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THE RING MODULATOR AS A DETECTOR OF CORRELATIONS

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National Institute of Oceanography, Wormley, Nr. Godalming, Surrey

Appendix to N.I.O. Internal Report No. A 1

"The Ring Modulator as a Detector of Correlations"

Discussion of the output noise levels when a low-pass filter is connected in the output

On re-reading this report the authors realise that the manner in which the performances of a true multiplier and of a ring modulator are compared may be misleading.

The noise levels used for the calculation of the comparative signal/noise ratios of a true multiplier and of a ring modulator (plotted in figure 6) are the unsmoothed noise levels: that is, with no filter whatsoever on the output of the circuit. It must not be assumed that the effect of a smoothing circuit connected in series with the output will reduce the noise by a similar proportion in both circuits. The authors have not been able to calculate the effect of smoothing the output of a ring modulator and this is why the comparison had to use the unsmoothed noise levels.

In the case of a true multiplier, if the mean square voltage contained in a bandwidth df at a frequency f is $E_x(f)\,df$ for the x signal $E_y(f)\,df$ for the y signal, the mean square noise output voltage V_s^2 which is obtained when the output is passed through a low-pass filter with a narrow bandwidth Δf_2 is given by

$$V_s^2 = \tfrac{1}{2} K^2 \Delta f_2 \int_0^\infty E_x(f)\, E_y(f)\, df$$

where K is the multiplier constant.

$E_x(f)$, $E_y(f)$, etc. will be called energy density functions.

This result is correct only if the bandwidths of $E_x(f)$ and $E_y(f)$ are much greater than Δf_2, since it assumes that the energy density of the output fluctuations is constant within the pass band of the low pass filter.

If $E_x(f)$ and $E_y(f)$ are similar "square" frequency bands of width Δf_1, the mean square voltage V_F^2 of the unsmoothed output voltage may be calculated by integrating the complete output energy spectrum. Combining this with the formula above gives:

$$V_S^2 / V_F^2 = \Delta f_2 / \Delta f_1$$

which is the result stated on page 3 of the report.

If the smoothing bandwidth Δf_2 is comparable with the signal bandwidths, the output noise may be calculated from the following formula for the energy density of the unsmoothed output fluctuations at a frequency p

$$E_V(p) = (K^2/4) \int_0^\infty [E_x(f)E_y(f+p) + E_x(f+p)E_y(f)]\,df$$

This equation takes account only of difference frequencies, so that there is still a limitation that Δf_2 must be less than the sum of the lowest frequencies present in x and y.

These formulae are correct whatever the correlation between x and y.

The fluctuations are measured about the mean D.C. output due to the correlated component.

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Distribution

The same as N.I.O. report A 1

The Ring Modulator as a Detector of Correlations

SUMMARY

The ring modulator has been used to detect correlations between two voltages. Its characteristics are an approximation to those of a true multiplier. In this paper its action is rigourously examined for inputs with Gaussian voltage distributions, that is, with voltages similar to that of random noise. It is shown that for small correlations the output "signal/noise" ratio is only 7% (0.6 db) down on that of a true multiplier over a certain range of working conditions, whilst for large correlations the signal/noise ratio is better in the ring modulator than the true multiplier.

Introduction

A ring modulator used symmetrically with input voltages of comparable magnitude (fig. 1) has been used to detect correlations between two radio signals received in a radio-telescope (ref. 1). The detection and measurement of correlations also has several important applications in Oceanographic research (ref. 2) and it is possible that the ring modulator might be useful for this purpose.

The ideal correlation detector is a device which multiplies the two inputs. If the multiplier has a characteristic equation $v = kxy$, then $\bar{v} = krxy$. (All definitions of symbols are listed on one page at the end of the report). No electronic multipliers which are both satisfactory and simple have so far been developed and the ring modulator offers a simple alternative that is satisfactory in some applications. Simple multipliers have large zero fluctuations whilst the output of a ring modulator is very stable compared with the magnitude of the input voltages that it will handle. The output drift can be kept considerably below 0.1% of the peak voltages. In order that the ring modulator may be used in this way it is necessary to know how to determine the correlation coefficient of the inputs from the mean output and also to know whether the ratio of the output due to correlated components to that due to random correlations in the input is as good as that in a true multiplier. This ratio will be called the output signal to noise ratio.

The action of the modulator depends upon the statistical distribution of voltages in the inputs and it will be assumed that this is of "normal" or Gaussian form, which is one of the most common met with in practice.

The Modulator Characteristics

The circuit is shown in its conventional form in fig. 1(a) and has been rearranged to make the analysis easier in fig. 1(b). The resistors in series with the rectifiers are necessary in the present application in order to make the modulator present a reasonably high impedance to the signal sources. It will be assumed that the rectifiers are perfect; that is, they act as switches which have zero resistance when a current is flowing through them in one direction, and an infinite resistance when there is a voltage across them in the reverse direction. It will also be assumed that the signal sources have zero internal impedance and that the output is connected to an infinite impedance.

There are eight possible states of conduction of the modulator but for convenience in subsequent calculation four of these have been split and twelve states are therefore shown in table 1.

It is interesting to compare the characteristics of the ring modulator with those of a true multiplier (fig. 2). It will be seen that the former may be regarded as a straight line approximation to the

/ latter

latter. An important difference is the spacing of the intersections of the characteristics with the line $x = y$. In the true multiplier the distance from the origin increases according to a square law, whereas, in the modulator the characteristics are equally spaced. This means that the modulator approximates most nearly to a multiplier with a characteristic equation $v = k\,xy/|\sqrt{xy}\,|$. It will be seen later that this fact emerges from the analysis.

The Detection of Correlations

Only an outline of the mathematics will be given in this section; certain calculations will be given in more detail in appendix 2. The analysis of the action of a true multiplier is not new, but is included for purposes of comparison. The results are presented in four stages in which the following are calculated.

(a) The mean d.c. output from a true multiplier with partly correlated inputs.
(b) The r.m.s. value of the fluctuations in the output voltage of a true multiplier with partly correlated inputs.
(c) The mean d.c. output and r.m.s. value of the fluctuations in the output voltage of a ring modulator with partially correlated inputs.
(d) Comparison of the signal to noise ratio of a ring modulator with that of a true multiplier.

For convenience in calculation it will be assumed that the voltage y is produced by adding a proportion c of the voltage x to another voltage z which is uncorrelated with x, that is, $y = cx + z$. This type of correlation is indistinguishable from correlations produced in any other way (appendix 1) and there is therefore no loss of generality in making this assumption.

The results obtained in this way will be in terms of c, X, Z but these may be expressed in terms of r, X, Y (X, Y can both be measured) for, by definition:

$$r = \frac{\overline{xy}}{XY} = \frac{\overline{x\,(cx + z)}}{XY} = \frac{cX}{Y} \qquad \text{(x and z being uncorrelated).}$$

also $y = cx + z.$

and $y^2 = c^2x^2 + 2cxz + z^2$

$\therefore\ Y^2 = c^2X^2 + Z^2 = r^2Y^2 + Z^2$

$$\therefore\ \left.\begin{aligned} c &= \frac{rY}{X} \\ Z^2 &= Y^2(1 - r^2) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

(a) The mean d.c. output from a true multiplier

$$v = kxy$$

Therefore it follows from the definition of r that:

$$\overline{v} = k\overline{xy} = krXY \approx krp \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2)$$

(b) The fluctuations in the output of a true multiplier due to partially correlated inputs

In what follows the factor k will be assumed to be unity. As it is only a scale parameter it may be re-introduced as necessary at any stage.

/ The probability

The probability distribution P(v) dv is defined as the probability of the output being between v and v + dv. The range of x and z corresponding to this range of output is shown by the shaded strips S_1, S_2 in fig. 3. The strips are bounded by the hyperbolas $v = (cx + z)x$ and $v + dv = (cx + z)x$. The joint probability of x, z falling in a small area dx dz is given by

$$P_X(x)\, P_Z(z)\, dxdz$$

$$\therefore\ P(v)dv = \int_{S_1 + S_2} P_X(x)\, P_Z(z)\, dxdz$$

On specifying the distributions P_X, P_Z the integral can be evaluated. P(v) then gives all the information that can be obtained about the output. In this article all random variables will be assumed normally distributed.

