Data-driven control: a behavioral approach (special issue JCW)

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Abstract

In this work, we study the design of a controller using system data. We present three data-driven approaches based on the notion of control as interconnection. In the first approach, we use both the data and representations to compute control variable trajectories that impose a prescribed path on the to-be-controlled variables. The second method is completely data-driven and we prove sufficient conditions for determining a controller directly from data. Finally, we show how to determine a controller directly from data in the case where the control and to-be-controlled variables coincide.

Keywords: Data-driven control, Behavioral approach, Interconnection, Annihilators.

1. Introduction

Over the years, several authors have proposed different methods for using system data in the design of a controller. For example, in [1, 2, 3] system data is used to find suitable control *inputs* and in [4] data is used to falsify a control law. Furthermore, data-driven control techniques have been applied in different applications and processes such as real-time, fault-tolerant controller design for electrical circuits [5], on-line data-driven control switching [4] and data-driven fault tolerant control design, see [6].

In this paper, we show how to find a controller directly using system data. Our solutions are based on the *behavioral framework* like in [3], but we do not assume a priori an input/output partition of variables. We use the *interconnection paradigm*, see [7, 8]. Most importantly, in our approach one can also identify a controller *representation* under suitable conditions which will be specified, while in [3] the aim was to design a control *input*. Furthermore, we do not have a prior assumption that the set of admissible control laws is known, as in [4]. Our solutions are off-line, non-iterative and summarised by a step-by-step algorithm.

This paper is organized as follows. In Section 2, we cover some relevant background material. In Section 3, we state formally the problems solved in this paper. In Sections 4, 5 and 6, we present our solutions. In Section 7, we provide some conclusions. All the necessary lemmas and proofs are gathered in Appendix A and Appendix B, respectively.

Notation. $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of real numbers, complex numbers, integers and positive integers, respectively. The space of $\mathtt{w}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{w}}$ and that of $\mathtt{g}\times\mathtt{w}$ real matrices by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}$. When both dimensions are not specified but finite,

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we write $\mathbb{R}^{\bullet\times\bullet}$. The space of real matrices with $\mathtt{g}$ rows and an infinite number of columns is denoted by $\mathbb{R}^{\mathtt{g}\times\infty}$. $I_{\mathtt{w}}$, $0_{\mathtt{w}\times\mathtt{w}}$ denotes $\mathtt{w}\times\mathtt{w}$ identity and zero matrices, respectively. colspan(A) and leftkernel(A) denotes the column span of $A\in\mathbb{R}^{\bullet\times\bullet}$ and the subspace spanned by all vectors v such that $vA=0$, respectively. $\mathrm{col}(A,B)$ is the matrix obtained by stacking $A\in\mathbb{R}^{\bullet\times\mathtt{w}}$ over $B\in\mathbb{R}^{\bullet\times\mathtt{w}}$, and $\mathrm{col}(A_i)_{i=1,\ldots,l}:=\mathrm{col}(A_1,\ldots,A_l)$. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$ and the set of $\mathtt{g}\times\mathtt{w}$ matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$. Let $R=R_0+\cdots+R_L\xi^L\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}$ with $R_L\neq 0$ then L is the degree of R and is denoted by $\deg(R)$. $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$, is closely associated with the *coefficient matrix* $\tilde{R}:=[R_0\ \ldots\ R_L\ 0_{\mathtt{g}\times\mathtt{w}}\ \ldots\ \ldots]$. $\tilde{R}$ has an infinite number of columns, which are zero everywhere except for a finite number of elements. Notice that $R=\tilde{R}\mathrm{col}(I_{\mathtt{w}}\ \ldots\ I_{\mathtt{w}}\xi^L\ 0\ \ldots)$. $\sigma_R\tilde{R}:=[0_{\mathtt{g}\times\mathtt{w}}\ R_0\ \ldots\ R_L\ 0_{\mathtt{g}\times\mathtt{w}}\ \ldots]$ is the *right shift* of $\tilde{R}$ and $\sigma_R^k\tilde{R}$ denotes k right shifts of $\tilde{R}$ where $k\in\mathbb{Z}_+$. The set of all maps from $\mathbb{Z}$ to $\mathbb{R}$ is denoted by $(\mathbb{R})^{\mathbb{Z}}$. The collection of all linear, closed, shift invariant subspaces of $(\mathbb{R}^{\bullet})^{\mathbb{Z}}$ equipped with the topology of pointwise convergence is denoted by $\mathscr{L}^{\bullet}$. The *backward shift* operator σ is defined by $(\sigma f)(t):=f(t+1)$.

2. Linear discrete complete system

We define a *dynamical system* by $\Sigma:=(\mathbb{Z},\mathbb{R}^{\mathtt{w}},\mathfrak{B})$ with $\mathbb{Z}$ the *time axis*, $\mathbb{R}^{\mathtt{w}}$ the *signal space* and $\mathfrak{B}\subseteq(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ the *behavior*. Let $\Delta\in\mathbb{Z}_+$, then the restriction of $\mathfrak{B}$ on the interval $[1,\Delta]$ is defined by

$$\mathfrak{B}_{|[1,\Delta]}:=\{w:[1,\Delta]\to\mathbb{R}^{\mathtt{w}}|\exists\, w'\in\mathfrak{B}\text{ s.t. } w(t)=w'(t)\text{ for all }1\leqslant t\leqslant\Delta\}.$$

Σ is *linear* if $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$, *time-invariant* if $\sigma\mathfrak{B}\subseteq\mathfrak{B}$ and *complete* if $[w\in\mathfrak{B}]\Leftrightarrow[w_{|[1,\Delta]}\in\mathfrak{B}_{|[1,\Delta]}$ for all $\Delta\in\mathbb{Z}]$. Moreover, $\mathfrak{B}\in\mathscr{L}^{\mathtt{w}}$ if and only if there exists $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that $\mathfrak{B}:=\{w:\mathbb{Z}\to\mathbb{R}^{\mathtt{w}}|R(\sigma)w=0\}$, i.e. $\mathfrak{B}=\ker(R(\sigma))$. R is called a *kernel representation* of $\mathfrak{B}$ and is *minimal* if no other kernel representation of $\mathfrak{B}$ has less than $\mathtt{g}$ rows. $\Sigma_{\mathtt{L}}:=(\mathbb{Z},\mathbb{R}^{\mathtt{w}},\mathbb{R}^{\mathtt{l}},\mathfrak{B}_{full})$ is a dynamical system with *latent variables*. $\mathfrak{B}_{full}$ is called the *full behavior* and consists of all trajectories (w,ℓ) with w a *manifest variable* trajectory and ℓ a *latent variable* trajectory. Let $R\in\mathbb{R}^{\mathtt{d}\times\mathtt{w}}[\xi]$ and $M\in\mathbb{R}^{\mathtt{d}\times\mathtt{l}}[\xi]$ then $\mathfrak{B}_{full}\in\mathscr{L}^{\mathtt{w}+\mathtt{l}}$ admits a representation of the form $R(\sigma)w=M(\sigma)\ell$, called a *hybrid representation*. It has been shown in [9] that $\mathfrak{B}_{full}$ induces a *manifest behavior* defined by $\mathfrak{B}:=\{w\in(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}\mid\exists\ell\in(\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}}\, s.t.\,(w,\ell)\in\mathfrak{B}_{full}\}$. $\mathfrak{B}$ is obtained by using the projection operator $\pi_w:(\mathbb{R}^{\mathtt{w}}\times\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}}\to(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ defined by $w:=\pi_w(w,\ell)$, hence $\mathfrak{B}=\pi_w(\mathfrak{B}_{full})$.

Let $w_1,w_2\in\mathfrak{B}$, then $\mathfrak{B}$ is *controllable* if there exists $t_1\geq 0$ and $w\in\mathfrak{B}$ such that $w(t)=w_1(t)$ for $t\leq 0$ and $w(t)=w_2(t-t_1)$ for $t\geq t_1$. Equivalently, $\mathfrak{B}=\ker(R(\sigma))$ is controllable if and only if $R(\lambda)$ is full row rank for all $\lambda\in\mathbb{C}$. We denote by $\mathscr{L}^{\mathtt{w}}_{contr}$ the collection of all controllable elements of $\mathscr{L}^{\mathtt{w}}$. Let $(w_1,w_2)\in\mathfrak{B}$, w_2 is *observable* from w_1 if there exists $f:(\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}}\to(\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}}$ such that $w_2=f(w_1)$. Let $\mathfrak{B}$ be described by $R_1(\sigma)w_1=R_2(\sigma)w_2$, with $R_1\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}_1}[\xi]$ and $R_2\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}_2}[\xi]$, then w_2 is observable from w_1 if and only if $R_2(\lambda)$ is full column rank for all $\lambda\in\mathbb{C}$, see [10].

$\mathfrak{B}$ is associated with a number of integer invariants, [10]. The following are of interest in this paper. Let $w\in\mathfrak{B}$, then a partition of $w:=(w_1,w_2)$ is an *input/output* partition if w_1 is *maximally free*, i.e. $\pi_{w_1}(\mathfrak{B})=(\mathbb{R}^{\bullet})^{\mathbb{Z}}$ and w_2 contains no free components. w_1 is the input and w_2 output. We denote by $\mathtt{p}(\mathfrak{B})$ and $\mathtt{m}(\mathfrak{B})$ the *output* and *input cardinality* (the number of outputs or inputs), respectively. The smallest integer L such that $[w_{|[t,t+L]}\in\mathfrak{B}_{|[t,t+L]}$ for all $t\in\mathbb{Z}]\Rightarrow[w\in\mathfrak{B}]$ is called the *lag* and denoted by $\mathtt{L}(\mathfrak{B})$. $\mathtt{n}(\mathfrak{B})$ denotes the *McMillan degree*, i.e. the smallest state-space dimension among all possible state representations of $\mathfrak{B}$. Finally, $\mathtt{l}(\mathfrak{B})$ denotes the *shortest lag* described as follows. Let $\mathfrak{B} = \ker(R(\sigma))$ and define the degree of each row of R to be the largest degree of the entries. Then the minimum of degrees of the rows of R is the minimal lag associated with R. $\mathtt{l}(\mathfrak{B})$ is smallest possible minimal lag over all R such that $\mathfrak{B} = \ker(R(\sigma))$.

