the more familiar complex Fourier series of a periodic wave form. The technique is to express a low duty cycle pulse train as a complex Fourier series, and examine this series as the duty cycle is reduced to a single pulse.

Let $f(t)$ be the amplitude of a pulse (e.g. a sound wave) defined during the interval $-T/2 < t < T/2$ and let $f(t) = f(t + T)$. $f(t)$ may be zero during most of the interval defined above.

$f(t)$ may be written as the complex Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jkw_0 t} \qquad (A1)$$

where the coefficients are given by

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jkw_0 t}dt \text{ and } w_0 = \frac{2\pi}{T} \qquad (A2)$$

In the interval $-T/2 < t < T/2$ the amplitude of the component of frequency nw_0 is C_n, and the power associated with this wave is $\frac{1}{2}[C_n C_{-n}]$. Thus the total energy of this harmonic in the interval T is

$$E_{nw_0} = T/2\,[C_n C_{-n}] \text{ where } C_n,\ C_{-n} \text{ are complex conjugates.}$$

Thus the energy density (in the frequency plane) in the bandwidth $(f_{n+1} - f_n)$ is E_{nw_0} divided by the frequency separation of the harmonics

$$J_{nw_0} = \frac{E_{nw_0} \times 2\pi}{(n+1)w_0 - nw_0} = \frac{2\pi T}{2w_0}\left[C_n C_{-n}\right] = \frac{2\pi T^2}{4\pi}\left[C_n C_{-n}\right]$$

$$= \frac{1}{2}\left\{ \int_{-T/2}^{T/2} f(t)\, e^{-j\frac{n\pi t}{T}}\, dt \int_{-T/2}^{T/2} f(t)\, e^{j\frac{n\pi t}{T}}\, dt \right\} \qquad (A3)$$

It is the behaviour of this expression as $T \to \infty$ which is of interest. But $2\pi/T$ hence as $T \to \infty$ $w_0 \to 0$ and unless $k \to \infty$ in such a way that kw_0 remains finite equation (A3) merely gives the square of the area under the curve $f(t)$. So let $kw_0 = w$ so that (A3) becomes

$$J_{nw_0} = \lim_{T\to\infty} J(w) = \lim_{T\to\infty} \frac{1}{2}\left\{ \int_{-T/2}^{T/2} f(t)\, e^{-jwt}dt \int_{-T/2}^{T/2} f(t)\, e^{jwt}dt \right\} \qquad (A4)$$

As $f(t) = 0$ outside the limits $\pm T_0$, where $2T_0$ is the actual length of the pulse and is finite, the right hand side of (A4) is independent of the value of T provided that $T > 2T_0$. This leads to the equation

$$J(w) = \frac{1}{2}\left\{ \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt \int_{-\infty}^{\infty} f(t)\, e^{jwt}\, dt \right\} \qquad (A5)$$

But $\int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$ is just the Fourier transform $F(jw)$ of $f(t)$, and $\int_{-\infty}^{\infty} f(t)\, e^{jwt}\, dt$ is its complex conjugate $F(-jw)$.

$$\text{Thus } \Delta E(w) = J(w)\,\Delta w = \tfrac{1}{2}|F(w)|^2 \Delta w \qquad (A6)$$

and this is the expression used in the text.