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The amplitude of waves reflected from a vertical circular cylinder

by

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SUMMARY

The problem of the diffraction of water waves by a vertical circular cylinder has been formally solved by Havelock (II). In this note we use the solution to calculate the amplitude of the reflected waves in different directions and at different distances from the cylinder, for a number of different values of the ratio of the radius of the cylinder to the wavelength of the waves.

The results are represented in Figures 1 to 8 and in Table 2. For small values of the radius, the amplitude at great distances is a minimum near $\Theta = 60°$; close to the cylinder there is a minimum near $\theta = 90°$.

For large values of the radius ($ka \gg 1$) the amplitude can be calculated by ray optical methods. It is found that the limiting case is approached in an oscillatory fashion.

The results are intended for use in estimating the effect of vertical piles on wave recorders nearby.

1. Notation

h = height (crest to trough) of incoming waves

σ = 2π divided by period of incoming waves

k = 2π divided by length of incoming waves

d = mean depth of water

a = radius of cylinder

x, z = horizontal, vertical coordinates, with origin O at intersection of axis of cylinder with mean water level, Ox in direction of incoming waves, Oz vertically upwards

r, θ = horizontal polar coordinates, with

2. Formal solution

It is assumed that all terms proportional to the square of the wave height can be neglected, i.e. that the problem is linear. Also that viscosity and surface tension are negligible. Under these conditions the velocity potential ϕ must satisfy the field equation

$$\nabla^2 \phi = 0 \tag{1}$$

and the boundary conditions

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{when} \quad z = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{when} \quad z = -d \tag{3}$$

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{when} \quad r = a \tag{4}$$

(see for example Ref. IV, Chapter 9). The elevation ζ of the free surface is given by

$$\zeta = \frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right)_{z=0} \tag{5}$$

Let the surface elevation of the incoming waves (in the absence of the cylinder) be

$$\eta^{(i)} = \frac{1}{2} h \, e^{i(kx - \sigma t)} \tag{6}$$

Then the corresponding velocity-potential is

$$\phi^{(i)} = \frac{gh}{2i\sigma} \frac{\cosh k(z+d)}{\cosh kd} e^{i(kx - \sigma t)} \tag{7}$$

which satisfies equations (1), (2) and (3) provided

$$\sigma^2 = gk \tanh kh \tag{8}$$

We now write

$$\begin{aligned} \phi &= \phi^{(i)} + \phi^{(r)} \\ \eta &= \eta^{(i)} + \eta^{(r)} \end{aligned} \tag{9}$$

where $\phi^{(r)}$ and $\eta^{(r)}$ are the velocity potential and surface elevation for the reflected waves. It is clear that $\phi^{(r)}$ must satisfy equations (1), (2) and (3) together with

$$\frac{\partial \phi^{(r)}}{\partial r} = -\frac{\partial \phi^{(i)}}{\partial r} \quad \text{when} \quad r = a \tag{10}$$

$$\phi^{(r)} \to 0 \quad \text{when} \quad r \to \infty \tag{11}$$

To convert to polar coordinates we use the identity

$$e^{ikr\cos\theta} \equiv J_0(kr) + 2\sum_{m=1}^{\infty} i^m J_m(kr)\cos m\theta \tag{12}$$

(see for example Ref. I, p. 32), so that equation (7) can be written

$$\phi^{(i)} = \frac{gh}{2i\sigma}\left[J_0(kr) + 2\sum_{1}^{\infty} i^m J_m(kr)\cos m\theta\right] \frac{\cosh k(z+d)}{\cosh kd} e^{-i\sigma t} \tag{13}$$

We choose for $\phi^{(r)}$ the following form:

$$\phi^{(r)} = \frac{-gh}{2i\sigma}\left[A_0 H_0^{(1)}(kr) + 2\sum_{1}^{\infty} A_m i^m H_m^{(1)}(kr)\cos m\theta\right] \frac{\cosh k(z+d)}{\cosh kd} e^{-i\sigma t} \tag{14}$$

where the A_m's are constant coefficients and

$$H_m^{(1)}(kr) = J_m(kr) + i\,Y_m(kr) \tag{15}$$

is a Hankel function of order m. This is a suitable expansion since $H_m^{(1)}(kr)\cos m\theta \cosh k(z+d)\, e^{-i\sigma t}$ satisfies equations (1), (2) and (3), and for large values of r

$$H_m^{(1)}(kr)\, e^{-i\sigma t} \sim \left(\frac{2}{\pi k r}\right)^{\frac{1}{2}} e^{i\left(kr - \sigma t + \frac{2m+1}{4}\pi\right)} \tag{16}$$

which represents a diverging wave. The boundary condition (4) at the cylinder can now be satisfied by taking

$$A_m = \frac{J_m'(ka)}{H_m^{(1)\prime}(ka)} \tag{17}$$

(where a dash denotes differentiation).

This solves the problem. The surface elevation in the reflected waves is given by

$$\zeta^{(r)} = \frac{1}{g}\frac{\partial \phi^{(r)}}{\partial t} \tag{18}$$

and so the "relative amplitude" $\left|\zeta^{(r)}/\zeta^{(i)}\right|$ is given by

$$\left|\frac{\zeta^{(r)}}{\zeta^{(i)}}\right| = \left|A_0 H_0^{(1)}(kr) + 2\sum_1^{\infty} A_m i^m H_m^{(1)}(kr)\cos m\theta\right| \tag{19}$$

where $A_m(ka)$ is given by (17).

3. Asymptotic values

For large values of kr we have

$$\frac{\zeta^{(r)}}{\zeta^{(i)}} \sim \left(\frac{2}{\pi k r}\right)^{\frac{1}{2}} e^{i\left(kr - \frac{\pi}{4}\right)}\left(A_0 + 2\sum_1^{\infty} A_m \cos m\theta\right) \tag{20}$$

so that at distances large compared with a wavelength the reflected wave dimishes like $r^{-\frac{1}{2}}$.

Further, when $ka \ll 1$ we have

$$\left.\begin{aligned} A_0(ka) &\sim i\pi\left(\tfrac{1}{2}ka\right)^2 \\ A_m(ka) &\sim -i\pi\frac{\left(\tfrac{1}{2}ka\right)^{2m}}{m!\,\overline{m-1}!} \qquad (m \geqslant 1) \end{aligned}\right\} \tag{21}$$

so that for small values of ka and large values of kr,

$$\left|\frac{\zeta^{(r)}}{\zeta^{(i)}}\right| \sim \left(\frac{2\pi}{kr}\right)^{\frac{1}{2}}\left(\tfrac{1}{2}ka\right)^2(1-2\cos\theta) + O(ka)^4 \tag{22}$$

Thus when the radius of the cylinder is small we should expect the reflection at great distances to be least when $\theta = \pm 60°$. This agrees with the result obtained by Rayleigh (Ref. VII, vol. 2, p.309) for the similar problem of diffraction of plane sound waves by a thin circular cylinder.