2.1. Annihilators and fundamental lemma

The module of *annihilators* associated with $\mathfrak{B}$ is defined by $\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{1\times \mathtt{w}}[\xi] | n(\sigma)\mathfrak{B} = 0\}$. If $\mathfrak{B} = \ker(R(\sigma))$ then $\mathfrak{N}_{\mathfrak{B}}$ equals the $\mathbb{R}[\xi]$-submodule of $\mathbb{R}^{1\times \mathtt{w}}[\xi]$ generated by the rows of R, see [11]. We denote the set of annihilators of $\mathfrak{B}$ of degree less than $j \in \mathbb{Z}_+$ by $\mathfrak{N}^j_{\mathfrak{B}} := \{r \in \mathbb{R}^{1\times \mathtt{w}}[\xi] | r \in \mathfrak{N}_{\mathfrak{B}}$ and r has degree $\leqslant j\}$. Let $r_1, \dots r_i \in \mathfrak{N}^j_{\mathfrak{B}}$ and $\tilde{r}_1 \dots \tilde{r}_i$ be the coefficients of $r_1, \dots r_i$; then $\tilde{\mathfrak{N}}^j_{\mathfrak{B}}$ denotes the set containing $\tilde{r}_1 \dots \tilde{r}_i$.

Definition 1. *Let $L \in \mathbb{Z}_+$. The* Hankel matrix *associated with a vectors $w(1), \dots w(T)$ for $T > L$ is defined by*

$$\mathfrak{H}_L(w) := \begin{bmatrix} w(1) & w(2) & \dots & w(T-L+1) \\ w(2) & w(3) & \dots & w(T-L+2) \\ \vdots & \vdots & \dots & \vdots \\ w(L) & w(L+1) & \dots & w(T) \end{bmatrix}.$$

$\mathfrak{H}_{L,J}(w)$ is the *Hankel matrix* with L block rows and J columns.

Definition 2. *A vector $\tilde{u} = \tilde{u}(1), \tilde{u}(2), \dots, \tilde{u}(T)$ is* persistently exciting *of order L if $\mathfrak{H}_L(\tilde{u})$ is full row rank.*

Now we state the *"fundamental lemma"* cf. [12].

Lemma 1. *Assume $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$. Let $\tilde{w} = \tilde{w}(1), \tilde{w}(2), \dots, \tilde{w}(T) := \mathrm{col}(\tilde{u}, \tilde{y}) \in \mathfrak{B}_{|[1,T]}$ such that $\tilde{u}(k) \in \mathbb{R}^{\mathtt{m}(\mathfrak{B})}$ is an input and $\tilde{y}(k) \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})}$ an output, for $1 \leqslant k \leqslant T$. Finally, let $L \in \mathbb{Z}_+$ be such that $L > \mathtt{L}(\mathfrak{B})$. If $\tilde{u}$ is persistently exciting of order at least $L + \mathtt{n}(\mathfrak{B})$, then colspan$(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}_{|[1,L]}$ and leftkernel$(\mathfrak{H}_L(\tilde{w})) = \tilde{\mathfrak{N}}^L_{\mathfrak{B}}$.*

Proof. See Theorem 1 of [12].

Under the conditions of Lemma 1, then for all $\tilde{w}' \in \mathfrak{B}_{|[1,L]}$ there exists $\tilde{v} \in \mathbb{R}^{T-L+1}$ such that $\tilde{w}' = \mathfrak{H}_L(\tilde{w})\tilde{v}$. Moreover, we can recover from $\tilde{w}$ the laws of the system that generated $\tilde{w}$. This leads us to the following definition.

Definition 3. *$\tilde{w} \in \mathfrak{B}$ is* sufficiently informative *about $\mathfrak{B}$ if colspan$(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}_{|[1,L]}$.*

2.2. Interconnection

We introduce some relevant concepts of control by interconnection, see [7, 8]. Let c and w denote the *control* and the *to-be-controlled* variables, respectively. Let the to-be-controlled *plant full behavior* be defined by

$$\mathcal{P}_{full} := \{(w, c) : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{c}} \mid (w, c) \text{ satisfies the plant equations}\}$$

and the *plant manifest behavior* by

$$\pi_w(\mathcal{P}_{full}) = \mathcal{P} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | \exists\, c \;\text{ s.t. } (w, c) \in \mathcal{P}_{full}\}.$$

Finally, let a controller acting on the control variables be described by the *control behavior*

$$\mathcal{C} := \{c : \mathbb{Z} \to \mathbb{R}^{\mathtt{c}} | c \text{ satisfies the controller equations}\}.$$

The interconnection of the plant and the controller through the control variables denoted by $\mathcal{P}_{full} \wedge_c \mathcal{C}$ is defined by the *full controlled bebavior*,

$$\mathcal{K}_{full} := \{(w, c) : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{c}} \, | (w, c) \in \mathcal{P}_{full} \text{ and } c \in \mathcal{C}\}.$$

$\mathcal{K}_{full}$ induce a *manifest controlled behavior* defined by

$$\mathcal{K} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | \exists\, c \in \mathcal{C} \text{ s.t. } (w, c) \in \mathcal{P}_{full}\} = \pi_w(\mathcal{P}_{full} \wedge_c \mathcal{C}).$$

$\mathcal{K}$ is said to be *implementable* with respect to $\mathcal{P}_{full}$ if there exists a controller $\mathcal{C}$ such that $\mathcal{K} = \mathcal{P}_{full} \wedge_c \mathcal{C}$. It has been proven in Theorem 1 of [13] that $\mathcal{C}$ such that $\mathcal{K} = \mathcal{P}_{full} \wedge_c \mathcal{C}$ exists if and only if $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, where $\mathcal{N} := \{w \in \mathcal{P} | (w, 0) \in \mathcal{P}_{full}\}$. In this paper we are interested in the case when $\mathcal{N} = 0$. Hence, we assume that any sub-behavior of $\mathcal{P}$ is implementable. Moreover, a special interconnection case of interest, called *full interconnection* arises when $w = c$. Under full interconnection the interconnection of the plant and the controller through w is denoted by $\mathcal{P} \wedge_w \mathcal{C}$ and induces a controlled behavior defined by $\mathcal{K} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \, | w \in \mathcal{P} \text{ and } w \in \mathcal{C}\}$.

3. Problem Statements

In this section, we define formally the problems solved in this paper. Let the to-be-controlled system full behavior be

$$\mathcal{P}_{full} = \{(w, c) | R_1(\sigma) w = M_1(\sigma) c\} \tag{1}$$

with $R_1 \in \mathbb{R}^{\mathtt{p} \times \mathtt{w}}[\xi]$ and $M_1 \in \mathbb{R}^{\mathtt{p} \times \mathtt{c}}[\xi]$. Assume $R_1(\sigma) w = M_1(\sigma) c$ is minimal and that c is observable from w. Let the manifest behavior and the desired controlled behavior be

$$\mathcal{P} = \{w | R_2(\sigma) w = 0\} \tag{2}$$

$$\mathcal{K} := \{w | D_1(\sigma) w = 0\} \tag{3}$$

respectively, with both $R_2 \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$ and $D_1 \in \mathbb{R}^{\mathtt{t} \times \mathtt{w}}[\xi]$ being minimal. Assume that $\mathcal{P} \in \mathscr{L}^{\mathtt{w}}_{contr}$ and let a to-be-designed controller that implements $\mathcal{K}$ be $\mathcal{C}$. We present a solution for the following problems.

Problem 1. "Prescribed path" case. *Given*

- *an observed infinite trajectory* $\text{col}(w, c) \in \mathcal{P}_{full}$*;*
- *a prescribed trajectory* $w_{pre} \in \mathcal{K}_{|[t_0, t_1]}$ *with* $t_0, t_1 \in \mathbb{N}$*,* $t_0 \leqslant t_1$*; and*
- R_1, M_1 *in* (1) *and* D_1 *in* (3).

Find a control variable trajectory $c_d \in \mathcal{C}$*, such that there exists* $w_d : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}}$ *such that*

a. $col(w_d, c_d) \in \mathcal{K}_{full}$

b. $w_{d|[t_0,t_1]} = w_{pre}$.

To find c_d, we use D_1 to compute w_d such that $w_{d|[t_0,t_1]} = w_{pre}$. Then, under the assumption that c is observable from w, we find c_d using R_1 and M_1.

Problem 2. General interconnection case. *Given observed infinite, sufficiently informative* $col(\tilde{w}, \tilde{c}) \in \mathcal{P}_{full}$ *and* $\tilde{w}_d \in \mathcal{K}$*, find a controller* $\mathcal{C}$ *such that* $\mathcal{P}_{full} \wedge_c \mathcal{C} = \mathcal{K}$.

To solve this problem, we find a control variable trajectory $\tilde{c}_d \in \mathcal{C}$ using the given trajectories and determine under which conditions $\tilde{c}_d$ is sufficiently informative about $\mathcal{C}$, such that standard procedures can be applied to find a representation of $\mathcal{C}$.