The derivation of P(v) is given in appendix 2. It is found to be

$$P(v) = \frac{e^{\frac{rv}{XY(1 - r^2)}}}{\pi XY\sqrt{1 - r^2}} K_o\left[\frac{|v|}{XY(1 - r^2)}\right] \quad \ldots\ldots\ (3)$$

where $K_o(x)$ is a modified Bessel Function of the second kind (ref.3)

The mean output is

$$\bar{v} = \int_{-\infty}^{+\infty} vP(v)dv$$

and this can be verified to be krXY in agreement with (a)

The mean square output about zero is

$$V_o^2 = \int_{-\infty}^{\infty} v^2P(v)dv$$

$$= k^2(1 + 2r^2)X^2Y^2$$

The mean square V_o^2 includes a contribution from the d.c. output $\bar{v}$ so the mean square of the fluctuations about the mean $\bar{v}$ is

$$V_F^2 = \int_{-\infty}^{\infty} (v - \bar{v})^2 P(v)$$

$$= V_o^2 - \bar{v}^2$$

$$= k^2(1 + r^2)X^2Y^2 = k^2(1 + r^2)\, p^2 \quad \ldots\ldots\ldots\ (4)$$

The effect of smoothing on the fluctuations about the mean can be shown to reduce V_F^2 by the ratio of the signal bandwidth Δf_1 to the smoothing circuit bandwidth Δf_2 so that the actual output fluctuations have a mean square given by $V_S^2 = V_F^2 \frac{\Delta f_2}{\Delta f_1}$. The smoothing also has the

/ effect

effect of centralising the fluctuations about the mean. In fact the greater the smoothing the more nearly are the fluctuations represented by a normal error curve about the mean.

(c) The mean output and r.m.s. voltage of the fluctuations about the mean in a ring modulator with partially correlated inputs

As in the case of the true multiplier the first aim is to derive the probability distribution function for the output.

It has already been explained that the ring modulator has twelve different states and as these have different characteristics the probability of the modulator being in any given state is evaluated separately. The total probability $P(v)dv$ of the output lying between v and v + dv will then be the sum of the probabilities $P_i(v)\,dv$ for the different states as these probabilities are exhaustive and exclusive. Whereas, in the true multiplier $P(v)dv$ was an integral over a strip between two hyperbolas, in the ring modulator the integration is over strips bounded by straight lines approximating to hyperbolas (fig. 4). The expression found for P(v) is:

$$P(v) = \sqrt{\frac{2}{\pi}}\left\{\frac{e^{-\frac{2v^2}{X^2}}}{X}\,\text{erfc}\,\frac{\sqrt{2}(2X - rY)v}{XY\sqrt{1-r^2}} + \frac{e^{-\frac{2v^2}{Y^2}}}{Y}\,\text{erfc}\,\frac{\sqrt{2}(2Y - rX)v}{XY\sqrt{1-r^2}}\right.$$

$$+ \frac{3e^{-\frac{18v^2}{X^2+Y^2+2rXY}}}{(X^2+Y^2+2rXY)^{\frac{1}{2}}}\left[\text{erf}\,\frac{\sqrt{2}\,(2Y^2 - X^2 + rXY)v}{XY\,(X^2+Y^2+2rXY)^{\frac{1}{2}}\sqrt{1-r^2}}\right. \quad \ldots\ldots \quad (5)$$

$$\left.\left. + \text{erf}\,\frac{\sqrt{2}\,(2X^2 - Y^2 + rXY)v}{XY\,(X^2+Y^2+2rXY)^{\frac{1}{2}}\sqrt{1-r^2}}\right]\right\}$$

for $v > 0$

For $v < 0$ the expression is the same except that v is replaced by $|v|$ and r by $-r$. erf x and erfc x are the error function and complimentary error function respectively.

The mean value is given in terms of the parameters by:

$$\bar{v} = \int_{-\infty}^{\infty} vP(v)dv = \frac{1}{6}\sqrt{\frac{p}{2\pi}}\left\{\left|4q + \frac{1}{q} + 4r\right|^{\frac{1}{2}} + \left|q + \frac{4}{q} + 4r\right|^{\frac{1}{2}} - \left|4q + \frac{1}{q} - 4r\right|^{\frac{1}{2}} - \left|q + \frac{4}{q} - 4r\right|^{\frac{1}{2}}\right\} \quad (6)$$

In a number of applications the correlation will be small and an approximation may be used:

$$\bar{v} = \frac{2}{3}\sqrt{\frac{p}{2\pi}}\; r\left\{\left(4q + \frac{1}{q}\right)^{-\frac{1}{2}} + \left(q + \frac{4}{q}\right)^{-\frac{1}{2}}\right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$\therefore\; r = \frac{\bar{v}}{\sqrt{p}}\left[3\sqrt{\frac{\pi}{2}}\;\frac{1}{\left\{\left(4q + \frac{1}{q}\right)^{-\frac{1}{2}} + \left(q + \frac{4}{q}\right)^{-\frac{1}{2}}\right\}}\right] \quad \ldots\ldots\ldots\ldots \quad (8)$$

/ The

The function in square brackets is plotted in fig. 5 so the correlation coefficient can easily be found by multiplication of the function by $\bar{v}/\sqrt{p}$. The general expression for r (eqn. (6)) can be put in the form

$$r = \frac{\bar{v}}{\sqrt{p}}\left[6\sqrt{2\pi}\,\frac{1}{\left\{\left|4q+\frac{1}{q}+4r\right|^{\frac{1}{2}}+\left|q+\frac{4}{q}+4r\right|^{\frac{1}{2}}-\left|4q+\frac{1}{q}-4r\right|^{\frac{1}{2}}-\left|q+\frac{4}{q}-4r\right|^{\frac{1}{2}}\right\}}\right] \quad (9)$$

The expression in square brackets gives rise to a series of curves for different values of r which are also shown in fig. 5. An approximate value of r may be assumed in order to locate the right curve to use to find a more accurate value of r.

In all the graphs presented in this report q is only given in the range 0 to 1. As q is merely the ratio of X to Y it is only necessary to call the smallest input X and the largest Y.

The mean square output about zero is:

$$V_o^2 = \int_{-\infty}^{\infty} v^2 P(v)\,dv.$$

$$= \frac{p}{2\pi}\left\{4\pi\cdot\frac{q+\frac{1}{q}}{9} - \frac{1}{2}\left(q - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\frac{4q}{(1-4q^2)}\right.$$

$$\left. -\frac{1}{2}\left(q - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\left(\frac{\frac{4}{q}}{q^2-4}\right) - \frac{2}{3}\right\} \quad \ldots\ldots \quad (10)$$

$$(0 < \tan^{-1} < \pi)$$

(r small).

For r small V_o^2 is approximately the same as V_F^2 so (10) may be used when considering the fluctuations about the mean. The general expression for V_o^2 is given in the appendix.

When $\frac{X}{Y}$ or q is $\gg 1$, $V_o^2 = \frac{p}{4q} = \frac{Y^2}{4}$. This expresses the fact that when the modulator is well out of balance one input merely acts as a switch for the other.

(d) Comparison of Signal to Noise Ratio in a True Multiplier and Ring Modulator

By signal to noise ratio is meant the ratio of $\bar{v}$ to V_F.

From 2, 4.

$$G_{r.m.} = \frac{r}{\sqrt{1+r^2}} \simeq r \text{ for small r.} \quad \ldots\ldots\ldots \quad (11)$$

and from 7, 10

$$G_{t.m.} = \frac{\frac{2r}{3}\left[\left(4q+\frac{1}{q}\right)^{-\frac{1}{2}} + \left(q+\frac{4}{q}\right)^{-\frac{1}{2}}\right]}{\left\{\frac{4\pi\left(q+\frac{1}{q}\right)}{9} - \frac{1}{2}\left(q - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\frac{4q}{1-4q^2} - \frac{1}{2}\left(q - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\left(\frac{\frac{4}{q}}{q^2-4}\right) - \frac{2}{3}\right\}^{\frac{1}{2}}} \quad \text{for small r.} \quad (12)$$

/ In fig. 6.