If, however, $kr \ll 1$ (and so $ka \ll 1$) we have

$$\left.\begin{aligned} H_0^{(1)}(kr) &\sim \frac{2i}{\pi}\log\left(\tfrac{1}{2}kr\right) \\ H_m^{(1)}(kr) &\sim \frac{-i\,(m-1)!}{\pi}\left(\tfrac{1}{2}kr\right)^{-m} \end{aligned}\right\} \tag{23}$$

and so

$$\begin{aligned} \left|\frac{\zeta^{(r)}}{\zeta^{(i)}}\right| &\sim 2\left(\tfrac{1}{2}ka\right)^2\left|\log\left(\tfrac{1}{2}kr\right) + i\left(\tfrac{1}{2}kr\right)^{-1}\cos\theta\right| \\ &\sim \frac{(ka)^2}{kr}\left|\cos\theta\right| \end{aligned} \tag{24}$$

Hence the reflected wave is proportional to r^{-1} near the cylinder, and is a minimum near $\theta = \pm 90°$.

4. Computed values

The series (19) and the asymptotic formula (20) were computed for the following values of ka, r/a and θ

ka = 0.02, 0.05, 0.1, 0.2, 0.5, 1.0;

r/a = 1.0, 1.5, 2.0, 2.5, 3, 4, 5, 6, 8, 10;

θ = 0°, 30°, 60°, 90°, 120°, 150°, 180°.

In some cases the computations were carried out for larger values of r/a until the exact and the asymptotic expressions approached sufficiently closely. Computations have also been made by Mathur, Ref. VI, for values of ka and kr equal to 3, 5, 7 and 10, and for some cylinders of shapes other than circular.

The series and the asymptotic formula are compared in Figures 1 and 2 for $\theta = 0°$ and $180°$. It will be seen that the asymptotic value is approached more rapidly for larger values of ka. On the other hand for the smaller values of ka the expression (24) is a surprisingly good approximation.

The general form of the reflected waves is shown in Figures 3 to 7, for ka = 0.05, 0.1, 0.2, 0.5 and 1.0. Contours of constant amplitude have been constructed by graphical interpolation between the computed points. The minimum near $\theta = 60°$ is especially marked in Figures 3, 4 and 5.

The diagrams in Figures 3 to 7 extend to $r/a = 15$. For greater values of r/a the asymptotic formula (20) is sufficiently accurate.

In Table 1 are tabulated the coefficients $C(ka, \theta)$ for the asymptotic formula for large kr, defined by

$$\left| \frac{\zeta^{(r)}}{\zeta^{(i)}} \right| = C(ka, \theta)\, (r/a)^{-1/2} \tag{25}$$

when ka = 0.05, 0.1, 0.2, 0.5, 1, 2 and 5. For smaller values of ka, the simple formula (22) can be used.

5. Very short waves

For short waves (large values of ka) the series (19) converges only slowly and in fact if the coefficients A_m are replaced by their asymptotic values for large ka the resulting series is not convergent.

However, the limiting case of very short waves may be obtained by the use of ray optics. Thus in Figure 9(B) AB and $A'B'$ represent two adjacent parallel rays striking the cylinder and reflected along BC and $B'C'$. If the angle BOB' is $\delta\theta'$, then the angle between BC and $B'C'$ is $2\delta\theta'$, and it is easily seen that BC and $B'C'$ intersect in a point X whose distance from B is $\frac{1}{2} a \cos\theta'$. The amplitude of the reflected waves diminishes as though the waves originated in a source at X. Since the relative amplitude is unity at the cylinder, at C it is n, where

$$n = \left(\frac{CX}{BX}\right)^{-\frac{1}{2}} = \left(1 + \frac{2\rho}{a\cos\theta'}\right) \tag{26}$$

where $BC = \rho$. Thus the curve of constant relative amplitude n is given by

$$\rho = \tfrac{1}{2} a \, (n^{-2} - 1) \cos \theta' \tag{27}$$

where ρ is measured along a line through the point $(r, \theta) = (a, \pi - \theta')$ in the direction $2\theta'$. These contours have been plotted in Figure 8. It will be seen that for values of θ near 180° $(\theta' \doteq 0)$ these curves are not very different from those of Figure 7 $(ka = 1)$. When θ is near zero, i.e. directly behind the cylinder, there is of course a shadow, where the relative amplitude is taken to be unity.

Writing $\theta' = 0$, $\rho = r - a$ in (26) we have

$$n = \left(\frac{2r}{a} - 1 \right)^{-1/2} \tag{28}$$

which gives the relative amplitude along the line immediately in front of the cylinder, for $ka = \infty$. Values of this function are given in Table 2. At great distances (i.e. $r/a \gg 1$) we have

$$n \sim \left(\frac{2r}{a} \right)^{-1/2} \tag{29}$$

so that the coefficient $C(\infty, \pi)$ in the asymptotic expansion (25) is

$$C(\infty, \pi) = \frac{1}{\sqrt{2}} = 0.707 \tag{30}$$

From Table 1 we see that the coefficient for $ka = 1.0$ is 0.738, so that $C(\infty, \pi)$ is less than $C(1, \infty)$, although for $ka \leq 1.0$ $C(ka, \pi)$ had apparently been increasing with ka. To investigate the behaviour of $C(ka, \pi)$ as ka increases still further, $C(ka, \pi)$ was calculated for $ka = 2.0$ and 5.0. It was found that

$$\left. \begin{aligned} C(2, \pi) &= 0.743 \\ C(5, \pi) &= 0.668 \end{aligned} \right\} \tag{31}$$

so that $C(ka, \pi)$ approaches its limit in an oscillatory fashion.