Problem 3. Full interconnection case. *Given observed* $\tilde{w} \in \mathcal{P}$ *and* $\tilde{w}_d \in \mathcal{K}$*, find a controller* $\mathcal{C}$ *such that* $\mathcal{P} \wedge_w \mathcal{C} = \mathcal{K}$ *from* $\tilde{w}$ *and* $\tilde{w}_d$.

Let $\mathfrak{N}_{\mathcal{C}}$ be the module of annihilators of $\mathcal{C}$. We aim to use $\tilde{w}$ and $\tilde{w}_d$ to find a set of generators for $\mathfrak{N}_{\mathcal{C}}$.

Remark 1. We assume that observed trajectories are infinitely long. In practical applications the observed trajectories have finite length. The problem of consistency, i.e. the convergence of the identified system to the "true system" as the length of observed trajectories tends to infinity, is of paramount importance. This is a matter for future research.

4. "Prescribed path" solution

We present a solution to Problem 1, which is summarized in Algorithm 1 on p. 7. To find a solution, it is necessary to verify that $\mathcal{K} \subseteq \mathcal{P}$ using the given information. Following from Theorem 2.5.4 in [14], $\mathcal{K} \subseteq \mathcal{P}$ if and only if there exists $F \in \mathbb{R}^{\mathtt{g}\times\mathtt{t}}[\xi]$ such that $R_2 = FD_1$, otherwise there is no solution to Problem 1. Notice that standard procedures can be applied to compute R_2 from (1), see for example elimination in [14].

Now, we show how to find w_d such that $w_{d|[t_0,t_1]} = w_{pre}$ using D_1 in (3) and $w \in \mathcal{P}$. First we introduce the following important results.

Theorem 1. *Let* $\mathcal{K} = \ker(D_1(\sigma))$*, with* D_1 *minimal, and* $w \in \mathcal{P}$*. Assume that* $\mathcal{K} \in \mathscr{L}^{\mathtt{w}}_{contr}$ *and let a left prime matrix* $Q \in \mathbb{R}^{\mathtt{w}\times\mathtt{t}}[\xi]$ *be such that* $D_1 Q = I_{\mathtt{t}}$*. Then* $Im((I_{\mathtt{w}} - QD_1)(\sigma)) = \ker(D_1(\sigma))$*. Define* w'_d *by*

$$w'_d := (I_{\mathtt{w}} - QD_1)(\sigma)w, \tag{4}$$

then $w'_d \in \mathcal{K}$.

In the following result we prove conditions under which w'_d is sufficiently informative about $\mathcal{K}$. Notice that $w \in \mathcal{P}$ and $w'_d \in \mathcal{K}$ need not necessarily have the same input/output structure, as we show in Lemma 2 in Appendix A. Therefore, we define $\mathrm{col}(u, y) =: \Pi w$ and $\mathrm{col}(u_i, y_i) =: \Pi_i w'_d$, where $\Pi, \Pi_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and u, u_i are inputs. Partition $\Pi_i = \mathrm{col}(\Pi_{iu}, \Pi_{iy})$ compatibly with the partition of $w'_d = \mathrm{col}(u_i, y_i)$ and define $F_u(\xi) := \Pi_{iu} - \Pi_{iu}QD_1$. Finally, Denote by $\mathcal{F}_u$ the $\mathbb{R}[\xi]$-submodule of $\mathbb{R}^{1\times\bullet}[\xi]$ generated by the rows of F_u, and by $\mathfrak{N}_{\mathcal{P}}$ the module of annihilators of $\mathcal{P}$.

Theorem 2. *Assume $\mathcal{P} \in \mathscr{L}^{\mathtt{w}}_{contr}$ and that $w \in \mathcal{P}$ is sufficiently informative about $\mathcal{P}$. If $\mathcal{F}_u \cap \mathfrak{N}_{\mathcal{P}} = \{0\}$ and u is persistently exciting of order at least $\mathtt{L}(\mathcal{P}) + \mathtt{n}(\mathcal{P})$ then u_i persistently exciting of order at least $\mathtt{L}(\mathcal{K}) + \mathtt{n}(\mathcal{K})$.*

Remark 2. The lags $\mathtt{L}(\mathcal{K})$, $\mathtt{L}(\mathcal{P})$ and McMillan degrees $\mathtt{n}(\mathcal{K})$, $\mathtt{n}(\mathcal{P})$ are not known a priori. Therefore, all observed trajectories must be generated with input variable trajectories persistently exciting of some sufficiently high order. Moreover, in the rest of the paper L greater than $\mathtt{L}(\mathcal{K})$ or $\mathtt{L}(\mathcal{P})$ is chosen to be "sufficiently large".

Let $L > \mathtt{L}(\mathcal{K})$, we find $w_d \in \mathcal{K}$ such that $w_{d_{|[t_0,t_1]}} = w_{pre}$ using w'_d. Recall that if w'_d is sufficiently informative then for all $w' \in \mathcal{K}_{|[1,L]}$ there exists a vector v such that $w' = \mathfrak{H}_L(w'_d)v$ (see Lemma 1). Therefore, given $w_{pre} \in (\mathbb{R}^{\mathtt{w}})^{[t_0,t_1]}$ with $1 \leqslant t_0 \leqslant t_1 \leqslant L$, the computation of w_d such that $w_{d_{|[t_0,t_1]}} = w_{pre}$ amounts to finding v if it exists such that $w_d = \mathfrak{H}_L(w'_d)v$. Define $H := \mathfrak{H}_{L,J}(w'_d)$ with $J \in \mathbb{Z}_+$ such that $J \gg L$ and H_1 as the block partition of the rows of H from row $\mathtt{w}t_0$ to row $\mathtt{w}t_1$. Then solve for v in

$$H_1 v = w_{pre}. \tag{5}$$

If (5) has no solution then $w_{pre} \notin \mathcal{K}_{|[t_0,t_1]}$, hence we can not compute $w_d \in \mathcal{K}$ such that $w_{d_{|[t_0,t_1]}} = w_{pre}$. Otherwise, $w_d \in \mathcal{K}$ such that $w_{d_{|[t_0,t_1]}} = w_{pre}$ is defined by

$$w_d := \mathfrak{H}(w'_d)v \tag{6}$$

where $\mathfrak{H}(w'_d) \in \mathbb{R}^{\infty \times J}$. Since $J \gg L$ then H_1 has more columns that rows, if v exists such that (5) holds then it is not unique. Let $\mathscr{A}$ be a matrix whose columns are a basis of $\ker(H_1)$ and $\bar{v}$ be a particular solution of (5). Then the set of all possible solutions for (5) is defined by $\mathscr{S} := \{\bar{v} + \mathscr{A}v | v \in \mathbb{R}^G\}$ where G is the number of columns of $\mathscr{A}$.

Theorem 3. *Assume that $w \in \mathcal{P}$ is sufficiently informative about $\mathcal{P}$ and that $\mathcal{F}_u \cap \mathfrak{N}_{\mathcal{P}} = \{0\}$. Then w'_d in (4) is sufficiently informative about $\mathcal{K}$. Moreover, if $w_{pre} \in \mathcal{K}_{|[t_0,t_1]}$ then w_d defined in (6) belongs to $\mathcal{K}$ with w_{pre} as the prescribed path.*

Now, we find a control variable trajectory c_d corresponding to w_d. Under the assumption that c is observable from w then there exists $O \in \mathbb{R}^{\mathtt{c} \times \mathtt{w}}[\xi]$ such that

$$\mathrm{col}(w, c) \in \mathcal{P}_{full} \Rightarrow c = O(\sigma)w. \tag{7}$$

Let M_1 and R_1 in (1) be minimal. Since c is observable from w, then $M_1(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$, hence M_1 admits a left inverse $K \in \mathbb{R}^{\mathtt{c} \times \mathtt{p}}[\xi]$. Define $O := KR_1$, then O satisfies (7). Consequently, c_d corresponding to w_d is defined by $c_d := O(\sigma)w_d$. Furthermore, if $\mathcal{C}$ implements $\mathcal{K}$ then $c_d \in \mathcal{C}$ as shown in Lemma 4 in Appendix A.

4.1. Example

Consider a system with a hybrid representation

$$\underbrace{\begin{bmatrix} \sigma + \frac{1}{2} & 1 & 0 & 1 \\ 0 & \sigma + \frac{1}{3} & 1 & 0 \\ 0 & 0 & \sigma + \frac{1}{4} & 1 \\ 0 & 0 & 0 & \sigma + \frac{1}{5} \end{bmatrix}}_{R_1} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M_1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{8}$$

Input : R_1,M_1,D_1, $\mathrm{col}(w,c) \in \mathcal{P}_{full}$, t_0, t_1 and w_{pre}
Output : c_d
Assumptions: Theorem 3

1 Verify $\mathcal{K} \subseteq \mathcal{P}$. If $\mathcal{K} \nsubseteq \mathcal{P}$, stop. Otherwise go to step 2.
2 Compute Q such that $D_1 Q = I_{\mathtt{t}}$.
3 Define $w'_d := (I_{\mathtt{w}} - QD_1)(\sigma)w$.
4 Choose L and J such that $L > \mathtt{L}(\mathcal{K})$ (see remark 2) and $J \gg L$.
5 Define $H := \mathfrak{H}_{L,J}(w'_d)$ and H_1 as a partition of rows of H from row $\mathtt{w}t_0$ to row $\mathtt{w}t_1$.
6 Solve $H_1 \upsilon = w_{pre}$ for υ.
7 **if** *no solution for* υ **then**
8     $w_{pre} \notin \mathcal{K}_{|[t_0,t_1]}$ [No Solution for c_d]. Stop.
9 **else**
10   Build $\mathfrak{H}(w'_d) \in \mathbb{R}^{\infty \times J}$;
11   Define $w_d := \mathfrak{H}(w'_d)\upsilon$;
12   Compute K such that $KM_1 = I_{\mathtt{c}}$;
13   Define $O := KR_1$;
14   Compute $c_d = O(\sigma)w_d$.
15 **end**