In fig. 6 the ratio of $G_{r.m.}$ is given in terms of the % and d.b. loss of the ring modulator compared with the true multiplier. It can be seen that, for r small, its loss when the inputs are balanced is only 0.6 db (7%). It is clear also that optimum conditions for detection occur when the inputs are approximately balanced. The value of $G_{r.m.}$ may be obtained from these curves by multiplication of the given ratio by $\frac{r}{\sqrt{1 + r^2}}$.

References.

1. Ryle M. Proc. Roy. Soc. A. Vol.211 P.351 (1952).

2. Tucker M.J. Proc. Roy. Soc. A. Vol. 202 P.565 (1950).

3. Gray & Mathews. Treatise on Bassel Functions. Macmillan 2nd ed.

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Symbols

x) y)	Instantaneous input voltages.
v	Instantaneous output voltage.
z	Instantaneous component of y that is uncorrelated with x.
X, Y, Z.	Root Mean Square Values of x, y, z.
V_o	R.M.S. value of v about zero.
V_F	R.M.S. value of v about the mean d.c. output before smoothing.
V_S	R.M.S. value of v about the mean d.c. output after smoothing.
$\bar{x}$	The bar over a symbol represents a time mean over a long period.
c	The proportion of x that is added to z to form y, that is, $y = cx + z$.
$p = XY$) $q = X/Y$)	Parameters specifying the performance of the modulator.
r	Correlation coefficient between x, y.
k	Scale constant in true multiplier.
$P_X(x)$, $P_Z(z)$	Probability distributions of x and z.
$P(v)$	Probability distribution of the output v.
$P_i(v)$	Probability distribution of v for the ring modulator in state i.
$G_{t.m.}$, $G_{r.m.}$	The output signal to noise ratio in a true multiplier and a ring modulator.
Δf_1, Δf_2.	Bandwidth of input signals (assumed equal) and of smoothing filter in output.

Appendix 1.

On determining the process producing a correlation between two signals

The problem is: given two time dependent signals which are partially correlated, is it possible to distinguish between the various possible ways in which the correlation could have been produced, assuming that no further information is available?

In practice the problem may be simplified to distinguishing between direct correlation in which the signals are:

$$x(t) = \alpha(t)$$
$$y(t) = c\alpha(t) + \beta(t)$$

and partial correlation in which the signals are:

$$x(t) = \gamma(t) + a\,\varepsilon(t).$$
$$y(t) = \delta(t) + b\,\varepsilon(t).$$

It is easiest to consider the signals as a series of discrete readings $x_1 \ldots\ldots x_n$ and $y_1 \ldots\ldots y_n$. Arguments applying to such series can clearly be generalized to apply to continuous signals.

In the first type of correlation this gives a series of pairs of simultaneous equations:-

$$x_1 = \alpha_1 \qquad y_1 = \beta_1 + c\,\alpha_1$$
$$x_2 = \alpha_2 \qquad y_2 = \beta_2 + c\,\alpha_2$$

etc.

Now c is unknown and at first sight there is insufficient information to solve the equations easily but if it is stipulated that $\alpha(t), \beta(t)$ are uncorrelated then there is an extra condition that $\frac{1}{N}\sum_1^N \alpha_n \beta_n \longrightarrow 0$ as $N \longrightarrow \infty$ which makes the equations unique.

In the second type of correlation there is a series of equations:

$$x_1 = \gamma_1 + a\,\varepsilon_1 \qquad y_1 = \delta_1 + b\,\varepsilon_1$$
$$x_2 = \gamma_2 + a\,\varepsilon_2 \qquad y_2 = \delta_2 + b\,\varepsilon_2$$

etc.

These equations are indeterminant even knowing that $\gamma, \delta, \varepsilon$ are uncorrelated. There is therefore nothing inconsistent with the known facts in putting $a\,\varepsilon(t) = x(t)$, in which case the equations are similar to the first.

This problem is a major difficulty in statistics. For example, because there is a strong correlation between the consumption of chocolate and the flow of water under Tower Bridge, it does not follow that a proportion of the population watch the Thames and decide to eat more chocolate when it is flowing fast. In fact, of course, the correlation is produced because both are correlated with the meridian altitude of the sun.

Appendix 2

The Probability Distribution of Output Voltage from a True Multiplier with partly correlated Inputs

As explained in section b) of the report, the probability distribution function is, for $v > 0$;

$$P(v)\,dv = \int_{S_1 + S_2} P_x(x)\, P_z(z)\, dx\, dz \qquad 1.$$

$$\text{where} \quad v = (cx + z)\,x \qquad 2.$$

Consider the transformation

$$x = \frac{1}{\sqrt{2}\sqrt{1+c^2}}\,(u - w)$$

$$z = \frac{1}{\sqrt{2}\sqrt{1+c^2}}\left(\left[\sqrt{1+c^2} - c\right]u + \left[\sqrt{1+c^2} + c\right]w\right) \qquad 3$$

Eqn. 3. has the effect of changing from rectangular axes to oblique axes s, t (shown in fig. 3) & then rotating them 45° anticlockwise to u, w.

On substituting from 3. into 2. we find that:

$$v = \frac{1}{2\sqrt{1+c^2}}\,(u^2 - w^2)$$

Now change from u, w to hyperbolic coordinates ρ, ϑ given by

$$\left.\begin{aligned} u &= \pm\,\rho\cosh\vartheta \\ w &= \pm\,\rho\sinh\vartheta \end{aligned}\right\} \qquad \pm \text{ for } \begin{matrix} S_1 \\ S_2 \end{matrix} \qquad 4.$$

Then,

$$\left.\begin{aligned} v &= \frac{\rho^2}{2\sqrt{1+c^2}} \\ dv &= \frac{\rho\, d\rho}{\sqrt{1+c^2}} \end{aligned}\right\} \qquad 5.$$

In these transformations the elements of area become $dx\,dz$, $\frac{du\,dw}{\sqrt{1+c^2}}$, $\frac{\rho\, d\rho\, d\vartheta}{\sqrt{1+c^2}}$

From 5. it is seen that the strips S_1, S_2 of constant v become two identical strips in the ρ, ϑ plane, parallel to the ϑ axis, and extending from $\vartheta = -\infty$ to $\vartheta = +\infty$

On substituting from 2., 3., 4. into 1. $P(v)\,dv$ becomes:

$$P(v)dv = 2\int_{-\infty}^{+\infty} P_x\left\{\frac{\rho}{\sqrt{2}\sqrt{1+c^2}}\left[\cosh\vartheta - \sinh\vartheta\right]\right\} P_z\left\{\frac{\rho}{\sqrt{2}\sqrt{1+c^2}}\left[\begin{matrix}(\sqrt{1+c^2}-c)\cosh\vartheta \\ +(\sqrt{1+c^2}+c)\sinh\vartheta\end{matrix}\right]\right\}\frac{\rho\, d\rho\, d\vartheta}{\sqrt{1+c^2}}$$

$$= 2\int_{-\infty}^{\infty} \exp\left\{\frac{-\rho^2}{4(1+c^2)X^2Z^2}\left[\{Z^2+X^2(1+2c^2)\}\cosh 2\vartheta + \{X^2-Z^2\}\sinh 2\vartheta - 2c\sqrt{1+c^2}X^2\right]\right\}\frac{d\vartheta\, dv}{2\pi XZ} \qquad 6.$$

Now let,

$$\left.\begin{aligned} Z^2 + (1+2c^2)X^2 &= k\cosh 2\varphi \\ X^2 - Z^2 &= k\sinh 2\varphi \end{aligned}\right\}$$

$$\therefore\ 4(1+c^2)X^2Y^2 = k^2 \qquad 7.$$

$\therefore$ 6. becomes:

$$P(v)\,dv = 2\int_{-\infty}^{+\infty}\frac{1}{2\pi XZ}\exp\left\{-\frac{\rho^2 k\cosh 2\psi}{4(1+c^2)X^2Z^2}\right\}\frac{d\psi}{2}\cdot\exp\frac{c\rho^2}{\sqrt{1+c^2}\,2Z^2}\,dv$$

where $\psi = \vartheta + \varphi$

$$\therefore P(v)dv = \frac{1}{\pi XZ}K_0\left(\frac{k\rho^2}{4(1+c^2)X^2Z^2}\right)\exp\frac{c\rho^2}{\sqrt{1+c^2}\cdot 2Z^2} \qquad \text{[ref. 3 p. 51]}$$

$$= \frac{1}{\pi XY\sqrt{1-r^2}}K_0\left(\frac{v}{XY(1-r^2)}\right)\exp\frac{rv}{XY(1-r^2)} \qquad 8.$$

where K_0 is a Bessel Function of the second kind, with imaginary argument. (Modified Bessel Function).