This oscillation is not clearly apparent from equation (19), but a better idea of the behaviour of $C(ka, \theta)$ can be obtained from an equivalent expression due to Imai, Ref. III. Imai showed that equation (19) holds for electromagnetic waves in the presence of a circular cylinder when the magnetic vector is oriented parallel to its axis, and by means of contour integration replaced (19) by another series, which in our notation can be expressed as follows:

$$\left|\frac{\zeta^{(r)}}{\zeta^{(i)}}\right| = \left(\frac{a}{2r}\right)^{1/2}\left|(\sin\theta)^{\frac{1}{2}} e^{-2ika\sin\frac{\theta}{2}}\left(1 - \frac{4k^2a^2\cos^2\frac{\theta}{2} - 1}{8ikr} + \frac{\frac{3}{8} + \operatorname{cosec}^2\frac{\theta}{2}}{2ika\sin\theta/2}\right) + (ka)^{-\frac{1}{6}} e^{i\pi/12}\sum_{s=1}^{\infty} B_s \frac{i\cos\beta_s(\pi-\theta)}{\sin\beta_s\pi}\left(1 - \frac{4\beta_s^2 - 1}{8ikr}\right)\right| \qquad (31)$$

where $$B_s = 2\sqrt{3\pi}\,\frac{(3y_s)^{-\frac{1}{3}} J_{2/3}(y_s)}{J_{1/3}(y_s) + J_{-1/3}(y_s)}\left(1 + O(ka)^{-2/3}\right),$$

$$\beta_s = ka\left(1 + \frac{1}{2}e^{i\pi/3}\left(\frac{3y_s}{ka}\right)^{2/3} - O\left(\frac{y_s}{ka}\right)^{4/3}\right),$$

and the y_s are the successive zeros of

$$J_{2/3}(y) - J_{-2/3}(y)$$

$$\left(y_1 = 0.6855, \quad y_2 = 3.9028, \quad y_s \sim \pi\left(s - \tfrac{3}{4}\right)\right)$$

Thus for any given value of ka, letting kr tend to infinity we obtain

$$C(ka, \theta) = 2^{-\frac{1}{2}}\left|\left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}} e^{-2ika\sin\frac{\theta}{2}}\left(1 + \frac{\frac{3}{8} + \operatorname{cosec}^2\frac{\theta}{2}}{2ika\sin\frac{\theta}{2}}\right) + (ka)^{-\frac{1}{6}} e^{i\pi/12}\sum_{s} B_s \frac{i\cos\beta_s(\pi-\theta)}{\sin\beta_s\pi}\right| \qquad (32)$$

which is oscillatory with respect to ka.

Since, however, β_s nearly equals ka for large ka and small s, the infinite series in (31) and (32) will converge very slowly whenever ka equals a large integer, owing to the term sin $\beta_s \pi$ in the denominator. In general, these formula do not seem to be any more convenient for purposes of computation than (19) and (20).

For very large ka and kr provided r/a is also large, (32) gives

$$C(ka, \theta) \sim \left(\frac{1}{2} \sin\frac{\theta}{2}\right)^{1/2} \tag{33}$$

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TABLE 1

Values of $C(ka, \theta)$ occuring in the asymptotic formula (25)

ka \ θ	0°	30°	60°	90°	120°	150°	180°	
.05	.070	.052	.000	.070	.141	.192	.216	x 10^{-1}
.10	.199	.145	.000	.199	.397	.544	.595	x 10^{-1}
.20	.060	.045	.004	.054	.101	.151	.166	
.5	.252	.201	.067	.197	.386	.526	.566	
1.0	.511	.408	.322	.506	.668	.728	.738	

TABLE 2

Values of relative wave amplitude n, given by (26), for infinitesimal wavelength, and θ = 180°

r/a	1	1.5	2	2.5	3	4	5
n	1.000	.707	.577	.500	.447	.378	.333
r/a	6	8	10	15	20	50	100
n	.302	.258	.229	.186	.160	.101	.071