Algorithm 1: Solution for Problem 1

the desired controlled behavior $\mathcal{K}$ with a representation

$$\underbrace{\begin{bmatrix} \sigma + \frac{1}{2} & 1 & -\sigma - \frac{1}{4} & 0 \\ 0 & s + \frac{1}{3} & 1 & -\sigma - \frac{1}{5} \\ 0 & 0 & \sigma + \frac{1}{6} & 1 \end{bmatrix}}_{D_1} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = 0 \tag{9}$$

and

$$w_{pre} = \begin{bmatrix} 0 & 0 & -0.3090 & -0.4256 & -0.7408 & -0.7841 \\ 0 & 0.1545 & 0.2733 & 0.5267 & 0.6490 & 0.7386 \\ 0 & 0 & -0.1545 & -0.2681 & -0.3598 & -0.4156 \\ 0 & 0.1545 & 0.2939 & 0.4045 & 0.4755 & 0.5000 \end{bmatrix}$$

with $t_0 = 1$ and $t_1 = 6$. By eliminating the control variables in (8) then $\mathcal{P} = \ker(R_2(\sigma))$ with $R_2 = \begin{bmatrix} \sigma + \frac{1}{2} & 1 & -\sigma - \frac{1}{4} & 0 \\ 0 & \sigma + \frac{1}{3} & 1 & -\sigma - \frac{1}{5} \end{bmatrix}$. Therefore, $F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ such that $R_2 = FD_1$ exists, hence $\mathcal{K}$ is implementable. We generate $\mathrm{col}(w,c)$ of length $T = 50000$ by simulation of (8) in `Matlab`, with input c_1, c_2 a realization of white Gaussian noise process to guarantee persistency of excitation (see [9] for details on how to determine transfer functions from (8)). Using `Singular`, rightInverse command we compute Q, then compute QD. In `Matlab` we compute $w'_d = w - \tilde{QD}w$ where $\tilde{QD}$ is the coefficient matrix of QD with 4 block columns. We chose $L = 100$ and $J = 4000$ and H_1 as the first 28 rows of $H := \mathfrak{H}_{L\times J}(w'_d)$. Then we solve υ as in step 6 of Algorithm 1. Continuing with the algorithm, we find w_d in step 11 with $\mathfrak{H}(w'_d) \in \mathbb{R}^{45000\times 4000}$. A left inverse of M_1

is $K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $O := \begin{bmatrix} \xi + \frac{1}{2} & 1 & 0 & 1 \\ 0 & \xi + \frac{1}{3} & 1 & 0 \end{bmatrix}$. We obtain c_d with $c_{d_{|[1,6]}}$ which imposes w_{pre} as

$$c_{d_{|[1,6]}} = \begin{bmatrix} 0 & 0 & -0.0129 & -0.0223 & -0.0300 & -0.0346 \\ 0.1545 & 0.3248 & 0.4633 & 0.5564 & 0.5951 & 0.5755 \end{bmatrix}.$$

5. General interconnection solution

In this section, we present a solution to Problem 2. The solution is summarized in Algorithm 2 on p. 9. The starting point is to verify $\mathcal{K} \subseteq \mathcal{P}$ using $\mathrm{col}(\tilde{w}, \tilde{c})$ and $\tilde{w}_d$. Let $L \in \mathbb{Z}_+$ such that $L > \mathtt{L}(\mathcal{P})$ and $L > \mathtt{L}(\mathcal{K})$. Since $\mathrm{col}(\tilde{w}, \tilde{c})$ and $\tilde{w}_d$ are sufficiency informative then $\mathrm{colspan}(\mathfrak{H}_L(\tilde{w})) = \mathcal{P}_{|[1,L]}$ and $\mathrm{colspan}(\mathfrak{H}_L(\tilde{w}_d)) = \mathcal{K}_{|[1,L]}$. Therefore, to verify that $\mathcal{K} \subseteq \mathcal{P}$ it is suffice to show that $\mathrm{colspan}(\mathfrak{H}_L(\tilde{w})) \supseteq \mathrm{colspan}(\mathfrak{H}_L(\tilde{w}_d))$. This is done by computing *principal angles* (see [15]) between $\mathrm{colspan}(\mathfrak{H}_L(\tilde{w}))$ and $\mathrm{colspan}(\mathfrak{H}_L(\tilde{w}_d))$. If the largest principal angle is equal to zero then $\mathcal{K} \subseteq \mathcal{P}$ (see Theorem 12.4.2 of [16]).

Now, to find a control variable trajectory that corresponds to the given $\tilde{w}_d \in \mathcal{K}$, we find an observability map $\mathsf{O} \in \mathbb{R}^{\bullet \times \bullet}$ using $\mathrm{col}(\tilde{w}, \tilde{c}) \in \mathcal{P}_{full}$. First we prove necessary and sufficient conditions for an observability map $Y \in \mathbb{R}^{\mathtt{c} \times \mathtt{w}}[\xi]$ such that given $w \in \mathcal{K}$ we can reconstruct a corresponding $c \in \mathcal{C}$.

Proposition 1. *Let* $\mathcal{P}_{full} = \ker([R_1(\sigma) \quad -M_1(\sigma)])$, $\mathcal{K} = \ker(D_1(\sigma))$ *and a controller that implements* $\mathcal{K}$ *be* $\mathcal{C} = \ker(C_1(\sigma))$. *Assume c is observable from w, then the following statements are equivalent*

1. $Y \in \mathbb{R}^{\mathtt{c} \times \mathtt{w}}[\xi]$ *defined by* $Y(\xi) := N(\xi)R_1(\xi) + G(\xi)D_1(\xi)$ *where* $G \in \mathbb{R}^{\mathtt{c} \times \bullet}[\xi]$ *and* $N \in \mathbb{R}^{\mathtt{c} \times \bullet}[\xi]$, *induces an observability map,*

2. *there exists* $F \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ *such that* $N(\xi)M_1(\xi) = I + F(\xi)C_1(\xi)$.

Let $Y \in \mathbb{R}^{\mathtt{c} \times \mathtt{w}}[\xi]$ satisfy the conditions of Proposition 1 and $L \in \mathbb{Z}_+$ satisfy $L > \mathtt{L}(\mathcal{P})$, $L > \mathtt{L}(\mathcal{K})$ and $L \gg \deg(Y)$. Let $\mathfrak{H}_L(\tilde{c})$, $\mathfrak{H}_L(\tilde{w})$ be the Hankel matrices associated with $\tilde{w}$ and $\tilde{c}$, respectively, both with L block rows and an infinite number of columns. Then a solution for $\mathsf{O} \in \mathbb{R}^{L \times L}$ in

$$\mathfrak{H}_L(\tilde{c}) = \mathsf{O}\mathfrak{H}_L(\tilde{w}) \tag{10}$$

induces an observability map, as we show in Lemma 5 in Appendix A. Consequently, the Hankel matrix of the control variable trajectory $\tilde{c}_d$ corresponding to $\tilde{w}_d$ is defined by

$$\mathfrak{H}_L(\tilde{c}_d) := \mathsf{O}\mathfrak{H}_L(\tilde{w}_d). \tag{11}$$

Furthermore, if a controller $\mathcal{C}$ implements $\mathcal{K}$ then $\tilde{c}_d \in \mathcal{C}$, see Lemma 6 in Appendix A.

In the following result we prove sufficient conditions for $\tilde{c}_d$ to be sufficiently informative about $\mathcal{C}$. Let $\Pi_d \in \mathbb{R}^{\mathtt{c} \times \mathtt{c}}$,$\Pi_1 \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ be such that $(\tilde{c}_u, \tilde{c}_y) = \Pi_d \tilde{c}_d$ and $(\tilde{w}_u, \tilde{w}_y) =: \Pi_1 \tilde{w}_d$ where $\tilde{c}_u$ and $\tilde{w}_u$ are inputs. Partition $\Pi_d := \mathrm{col}(\Pi_{du}, \Pi_{dy})$ compatibly with partitions of $\tilde{c}_d$. Now, let Y satisfying conditions of Proposition 1. Define $Y_u := \Pi_{du} Y$ and denote by $\mathcal{Y}_u$ the $\mathbb{R}[\xi]$-submodule of $\mathbb{R}^{1 \times \bullet}[\xi]$ generated by the rows of Y_u and by $\mathfrak{N}_{\mathcal{K}}$ the module of annihilators of $\mathcal{K}$. Finally, let $\tilde{Y}$ be the coefficient of matrix of Y with finite number L block-columns where $L > \mathtt{L}(\mathcal{C})$.