Similarly for $v<0$ it is found that

$$P(v)dv = \frac{1}{\pi XY\sqrt{1-r^2}}K_0\left(-\frac{v}{XY(1-r^2)}\right)\exp\frac{rv}{XY(1-r^2)} \qquad 9$$

Finally 8, 9, can be combined to give, for all v

$$P(v) = \frac{1}{\pi XY\sqrt{1-r^2}}K_0\left(\frac{|v|}{XY(1-r^2)}\right)\exp\frac{rv}{XY(1-r^2)} \qquad 10.$$

The mean is given by:

$$\bar{v} = \int_{-\infty}^{\infty} v P(v)\,dv$$

$$= \frac{2}{\pi XY\sqrt{1-r^2}}\int_0^{\infty} v\sinh\frac{rv}{XY(1-r^2)}\,K_0\left(\frac{v}{XY(1-r^2)}\right)dv$$

$$= rXY \qquad \text{[ref. 3. pp. 75-77]}$$

The mean square about zero is given by:

$$V_0^2 = \int_{-\infty}^{\infty} v^2 P(v)\,dv$$

$$= 2\int_0^{\infty} \frac{v^2}{\pi XY\sqrt{1-r^2}} \cosh\frac{rv}{XY(1-r^2)} K_0\left(\frac{v}{XY(1-r^2)}\right) dv$$

$$= (1+2r^2)X^2Y^2$$

[ref 3 p. 77]

$$\therefore V_F^2 = V_0^2 - \bar{v}^2$$

$$= (1+r^2)X^2Y^2$$

$$\therefore G_{t.m} = \frac{r}{\sqrt{1+r^2}}$$

The Probability Distribution of the Output Voltage in a Ring Modulator with partly correlated Inputs.

The strips over which $\int P_X(x)\,P_Z(z)\,dx\,dz$ is to be evaluated to give $P(v)dv$ are shown in fig. 4. The output v in each of the states is listed in Table 1.

The calculations for $P_2(v)\,dv$ are given in some detail whilst the results for the other states are only quoted.

In state 2 $\qquad v = \dfrac{(c+1)x + z}{6}$

$$\therefore P_2(v)dv = \iint P_X(x) P_Z(z)\,dx\,dz$$

$$= \int_{x=2v}^{x=4v} P_X(x)\,P_Z(6v - \overline{c+1}\,x)\,dx\;6\,dv$$

The limits are found from the inequalities in table 1

$$P_2(v)dv = \frac{6\,dv}{2\pi XZ}\int_{2v}^{4v} \exp\left\{-\frac{1}{2X^2Z^2}\left[x^2(Z^2+\overline{c+1}^2X^2) - 2(c+1)x\,6vX^2 + 36v^2X^2\right]\right\}dx$$

$$= \frac{6\,dv}{2\pi XZ}\int_{2v}^{4v} \exp\left\{\frac{-1}{2X^2Z^2}\left[\left(x\{Z^2+\overline{c+1}^2X^2\}^{\frac12} - \frac{6v.\overline{c+1}X^2}{\{Z^2+\overline{c+1}^2X^2\}^{1/2}}\right)^2 + \frac{36Z^2X^2v^2}{(Z^2+\overline{c+1}^2X^2)}\right]\right\}dx$$

$$= \frac{6\exp\left\{-\frac{18v^2}{(Z^2+\overline{c+1}^2X^2)}\right\}}{2\pi XZ}\;\frac{\sqrt{2}\,XZ\,dv}{(Z^2+\overline{c+1}^2X^2)^{1/2}}\int_{av}^{bv} e^{-u^2}du$$

where,

$$v = \frac{1}{\sqrt{2}\,XZ}\left\{ x\left(Z^2+\overline{c+1}^2X^2\right)^{1/2} - \frac{6v(c+1)X^2}{\left(Z^2+\overline{c+1}^2X^2\right)^{1/2}}\right\}$$

$$a = \frac{\sqrt{2}}{XZ}\;\frac{\left(Z^2+\overline{c-2}\;\overline{c+1}\,X^2\right)}{\left(Z^2+\overline{c+1}^2X^2\right)^{1/2}}$$

$$b = \frac{\sqrt{2}}{XZ}\;\frac{\left(2Z^2+\overline{c+1}\;\overline{2c-1}\,X^2\right)}{\left(Z^2+\overline{c+1}^2X^2\right)^{1/2}}$$

If r, X, Y are substituted for c, X, Z it is found that

$$P_2(v) = \frac{3}{\sqrt{2\pi}}\,\frac{\exp\left\{-\frac{18v^2}{X^2+Y^2+2rXY}\right\}}{(X^2+Y^2+2rXY)^{1/2}}\left[\operatorname{erf}\frac{\sqrt{2}(2Y^2-X^2+rXY)v}{XY(X^2+Y^2+2rXY)^{1/2}\sqrt{1-r^2}} + \operatorname{erf}\frac{\sqrt{2}(2X^2-Y^2+rXY)v}{XY(X^2+Y^2+2rXY)^{1/2}\sqrt{1-r^2}}\right]$$

Similary

$$P_1(v) = \frac{1}{\sqrt{2\pi}}\,\frac{\exp\left\{-\frac{2v^2}{X^2}\right\}}{X}\operatorname{erfc}\frac{\sqrt{2}(2X-rY)v}{XY\sqrt{1-r^2}}$$

$$P_3(v) = \frac{1}{\sqrt{2\pi}}\,\frac{\exp\left\{-\frac{2v^2}{Y^2}\right\}}{Y}\operatorname{erfc}\sqrt{2}\,\frac{(2Y-rX)v}{XY\sqrt{1-r^2}}$$

where erf and erfc are the error function and complimentary error function.

It is clear from the symmetry of the problem that the other positive states 7, 8, 9 contribute similar terms to $P(v)$, and so for $v>0$

$$P(v) = 2\left(P_1(v)+P_2(v)+P_3(v)\right)$$

It can be shown that, for $v<0$, $P(v)$ is the same except that v is replaced by $|v|$ and r by $-r$.