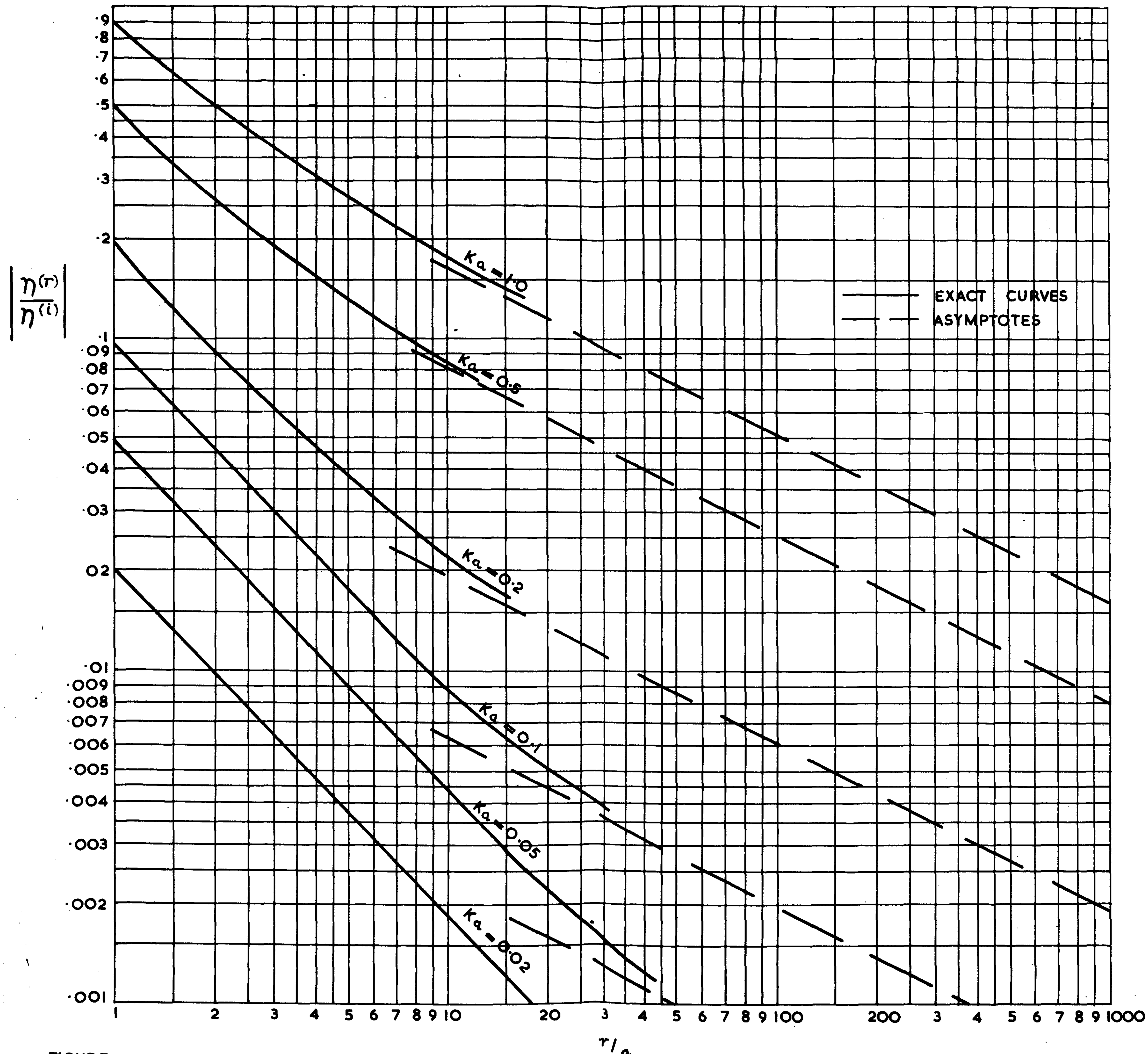

FIGURE 1

RELATIVE AMPLITUDES OF REFLECTED WAVES DIRECTLY BEHIND CYLINDER $\theta = 0$

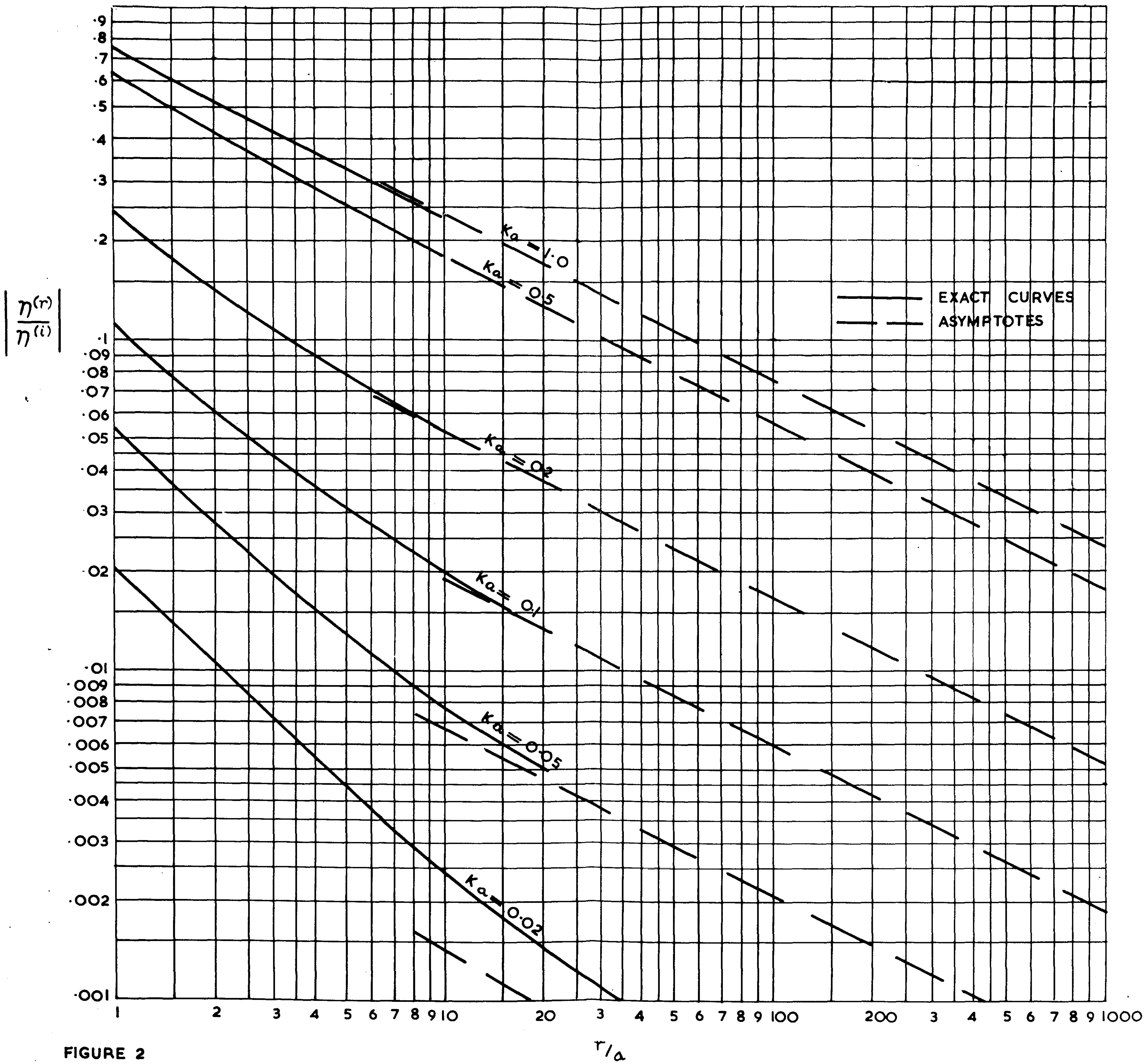

FIGURE 2

RELATIVE AMPLITUDES OF REFLECTED WAVES DIRECTLY IN FRONT OF CYLINDER ($\theta = 180°$)