Theorem 4. *Assume that a controller $\mathcal{C}$ implements $\mathcal{K}$, that c is observable from w and that $\mathtt{O}$ induces an observability map. Let $\mathcal{K} \in \mathscr{L}^{\mathtt{w}}_{contr}$, $\tilde{w}_d \in \mathcal{K}$ and $\tilde{c}_d \in \mathcal{C}$ whose Hankel matrix is defined in* (11). *If $\mathcal{Y}_u \cap \mathfrak{N}_{\mathcal{K}} = \{0\}$ and $\tilde{w}_u$ is persistently exciting of order at least $\mathtt{L}(\mathcal{K}) + \mathtt{n}(\mathcal{K})$ then $\tilde{c}_u$ is persistently exciting of order at least $\mathtt{L}(\mathcal{C}) + \mathtt{n}(\mathcal{C})$.*

Remark 3. Note that it is not straightforward to verify the assumption of Theorem 4 from data. Therefore verifying that $\tilde{c}_u$ is persistently exciting can be done by determining which rows of $\mathfrak{H}_L(\tilde{c}_d)$ corresponds to the input variables (see steps 1)-3) of Algorithm 2 of [17]). Now let $\mathfrak{H}_{\bullet\mathtt{m}(\mathcal{C})}(\tilde{c}_u)$ be the rows of $\mathfrak{H}_L(\tilde{c}_d)$ corresponding to the input variables, if $\mathfrak{H}_{\bullet\mathtt{m}(\mathcal{C})}(\tilde{c}_u)$ is full row rank then $\tilde{c}_u$ is persistently exciting.

Input : $\mathrm{col}(\tilde{w}, \tilde{c}) \in \mathcal{P}_{full}$ and $\tilde{w}_d \in \mathcal{K}$.
Output : $\tilde{c}_d \in \mathcal{C}$.
Assumptions: Theorem 4.

Choose L to be sufficiently large (see remark 2).
Build the Hankel matrices: $\mathfrak{H}_L(\tilde{w}), \mathfrak{H}_L(\tilde{c}), \mathfrak{H}_L(\tilde{w}_d)$.
Verify $\mathcal{K} \subseteq \mathcal{P}$
if $\mathcal{K} \subseteq \mathcal{P}$ **then**
    Solve $\mathfrak{H}_L(\tilde{c}) = \mathtt{O}\mathfrak{H}_L(\tilde{w})$ for $\mathtt{O}$;
    Compute $\mathfrak{H}_L(\tilde{c}_d) = \mathtt{O}\mathfrak{H}_L(\tilde{w}_d)$.
else
    $\mathcal{K} \nsubseteq \mathcal{P}$ {No solution for $\tilde{c}_d$}.
end

Algorithm 2: Solution of Problem 2

5.1. Example

Consider a system in subsection 4.1. We generate $\mathrm{col}(\tilde{w}, \tilde{c})$ and $\tilde{w}_d$ both of length $T = 50000$ by simulation of (8) and (9) in `Matlab`, with inputs (c_1, c_2 and w_4 in (8) and (9), respectively) a realization of white Gaussian noise process to guarantee persistency of excitation. We choose $L = 100$ and compute the largest principal angle to be 1.2363×10^{-14} which is approximately zero, therefore we continue with the rest of the Algorithm. Under the assumption of the algorithm, $\tilde{c}_d$ is sufficiently informative and can be used to find representations of $\mathcal{C}$.

To find a representation of $\mathcal{C}$, we build $\mathfrak{H}_{l+1}(\tilde{c}_d)$, where $l \in \mathbb{Z}_+$ is the lag of $\mathcal{C}$, $l = 2$. Then we compute the *singular value decomposition* (SVD) of $\mathfrak{H}_{l+1}(\tilde{c}_d) := U\Sigma V^\top$. Let r be the rank of $\mathfrak{H}_{l+1}(\tilde{c}_d)$. Partition U into $\begin{bmatrix} U_1 & U_2 \end{bmatrix}$ where U_1 has r columns then $U_2^\top$ is the left kernel of $\mathfrak{H}_{l+1}(\tilde{c})$, and we obtain a kernel representation

$$\begin{bmatrix} -0.9356\sigma^2 - 0.3430\sigma - 0.0312 & -0.0780 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0. \tag{12}$$

For comparison, we use polynomial operations to compute a controller representation from (8) and (9). This is done by computing the syzygy of $\mathrm{col}(R_1, D_1')$ where $D_1' = [0\ 0\ \xi + \frac{1}{6}\ 1]$. We obtain

$$\underbrace{\begin{bmatrix} -\sigma^2 - 0.3667\sigma - 0.0333 & -0.0833 \end{bmatrix}}_{C_2} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0. \tag{13}$$

Equations (12) and (13) represent the same behavior because there exists a nonsingular, square matrix with constant determinant $U \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $C_2 = UC_1$, see [8]. In this case $U = [1.0688]$.

6. Full interconnection solution

Finally, we present a solution to Problem 3. As in Section 5 we verify that $\mathcal{K} \subseteq \mathcal{P}$ using principal angles. Now, let $\mathfrak{N}_{\mathcal{P}}, \mathfrak{N}_{\mathcal{K}}$ and $\mathfrak{N}_{\mathcal{C}}$ denote the module of annihilators of $\mathcal{P}, \mathcal{K}$ and $\mathcal{C}$, respectively. To find a set of generators of $\mathfrak{N}_{\mathcal{C}}$ using $\tilde{w}$ and $\tilde{w}_d$, consequently finding $\mathcal{C}$, under conditions of Lemma 7 in Appendix A we find bases generators of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{K}}$ using $\tilde{w} \in \mathcal{P}$ and $\tilde{w}_d \in \mathcal{K}$, respectively. Then determining basis generators of $\mathfrak{N}_{\mathcal{C}}$ using bases generators of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{K}}$. This procedure has been summarized in Algorithm 3 on p. 11. Note that in Algorithm 3 we denote by $\mathfrak{N}_{\mathcal{C}}^n$ a set of annihilators of $\mathcal{C}$ of degree n.

In the following result we prove the correctness of Algorithm 3.

Proposition 2. *Let $\tilde{w} \in \mathcal{P}$ and $\tilde{w}_d \in \mathcal{K}$ be sufficiently informative about $\mathcal{P}$ and $\mathcal{K}$. Assume that $\mathcal{P}, \mathcal{K} \in \mathscr{L}^{\mathtt{w}}_{contr}$. Also assume that $r_1, \ldots, r_{\mathtt{g}}$ and $a_1, \ldots, a_{\mathtt{t}}$ in Algorithm 3 are minimum lag bases of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{K}}$, respectively. If a controller $\mathcal{C}$ implements $\mathcal{K}$ via full interconnection, then $\mathfrak{N}_{\mathcal{C}}$ in Algorithm 3 is the module of annihilators of $\mathcal{C}$.*

7. Conclusions

We have shown how to compute control variable trajectories that impose a "prescribed path" on the to-be-controlled variables, using both observed trajectories and system representations, this is summarized in Algorithm 1. We also presented a method of computing control variable trajectory, corresponding to an "example" trajectory of the desired controlled behavior (Algorithm 2). We proved sufficient conditions for such control variable trajectory to be sufficiently informative about a controller, consequently using them to find a controller representation . Finally, we showed how to find generators of the module of annihilators of the controller given to-be-controlled variable trajectory and an "example" trajectory from the desired controlled system (Algorithms 3).

Appendix A. Lemmas

Lemma 2. *Let $\mathcal{P} = \ker(R_2(\sigma))$, where $R_2 \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ induces a minimal representation. Assume $\mathcal{K} \subseteq \mathcal{P}$. Then there exists $D' \in \mathbb{R}^{(\mathtt{t}-\mathtt{g})\times\mathtt{w}}[\xi]$ such that $D_1 = col(R_2, D')$ induces a minimal representation of $\mathcal{K}$. Moreover, $\mathtt{p}(\mathcal{K}) \geq \mathtt{p}(\mathcal{P})$.*

Lemma 3. *Let $\Pi_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ such that $\operatorname{col}(u_i, y_i) =: \Pi_i w'_d$ where $w'_d \in \mathcal{K}$. Define a partition of $\Pi_i := \operatorname{col}(\Pi_{iu}, \Pi_{iy})$ compatibly with the partition of w'_d. Then $\Pi_{iu}(I_{\mathtt{w}} - QD_1)$ is full row rank.*

Lemma 4. *Let $w_d \in \mathcal{K}$ defined in (6) and define $c_d := O(\sigma)w_d$. Under the observability assumption, if a controller $\mathcal{C}$ implements $\mathcal{K}$ then c_d belongs $\mathcal{C}$. Moreover, c_d imposes the prescribed path w_{pre} on the to-be-controlled variable trajectory for the time interval $[t_0, t_1]$.*

Lemma 5. *Assume that $\operatorname{col}(\tilde{w}, \tilde{c}) \in \mathcal{P}_{full}$ is sufficiently informative about $\mathcal{P}_{full}$ and that $Y \in \mathbb{R}^{\mathtt{c}\times\mathtt{w}}[\xi]$ satisfy the conditions of Prop. 1. Then O in (10) is an observability map.*

Input : $\tilde{w} \in \mathcal{P}$ and $\tilde{w}_d \in \mathcal{K}$
Output : $\mathfrak{N}_{\mathcal{C}}$
Assumptions: Lemma 7

1 Verify $\mathcal{K} \subseteq \mathcal{P}$. If $\mathcal{K} \subseteq \mathcal{P}$ continue to step 2 else end.

2 Determinations of bases of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{K}}$

i. Using Algorithm 2 of [17] determine minimum lag bases $r_1, \dots, r_{\mathtt{g}}$ and $a_1, \dots, a_{\mathtt{t}}$ of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{K}}$, respectively.

ii. Define $d_m := \deg(a_m)$ for $m = 1, \dots, \mathtt{t}$, $\mathtt{t} := \{1, 2, \dots, \mathtt{t}\}$ and $\mathtt{g} := \{1, 2, \dots, \mathtt{g}\}$. Let $d = \max(d_1, \dots d_m)$.