The mean value of v is $\int_{-\infty}^{+\infty} vP(v)\,dv$ and from the expressions for $P_i(v)$ this integral is seen to involve integrals of the form

$$\int_0^\infty v e^{-\alpha^2v^2}\operatorname{erf}\beta v\,dv = \frac{\beta}{2\alpha^2}\,\frac{1}{(\alpha^2+\beta^2)^{1/2}}$$

$$\int_0^\infty v e^{-\alpha^2v^2}\operatorname{erfc}\beta v\,dv = \frac{1}{2\alpha^2}\left(1-\frac{\beta}{(\alpha^2+\beta^2)^{1/2}}\right)$$

Upon substituting for α, β from the $P_i(v)$ it is found that

$$\bar{v} = \frac{1}{6\sqrt{2\pi}}\left\{|4X^2+Y^2+4rXY|^{1/2} + |4Y^2+X^2+4rXY|^{1/2} - |4X^2+Y^2-4rXY|^{1/2} - |4Y^2+X^2-4rXY|^{1/2}\right\}$$

$$= \sqrt{\frac{P}{2\pi}}\,\frac{1}{6}\left\{\left|4q+\frac{1}{q}+4r\right|^{1/2} + \left|\frac{4}{q}+q+4r\right|^{1/2} - \left|4q+\frac{1}{q}-4r\right|^{1/2} - \left|\frac{4}{q}+q-4r\right|^{1/2}\right\}$$

The mean square of the output about zero is $\int_{-\infty}^{+\infty} v^2 P(v)\,dv$

This involves integrals of the form:

$$\int_0^\infty v^2 e^{-\alpha^2 v^2}\,\mathrm{erf}\,\beta v\,dv = \frac{1}{2\alpha^3\sqrt{\pi}}\left\{\tan^{-1}\frac{\beta}{\alpha} + \frac{\alpha\beta}{(\alpha^2+\beta^2)}\right\}$$

$$\int_0^\infty v^2 e^{-\alpha^2 v^2}\,\mathrm{erfc}\,\beta v\,dv = \frac{1}{2\alpha^3\sqrt{\pi}}\left\{\frac{\pi}{2} - \tan^{-1}\frac{\beta}{\alpha} - \frac{\alpha\beta}{(\alpha^2+\beta^2)}\right\}$$

In these two integrals the values of the $\tan^{-1}$ terms lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

On substitution for α, β from the $P_i(v)$ & after combining the various $\tan^{-1}$ terms together (paying careful attention to the interval in which each term lies) it is found that:

$$V_0^2 = \frac{1}{4\pi}\left\{\pi(X^2+Y^2) - X^2\tan^{-1}\frac{4XY\sqrt{1-r^2}}{Y^2-4X^2} - Y^2\tan^{-1}\frac{4XY\sqrt{1-r^2}}{X^2-4Y^2}\right.$$
$$+\frac{(X^2+Y^2)}{9}\left[\tan^{-1}\frac{4XY\sqrt{1-r^2}}{Y^2-4X^2} + \tan^{-1}\frac{4XY\sqrt{1-r^2}}{X^2-4Y^2} - \pi\right]$$
$$\left.+\frac{2rXY}{9}\tan^{-1}\frac{30r\sqrt{1-r^2}\,X^2Y^2}{4Y^4+4X^4+17(1-2r^2)X^2Y^2} - \frac{4}{3}\sqrt{1-r^2}\,XY\right\}$$

$$= \frac{P}{2\pi}\left\{\frac{4\pi\left(q+\frac{1}{q}\right)}{9} - \frac{1}{2}\left(q - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\frac{4\sqrt{1-r^2}}{\frac{1}{q}-4q} - \frac{1}{2}\left(\frac{1}{q} - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\frac{4\sqrt{1-r^2}}{q-\frac{4}{q}}\right.$$
$$\left.+\frac{r}{9}\tan^{-1}\frac{30r\sqrt{1-r^2}}{\frac{4}{q^2}+4q^2+17(1-2r^2)} - \frac{2}{3}\sqrt{1-r^2}\right\}$$

In this expression for V_0^2, $\tan^{-1}$ lies in the range 0 to π not $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

For r small, V_0^2 becomes

$$V_0^2 = \frac{P}{2\pi}\left\{\frac{4\pi}{9}\left(q+\frac{1}{q}\right) - \left(q - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}2q - \left(\frac{1}{q} - \frac{q+\frac{1}{q}}{9}\right)\tan^{-1}\frac{2}{q} - \frac{2}{3}\right\}$$

For $r = 1$, there are three different states

1. $\frac{X}{Y} = q > 2$ $\quad V_0^2 = \frac{Y^2}{4} = \frac{P}{4q}$ $\quad \bar{V} = \frac{Y}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{P}{q}}$
2. $2 > \frac{X}{Y} = q > \frac{1}{2}$ $\quad V_0^2 = \frac{(X+Y)^2}{4\cdot 9} = \frac{P\left(\sqrt{q}+\frac{1}{\sqrt{q}}\right)^2}{4\cdot 9}$ $\quad \bar{V} = \frac{X+Y}{3\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{P}}{3}\left(\sqrt{q}+\frac{1}{\sqrt{q}}\right)$
3. $\frac{1}{2} > \frac{X}{Y} = q$ $\quad V_0^2 = \frac{X^2}{4} = \frac{Pq}{4}$ $\quad \bar{V} = \frac{X}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\sqrt{Pq}$

STATE			RELATIONSHIP: x, y	CONDUCTING DIODES	OUTPUT VOLTAGE v
1	$x>0$	$y>0$	$y > 2x$	D_2, D_3	$\frac{x}{2}$
2	$x>0$	$y>0$	$2x > y > \frac{x}{2}$	D_1, D_2, D_3	$\frac{x+y}{6}$
3	$x>0$	$y>0$	$\frac{x}{2} > y$	D_2, D_1	$\frac{y}{2}$
4	$x>0$	$y<0$	$\frac{x}{2} > -y$	D_2, D_1	$\frac{y}{2}$
5	$x>0$	$y<0$	$2x > -y > \frac{x}{2}$	D_1, D_2, D_4	$\frac{y-x}{6}$
6	$x>0$	$y<0$	$-y > 2x$	D_1, D_4	$-\frac{x}{2}$
7	$x<0$	$y<0$	$-y > -2x$	D_1, D_4	$-\frac{x}{2}$
8	$x<0$	$y<0$	$-2x > -y > -\frac{x}{2}$	D_1, D_3, D_4	$-\frac{x+y}{6}$
9	$x<0$	$y<0$	$-\frac{x}{2} > -y$	D_3, D_4	$-\frac{y}{2}$
10	$x<0$	$y>0$	$-\frac{x}{2} > y$	D_3, D_4	$-\frac{y}{2}$
11	$x<0$	$y>0$	$-2x > y > -\frac{x}{2}$	D_2, D_3, D_4	$\frac{x-y}{6}$
12	$x<0$	$y>0$	$y > -2x$	D_2, D_3	$\frac{x}{2}$

TABLE I

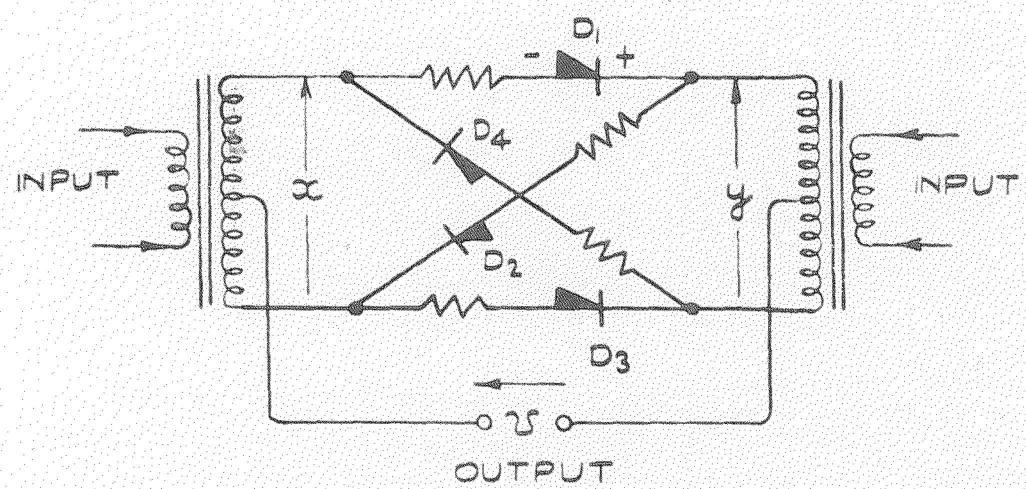


FIG. 1(a)