Ka = 0·05

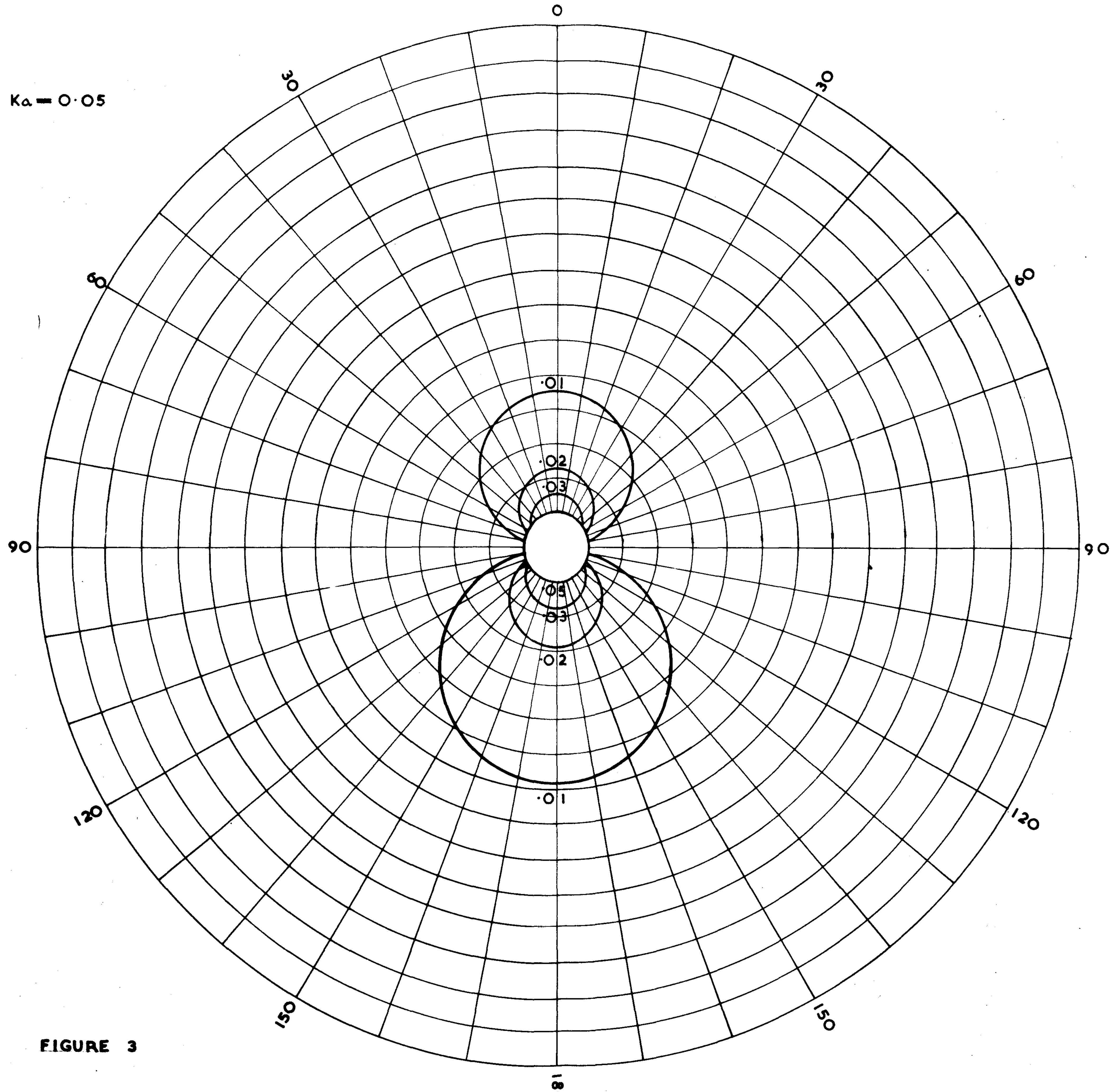

FIGURE 3

CURVES OF CONSTANT RELATIVE AMPLITUDE OF REFLECTED WAVES

(Ka = 0·05)

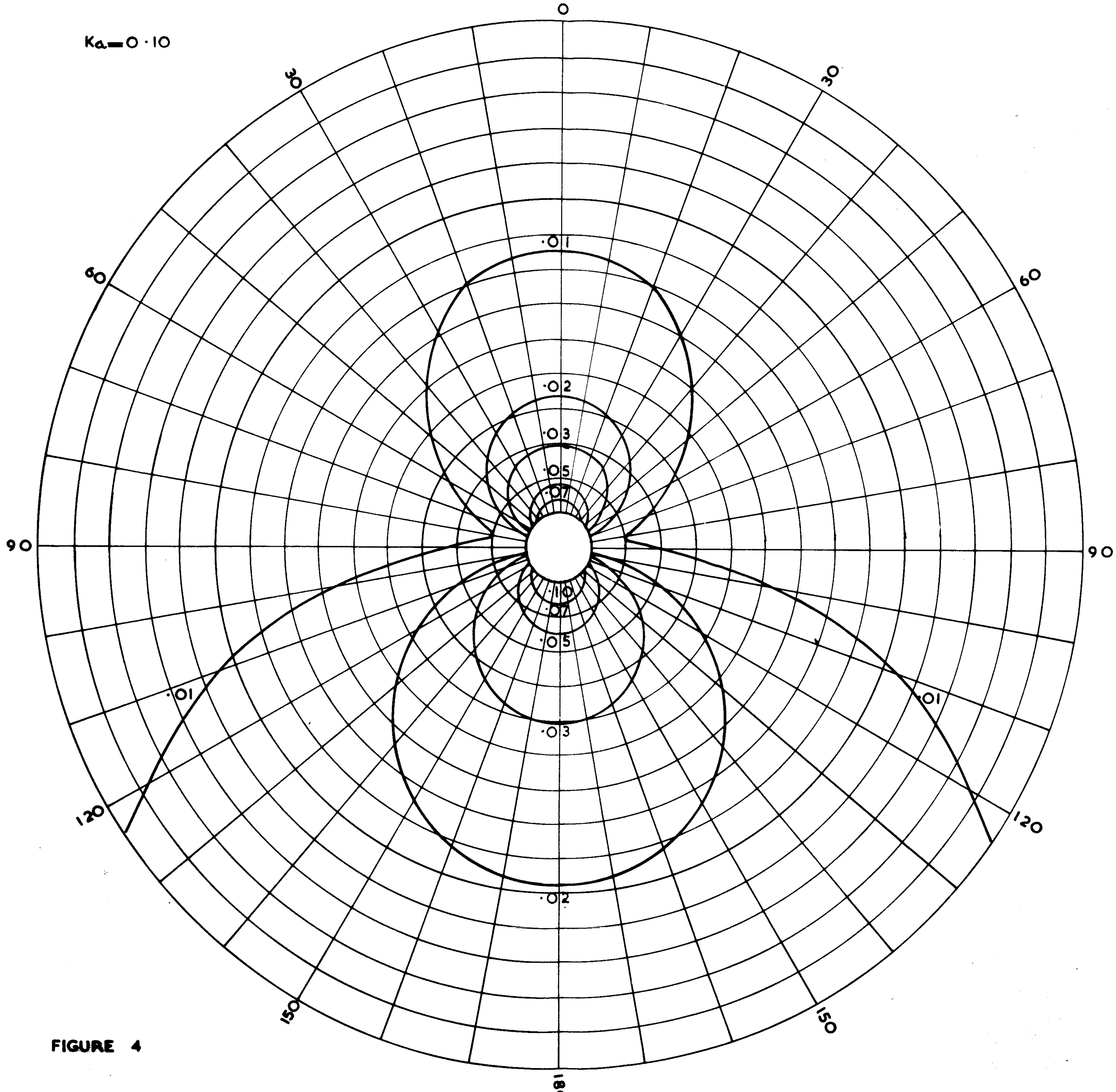

FIGURE 4

CURVES OF CONSTANT RELATIVE AMPLITUDE OF REFLECTED WAVES
(Ka = 0·10)