3 Compute steps 4-5 recursively starting from $n = 0$ to d.

4 Classifying $r_1, \dots, r_{\mathtt{g}}$ and $a_1, \dots, a_{\mathtt{t}}$ by their lags

i. choose $l_1, \dots l_k \in \mathtt{g}$ such that $r_{l_1}, \dots, r_{l_k}$ are all of lag n. If there is no $r_{l_1}, \dots, r_{l_k}$ of lag n set $k = 0$. Choose $l'_1, \dots l'_q \in \mathtt{t}$ such that $a_{l'_1}, \dots, a_{l'_q}$ are all of lag n. If there is no $a_{l'_1}, \dots, a_{l'_q}$ of lag n set $q = 0$.

5 Compute $\mathfrak{N}_{\mathcal{C}}^n$

if $k = q$ **then**
$\mathfrak{N}_{\mathcal{C}}^n := \{0\}$
else if $k = 0$ *and* $q \neq 0$ **then**
$a_{l'_1}, \dots, a_{l'_q}$ are annihilators of $\mathcal{C}$ of degree n hence $\mathfrak{N}_{\mathcal{C}}^n := \{a_{l'_1}, \dots, a_{l'_q}\}$.
else if $k < q$ **then**
Define the matrix A whose columns are the coefficient of $r_{l_1} \dots r_{l_k}$ by

$$A := \begin{bmatrix} \tilde{r}_{0_{l_1}} & \cdots & \tilde{r}_{0_{l_k}} \\ \vdots & \cdots & \vdots \\ \tilde{r}_{n_{l_1}} & \cdots & \tilde{r}_{n_{l_k}} \end{bmatrix};$$

Define a projection matrix $P := A[A^\top A]^{-1} A^\top$;
Define $H := [\tilde{a}_{l'_1} - P\tilde{a}_{l'_1}, \dots, \tilde{a}_{l'_q} - P\tilde{a}_{l'_q}]$;
Compute x rank of H and compute the SVD of $H = U \Sigma V^\top$;
Partition $U = [U_1 \; U_2]$ where U_1 has x columns;
The columns of U_1, $\tilde{u}_1^\top, \dots \tilde{u}_x^\top$ defines the coefficients of annihilators of $\mathcal{C}$ of degree n hence $\mathfrak{N}_{\mathcal{C}}^n := \{u_1, \dots, u_x\}$.

6 Specification of $\mathfrak{N}_{\mathcal{C}}$

i. Define $\mathfrak{N}_{\mathcal{C}} := \bigcup_{k=0}^{d} \mathfrak{N}_{\mathcal{C}}^k$

Algorithm 3: Solution of Problem 3

Lemma 6. *Let* $\mathrm{col}(\tilde{w}, \tilde{c}) \in \mathcal{P}_{full}$ *and* $\tilde{w}_d \in \mathcal{K}$ *be sufficiently informative about their respective behaviors. Assume* $\mathcal{K} \subseteq \mathcal{P}$ *and let* $\mathtt{O}$ *satisfy conditions of Lemma 5. Under the observability assumption if a controller* $\mathcal{C}$ *implements* $\mathcal{K}$*, then the control variable trajectory* $\tilde{c}_d$ *in* (11) *belongs to* $\mathcal{C}$.

Lemma 7. *Let* $r_1, \ldots, r_\mathtt{t}$ *and* $c_1, \ldots, c_\mathtt{j}$ *be bases generators of* $\mathfrak{N}_\mathcal{P}$ *and* $\mathfrak{N}_\mathcal{C}$*, respectively, where* $\mathtt{t}, \mathtt{j} \in \mathbb{Z}_+$*. If* $\mathcal{C}$ *implements* $\mathcal{K}$ *via full interconnection, then* $r_1, \ldots, r_\mathtt{t}, c_1, \ldots, c_\mathtt{j}$ *is a set of generators of* $\mathfrak{N}_\mathcal{K}$*. Moreover,* $r_1, \ldots, r_\mathtt{t}, c_1, \ldots, c_\mathtt{j}$ *is a basis generators of* $\mathfrak{N}_\mathcal{K}$ *if and only if* $\mathfrak{N}_\mathcal{P} \cap \mathfrak{N}_\mathcal{C} = \{0\}$.

Appendix B. Proofs

Proof of Theorem 1. The existence of Q such that $D_1 Q = I_\mathtt{t}$ follows from the fact that D_1 is minimal and $\mathcal{K} \in \mathscr{L}^\mathtt{w}_{contr}$, consequently $D_1(\lambda)$ is full row rank for all $\lambda \in \mathbb{C}$, therefore D_1 admits a right inverse. To show the inclusion $\mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma)) \subseteq \ker(D_1(\sigma))$, for all $w \in \mathcal{P}$ define $w' := (I_\mathtt{w} - QD_1)(\sigma)w$. Now compute $D_1(\sigma)w' = D_1(\sigma)((I_\mathtt{w} - QD_1)(\sigma)w) = D_1(\sigma)w - D_1QD_1(\sigma)w$. Since $D_1Q = I_\mathtt{t}$ it follows that $D_1(\sigma)w' = D_1(\sigma)w - D_1(\sigma)w = 0$. Hence $\mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma)) \subseteq \ker(D_1(\sigma))$. To prove the converse inclusion, assume by contradiction that there exists $w' \in \mathcal{K}$ such that $w' \notin \mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma))$. Now $(I_\mathtt{w} - QD_1)(\sigma)w' = w' - (QD_1)(\sigma)w' = w'$, which implies that $w' \in \mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma))$. Therefore, $\mathrm{Im}((I - QD_1)(\sigma)) = \ker(D_1(\sigma))$. To prove $w'_d \in \mathcal{K}$, notice that since $\mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma)) = \ker(D_1(\sigma))$ and $\mathcal{K} = \ker(D_1(\sigma))$ it follows that $w'_d \in \mathcal{K}$.

Proof of Lemma 2. Let $\mathfrak{N}_\mathcal{K}$ and $\mathfrak{N}_\mathcal{P}$ denote the module of annihilators of $\mathcal{K}$ and $\mathcal{P}$, respectively. By the assumption that $\mathcal{K} \subseteq \mathcal{P}$ then $\mathfrak{N}_\mathcal{P} \subseteq \mathfrak{N}_\mathcal{K}$. Define $R_2 := \mathrm{col}(r_1, \ldots, r_\mathtt{g})$. Since R_2 is minimal then $r_1, \ldots, r_\mathtt{g}$ is a basis of $\mathfrak{N}_\mathcal{P}$. Now since $\mathfrak{N}_\mathcal{P} \subseteq \mathfrak{N}_\mathcal{K}$, then there exists $r'_{\mathtt{g}+1}, \ldots r'_\mathtt{t}$ such that $r_1, \ldots, r_\mathtt{g}, r'_{\mathtt{g}+1}, \ldots r'_\mathtt{t}$ is a basis of $\mathfrak{N}_\mathcal{K}$. Define $D' := \mathrm{col}(r'_{\mathtt{g}+1}, r'_{\mathtt{g}+2}, \ldots r'_\mathtt{t})$. Now the rows of $D_1 = \mathrm{col}(R_2, D')$ span $\mathfrak{N}_\mathcal{K}$ and are a basis of $\mathfrak{N}_\mathcal{K}$ hence D_1 is minimal. Now notice that $\mathtt{p}(\mathcal{P}) = \mathtt{g}$ and $\mathtt{p}(\mathcal{K}) = \mathtt{g} + (\mathtt{t} - \mathtt{g}) = \mathtt{t}$ hence $\mathtt{t} > \mathtt{g}$ means that $\mathcal{K}$ has more output variables.

Proof of Theorem 2. Let $L \in \mathbb{Z}_+$ be such that $L - \deg(F_u) \geqslant \mathtt{L}(\mathcal{K}) + \mathtt{n}(\mathcal{K})$. Denote by $\tilde{F}_u$ the coefficient matrix of F_u with a finite number L of block-columns. Define $\mathfrak{H}_{L-\deg(F_u)}(u_i) := \mathrm{col}(\sigma_R^k \tilde{F}_u)_{k=0,\ldots,L-1-\deg(F_u)} \mathfrak{H}_L(w)$. Assume by contradiction that u_i is not persistently exciting, then there exists a non-zero vector $\tilde{\alpha} \in \mathbb{R}^{1 \times (L-\deg(F_u))\mathtt{m}(\mathcal{K})}$ such that $\tilde{\alpha}\mathfrak{H}_{L-\deg(F_u)}(u_i) = 0$. Consequently, $\tilde{\alpha}\mathrm{col}(\sigma_R^k \tilde{F}_u)_{k=0,\ldots,L-1-\deg} \in \mathrm{leftkernel}(\mathfrak{H}_L(w))$. Now let $\alpha \in \mathbb{R}^{1\times\bullet}[\xi]$ to be the polynomial vector whose coefficient matrix is $\tilde{\alpha}$. Since u is persistently exciting and $\mathcal{P} \in \mathscr{L}^\mathtt{w}_{contr}$ then $\mathrm{leftkernel}(\mathfrak{H}(w)) = \mathfrak{N}_\mathcal{P}$. Therefore, $\alpha F_u \in \mathfrak{N}_\mathcal{P}$ moreover, $\alpha F_u \neq 0$ (see Lemma 3), which contradicts $\mathcal{F}_u \cap \mathfrak{N}_\mathcal{P} = \{0\}$.

Proof of Lemma 3. From Theorem 1 the fact that $\mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma)) = \ker(D_1(\sigma))$ implies that $\Pi_{iu}\mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma)) = (\mathbb{R}^{\mathtt{m}(\mathcal{K})})^\mathbb{Z}$. Now, since $\Pi_{iu}\mathrm{Im}((I_\mathtt{w} - QD_1)(\sigma)) = \mathrm{Im}(\Pi_{iu}(I_\mathtt{w} - QD_1)(\sigma))$ then $\Pi_{iu}(I_\mathtt{w} - QD_1)$ is full row rank. Furthermore, $\Pi_{iu}(I_\mathtt{w} - QD_1)(\sigma)$ is surjective (see [10]).