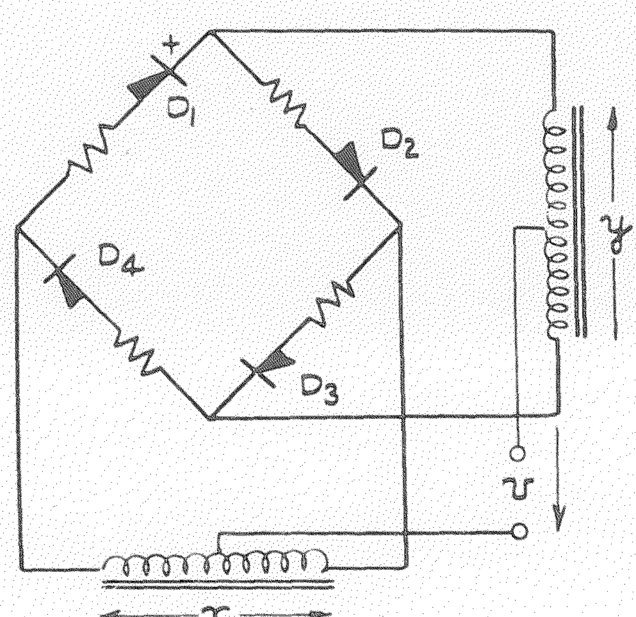
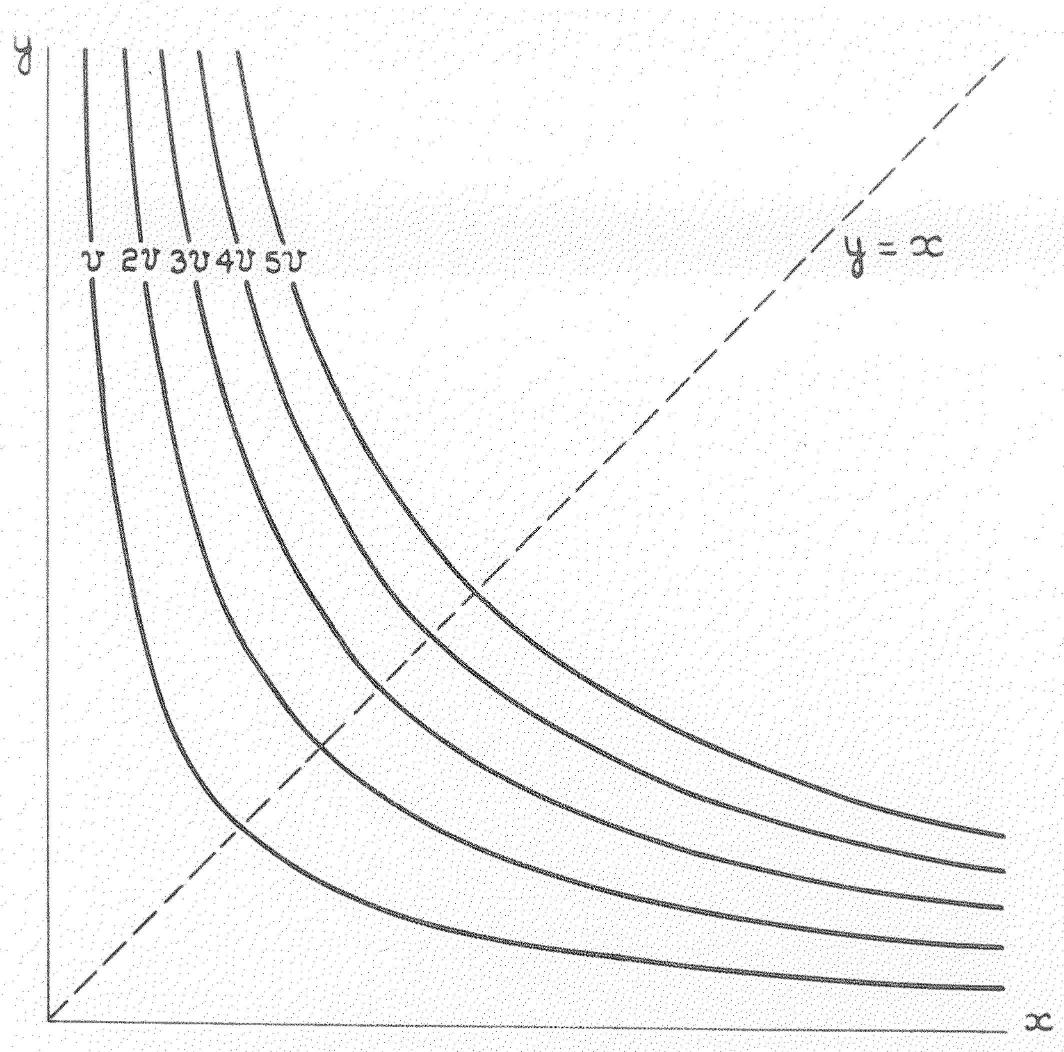
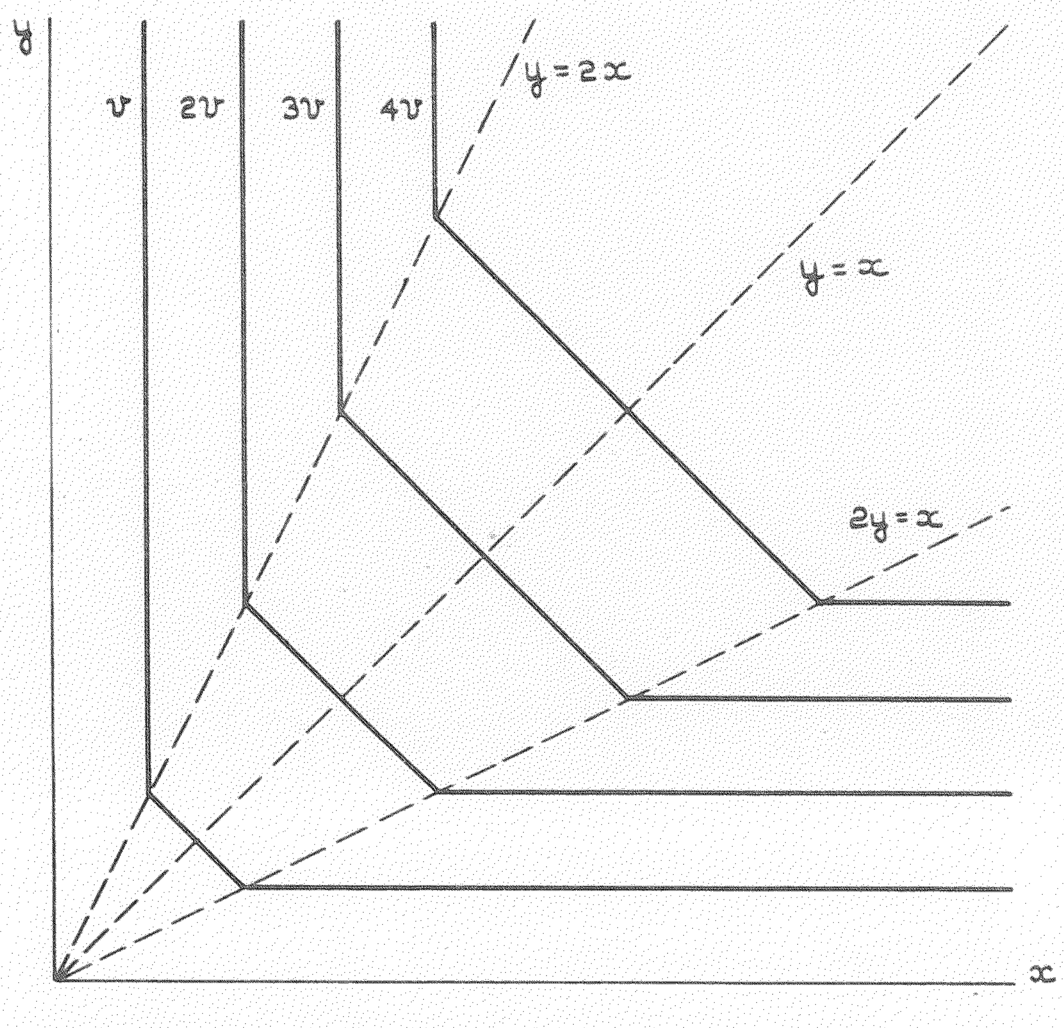


FIG. 1(b)