Ka = 0·20

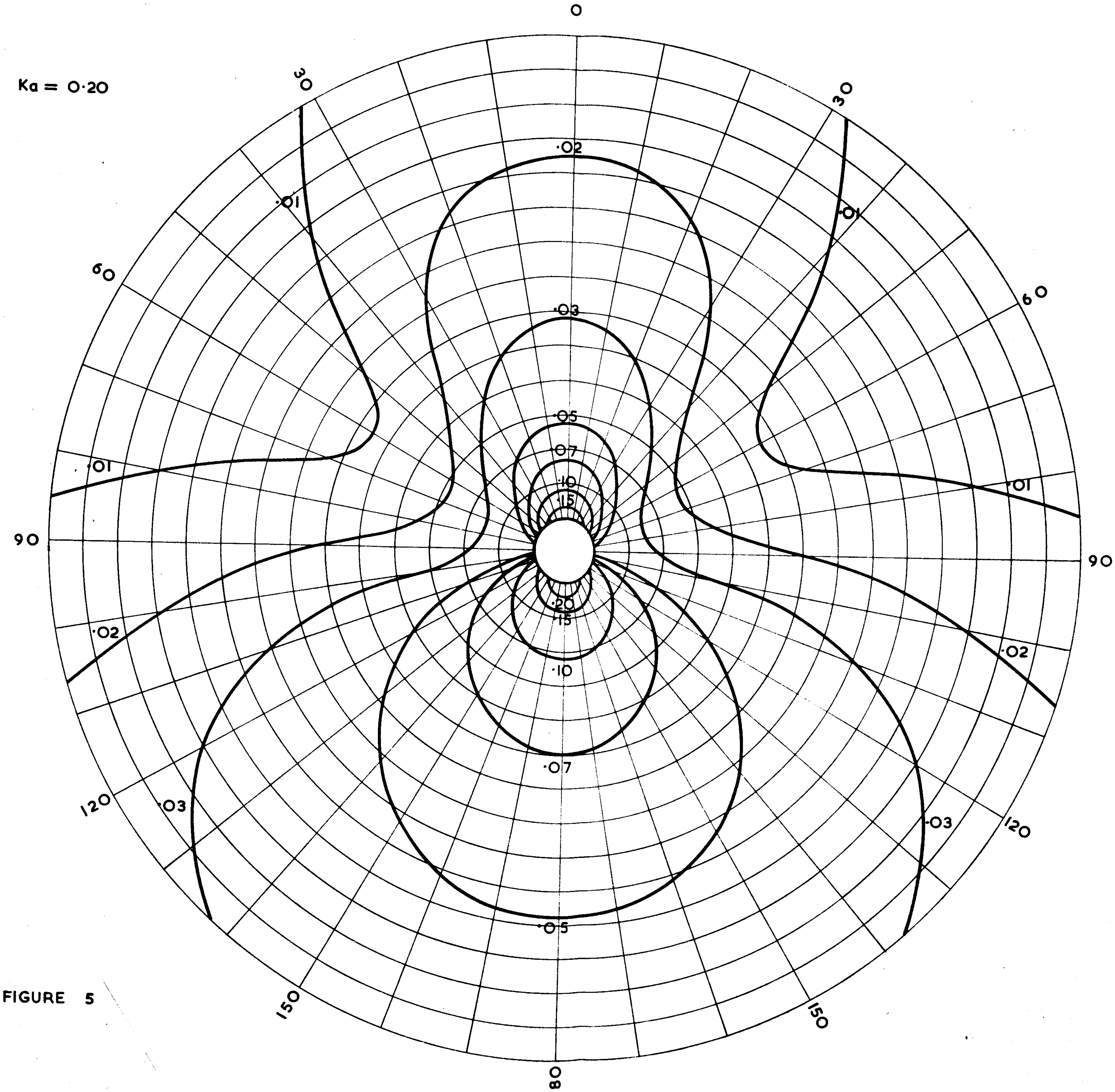

FIGURE 5

CURVES OF CONSTANT RELATIVE AMPLITUDE OF REFLECTED WAVES

(Ka = 0·20)

Ka = 0·50

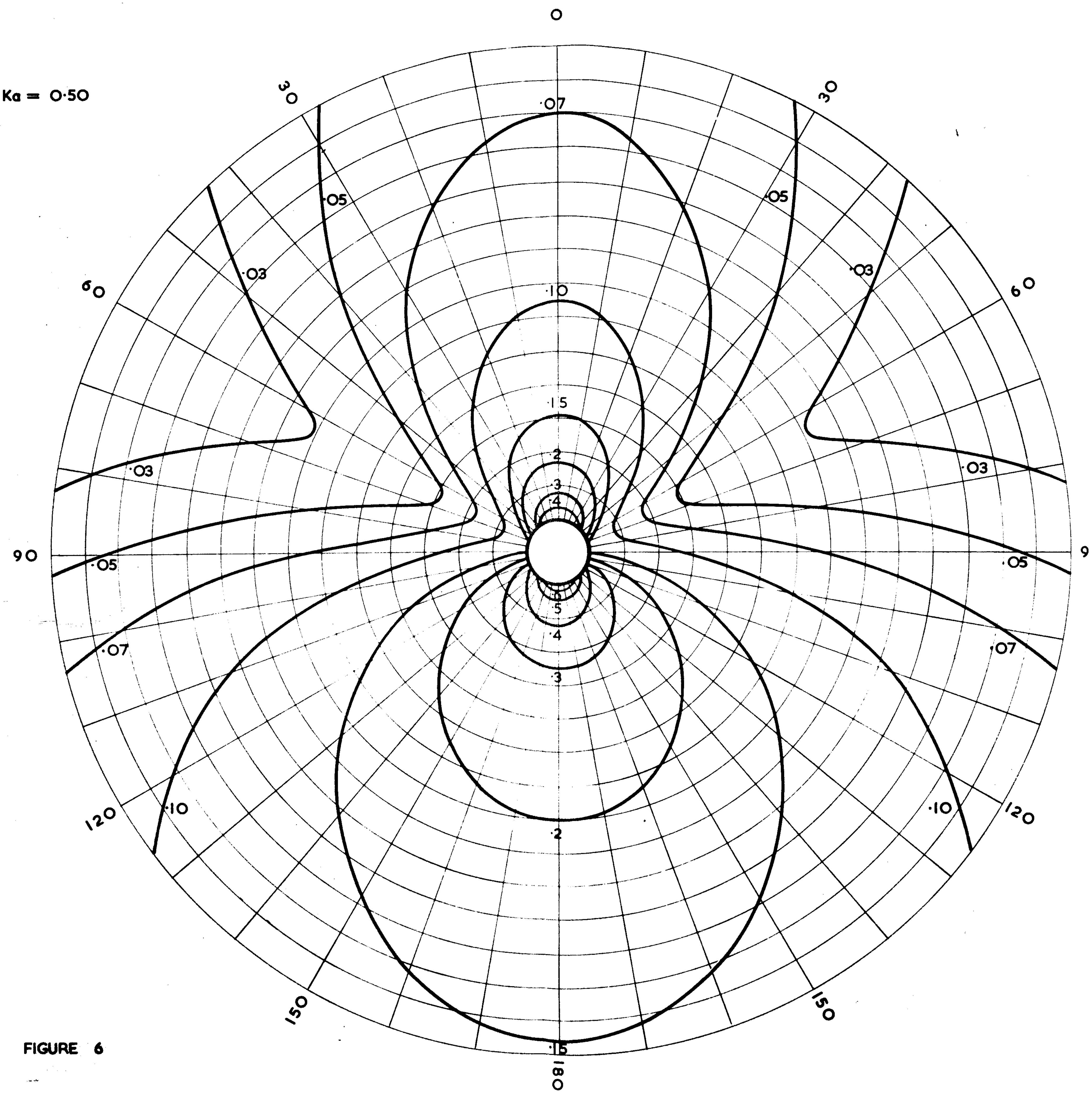

FIGURE 6

CURVES OF CONSTANT RELATIVE AMPLITUDE OF REFLECTED WAVES
(Ka = ·50)