Proof of Theorem 3. The fact that w'_d in (4) is sufficiently informative about $\mathcal{K}$ follows from Theorem 2, therefore $\mathrm{colspan}(\mathfrak{H}(w'_d)) = \mathcal{K}$. Now since $w_{pre} \in \mathcal{K}_{|[t_0,t_1]}$ then $\upsilon \in \mathscr{S}$

exists such that (5) holds, therefore $\mathfrak{H}(w'_d)v = w_d$. Let H_1 as in (5) and $\mathfrak{H}(w'_d)_{|[t_0,t_1]}$ be the block rows of $\mathfrak{H}(w'_d)$ from row $\mathtt{w}t_0$ to row $\mathtt{w}t_1$. Then $\mathfrak{H}(w'_d)_{|[t_0,t_1]} = H_1$ which implies that $w_{d_{|[t_0,t_1]}} = w_{pre}$.

Proof of Lemma 4. The fact that $w_d \in \mathcal{K}$ follows from Theorem 3. By observability, c_d corresponds to $w_d \in \mathcal{K}$ and since $\mathcal{C}$ implements $\mathcal{K}$ then $c_d \in \mathcal{C}$. c_d imposing w_{pre} follows from the fact that c_d corresponds to w_d such that $w_{d_{[t_0,t_1]}} = w_{pre}$.

Proof of Proposition 1. To show 2) $\Rightarrow$ 1) let $(w,c) \in \mathcal{P}_{full}$ such that $w \in \mathcal{K}$ then $R_1(\sigma)w = M_1(\sigma)c$, moreover $C_1(\sigma)c = 0$. Now,

$$\begin{aligned} R_1(\sigma)w &= M_1(\sigma)c \\ (NM_1)(\sigma)c &= (NR_1)(\sigma)w \\ &= (NR_1)(\sigma)w + 0 \\ (NM_1)(\sigma)c &= (NR_1)(\sigma)w + (GD_1)(\sigma)w. \end{aligned}$$

Since $N(\xi)M_1(\xi) = I + F(\xi)C_1(\xi)$ then

$$\begin{aligned} c + (FC_1)(\sigma)c &= (NR_1)(\sigma)w + (GD_1)(\sigma)w \\ c + 0 &= (NR_1)(\sigma)w + (GD_1)(\sigma)w \\ c &= (NR_1)(\sigma)w + (GD_1)(\sigma)w \end{aligned}$$

Therefore, Y induces an observability map. To prove 1) $\Rightarrow$ 2), let $(w,c) \in \mathcal{P}_{full}$ such that $w \in \mathcal{K}$. By the assumptions that c is observable from w and Y induces an observability map, it follows that $c = (NR_1)(\sigma)w + (GD_1)(\sigma)w$. Since $w \in \mathcal{K}$ then $D_1(\sigma)w = 0$. Hence $c = (NR_1)(\sigma)w$. Now since $(w,c) \in \mathcal{P}_{full}$ then $R_1(\sigma)w = M_1(\sigma)c$. It follows that $c = (NR_1)(\sigma)w = (NM_1)(\sigma)c$ hence $c = (NM_1)(\sigma)c$. Consequently $(NM_1 - I)(\sigma)c = 0$. Now recall that the controller $\mathcal{C} = \ker(C_1(\sigma))$ implements $\mathcal{K}$, therefore $C_1(\sigma)c = 0$. Since $(NM_1 - I)(\sigma)c = 0$ and $C_1(\sigma)c = 0$, this implies that F exists such that $NM_1 - I = FC$.

Proof of Lemma 5. Let $L \in \mathbb{Z}_+$ satisfy $L > \mathtt{L}(\mathcal{P})$, $L > \mathtt{L}(\mathcal{K})$ and $L \gg \deg(Y)$ and denote by $\tilde{Y}$ the coefficient matrix of Y with finite number L of block-columns. Under the assumption that Y induces an observability map, then $\bar{\mathsf{O}} := \mathrm{col}(\sigma_R^k \tilde{Y})_{k=0,\dots,L-1}$ is a solution of (10), therefore $\mathfrak{H}_L(\tilde{c}) := \bar{\mathsf{O}}\mathfrak{H}_L(\tilde{w})$. Now since $\mathrm{col}(\tilde{w},\tilde{c})$ is sufficiently informative then leftkernel$(\mathfrak{H}_L(\tilde{w})) \neq 0$. Therefore, (10) has infinitely many solutions. Let $\mathscr{K} \in \mathbb{R}^{L\times\bullet}$ be a matrix whose columns are a basis of leftkernel$(\mathfrak{H}_L(\tilde{w}))$. Then the set of solutions of (10) is defined by $\mathsf{G} := \{\bar{\mathsf{O}} + \mathscr{K}\mathscr{T} | \mathscr{T} \in \mathbb{R}^{\bullet\times L}\}$. Let $\mathsf{O} \in \mathsf{G}$, then $\mathsf{O} := \bar{\mathsf{O}} + \mathscr{K}\mathscr{T}$. Compute $\mathsf{O}\mathfrak{H}_L(\tilde{w}) = (\bar{\mathsf{O}} + \mathscr{K}\mathscr{T})\mathfrak{H}_L(\tilde{w}) = \bar{\mathsf{O}}\mathfrak{H}_L(\tilde{w}) + \mathscr{K}\mathscr{T}\mathfrak{H}_L(\tilde{w})$. Notice that $\mathscr{K}\mathscr{T}\mathfrak{H}_L(\tilde{w}) = 0$. Therefore, $\mathsf{O}\mathfrak{H}_L(\tilde{w}) = \bar{\mathsf{O}}\mathfrak{H}_L(\tilde{w}) + 0 = \mathfrak{H}_L(\tilde{c})$. Hence, O induce an observability map.

Proof of Lemma 6. Let $\bar{\mathsf{O}}$ and $\mathscr{K}$ as in Lemma 5. Now since O satisfies conditions of lemma 5 then $\mathsf{O} := \bar{\mathsf{O}} + \mathscr{K}\mathscr{T}$ where $\mathscr{T} \in \mathbb{R}^{\bullet\times L}$. Compute $\mathsf{O}\mathfrak{H}_L(\tilde{w}_d) = \bar{\mathsf{O}}\mathfrak{H}_L(\tilde{w}_d) + \mathscr{K}\mathscr{T}\mathfrak{H}_L(\tilde{w}_d)$. Now since $\mathcal{K} \subseteq \mathcal{P}$, then leftkernel$(\mathfrak{H}_L(\tilde{w})) \subseteq$ leftkernel$(\mathfrak{H}_L(\tilde{w}_d))$. Therefore, $\mathscr{K}\mathscr{T} \in$ leftkernel$(\mathfrak{H}_L(\tilde{w}_d))$. Consequently, $\mathsf{O}\mathfrak{H}_L(\tilde{w}_d) = \bar{\mathsf{O}}\mathfrak{H}_L(\tilde{w}_d) + 0 = \mathfrak{H}_L(\tilde{c}_d)$. Now under the observability assumption and the fact that $\bar{\mathsf{O}} := \mathrm{col}(\sigma_R^k \tilde{Y})_{k=0,\dots,L-1}$ where Y induces an observability map, then $\tilde{c}_d$ belong to a controller $\mathcal{C}$ that implements $\mathcal{K}$.

Proof of Theorem 4. The fact that O induce an observability map follows from Lemma 5 and that $\tilde{c}_d \in \mathcal{C}$ follows from Lemma 6. Therefore, $\mathfrak{H}_L(\tilde{c}_d) = \mathsf{O}\mathfrak{H}_L(\tilde{w}_d) = \bar{\mathsf{O}}\mathfrak{H}_L(\tilde{w}_d) + 0$ where $\bar{\mathsf{O}} := \text{col}(\sigma_R^k \tilde{Y})_{k=0,\dots,L-1}$. Now define $\mathsf{O}_u \in \mathbb{R}^{\bullet \mathtt{m}(\mathcal{C}) \times L}$ by $\mathsf{O}_u := \text{col}(\sigma_R^k \tilde{Y}_u)_{k=0,\dots,L-1}$, furthermore define $\mathfrak{H}_{\bullet \mathtt{m}(\mathcal{C})}(\tilde{c}_u) := \mathsf{O}_u \mathfrak{H}_L(\tilde{w}_d)$. Assume to the contrary that $\tilde{c}_u$ is not persistently exciting, then there exists $\tilde{\alpha} \in \mathbb{R}^{1 \times \bullet \mathtt{m}(\mathcal{C})}$ such that $\tilde{\alpha}\mathfrak{H}_{\bullet \mathtt{m}(\mathcal{C})}(\tilde{c}_u) = 0$. Therefore, $\tilde{\alpha}\mathsf{O}_u \in \text{leftkernel}(\mathfrak{H}_L(\tilde{w}_d))$. Now since $\mathcal{K} \in \mathscr{L}^{\mathtt{w}}_{contr}$ and $\tilde{w}_u$ is persistently exciting then $\text{leftkernel}(\mathfrak{H}_L(\tilde{w}_d)) = \tilde{\mathfrak{N}}^L_{\mathcal{K}}$, hence $\tilde{\alpha}\mathsf{O}_u \in \tilde{\mathfrak{N}}^L_{\mathcal{K}}$. Let $\alpha \in \mathbb{R}^{1\times \bullet}[\xi]$ be the polynomial vector whose coefficient matrix is $\tilde{\alpha}$ then $\alpha Y_u \in \mathfrak{N}_{\mathcal{K}}$. Since $\text{Im}(Y_u(\sigma)) = (\mathbb{R}^{\mathtt{m}(\mathcal{C})})^{\mathbb{Z}}$ then Y_u is full row rank, hence $\alpha Y_u \neq 0$. Consequently, $\alpha Y_u \in \mathfrak{N}_{\mathcal{K}}$ and $\alpha Y_u \neq 0$ hence a contradiction.