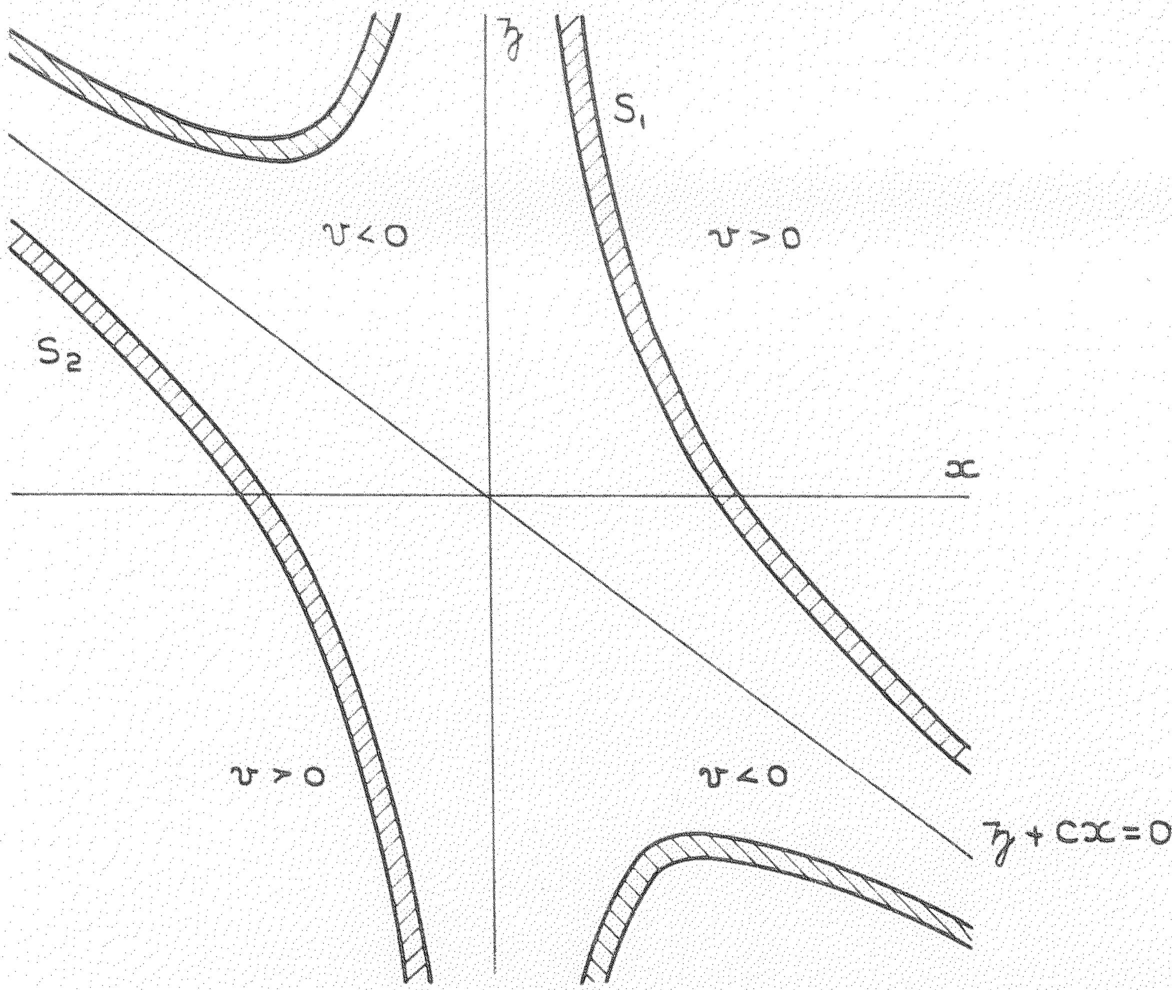
CHARACTERISTICS OF TRUE MULTIPLIER

FIG. 2a.



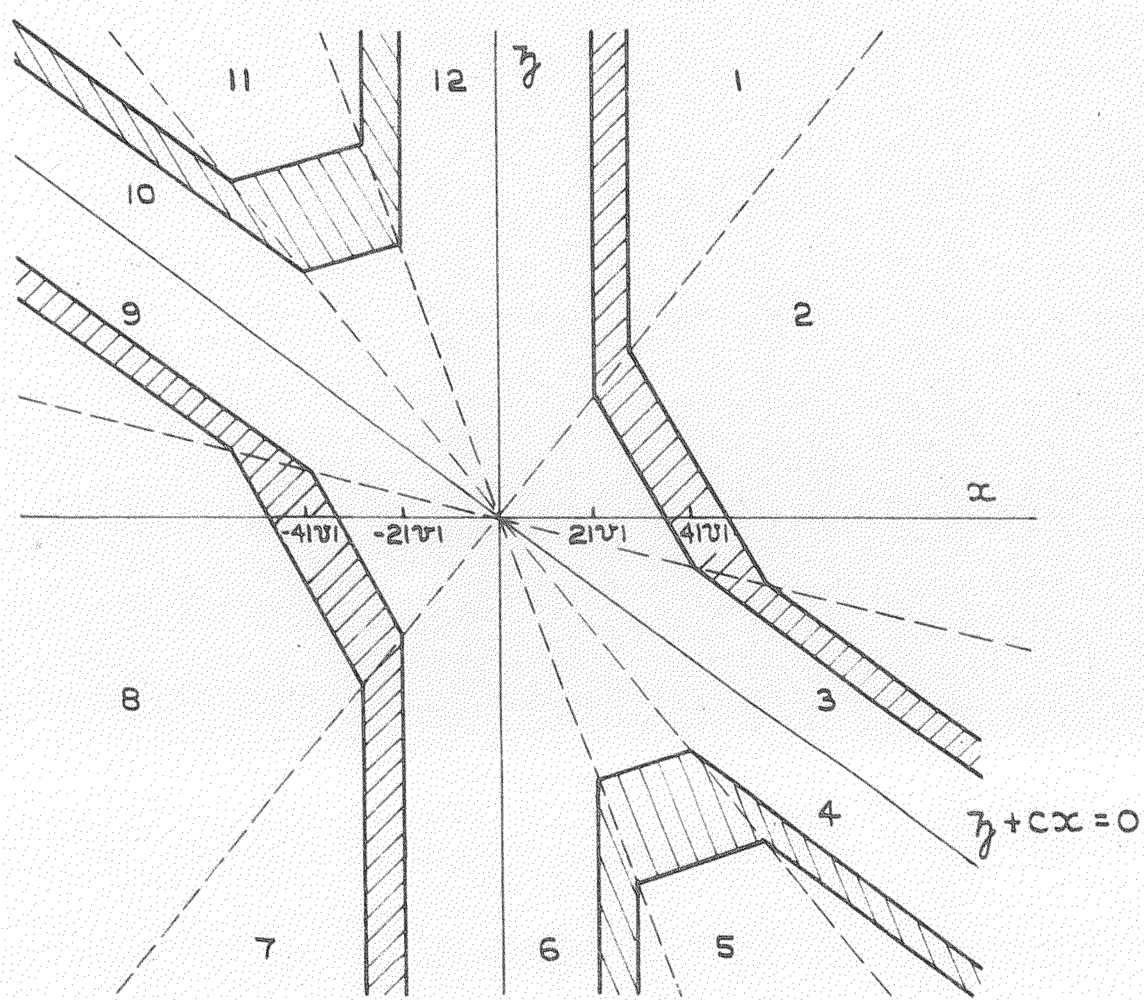
CHARACTERISTICS OF RING MODULATOR

FIG. 2b.