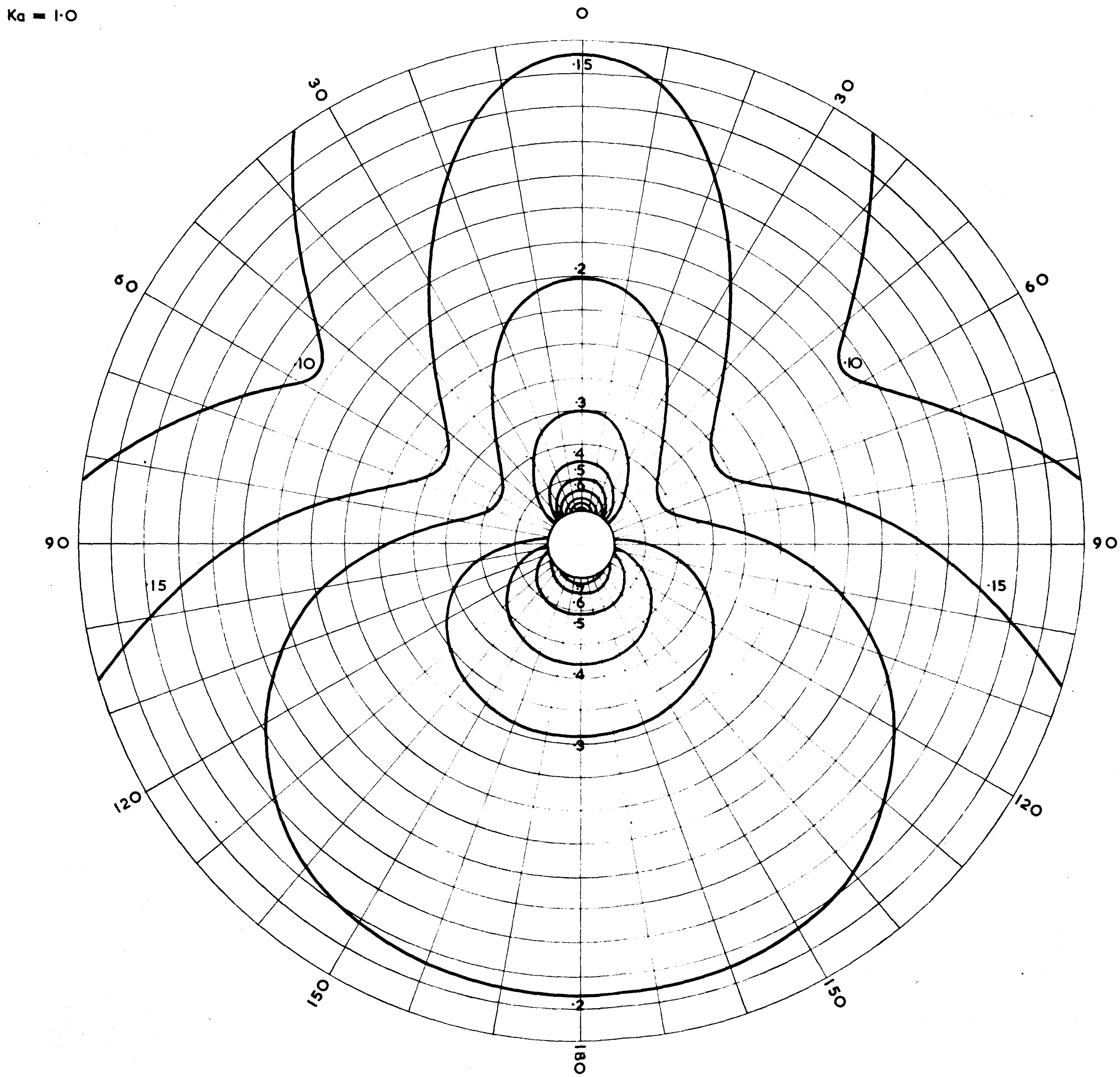

FIGURE 7.

CURVES OF CONSTANT RELATIVE AMPLITUDE OF REFLECTED WAVES.

(Ka = 1·0)