Proof of Lemma 7. Define $R_1 := \text{col}(r_1, \dots, r_{\mathtt{t}})$ and $C_1 := \text{col}(c_1, \dots, c_{\mathtt{j}})$ then $\mathcal{P} = \ker(R_1(\sigma))$ and $\mathcal{C} = \ker(C_1(\sigma))$. Under full interconnection $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$, therefore $\mathcal{K} = \ker(R_1(\sigma)) \cap \ker(C_1(\sigma))$. Consequently, $r_1, \dots, r_{\mathtt{t}}, c_1, \dots, c_{\mathtt{j}}$ is generators of $\mathfrak{N}_{\mathcal{K}}$. Furthermore $\mathfrak{N}_{\mathcal{K}} = \mathfrak{N}_{\mathcal{P}} + \mathfrak{N}_{\mathcal{C}}$. Now to prove (IF), let $r_1, \dots, r_{\mathtt{t}}, c_1, \dots, c_{\mathtt{j}}$ be a basis generators of $\mathfrak{N}_{\mathcal{K}}$. Assume to the contrary that there exists a non-zero $\alpha \in \mathbb{R}^{1\times\bullet}[\xi]$ such that $\alpha \in \mathfrak{N}_{\mathcal{P}} \cap \mathfrak{N}_{\mathcal{C}}$. Now since $r_1, \dots, r_{\mathtt{t}}$ and $c_1, \dots, c_{\mathtt{j}}$ are bases generators of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{C}}$, respectively, then $\alpha = \beta_1 r_1 + \dots + \beta_{\mathtt{t}} r_{\mathtt{t}}$ moreover, $\alpha = \beta'_1 c_1 + \dots + \beta'_{\mathtt{j}} c_{\mathtt{j}}$ where $\beta_{1,\dots,\mathtt{t}}, \beta'_{1,\dots,\mathtt{j}} \in \mathbb{R}[\xi]$. Therefore $\beta_1 r_1 + \dots + \beta_{\mathtt{t}} r_{\mathtt{t}} = \beta'_1 c_1 + \dots + \beta'_{\mathtt{j}} c_{\mathtt{j}} \Rightarrow \beta_1 r_1 + \dots + \beta_{\mathtt{t}} r_{\mathtt{t}} - \beta'_1 c_1 - \dots - \beta'_{\mathtt{j}} c_{\mathtt{j}} = 0$. Now by the assumption that $r_1, \dots, r_{\mathtt{t}}, c_1, \dots, c_{\mathtt{j}}$ is a basis generators of $\mathfrak{N}_{\mathcal{K}}$ then $\beta_1 r_1 + \dots + \beta_{\mathtt{t}} r_{\mathtt{t}} - \beta'_1 c_1 - \dots - \beta'_{\mathtt{j}} c_{\mathtt{j}} = 0$ implies that $\beta_{1,\dots,\mathtt{t}}, \beta'_{1,\dots,\mathtt{j}} = 0$. Consequently $\alpha = 0$, therefore $\mathfrak{N}_{\mathcal{P}} \cap \mathfrak{N}_{\mathcal{C}} = \{0\}$. To prove the converse, assume $\mathfrak{N}_{\mathcal{P}} \cap \mathfrak{N}_{\mathcal{C}} = \{0\}$. Suppose $r_1, \dots, r_{\mathtt{t}}, c_1, \dots, c_{\mathtt{j}}$ is not a basis generators of $\mathfrak{N}_{\mathcal{K}}$ then there exist non-zero $\beta_{1,\dots,\mathtt{t}}, \beta'_{1,\dots,\mathtt{j}} \in \mathbb{R}[\xi]$ such that $\beta_1 r_1 + \dots + \beta_{\mathtt{t}} r_{\mathtt{t}} + \beta'_1 c_1 + \dots + \beta'_{\mathtt{j}} c_{\mathtt{j}} = 0$. Now since $r_1, \dots, r_{\mathtt{t}}$ and $c_1, \dots, c_{\mathtt{j}}$ are bases generators of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{C}}$, respectively, and by the assumption that $\mathfrak{N}_{\mathcal{P}} \cap \mathfrak{N}_{\mathcal{C}} = \{0\}$ then $\beta_1 r_1 + \dots + \beta_{\mathtt{t}} r_{\mathtt{t}} + \beta'_1 c_1 + \dots + \beta'_{\mathtt{j}} c_{\mathtt{j}} = 0 \Rightarrow \beta_{1,\dots,\mathtt{t}}, \beta'_{1,\dots,\mathtt{j}} = 0$. Hence $r_1, \dots, r_{\mathtt{t}}, c_1, \dots, c_{\mathtt{j}}$ is a basis of $\mathfrak{N}_{\mathcal{K}}$.

Proof of Proposition 2. The fact that $r_1, \dots, r_{\mathtt{g}}$ and $a_1, \dots, a_{\mathtt{t}}$ in Algorithm 3 are minimum lag bases of $\mathfrak{N}_{\mathcal{P}}$ and $\mathfrak{N}_{\mathcal{K}}$, respectively follows from Theorem 14 of [17]. Denote by $\mathfrak{N}^n_{\mathcal{K}}$, $\mathfrak{N}^n_{\mathcal{P}}$ and $\mathfrak{N}^n_{\mathcal{C}}$ the set of annihilators of degree n. From Algorithm 3 let $a_{l'_1}, \dots, a_{l'_q} \in \mathfrak{N}^n_{\mathcal{K}}$ and $r_{l_1}, \dots, r_{l_k} \in \mathfrak{N}^n_{\mathcal{P}}$. Since $r_1, \dots, r_{\mathtt{g}}$ and $a_1, \dots, a_{\mathtt{t}}$ are bases generators of their respective modules then $a_{l'_1}, \dots, a_{l'_q}$ and $r_{l_1}, \dots, r_{l_k}$ are bases generators of $\mathfrak{N}^n_{\mathcal{K}}$ and $\mathfrak{N}^n_{\mathcal{P}}$, respectively. Moreover, the fact that $a_1, \dots, a_{\mathtt{t}}$ is a basis implies that $\mathfrak{N}_{\mathcal{P}} \cap \mathfrak{N}_{\mathcal{C}} = \{0\}$. Consequently, $\mathfrak{N}^n_{\mathcal{P}} \cap \mathfrak{N}^n_{\mathcal{C}} = \{0\}$ and $\mathfrak{N}^n_{\mathcal{P}} + \mathfrak{N}^n_{\mathcal{C}} = \mathfrak{N}^n_{\mathcal{K}}$. Therefore, in Algorithm 3 if $k = q$ then $a_{l'_1}, \dots, a_{l'_q} \in \mathfrak{N}^n_{\mathcal{P}}$, hence $\mathfrak{N}^n_{\mathcal{C}} = \{0\}$. Furthermore, if $k = 0$ and $q \neq 0$, then $a_{l'_1}, \dots, a_{l'_q} \in \mathfrak{N}^n_{\mathcal{K}}$ such that $a_{l'_1}, \dots, a_{l'_q} \notin \mathfrak{N}^n_{\mathcal{P}}$ implies that $a_{l'_1}, \dots, a_{l'_q} \in \mathfrak{N}^n_{\mathcal{C}}$, therefore $\mathfrak{N}^n_{\mathcal{C}} = \{a_{l'_1}, \dots, a_{l'_q}\}$. Finally $k < q$ means $\mathfrak{N}^n_{\mathcal{K}}$ has more annihilators of degree n than $\mathfrak{N}^n_{\mathcal{P}}$, therefore some of them belong to $\mathfrak{N}^n_{\mathcal{C}}$. Denote by $\tilde{\mathfrak{N}}^n_{\mathcal{P}}$ and $\tilde{\mathfrak{N}}^n_{\mathcal{K}}$ the sets containing $\tilde{a}_{l'_1}, \dots, \tilde{a}_{l'_q}$ and $\tilde{r}_{l_1}, \dots, \tilde{r}_{l_k}$, respectively. Now $\mathfrak{N}^n_{\mathcal{C}} \cap \mathfrak{N}^n_{\mathcal{P}} = \{0\}$ and $\mathfrak{N}^n_{\mathcal{P}} + \mathfrak{N}^n_{\mathcal{C}} = \mathfrak{N}^n_{\mathcal{K}}$ implies that $\tilde{\mathfrak{N}}^n_{\mathcal{C}} \cap \tilde{\mathfrak{N}}^n_{\mathcal{P}} = \{0\}$ and $\tilde{\mathfrak{N}}^n_{\mathcal{P}} + \tilde{\mathfrak{N}}^n_{\mathcal{C}} = \tilde{\mathfrak{N}}^n_{\mathcal{K}}$. Moreover, since $\tilde{r}_{l_1}, \dots, \tilde{r}_{l_k}$ is a basis of $\tilde{\mathfrak{N}}^n_{\mathcal{P}}$ then the projection matrix P exists. Consequently, $\tilde{u}_1^\top, \dots \tilde{u}_x^\top$ are the coefficient vectors of annihilators of $\mathcal{C}$ of lag n. Hence $\mathfrak{N}^n_{\mathcal{C}} = \{u_1, \dots, u_x\}$.

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