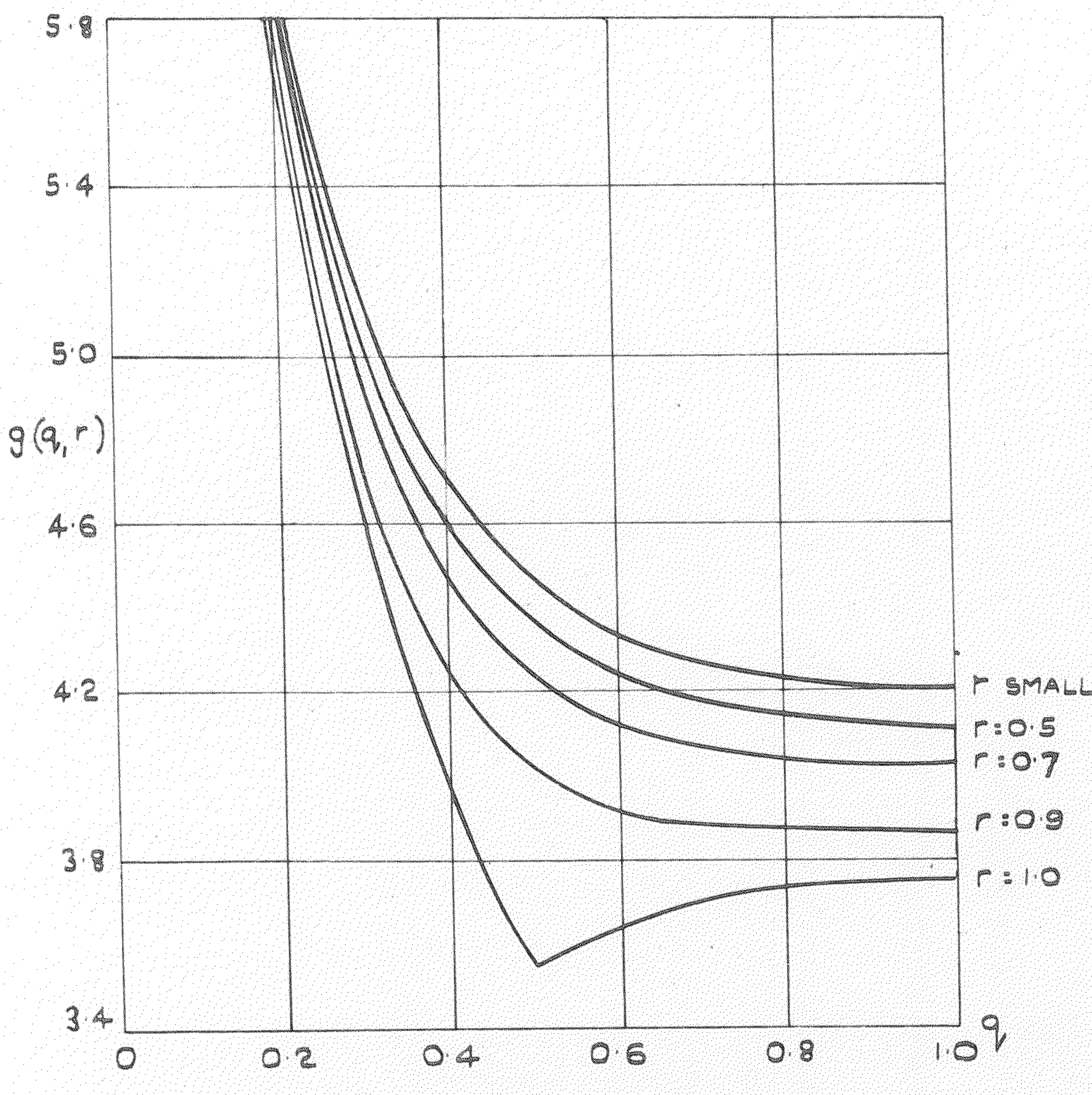
RANGE OF INTEGRATION FOR TRUE MULTIPLIER

FIG. 3.



RANGE OF INTEGRATION FOR RING MODULATOR

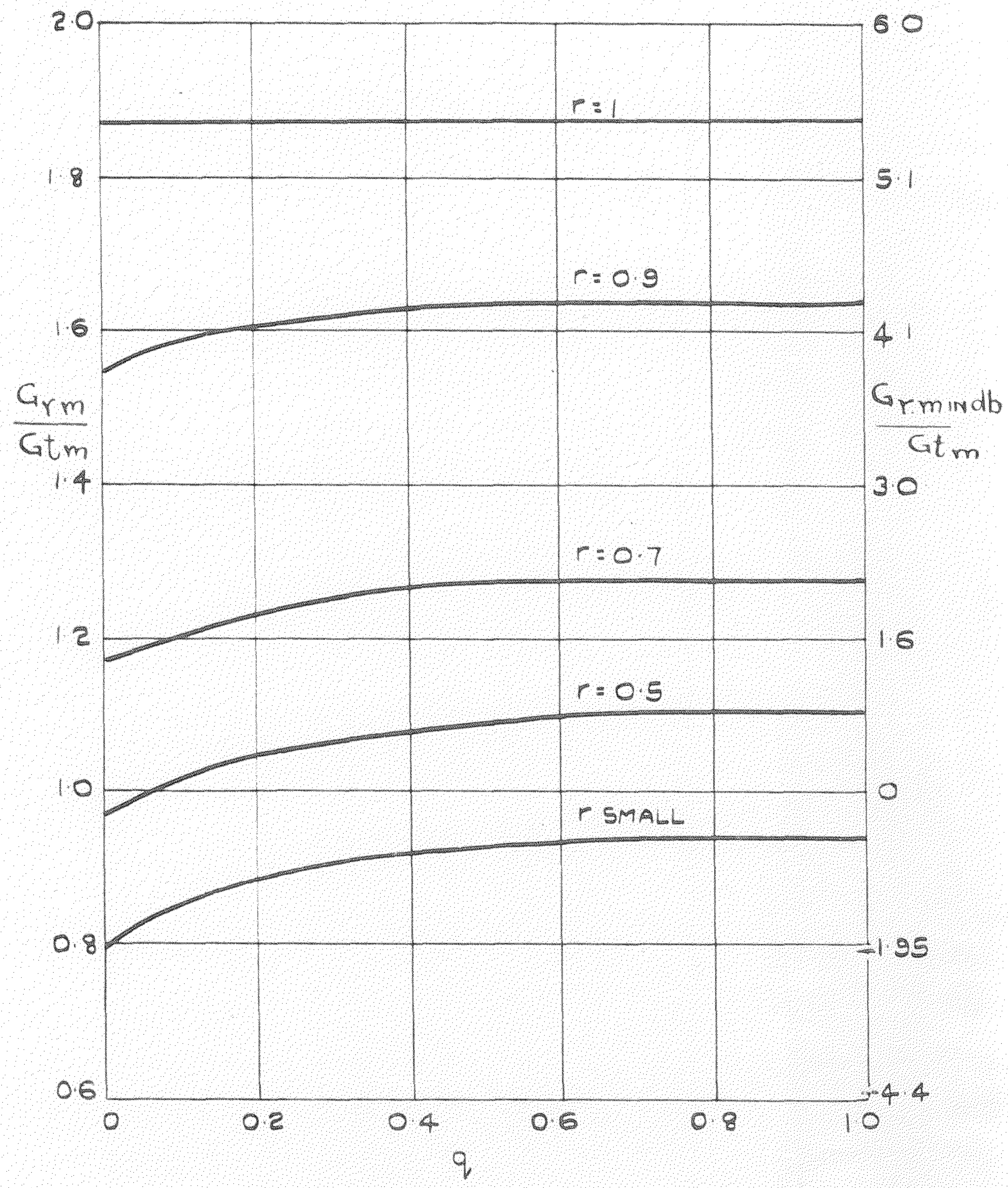
FIG. 4.



CURVES FOR DETERMINATION OF r

$$r = \frac{\bar{v}}{\sqrt{p}}\, g(q, r)$$

FIG. 5



COMPARISON OF SIGNAL/NOISE
IN RING MODULATOR AND TRUE MULTIPLIER.

FIG. 6

Distribution List.

	Copy No.
D.P.R.	1
D.R.P.P.	2-11
Hydrographer	12
A.R.L.	13-14
The Capt., H.M.U.D.E.	15
Captain Superintendent, A.S.R.E.	16
The Capt., H.M.U.C.W.E.	17
Captain Superintendent, A.E.W.	18
D.S.I.R.	19
The Director, N.P.L.	20
U.S. Naval Attache.	21
N.I.O. File.	22-50