Ka = ∞

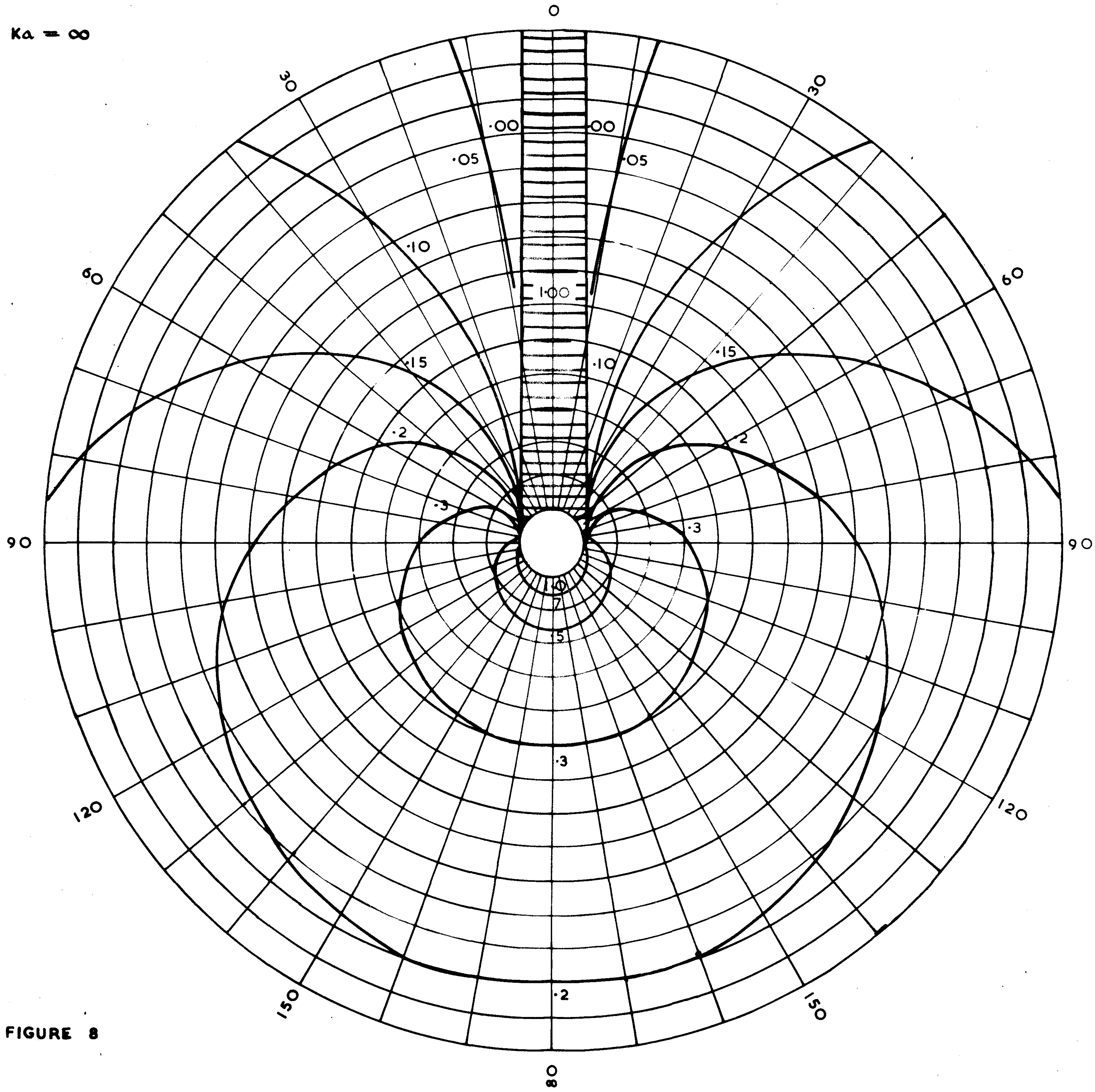

FIGURE 8

CURVES OF CONSTANT RELATIVE AMPLITUDE OF REFLECTED WAVES — LIMITING CASE

(Ka = ∞)

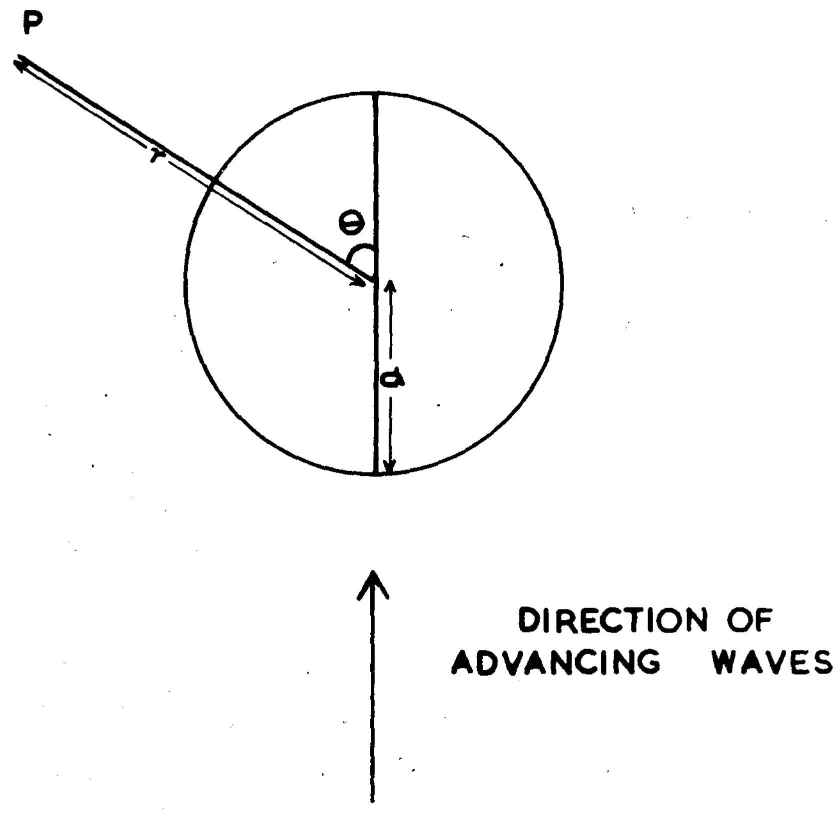

FIGURE 9 (A)

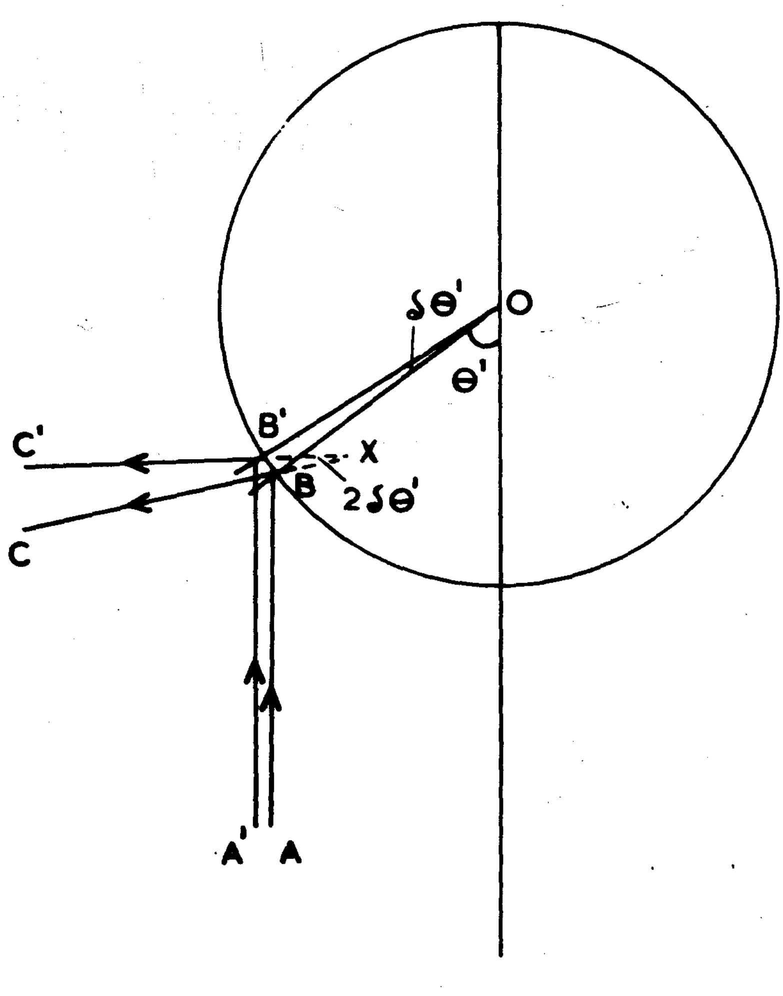

FIGURE 9 (B)